SIGNALS and SYSTEMS

reference sheet

by

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Continuous-time Signals and Systems

General / Miscellaneous

1. $\omega_o = \frac{2\pi}{T_o} = 2\pi f_o$ angular or fundamental frequency
2. $f(\Leftrightarrow t) = f(t)$ even function (e.g. $cos\,t$)
3. $f(\Leftrightarrow t) = \Leftrightarrow f(t)$ odd function (e.g. $sin\,t$)
4. $sinc\,z = \frac{sin\,\pi z}{\pi z}$ sinc-function
5. $\int_{\infty}^{\infty} \frac{sin\,ax}{bx} dx = \frac{\pi}{b}$ area under a sinc-function
6. $e^{-t}\delta(t) = e^{-0}\delta(t) = \delta(t)$
7. $\sqrt{\omega^2} = |\omega|$
8. $H_{\text{lpf}}(\omega) = 1 \Leftrightarrow H_{\text{hpf}}(\omega)$
9. $arctan(\frac{\omega}{0}) = \frac{\pi}{2} sgn(\omega)$

Trigonometry

1. $sinx = cos(x \Leftrightarrow \frac{1}{2}\pi)$
2. $sin^2 x + cos^2 x = 1$
3. $cos2x = 2cos^2 x \Leftrightarrow 1 = 1 \Leftrightarrow 2sin^2 x$
4. $sin2x = 2sinxcosx$
5. $cos(\alpha + \beta) = cos\alpha cos\beta \Leftrightarrow sin\alpha sin\beta$
6. $sin(\alpha + \beta) = sin\alpha cos\beta + cos\alpha sin\beta$
7. $2\,cos\alpha\,cos\beta = cos(\alpha + \beta) + cos(\alpha \Leftrightarrow \beta)$
8. $\Leftrightarrow 2\,sin\alpha\,sin\beta = cos(\alpha + \beta) \Leftrightarrow cos(\alpha \Leftrightarrow \beta)$
9. $\frac{d}{d\omega} sin\omega = cos\omega$
10. $\frac{d}{d\omega} cos\omega = \Leftrightarrow sin\omega$
11. $\frac{d}{d\omega} arctan(\omega) = \frac{1}{1+\omega^2}$

Complex Numbers

1. $j = \sqrt{\Leftrightarrow 1}$ imaginary constant
2. $z = a + bj = Re[z] + jIm[z]$ complex number (**rectangular** or **Cartesian** form)
3. $z^* = a \Leftrightarrow bj = Re[z] \Leftrightarrow jIm[z]$ complex conjugate of z
4. $z = |z|e^{j\angle z}$ complex number (**polar** form)
5. $|z| = \sqrt{Re^2[z] + Im^2[z]}$ **magnitude** of complex number z
6. $\angle z = arctan(\frac{Im[z]}{Re[z]})$ **angle** or **phase** of complex number z

7. $e^{jx} = cosx + jsinx$ **Euler's formula**
8. in $\mathbb{C}$: $1 = e^{j0}$, $\Leftrightarrow 1 = e^{j\pi}$, $j = e^{\frac{1}{2}j\pi}$, $\Leftrightarrow j = e^{-\frac{1}{2}j\pi}$ basic points on unit circle in $\mathbb{C}$
9. $e^{jn\pi} = (\Leftrightarrow 1)^n$
10. $cos\,\omega_o t = \frac{1}{2}(e^{j\omega_o t} + e^{-j\omega_o t})$
11. $sin\,\omega_o t = \frac{1}{2j}(e^{j\omega_o t} \Leftrightarrow e^{-j\omega_o t})$
12. $\frac{1}{a+bj} = \frac{a-bj}{a^2+b^2}$
13. $|(a+bj)\cdot(c+dj)| = |(a+bj)|\cdot|(c+dj)|$
14. $z + z^* = 2Re[z]$
15. $zz^* = |z|^2$
16. $\frac{1}{z^*} = (\frac{1}{z})^*$
17. $\angle z^* = \Leftrightarrow \angle z$
18. $\angle\{(a+bj)\cdot(c+dj)\} = \angle\{a+bj\} + \angle\{c+dj\}$
19. $z \Leftrightarrow z^* = 2jIm[z]$
20. $\angle z^{-1} = \Leftrightarrow \angle z$
21. $|z| = |z^*|$
22. $|z^{-1}| = |z|^{-1}$

Singularity Signals

1. $u(t) \;=\; \int_{-\infty}^{t} \delta(s)ds \;=\; 1\; t \geq 0,$ and 0 elsewhere — unit **step** function
2. $r(t) \;=\; \int_{-\infty}^{t} u(s)ds \;=\; tu(t)$ — unit **ramp** function
3. $p(t) \;=\; \int_{-\infty}^{t} r(s)ds \;=\; \frac{1}{2}t^2u(t)$ — unit **parabola** function
4. $\Pi(t) \;=\; u(t+\frac{1}{2}) \Leftrightarrow u(t \Leftrightarrow \frac{1}{2}) \;=\; 1,\; |t| \leq \frac{1}{2},$ and 0 elsewhere — unit **pulse** function
5. $\Lambda(t) \;=\; r(t+1) \Leftrightarrow 2r(t) + r(t \Leftrightarrow 1)$ — unit **triangle** function
6. $\delta(t) \;=\; \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\Pi(\frac{t}{2\epsilon})$ — unit **impulse** or delta function (a.k.a. the "spike")
7. $\dot{\delta}(t) \;=\; \frac{d}{dt}\delta(t)$ — **doublet**

8. $\delta(at) \;=\; \frac{1}{|a|}\delta(t)$, for $a \in \mathbb{R}\backslash\{0\}$
9. $u(at) \;=\; u(t)$
10. $r(at) \;= a\, r(t)$

11. $\int_{-\infty}^{\infty} \delta(t)dt \;=\; 1$
12. $x(t)\delta(t \Leftrightarrow a) \;=\; x(a)\delta(t \Leftrightarrow a)$
13. $\int_{a}^{b} x(t)\delta(t)dt \;=\; x(0)$, if $a < 0 < b$, and 0, elsewhere
14. $\int_{-\infty}^{\infty} x(s)\delta(t \Leftrightarrow s)ds \;=\; x(t)$ — sifting or convolution property (integral)
15. $\int_{-\infty}^{\infty} x(s)\delta^{(n)}(t \Leftrightarrow s)ds \;=\; (\Leftrightarrow 1)^n x^{(n)}(t)$

Power and Energy

1. $E \;=\; \lim_{\tau\to\infty} \int_{-\tau}^{\tau} |x(t)|^2 dt$ — average **energy** (in joules)
2. $P \;=\; \lim_{\tau\to\infty} \frac{1}{2\tau}\int_{-\tau}^{\tau} |x(t)|^2 dt$ — average **power** (in watts)

- **if $0 \leq E < \infty$ then $P = 0$ and the signal is called a energy signal.**
- **if $0 < P < \infty$ then $E = \infty$ and the signal is called a power signal.**
- **the energy of a periodic signal is infinite and the power equals $P = \frac{1}{T_o}\int_{t_o}^{t_o+T_o} |x(t)|^2 dt$, with t_o arbitrary.**

Convolution

- **definition:** $h(t) \otimes x(t) \;=\; \int_{-\infty}^{\infty} h(t \Leftrightarrow \tau)x(\tau)d\tau \;=\; \int_{-\infty}^{\infty} h(\tau)x(t \Leftrightarrow \tau)d\tau$

1. $x_1(t) \otimes x_2(t) \;=\; x_2(t) \otimes x_1(t)$ — **commutativity** property of convolution
2. $x_1(t) \otimes (x_2(t) \otimes x_3(t)) \;=\; (x_1(t) \otimes x_2(t)) \otimes x_3(t))$ — **associativity** property of convolution
3. $x_1(t) \otimes (x_2(t) + x_3(t)) \;=\; x_1(t) \otimes x_2(t) + x_1(t) \otimes x_3(t)$ — **distributivity** property of convolution

4. **trick 1:** $x(t \Leftrightarrow \tau) \;=\; x(t) \otimes \delta(t \Leftrightarrow \tau)$ — delay system
5. **trick 2:** $x(t) \;=\; x(t) \otimes \delta(t),\; \forall x(t)$
6. **trick 3:** convolving a signal $x(t)$ with $u(t)$ means taking the running time integral of $x(t)$.
7. **trick 4:** convolving a signal $x(t)$ with $\frac{d}{dt}\delta(t)$ means taking the derivative of $x(t)$.

8. $u(t) \;=\; u(t) \otimes \delta(t) \;=\; \int_{-\infty}^{t} \delta(s)ds$
9. $r(t) \;=\; u(t) \otimes u(t) \;=\; \int_{-\infty}^{t} u(s)ds$
10. $p(t) \;=\; u(t) \otimes r(t) \;=\; \int_{-\infty}^{t} r(s)ds$

$\uparrow$

and conversely

$\downarrow$

8*. $\delta(t) \;=\; \frac{d}{dt}\delta(t) \otimes u(t) \;=\; \frac{d}{dt}u(t)$

9*. $u(t) \;=\; \frac{d}{dt}\delta(t) \otimes r(t) \;=\; \frac{d}{dt}r(t)$

10*. $r(t) \;=\; \frac{d}{dt}\delta(t) \otimes p(t) \;=\; \frac{d}{dt}p(t)$

General Systems

1.	$y(t) \;=\; H[x(t)]$	system $H[\cdot]$
2.	$H[\alpha_1 x_1(t) \,+\, \alpha_2 x_2(t)] \;=\; \alpha_1 H[x_1(t)] \,+\, \alpha_2 H[x_2(t)]$	condition for **linearity** of $H[\cdot]$
3.	$H[D_\tau[x(t)]] \;=\; D_\tau[H[x(t)]]$	condition for **time-invariance** of $H[\cdot]$
4.	$y(t_o) \;=\; H[x(t_0)]$ depends only on $x(t)$ for $t = t_o$	condition for **memoryless** property of $H[\cdot]$
5.	$y(t_o) \;=\; H[x(t_0)]$ depends only on $x(t)$ for $t \leq t_o$	condition for **causality** of $H[\cdot]$

- **note that $H[x(t)] \;=\; x(t) + 1$ is non-linear.**
- **memoryless systems are causal.**

Linear Systems

1. $g(t,s) \;=\; H[\delta(t \Leftrightarrow s)]$
2. $y(t) \;=\; \int_{-\infty}^{\infty} g(t,\tau)x(\tau)d\tau$
3. check for causality: input $\delta(t \Leftrightarrow s)$ and check if for any s the output is zero $\forall t > s$.
4. H is time-invariant $\Longleftrightarrow$ $g(t+\Delta, s+\Delta) \;=\; g(t,s)$

- **if $H[\cdot]$ is linear and memoryless, then $H[\cdot]$ will be of the form $H[\cdot] \;=\; \alpha(t)x(t)$.**

Linear and Time-Invariant (LTI) Systems

1. $H_1[\cdot]$, $H_2[\cdot]$ LTI $\Leftrightarrow$ $H_1[H_2[\cdot]]$ LTI
2. $H_1[\cdot]$, $H_2[\cdot]$ LTI $\Leftrightarrow$ $(H_1 + H_2)[\cdot]$ LTI

1.	$h(t) \;=\; H[\delta(t)]$	**impulse response** of $H[\cdot]$
2.	$y_{step} \;=\; H[u(t)]$	**step response** of $H[\cdot]$
3.	$y_{ramp} \;=\; H[r(t)]$	**ramp response** of $H[\cdot]$
4.	$y_{parabola} \;=\; H[p[(t)]$	**parabola response** of $H[\cdot]$
5.	$H(\omega) \;=\; \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt \;=\; \lvert H(\omega)\rvert\, e^{j\angle H(\omega)}$	**frequency response** of $H[\cdot]$ (Fourier transform of $h(t)$)
6.	$H(s) \;=\; \int_{0^-}^{\infty} h(t)\, e^{-st}\, dt$	**transfer function** of $H[\cdot]$ (Laplace transform of $h(t)$)
7.	$h(t)$ real $\Rightarrow$ $H(\Leftrightarrow\omega) \;=\; H^*(\omega)$	

8.	$y(t) \;=\; h(t) \otimes x(t) \;=\; \int_{-\infty}^{\infty} h(t \Leftrightarrow \tau)x(\tau)d\tau \;=$	
	$=\; \int_{-\infty}^{\infty} h(\tau)x(t \Leftrightarrow \tau)d\tau \;=\; h(t) \otimes x(t)$	**superposition** or **convolution integral**
9.	$H[e^{j\omega t}] \;=\; H(\omega)\, e^{j\omega t}$	for periodic signals (scaling; eigenfunctions / -value)
10.	$Y(\omega) \;=\; X(\omega)\, H(\omega)$	Fourier transform of (8)
11.	$Y(s) \;=\; X(s)\, H(s)$	Laplace transform of (8)

12. $h(t) \;=\; \frac{d}{dt}\delta(t) \otimes y_{step}(t) \;=\; \frac{d}{dt}y_{step}(t)$

13. $y_{step} \;=\; \frac{d}{dt}\delta(t) \otimes y_{ramp}(t) \;=\; \frac{d}{dt}y_{ramp}(t)$

14. $y_{ramp} = \frac{d}{dt}\delta(t) \otimes y_{parabola}(t) = \frac{d}{dt} y_{parabola}(t)$

$\uparrow$

and conversely

$\downarrow$

12*. $y_{step} = h(t) \otimes u(t) = \int_{-\infty}^{t} h(s)ds$

13*. $y_{ramp} = h(t) \otimes r(t) = h(t) \otimes u(t) \otimes u(t) = y_{step} \otimes u(t) = \int_{-\infty}^{t} y_{step}(s)ds$

14*. $y_{parabola} = h(t) \otimes p(t) = h(t) \otimes r(t) \otimes u(t) = y_{ramp} \otimes u(t) = \int_{-\infty}^{t} y_{ramp}(s)ds$

Frequency, Phase and Magnitude Response

1. $H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$ frequency response is $\mathcal{F}[h(t)]$
2. from an LDE with CC: let $x(t) = e^{j\omega t}$ and $y(t) = H(\omega)e^{j\omega t}$; solve for $H(\omega)$
3. from an LDE with CC: let $X(\omega) = \mathcal{F}[x(t)]$ and $Y(\omega) = \mathcal{F}[y(t)]$; solve for $H(\omega) = \frac{Y(\omega)}{X(\omega)}$, using $\mathcal{F}[\dot{x}(t)] = j\omega X(\omega)$
4. if $H(\omega) = K\frac{1+aj\omega}{1+bj\omega}$, then $|H(\omega)| = |K|\frac{|1+aj\omega|}{|1+bj\omega|} = |K|\frac{\sqrt{1+(a\omega)^2}}{\sqrt{1+(b\omega)^2}}$
5. if $H(\omega) = K\frac{1+aj\omega}{1+bj\omega}$, then $\angle H(\omega) = \angle K + \angle(1+aj\omega) \Leftrightarrow \angle(1+bj\omega) = \angle K + arctan(a\omega) \Leftrightarrow arctan(b\omega)$

BIBO-Stability

- **definition:** system $H[\cdot]$ is BIBO-stable **iff** every bounded input results in a bounded output

1. system $H[\cdot]$ is BIBO-stable **iff** $\int_{-\infty}^{\infty} |h(t)| < \infty$
2. **if** system $H[\cdot]$ is BIBO-stable **then** $H(s) = \frac{N(s)}{D(s)}$ must be proper, i.e. degree $N(s) \leq$ degree $D(s)$
3. **if** causal system $H[\cdot]$ is BIBO-stable **then** all the poles of $H(s)$ must lie in the **open** left-half plane

- **necessary conditions 2 and 3 for stability are sufficient if $H(s)$ is rational (generally the case)**
- **for all stable causal systems, the frequency response exists**

Distortion

- an LTI system is **distortionless** if the output $y(t) = H[x(t)] = K\, x(t \Leftrightarrow \tau)$, a scaled and shifted replica of the input $x(t) \iff$
  $\iff H(\omega) = K\, e^{-j\omega\tau} \iff |H(\omega)|$ is constant and $\angle H(\omega) = \Leftrightarrow\omega\tau$ is linear in ω.
- an LTI system shows **amplitude distortion** when $|H(\omega)|$ varies with ω.
- an LTI system shows **phase distortion** when $\angle H(\omega)$ is non-linear in ω, and then we define $\tau_g(\omega) = \Leftrightarrow\frac{d\angle H(\omega)}{d\omega}$ as the **group delay**.

Minimum Phase

A system $H[\cdot]$ is said to be **minimum phase**, if all poles and zeros of $H(s)$ lie in the open left half plane, so that $H^{-1}(s)$ is also minimum phase; hence a causal system that is minimum phase, is also causally invertible.

Feedback

Given a feedback system block diagram, we can write the equations:

- $e(t) = x(t) \pm y(t) \otimes h_2(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad E(s) = X(s) \pm Y(s)H_2(s)$ (error function);
- $y(t) = e(t) \otimes h_1(t) \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad Y(s) = E(s)H_1(s)$;
- combine: $y(t) = x(t) \otimes h_1(t) \pm y(t) \otimes h_2(t) \otimes h_1(t)$; $\overset{\mathcal{L}}{\longleftrightarrow} \quad Y(s) = X(s)H_1(s) \pm Y(s)H_1(s)H_2(s) \quad \Leftrightarrow \quad \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 \mp H_1(s)H_2(s)}$.
- **in control problems, where we try to optimize feedback, we often use the final value theorem for Laplace transforms;**

Finding the Output of LTI Systems

1. $y(t) = x(t) \otimes h(t)$ **straight convolution**
2. $y(t) = \mathcal{F}^{-1}[X(\omega)H(\omega)]$ **using Fourier transforms**
2. $y(t) = \mathcal{L}^{-1}[X(s)H(s)]$ **using Laplace transforms**
3. if $x(t) = e^{j\omega_o t}$, then $y(t) = H(\omega_o)e^{j\omega_o t}$ **using Fourier eigenfunctions**
4. if $x(t) = e^{st}$, then $y(t) = H(s)e^{st}$ **using Laplace eigenfunctions**
5. if $x(t) = cos(\omega_o t + \phi)$ and $h(t)$ is real, then $y(t) = |H(\omega_o)|cos(\omega_o t + \phi + \angle H(\omega_o))$
   steady-state sinusoidal response
6. if $x(t) = sin(\omega_o t + \phi)$ and $h(t)$ is real, then $y(t) = |H(\omega_o)|sin(\omega_o t + \phi + \angle H(\omega_o))$
   steady-state sinusoidal response
7. if $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_o t}$, then $y(t) = \sum_{n=-\infty}^{\infty} X_n H(n\omega_o)e^{jn\omega_o t} = \sum_{n=-\infty}^{\infty} X_n |H(n\omega_o)| e^{j(n\omega_o t + \angle H(n\omega_o))}$
   using Fourier series

- **the transfer function and consequently the impulse response can be derived from any LDE by assuming zero initial conditions.**

Remarks

- **in LTI systems it makes no difference if we see $h(t)$ as the input signal and $x(t)$ as the impulse response of the LTI system (follows directly from the commutativity property of convolution).**
- **both $h(t)$ and $H(\omega)$ completely characterize an LTI system.**
- **for an LTI system $H[\cdot] : h(t) = 0$ for $t < 0 \Leftrightarrow H[\cdot]$ is causal.**
- **a differentiator ($h(t) = \dot{\delta}(t)$) is causal.**
- **an ideal differentiator $\dot{\delta}(t)$ is a causal system.**
- **$H(\omega)$ real $\Leftrightarrow h(\Leftrightarrow t) = h(t) \Rightarrow H[\cdot]$ is non-causal (unless $h(t) = 0 \; \forall t \neq 0$) $\Longrightarrow$ filters are non-causal**
- **an LTI system converts a periodic input signal into a periodic output signal.**
- **if $H[\cdot]$ is LTI and memoryless, then $h(t) = 0$ for $t \neq 0$ and $H[\cdot]$ will be of the form $H[\cdot] = \alpha x(t)$.**

Fourier Series

Definition of Fourier Series

We can decomposing any *periodic power signal* (with period T_o) into *discrete* frequencies, each a multiple of ω_o, yielding the **Fourier series** representation of that signal:

- $x(t) \;=\; \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_o t}$ **synthesis** equation
- $X_n \;=\; \frac{1}{T_o}\int_{T_o} x(t)e^{-jn\omega_o t}dt$ **analysis** equation

Properties of Fourier Series (given $x(t) \overset{\mathcal{FS}}{\longleftrightarrow} X_n$):

- **real signals:** if $x(t)$ is real, then $X_{-n} \;=\; X_n^*$
- **real and even signals:** if $x(t)$ is real and even in t, then X_n is real and even in n
  $\mathrm{Ev}[x(t)] \overset{\mathcal{FS}}{\longleftrightarrow} \mathrm{Re}[X(\omega)]$ **and** $\mathrm{Re}[x(t)] \overset{\mathcal{FS}}{\longleftrightarrow} \mathrm{Ev}[X(\omega)]$
- **real and odd signals:** if $x(t)$ is real and odd in t, then X_n is imaginary and odd in n
  $\mathrm{Od}[x(t)] \overset{\mathcal{FS}}{\longleftrightarrow} \mathrm{Im}[X(\omega)]$ **and** $\mathrm{Im}[x(t)] \overset{\mathcal{FS}}{\longleftrightarrow} \mathrm{Od}[X(\omega)]$
- **half-wave odd symmetry:** if $x(t \pm \frac{1}{2}T_o) = \Leftrightarrow x(t)$, $\forall t$ (e.g. $x(t) = sint$), then $X_n = 0$ for n even (in particular, $X_0 = 0$)
- **time reversal:** $x(\Leftrightarrow t) \overset{\mathcal{FS}}{\longleftrightarrow} X_{-n}$
- **complex conjugate:** $x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} X_{-n}^*$ **and** $x^*(\Leftrightarrow t) \overset{\mathcal{FS}}{\longleftrightarrow} X_n^*$
- **delay:** $x(t \Leftrightarrow \tau) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jn\omega_o\tau}X_n$
- **differentiation:** $\dot{x}(t) \overset{\mathcal{FS}}{\longleftrightarrow} jn\omega_o X_n$
- **integration:** if $X_o = 0$ then $u(t) \otimes x(t) \overset{\mathcal{FS}}{\longleftrightarrow} \frac{X_n}{jn\omega_o}$, for $n \neq 0$
- **linearity:** $\alpha x(t) + \beta y(t) \overset{\mathcal{FS}}{\longleftrightarrow} \alpha X_n + \beta Y_n$
- **PARSEVAL's identity:** $P \;=\; \sum_{n=-\infty}^{\infty} X_n^* X_n \;=\; \sum_{n=-\infty}^{\infty} |X_n|^2$
- **power at fundamental frequency** of $x(t) \;=\; \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_o t}$ equals $|X_{-1}|^2 \;+\; |X_1|^2$

Some Examples

- **DC:** $x(t) \;=\; 1 \overset{\mathcal{FS}}{\longleftrightarrow} X_o \;=\; 1,\; X_n \;=\; 0$ otherwise
- **cosine:** $cos\,\omega_o t \;=\; \frac{1}{2}(e^{j\omega_o t} \;+\; e^{-j\omega_o t})$
- **sine:** $sin\,\omega_o t \;=\; \frac{1}{2j}(e^{j\omega_o t} \Leftrightarrow e^{-j\omega_o t})$
- **even square wave: period T_o, $x_1(t) = 1$ for $|t| < T_1$ and $x_1(t) = 0$ elsewhere in that period:**
  $x_1(t) \overset{\mathcal{FS}}{\longleftrightarrow} X_n \;=\; \frac{1}{n\pi}\,sin\,n\omega_o T_1$
- **odd square wave: $x_2(t)$ is shifted $x_1(t)$**
- **even triangle wave: $x_3(t)$ is running time integral of $x_1(t)$**
  $x_3(t) \overset{\mathcal{FS}}{\longleftrightarrow} X_n \;=\; \Leftrightarrow\frac{4}{n^2\pi^2}$, for n odd, and $X_n \;=\; 0$, for n even
- **pulse train: $x_4(t)$ is same as $x_1(t)$ but shifted by τ and with added DC (this changes X_0 only):**
  $x_4(t) \;=\; \sum_{k=-\infty}^{\infty} \Pi(\frac{t-kT_o-\tau}{t_o}) \overset{\mathcal{FS}}{\longleftrightarrow} X_n \;=\; \frac{t_o}{T_o}e^{-jn\omega_o\tau}sinc(\frac{nt_o}{T_o})$
- **even impulse train:** $x_5(t) \;=\; \sum_{k=-\infty}^{\infty} \delta(at \Leftrightarrow kT) \overset{\mathcal{FS}}{\longleftrightarrow} X_n \;=\; \frac{1}{|a|T}$

Fourier Transform

Definition and Interpretation

The **Fourier transform** $X(\omega)$ is a decomposition of an **(a)periodic signal** $x(t)$ into $e^{j\omega t}$-type signals with ω real ($x(t)$ and $X(\omega)$ uniquely define each other); we write $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$:

- $x(t) \ = \ \mathcal{F}^{-1}[X(\omega)] \ = \ \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$ **synthesis** equation
- $X(\omega) \ = \ \mathcal{F}[x(t)] \ = \ \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ **analysis** equation

The Fourier transform $X(\omega)$ exists when it's finite for all ω. Allowing δ-functions, we can accept a broader class of signals: sufficient conditions for the existence of the **generalized Fourier transform** of $x(t)$ are (1) $x(t)$ is bounded; and (2) $x(t)$ is periodic ($E = \infty$) with finite power.

The interpretation of the Fourier transform is that $X(\omega)$ contains the frequency content of the signal $x(t)$ for each ω. For example, consider $x(t) \ = \ 1 \stackrel{\mathcal{F}}{\longleftrightarrow} \delta(\omega)$ (DC only), $x(t) \ = \ e^{j\omega_o t} \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega \Leftrightarrow \omega_o)$ (single frequency at ω_o), and finally $x(t) \ = \ cos\,\omega_o t \stackrel{\mathcal{F}}{\longleftrightarrow} \pi[\delta(\omega + \omega_o) \ + \ \delta(\omega \Leftrightarrow \omega_o)]$ (contains two frequencies: ω_o and $\Leftrightarrow\omega_o$). Note: **if the signal $x(t)$ is time-limited, $X(\omega)$ has infinite bandwidth.**

Properties of Fourier Transforms (given $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)$):

• **real signals:**	if $x(t)$ is real, then $X(\omega) \ = \ X^*(\Leftrightarrow\omega)$	• check results
	$\mathcal{R}e[X(\omega)] \ = \ \mathcal{R}e[X(\Leftrightarrow\omega)]$ **and** $\mathcal{I}m[X(\omega)] \ = \ \Leftrightarrow\mathcal{I}m[X(\Leftrightarrow\omega)]$	• check results
• **real and even signals:**	if $x(t)$ is real and even in t, then $X(\omega)$ is real and even in ω	• check results
	$\mathcal{E}v[x(t)] \stackrel{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e[X(\omega)]$ **and** $\mathcal{R}e[x(t)] \stackrel{\mathcal{F}}{\longleftrightarrow} \mathcal{E}v[X(\omega)]$	• check results
• **real and odd signals:**	if $x(t)$ is real and odd in t, then $X(\omega)$ is imaginary and odd in ω	• check results
	$\mathcal{O}d[x(t)] \stackrel{\mathcal{F}}{\longleftrightarrow} j\mathcal{I}m[X(\omega)]$ **and** $j\mathcal{I}m[x(t)] \stackrel{\mathcal{F}}{\longleftrightarrow} \mathcal{O}d[X(\omega)]$	• check results
• **time reversal:**	$x(\Leftrightarrow t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\Leftrightarrow\omega)$	
• **complex conjugate:**	$x^*(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(\Leftrightarrow\omega)$ **and** $x^*(\Leftrightarrow t) \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(\omega)$	
• **linearity:**	$\alpha_1 x_1(t) + \alpha_2 x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \alpha_1 X_1(\omega) + \alpha_2 X_2(\omega)$	
• **delay:**	$x(t \Leftrightarrow \tau) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega)e^{-j\omega\tau}$	• phase change (of $\Leftrightarrow\omega\tau$) only
• **scale:**	$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{\lvert a\rvert}X(\frac{\omega}{a})$	• wide (narrow) in t $\Leftrightarrow$ narrow (wide) in ω
• **duality:**	$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) \ \Leftrightarrow \ X(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi x(\Leftrightarrow\omega)$	
• **frequency translation:**	$x(t)e^{j\omega_o t} \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega \Leftrightarrow \omega_o)$	• used for modulation
• **differentiation:**	$\frac{d^n}{dt^n}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} (j\omega)^n X(\omega)$	• help solve LDEs
• **integration:**	$u(t) \otimes x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega}X(\omega) \ + \ \pi X(0)\delta(\omega)$	• δ-function accounts for DC in $x(t)$
• **convolution:**	$x_1(t) \otimes x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(\omega) \cdot X_2(\omega)$	
• **multiplication:**	$x_1(t) \cdot x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}X_1(\omega) \otimes X_2(\omega)$	
• **t-multiplication:**	$t\,x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega)$	
• **modulation:**	$x(t)cos\,\omega_o t \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}[X(\omega \Leftrightarrow \omega_o) + X(\omega + \omega_o)]$	
• **PARSEVAL's identity:**	$E \ = \ \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt \ = \ \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$	

Some Examples and Facts

- $X(0) \ = \ \int_{-\infty}^{\infty} x(t)dt$ is the **DC level/power** of signal $x(t)$
- $x(0) \ = \ \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)d\omega$ is the **total power** in $x(t)$

- $h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} H(\omega)$ **frequency response**
- $\frac{1}{\pi t} \stackrel{\mathcal{F}}{\longleftrightarrow} \Leftrightarrow j\ sgn(\omega) \ = \ e^{-j\frac{\pi}{2}sgn(\omega)}$ **Hilbert transform**; FT exists, even though $x(t)$ is unbounded
  - $h_{HT} \otimes x(t)$ phase shifts $x(t)$ only

- $H_{HT}[cos\, t] = sin\, t$ and $H_{HT}[sin\, t] = cos\, t$
- $H_{HT}[H_{HT}[x(t)]] = \Leftrightarrow x(t)$

- $sgn(t) \xleftrightarrow{\mathcal{F}} \frac{2}{j\omega}$
- $\Pi(\frac{t}{t_o}) \xleftrightarrow{\mathcal{F}} t_o sinc(\frac{\omega t_o}{2\pi})$
- $sinc(\frac{t}{\alpha}) \xleftrightarrow{\mathcal{F}} \alpha\Pi(\frac{\alpha\omega}{2\pi})$
- for $\alpha > 0$, $\beta e^{-\alpha t}u(t) \xleftrightarrow{\mathcal{F}} \frac{\beta}{\alpha + j\omega}$
- for $\alpha > 0$, $\beta e^{-\alpha|t|} \xleftrightarrow{\mathcal{F}} \frac{2\alpha\beta}{\alpha^2+\omega^2}$
- $e^{-\pi t^2} \xleftrightarrow{\mathcal{F}} e^{-\pi f^2} = e^{-\frac{\omega^2}{4\pi}}$
- $sin\,\omega_o t \xleftrightarrow{\mathcal{F}} \frac{\pi}{j}[\delta(\omega \Leftrightarrow \omega_o) \Leftrightarrow \delta(\omega + \omega_o)]$
- $cos\,\omega_o t \xleftrightarrow{\mathcal{F}} \pi[\delta(\omega \Leftrightarrow \omega_o) + \delta(\omega + \omega_o)]$
- $0 \xleftrightarrow{\mathcal{F}} 0$
- $1 \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega)$
- $\delta(t) \xleftrightarrow{\mathcal{F}} 1$
- $\delta(t \Leftrightarrow t_o) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_o}$
- $\dot{\delta}(t) \xleftrightarrow{\mathcal{F}} j\omega$
- $u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} + \pi\delta(\omega)$
- $\Lambda(\frac{t}{T}) = \frac{1}{T}\Pi(\frac{t}{T}) \otimes \Pi(\frac{t}{T}) \xleftrightarrow{\mathcal{F}} T sinc^2(\frac{\omega T}{2\pi})$
- $e^{j\omega_o t} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega \Leftrightarrow \omega_o)$
- $\sum_{n=-\infty}^{\infty} \delta(t \Leftrightarrow nT) \xleftrightarrow{\mathcal{F}} \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega \Leftrightarrow n\omega_o)$
- $\dot{\delta}(t) + \delta(t) + \frac{1}{2}sgn(t) \xleftrightarrow{\mathcal{F}} j\omega + 1 + \frac{1}{j\omega}$
- the Fourier transform of a Gaussian $e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$ is another Gaussian of the same form
- the Fourier transform of a Fourier Series $x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t} \xleftrightarrow{\mathcal{F}} 2\pi\sum_{k=-\infty}^{\infty} X_k\delta(\omega \Leftrightarrow k\omega_o)$

Laplace Transform

Definition of Laplace Transform

The **Laplace transform** $X(s)$ is a decomposition of an **(a)periodic signal** $x(t)$ into signals of the type $e^{st} = e^{\sigma t}e^{j\omega t}$ with $s = \sigma + j\omega$ complex. Compared to the Fourier transform, the extra (real) exponential $e^{\sigma t}$ allows us to treat a broader class of signals. The $e^{\sigma t}$ part is used to model *growth* or *decay,* while the $e^{j\omega t}$ part models the *oscillatory* behavior of $x(t)$. Again, $x(t)$ and $X(s)$ uniquely define each other and here we write $x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)$:

- unilateral LT: $X(s) \;=\; \mathcal{L}[x(t)] \;=\; \int_{0-}^{\infty} x(t)e^{-st}dt \;=\; \int_{0-}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}$

(• bilateral LT: $X(s) \;=\; \mathcal{L}[x(t)] \;=\; \int_{-\infty}^{\infty} x(t)e^{-st}dt \;=\; \int_{-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}$)

Note that the ULT only works for (real world) **"causal" signals** while the BLT permits us to consider any signal; furthermore, if $x(t) = 0$ for $t < 0$ (causal) then the ULT and the BLT are the same. Both the ULT and the BLT have **regions of convergence (ROC)** over which the transform exists. A ROC is the range of values of s over which the integral converges. A ROC cannot contain a pole, in fact for causal signals/systems, the ROC contains all s to the right of the rightmost pole. **If $x(t)$ is of finite duration,** $ROC_x = \mathbb{C}$.

Any signal $x(t)$ that is **of exponential order,** i.e. $|x(t)|$ does not increase faster than $A\,e^{ct}$ with A and c real constants, can be transformed into $X(s)$. Examples of signals that are not of exponential order, consider $x(t) = e^{t^2}u(t)$ and $x(t) = t^t u(t) = e^{t\,ln\,t}u(t)$.

Properties of Unilateral Laplace Transforms (given $x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)$ with ROC_x):

• **linearity:**	$\alpha_1 x_1(t) + \alpha_2 x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \alpha_1 X_1(s) + \alpha_2 X_2(s)$	**at least** $ROC_{x_1} \cap ROC_{x_2}$
• **delay** ($\tau > 0$)**:**	$x(t \Leftrightarrow \tau)u(t \Leftrightarrow \tau) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s)e^{-s\tau}$	ROC_x
• **scale:**	for real $a > 0:\; x(at) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{a}X(\frac{s}{a})$	$\{s \mid \frac{s}{a} \in ROC_x\}$
• **frequency translation:**	$x(t)e^{-\alpha t} \stackrel{\mathcal{L}}{\longleftrightarrow} X(s+\alpha)$	$\{s \mid (s+\alpha) \in ROC_x\}$
• **differentiation:**	$\frac{d^n}{dt^n}x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} s^n X(s) \Leftrightarrow s^{n-1}x(0^-) \Leftrightarrow ... \Leftrightarrow s x^{(n-2)}(0^-) \Leftrightarrow x^{(n-1)}(0^-)$	**at least** ROC_x
• **integration:**	for $t > 0:\; u(t) \otimes x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s) + \frac{1}{s}\int_{-\infty}^{0^-} x(\tau)d\tau$	**at least** $ROC_x \cap \{s \mid Re(s) > 0\}$
• **convolution:**	$x_1(t) \otimes x_2(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X_1(s) \cdot X_2(s)$	**at least** $ROC_{x_1} \cap ROC_{x_2}$
• **t-multiplication:**	$t \cdot x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \Leftrightarrow \frac{dX(s)}{ds}$	ROC_x
• **initial value:**	$lim_{s\to\infty}\, sX(s) = lim_{t\to 0^+}\, x(t)$, if this limit exists	
• **final value:**	$lim_{s\to 0}\, sX(s) = lim_{t\to\infty}\, x(t)$, **if this limit exists**	
	IMPORTANT: $sin\,t\,u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^2+1}$, so $\lim_{s\to 0}\; sX(s) = 0$, but $\lim t \to \infty\; sin\,t\,u(t)$ clearly does not exist!!!	

Some Examples

- $h(t) \stackrel{\mathcal{L}}{\longleftrightarrow} H(s)$ **transfer function**

• $0 \stackrel{\mathcal{L}}{\longleftrightarrow} 0$		• $\dot{\delta}(t) \stackrel{\mathcal{L}}{\longleftrightarrow} s$	all s
• $\delta(t) \stackrel{\mathcal{L}}{\longleftrightarrow} 1$	all s	• $\delta(t \Leftrightarrow \tau) \stackrel{\mathcal{L}}{\longleftrightarrow} e^{-s\tau}$	all s
• $u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$	$\mathcal{R}e[s] > 0$	• $\Leftrightarrow u(\Leftrightarrow t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$	$\mathcal{R}e[s] < 0$
• $r(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^2}$	$\mathcal{R}e[s] > 0$	• $\Leftrightarrow r(\Leftrightarrow t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$	$\mathcal{R}e[s] < 0$
• $\frac{t^{n-1}}{(n-1)!}\,u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^n}$	$\mathcal{R}e[s] > 0$	• $\Leftrightarrow \frac{t^{n-1}}{(n-1)!}\,u(\Leftrightarrow t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s^n}$	$\mathcal{R}e[s] < 0$
• $e^{-\alpha t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+\alpha}$	$\mathcal{R}e[s] > \Leftrightarrow \mathcal{R}e[\alpha]$	• $\Leftrightarrow e^{-\alpha t}u(\Leftrightarrow t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+\alpha}$	$\mathcal{R}e[s] < \Leftrightarrow \mathcal{R}e[\alpha]$
• $\frac{t^{n-1}}{(n-1)!}\,e^{-\alpha t}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+\alpha)^n}$	$\mathcal{R}e[s] > \Leftrightarrow \mathcal{R}e[\alpha]$	• $\Leftrightarrow \frac{t^{n-1}}{(n-1)!}\,e^{-\alpha t}u(\Leftrightarrow t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{(s+\alpha)^n}$	$\mathcal{R}e[s] < \Leftrightarrow \mathcal{R}e[\alpha]$
• $(sin\,\omega_o t)u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{\omega_o}{s^2+\omega_o^2}$	$\mathcal{R}e[s] > 0$	• $(e^{-\alpha t}\,sin\,\omega_o t)u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{\omega_o}{(s+\alpha)^2+\omega_o^2}$	$\mathcal{R}e[s] > \Leftrightarrow \mathcal{R}e[\alpha]$
• $(cos\,\omega_o t)u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+\omega_o^2}$	$\mathcal{R}e[s] > 0$	• $(e^{-\alpha t}\,cos\,\omega_o t)u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{s+\alpha}{(s+\alpha)^2+\omega_o^2}$	$\mathcal{R}e[s] > \Leftrightarrow \mathcal{R}e[\alpha]$

Linear Differential Equations

Solving LDE w/CC using Fourier transforms

One can solve LDE w/CC using Fourier transforms or Laplace transforms. Below we show how to do it with Laplace transform. For Fourier transforms, the procedure is identical: find $h(t) = \mathcal{F}^{-1}[H(\omega)]$ with $H(\omega) = \frac{Y(\omega)}{X(\omega)}$ and $X(\omega) = \mathcal{F}[\delta(t)] = 1$. $Y(\omega)$ follows from using the differentiation property for Fourier transforms and some algebraic manipulations.

Example: if $\ddot{y}(t) \Leftrightarrow \dot{y}(t) \Leftrightarrow 30y(t) = x(t)$, then for $x(t) = \delta(t)$ we find $(j\omega)^2 H(\omega) \Leftrightarrow (j\omega)H(\omega) \Leftrightarrow 30H(\omega) = 1 \iff H(\omega) = \frac{1}{(j\omega)^2 - (j\omega) - 30}$, so $h(t) = \mathcal{F}^{-1}[\frac{1}{(j\omega+5)(j\omega-6)}] = \mathcal{F}^{-1}[\frac{-1/11}{j\omega+5}] + \mathcal{F}^{-1}[\frac{1/11}{j\omega-6}] = \Leftrightarrow \frac{1}{11}e^{-5t}u(t) + \frac{1}{11}e^{6t}u(t)$. Note that this system is BIBO unstable.

Solving LDE w/CC using Laplace transforms

We show how the unilateral Laplace transform can be used to solve linear differential equations with constant coefficients. In particular, the integration property of the unilateral Laplace transform is utilized. The procedures are not shown for the general case, but rather for an example.

Example: Assume we are given the following LDE w/CC that we are trying to solve:

$$\ddot{y}(t) + 2\dot{y}(t) + y(t) = 2\dot{x}(t)$$

with initial conditions $\dot{y}(0^-) = 2,\ y(0^-) = 2$ and $x(0^-) = 0$.

First, take the unilateral Laplace transform of both sides, using the differentiation property:

$$[s^2Y(s) \Leftrightarrow sy(0^-) \Leftrightarrow \dot{y}(0^-)] + a[sY(s) \Leftrightarrow y(0^-)] + Y(s) = c[sX(s) \Leftrightarrow x(0^-)] \iff Y(s) = \frac{s(2+2X(s)) + 2 + 4}{s^2 + 2s + 1}$$

For input signal $x(t) = cost\, u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{s}{s^2+1}$:

$$Y(s) = \frac{2s + 6 + 2s \cdot \frac{s}{s^2+1}}{s^2+2s+1} = \frac{2s^3 + 8s^2 + 2s + 6}{(s+1)^2(s^2+1)} = \frac{1}{s+1} + \frac{5}{(s+1)^2} + \frac{s}{s^2+1} \iff y(t) = \mathcal{L}^{-1}[Y(s)] = (e^{-t} + 5te^{-t} + cost)u(t)$$

ZIR and ZSR

The **zero-input response** $y_{zir}(t)$ is $y(t)$ when the non-zero intial conditions are assumed to contain all information about what happened in the past, and for $t \geq t_o$, there is no input to the system: $x(t) = 0$, for $t \geq t_o$. So $y_{zir}(t)$ is the system response to initial conditions only. In the example above, $y_{zir}(t) = (2e^{-t} + 4te^{-t})u(t)$.

The **zero-state response** $y_{zsr}(t)$ is $y(t)$ when zero initial conditions are assumed and for $t \geq t_o$, the input $x(t)$ is known. In the example, $y_{zsr}(t) = (\Leftrightarrow e^{-t} + te^{-t} + cost)u(t)$.

Note that $y(t) = y_{zir}(t) + y_{zsr}(t)$ always holds. In the example: $y_{zir}(t) + y_{zsr}(t) = (2e^{-t} + 4te^{-t})u(t) + (\Leftrightarrow e^{-t} + te^{-t} + cost)u(t) = (e^{-t} + 5te^{-t} + cost)u(t) = y(t)$.

Partial Fraction Expansions

Sorry, I never found the time to do this section: see your notes!

Discrete-time Signals and Systems

General Remarks

Since in discrete time, when observing some continuous-time signal at times $t = nT$, complex exponentials $e^{j\omega_1 nT}$ and $e^{j\omega_2 nT}$ with $\omega_2 = \omega_1 + \frac{2\pi}{T}$, look exactly identical, we introduce the concept of **normalized frequency:** $\Omega \equiv \omega T$ (note that we basically only rescaled the time axis). All of this implies that **DISCRETE-TIME SIGNALS HAVE PERIODIC DTFTs**, with period $\frac{2\pi}{T}$ in unnormalized frequency and period 2π in normalized frequency.

In discrete-time, **high frequencies** lie around odd multiples of π, while low frequencies lie around even multiples of π. Consequently, an ideal **low-pass filter** with cut-off frequency $\Omega_1 < \pi$ has impulse response $h_{\mathrm{lpf}}[n] = \frac{\Omega_1}{\pi} sinc\, \frac{n\Omega_1}{\pi}$ and a pulse train as frequency response. The frequency response of a **high-pass, bandpass or notch filter** is simply a shifted and possibly duplicated version of the frequency response of a low-pass filter.

Singularity Signals

1.	$\delta[n] \;=\; u[n] \Leftrightarrow u[n \Leftrightarrow 1] \;=\; 1, \;\; n = 0,$ and 0 elsewhere	**delta** function
2.	$u[n] \;=\; \sum_{k=-\infty}^{n} \delta[k] \;=\; \sum_{k=0}^{\infty} \delta[n \Leftrightarrow k] \;=\; 1 \; n \geq 0,$ and 0 elsewhere	unit **step** function
3.	$x[n] \;=\; \sum_{k=-\infty}^{\infty} x[k]\, \delta[n]$	sifting property (sum)

Convolution

- **definition:** $h[n] \otimes x[n] \;=\; \sum_{k=-\infty}^{\infty} x[k]\, h[n \Leftrightarrow k] \;=\; \sum_{k=-\infty}^{\infty} x[n \Leftrightarrow k]\, h[k]$ **often very easy to do!**

1.	**trick 1:** $x[n \Leftrightarrow n_o] \;=\; x[n] \otimes \delta[n \Leftrightarrow n_o]$	delay system
2.	**trick 2:** $x[n] \;=\; x[n] \otimes \delta[n], \; \forall x[n]$	
3.	**trick 3:** $\sum_{k=-\infty}^{n} x[n] \;=\; u[n] \otimes x[n]$	convolving with $u[n]$ means taking the running sum

- **example:** $u[n] \otimes u[n] \;=\; \sum_{k=-\infty}^{\infty} u[k]\, u[n \Leftrightarrow k] \;=\; \sum_{k=0}^{n} 1 \;=\; (n+1)u[n]$

BIBO-Stability

- **definition:** system $H[\cdot]$ is BIBO-stable **iff** every bounded input results in a bounded output

1. system $H[\cdot]$ is BIBO-stable **iff** $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$
2. causal system $H[\cdot]$ is BIBO-stable **iff** all the poles of $H(z)$ are **strictly** inside the unit circle

- **for all stable causal systems, the frequency response exists**

Finding the Output of LTI Systems

1.	$h[n] \;=\; H[\delta[n]]$	**unit sample response**
2.	$y[n] \;=\; x[n] \otimes h[n]$	**straight convolution**
3.	$y[n] \;=\; \mathcal{F}^{-1}[X(e^{j\Omega})H(e^{j\Omega})]$	**using Discrete-time Fourier transforms**
4.	$y[n] \;=\; \mathcal{L}^{-1}[X(z)H(z)]$	**using Z-transforms**
5.	if $x[n] \;=\; e^{j\Omega_o n}$, then $y[n] \;=\; H(e^{j\Omega_o})e^{j\Omega_o n}$	**using Fourier eigenfunctions**

6. if $x[n] = z^n$, then $y[n] = H(z)z^n$ **using Z-eigenfunctions**

7. if $x[n] = cos(\Omega_o n + \phi)$ and $h[n]$ is real, then $y[n] = |H(e^{j\Omega_o})| cos(\Omega_o n + \phi + \angle H(e^{j\Omega_o}))$

   steady-state sinusoidal response

8. if $x[n] = sin(\Omega_o n + \phi)$ and $h[n]$ is real, then $y[n] = |H(e^{j\Omega_o})| sin(\Omega_o n + \phi + \angle H(e^{j\Omega_o}))$

   steady-state sinusoidal response

- **the transfer function and consequently the impulse response can be derived from any LDE by assuming zero initial conditions.**

Discrete-time Fourier Transform

Definition of Discrete-time Fourier Transform:

The **Discrete-time Fourier transform** $X(\Omega)$ is a decomposition of an **(a)periodic signal** $x[n]$ into $e^{j\Omega n}$-type signals with Ω real ($x[n]$ and $X(\Omega)$ uniquely define each other). $X(\Omega)$ is periodic in Ω with period 2π. We write $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega)$:

- $x[n] = \mathcal{F}^{-1}[X(\Omega)] = \frac{1}{2\pi}\int_{\Omega_o}^{\Omega_o+2\pi} X(\Omega)e^{jn\Omega}d\Omega$ **synthesis** equation
- $X(\Omega) = \mathcal{F}[x(t)] = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega}$ **analysis** equation

The discrete-time Fourier transform $X(\Omega)$ exists when it's finite for all Ω. The interpretation of the discrete-time Fourier transform is that $X(\omega)$ contains the frequency content of the signal $x[n]$. **The DTFT is always periodic.** Often, the DTFT of a discrete-time signal can be found more easily, by writing $x[n]$ as a sampled continuous-time signal $x[n] = x(nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t \Leftrightarrow nT)$, which has Fourier transform $\sum_{n=-\infty}^{\infty} x(nT)e^{-j\omega nT}$; the change of variables $\omega T = \Omega$ gives the DTFT of $x[n]: X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega}$ (**EXTREMELY HELPFUL!**).

Properties of Discrete-time Fourier Transforms (given $x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega)$):

• **real signals:**	if $x[n]$ is real, then $X(\Omega) = X^*(\Leftrightarrow\Omega)$	• check results
	$\mathcal{R}e[X(\Omega)] = \mathcal{R}e[X(\Leftrightarrow\Omega)]$ **and** $\mathcal{I}m[X(\Omega)] = \Leftrightarrow\mathcal{I}m[X(\Leftrightarrow\Omega)]$	• check results
• **real and even signals:**	if $x[n]$ is real and even in n, then $X(\Omega)$ is real and even in Ω	• check results
	$\mathcal{E}v[x[n]] \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{R}e[X(\Omega)]$ **and** $\mathcal{R}e[x[n]] \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{E}v[X(\Omega)]$	• check results
• **real and odd signals:**	if $x[n]$ is real and odd in n, then $X(\Omega)$ is imaginary and odd in Ω	• check results
	$\mathcal{O}d[x[n]] \overset{\mathcal{F}}{\longleftrightarrow} j\mathcal{I}m[X(\Omega)]$ **and** $j\mathcal{I}m[x[n]] \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{O}d[X(\Omega)]$	• check results
• **time reversal:**	$x[\Leftrightarrow n] \overset{\mathcal{F}}{\longleftrightarrow} X(\Leftrightarrow\Omega)$	
• **complex conjugate:**	$x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(\Leftrightarrow\Omega)$	
• **linearity:**	$\alpha_1 x_1[n] + \alpha_2 x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} \alpha_1 X_1(\Omega) + \alpha_2 X_2(\Omega)$	
• **delay:**	$x[n \Leftrightarrow n_o] \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega)e^{-jn_o\Omega}$	• phase change (of $\Leftrightarrow n_o\Omega$) only
• **frequency translation:**	$x[n]e^{j\Omega_o n} \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega \Leftrightarrow \Omega_o)$	• used for modulation
• **first difference:**	$x[n] \Leftrightarrow x[n \Leftrightarrow 1] \overset{\mathcal{F}}{\longleftrightarrow} X(\Omega)(1 \Leftrightarrow e^{-j\Omega})$	
• **running sum:**	$u[n] \otimes x[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\Omega}}X(\Omega) + \pi X(0)\sum_{k=-\infty}^{\infty}\delta(\Omega \Leftrightarrow 2\pi k)$	• δ-functions account for DC in $x[n]$
• **convolution:**	$x_1[n] \otimes x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} X_1(\Omega) \cdot X_2(\Omega)$	
• **multiplication:**	$x_1[n] \cdot x_2[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{\Omega_o}^{\Omega_o+2\pi} X_1(\theta) \otimes X_2(\Omega \Leftrightarrow \theta)d\theta$	• periodic convolution
• **n-multiplication:**	$n\,x[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\Omega}X(\Omega)$	
• **upsampling:**	$x_{(k)}[n] \overset{\mathcal{F}}{\longleftrightarrow} X(k\Omega)$	• $x_{(k)}[n] = x[\frac{n}{k}]$, if n is a multiple of k and 0 otherwise
• **PARSEVAL's identity:**	$E = \sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{\Omega_o}^{\Omega_o+2\pi} \lvert X(\Omega)\rvert^2 d\Omega$	

Some Examples

- $h[n] \overset{\mathcal{F}}{\longleftrightarrow} H(\Omega)$ **frequency response**
- $sin\,\Omega_o n \overset{\mathcal{F}}{\longleftrightarrow} \frac{\pi}{j}\sum_{k=-\infty}^{\infty}[\delta(\Omega \Leftrightarrow \Omega_o \Leftrightarrow 2\pi k) \Leftrightarrow \delta(\Omega + \Omega_o \Leftrightarrow 2\pi k)]$
- $cos\,\Omega_o n \overset{\mathcal{F}}{\longleftrightarrow} \pi\sum_{k=-\infty}^{\infty}[\delta(\Omega \Leftrightarrow \Omega_o \Leftrightarrow 2\pi k) + \delta(\Omega + \Omega_o \Leftrightarrow 2\pi k)]$

- $0 \overset{\mathcal{F}}{\longleftrightarrow} 0$
- $\delta[n] \overset{\mathcal{F}}{\longleftrightarrow} 1$
- $u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{j\Omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega \Leftrightarrow 2\pi k)$
- $\frac{\Omega_1}{\pi}sinc(\frac{\Omega_1 n}{\pi}) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty}\Pi(\frac{\Omega-2\pi k}{2\Omega_1})$

- $1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\sum_{k=-\infty}^{\infty}\delta(\Omega \Leftrightarrow 2\pi k)$
- $\delta[n \Leftrightarrow n_o] \overset{\mathcal{F}}{\longleftrightarrow} e^{-jn_o\Omega}$
- $e^{j\Omega_o n} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\sum_{k=-infty}^{\infty}\delta(\Omega \Leftrightarrow \Omega_o \Leftrightarrow 2\pi k)$

Z-Transform or DT Laplace Transform

Definition of Z-Transform:

The **Z-transform** $X(z)$ is a decomposition of an **(a)periodic signal** $x(t)$ into signals of the type $z^{-n} = r^{-n}e^{-jn\Omega}$ with $z = re^{j\Omega}$ complex. Compared to the discrete-time Fourier transform, the extra (real) exponential r^{-n} allows us to treat a broader class of signals. The r^{-n} part is used to model *growth* or *decay,* while the $e^{-jn\omega}$ part models the *oscillatory* behavior of $x(t)$. Again, $x[n]$ and $X(z)$ uniquely define each other and here we write $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$:

- unilateral ZT: $X(z) \;=\; \mathcal{Z}[x[n]] \;=\; \sum_{k=0}^{\infty} x[n]z^{-n}$

(• bilateral ZT: $X(z) \;=\; \mathcal{Z}[x[n]] \;=\; \sum_{k=-\infty}^{\infty} x[n]z^{-n}$)

Note that the UZT only works for (real world) "causal" signals while the BZT permits us to consider any signal; furthermore, if $x[n] = 0$ for $n < 0$ (causal) then the UZT and the BZT are the same. Both the UZT and the BZT have **regions of convergence (ROC)** over which the transform exists. A ROC is the range of values of z over which the integral converges. A ROC cannot contain a pole, in fact for causal signals/ systems, the ROC contains all z on the outside of the circle that crosses the outermost pole. **If** $x[n]$ **is of finite duration,** $ROC_x = \mathbb{C}$, except possibly for $z = 0$ and $z = \infty$.

Any signal $x[n]$ that is **of exponential order,** i.e. $|x[n]|$ does not increase faster than $A\,c^n$ with A and c real constants, can be transformed into $X(z)$. As an example of a signal that is not of exponential order, consider for instance $x[n] \;=\; 2^{n^2}u[n]$

Properties of Unilateral Z-Transforms (given rightsided $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ with ROC_x):

• **linearity:**	$\alpha_1 x_1[n] + \alpha_2 x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} \alpha_1 X_1(z) + \alpha_2 X_2(z)$	**at least** $ROC_{x_1} \cap ROC_{x_2}$
• **time reversal:**	$x[\Leftrightarrow n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z^{-1})$	$\{z \,\vert\, z^{-1} \in ROC_x\}$
• **delay** ($M > 0$):	$x[n \Leftrightarrow M] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-M}X(z)$	ROC_x
• **advance** ($M > 0$):	$x[n + M] \overset{\mathcal{Z}}{\longleftrightarrow} z^M\{X(z) \Leftrightarrow x[M \Leftrightarrow 1]z^{-(M-1)} \Leftrightarrow \cdots \Leftrightarrow x[0]\}$	ROC_x
• **frequency translation:**	$x[n]p^{-n} \overset{\mathcal{Z}}{\longleftrightarrow} X(pz)$	$\{z \,\vert\, pz \in ROC_x\}$
• **running sum:**	$u[n] \otimes x[n] \;=\; \sum_{k=-\infty}^{n} x[k] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}X(z)$	at least $ROC_x \;\cap\; \{z \,\vert\, \vert z\vert > 1\}$
• **convolution:**	$x_1[n] \otimes x_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) \cdot X_2(z)$	**at least** $ROC_{x_1} \cap ROC_{x_2}$
• **n-multiplication:**	$n \cdot x[n] \overset{\mathcal{Z}}{\longleftrightarrow} \Leftrightarrow z\frac{dX(z)}{dz}$	ROC_x
• **initial value:**	$lim_{z\to\infty} X(z) \;=\; x[0]$, which always exists	
• **final value:**	$lim_{z\to 1}(1 \Leftrightarrow z^{-1})X(z) \;=\; lim_{n\to\infty} x[n]$, if this limit exists	

FOR A TWOSIDED $x[n] \overset{\mathcal{Z}}{\longleftrightarrow} X(z)$ **(useful for INITIAL CONDITIONS in LDEs):**

- **delay** ($M > 0$): $x[n \Leftrightarrow M] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-M}X(z) + \{x[\Leftrightarrow M] + x[\Leftrightarrow M+1]z^{-1} + \cdots + x[\Leftrightarrow 1]z^{-M+1}\}$

Some Examples

- $h[n] \overset{\mathcal{Z}}{\longleftrightarrow} H(z)$ **transfer function**

• $0 \overset{\mathcal{Z}}{\longleftrightarrow} 0$			
• $\delta[n] \overset{\mathcal{Z}}{\longleftrightarrow} 1$	all z	• $\delta[n \Leftrightarrow M] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-M}$	all s except 0 (∞) if $m > 0$ ($m < 0$)
• $u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}$	$\vert z\vert > 1$	• $u[\Leftrightarrow n \Leftrightarrow 1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-z^{-1}}$	$\vert z\vert < 1$
• $\alpha^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}$	$\vert z\vert > \vert\alpha\vert$	• $\Leftrightarrow\alpha^n u[\Leftrightarrow n \Leftrightarrow 1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-\alpha z^{-1}}$	$\vert z\vert < \vert\alpha\vert$
• $n\alpha^n u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\vert z\vert > \vert\alpha\vert$	• $\Leftrightarrow n\alpha^n u[\Leftrightarrow n \Leftrightarrow 1] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\vert z\vert < \vert\alpha\vert$
• $[cos\Omega_o n]u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1-[cos\Omega_o]z^{-1}}{1-[2cos\Omega_o]z^{-1}+z^{-2}}$	$\vert z\vert > 1$	• $[sin\Omega_o n]u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{[sin\Omega_o]z^{-1}}{1-[2cos\Omega_o]z^{-1}+z^{-2}}$	$\vert z\vert > 1$
• $[r^n cos\Omega_o n]u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1-[rcos\Omega_o]z^{-1}}{1-[2rcos\Omega_o]z^{-1}+r^2z^{-2}}$	$\vert z\vert > \vert r\vert$	• $[r^n sin\Omega_o n]u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{[rsin\Omega_o]z^{-1}}{1-[2rcos\Omega_o]z^{-1}+r^2z^{-2}}$	$\vert z\vert > \vert r\vert$

Linear Difference Equations

Solving LDE w/CC using Z-transforms

We show how the unilateral Z-transform can be used to solve linear differential equations with constant coefficients. In particular, the delay property of the unilateral Z-transform is utilized. The procedures are not shown for the general case, but rather for an example.

Example: Assume we are given the following LDE w/CC that we are trying to solve:

$$y[n] \Leftrightarrow 2y[n \Leftrightarrow 1] \;=\; x[n] \;+\; x[n \Leftrightarrow 1]$$

with initial conditions $y[\Leftrightarrow 1] = 1$ and $x[\Leftrightarrow 1] = 0$.

First, take the unilateral Z-transform of both sides, using the delay property:

$$Y(z) \Leftrightarrow [2z^{-1}Y(z) + 2y[\Leftrightarrow 1]] \;=\; X(z) \;+\; [z^{-1}X(z) + x[\Leftrightarrow 1]] \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad Y(z) \;=\; \frac{1 + z^{-1}}{(1 \Leftrightarrow z^{-1})(1 \Leftrightarrow 2z^{-1})} \;+\; \frac{x[\Leftrightarrow 1] + 2y[\Leftrightarrow 1]}{1 \Leftrightarrow 2z^{-1}},$$

wherein inverse Z-transform of the first part equals the **zero-state-response:**

$$y_{zsr}[n] \;=\; \mathcal{Z}[\frac{1 + z^{-1}}{(1 \Leftrightarrow z^{-1})(1 \Leftrightarrow 2z^{-1})}] \;=\; \mathcal{Z}[\frac{1}{(1 \Leftrightarrow z^{-1})(1 \Leftrightarrow 2z^{-1})}] \;+\; \mathcal{Z}[\frac{z^{-1}}{(1 \Leftrightarrow z^{-1})(1 \Leftrightarrow 2z^{-1})}] \;=\; (\delta[n] + \delta[n \Leftrightarrow 1]) \otimes \mathcal{Z}[\frac{1}{(1 \Leftrightarrow z^{-1})(1 \Leftrightarrow 2z^{-1})}] \;=$$

$$=\; (\delta[n] + \delta[n \Leftrightarrow 1]) \otimes \mathcal{Z}[\frac{\Leftrightarrow 1}{1 \Leftrightarrow z^{-1}} \;+\; \frac{2}{1 \Leftrightarrow 2z^{-1}}] \;=\; (\delta[n] + \delta[n \Leftrightarrow 1]) \otimes (\Leftrightarrow u[n] + 2 \cdot 2^n u[n]) \;=\; \Leftrightarrow u[n] \Leftrightarrow u[n \Leftrightarrow 1] + 2^{n+1}u[n] + 2^n u[n \Leftrightarrow 1].$$

Secondly, we have the **zero-input-response:**

$$y_{zir}[n] \;=\; \mathcal{Z}[\frac{2}{(1 \Leftrightarrow 2z^{-1})}] \;=\; 2 \cdot 2^n u[n] \;=\; 2^{n+1}u[n].$$

The actual output $y[n]$ is the sum of the two:

$$y[n] \;=\; y_{zir}[n] + y_{zir}[n] \;=\; \Leftrightarrow u[n] \Leftrightarrow u[n \Leftrightarrow 1] + 2^{n+1}u[n] + 2^n u[n \Leftrightarrow 1] + 2^{n+1}u[n] \;=\; \Leftrightarrow u[n] \Leftrightarrow u[n \Leftrightarrow 1] + 2^{n+2}u[n] + 2^n u[n \Leftrightarrow 1].$$

Partial Fraction Expansions

Suppose we are given $Y(z) = \frac{1}{(1-z^{-1})(1-2z^{-1})}$ (see also above) that we wish to expand into partial fractions. The procedure for doing this, is to first factor and expand into partial fractions $\frac{Y(z)}{z}$, expressed in terms of z :

$$\frac{Y(z)}{z} \;=\; \frac{z}{z} \cdot \frac{1}{(1 \Leftrightarrow z^{-1})(1 \Leftrightarrow 2z^{-1})} \;=\; \frac{z}{(z \Leftrightarrow 1)(z \Leftrightarrow 2)} \;=\; \frac{\Leftrightarrow 1}{z \Leftrightarrow 1} + \frac{2}{z \Leftrightarrow 2},$$

whereafter we multiply both sides again by z and write the partial fractions in terms of z^{-1}, so that we can use the Z-transform tables in order to find $y[n]$:

$$Y(z) \;=\; \frac{\Leftrightarrow 1}{1 \Leftrightarrow z^{-1}} + \frac{2}{1 \Leftrightarrow 2z^{-1}} \quad \Longrightarrow \quad y[n] \;=\; \Leftrightarrow u[n] + 2 \cdot 2^n u[n] \;=\; \Leftrightarrow u[n] + 2^{n+1}u[n].$$

Comparison of FS and FT

Decomposing a *periodic* signal into *discrete* frequencies, each a multiple of some basic frequency ω_o, gives us the **Fourier series** representation of that signal. With **Fourier transforms,** we can decompose *aperiodic* signals into its component frequencies, which, however, are not multiples of some frequency, but rather range from $\omega = \Leftrightarrow\infty$ to $\omega = \infty$. For periodic signals, $x(t)$, $\mathcal{F}[x(t)$ and the FS represenation all carry the same information.

How to get from FS to FT? If a periodic signal $x(t)$ is written as a Fourier Series, $x(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_o t}$, then its Fourier Transform is

$$X(\omega) = \mathcal{F}[x(t)] = \mathcal{F}[\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_o t}] = \sum_{n=-\infty}^{\infty} X_n \mathcal{F}[e^{jn\omega_o t}] = 2\pi \sum_{n=-\infty}^{\infty} X_n \delta(\omega \Leftrightarrow n\omega_o).$$

How to get from FT to FS (if the signal is periodic)? If a periodic signal $x(t)$ is written as a Fourier Series,

$$X_n = \frac{1}{T_o} X(\omega)|_{\omega = n\omega_o}.$$

THIS SECTION IS FINISHED NOR CLEAR TO MYSELF; APOLOGIES.

Comparison of FT and ULT

The Laplace transform is a more general transform than the Fourier transform, meaning that it can be used for all signals that can be Fourier transformed, and more. From the definitons, the relationship between the Laplace and Fourier transform seems to follow relatively straightforwardly:

$$X(s) = \mathcal{F}[x(t)\, e^{-\sigma t}] \quad \text{and} \quad \mathcal{F}[x(t)] = X(j\omega)$$

However, we should distinguish between three cases:

Case (a): when ROC_x **includes the** $j\omega$**-axis** then $X(s)|_{s=j\omega} = X(j\omega) = \mathcal{F}[x(t)]$ and $\mathcal{F}^{-1}[X(j\omega)] = x(t)$.

Case (b): when ROC_x **is just bounded by the** $j\omega$**-axis** then $\mathcal{F}[x(t)]$ can sometimes be computed, but if so, then $\mathcal{F}[x(t)] \neq X(j\omega)$ and $\mathcal{F}^{-1}[X(j\omega)] \neq x(t)$. For example, $u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$ with $ROC = \{s\,|\,\text{Re}[s] > 0\}$, but $u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} + \pi\delta(\omega) \neq \frac{1}{j\omega} = X(j\omega)$. Furthermore, $\mathcal{F}^{-1}[\frac{1}{j\omega}] = \frac{1}{2} sgn(t)$.

Case (c): when ROC_x **doesn't include the** $j\omega$**-axis** then $\mathcal{F}[x(t)]$ cannot be computed. Although $X(j\omega)$ may be defined, $\mathcal{F}^{-1}[X(j\omega)] \neq x(t)$. For example, $e^t u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-1}$ with $ROC = \{s\,|\,\mathcal{R}][s] > 1\}$; $\mathcal{F}[e^t u(t)]$ is undefined and $\mathcal{F}^{-1}[\frac{1}{j\omega-1}] = \Leftrightarrow e^{-t} u(\Leftrightarrow t)$.

- $\mathcal{F}^{-1}[\frac{1}{j\omega-\alpha}] = \Leftrightarrow e^{\alpha t} u(\Leftrightarrow t)$ for complex α
- $\mathcal{F}^{-1}[\frac{1}{1-\omega^2}] = \mathcal{F}^{-1}[\frac{1}{(j\omega+j)(j\omega-j)}] = \mathcal{F}^{-1}[\frac{\frac{j}{2}}{(j\omega+j)}] \Leftrightarrow \mathcal{F}^{-1}[\frac{\frac{j}{2}}{(j\omega-j)}] = \frac{j}{4} sgn(t)(e^{-jt} + e^{jt}) = \frac{1}{2} sgn(t) sin(t)$

Comparison of DTFT and UZT

The Z-transform is a more general transform than the discrete-time Fourier transform, meaning that it can be used for all signals that can be Fourier transformed, and more. From the definitons, the relationship between the Laplace and Fourier transform seems to follow relatively straightforwardly:

$$X(z) = \mathcal{F}[x[n]\, r^{-n}] \quad \text{and} \quad \mathcal{F}[x[n]] = X(e^{j\Omega})$$

However, as above, we should distinguish between the cases where the ROC contains the unit circle (given by $|z| = 1 \;\Leftrightarrow\; z = e^{j\Omega}$), is just bounded by it, or does not contain it at all. These three cases lead to perfectly similar conclusion about the existence of the Fourier transform as in the continuous-time case above.

Modulation

NOTE:
- modulators are generally **time-variant systems**;
- some modulators are **linear**, some **non-linear**;
- **asynchronous** demodulators can only demodulate DSB-AM-LC;
- **synchronous** demodulators can demodulate all linear schemes discussed;
- (with **synchronous** we mean that the received signal is multiplied with a copy of the carrier;)

- WB-FM is much more **noise immune** than AM, since noise mainly effects the amplitude of a signal and not so much the (instantaneous) frequency; furthermore, the amount of used bandwidth is much larger for WB-FM, leading to better noise immunity.

Motivation for Modulation

- baseband signals will not propagate, but higher frequency signals will;
- in order to simultaneously send multiple signals, one must use (distinct) carrier frequencies to prevent (limit) interference.

Assumptions and Terminology

- In $H[m(t)] = x(t)$, $H[\cdot]$ is the *modulation system*, $m(t)$ the *message* or *modulating signal* and $x(t)$ the transmitted *modulated signal;*
- the message signal, $m(t) \overset{\mathcal{F}}{\longleftrightarrow} M(\omega)$, is **bandlimited** to $|\omega| \leq 2\pi B_M \; rad/s$; furthermore, we assume $|m(t)| \leq 1, \forall t$;
- the modulated signal, $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, is a **narrowband bandpass signal,** centered at ω_c and has bandwidth $2\pi B_X \ll \omega_c \; rad/s$;

Underlying Fourier Transform Properties

- $e^{j\omega_o t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega \Leftrightarrow \omega_o)$
- $x(t)e^{j\omega_o t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega \Leftrightarrow \omega_o)$ • frequency translation

Overview of Analog Modulation Systems

Amplitude Modulation (linear):
- Double-Sideband Amplitude Modulation (DSB-AM)
- Double-Sideband Amplitude Modulation with Large Carrier (DSB-AM-LC; a.k.a. broadcast AM)
- Single-Sideband Amplitude Modulation (SSB-AM)
- Quadrature Amplitude Modulation (QAM; see notes 15)

Angle Modulation (non-linear):
- Narrowband Frequence Modulation (NB-FM)
- Narrowband Phase Modulation (NB-PM)
- Wideband Frequency Modulation (WB-FM)
- Wideband Phase Modulation (WB-PM)

Double-Sideband Amplitude Modulation (DSB-AM)

- **modulation:** $H[m(t)] = x(t) = m(t)\, cos\, \omega_c t \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \frac{1}{2\pi} M(\omega) \otimes \pi[\delta(\omega + \omega_c) + \delta(\omega \Leftrightarrow \omega_c)] = \frac{1}{2}[M(\omega + \omega_c) + M(\omega \Leftrightarrow \omega_c)]$;
- **demodulation:** $y(t) = x(t)\, cos\, \omega_c t = m(t)\, cos^2\, \omega_c t = \frac{1}{2} m(t) + \frac{1}{2} m(t) cos\, 2\omega_c t \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = \frac{1}{2} M(\omega) + \frac{1}{4}[M(\omega + 2\omega_c) + M(\omega \Leftrightarrow 2\omega_c)]$;
  subsequently, an ideal LPF with $H_{lpf}(\omega) = 1(0)$, $|\omega| \leq (>) 2\pi B_M$ passes $\frac{1}{2} m(t)$ only;
- **remarks:**
  - demodulator must have the $cos\, \omega_c t$ carrier in phase lock to carrier at modulator, which requires a **phase locked loop;**
  - you can demodulate SSB-AM using imperfect carrier $cos(\omega_c t + \theta)$: received signal is $\frac{1}{2} cos(\theta)\, m(t)$: fine as long as $cos(\theta) \neq 0$;
  - you can demodulate SSB-AM using imperfect carrier $cos([\omega_c + \Delta\omega] t)$;
  - the modulated signal $x(t)$ takes up twice the bandwidth $2\pi B_M$ of the message signal $m(t)$: $B_X = 2B_M$;
  - DSB-AM: **power-efficient, wastes bandwidth, requires synchronized reception.**

Double-Sideband Amplitude Modulation w/ Large Carrier (DSB-AM-LC)

- **modulation:** $H[m(t)] = x(t) = [1 + \mu m(t)]\, cos\, \omega_c t \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \pi[\delta(\omega + \omega_c) + \delta(\omega \Leftrightarrow \omega_c)] + \frac{\mu}{2}[M(\omega \Leftrightarrow \omega_c) + M(\omega + \omega_c)]$;
- **demodulation:** **(sync)** $y(t) = x(t)\, cos\, \omega_c t = [1 + \mu m(t)]\, cos^2\, \omega_c t = \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = \mu Y_{\text{DSB-AM}}(\omega) + \frac{\pi}{4}[\delta(\omega \Leftrightarrow 2\omega_c) + 2\delta(\omega) + \delta(\omega + 2\omega_c)]$;
  subsequently, an ideal LPF with $H_{lpf}(\omega) = 1(0)$, $|\omega| \leq (>) 2\pi B_M$ passes $\frac{1}{4} + \frac{\mu}{2} m(t)$ only;
  (async) $x(t)$ $\Leftrightarrow\!\rightarrow$ rectifier $\Leftrightarrow\!\rightarrow$ $y(t) = |x(t)|$ $\Leftrightarrow\!\rightarrow$ LPF $\Leftrightarrow\!\rightarrow$ $z(t) = 1 + \mu m(t)$ $\Leftrightarrow\!\rightarrow$ capacitor $\Leftrightarrow\!\rightarrow$ $\mu m(t)$;
- **remarks:**
  - **demodulation of DSB-AM-LC can be done asynchronously, i.e. there is no need for synchronization of demodulator to modulator;**
  - μ is the **modulation index;**
  - if the amplitude of $\mu m(t) = (>) 1$ we have **full modulation (overmodulation** $\Leftrightarrow\!\rightarrow$ distorted recovery);
  - power efficiency is poor $(= \frac{\frac{1}{2}\mu^2 P_M}{\frac{1}{2}(1 + \mu^2 P_M)} < 50\%)$, because of large carrier;
  - the modulated signal $x(t)$ takes up twice the bandwidth $2\pi B_M$ of the message signal $m(t)$: $B_X = 2B_M$;
  - DSB-AM-LC: **power-inefficient, wastes bandwidth, does not require synchronized reception.**

Single-Sideband Amplitude Modulation (SSB-AM)

Upper-Sideband Amplitude Modulation (USB-AM)

- **modulation:** $H[m(t)] = x_u(t) = m(t)\, cos\, \omega_c t \Leftrightarrow \hat{m}(t)\, sin\, \omega_c t \overset{\mathcal{F}}{\longleftrightarrow} X_u(\omega) = M(\omega + \omega_c)\, u(\Leftrightarrow \omega \Leftrightarrow \omega_c) + M(\omega \Leftrightarrow \omega_c)\, u(\omega \Leftrightarrow \omega_c)$;
- **demodulation:** $y(t) = x_u(t)\, cos\, \omega_c t = \cdots = \frac{1}{2} m(t) + \frac{1}{2} m(t)\, cos\, 2\omega_c t \Leftrightarrow \frac{1}{2} \hat{m}(t)\, sin\, 2\omega_c t$ $\Leftrightarrow\!\rightarrow$ LPF $\Leftrightarrow\!\rightarrow$ $z(t) = \frac{1}{2} m(t)$;

Lower-Sideband Amplitude Modulation (LSB-AM)

- **modulation:** $H[m(t)] = x_l(t) = m(t)\, cos\, \omega_c t + \hat{m}(t)\, sin\, \omega_c t \overset{\mathcal{F}}{\longleftrightarrow} X_l(\omega) = M(\omega + \omega_c)\, u(\omega + \omega_c) + M(\omega \Leftrightarrow \omega_c)\, u(\Leftrightarrow \omega + \omega_c)$;
- **demodulation:** $y(t) = x_l(t)\, cos\, \omega_c t = \cdots = \frac{1}{2} m(t) + \frac{1}{2} m(t)\, cos\, 2\omega_c t + \frac{1}{2} \hat{m}(t)\, sin\, 2\omega_c t$ $\Leftrightarrow\!\rightarrow$ LPF $\Leftrightarrow\!\rightarrow$ $z(t) = \frac{1}{2} m(t)$;

Remarks on SSB-AM

- Hilbert transformer is a system with $h_{HT}(t) = \frac{1}{\pi t}$ and $H_{HT}(\omega) = \Leftrightarrow j\, sgn(\omega) = e^{-j\frac{\pi}{2} sgn(\omega)}$ (phase shift); $\hat{x}(t) = x(t) \otimes h_{HT}$;
- **power efficient:** no carrier transmitted;

- **bandwidth efficient:** now the modulated signal $x(t)$ takes up the same amount of bandwidth as the message signal: $B_X = B_M$.
- you can demodulate SSB-AM using imperfect carrier $cos(\omega_c t + \theta)$: received signal is $\frac{1}{2} cos(\theta)\, m(t)$: fine as long as $cos(\theta) \neq 0$ (see notes 15);
- you can demodulate SSB-AM using imperfect carrier $cos([\omega_c + \Delta\omega]t)$ (see notes 15);
- SSB-AM: **power-efficient, no waste of bandwidth, requires synchronized reception.**

Design of AM Receivers (DSB-AM-LC)

- total received signal is $x_{tot} = \sum_{n=1}^{N} [1 + \mu m_n(t)] \cos \omega_{cn} t$, the sum of N DSB-AM-LC signals; we need message signal $m_n(t)$ only;
- solution A : use a tunable BPF; however it is hard to make a (multi-stage) filter with sufficiently shapr cutoff;
- solution B : heterodyne receiver;
- solution C : superheterodyne receiver;
- **SEE NOTES - SEE NOTES - SEE NOTES - SEE NOTES - SEE NOTES**

Angle Modulation (FM and PM)

- **A: PM:** $x(t) = A\, cos[\omega_c t + \theta(t)]$, with $\theta(t) = \phi_\Delta m(t)$, with $|\phi_\Delta| \leq \pi$ the phase deviation constant;
  $\theta(t) = \phi_\Delta m(t)$ is the **instantaneous phase**;
- **B: FM:** $x(t) = A\, cos[\omega_c t + \theta(t)]$, with $\theta(t) = \omega_\Delta \int_{-\infty}^{t} m(s)ds$, with ω_Δ the frequency deviation constant;
  $\omega(t) = \frac{d}{dt}[\omega_c t + \theta(t)] = \omega_c + \omega_\Delta m(t$ is the **instantaneous frequency**;

- we speak of **NARROWBAND** angle modulation when $|\theta(t)| \ll 1$, hence $e^{j\theta(t)} \approx 1 + j\theta(t)$, so $x(t) \approx cos\, \omega_c t \Leftrightarrow \theta(t)\, sin\, \omega_c t$;
- note that in narrowband angle modulation a **LARGE CARRIER** $cos\, \omega_c t$ is transmitted (inefficient!)

NARROWBAND PHASE MODULATION (NB-PM)

- **modulation:** $\theta = \phi_\Delta m(t)$, so $H[m(t)] = x(t) = cos\, \omega_c t \Leftrightarrow \phi_\Delta m(t)\, sin\, \omega_c t \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) = \pi[\delta(\omega \Leftrightarrow \omega_c) + \delta(\omega + \omega_c)] +$
  $+ \frac{j}{2} \phi_\Delta [M(\omega \Leftrightarrow \omega_c) \Leftrightarrow M(\omega + \omega_c)]$ (similar to DSB-AM-LC);

NARROWBAND FREQUENCY MODULATION (NB-FM)

- **modulation:** $\theta = \omega_\Delta \int_{-\infty}^{t} m(s)ds$, with $|\theta(t)| \ll 1$, hence $m(t)$ cannot have a DC component, hence $M(\omega)|_{\omega=0} = 0$;
  $H[m(t)] = x(t) = cos\, \omega_c t \Leftrightarrow \theta(t)\, sin\, \omega_c t \stackrel{\mathcal{F}}{\longleftrightarrow} X(\omega) = \pi[\delta(\omega \Leftrightarrow \omega_c) + \delta(\omega + \omega_c)] + \frac{j}{2}[\Theta(\omega \Leftrightarrow \omega_c) \Leftrightarrow \Theta(\omega + \omega_c)]$,
  with $\Theta(\omega) = \omega_\Delta \frac{M(\omega)}{j\omega}$, so $X(\omega) = \pi[\delta(\omega \Leftrightarrow \omega_c) + \delta(\omega + \omega_c)] + \frac{\omega_\Delta}{2}[\frac{M(\omega - \omega_c)}{\omega - \omega_c} \Leftrightarrow \frac{M(\omega + \omega_c)}{\omega + \omega_c}]$;

Examples of Systems and their Properties

			time			**BIBO**
	system	**linear**	**invariant**	**causal**	**memoryless**	**stable**
(CT-1)	$y(t) = x(t) \otimes sin(t)u(t)$	yes	yes	yes	no	
(CT-2)	$y(t) = x(t) \cdot cos(t)u(t)$	yes	no	yes	yes	
(CT-3)	$y(t) = \int_{-\infty}^{t} \sqrt{x(s)}ds$	no	yes	yes	no	
(CT-4)	$y(t) = x(2t) + 1$	no	no	no	no	
(CT-5)	$y(t) = x(t) \otimes e^t u(t+1)$	yes	yes	no	no	
(CT-6)	$y(t) = x(\Leftrightarrow t)$	yes	no	no	no	
(CT-7)	$y(t) = \int_{-\infty}^{t-1} x(s)ds$	yes	yes	yes	no	
(CT-8)	$y(t) = \int_{-\infty}^{t} e^{-(t-s)} x^2(s)ds$	no	yes	yes	no	
(CT-9)	$y(t) = \lvert x(t) \rvert$	no	yes	yes	yes	
(CT-10)	$y(t) = \int_{-\infty}^{\infty} e^{-\lvert t-s \rvert} x(s)ds$	yes	yes	no	no	
(DT-1)	$y[n] = 0$	yes	yes	yes		yes
(DT-2)	$y[n] = cos(n)2^{x[n]}$	no	no	yes		yes
(DT-3)	$y[n] + y[n \Leftrightarrow 1] = x[n]$	yes	yes	yes		no
(DT-4)	$y[n] = \sum_{k=-\infty}^{n} x[k]$	yes	yes	yes		no

Electronics

1.	$v(t) = i(t)R$	**RESISTOR**
2.	R_1 and R_2 in series equals $R_1 + R_2$	I same; $V = V_1 + V_2$ (splits)
3.	R_1 and R_2 in parallel equals $(R_1^{-1} + R_2^{-1})^{-1}$	V same; $I = I_1 + I_2$ (splits)
4.	$i(t) = C\frac{dv(t)}{dt}$	**CAPACITOR**
5.	C_1 and C_2 in series equals $(C_1^{-1} + C_2^{-1})^{-1}$	q (charge) same; $V = qC^{-1}$
6.	C_1 and C_2 in parallel equals $C_1 + C_2$	V same; $q = CV$
7.	$v(t) = L\frac{di(t)}{dt}$	**INDUCTOR**
8.	L_1 and L_2 in series equals	
9.	L_1 and L_2 in parallel equals	
10.	KCL	Kirchoff's Current Law (**KCL**)
11.	KVL	Kirchoff's Voltage Law (**KVL**)

- **inductors and capacitors (amount of charge) are components with memory.**
- **resistors and opamps are memoryless components.**
- **a circuit with an L and a C in parallel is of order 2; a circuit with two L's or C's in parallel is of order 1.**

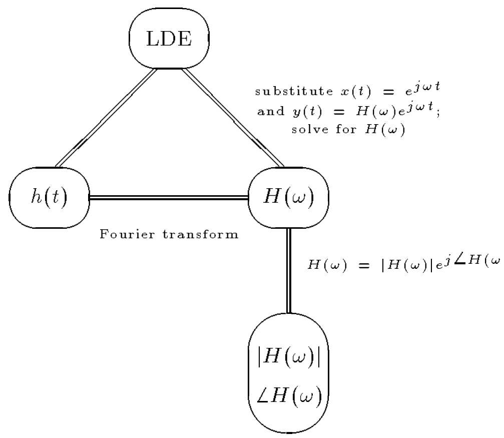

Figure 0.1: *Relations between System Descriptors.*

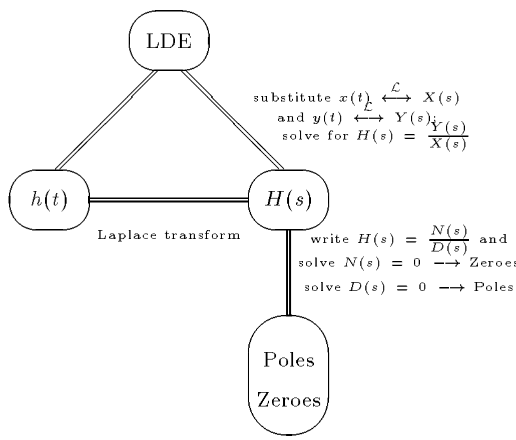

Figure 0.2: *Relations between System Descriptors.*