Discrete Mathematics: Homework 5

Name ID: Number

2020.4.8

1. Let p be an odd prime and let $\mathbb{Z}_p{}^* = \{[1]_p, [2]_p, \ldots, [p-1]_p\}$.

   (a) Show that $([a]_p)^2 = [1]_p$ iff $[a]_p \in \{[1]_p, [p-1]_p\}$.

   (b) Show that $[1]_p \cdot [2]_p \cdot \ldots [p-1]_p = [-1]_p$ and thus conclude that $(p-1)! \equiv -1\ (mod\ p)$.

   (a) 证明. $[1]_p = 1 + np, n \in \mathbb{N}^*$. $[p-1]_p = p - 1 + np = (n+1)p - 1$. $[a]_p = a + np$

   prove **IF**:

   when $a + np = 1 + np$, $([a]_p)^2 = (1+np)^2 = 1 + (2n + n^2p)p = [1] + p$

   when $a + np = (n+1)p - 1$, $([a]_p)^2 = (n^2 - 1)p + 1 = [1] + p$

   prive **ONLY IF**:

   $[a]_p^2 = a^2 + 2anp + n^2p^2 = a^2 + (2an + n^2p)p = 1 + mp = [1]_p$ where $m, n \in \mathbb{N}^*$

   so, $a^2 = [1]_p$,$a = [\pm 1]_p$

   □

   (b) 证明. Because p is an odd prime number,

   $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$

   So $\forall a \in \mathbb{N}^*, 0 < a < p, \exists a^{-1} \in [1, p-1], s.t.\ aa^{-1} \equiv 1\ (mod\ p)$

   $\forall a \in (1, p-1)$, we can find a unique $a^{-1} \in (1, p-1)$

   By the muplication of $\mathbb{Z}_n$, $[2]_p \cdot \ldots [p-2]_p = [1]_p$

   And obviously, $[1]_p \cdot [p-1]_p = [-1]_p$

   So, $[1]_p \cdot [2]_p \cdot \ldots [p-1]_p = [(p-1)!]_p = [-1]_p$

   Which equals to $(p-1)! \equiv -1\ (mod\ p)$

   □

2. Let x, y, z be integers such that $x^2 + y^2 \equiv 3z^2$ $(mod\ 4)$. Show that x, y, z must be all even. Based on this result, show that the equation $x^2 + y^2 = 3z^2$ has no other integersolutions except $(x, y, z) = (0, 0, 0)$.

证明. $x^2 + y^2 \equiv 3z^2$ $(mod\ 4)$, we have$x^2 + y^2 = 3z^2 + 4n, n \in \mathbb{Z}$
so, $\frac{x^2+y^2-3z^2}{4}$ is an integer. $4 \mid x^2 + y^2 - 3z^2$
if x, y are an odd and an even, z is an even.
Suppose $x = 2k + 1, y = 2q, z = 2r$, we have $(2k + 1)^2 + (2q)^2 - 3(2r)^2\ mod\ 4 = 1$
if x, y are an odd and an even, z is an odd.
Suppose $x = 2k + 1, y = 2q, z = 2r + 1$, we have $(2k + 1)^2 + (2q)^2 - 3(2r + 1)^2\ mod\ 4 = 2$
if x, y are odds, z is an even.
Suppose $x = 2k + 1, y = 2q + 1, z = 2r$, we have $(2k + 1)^2 + (2q)^2 - 3(2r)^2\ mod\ 4 = 2$
if x, y are odds, z is an odd.
Suppose $x = 2k + 1, y = 2q + 1, z = 2r + 1$, we have $(2k + 1)^2 + (2q)^2 - 3(2r)^2\ mod\ 4 = 3$
where $k, q, r \in \mathbb{Z}$
So, x, y, z must be all even.

let $x^2 + y^2 \equiv 3z^2$ $(mod\ 4)$
let $x = 2k, y = 2q, z = 2r$, so we have $x^2 + y^2 = 4(k^2 + q^2) = 12(r^2) = 3z^2$
and we have $k^2 + q^2 = 3r^2$
and $|k| < |x|, |q| < |y|, |r| < |z|$ or $k = x = 0, q = y = 0, r = z = 0$
Assume the smallest integer positive solution is x_0, y_0, z_0, but there exists $k_0 = \frac{1}{2}x_0, q_0 = \frac{1}{2}y_0, r_0 = \frac{1}{2}z_0$.
So nonzero integer solution doesn't exists.
$(x, y, z) = (0, 0, 0)$

□

3. Let a_1, a_2, a_3, a_4 be arbitrary integers. Find ALL integer solutions of the following equation system.

$$\begin{cases} x \equiv a_1 \pmod{11}; \\ x \equiv a_2 \pmod{13}; \\ x \equiv a_3 \pmod{17}; \\ x \equiv a_4 \pmod{19}; \end{cases}$$

Solution:

$n = 11 \cdot 13 \cdot 17 \cdot 19 = 46189$

$m_1 = 4199 \quad m_2 = 3553 \quad m_3 = 2717 \quad m_4 = 2413$

$m_1^{-1} = 7 \quad m_2^{-1} = 10 \quad m_3^{-1} = 11 \quad m_4^{-1} = 18$

$c_1 = 29392 \quad c_2 = 35530 \quad c_3 = 29887 \quad c_4 = 43758$

$a = \sum_{i=1}^{4} a_i c_i \pmod{n} = 29392a_1 + 35530a_2 + 29887a_3 + 43758a_4 \pmod{46189}$

$x \in [29392a_1 + 35530a_2 + 29887a_3 + 43758a_4]_{46189}$

4. A composite integer N that satisfies the congruence $b^{N-1} \equiv 1\ (mod\ N)$ for all positive integers b with $gcd(b, N) = 1$ is called a Carmichael number. Suppose that $N = p_1p_2p_3$ is an integer, where p_1, p_2, p_3 are primes such that $(p_i-1)|(N-1)$ for $i = 1, 2, 3$. Show that N is a Carmichael number.

证明. By Fermat's Little Theorem, we have,

$b^{p_1-1} \equiv 1 \pmod{p_1}$

$b^{p_2-1} \equiv 1 \pmod{p_2}$

$b^{p_3-1} \equiv 1 \pmod{p_3}$

$N - 1 = \mathbb{Z}(p_i - 1)$, $b^{N-1} = b^{\mathbb{Z}(p_i-1)}$

$\forall x \equiv 1 \pmod{y}$, we have $x^n \equiv 1 \pmod{y}$ Because $x = 1 + \mathbb{Z}y$ and $x^n = 1 + (2y + \mathbb{Z}y^2)\mathbb{Z}$

$\forall x \equiv 1 \pmod{y}$, we have $x \equiv 1 \pmod{ky}$ Because $x = 1 + \mathbb{Z}y$ and $x = 1 + \mathbb{Z}y + \mathbb{Z}y$

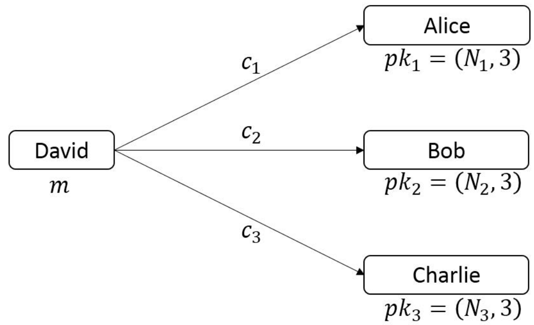
So, $b^{N-1} \equiv b^{3(N-1)} \equiv b^{p_1-1}b^{p_2-1}b^{p_3-1} \pmod{p_1p_2p_3}$

Which has the same reminder 1 as $b^{p_1-1} \equiv 1 \pmod{p_1}$

So, $b^{N-1} \equiv 1\ (mod\ N)$

So, N is a Carmichael number. □

5. See the following figure. The RSA public keys of Alice, Bob and Charlie are $pk1 = (N_1, 3)$, $pk2 = (N_2, 3)$ and $pk3 = (N_3, 3)$, respectively. David wants to send a private message m to Alice, Bob and Charlie, where m is an integer and $0 < m < N_i$ for $i = 1, 2, 3$. In order to keep m secret from an eavesdropper Eve, David encrypts m as $c_1 = m^3 \bmod N_1$, $c_2 = m^3 \bmod N_2$ and $c_3 = m^3 \bmod N_3$; and then sends $c_1$to Alice, c_2 to Bob and c_3 to Charlie.



Suppose that N_1, N_2, N_3 are pairwise relatively prime. Show that with the knowledge of all public keys and all ciphertexts, Eve can decide the value of m.

证明. By RSA Theorem, we have

$$\begin{cases} x^3 \equiv c_1 \pmod{N_1} \\ x^3 \equiv c_2 \pmod{N_2} \\ x^3 \equiv c_3 \pmod{N_3} \end{cases}$$

so, $n = N_1N_2N_3$

$m_1 = N_2N_3 \quad m_2 = N_1N_3 \quad m_3 = N_1N_2$

$c_1 = m_1m_1^{-1} \quad c_2 = m_2m_2^{-1} \quad c_3 = m_3m_3^{-1}$

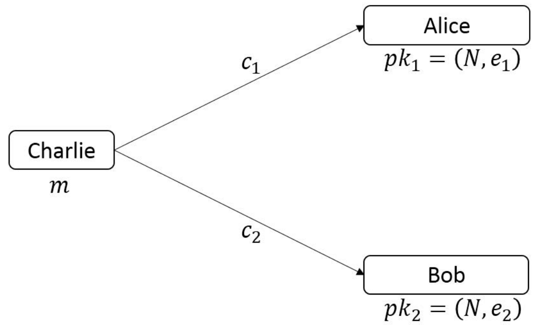
$a = \sum_{i=1}^{k} a_ic_i \pmod{n}$

$x^3 \in [\sum_{i=1}^{3} a_ic_i]_n$

so we can try to solve every x as a message.

□

6. See the following figure. Alice and Bob trust each other very much. They set their RSA public keys as $pk1 = (N, e1)$ and $pk2 = (N, e2)$, respectively. Charlie wants to send aprivate message m to Alice and Bob, where $0 \leq m < N$ is an integer and $gcd(m, N) = 1$. In order to keep m secret from an eavesdropper Eve, Charlie encrypts m as $c_1 = m^{e1} \bmod N$ and $c_2 = m^{e2} \bmod N$; and then sends $c1$ to Alice and $c2$ to Bob.



Suppose that $gcd(e1, e2) = 1$. Show that with the knowledge of all public keys and all ciphertexts, Eve can decide the value of m.

证明. By RSA Theorem, we have

$$\begin{cases} x^{e_1} \equiv c_1 \pmod N \\ x^{e_2} \equiv c_2 \pmod N \end{cases}$$

□

Because $gcd(e_1, e_2) = 1$, we have $x_1e_1 + y_1e_2 = gcd(e_1, e_2) = 1$ by EXGCD.
so, $x^{x_1e_1+y_1e_2} \equiv c_1^{x_1} + c_2^{x_2} \equiv x^1 \equiv \pmod N$, and N is a prime. so, Eve can solve m by solving $m \equiv c_1^{x_1} + c_2^{x_2} \pmod N$