Discrete Mathematics: Homework 5

(Deadline: 10:00am, April 10, 2020)

1. (25 points) Let $a \in \mathbb{Z}, b \in \mathbb{Z}^+$ and $x \in \mathbb{R}$. Show that there exist unique $q, r \in \mathbb{Z}$ such that $a = bq + r$ and $x \leq r < x + b$.

   (Hint: need to show existence and uniqueness; apply the division algorithm to $(a - \lceil x \rceil, b)$)

2. (15 points) Let $a, b > 1$ be relatively prime integers. Show that if $a|n$ and $b|n$, then $ab|n$.

   (Hint: Bézout's theorem)

3. (25 points) Let $a, b_1, b_2, \ldots, b_k \in \mathbb{Z}^+$. Show that $\gcd(a, b_1 b_2 \cdots b_k) = 1$ if and only if $\gcd(a, b_i) = 1$ for every $i \in [k]$.

   (Hint: fundamental theorem of arithmetic or Bézout's theorem)

4. (25 points) Let $x \in \mathbb{R}$ and $n \in \mathbb{Z}^+$. Show that $\left\lfloor \frac{\lfloor x \rfloor}{n} \right\rfloor = \left\lfloor \frac{x}{n} \right\rfloor$.

   (Hint: division algorithm)

5. (15 points) Let $a, b \in \mathbb{Z}, n \in \mathbb{Z}^+$ and $a \equiv b \pmod{n}$. Let $c_0, c_1, \ldots, c_k \in \mathbb{Z}$, where $k \in \mathbb{Z}^+$. Show that $c_0 + c_1 a + \cdots + c_k a^k \equiv c_0 + c_1 b + \cdots + c_k b^k \pmod{n}$.

   (Hint: show that $a^i - b^i$ is a multiple of n)

6. (15 points) Let p be a prime and $p \notin \{2, 5\}$. Show that p divides infinitely many elements of the set $\{9, 99, 999, 9999, 99999, \ldots\}$.

   (Hint: Euler's Theorem, consider $([10]_p)^{p-1}$)