Discrete Mathematics: Homework 11

Name ID: Number

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1. A group of n students is assigned seats for each of two classes in the same classroom. How many ways can these seats be assigned if no student is assigned the same seat for both classes?

   Solution:

   For n elements, use the ***Derangement Fomular****:*

   $$D_n = \lfloor \frac{n!}{e} + 0.5 \rfloor$$

   consider the first seating, we have $n!$ ways to sit.
   So, the answer is $n! \cdot \lfloor \frac{n!}{e} + 0.5 \rfloor$

2. Suppose that a park has 10 ponds and 17 birds sitting on the ponds. No two ponds are exactly the same and no two birds are exactly the same, so we countthem separately. Each pond has at least one bird. How many possibilities? You may leave your answer in reduced form.

   Solution:

   When considering put 17 different birds into 10 same ponds, ww can use ***Stirling Number of the Second Kind****.*

   $$S(n, r) = S(n-1, r-1) + S(n-1, r) \cdot r = \frac{1}{m!} \sum_{k=0}^{m} C_m^k (m-k)^r (-1)^k, n > r > 1$$

   And, $S(16, 10) = \frac{1}{16!} \sum_{k=0}^{16} C_{16}^k (16-k)^{10} (-1)^k$
   Using ***multiply principle****, we have the answer $10! \frac{1}{16!} \sum_{k=0}^{16} C_{16}^k (16-k)^{10} (-1)^k$*

3. Suppose we have just 3 yuan coins, 4 yuan coins , 7 yuan coins and 9 yuan coins. How many ways to make 23 yuan?

   Solution:

   Consider $f(x) = (1 + x^3 + x^6 + \cdots)(1 + x^4 + x^12 + \cdots)(1 + x^7 + x^14 + \cdots)(1 + x^9 + x^18 + \cdots)$
   The solution is r where $f(x) = \cdots + rx^23 + \ldots$
   By ***Wolfram Alpha****, we have $r = 8$*

4. Solve the recurrence relation $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}, h_0 = 0, h_1 = 1$ and $h_2 = 2$

Solution:

Consider the equation $x^3 - x^2 - 9x + 9 = 0$

then we have $x_1 = 1, x_2 = 3, x_3 = -3$

let $h_n = c_1 + c_2 3^n + c_3(-3)^n$

$h_0 = c_1 + c_2 + c_3 = 0$

$h_1 = c_1 + 3c_2 - 3c_3 = 1$

$h_2 = c_1 + 9c_2 + 9c_3 = 2$

we have the solution:

$$f(x) = \begin{cases} c_1 = -\dfrac{1}{4} \\ c_2 = \dfrac{1}{3} \\ c_3 = -\dfrac{1}{12} \end{cases}$$

so the answer is, $h_n = -\frac{1}{4} + 3^n - \frac{1}{12}(-3)^n$

5. Solve the recurrence relation $h_n = 3h_{n-2} - 2h_{n-3}.h_0 = 1, h_1 = 0$ and $h_2 = 0$

Solution:

$h_n - 3h_{n-2} + 2h_{n-3} = 0$

we can solve the equation:

$x^2 - 3x + 2 = 0$ *we have* $x_1 = 1$ *and* $x_2 = -2$

so, $h_n = c_1(1)^n + c_2(-2)^n + c_3 n$

we have $1 = h_0 = c_1 + c_2$ *and* $0 = h_1 = c_1 - 2c_2 + c_3$ *and* $0 = h_2 = c_1 + 4c_2 + 2c_3$

$$\textit{the solution is } f(x) = \begin{cases} c_1 = \dfrac{8}{9} \\ c_2 = \dfrac{1}{9} \\ c_3 = -\dfrac{2}{3} \end{cases}$$

so, the answer is $h_n = \frac{8}{9} + \frac{1}{9}(-2)^n - \frac{2}{3}n$

6. Suppose $a_1, \ldots, a_{65}$ is a seqence of numbers. Suppose that for $i \neq j$ then $a_i \neq a_j$. Then prove that either there is an increasing subsequence of length 10 or a decreasing subsequence of length 8.

证明. By THEOREM, Every sequence of $n^2 + 1$ distinct real numbers contains a subsequence of length $n + 1$ that is either strictly increasing or strictly decreasing.

So there are a seqence of length 9 either increasing or decreasing. □

7. Consider the set $S = \{1, \ldots, n\}$, and suppose S_1 and S_2 are non-emptysubsets with $S = S_1 \cup S_2$ with $S_1 \cap S_2 = \emptyset$. Let $A \subset S \times S \times S$ be the subset consisting of elements $(x, y, z) \in S \times S \times S$ satisfying both of the following 2 conditions:

   (a) All of the elements x, y, z are in S_1 or all of the elements x, y, z are in S_2.

   (b) $x + y + z$ is divisible by n

   Question: Determine all the possible values of n for which there exists non-empty subsets S_1, S_2 such that$\{1, \ldots, n\} = S_1 \cup S_2$,$S_1 \cap S_2 = \emptyset$ and with the above definition $|A| = 36$?