Discrete Mathematics: Homework 4

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1. Show that a set S is infinite if and only if there is a proper subset A $\subset$ S such that |A|=|S|

   Solution:

   Prove ***ONLY IF****:*

   If S is infinite and countable, there exists an infinite subset $\{S_1, S_2, S_3, \dots\} \subset S$

   Let $A = \{S_{i+1}\}$ *and* $A \subset S$

   $|A| = \infty - 1 = |S| = \infty$

   Prove ***IF****:*

   Suppose S is finite, $\forall i \in \mathbb{N}^*, A = a_i, S = s_i, a_i = s_i, |S| = |A|$.

   Because $A \subset S$, $\exists b \in (S - A) \neq \emptyset$, *but b doesn't exist.*

   So S is not finite. S is infinite.

2. Prove or disprove$|\{(x, y) \in R^2 : x^2 + y^2 = 1\}| = |R|$.

   Solution:

   f: (x,y) $\to \mathbb{R} : (x, y) \mapsto \frac{y}{x}$ *is a bijection.*

3. Prove or disprove|{(x, y)$\in \mathbb{R}^2 : x^2 + y^2 < 1$}|=|R| .

   Solution:

   f: (x,y) $\to \mathbb{R} : (x, y) \mapsto \tan(\frac{1}{2}\pi xy)$

   f: $\mathbb{R} \to (x, y) : (a \in \mathbb{R}, b \in \mathbb{R}) \mapsto (\frac{2\arctan(a)}{\pi}, \frac{2\arctan(b)}{\pi})$

4. Prove or disprove $|\{X \subseteq \mathbb{Z}^+ : X \ is\ a\ finite\ set\}| = |\mathbb{Z}^+|$

Solution:

$A = \{X \subseteq \mathbb{Z}^+ : X \ is\ a\ finite\ set\}/= a_i$

(a) $f : A \to \mathbb{Z}^+ : a_1, a_2, \ldots, a_n \mapsto N_i = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$ *is an injection.*

(b) $\mathbb{Z}^+$ *is an infinite countable set. so there exists* $A = \{z_1, z_2, z_3, \ldots\} \in \mathbb{Z}^+$

5. Prove or disprove $|\{(a_1, a_2, a_3, \ldots) : a_1, a_2, a_3, \cdots \in \mathbb{Z}^+\}| = |\mathbb{Z}^+|$

Solution:

Let $A = \{(a_1, a_2, a_3, \ldots) : a_1, a_2, a_3, \cdots \in \mathbb{Z}^+\}$

$A \to \mathbb{Z}^+ : a_i \mapsto i$

$\mathbb{Z}^+ \to A : i \mapsto a_i$

So, $A \to \mathbb{Z}^+$ *is a bijection.*

6. Find a countably infinite number of subsets of $\mathbb{Z}^+$, say $A_1, A_2, \cdots \in \mathbb{Z}^+$, such that the following requirements are simultaneously satisfied:

(a) A_i is countably infinite for every $i = 1, 2, \ldots;$

(b) $A_i \cap A_j = \emptyset, \ \forall i \neq j;$

(c) $\cup_{i=1}^{\infty} A_i = \mathbb{Z}^+$.

Solution:

Let $P = \{x | xisprimenumber\}$

$A_i = \{y | y = \prod_{j=0}^{j \leq \infty} {p_j}^{e^j}, \sum e^j = i\}$

e.g. $A_1 = \{1, 2, 3, \ldots\},\ A_2 = \{1 \times 2, 1 \times 3, \ldots, 2 \times 2, 2 \times 3, \ldots\}$

(1) *It's obvius that* A_i *is countably infinite for* $i \in \mathbb{N}^*$

(2) *By Fundamental Theorem of Arithmetic, Every integer* $n > 1$ *can be uniquely written as* $n = {p_1}^{e_1} {p_2}^{e_2} \ldots {p_r}^{e_r}$, *where* $p_1 < p_2 < \cdots < p_r$ *are primes and* $e_1, e_2, \ldots, e_r \geq 1$. *Uniqueness gets (b) and existence gets (c).*