3D Renderer of Implicit Surfaces

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Signed Distance Functions : Distance Surfaces

From *“Sphere tracing: a geometric method for antialiased ray tracing of implicit surfaces”* an implicit surface is defined by a function that, given a point in space, indicates whether the point is inside, on or outside the surface.

Definition (Distance Surface)

A distance surface is implicitly defined by a function $f : \mathbb{R}^3 \to \mathbb{R}$ that characterizes $A \subset \mathbb{R}^3$, set of points that are on or inside the implicit surface:

$$A = \{x : f(x) \leq 0\}$$

The surface can be also defined with $f^{-1}(0)$, which gives exactly the points on the surface.

Signed Distance Functions : Point-To-Set Distance

We can define the surface from the outside using a *point-to-set* distance:

$$d(x, A) = min_{y \in A} ||x - y||$$

Thus $d(x, A)$, given a point $x \in \mathbb{R}^3$, returns the shortest distance to the surface defined by A.
We can interchange $d(x, A)$ with $d(x, f^{-1}(0))$ even if they are slightly different, because here we'll handle points that aren't in surfaces interiors.

Signed Distance Functions : Bound and SDF

Definition (Signed Distance Bound)

We say that $f : \mathbb{R}^3 \to \mathbb{R}$ is a signed distance bound of its implicit surface $f^{-1}(0)$ if and only if

$$|f(x)| \leq d(x, f^{-1}(0)) \tag{1}$$

Definition (Signed Distance Function)

We say that f is a signed distance function (SDF) when holds

$$|f(x)| = d(x, f^{-1}(0)) \tag{2}$$

The first definition says that a signed distance bound is always at least as cautious as the true distance function $d(x, f^{-1}(0))$ or the SDF.

Signed Distance Functions : DUFs and DIFs

Two other names for the concepts of the previous slide are *distance underestimate (implicit) functions* (DUFs) and *distance implicit functions* (DIFs).
Remembering that $DUF(x) \leq DIF(x)$ for every DUFs and DIFs respecting the definitions above.
We will interchange Signed Distance Bound with DUF and SDF with DIF, because they express the same concepts.

Signed Distance Functions : Lipschitz

Definition (Lipschitz Function)

We say that a function $f : \mathbb{R}^3 \to \mathbb{R}$ is Lipschitz over a domain D if and only if there is a positive, finite, constant λ such that

$$|f(x) - f(y)| \leq \lambda ||x - y|| \tag{3}$$

Such λ is called the Lipschitz constant.

There is no upper limit to λ, but there is a lower bound. Let $Lip(f)$ be the function returning such minimum value.

Signed Distance Functions : Lipschitz

Manipulating 3 we can observe that

$$\lambda \geq \frac{|f(x) - f(y)|}{||x - y||}$$
$$\lambda \geq \frac{f(x) - f(y)}{||x - y||}$$
$$\lambda \geq \lim_{||x-y|| \to 0} \frac{f(x) - f(y)}{x - y} = f'(x)$$

the last step, given the continuity of f, permits us to estimate a safe lower bound for the Lipschitz constant.

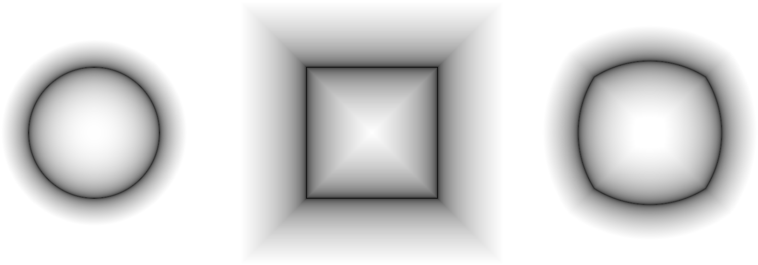
Theorem

Let f be Lipschitz with Lipschitz constant λ. Then the function f/λ is a SDF of its implicit surface.

Proof.

In the assignment. $\square$

Signed Distance Functions : Examples



The images can be seen as 2D section of 3D objects done with a yz-aligned plane passing through the origin.

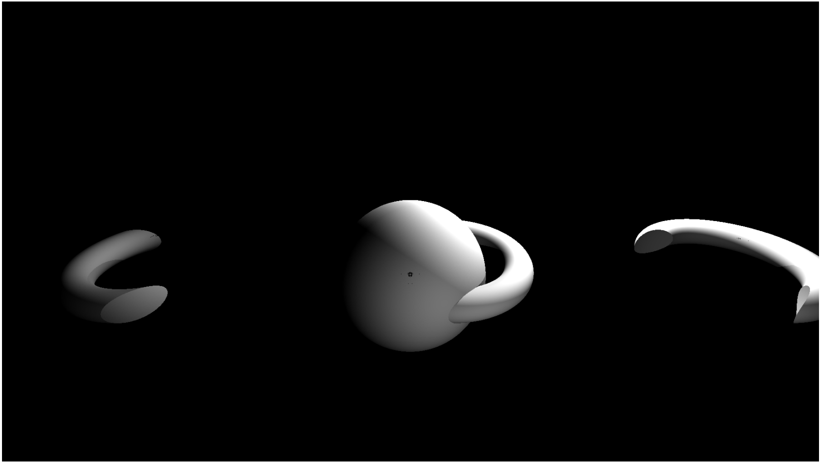
$$\begin{aligned} \text{circle} &: x^2 + y^2 - 1 \\ \text{square} &: max(|x|, |y|) - 1 \\ \text{squircle} &: k * (max(|x|, |y|) - 1) + (1 - k) * (x^2 + y^2 - 1) \\ &\quad \text{where } k \in [0, 1] \end{aligned}$$

Signed Distance Functions : Constructive Solid Geometry

SDFs make easy to create complex shapes from few simple primitives. This technique it known as Constructive Solid Geometry (CSG).

- union $min(f_1, f_2)$
- intersection $max(f_1, f_2)$
- subtraction $max(f_1, -f_2)$
- mixing $k * f_1 + (1 - k) * f_2$ with $k \in [0, 1]$

Signed Distance Functions : Constructive Solid Geometry



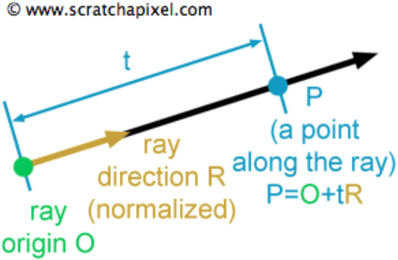
From left to right: intersection, union and subtraction of a sphere and a torus.

Ray Tracing : Rays and Cameras

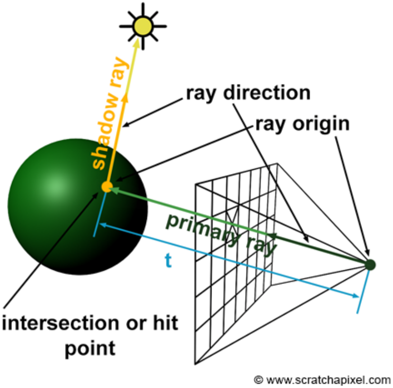
A ray is defined as a point and a direction in formulas:

$$r(t) = O + tR$$

where $O, R \in \mathbb{R}^3$ and R is a unit vector.
At each time t we can compute a point position on the ray.



Ray Tracing : Rays and Cameras



A pinhole camera inside a scene with a sphere and a light source.

Ray Tracing : Sphere Tracing

Ray Marching proceeds along the rays with a fixed step, with sphere tracing an adaptive step is used.