Grolmusz Verification

Nicole Tian

May 2025

0.1 Theorem 1.2.

Let m be a positive integer, and suppose that m has $r > 1$ different prime divisors: $m = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$. Then there exists $c = c(m) > 0$, such that for every integer $h > 0$, there exists an explicitly constructible uniform set-system $\mathcal{H}$ over a universe of h elements, such that

1. $|\mathcal{H}| \geq exp(c\frac{(\log h)^r}{(\log\log h)^{r-1}}$
2. $\forall H \in \mathcal{H} : |H| \equiv 0 \pmod{m}$
3. $\forall G, H \in \mathcal{H}, G \neq H : |G \cap H| \not\equiv 0 \pmod{m}$

The value of c is roughly p_r^{-r} where p_r is the largest prime factor.

0.2 Lemma 3.2.

For every integer $n > 0$, there exists a uniform set-system $\mathcal{H}$ over a universe of $2(m-1)n^{2d}/d!$ elements which is explicitly constructible from the polynomial Q and satisfies

1. $|\mathcal{H}| = n^n$
2. $\forall H \in \mathcal{H} : \ |H| \equiv 0 \pmod{m}$
3. $\forall G, H \in \mathcal{H}, G \neq H : |G \cap H| \not\equiv 0 \pmod{m}$

(Point of Confusion) Lemma 3.2 easily yields Theorem 1.2 setting $d = \Theta(n^{\frac{1}{r}})$ and using elementary estimations for the binomial coefficients.

0.3 Idea

Assume we want a universe of $h = \frac{n^{2\sqrt{n}}}{(\sqrt{n})!}$ elements. WLOG let $m = p_1^{\alpha_1} p_2^{\alpha_2}$ and by Grolmusz's assertion, let $d = \Theta(n^{\frac{1}{2}})$.

By Lemma 3.2, for any $n > 0$, there exists $\mathcal{H}$ over a universe of $2(m-1)\frac{n^{2(\sqrt{n})}}{(\sqrt{n})!}$ elements. Since we only care about the asymptotes, we can ignore the constant, leaving us with a universe of $\frac{n^{2(\sqrt{n})}}{(\sqrt{n})!}$ elements.

Therefore, we have found a n that gives us the h we're looking for. We can repeat this process for any arbitrary $h > 0$ and find such n. Now we can build a set-system using that n, and by the properties of Lemma 3.2, that set-system already satisfies (2) and (3) of Theorem 1.2.

The only thing left is to prove (1):

$$|H| \geq exp(c\frac{(\log h)^r}{(\log\log h)^{r-1}}$$

By Lemma 3.2, $|\mathcal{H}| = n^n$, and using Stirling's approximation: $\ln(n!) = n\ln(n) - n + O(\ln n)$, we can verify that this set system holds under (1) at the asymptotes:

$$\begin{aligned}
\log h &= 2\sqrt{n}\log n - \log(\sqrt{n}!) &(1)\\
&= 2\sqrt{n}\log n - \left(\frac{1}{2}\sqrt{n}\log n + \sqrt{n}\right) &(2)\\
&= 1.5\sqrt{n}\log n &(3)
\end{aligned}$$

$$\log\log h = \log(1.5) + \frac{1}{2}\log\ n + \log\log n$$

Our claim: $|\mathcal{H}| = n^n \geq \exp(c\frac{(\log h)^2}{(\log\log h})$. By applying log to both sides we get:

$$\begin{aligned}
n\log n &\geq c\frac{(\log h)^2}{(\log\log h} &(4)\\
&= c\frac{[1.5\sqrt{n}\ \log n]^2}{\log(1.5) + \frac{1}{2}\log\ n + \log\log n} &(5)\\
&\geq \frac{c * 1.5^2 * n\log^2 n}{\frac{1}{2}\log\ n + \log n} &(6)\\
&= c * n\log n &(7)
\end{aligned}$$

which holds due to $c < 1$ always.