1 Grolmusz's Construction

Theorem 1.1. *For each $n \in \mathbb{N}$ there exists a family $\mathcal{A}$ of n^n subsets over a universe of size $n^{O(\sqrt{n})}$ with the following properties:*

1. *For all $A \in \mathcal{A}$, $|A| \equiv 0 \pmod 6$.*

2. *For all $A \neq A' \in \mathcal{A}$, $|A \cap A'| \not\equiv 0 \pmod 6$.*

Proof.

Define the function $\delta(u, v) = \begin{cases} 0 & u = v \\ 1 & u \neq v \end{cases}$

Using δ we build a square matrix $\mathcal{A} \in M_{n^n \times n^n}$ of vectors $x \in [n]^n$ such that each entry is denoted by

$$a_{xy} = Q(\delta(x_1, y_n), \delta(x_2, y_2), \ldots, \delta(x_n, y_n))$$

Apply the previous corollary to produce a multiset $\mathcal{S} = \{S_1, \ldots, S_D\}$ satisfying the conditions of Corollary 1.3. We see that $\mathcal{A}$ is composed of matrices created with respects to correlating elements of $\mathcal{S}$: $B_{i_1, i_2, \ldots, i_l}$ where $S = \{i_1, i_2, \ldots, i_l\}$ such that each entry

$$B_{xy}^{i_1, i_2, \ldots, i_l} = \delta(x_1, y_1) \times \delta(x_2, y_2) \times \ldots \times \delta(x_l, y_l)$$

We can now see that the diagonals of $\mathcal{A} = a_{xx} = 0 \pmod m$ due to all l-sized blocks of B having a value of 1, giving us the full dimension.

Now we can construct a hypergraph $\mathcal{G}$. Let the vertices be the vectors $x \in [n]^n$. Let the edges be the B-blocks. The maximum number of edges is represented by the equation (assuming $n \geq 2d$)

$$Q(n, n, \ldots, n) = \sum_{l < \leq d} \sum a_{i_1, i_2, \ldots, i_n} n^l < (m-1) \sum_{l \leq d} \binom{n}{l} n^l < \frac{2(m-1)n^{2d}}{d!}$$

This equations represents both the number of variations, $\binom{n}{l}$, and the unique variations of each, n^l

Now we consider the dual graph of $\mathcal{G}$ which we will denote as $\mathcal{H}$. Let the universe be the set of B-blocks and the members of the set be the vectors $x \in [n]^n$.

$$|\mathcal{H}| = n^n$$

$\forall H \in \mathcal{H}, |H| = D$, the number of multisets, making this a uniform system with $Q(1, 1, \ldots, 1)$ elements.

Overlap between any two vertices: x and y, can we represented by the number of B-Blocks the two have in common by construction of the original matrix $\mathcal{A}$:

$$\forall H_x \neq H_y \in \mathcal{H},\ |H_x \cap H_y| = a_{xy} \neq 0 \pmod m$$

The intersection of the same vector $x = y$ can be represented the same way

$$\forall H_x \in \mathcal{H},\ |H_x| = |H_x \cap H_x| = a_{xx} = 0 \pmod{m}$$

$\square$