

# Functions

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## 0.1 Solutions

1 Prove Proposition 3.6.

*Proof.* By proposition 3.5, if there exists function  $G$  such that  $G \circ (g \circ f) = id_{\mathbf{X}}$  and  $(g \circ f) \circ G = id_{\mathbf{Y}}$  then  $(g \circ f)$  is bijective. And by definition of Inverse Functions we know  $G$  is the reverse function of  $(g \circ f)$ .

Due to  $f$  and  $g$  are bijective so exist  $f^{-1}$  and  $g^{-1}$  and due to  $\circ$  is associative so we have

$$f^{-1} \circ g^{-1} (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ id_{\mathbf{Y}} \circ f = f^{-1} \circ f = id_{\mathbf{X}}$$

similarly

$$(g \circ f) \circ f^{-1} \circ g^{-1} = id_{\mathbf{Y}}$$

Hence,  $(g \circ f)$  is bijective and  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$  □

3 (a) *Proof.* Because of  $f : \mathbf{X} \rightarrow \mathbf{Y}$  and  $g : \mathbf{Y} \rightarrow \mathbf{V}$  so

$$\exists \tilde{f} : im(f) \rightarrow \mathbb{X} : \tilde{f} \circ f = id_{\mathbf{X}}$$

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which implies that

$$(\tilde{f} \circ \tilde{g}) \circ (g \circ f) = id_{\mathbf{X}}$$

so  $(f \circ g)$  is injective. Prove of surjective is similar. □

(b) *Proof.* If  $f$  is injective then there exists a unique function  $\tilde{f} : im(f) \rightarrow X$  such that  $\tilde{f} \circ f = id_{\mathbf{X}}$ . Then define function  $\tilde{f}$  in a way that  $\tilde{f}|_{im(f)} = \tilde{f}$ . Finally, we got a function that  $\tilde{f} \circ f = id_{\mathbf{X}}$  □

(c) *Proof.* Similar to (b). □

6 *Proof.* Let  $f$  be a function that

$$f : \mathcal{P}(\mathbf{X}) \rightarrow \{0, 1\}^{\mathbf{X}}, \mathbf{A} \mapsto \mathcal{X}_{\mathbf{A}}$$

$\mathcal{X}$  is a function on  $\mathbf{X}$  taking the values is  $\{0, 1\}$ . Exists  $\mathbf{A} \subseteq \mathbf{X}$  that  $\mathcal{X}_{\mathbf{A}}$  send all element of  $\mathbf{A}$  to 1. Due to domain of  $f$  is all subset of  $\mathbf{X}$  we must have  $\mathbf{A} \in \mathbf{X}$  so  $f$  is surjective.

Define a function

$$\tilde{f} : \{0, 1\}^{\mathbf{X}} \rightarrow \mathcal{P}(\mathbf{X}), \mathcal{X}_{\mathbf{A}} \mapsto \mathbf{A}$$

then  $\tilde{f} \circ f = id_{\mathcal{P}(\mathbf{X})}$  so  $f$  is injective. Hence  $f$  is bijective. □

7 (a) Both domain and codomain of those two functions are same. Then if we prove that their graph are same then they are equal.

Choose arbitrary element  $x \in \mathbf{A}$  we have  $f|_{\mathbf{A}}(x) = (f \circ i)(x)$  directly from definition of restriction and inclusion.

(b)