Functions

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0.1 Solutions

1 Prove Proposition 3.6.

Proof. By proposition 3.5, if there exists function G such that $G\circ(g\circ f) = id_{\mathbf{X}}$ and $(g\circ f)\circ G = id_{\mathbf{V}}$ then $(g \circ f)$ is bijective. And by definition of Inverse Functions we know G is the reverse function of $(g \circ f)$.

Due to f and g are bijective so exist f^{-1} and g^{-1} and due to $\circ$ is associative so we have

$$f^{-1} \circ g^{-1}(g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ id_{\mathbf{Y}} \circ f = f^{-1} \circ f = id_{\mathbf{X}}$$

similarly

$$(g \circ f) \circ f^{-1} \circ g^{-1} = id_{\mathbf{V}}$$

.

Hence, $(g \circ f)$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ □

3 (a) *Proof.* Because of $f : \mathbf{X} \to \mathbf{Y}$ and $g : \mathbf{Y} \to \mathbf{V}$ so

$$\exists \tilde{f} : im(f) \to \mathbb{X} : \tilde{f} \circ f = id_{\mathbf{X}}$$

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which implies that

$$(\tilde{f} \circ \tilde{g}) \circ (g \circ f) = id_{\mathbf{X}}$$

so $(f \circ g)$ is injective. Prove of surjective is similar. □

(b) *Proof.* If f is injective then there exists a unique function $\tilde{f} : im(f) \to X$ such that $\tilde{f} \circ f = id_{\mathbf{X}}$. Then define function $\tilde{f}$ in a way that $\tilde{f}|_{im(f)} = \tilde{f}$. Finally, we got a function that $\tilde{f} \circ f = id_{\mathbf{x}}$ □

(c) *Proof.* Similar to (b). □

6 *Proof.* Let f be a function that

$$f : \mathcal{P}(\mathbf{X}) \to \{\mathbf{0}, \mathbf{1}\}^{\mathbf{X}}, \mathbf{A} \mapsto \mathcal{X}_{\mathcal{A}}$$

$\mathcal{X}$ is a function on $\mathbf{X}$ taking the values is $\{0, 1\}$. Exists $\mathbf{A} \subseteq \mathbf{X}$ that $\mathcal{X}_{\mathbf{A}}$ send all element of $\mathbf{A}$ to 1. Due to domain of f is all subset of $\mathbb{X}$ we must have $\mathbf{A} \in \mathbf{X}$ so f is surjective.

Define a function

$$\tilde{f} : \{0, 1\}^{\mathbf{X}} \to \mathcal{P}(\mathbf{X}), \mathcal{X}_{\mathbf{A}} \mapsto \mathcal{A}$$

then $\tilde{f} \circ f = id_{\mathcal{P}(\mathbf{X})}$ so f is injective. Hence f is bijective. □

7 (a) Both domain and codomain of those two functions are same. Then if we prove that their graph are same then they are equal.

Choose arbitary element $x \in \mathbf{A}$ we have $f|_A(x) = (f \circ i)(x)$ directly from definition of restriction and inclusion.

(b)