Sets

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0.1 Solutions

1 Let $\mathbb{X}$,$\mathbb{Y}$ and $\mathbb{Z}$ bet sets. Prove the transitivity of inclusion, that is,

$$(\mathbb{X} \subseteq \mathbb{Y}) \wedge (\mathbb{Y} \subseteq \mathbb{Z}) \Rightarrow \mathbb{X} \subseteq \mathbb{Z}$$

Proof. Suppose exists an arbitary element $x \in \mathbb{X}$ then $x \in \mathbb{Y}$ by the definition of $\subseteq$, Hence, $x \in \mathbb{Z}$ cause $\mathbb{Y} \subseteq \mathbb{Z}$. $\square$

2 Verify the claims of Proposition 2.4

Proof. Suppose that $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$ are subset of $\mathbb{U}$.

(i) $\mathbb{X} \cup \mathbb{Y} := \{x \in \mathbb{U} : (x \in \mathbb{X}) \vee (x \in \mathbb{Y})\}$ and $\mathbb{Y} \cup \mathbb{Y} := \{x \in \mathbb{U} : (x \in \mathbb{Y}) \vee (x \in \mathbb{X})\}$. Clearly, those two expression are equal by their definition. It's similar to $\mathbb{X} \cap \mathbb{Y}$.

(ii) $\cap$ and $\cup$ are associativity due to associativity of $\wedge$ and $\vee$

(iii) First to prove

$$\mathbb{X} \cup (\mathbb{Y} \cap \mathbb{Z}) \Rightarrow (\mathbb{X} \cup \mathbb{Y}) \cap (\mathbb{X} \cup \mathbb{Z})$$

By definition,

$$\mathbb{X} \cup (\mathbb{Y} \cap \mathbb{Z}) \Rightarrow \forall x \in \mathbb{U} : (x \in \mathbb{X}) \vee (x \in \mathbb{Y} \wedge x \in \mathbb{Z}) \Rightarrow \forall x \in \mathbb{U} : (x \in \mathbb{X} \vee x \in \mathbb{Y}) \wedge (x \in \mathbb{X} \vee x \in \mathbb{Z})$$

prove of another direction is similarly.

(iv) Suppose that $\mathbb{X} \neq \mathbb{Y}$ because the case $\mathbb{X} = \mathbb{Y}$ is trivial. Under this assumption have $\mathbb{X} \subset \mathbb{Y}$. So the proposition can be rewrited as

$$\mathbb{X} \subset \mathbb{Y} \iff \mathbb{X} \cup \mathbb{Y} = \mathbb{Y} \iff \mathbb{X} \cap \mathbb{Y} = \mathbb{X}$$

which is clearly definite true.

$\square$

3 Provide a complete proof of Proposition 2.7.

(i) $(\bigcap_\alpha \mathbf{A}_\alpha) \cap (\bigcap_\beta \mathbf{B}_\beta) = \bigcap_{(\alpha,\beta)} \mathbf{A}_\alpha \cap \mathbf{B}_\beta$.
$(\bigcup_\alpha \mathbf{A}_\alpha) \cup (\bigcup_\beta \mathbf{B}_\beta) = \bigcup_{(\alpha,\beta)} \mathbf{A}_\alpha \cup \mathbf{B}_\beta$ (associativity)

Proof. by definition we have

$$\begin{aligned}(\bigcap_\alpha \mathbf{A}_\alpha) \cap (\bigcap_\beta \mathbf{B}_\beta) &= \{x \in \mathbb{X} : \forall \alpha \in \mathsf{A} : \mathsf{x} \in \mathbf{A}_\alpha\} \cap \{\mathbf{x} \in \mathbb{X} : \forall \beta \in \mathsf{B} : \mathsf{x} \in \mathbf{B}_\beta\} \\ &= \{x \in \mathbb{X} : \forall \alpha \in \mathsf{A}, \forall \beta \in \mathsf{B} : \mathsf{x} \in \mathbf{A}_\alpha \wedge \mathbf{x} \in \mathbf{B}_\beta\} \\ &= \bigcap_{(\alpha,\beta)} \mathbf{A}_\alpha \cap \mathbf{B}_\beta\end{aligned}$$

Prove of $\bigcup$ is similarly. $\square$

(ii) $(\bigcap_\alpha \mathbf{A}_\alpha) \cup (\bigcap_\beta \mathbf{B}_\beta) = \bigcap_{(\alpha,\beta)} \mathbf{A}_\alpha \cup \mathbf{B}_\beta$
$(\bigcup_\alpha \mathbf{A}_\alpha) \cap (\bigcup_\beta \mathbf{B}_\beta) = \bigcup_{(\alpha,\beta)} \mathbf{A}_\alpha \cap \mathbf{B}_\beta$ (distribuitivity)

Proof. By Proposition 2.4(iii) and the definition of $\bigcap$ and $\bigcup$ we have

$$(\mathsf{A}_{\alpha_0} \cap \mathsf{A}_{\alpha_1} ...) \cup (\mathsf{B}_{\beta_0} \cap \mathsf{B}_{\beta_1} ...) = ((\mathsf{A}_{\alpha_0} \cap \mathsf{A}_{\alpha_1} ...) \cup \mathsf{B}_{\beta_0} \cap (\mathsf{A}_{\alpha_0} \cap \mathsf{A}_{\alpha_1} ...) \cup \mathsf{B}_{\beta_1} \cap ...)$$

Then $\bigcap_{(\alpha,\beta)} \mathsf{A}_\alpha \cup \mathsf{B}_\beta$ is got by apply Proposition 2.4(iii) again to right side of this equation. This method can be apply to the second equation of this proposition. ☐

(iii) $(\bigcap_\alpha \mathsf{A}_\alpha)^{\mathsf{c}} = \bigcup_\alpha \mathsf{A}_\alpha^{\mathsf{c}}$
$(\bigcup_\alpha \mathsf{A}_\alpha)^{\mathsf{c}} = \bigcap_\alpha \mathsf{A}_\alpha^{\mathsf{c}}$ (de Morgan's laws)

Proof. Write your proof here. ☐

4 Let $\mathbf{X}$ and $\mathbf{Y}$ be nonempty sets. Show that

$$\mathbf{X} \times \mathbf{Y} = \mathbf{Y} \times \mathbf{X} \iff \mathbf{X} = \mathbf{Y}$$

Proof. ☐