Introduction to Matrix Algebra

To continue our exploration to different forms of algebraic operations (stuff that are often skipped over in your algebra classes), we will dig into a topic that is vital for all forms of data processing and analysis: matrix algebra.

In everything we have explored so far, we have always stayed in realm of operations and solutions that exist in either a single answer ($x = 5$) or a string of answers – like coordinates such as (x, y, z). However, in the real world, we rarely deal with such simple restrictions and constraints. We may have thousands of data points or information that spans numerous dimensions, and the process by which we have been analyzing information becomes extremely muddled tedious through the techniques we have been presented. Instead, we will construct a framework that deals with large sets of information, in what we will call a *matrix*.

Definition:
A **matrix** is a $m \times n$ dimensional collection of numbers (or symbols or expressions), arranged in m rows and n columns.
It is presented in the form $\boldsymbol{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$, where a_{ij} is the entry in the i^{th} row and j^{th} column.

For example, the matrix **A** to our right has 3 rows and 2 columns, and so it is what we would consider to be 3× 2 matrix. Based on the construction we have above, we can even name our entries. The a_{13} entry is 0, while $a_{22} = -2$.

$$\boldsymbol{A} = \begin{pmatrix} 3 & 5 \\ 4 & -2 \\ 0 & 1 \end{pmatrix}$$

Example:
For the following matrices, determine the $m \times n$ dimensions & identify the value for our given entry, a_{ij}.

[[1]] $\begin{pmatrix} 6 & -2 \\ 5 & 1 \end{pmatrix}, a_{12}$ [[2]] $\begin{pmatrix} -1 \\ -2 \\ 8465 \end{pmatrix}, a_{11}$ [[3]] $\begin{pmatrix} 12 & 3 & -1 & 2 & -1 \\ 5 & -2 & 0 & 1 & 0 \\ 0 & -6 & 4 & 1 & 1 \end{pmatrix}, a_{24}$

Despite how scary it may sound, the basics of matrix algebra are fairly straightforward. For example, the concept of addition and subtraction feels very similar. We can add and subtract two matrices if they have the *same dimensions* (*the same number of rows and same number of columns*). If we have that, we can add/subtract by adding and subtracting each respective entry. Formula-wise, we do it like what we have below:

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \boldsymbol{B} = \begin{pmatrix} b_{11} & b_{21} \\ b_{21} & b_{22} \end{pmatrix},$$

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} a_{11} + b_{11} & a_{21} + b_{21} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}, \qquad \boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} a_{11} - b_{11} & a_{21} - b_{21} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{pmatrix}$$

The table below outlines the steps we need.

How to Perform Matrix Addition / Subtraction	
Step 1: Check to see if all matrices added and subtracted are of the same dimension. (Each are $m \times n$ matrices with the same m and n.)	$\begin{pmatrix} 6 & 4 & -3 & 0 \\ 2 & -5 & 1 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 3 & 1 & -2 \\ 0 & 4 & -6 & 7 \end{pmatrix}$ 2× 4 Matrix 2× 4 Matrix ✓Same Dimensions ✓
Step 2: Perform the addition/subtraction operation to each set of corresponding entries, a_{ij}.	$= \begin{pmatrix} (6-2) & (4+3) & (-3+1) & (0-2) \\ (2+0) & (-5+4) & (1-6) & (-1+7) \end{pmatrix}$
Step 3: Simplify each expression.	$= \begin{pmatrix} 4 & 7 & -2 & -2 \\ 2 & -1 & -5 & 6 \end{pmatrix}$

Example:
Perform the following matrix operations, if possible.

[[4]] $\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-5} \\ \mathbf{3} & \mathbf{1} & \mathbf{7} \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-2} & \mathbf{3} \end{pmatrix}$ [[5]] $\begin{pmatrix} \mathbf{3} & \mathbf{5} \\ \mathbf{-6} & \mathbf{15} \end{pmatrix} - \begin{pmatrix} \mathbf{8} & \mathbf{-2} \\ \mathbf{-7} & \mathbf{11} \end{pmatrix}$ [[6]] $\begin{pmatrix} \mathbf{8} & \mathbf{2} \\ \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} - \begin{pmatrix} \mathbf{5} & \mathbf{5} \\ \mathbf{5} & \mathbf{3} \end{pmatrix}$

We can also perform a couple other operation without much extra work. These operations are called ***scalar multiplication*** and ***transposition*** (taking a ***transpose***).

Definition:
The operation of **scalar multiplication** to a matrix takes a 1×1 value (known as a **scalar,** $\boldsymbol{c}$) and multiplies it to a matrix A by multiplying each entry by that scalar c.

$$\boldsymbol{cA} = c\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} c * a_{11} & \cdots & c * a_{1n} \\ \vdots & \ddots & \vdots \\ c * a_{m1} & \cdots & c * a_{mn} \end{pmatrix}$$

Definition:
The **transpose** operation of a matrix (or the **transpose** of a matrix) **A** turns every row of the matrix into a column and vice-versa. It makes every entry a_{ij} and *transposes* it to now be the entry a_{ji}. It is represented by the symbol $\boldsymbol{A}^T$ (or sometimes seen as $\boldsymbol{A'}$ or $\boldsymbol{A}^{tr}$).

Example:
Perform the following matrix operations.

[[7]] $\boldsymbol{A} = \begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \mathbf{0} & \mathbf{-2} & \mathbf{-5} \end{pmatrix}$**;** $\boldsymbol{A}^T$ [[8]] $\mathbf{4}\begin{pmatrix} \mathbf{-1} & \mathbf{-5} \\ \mathbf{3} & \mathbf{4} \end{pmatrix}$ [[9]] $\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$; $\boldsymbol{A}^T$

Matrix multiplication is where operations get a bit tricky. We will start with a special case of matrices that called ***vectors***, which uses the ***dot product*** operation.

Definition:
A **(row) vector** is matrix that has a single row with multiple columns.

- It looks like $\boldsymbol{v} = (a_1 \quad a_2 \quad \cdots \quad a_n)$, and a column vector is the *transpose* of this row vector.

Row (and column) vectors can perform the operations of addition, subtraction, transposition and scalar multiplication the same way that any regular matrix can. Furthermore, we can *stack* these vectors one on top of another to make rectangular matrix – this process is called *concatenation*. We will see this stacking come into place when we get to the programming part.

When it comes to other forms of multiplications using matrices, these vectors have a few products we can explore. We will cover two forms today, starting with the *dot product*.

Definition:
For two (row) vectors with the same number of columns, say

$$\boldsymbol{a} = (a_1 \quad a_2 \quad \cdots \quad a_k), \text{ and } \boldsymbol{b} = (b_1 \quad b_2 \quad \ldots \quad b_k),$$

the **dot product** of two vectors multiplies the respective column entries and adds them up, giving us

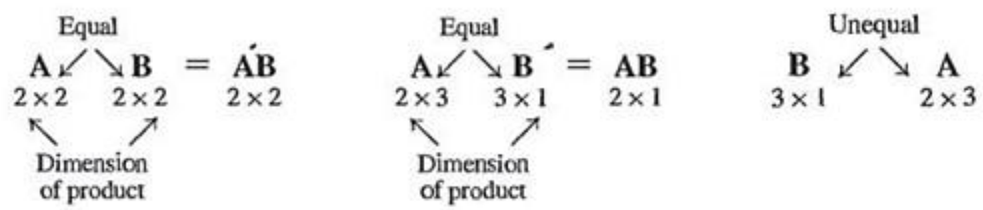
$$\boldsymbol{a}.\boldsymbol{b} = a_1 * b_1 + a_2 * b_2 + \cdots + a_k b_k \ .$$

The dot product has some neat applications that are applied to physics, computer science, statistics, and a whole bunch of things. Also note that we can use the dot product on two column vectors as well 😊.

To perform the general form of ***matrix multiplication***, we have to figure out how to *multiply* these matrices. Typically, we do not just multiply the corresponding entries together like we did with addition and subtraction. Instead, we have to make sure a few categories are realized.

First, for two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, we can multiply them, using the notation $\boldsymbol{AB}$, by using these dot products in a neat way. However, we have one major condition: *the number of columns of* $\boldsymbol{A}$ must equal *the number of rows of* $\boldsymbol{B}$. The reason is that for each entry in $\boldsymbol{AB}$, we will compute it by taking the dot product of a row from $\boldsymbol{A}$ and a column from $\boldsymbol{B}$.

(Case 1) (Case 2) (Case 3)



A great way to determine if two matrices can be multiplied together is to draw it out. If the dimensions on the inside (we'll called them "Inner" dimensions) are equal, like in Case 1 and Case 2, we can multiply them together. If they are not equal, like in Case 3, they just are compatible to be multiplied.

- Note that the dimension resulting matrix $\boldsymbol{AB}$ is the same as the "Outer" dimensions!

Now, time to get to the multiplication part! Because the "Inner" dimensions are the same, each row of $\boldsymbol{A}$ has the same number of entries as each column of $\boldsymbol{B}$. We will then multiply matrices by taking the dot products of these rows and columns!

Okay, y'all are probably reading that comment and saying, "Huh??" Let's break it down:
Say we can to calculate $\boldsymbol{DB}$ and our matrices are the following:

$$\boldsymbol{A} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{pmatrix}, \text{and } \boldsymbol{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}.$$

Note that $\boldsymbol{D}$ is a 3×2 matrix, and $\boldsymbol{B}$ is a 2×3 matrix. That's great – they have the same "Inner" dimensions, and we know that $\boldsymbol{DB}$ will end up being a 3×3 matrix. So we have that,

$$\boldsymbol{DB} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

To get each entry a_{ij}, we will then do the dot product of $row\ i$ from the first matrix, and $column\ j$ from the second matrix. Below, we have come color-coding that gives us three of those entries:

$$\boldsymbol{DB} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

- $a_{11} = (d_{11} \quad d_{12}).(b_{11} \quad b_{21})$
- $a_{23} = (d_{21} \quad d_{22}).(b_{13} \quad b_{23})$
- $a_{32} = (d_{31} \quad d_{32}).(b_{12} \quad b_{22})$

In general, we compute each entry as $\boxed{a_{ij} = [Row\ i\ of\ first\ matrix].[Column\ j\ of\ second\ matrix\ ]}$.

How to Perform Matrix Addition / Subtraction	
Step 1: Check to see if all matrices being multiplied are of the same "inner" dimensions. (We have an $m \times n$ matrix multiplied by an $n \times p$ matrix, where m and p can differ.)	$\begin{pmatrix} 0 & 4 & -3 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 0 & 3 \\ -1 & 1 \end{pmatrix}$ 2×3 Matrix $\qquad$ 3×2 Matrix ✓Same "Inner" Dimensions ✓
Step 2: Our result will be an $m \times p$ matrix. Perform the dot product operation to each set of corresponding entries, a_{ij}. Recall: For entry a_{ij}, we will do the dot product of $row\ i$ from the first matrix, and $column\ j$ from the second matrix.	So, $\begin{pmatrix} [Row1].[Column1] & [Row1].[Column2] \\ [Row2].[Column1] & [Row2].[Column2] \end{pmatrix}$ $= \begin{pmatrix} (0*5+4*0-3(-1)) & (0*2+4*3-3*1) \\ (2*5-1*0+0(-1)) & (2*2-1*3+0*1) \end{pmatrix}$
Step 3: Simplify each expression.	$= \begin{pmatrix} 3 & 9 \\ 10 & 1 \end{pmatrix}$

- **Fun Fact**: The dot product $\boldsymbol{a}.\boldsymbol{b}$ for row vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ will be the same as computing $\boldsymbol{ab^T}$

Example:
Perform the following matrix operations, if possible.

[[10]] $\begin{pmatrix} \mathbf{2} & \mathbf{4} & \mathbf{1} \end{pmatrix}.\begin{pmatrix} \mathbf{0} & \mathbf{3} & \mathbf{-2} \end{pmatrix}$

[[11]] $\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{-2} \\ \mathbf{6} \\ \mathbf{4} \end{pmatrix}$

[[12]] $\begin{pmatrix} \mathbf{3} & \mathbf{5} \\ \mathbf{-6} & \mathbf{15} \end{pmatrix}\begin{pmatrix} \mathbf{8} & \mathbf{-2} \\ \mathbf{-7} & \mathbf{11} \end{pmatrix}$

[[13]] $\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-5} \\ \mathbf{3} & \mathbf{1} & \mathbf{7} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbf{2} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-2} & \mathbf{3} \end{pmatrix}$

[[14]] $\begin{pmatrix} \mathbf{8} & \mathbf{2} \\ \mathbf{0} & \mathbf{3} \\ \mathbf{1} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{5} & \mathbf{5} \\ \mathbf{5} & \mathbf{3} \end{pmatrix}$

[[15]] $\begin{pmatrix} \mathbf{5} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$

For example [[15]], we get a neat result. (*What is it?*) This result comes from multiplying something called the ***identity matrix***.

Definition:

The $\boldsymbol{n} \times \boldsymbol{n}$ **identity matrix**, written as $\boldsymbol{I_n} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$, is a matrix with an entry of 1 along the upper-left to lower-right diagonal, and 0 everywhere else.

- The 2×2 identity matrix is of the form $\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$.
- The 3×3 identity matrix is of the form $\begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{pmatrix}$.

Example: As a follow-up to [[15]], solve the following:

[[16]] $\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}\begin{pmatrix} \mathbf{5} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}$

The last piece we will cover today is the idea of ***inverses***. Note how in the previous section, we didn't go into *dividing* matrices. Well… that gets a little more complicated, and we have to buff up our math terms to get to this level. Don't worry – it isn't too scary 😉.

We already use a couple of *inverses* in basic arithmetic: *additive inverses* and *multiplicative inverse*.
In fancy math terms, an *inverse* to a number is something that when you put it together with your original number (either through adding, multiplying, or whatnot), you will get the identity. So:

(1) The *additive inverse* of a number x is $(-x)$, since $x + (-x) = x - x = 0$, our *additive identity*.
(2) The *multiplicative inverse* of a number x (also known as the *reciprocal*) is $(1/x)$, since $x * (1/x) = x/x = 1$, our *multiplicative identity*.

So, for an $n \times n$ matrix $\boldsymbol{A}$, our inverse $\boldsymbol{A^{-1}}$ is a matrix is that gives us $\boldsymbol{AA^{-1}} = \boldsymbol{I_n} = \boldsymbol{A^{-1}A}$.

- Sadly, matrices that don't have the same number of rows and columns (we call them *square matrices*) do not have inverses ☹.

For matrices that are super-big, finding this inverse gets pretty gross. For now, we will look at only 2×2 matrices and use computers to get the inverse for larger ones.

For 2×2 matrices, there is a really nice formula to work with that gets us our solution:

- For a 2×2 matrix $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its inverse $\boldsymbol{A^{-1}}$ can be calculated by the formula:

$$\boldsymbol{A^{-1}} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

<u>Example</u>:

Find the inverse for the following matrices.

[[17]] $\begin{pmatrix} \mathbf{3} & \mathbf{5} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}$

[[18]] $\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$

[[19]] Marcus has a matrix $\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{0} \end{pmatrix}$. Naomi says the inverse is $\boldsymbol{A^{-1}} = \begin{pmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix}$.

Check to see if Naomi is correct.