

Properties of Determinant and Eigen Value

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For example:

I. PROPERTIES AND PROOFS

A. Properties of Determinant

1) **Property1:** If A is an $n \times n$ matrix, then

$$|A^T| = |A|$$

Proof :

From the definition of determinant

If A is an $n \times n$ matrix, then the determinant of A can be computed by multiplying the entries of any row (or column) by their cofactors and summing the resulting products:

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$$

(cofactor expansion by row i)

$$|A| = a_{j1}C_{j1} + a_{j2}C_{j2} + \dots + a_{jn}C_{jn}$$

(cofactor expansion by column j)

The cofactor expansion by row i of $|A^T|$ is precisely the same number as the cofactor expansion by column i of $|A|$. Hence the $|A|$ and $|A^T|$ will be having the same value.

2) **Property2:** Effect of a row (column) operation on $|A|$

- If a row is multiplied by a constant α , then $|A|$ changes to $\alpha|A|$.
- If two rows are interchanged, then $|A|$ changes to $-|A|$
- If a multiple of one row is added to another, then there is no change in $|A|$.

Proof :

Row Operation1:

Suppose the matrix B is obtained from a matrix A by multiplying row i by a non-zero constant α

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{21} & a_{22} & \dots a_{2n} \\ \vdots & \vdots & \dots \\ a_{n1} & a_{n2} & \dots a_{nn} \end{vmatrix}$$

$$|B| = \begin{vmatrix} a_{11} & a_{12} & \dots a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \dots \alpha a_{2n} \\ \vdots & \vdots & \dots \\ a_{n1} & a_{n2} & \dots a_{nn} \end{vmatrix}$$

If we evaluate $|B|$ using the cofactor expansion by row i , we obtain

$$\begin{aligned} |B| &= \alpha a_{i1}C_{i1} + \alpha a_{i2}C_{i2} + \dots + \alpha a_{in}C_{in} \\ &= \alpha(a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}) \\ &= \alpha|A| \end{aligned}$$

So we have:

The effect of multiplying a row of A by α is to multiply $|A|$ by α , $|B| = \alpha|A|$.

Row Operation2:

Suppose the matrix B is obtained from a matrix A by interchanging two row

Let A is a 2×2 matrix and matrix B is obtained by interchanging 2 rows then,

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$|B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

so $|B| = -|A|$

Now let A be a 3×3 matrix and let B be a matrix obtained from A by interchanging two rows. Then if we expand $|B|$ using a different row, each cofactor contains the determinant of a 2×2 matrix which is a cofactor of A with two rows interchanged, so each will be multiplied by -1 , and $|B| = -1|A|$.

To visualise this, consider for example

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad |B| = \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$$

$$|A| = -d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + e \begin{vmatrix} a & c \\ g & i \end{vmatrix} - f \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

$$|B| = -d \begin{vmatrix} h & i \\ b & c \end{vmatrix} + e \begin{vmatrix} g & i \\ a & c \end{vmatrix} - f \begin{vmatrix} g & h \\ a & b \end{vmatrix} = -|A|$$

Since all the 2×2 determinants change sign. In the same way, if this holds for $(n-1) \times (n-1)$ matrices, then it holds for $n \times n$ matrices.

So we have the effect of interchanging two rows of a matrix is to multiply the determinant by -1 : $|B| = -|A|$.

Row Operation3:

Suppose the matrix B is obtained from the matrix A by replacing row j of A by row j plus k times row i of A , $j \neq i$.

For example, consider the case in which B is obtained from A by adding 4 times row 1 of A to row 2. Then

$$|A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

$$|B| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} + 4a_{11} & a_{22} + 4a_{12} & \dots & a_{2n} + 4a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

In general, in a situation like this, we can expand $|B|$ by row j :

$$|B| = (a_{j1} + ka_{i1})C_{j1} + (a_{j2} + ka_{i2})C_{j2} + \dots + (a_{jn} + ka_{in})C_{jn}$$

$$= a_{j1}C_{j1} + a_{j2}C_{j2} + \dots + a_{jn}C_{jn} + k(a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn})$$

$$= |A| + 0$$

The last expression in brackets is 0 because it consists of the cofactors of one row multiplied by the entries of another row. So this row operation does not change the value of $|A|$. So there is no change in the value of the determinant if a multiple of one row is added to another.

3) **Property3:** If A is upper triangular or lower triangular of order $n \times n$ then

$$|A| = a_{11}a_{22}\dots a_{nn} = \prod_{i=1}^n a_{ii}$$

Proof :

Let A be an upper triangular matrix of order $n \times n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}_{n \times n}$$

Expanding the left most column, the cofactor expansion formula tells you that the determinant of A is

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} & \dots & a_{2n} \\ 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix}_{(n-1) \times (n-1)}$$

Now this smaller $(n-1)$ by $(n-1)$ matrix is also upper triangular, so you can compute it as a_{22} times an $(n-2)$ by $(n-2)$ upper triangular determinant:

$$|A| = a_{11}a_{22} \begin{vmatrix} a_{33} & a_{34} & \dots & a_{3n} \\ 0 & a_{44} & \dots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{vmatrix}_{(n-2) \times (n-2)}$$

Iterating this argument, you're eventually going to get

$$\begin{aligned} \det A &= a_{11} \dots a_{n-2,n-2} \cdot \det \begin{pmatrix} a_{n-1,n-1} & a_{n-1,n} \\ 0 & a_{nn} \end{pmatrix} \\ &= a_{11} \cdot a_{22} \dots a_{nn} \end{aligned}$$

4) **Property4:** If A and B are $n \times n$ matrices, then

$$|AB| = |A||B|$$

Proof :

We first prove the theorem in the case when the matrix A is an elementary matrix.

Let E_1 be an elementary matrix that multiplies a row by a non-zero constant k . Then E_1B is the matrix B obtained by performing that row operation on B . From the property effects of row operations on determinant $|E_1B| = k|B|$, as the same $|E_1| = |E_1I| = k|I| = k$. Therefore

$$|E_1B| = k|B| = |E_1||B|$$

Similarly it is true for other two types of elementary row operations. So we assume that the theorem is true when A is any elementary matrix.

Now recall that every matrix is row equivalent to a matrix in reduced row echelon form, so if R denotes the reduced row echelon form of the matrix A , then we can write

$$A = E_r E_{r-1} \dots E_1 R$$

where the E_i are elementary matrices. Since A is a square matrix, R is either the identity matrix or a matrix with a row

of zeros.

Applying the result for an elementary matrix repeatedly

$$|A| = |E_r E_{r-1} \dots E_1 R| = |E_r| |E_{r-1}| \dots |E_1| |R|$$

where $|R|$ is either 1 or 0. In fact, since the determinant of an elementary matrix must be non-zero, $|R| = 0$ if and only if $|A| = 0$.

If $R = I$, then by repeated application of the result for elementary matrices, this time with the matrix B ,

$$\begin{aligned} |AB| &= |(E_r E_{r-1} \dots E_1 I)B| \\ &= |E_r E_{r-1} \dots E_1 B| \\ &= |E_r| |E_{r-1}| \dots |E_1| |B| \\ &= |E_r E_{r-1} \dots E_1| |B| \\ &= |A| |B| \end{aligned}$$

If $R = I$, then

$$|AB| = |E_r E_{r-1} \dots E_1 RB| = |E_r| |E_{r-1}| \dots |RB|$$

Since the product matrix RB must also have a row of zeros, $|RB| = 0$.

Therefore, $|AB| = 0 = 0|B|$ and the theorem is proved.

B. Eigen Values Properties

1) **Property1:** A and A^T will have same Eigen value
Proof :

We know that the eigenvalues of a matrix are roots of its characteristic polynomial. Hence if the matrices A and A^T have the same characteristic polynomial, then they have the same eigenvalues.

So we show that the characteristic polynomial $p_A(t) = \det(A - tI)$ of A is the same as the characteristic polynomial $p_{A^T}(t) = \det(A^T - tI)$ of the transpose A^T .

We have

$$\begin{aligned} p_{A^T}(t) &= \det(A^T - tI) \\ &= \det(A^T - tI^T) \quad \text{since } I^T = I \\ &= \det((A - tI)^T) \\ &= \det((A - tI)) \\ &= p_A(t) \end{aligned}$$

(since $\det(B^T) = \det(B)$ for any square matrix B) Therefore we obtain $p_{A^T}(t) = p_A(t)$, and we conclude that the eigenvalues of A and A^T are the same.

2) **Property2:** Determinant of a matrix is equal to product of its Eigen values

$$\det(A) = \prod_{i=1}^n \lambda_i = \lambda_1 \lambda_2 \dots \lambda_n$$

Proof :

Suppose that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A . Then the λ s are also the roots of the characteristic polynomial, i.e.

$$\begin{aligned} \det(A - \lambda I) &= p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n) \\ &= (-1)(\lambda - \lambda_1)(-1)(\lambda - \lambda_2) \dots (-1)(\lambda - \lambda_n) \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n) \end{aligned}$$

The first equality follows from the factorization of a polynomial given its roots; the leading (highest degree) coefficient $(-1)^n$ can be obtained by expanding the determinant along the diagonal.

Now, by setting λ to zero (simply because it is a variable) we get on the left side $\det(A)$, and on the right side $\lambda_1, \lambda_2, \dots, \lambda_n$, that is we indeed obtain the desired result

$$\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$$

So the determinant of the matrix is equal to the product of its eigenvalues.

3) **Property3:** The trace of an $n \times n$ matrix A is equal to the sum of its eigenvalues.

$$\text{tr}(A) = \sum_{i=1}^n a_{ii} = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

Proof :

Suppose that $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of A . Then the λ s are also the roots of the characteristic polynomial, i.e.

$$|A - \lambda I| = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n) \quad (1)$$

$$|A - \lambda I| = (-1)^n (\lambda^n + a_{n-1} \lambda^{n-1} \dots + a_0) \quad (2)$$

The coefficient of λ^{n-1} is $a_{n-1}(-1)^{n-1}$ from (1). From (2) to obtain the coefficient of λ^{n-1} , if we multiply all the λ s together, one from each other, we obtain the term λ^n . So to obtain the term with λ^{n-1} , we need to multiply first $-\lambda_1$ times the λ s in all the remaining factors, then if we multiply all the λ s together, one from each factor, then $-\lambda_2$ times the λ s and so on. Putting back the factor $(-1)^n$, the term involving λ^{n-1} is

$$(-1)^n (-\lambda_1 - \lambda_2 - \dots - \lambda_n) \lambda^{n-1} = (-1)^{n-1} (\lambda_1 + \lambda_2 + \dots + \lambda_n) \lambda^{n-1} \quad (3)$$

Now suppose $A = (a_{ij})$ is an $n \times n$ matrix and look at the coefficient of λ^{n-1} in the cofactor expansion of $|A - \lambda I|$ by row 1:

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}_{n \times n}$$

$$= (a_{11} - \lambda)C_{11} + a_{12}C_{12} + \dots$$

Only the first term of the cofactor expansion, $(a_{11} - \lambda)C_{11}$, contains higher power of λ than λ^{n-1} .

Let us see why C_{11} is the determinant of the $(n-1) \times (n-1)$ matrix obtained from matrix $(A - \lambda I)$ by crossing out the first row and first column, so it is of the form $|C - \lambda I|$, where C is the $(n-1) \times (n-1)$ matrix obtained from A by crossing out the first row and first column.

Therefore by our assumption,

$$\begin{aligned} |A - \lambda I| &= (a_{11} - \lambda)C_{11} \\ &= (a_{11} - \lambda)((-1)^{n-1} + (-1)^{n-2}(a_{22} + \dots + a_{nn})\lambda^{n-2} + \dots) \\ &= (-1)^n \lambda^n + (-1)^{n-1}(a_{11} + a_{22} + \dots + a_{nn})\lambda^{n-1} + \dots \end{aligned}$$

We can now conclude that the term involving λ^{n-1} in the expansion of $|A - \lambda I|$ for any $n \times n$ matrix A is equal to

$$(-1)^{n-1}(a_{11} + a_{22} + \dots + a_{nn})\lambda^{n-1} \quad (4)$$

Comparing the coefficients of λ^{n-1} in the two expressions (3) and (4), we see that

$$a_{11} + a_{22} + a_{33} + \dots + a_{nn} = \lambda_1 + \lambda_2 + \lambda_3 + \dots + \lambda_n$$

that is, the trace of A is equal to the sum of the eigenvalues

4) Property4: A is an invertible matrix with eigenvalue λ corresponding to eigenvector x , then A^{-1} has eigenvalue λ^{-1} corresponding to the same eigenvector x .

Proof :

We know that for a matrix A the Eigen polynomial is $Ax = \lambda x$
Now multiply the polynomial with $\lambda^{-1}A^{-1}$ on both sides, we get

$$\lambda^{-1}A^{-1}(Ax) = \lambda^{-1}A^{-1}\lambda x$$

$$\lambda^{-1}x = A^{-1}x$$

Thus A^{-1} has Eigen value λ^{-1} corresponding to the same Eigen vector x

So if A is an invertible matrix with eigenvalue λ corresponding to eigenvector x , then A^{-1} has eigenvalue λ^{-1} corresponding to the same eigenvector x

5) Property5: If a matrix A has eigenvalue λ with corresponding eigenvector x , then for any $k = 1, 2, \dots, A^k$ has eigenvalue λ^k corresponding to the same eigenvector x .

proof :

Suppose the matrix A has eigenvalue λ with eigenvector x

$$Ax = \lambda x$$

Now

$$A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2x$$

Again multiplying with A

$$A^3x = \lambda^3x, A^4x = \lambda^4x, \dots$$

Similarly for matrix A^k the Eigen values will be $\lambda_1^k, \lambda_2^k, \dots$

REFERENCES

- [1] Linear Algebra Concepts and Methods by Martin Anthony and Michele Harvey