Matrix Theory

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Learning Outcomes:

What is a matrix,Different types of matrices,Basic operations on matrices,Trace, Transpose, Determinant and Inverse of a matrix, REF,Rank, Eigen values.

What is a Matrix?

Definition of a Matrix

A **matrix** is a rectangular array of numbers,symbols or expressions, arranged in rows and columns.

$\mathrm{m} \times \mathrm{n}$ matrix

$$\begin{pmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \cdots & \mathrm{a}_{1\mathrm{n}} \\ \mathrm{a}_{21} & \mathrm{a}_{22} & \cdots & \mathrm{a}_{2\mathrm{n}} \\ . & . & \cdots & . \\ \mathrm{a}_{\mathrm{m}1} & \mathrm{a}_{\mathrm{m}2} & \cdots & \mathrm{a}_{\mathrm{mn}} \end{pmatrix}_{mxn}$$

m -is the number of rows

n - is number of columns

$m \times n$ -order of matrix

a_{ij} $i = j$ Diagonal element

$i \neq j$ Off diagonal element

Types of Matrices

Row matrix

$$\begin{pmatrix} a & b & \cdots & n \end{pmatrix}_{1\times n}$$

Column matrix

$$\begin{pmatrix} a \\ b \\ m \end{pmatrix}_{m\times 1}$$

Null matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3\times 3}$$

Diagonal matrix

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{2\times 2}$$

Scalar matrix

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}_{2\times 2}$$

Unit matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$$

Lower triangular

$$\begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{pmatrix}_{3\times 3}$$

Upper triangular

$$\begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix}_{3\times 3}$$

Square matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}_{3\times 3}$$

Types of Matrices

Row matrix

$$\begin{pmatrix} a & b & \cdots & n \end{pmatrix}_{1\times n}$$

$\rightarrow a_{ij} - \; iis \; row \; number,$
$- \; j \; is \; column \; number$

$\rightarrow$ no of rows- $i = 1$

Column matrix

$$\begin{pmatrix} a \\ b \\ m \end{pmatrix}_{m\times 1}$$

$\rightarrow$ no of columns- $j = 1$

Square matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}_{3\times 3}$$

$\rightarrow$ no of rows- $i = m$
$\rightarrow$ no of columns- $j = m$
$\rightarrow$ where $i = j$

Types of Matrices

Rectangular matrix

$$\begin{pmatrix} a & b \\ d & e \\ g & h \end{pmatrix}_{3\times 2}$$

$\rightarrow$ no of rows- $i = m$

$\rightarrow$ no of columns- $j = n$

Diagonal matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}_{3\times 3}$$

$\rightarrow a_{ij} =$x for $i = j$

$a_{ij} =$0 for $i \neq j$

Scalar matrix

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & f \\ 0 & 0 & a \end{pmatrix}_{3\times 3}$$

$\rightarrow a_{ij} =$x for $\forall i = j$

$a_{ij} =$0 for $i \neq j$

Types of Matrices

Unit matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{3\times 3} \qquad \begin{aligned} \rightarrow a_{ij} =& 1 \quad \text{for } \forall i = j \\ a_{ij} =& 0 \quad \text{for } i \neq j \end{aligned}$$

Lower triangular matrix

$$\begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{pmatrix}_{3\times 3} \qquad \begin{aligned} \rightarrow a_{ij} =& \mathrm{x} \quad \text{for } i > j \\ a_{ij} =& 0 \quad \text{for } i < j \end{aligned}$$

Upper triangular matrix

$$\begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix}_{3\times 3} \qquad \begin{aligned} \rightarrow a_{ij} =& \mathrm{x} \quad \text{for } i < j \\ a_{ij} =& 0 \quad \text{for } i > j \end{aligned}$$

$$[c_{ij}] = [a_{ij}] + [b_{ij}]$$

$$[c_{ij}] = [a_{ij}] - [b_{ij}]$$

where i-row number
j-column number

Addition/Subtraction

Order of two matrices should be equal

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 5 \\ 5 & 9 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$$

Properties

- Commutative:
  A+B=B+A
- Associative:
  A+B)+C=A+(B+C)

Scalar Multiplication

$$[b_{ij}] = \mathrm{s} * [a_{ij}]$$

where i-row number

j-column number

s-scalar value

Multiplication(scalar)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}_{2\times 2} * \quad 2 = \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}_{2\times 2}$$

Properties

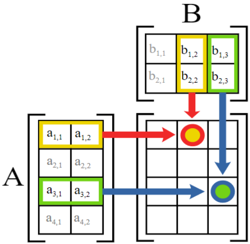
- Distributive: $(c + d)M = M(c + d)$
- Associative: $(cd)M = c(dM)$
- Identity: $IM = M$
- Null: $0M = 0$

Matrix Multiplication

Multiplication(matrix)

$$[\mathrm{AB}]_{ij} = \sum_{k=1}^{m} \mathrm{A}_{ik}\mathrm{B}_{kj}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ \cdot & \cdot \\ a_{31} & a_{32} \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & b_{12} & b_{13} \\ \cdot & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} \cdot & x_{12} & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & x_{33} \\ \cdot & \cdot & \cdot \end{bmatrix}$$



For A(1×2) B(2×1)

$i = 1, k = 2, j = 1$

$AB_{11} = a_{11}b_{11} + a_{12}b_{21}$

$A \rightarrow m \times k$

$B \rightarrow k \times n$

$AB \rightarrow m \times n$

where i-row number

j-column number

Properties

- $AB \neq BA$
- $(AB)C = A(BC)$
- $(AB)C = A(BC)$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$

Trace of Matrices

Trace of a matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}_{3\times 3}$$

$$\mathrm{Trace} = \sum_{k=1}^{m} \mathrm{a}_{kk}$$

$A \rightarrow m \times m$

where k-row, column number

Properties

- $tr(kA) = k.tr(A)$
- $tr(AT) = tr(A)$
- $tr(A + B) = tr(A) + tr(B)$
- $tr(AB) = tr(BA)$

$$[a_{ij}] = [a_{ji}]$$

$A \rightarrow m \times n$

where i-row number

j-column number

Transpose of a Matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}_{m \times n} \quad \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}_{n \times m}$$

Properties

- $(\mathrm{A}^{\mathrm{T}})^{\mathrm{T}} = \mathrm{A}$
- $(\mathrm{kA}^{\mathrm{T}}) = \mathrm{k}(\mathrm{A}^{\mathrm{T}})$
- $(\mathrm{A} + \mathrm{B})^{\mathrm{T}} = \mathrm{A}^{\mathrm{T}} + \mathrm{B}^{\mathrm{T}}$
- $(\mathrm{ABC})^{\mathrm{T}} = \mathrm{C}^{\mathrm{T}}\mathrm{B}^{\mathrm{T}}\mathrm{A}^{\mathrm{T}}$

Determinant of a Matrix

$$\det(\mathrm{A})_{\mathrm{ij}} = \sum_{\mathrm{j=1}}^{\mathrm{n}} \mathrm{a_{ij}A_{ij}} \qquad \text{Cofactor} \rightarrow \mathrm{A_{ij}} = (-1)^{\mathrm{i+j}}\det(\mathrm{A_{ij}})$$

$A \rightarrow m \times n$

where i-row number

j-column number

$$|\mathrm{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}_{2\times 2} = \mathrm{ad} - \mathrm{bc}$$

$$|\mathrm{A}| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \mathrm{a}\begin{vmatrix} \square & \square & \square \\ d & \square & f \\ g & \square & i \end{vmatrix} \text{-b}\begin{vmatrix} \square & \square & \square \\ d & e & \square \\ g & h & \square \end{vmatrix} +\mathrm{c}\begin{vmatrix} \square & \square & \square \\ \square & e & f \\ \square & h & i \end{vmatrix}$$

$$= \mathrm{a}\begin{vmatrix} e & f \\ h & i \end{vmatrix} \text{-b}\begin{vmatrix} d & f \\ g & i \end{vmatrix} +\mathrm{c}\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Cramer's Rule

$\mathrm{ax + by = e}$

$\mathrm{cx + dy = f}$

$$\text{Coeff Matrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\mathrm{detA}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\mathrm{detA}} \qquad \mathrm{detA} \neq 0$$

Properties of Determinant

- $\det(A) = \det(A^T)$
- $A -$ triangular matrix then $\det(A) = \prod_{i=1}^{n} a_{ii}$
- $\det(AB) = \det(A).\det(B) \;\; \forall\; n \times n$ matrices
- B is obtained by applying any one of row operations on A
    - $R_i \leftrightarrow R_j$ then $\det(B) = \det(A)$
    - $sR_i \dashrightarrow R_i$ then $\det(B) = s\det(A)$
    - $sR_i + R_j \dashrightarrow R_j$ then $\det(B) = \det(A)$

Inverse of a matrix

$\mathrm{A}^{-1} = \frac{1}{\mathrm{detA}}\mathrm{adj}(\mathrm{A}) \qquad A \rightarrow m \times n$

$$\text{Matrix A=} \begin{pmatrix} \mathrm{a_{11}} & \mathrm{a_{12}} & \cdots & \mathrm{a_{1n}} \\ \mathrm{a_{21}} & \mathrm{a_{22}} & \cdots & \mathrm{a_{2n}} \\ . & . & \cdots & . \\ \mathrm{a_{m1}} & \mathrm{a_{m2}} & \cdots & \mathrm{a_{mn}} \end{pmatrix}_{m\times n} \qquad AA^{-1} = A^{-1}A = I$$

$$\text{Adj A= Traspose of } \begin{pmatrix} \mathrm{A_{11}} & \mathrm{A_{12}} & \cdots & \mathrm{A_{1n}} \\ \mathrm{A_{21}} & \mathrm{A_{22}} & \cdots & \mathrm{A_{2n}} \\ . & . & \cdots & . \\ \mathrm{A_{m1}} & \mathrm{A_{m2}} & \cdots & \mathrm{A_{mn}} \end{pmatrix}_{m\times n}$$

where $A_{ij} = \ (-1)^{i+j}det(M_{ij}) \qquad M_{ij}$ is Minor of a_{ij}

$[A][X] = [B] \qquad A^{-1}AX = A^{-1}B$

Rank of a matrix

Row Operations $\rightarrow \; R_i \leftrightarrow R_j$

$\rightarrow \; sR_i \dashrightarrow R_i$

$\rightarrow \; sR_i + R_j \dashrightarrow R_j$

Row Echelon Form

$$\begin{pmatrix} 1 & -2 & 1 & 5 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & 6 \end{pmatrix} \quad \rightarrow \text{Gaussian Elimination}$$

Reduced Row Echelon Form

$$\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 6 \end{pmatrix} \quad \rightarrow \text{Gaussian Jordan Elimination}$$

Rank=number of non zero rows

Eigen value and Eigen vector

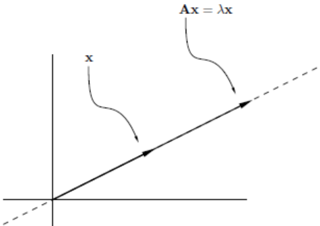
$A \rightarrow n \times n$

$$AX = \lambda X$$

where- $\lambda -$ a scalar called Eigenvalue

$X_{n\times 1}$ $-$ Eigen vector

$$(A - \lambda I)X = 0 \text{ -Characteristic equation}$$
$$|A - \lambda I| = 0$$



Properties of Eigen Values

- $\mathrm{A and A^T}$ will have same Eigen value.
- $\mathrm{A - Eigenvalues} \lambda_1, \lambda_2 ..., \lambda_\mathrm{n}$ for $\mathrm{kA} - \mathrm{k}\lambda_1, \mathrm{k}\lambda_2 ..., \mathrm{k}\lambda_\mathrm{n}$
- $\mathrm{A - Eigenvalues} \lambda_1, \lambda_2 ..., \lambda_\mathrm{n}$ for $\mathrm{A^k} - \lambda_1^\mathrm{k}, \lambda_2^\mathrm{k} ..., \lambda_\mathrm{n}^\mathrm{k}$
- $\mathrm{A - Eigenvalues} \lambda_1, \lambda_2 ..., \lambda_\mathrm{n}$ for $\mathrm{A^{-1}} - 1/\lambda_1, 1/\lambda_2 ..., 1/\lambda_\mathrm{n}$
- $\det(\mathrm{A}) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 ... \lambda_n$
- $\mathrm{tr}(\mathrm{A}) = \sum_{i=1}^{n} a_{ii} = \lambda_1 + \lambda_2 + ... + \lambda_n$