Vector Space

July 2, 2020

Learning Outcomes:

Vector spaces, Null Space,Basis,Normal Equation,Linear Dependence,Similarity.

Vector Space

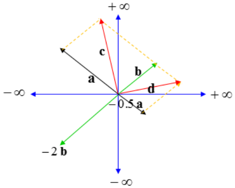
A vector space involves four things—two sets V and F, and two algebraic operations called vector addition and scalar multiplication.

- V a nonempty set of objects called vectors.We consider V as a matrices or tuples.

- F is a scalar field , for us F is either the field R of real numbers or the field C of complex numbers.

- Vector addition denoted by $x + y$ is an operation between elements of V.

- Scalar multiplication denoted by αx is an operation between elements of F and V.



Definition of a Vector space

The set V is called a vector space over scalar field F when the vector addition and scalar multiplication operations satisfy the following properties.

(A1) $x + y \in V \;\; \forall\, x, y \in V.$

(A2) $(x + y) + z = x + (y + z)$ forevery $x, y, z \in V$.

(A3) $x + y = y + x$ forevery $x, y \in V$.

(A4) $0 \in V$ such that $0 + x = x$ forevery $x \in V$.

(A5) $x \in V$, $\exists\, (-x) \in V$ such that $x + (-x) = 0$.

(M1) $\alpha x \in V\; \forall \alpha \in F$ and $x \in V$.

(M2) $(\alpha\beta)x = \alpha(\beta x)\; \forall\, \alpha, \beta \in V$ and every $x \in V$.

(M3) $\alpha(x + y) = \alpha x + \alpha y\; \forall\, \alpha \in V$ and every $x, y \in V$.

(M4) $(\alpha + \beta)x = \alpha x + \beta x\; \forall\, \alpha, \beta \in V$ and every $x \in V$.

(M5) $1x = x$ for every $x \in V$

Vector Space

(A1) - (A5) are generalized versions of additive properties of matrix addition

(M1) - (M5) are generalized versions of scalar multiplications of matrix

The set $\mathbb{R}^{mxn}$ of $m \times n$ real matrices is a vector space over $\mathbb{R}$

The set $\mathbb{C}^{mxn}$ of $m \times n$ complex matrices is a vector space over $\mathbb{C}$

$$\mathbb{R}^{1xn} = \{(x_1, x_2, ..., x_n), x_i \in \mathbb{R}\}$$

Definition of a Sub space

Let S be a nonempty subset of a vector space V over F (symbolically, $S \subseteq V$). If S is also a vector space over F using the same addition and scalar multiplication operations, then S is said to be a subspace of V.

S need to satisfy A1 and M1 properties.

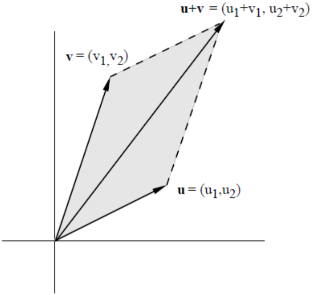
$Z = \{0\}$ - is a trivial subspace

Definition of a Spane

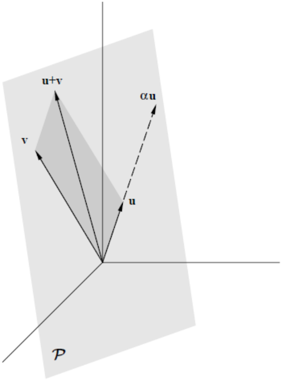
For a set of vectors $S = \{v_1, v_2, ..., v_3\}$, the subspace

$$span(S) = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_r v_r$$

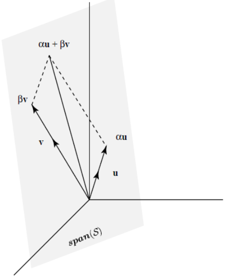
generated by forming all linear combinations of vectors from S is called the *space spanned* by S



$\mathbf{u+v} = (u_1+v_1, u_2+v_2)$
$\mathbf{v} = (v_1, v_2)$
$\mathbf{u} = (u_1, u_2)$



$\mathbf{u+v}$
$\alpha\mathbf{u}$
$\mathbf{v}$
$\mathbf{u}$
$\mathcal{P}$



$\alpha\mathbf{u} + \beta\mathbf{v}$
$\beta\mathbf{v}$
$\mathbf{v}$
$\alpha\mathbf{u}$
$\mathbf{u}$
$span(S)$

Suppose that $v_1, v_2, ..., v_n$ is a spanning set for V

$$V = \alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n | \alpha_i \in \mathbb{R}$$

Stack the v_i 's as columns in a matrix $A_{mxn} = \{v_1 | v_2 | ... | v_n\}$ and put the α_i 's in an n x 1 column $x = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ to write

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = (v_1 | v_2 | ... | v_n) \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n \end{pmatrix} = AX$$

Range, Row, Column Space

Range Space

The range of a matrix $A \in \mathbb{R}^{mxn}$ is defined to be the subspace $R(A)$ of $\mathbb{R}^m$ that is generated by the range of $f(x) = Ax$. That is

$$R(A) = \{Ax | x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$$

Similarly range of A^T is the subspace of $\mathbb{R}^n$ is

$$R(A^T) = \{A^T y | y \in \mathbb{R}^m\} \subseteq \mathbb{R}^n$$

Row, Column Space

$R(A) =$ the space spanned by the columns of $A \rightarrow$ column space
$R(A^T) =$ the space spanned by the rows of $A \rightarrow$ row space
Note:U be row Echelon form of A then non zero rows of U span $R(A^T)$ and basic columns in A span $R(A)$

Null Space For an m x n matrix $N(A)$ is solutions to the homogeneous system $AX = 0$.

$$N(A) = \{x_{nx1} \mid Ax = 0\} \subseteq \mathbb{R}^n$$

Left Hand Null Space The set $N(A^T) = \{y_{mx1} | A^T y = 0\} \subseteq \mathbb{R}^m$

is called the left hand null space of A because $N(A^T)$ is the set of all solutions to the left hand homogeneous system $y^T A = 0^T$.

Zero Null Space and Spanning of N:

For matrix A_{mxn}

$N(A) = \{0\}$ if and only if rank (A) = n;

$N(A^T) = \{0\}$ if and only if rank (A) = m;

Spanning set for N (A) =the hi's in the general solution of Ax = 0.
Spanning set for $N(A^T)$ =last m - r rows of P.
where P is a non - singular matrix and PA = U
U is Row Echelon form of A

Problem: Determine a spanning set for $N(\mathbf{A}^T)$, where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}$.

Solution: To find a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{U}$ is in row echelon form, proceed as described in Exercise 3.9.1 and row reduce the augmented matrix $(\mathbf{A} \mid \mathbf{I})$ to $(\mathbf{U} \mid \mathbf{P})$. It must be the case that $\mathbf{PA} = \mathbf{U}$ because $\mathbf{P}$ is the product of the elementary matrices corresponding to the elementary row operations used. Since any row echelon form will suffice, we may use Gauss–Jordan reduction to reduce $\mathbf{A}$ to $\mathbf{E_A}$ as shown below:

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 3 & 0 & 1 & 0 \\ 3 & 6 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|ccc} 1 & 2 & 0 & 1 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & -5/3 & 1 \end{array}\right)$$

$$\mathbf{P} = \begin{pmatrix} -1/3 & 2/3 & 0 \\ 2/3 & -1/3 & 0 \\ 1/3 & -5/3 & 1 \end{pmatrix}, \text{ so (4.2.12) implies } N(\mathbf{A}^T) = span\left\{ \begin{pmatrix} 1/3 \\ -5/3 \\ 1 \end{pmatrix} \right\}.$$

Linear Independence and Dependence:

Linear Independence A set of vectors $S = \{v_1, v_2, ..., v_n\}$ is said to be a linearly independent set whenever the only solution for the scalars α_i in the homogeneous equation

$$\alpha_1 v_1 + \alpha_2 v_2 + ... + \alpha_n v_n = 0$$

is the trivial solution. Whenever there is a non-trivial solution for the α_i 's (atleast one $\alpha_i \neq 0$) then S is said to be linearly dependent set

Linear Independence and Dependence:

Let A be m x n matrix

Each of the following statements is equivalent to saying that the columns of A form a linearly independent set.

- $N(A) = 0$
- $rank(A) = n$

Each of the following statements is equivalent to saying that the rows of A form a linearly independent set.

- $N(A^T) = 0$
- $rank(A) = m$

When A is a square matrix, each of the following statements is equivalent to saying that A is nonsingular.

- The columns of A form a linearly independent set.
- The rows of A form a linearly independent set.

Linear Independence and Dependence:

For a nonempty set of vectors $S = u_1, u_2, ..., u_n$ in a space V, the following statements are true.
- If S contains a linearly dependent subset, then S itself must be linearly dependent.
- If S is linearly independent, then every subset of S is also linearly independent.
- If S is linearly independent and if $v \in V$, then the extension set $S_{ext} = S \cup v$ is linearly independent if and only if $v \notin span(S)$. - If $S \subseteq \mathbb{R}_m$ and if $n > m$, then S must be linearly dependent.

Basis

A linearly independent spanning set for a vector space V is called a basis for V.

Characteristics of a Basis: Let V be a subspace of $\mathbb{R}^m$, and let $B = b_1, b_2, ..., b_n \subseteq V$. The following statements are equivalent.

- B is a basis for V.
- B is a minimal spanning set for V.
- B is a maximal linearly independent subset of V. The unit vectors

$S = e_1, e_2, ..., e_n$ in $\mathbb{R}^n$ are a basis for $\mathbb{R}^n$. This is called the standard basis for $\mathbb{R}^n$.

Dimension

The dimension of a vector space V is defined to be

- number of vectors in any basis for V
- number of vectors in any minimal spanning set for V
- number of vectors in any maximal independent subset of V.

For vector spaces M and N such that $M \subseteq N$,

- $dimM \leqslant dimN$.
- If $dimM = dimN$, then $M = N$.

For an m x n matrix of real numbers such that $rank(A) = r$,

- $dimR(A) = r$
- $dimN(A) = n - r$
- $dimR(A^T) = r$
- $dimN(A^T) = m - r$

Rank Plus Nullity Theorem: $dimR(A) + dimN(A) = n$ for all $m \times n$ matrices.

Dimension, Rank and Connectivity

Let G be a graph containing m nodes. If G is undirected, arbitary assign directions to the edges to make G directed, and let E be the corresponding incidence matrix.
G is connected if and only if $rank(E) = m - 1$

If X and Y are subspaces of a vector space V then
$dim(X + Y) + dinX + dimY - dim(X \cap Y)$

Rank

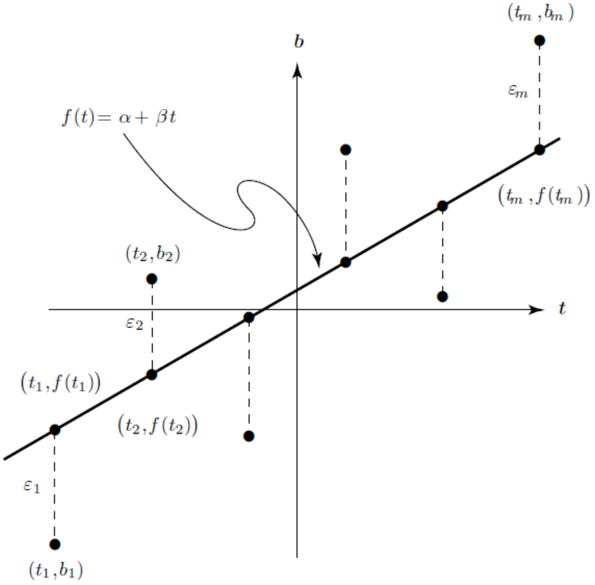
For $A \in \mathbb{R}^{mxn}$,

- $rank(A) =$ The number of nonzero rows in any row echelon form that is row equivalent to A.
- $rank(A) =$ The number of basic columns in A (as well as the number of basic columns in any matrix that is row equivalent to A).
- $rank(A) =$ The number of independent columns in A — i.e., the size of a maximal independent set of columns from A.
- $rank(A) =$ The number of independent rows in A — i.e., the size of a maximal independent set of rows from A.
- $rank(A) = dimR(A)$
- $rank(A) = dimR(A^T)$
-$rank(A) = n - dimN(A)$
- $rank(A) = m - dimN(A^T)$

Normal Equations

- For an $m \times n$ system $\mathbf{Ax} = \mathbf{b}$, the associated system of ***normal equations*** is defined to be the $n \times n$ system $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$.
- $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ is always consistent, even when $\mathbf{Ax} = \mathbf{b}$ is not consistent.
- When $\mathbf{Ax} = \mathbf{b}$ is consistent, its solution set agrees with that of $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$. As discussed in §4.6, the normal equations provide least squares solutions to $\mathbf{Ax} = \mathbf{b}$ when $\mathbf{Ax} = \mathbf{b}$ is inconsistent.
- $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ has a unique solution if and only if $rank\,(\mathbf{A}) = n$, in which case the unique solution is $\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$.
- When $\mathbf{Ax} = \mathbf{b}$ is consistent and has a unique solution, then the same is true for $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$, and the unique solution to both systems is given by $\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$.

Least Square Problem



General Least Squares Problem

For $\mathbf{A} \in \Re^{m \times n}$ and $\mathbf{b} \in \Re^{m}$, let $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b}$. The general least squares problem is to find a vector $\mathbf{x}$ that minimizes the quantity

$$\sum_{i=1}^{m} \varepsilon_i^2 = \boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} = (\mathbf{A}\mathbf{x} - \mathbf{b})^T (\mathbf{A}\mathbf{x} - \mathbf{b}).$$

Any vector that provides a minimum value for this expression is called a ***least squares solution.***

- The set of all least squares solutions is precisely the set of solutions to the system of normal equations $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$.
- There is a unique least squares solution if and only if $rank\,(\mathbf{A}) = n$, in which case it is given by $\mathbf{x} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$.
- If $\mathbf{A}\mathbf{x} = \mathbf{b}$ is consistent, then the solution set for $\mathbf{A}\mathbf{x} = \mathbf{b}$ is the same as the set of least squares solutions.

Linear Transformation

Let U and V be vector spaces over a field F ($\mathbb{R} or C$). A linear transformation from U into V is defined to be a linear function T mapping U into V. That is,

$$T(x + y) = T(x) + T(y)$$

$$T(\alpha x) = \alpha T(x)$$

$$T(\alpha x + y) = \alpha T(x) + T(y) \text{ \textit{forall} } x, y \in U, \alpha \in F$$

A linear operator on U is defined to be a linear transformation from U into itself — i.e., a linear function mapping U back into U.

Linear Transformation

$\rightarrow$The projector P that maps each point $v = (x, y, z) \in \mathbb{R}^3$ to its orthogonal projection $(x, y, 0)$ in the xy - plane, as depicted in Figure is a linear operator on $\mathbb{R}^3$ because if $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, then

$$P(\alpha u + v) = (\alpha u_1 + v_1, \alpha u_2 + v_2, 0) = \alpha(u_1, u_2, 0) + (v_1, v_2, 0)$$

$\rightarrow$ The reflector R that maps each vector $v = (x, y, z) \in \mathbb{R}^3$, to its reflector $R(v) = (x, y, -z)$ about the xy - plane, as shown , is a linear operator on $\mathbb{R}^3$

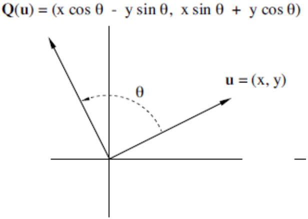
Just as the rotator $\mathbf{Q}$ is represented by a matrix $[\mathbf{Q}] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, the projector $\mathbf{P}$ and the reflector $\mathbf{R}$ can be represented by matrices

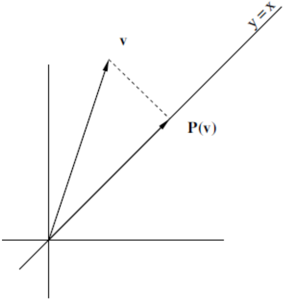
$$[\mathbf{P}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad [\mathbf{R}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

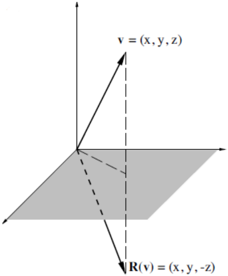
in the sense that the "action" of $\mathbf{P}$ and $\mathbf{R}$ on $\mathbf{v} = (x, y, z)$ can be accomplished with matrix multiplication using $[\mathbf{P}]$ and $[\mathbf{R}]$ by writing

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}.$$

Linear Transformation







Coordinates of a Vector

Let$B = u1, u2, ..., un$ be a basis for a vector space U, and let $v \in U$. The coefficients α_i in the expansion $v = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$ are called the coordinates of v with respect to B, and, from now on,$[v]_B$ will denote the column vector

$$[v]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_n \end{pmatrix}$$

B' is the permutation of B then $[v]_{B'}$ is the corresponding permutation of $[v]_B$.

Space of Linear Transformations

- For each pair of vector spaces $\mathcal{U}$ and $\mathcal{V}$ over $\mathcal{F}$, the set $\mathcal{L}(\mathcal{U}, \mathcal{V})$ of all linear transformations from $\mathcal{U}$ to $\mathcal{V}$ is a vector space over $\mathcal{F}$.
- Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\mathcal{B}' = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be bases for $\mathcal{U}$ and $\mathcal{V}$, respectively, and let $\mathbf{B}_{ji}$ be the linear transformation from $\mathcal{U}$ into $\mathcal{V}$ defined by $\mathbf{B}_{ji}(\mathbf{u}) = \xi_j \mathbf{v}_i$, where $(\xi_1, \xi_2, \ldots, \xi_n)^T = [\mathbf{u}]_{\mathcal{B}}$. That is, pick off the j^{th} coordinate of $\mathbf{u}$, and attach it to $\mathbf{v}_i$.
  - ▷ $\mathcal{B}_{\mathcal{L}} = \{\mathbf{B}_{ji}\}_{j=1\ldots n}^{i=1\ldots m}$ is a basis for $\mathcal{L}(\mathcal{U}, \mathcal{V})$.
  - ▷ $\dim \mathcal{L}(\mathcal{U}, \mathcal{V}) = (\dim \mathcal{U})(\dim \mathcal{V})$.