Properties of Determinant and Eigan Value

1st Naga Ramya Ogireddy
dept. Electronics and Communication Engineering
RGUKT Nuzvid
Nuzvid, India
ogireddynagaramya@gmail.com

2nd Sai Lahari Ogireddy
dept. Electronics and Communication Engineering
RGUKT Nuzvid
Nuzvid, India
ogireddysailahari@gmail.com

***Index Terms*—matrix, determinant, inverse, Eigen value, Eigen vector**

I. PROPERTIES AND PROOFS

A. *Properties of Determinant*

*1) **Property1**:* If A is an n × n matrix, then

$$|A^T| = |A|$$

Proof :
From the definition of determinant
If A is an n × n matrix, then the determinant of A can be computed by multiplying the entries of any row (or column) by their cofactors and summing the resulting products:

$$|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}$$

(cofactor expansion by row i)

$$|A| = a_{j1}C_{j1} + a_{j2}C_{j2} + ... + a_{jn}C_{jn}$$

(cofactor expansion by column j)

The cofactor expansion by row i of $|A^T|$ is precisely the same number as the cofactor expansion by column i of $|A|$. Hence the $|A|$ and $|A^T|$ will be having the same value.

*2) **Property2**:* **Effect of a row (column) operation on** $|A|$

- If a row is multiplied by a constant α,then $|A|$ changes to $\alpha|A|$.
- If two rows are interchanged, then $|A|$ changes to $-|A|$
- If a multiple of one row is added to another,then there is no change in $|A|$.

Proof :

Row Operation1:

Suppose the matrix B is obtained from a matrix A by multiplying row i by a non-zero constant α

For example:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & ...a_{1n} \\ a_{21} & a_{22} & ...a_{2n} \\ . & . & .... \\ a_{n1} & a_{n2} & ...a_{nn} \end{vmatrix}$$

$$|B| = \begin{vmatrix} a_{11} & a_{12} & ...a_{1n} \\ \alpha a_{21} & \alpha a_{22} & ...\alpha a_{2n} \\ . & . & .... \\ a_{n1} & a_{n2} & ...a_{nn} \end{vmatrix}$$

If we evaluate $|B|$ using the cofactor expansion by row i, we obtain

$$\begin{aligned} |B| &= \alpha a_{i1}C_{i1} + \alpha a_{i2}C_{i2} + ... + \alpha a_{in}C_{in} \\ &= \alpha(a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}) \\ &= \alpha|A| \end{aligned}$$

So we have:
The effect of multiplying a row of A by α is to multiply $|A|$ by α,$|B| = \alpha|A|$.

Row Operation2:

Suppose the matrix B is obtained from a matrix A by interchanging two row
Let A is a 2 x 2 matrix and matrix B is obtained by interchanging 2 rows then,

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$|B| = \begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad$$

so $|B| = -|A|$

Now let A be a 3 x 3 matrix and let B be a matrix obtained from A by interchanging two rows. Then if we expand $|B|$ using a different row, each cofactor contains the determinant of a 2 x 2 matrix which is a cofactor of A with two rows interchanged, so each will be multiplied by -1, and $|B| = -1|A|$.

To visualise this, consider for example

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \quad |B| = \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$$

$$|A| = -d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + e \begin{vmatrix} a & c \\ g & i \end{vmatrix} - f \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

$$|B| = -d \begin{vmatrix} h & i \\ b & c \end{vmatrix} + e \begin{vmatrix} g & i \\ a & c \end{vmatrix} - f \begin{vmatrix} g & h \\ a & b \end{vmatrix} = -|A|$$

Since all the 2 x 2 determinants change sign. In the same way, if this holds for $(n-1)$ x $(n-1)$ matrices, then it holds for n x n matrices.
So we have the effect of interchanging two rows of a matrix is to multiply the determinant by -1: $|B| = -|A|$.

Row Operation3:

Suppose the matrix B is obtained from the matrix A by replacing row j of A by row j plus k times row i of A, $j = i$.
For example, consider the case in which B is obtained from A by adding 4 times row 1 of A to row 2. Then

$$|A| = \begin{vmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & ... & . \\ a_{n1} & a_{n2} & ... & a_{nn} \end{vmatrix}$$

$$|B| = \begin{vmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} + 4a_{11} & a_{22} + 4a_{12} & ... & a_{2n} + 4a_{1n} \\ . & . & ... & . \\ a_{n1} & a_{n2} & ... & a_{nn} \end{vmatrix}$$

In general, in a situation like this, we can expand
$|B|$ by row j :

$$|B| = (a_{j1}+ka_{i1})C_{j1}+(a_{j2}+ka_{i2})C_{j2}+...+(a_{jn}+ka_{in})C_{jn}$$

$$= a_{j1}C_{j1} + a_{j2}C_{j2} + ... + a_{jn}C_{jn}$$
$$+ k(a_{i1}C_{j1} + a_{i2}C_{j2} + ... + a_{in}C_{jn})$$

$$= |A| + 0$$

The last expression in brackets is 0 because it consists of the cofactors of one row multiplied by the entries of another row. So this row operation does not change the value of $|A|$.
So there is no change in the value of the determinant if a multiple of one row is added to another.

3) ***Property3**:* If A is upper triangular or lower triangular of order n x n then

$$|A| = a_{11}a_{22}...a_{nn} = \prod_{i=1}^{n} a_{ii}$$

Proof :
Let A be an upper triangular matrix of order n x n

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2n} \\ . & . & ... & . \\ 0 & 0 & ... & a_{nn} \end{pmatrix}_{n \times n}$$

Expanding the left most column, the cofactor expansion formula tells you that the determinant of A is

$$|A| = a_{11} \begin{pmatrix} a_{22} & a_{23} & ... & a_{2n} \\ 0 & a_{33} & ... & a_{3n} \\ . & . & ... & . \\ 0 & 0 & ... & a_{nn} \end{pmatrix}_{n-1 \times n-1}$$

Now this smaller (n-1) by (n-1) matrix is also upper triangular, so you can compute it as a_{22} times an (n-2) by (n-2) upper triangular determinant:

$$|A| = a_{11}a_{22} \begin{pmatrix} a_{33} & a_{34} & ... & a_{3n} \\ 0 & a_{44} & ... & a_{4n} \\ . & . & ... & . \\ 0 & 0 & ... & a_{nn} \end{pmatrix}_{n-2 \times n-2}$$

Iterating this argument, you're eventually going to get

$$DetA = a_{11}...a_{n-2,n-2}.det \begin{pmatrix} a_{n-1,n-1} & a_{n-1,n} \\ 0 & a_{nn} \end{pmatrix}$$
$$= a_{11}.a_{22}...a_{nn}$$

4) ***Property4**:* If A and B are n x n matrices,then

$$|AB| = |A||B|$$

Proof :
We first prove the theorem in the case when the matrix A is an elementary matrix.

Let E_1 be an elementary matrix that multiplies a row by a non-zero constant k.Then E_1B is the matrix B obtained by performing that row operation on B, From the property effects of row operations on determinant $|E_1B| = k|B|$, as the same $|E_1| = |E_1I| = k|I| = k$.Therefore

$$|E_1B| = k|B| = |E_1||B|$$

Similarly it is true for other two types of elementary row operations. So we assume that the theorem is true when A is any elementary matrix.

Now recall that every matrix is row equivalent to a matrix in reduced row echelon form,so if R denotes the reduced row echelon form of the matrix A,then we can write

$$A = E_rE_{r-1}...E_1R$$

where the $E_i$are elementary matrices.Since A is a square matrix,R is either the identity matrix or a matrix with a row

of zeros.
Applying the result for an elementary matrix repeatedly

$$|A| = |E_r E_{r-1} ... E_1 R| = |E_r||E_{r-1}|...|E_1||R|$$

where $|R|$ is either 1 or 0. In fact, since the determinant of an elementary matrix must be non-zero, $|R| = 0$if and only if $|A| = 0$.

If $R = I$, then by repeated application of the result for elementary matrices, this time with the matrix B,

$$\begin{aligned} |AB| &= |(E_r E_{r-1}...E_1 I)B| \\ &= |E_r E_{r-1}...E_1 B| \\ &= |E_r||E_{r-1}|...|E_1||B| \\ &= |E_r E_{r-1}...E_1||B| \\ &= |A||B| \end{aligned}$$

If $R = I$, then

$$|AB| = |E_r E_{r-1}...E_1 RB| = |E_r||E_{r-1}|...|RB|$$

Since the product matrix RB must also have a row of zeros, $|RB| = 0$.

Therefore, $|AB| = 0 = 0|B|$ and the theorem is proved.

B. *Eigen Values Properties*

*1) **Property1**:* A and A^T will have same Eigen value
Proof :

We know that the eigenvalues of a matrix are roots of its characteristic polynomial. Hence if the matrices A and A^T have the same characteristic polynomial,then they have the same eigenvalues.

So we show that the characteristic polynomial
$p_A(t) = det(A - tI)$ of A is the same as the characteristic polynomial $p_{A^T}(t) = det(A^T - tI)$ of the transpose A^T.

We have

$$\begin{aligned} p_{A^T}(t) &= det(A^T - tI) \\ &= det(A^T - tI^T) \qquad since I^T = I \\ &= det((A - tI)^T) \\ &= det((A - tI)) \\ &= p_A(t) \end{aligned}$$

(since $det(B^T) = det(B)$ for any square matrix$B$) Therefore we obtain $p_{A^T}(t) = p_A(t)$, and we conclude that the eigenvalues of A and A^T are the same.

*2) **Property2**:* Determinant of a matrix is equal to product of its Eigen values

$$\det(\mathrm{A}) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 ... \lambda_n$$

Proof :
Suppose that$\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A. Then the λs are also the roots of the characteristic polynomial,i.e.

$$\begin{aligned} det(A - \lambda I) = p(\lambda) &= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n) \\ &= (-1)(\lambda - \lambda_1)(-1)(\lambda - \lambda_2)...(-1)(\lambda - \lambda_n) \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n) \end{aligned}$$

The first equality follows from the factorization of a polynomial given its roots;the leading (highest degree) coefficient $(-1)^n$ can be obtained by expanding the determinant along the diagonal.

Now,by setting λ to zero (simply because it is a variable) we get on the left side $det(A)$, and on the right side $\lambda_1, \lambda_2, ...\lambda_n$, that is we indeed obtain the desired result

$$det(A) = \lambda_1 \lambda_2 ... \lambda_n$$

So the determinant of the matrix is equal to the product of its eigenvalues.

*3) **Property3**:* The trace of an n x n matrix A is equal to the sum of its eigenvalues.

$$\mathrm{tr}(\mathrm{A}) = \sum_{i=1}^{n} a_{ii} = \lambda_1 + \lambda_2 + ... + \lambda_n$$

Proof :
Suppose that$\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A. Then the λs are also the roots of the characteristic polynomial,i.e.

$$|A - \lambda I| = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n) \qquad (1)$$

$$|A - \lambda I| = (-1)^n (\lambda^n + a_{n-1}\lambda^{n-1} ... + a_0) \qquad (2)$$

The coefficient of λ^{n-1} is $a_{n-1}(-1)^{n-1}$ from (1).From (2) to obtain the coefficient of λ^{n-1}. if we multiply all the λs together, one from each other,we obtain the tern λ^n. So to obtain the term with λ^{n-1},we need to multiply first $-\lambda_1$ times the λs in all the remaining factors,then if we multiply all the λs together, one from each factor, then $-\lambda_2$ times the λs and so on.Putting back the factor $(-1)^n$,the term involving λ^{n-1} is

$$(-1)^n(-\lambda_1 - \lambda_2 - ... - \lambda_n)\lambda^{n-1} = (-1)^{n-1}(\lambda_1 + \lambda_2 + ... + \lambda_n)\lambda^{n-1} \qquad (3)$$

Now suppose $A = (a_{ij})$ is an n x n matrix and look at the coefficient of λ^{n-1} in the cofactor expansion of $|A - \lambda I|$ by row 1:

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} - \lambda & ... & a_{2n} \\ . & . & ... & . \\ a_{n1} & a_{n2} & ... & a_{nn} - \lambda \end{vmatrix}_{n \times n}$$

$$= (a_{11} - \lambda)C_{11} + a_{12}C_{12} + ...$$

Only the first term of the cofactor expansion,$(a_{11} - \lambda)C_{11}$, contains higher power of λ than λ^{n-1}.

Let us see why C_{11} is the determinant of the $(n-1)$ x $(n-1)$ matrix obtained from matrix $(A-\lambda I)$ by crossing out the first row and first column, so it is of the form $|C - \lambda I|$, where C is the $(n-1)$ x $(n-1)$ matrix obtained from A by crossing out the first row and first column.

Therefore by our assumption,

$|A - \lambda I|$
$= (a_{11} - \lambda)C_{11}$
$= (a_{11} - \lambda)((-1)^{n-1} + (-1)^{n-2}(a_{22} + ... + a_{nn})\lambda^{n-2} + ...)$
$= (-1)^n\lambda^n + (-1)^{n-1}(a_{11} + a_{22} + ... + a_{nn})\lambda^{n-1} + ...$

We can now conclude that the term involving λ^{n-1} in the expansion of $|A - \lambda I|$ for any n x n matrix A is equal to

$$(-1)^{n-1}(a_{11} + a_{22} + ... + a_{nn})\lambda^{n-1} \quad (4)$$

Comparing the coefficients of λ^{n-1} in the two expressions (3) and (4), we see that

$$a_{11} + a_{22} + a_{33} + ... + a_{nn} = \lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n$$

that is, the trace of A is equal to the sum of the eigenvalues

*4) **Property4**:* A is an invertible matrix with eigenvalue λ corresponding to eigenvector x, then A^{-1} has eigenvalue λ^{-1} corresponding to the same eigenvector x.

Proof :
We know that for a matrix A the Eigen polynomial is $Ax = \lambda x$
Now multiply the polynomial with $\lambda^{-1}A^{-1}$ on both sides,we get

$$\lambda^{-1}A^{-1}(Ax) = \lambda^{-1}A^{-1}\lambda x$$

$$\lambda^{-1}x = A^{-1}x$$

Thus A_{-1} has Eigen value λ^{-1} corresponding to the same Eigen vector x
So if A is an invertible matrix with eigenvalue λ corresponding to eigenvector x, then A^{-1} has eigenvalue λ^{-1} corresponding to the same eigenvector x

*5) **Property5**:* If a matrix A has eigenvalue λ with corresponding eigenvector x, then for any $k = 1, 2, ..., A^k$ has eigenvalue λk corresponding to the same eigenvector x.

proof :
Suppose the matrix A has eigenvalue λ with eigenvector x

$$Ax = \lambda x$$

Now

$$A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$$

Again multiplying with A

$$A^3x = \lambda^3x, A^4x = \lambda^4x, ..$$

Similarly for matrix A^k the Eigen values will be $\lambda_1^k, \lambda_2^k..$

References

[1] Linear Algebra Concepts and Methods by Martin Anthony and Michele Harvey