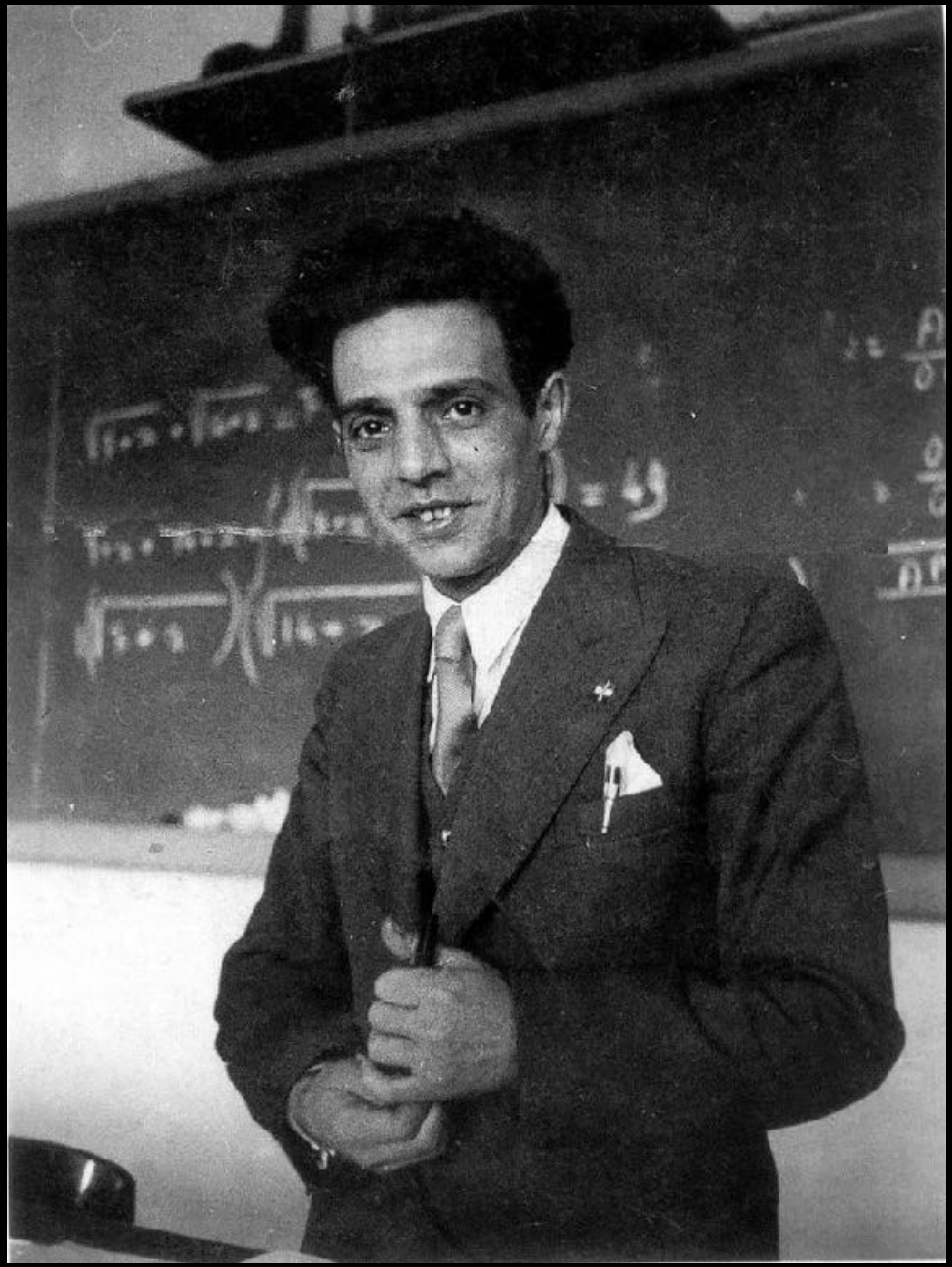
THE MOVING POINT FILLS THE WHOLE SPACE



Selected works of Giuseppe Peano

Translated and edited by Hubert C. Kennedy

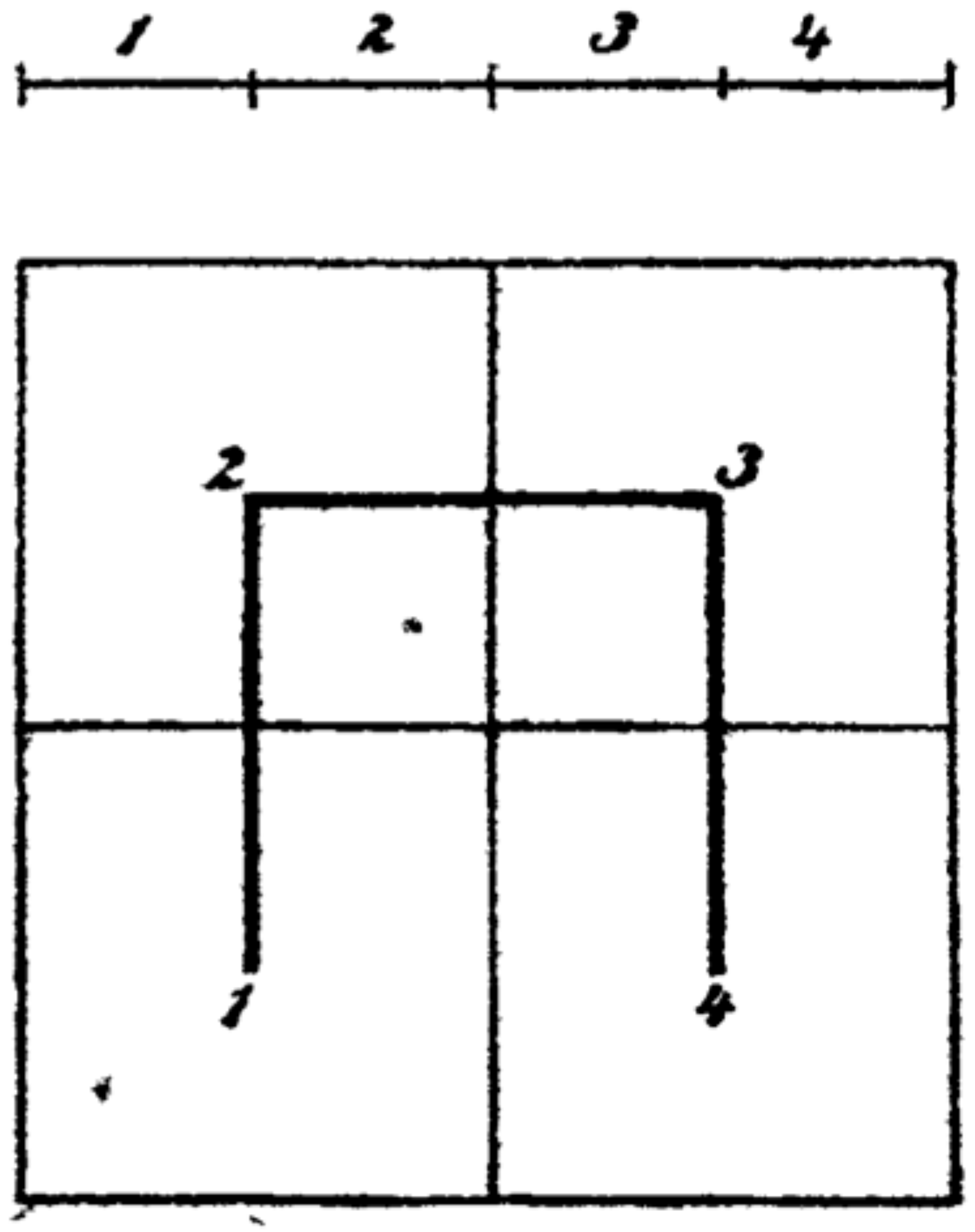


Fig. 1.

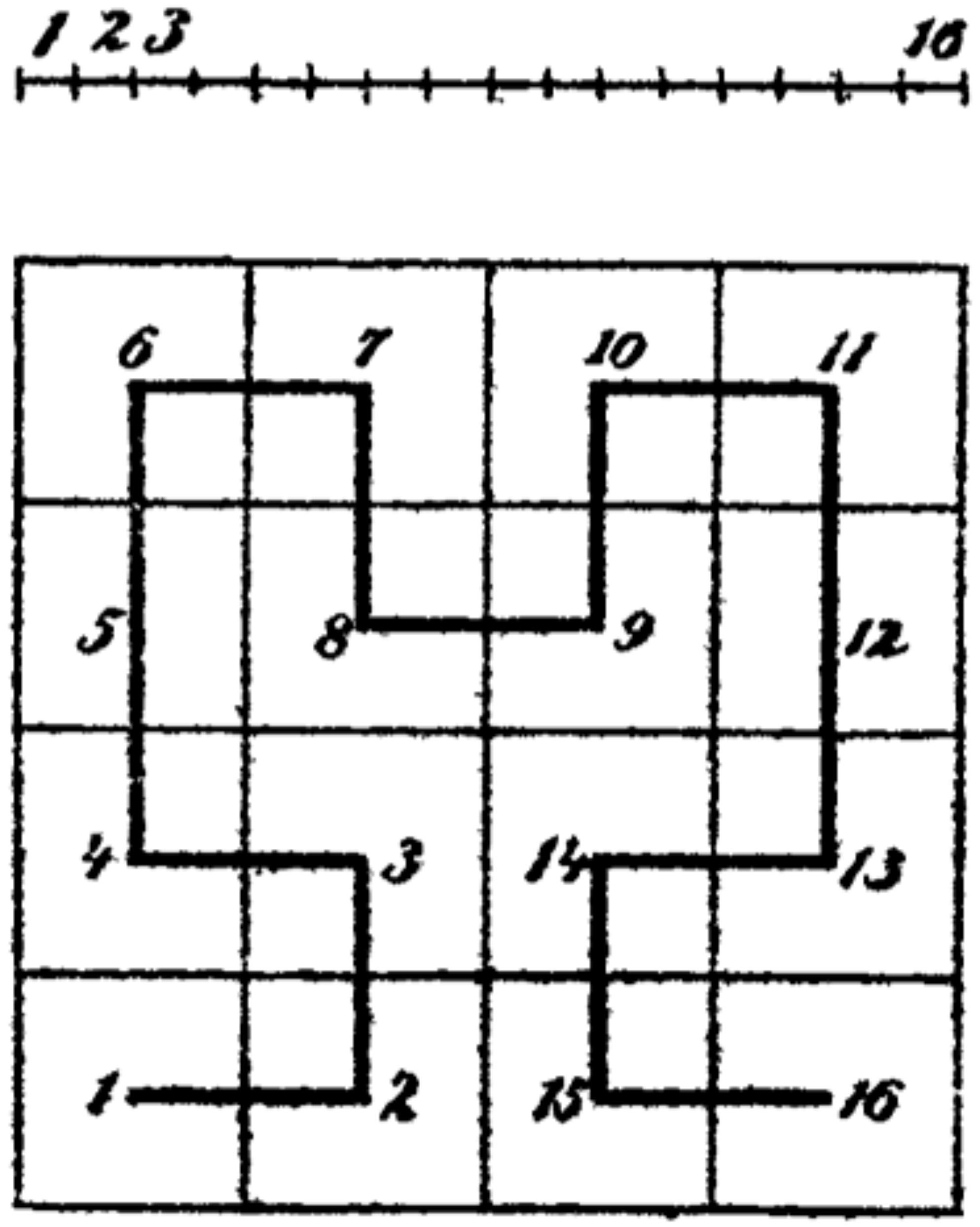


Fig. 2.

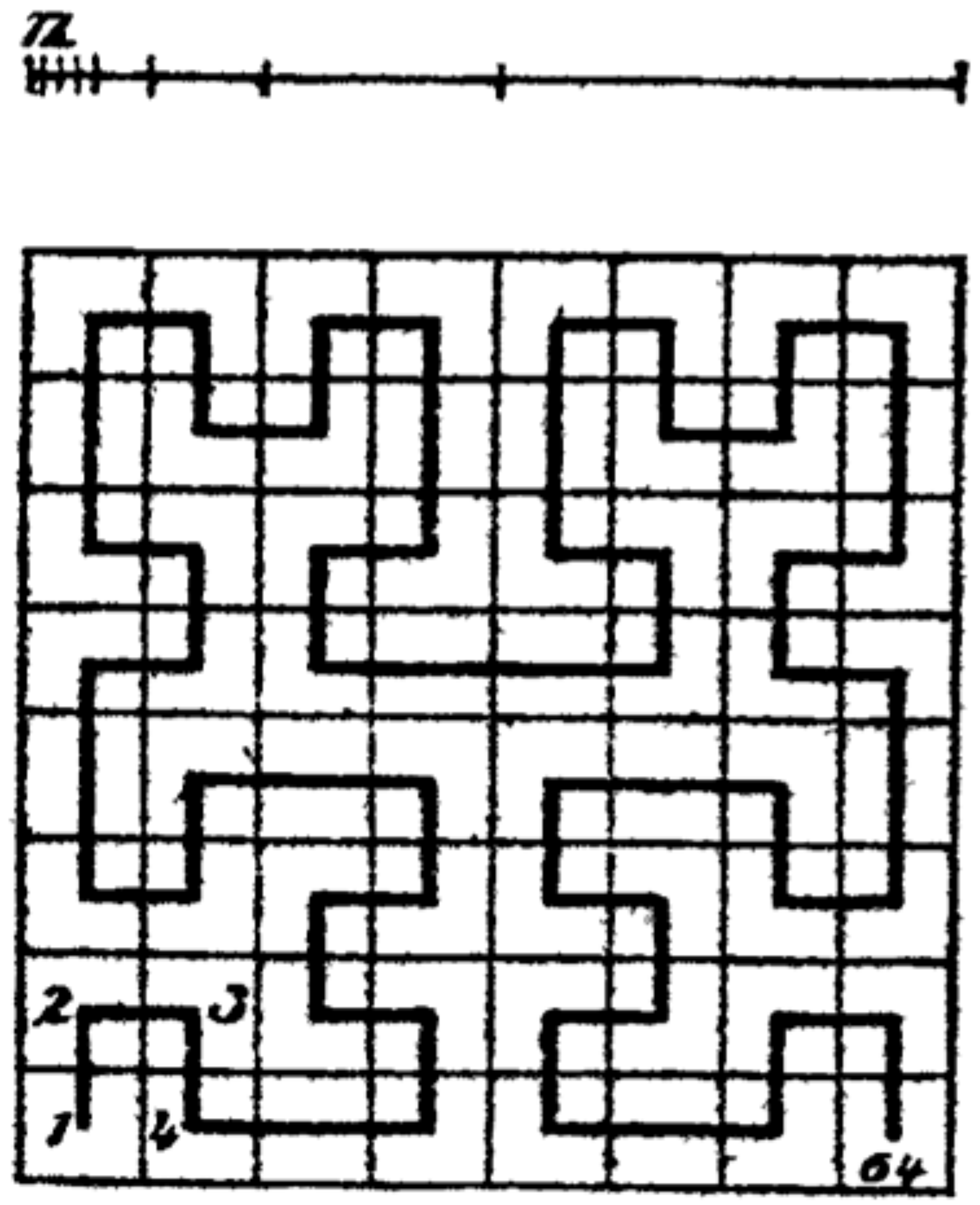


Fig. 3.

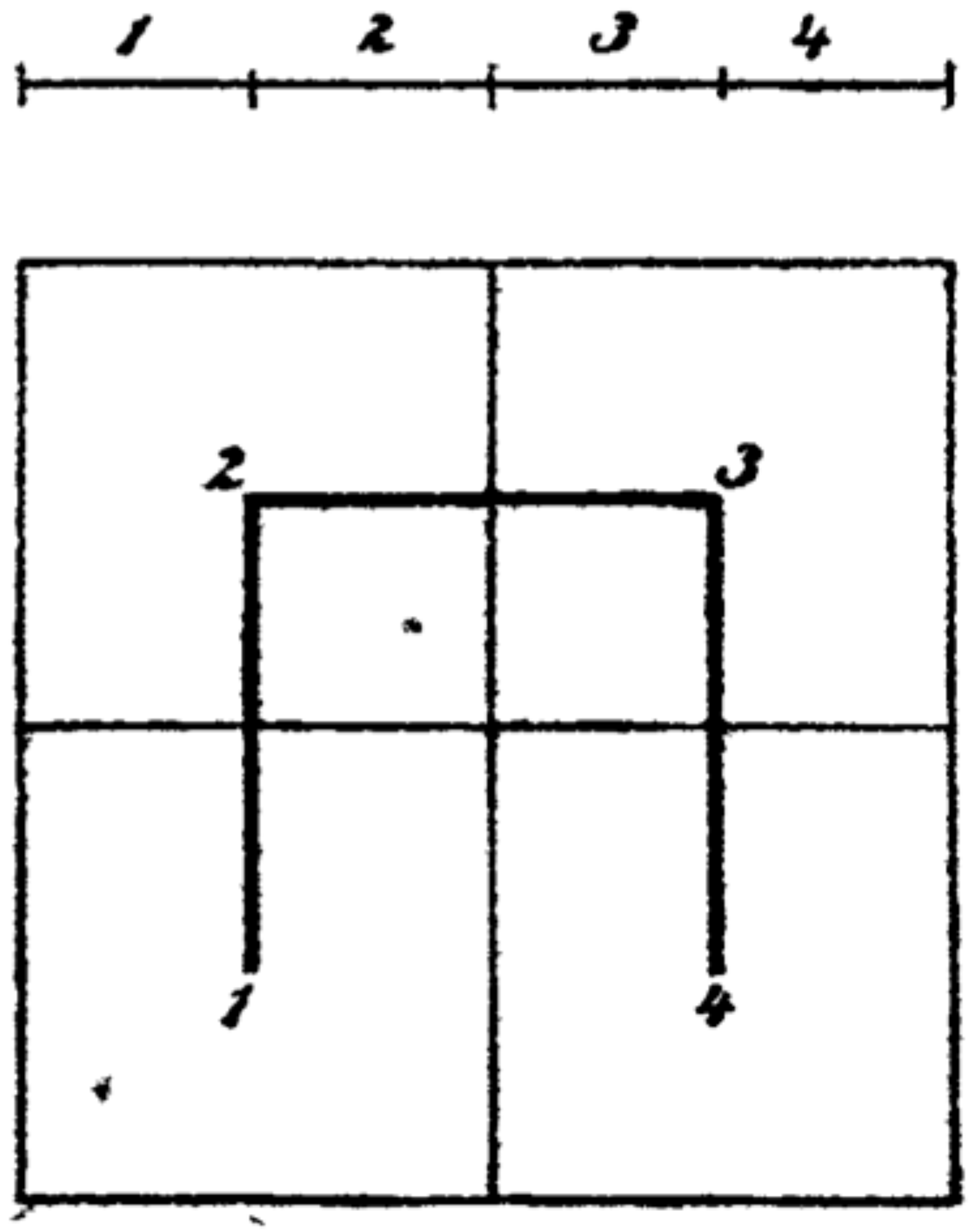


Fig. 1.

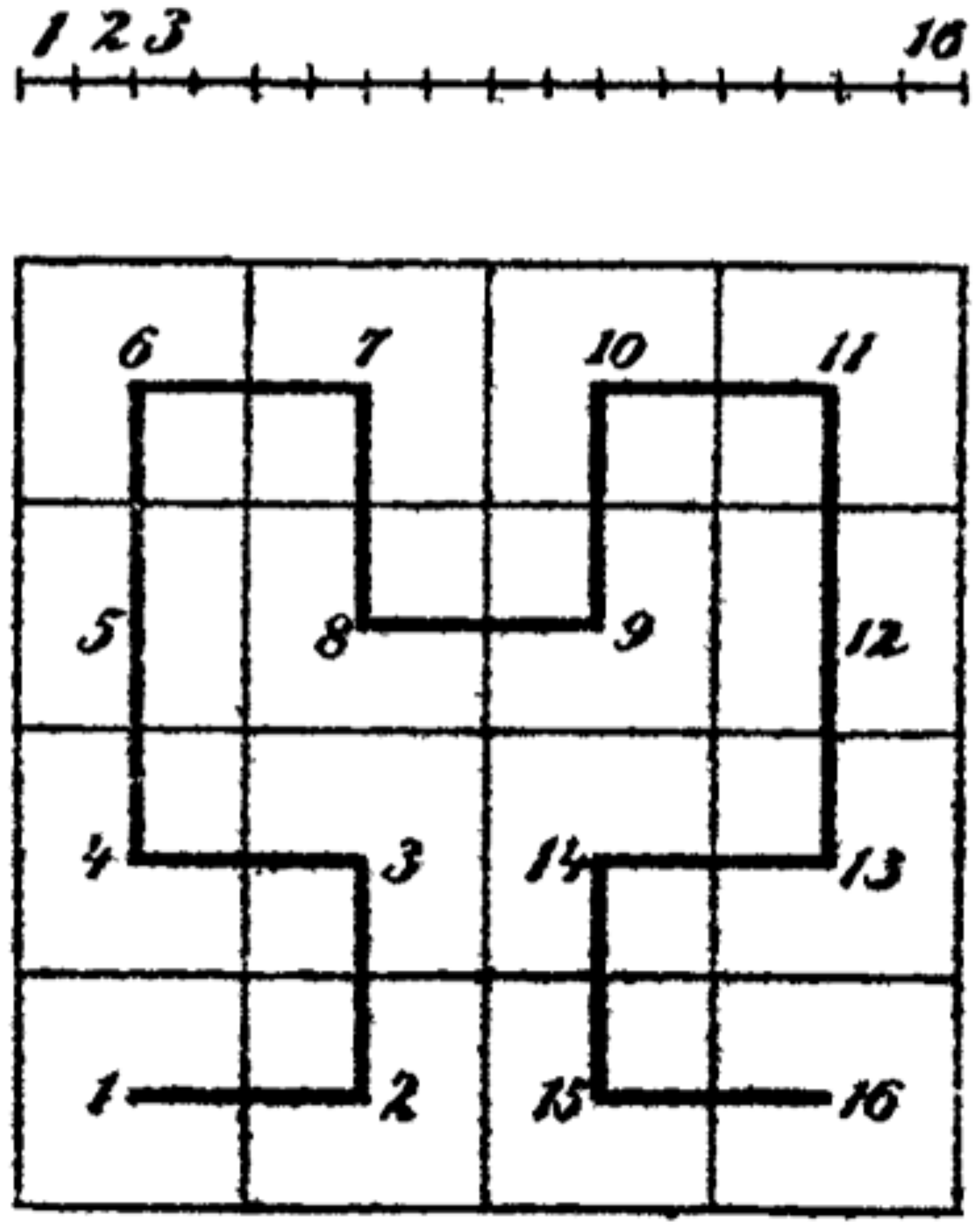


Fig. 2.

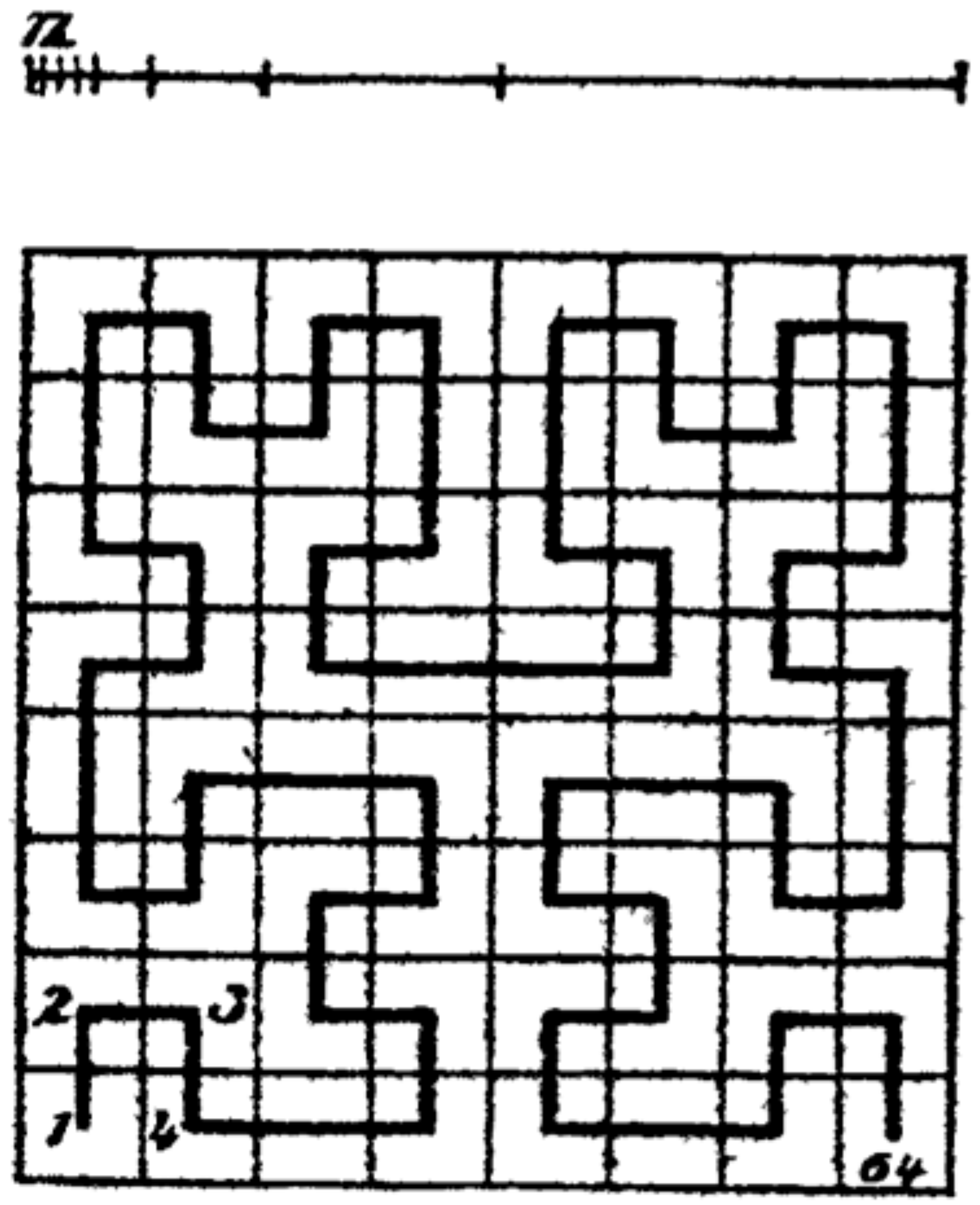
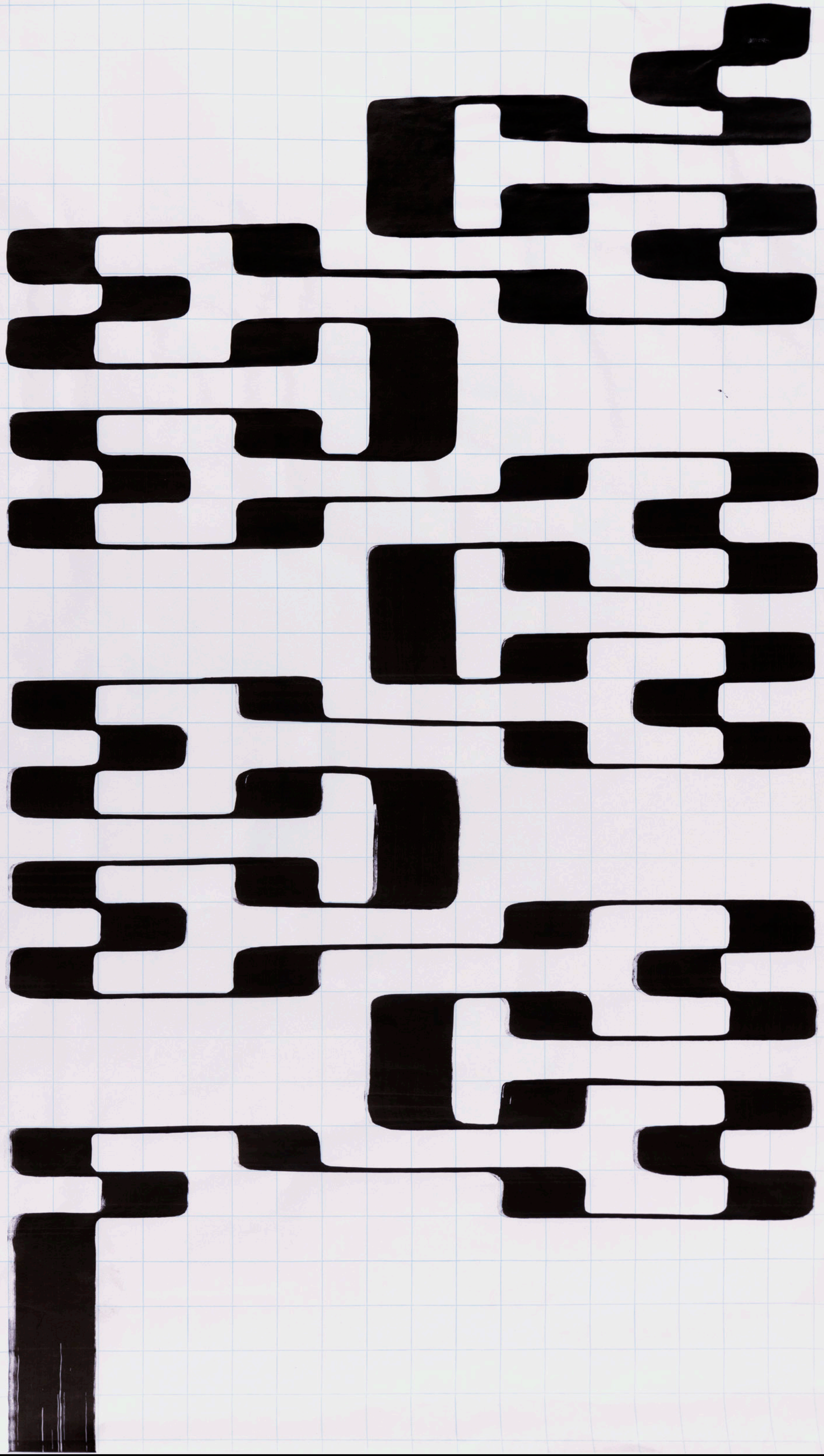
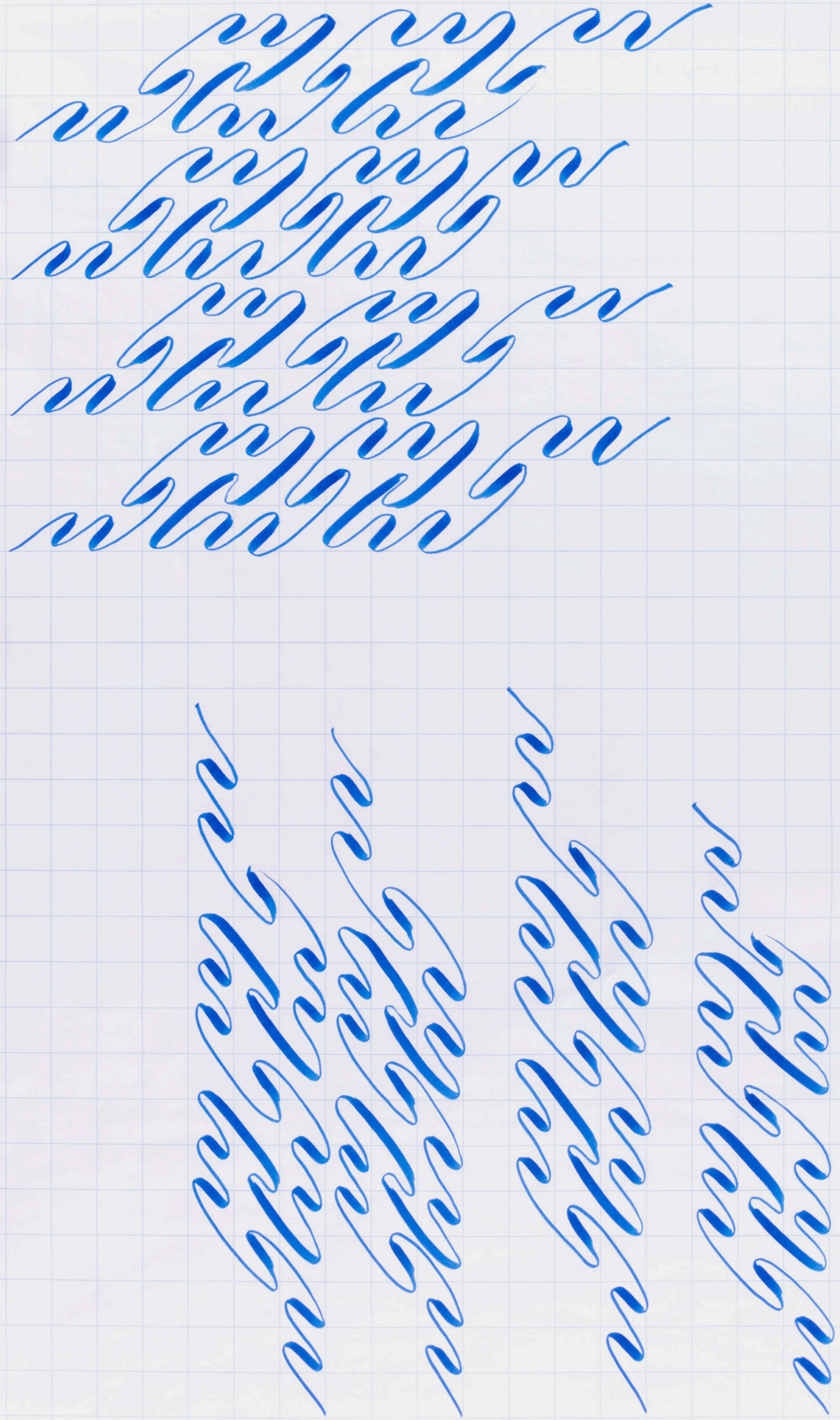
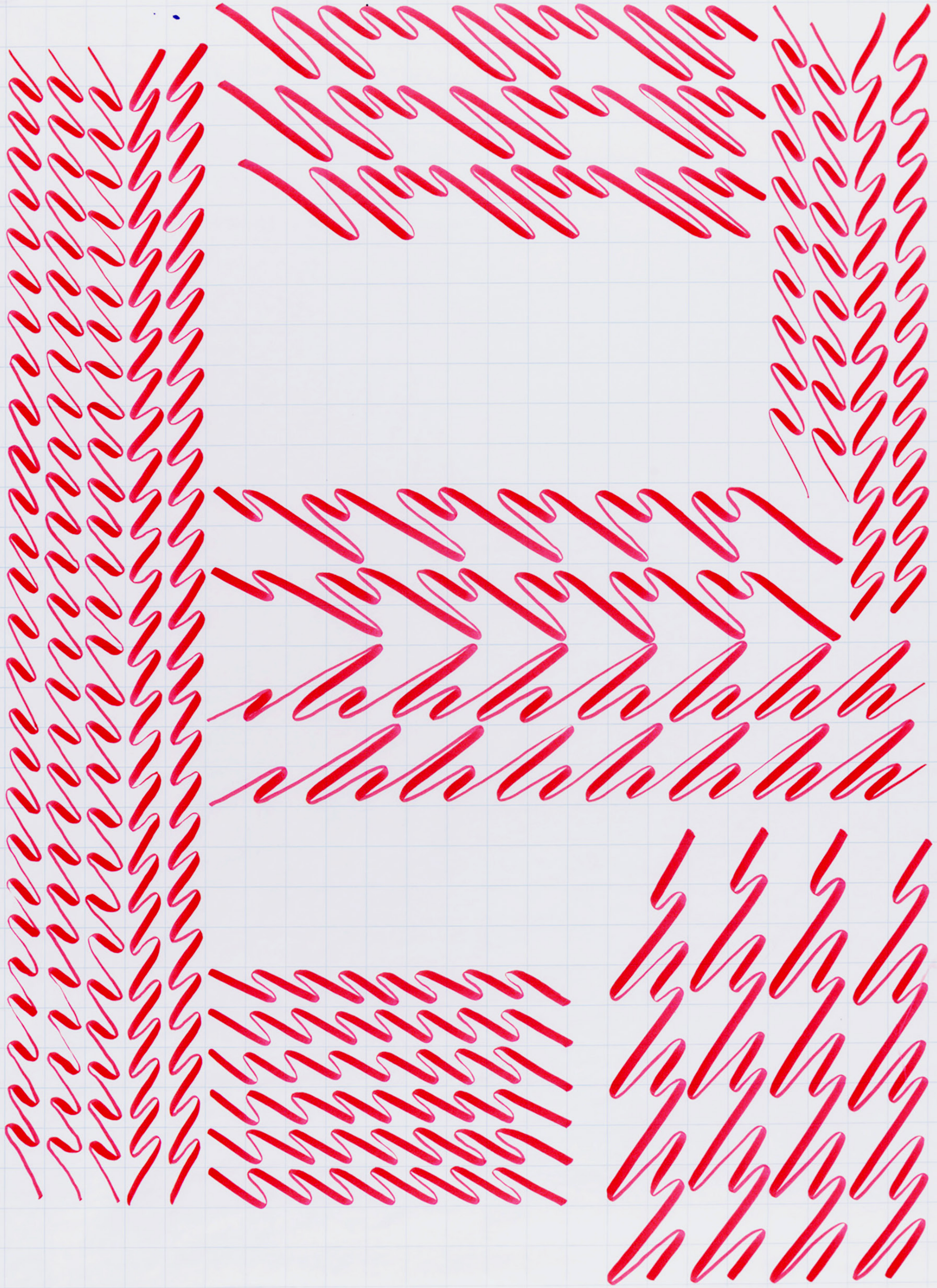


Fig. 3.

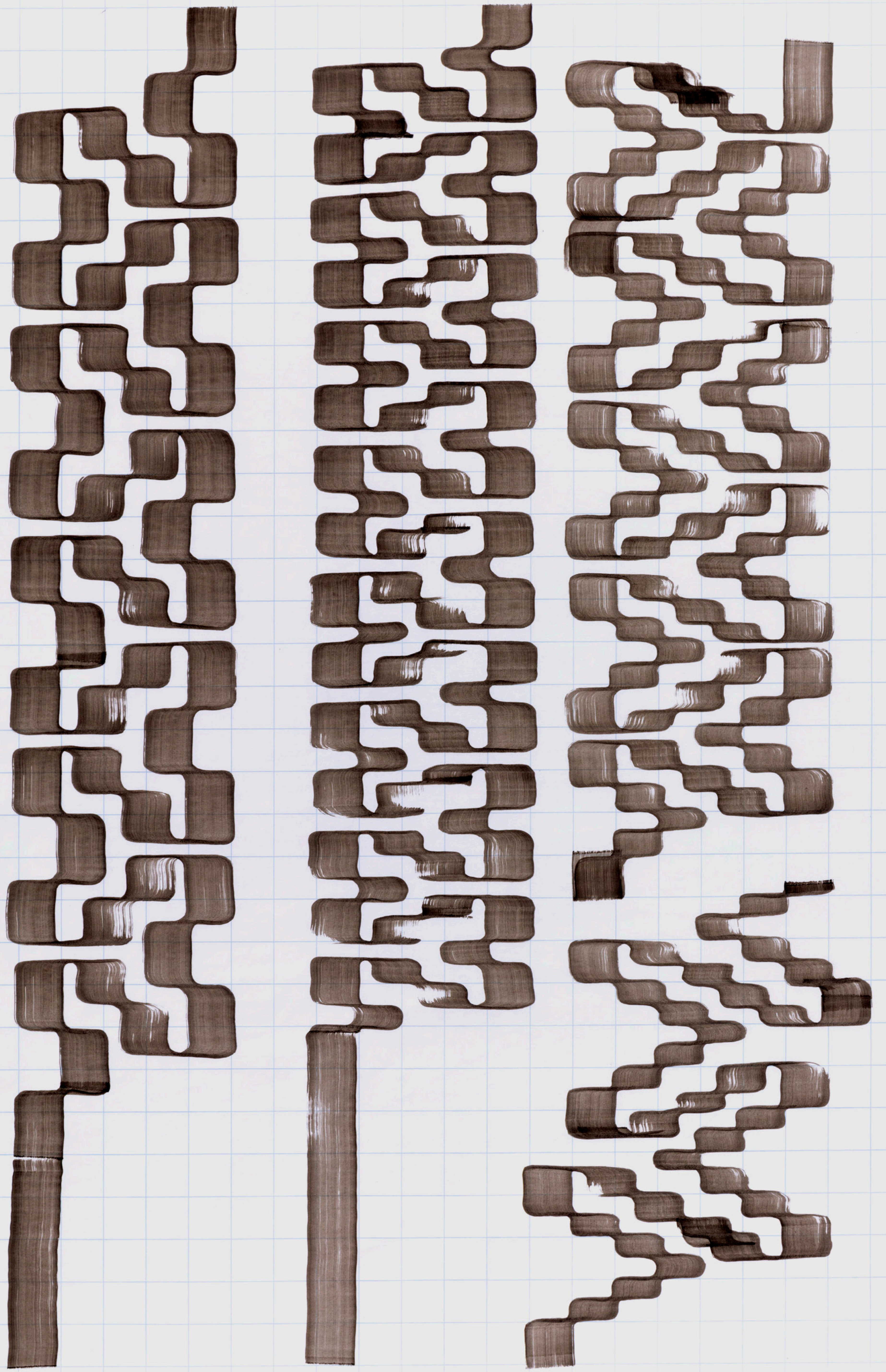
(a) (b) (c) (d) (e)



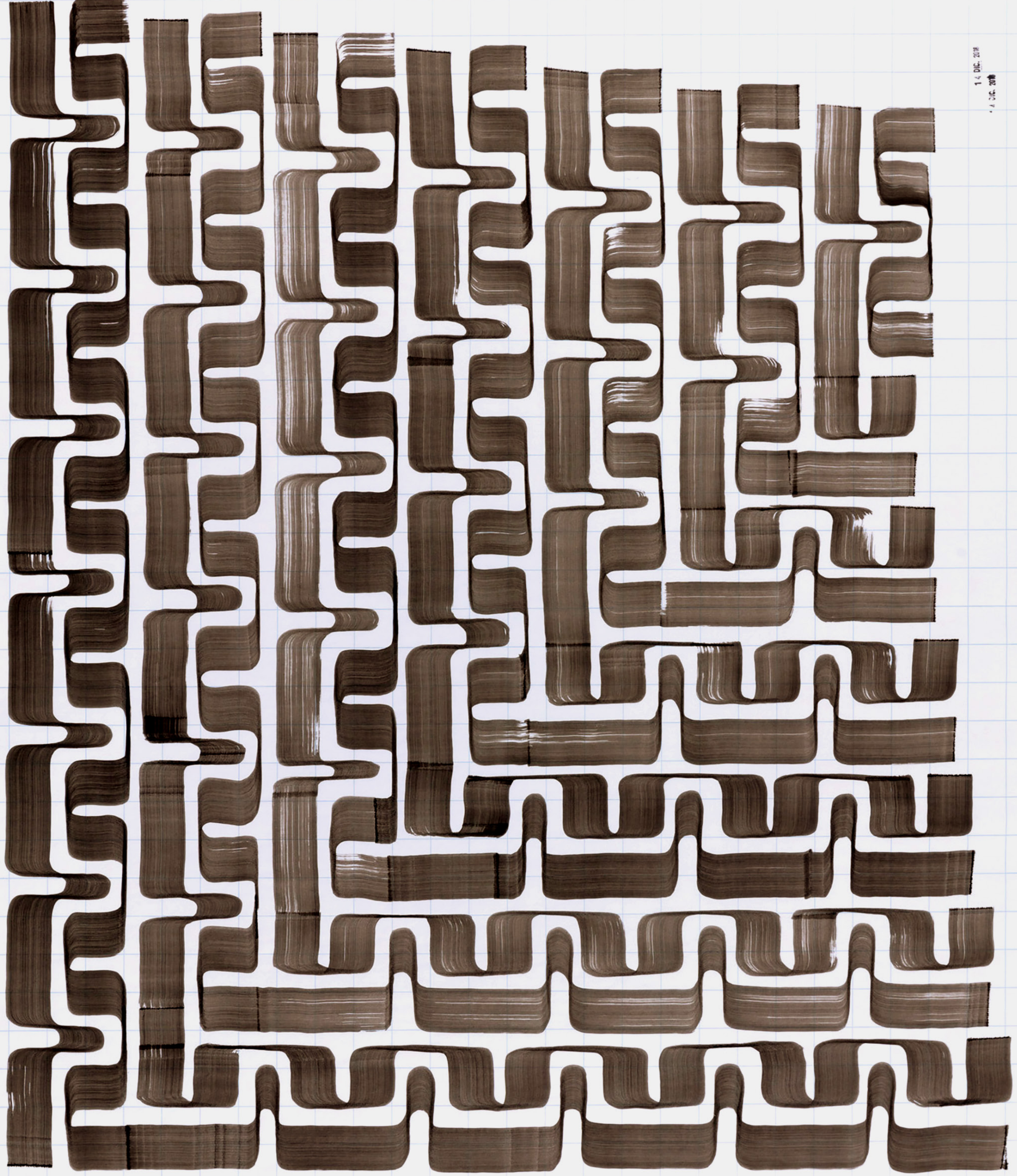




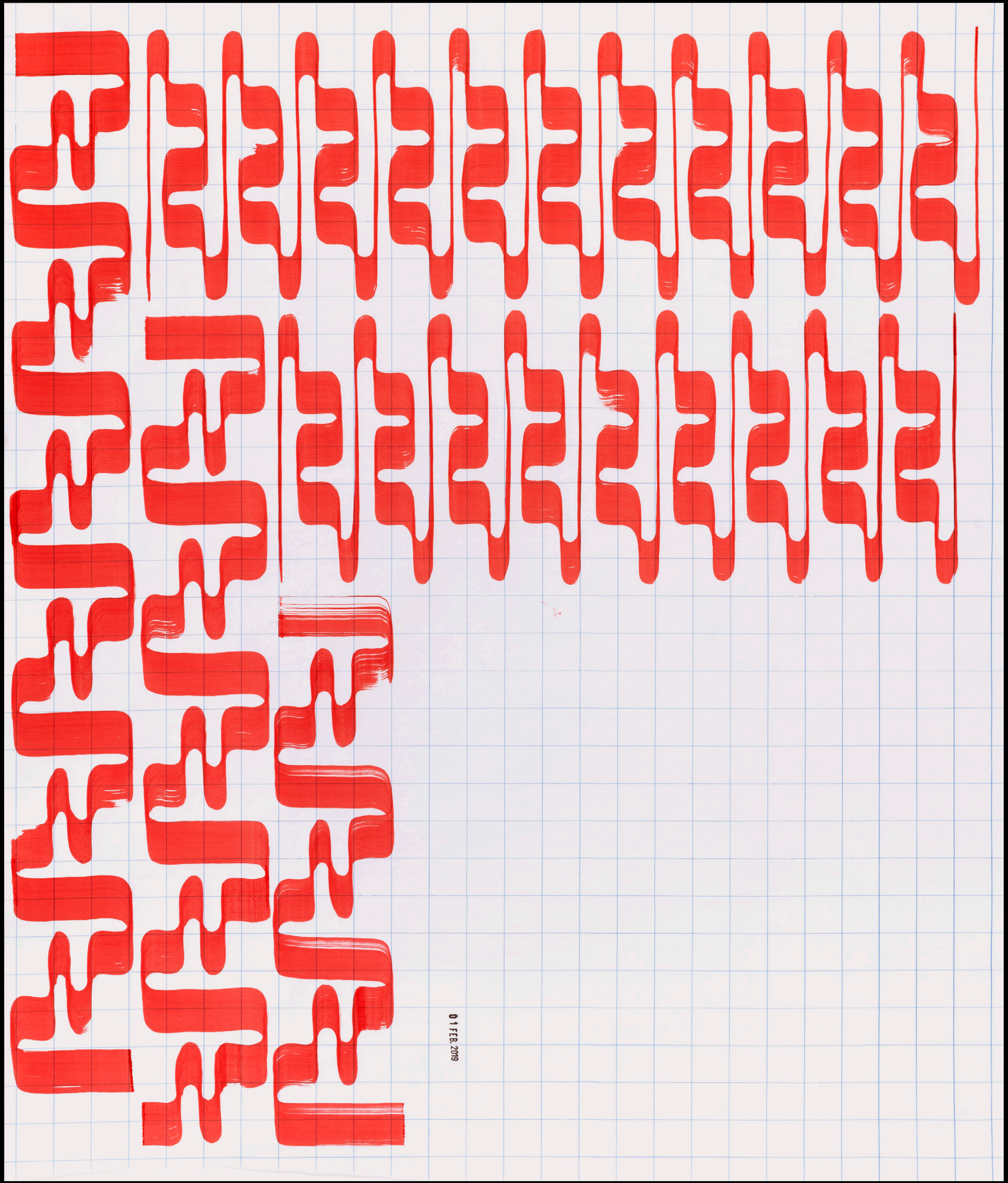
12 DIC. 2018

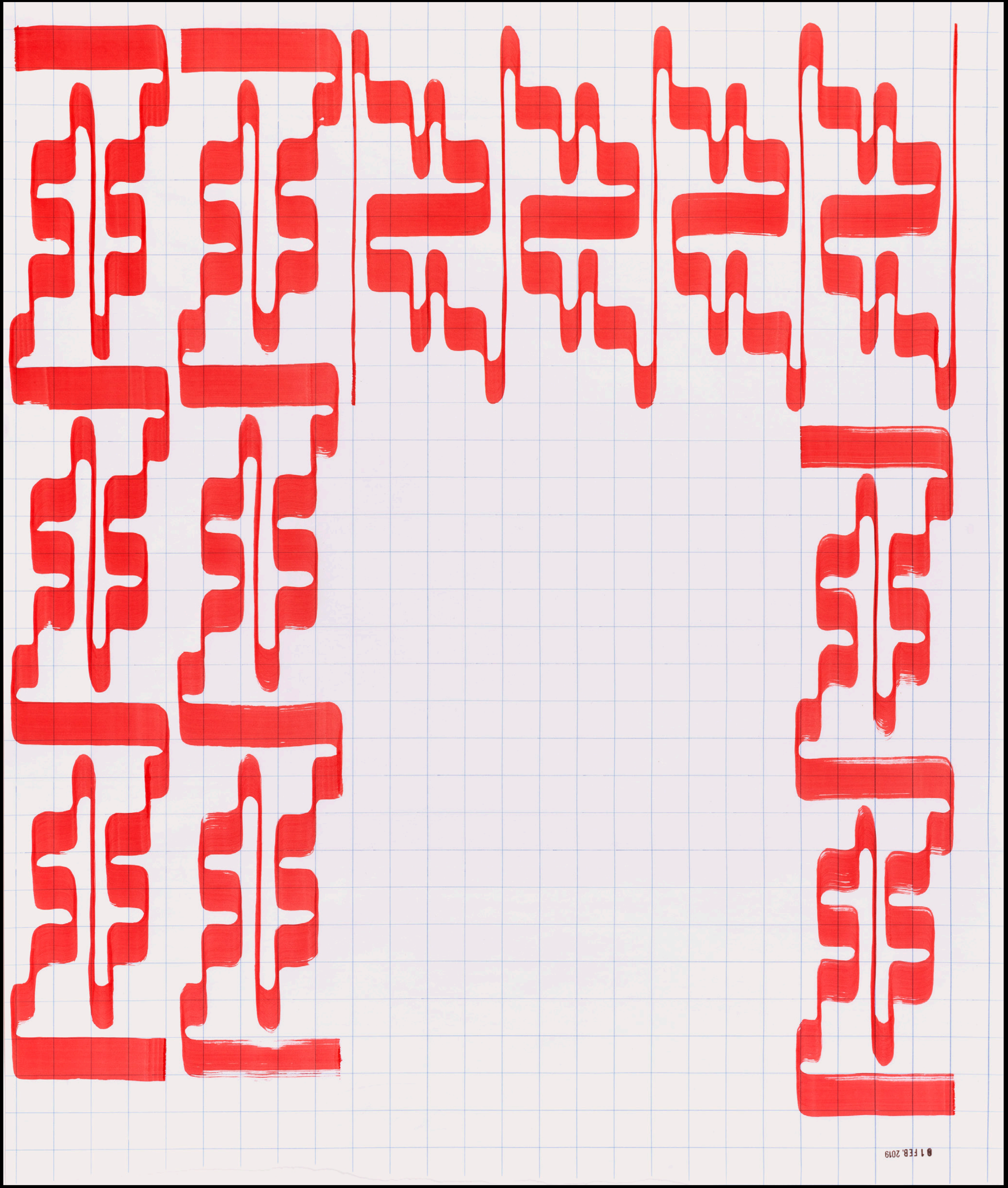


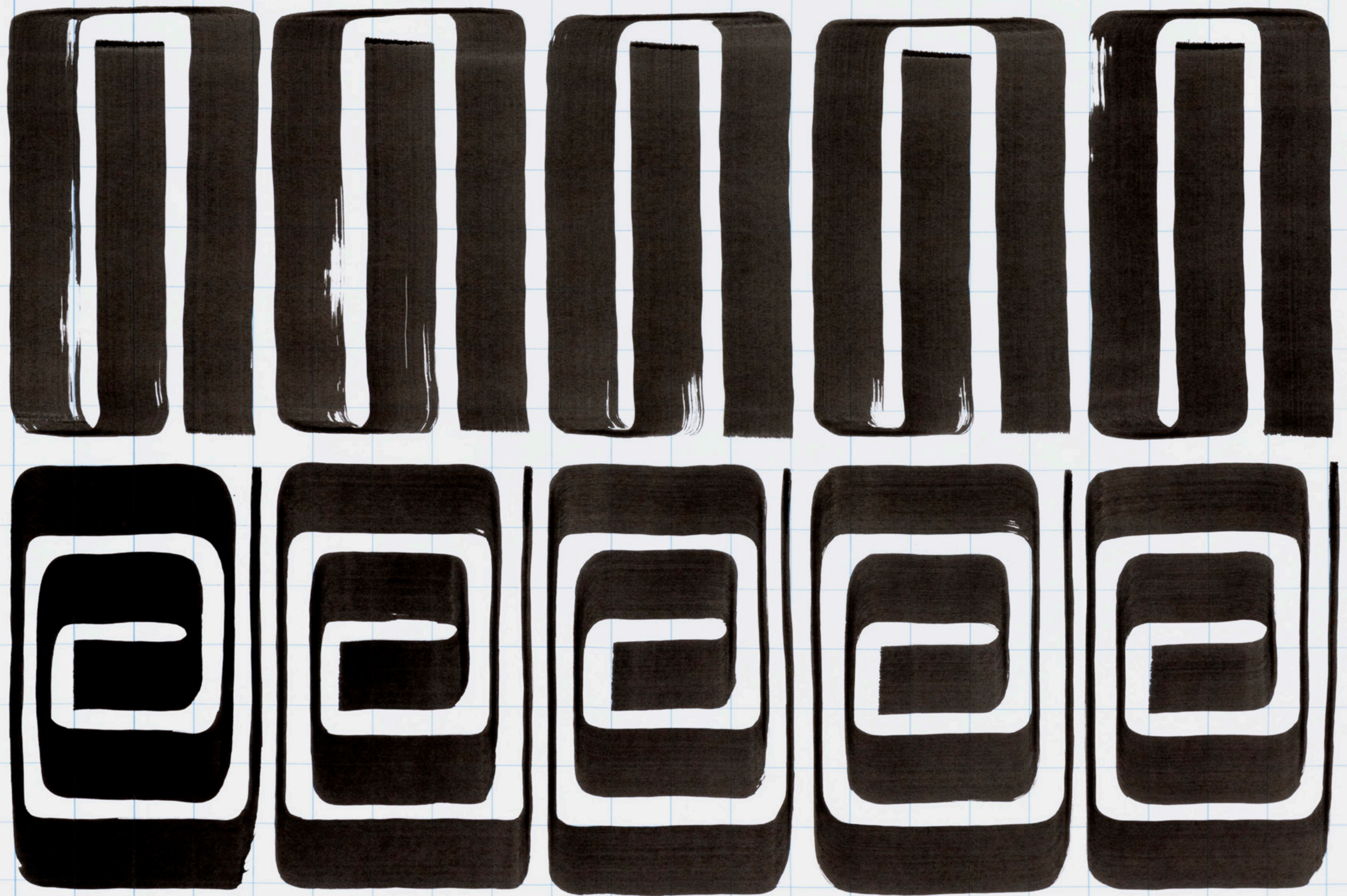
14 DIC. 2018

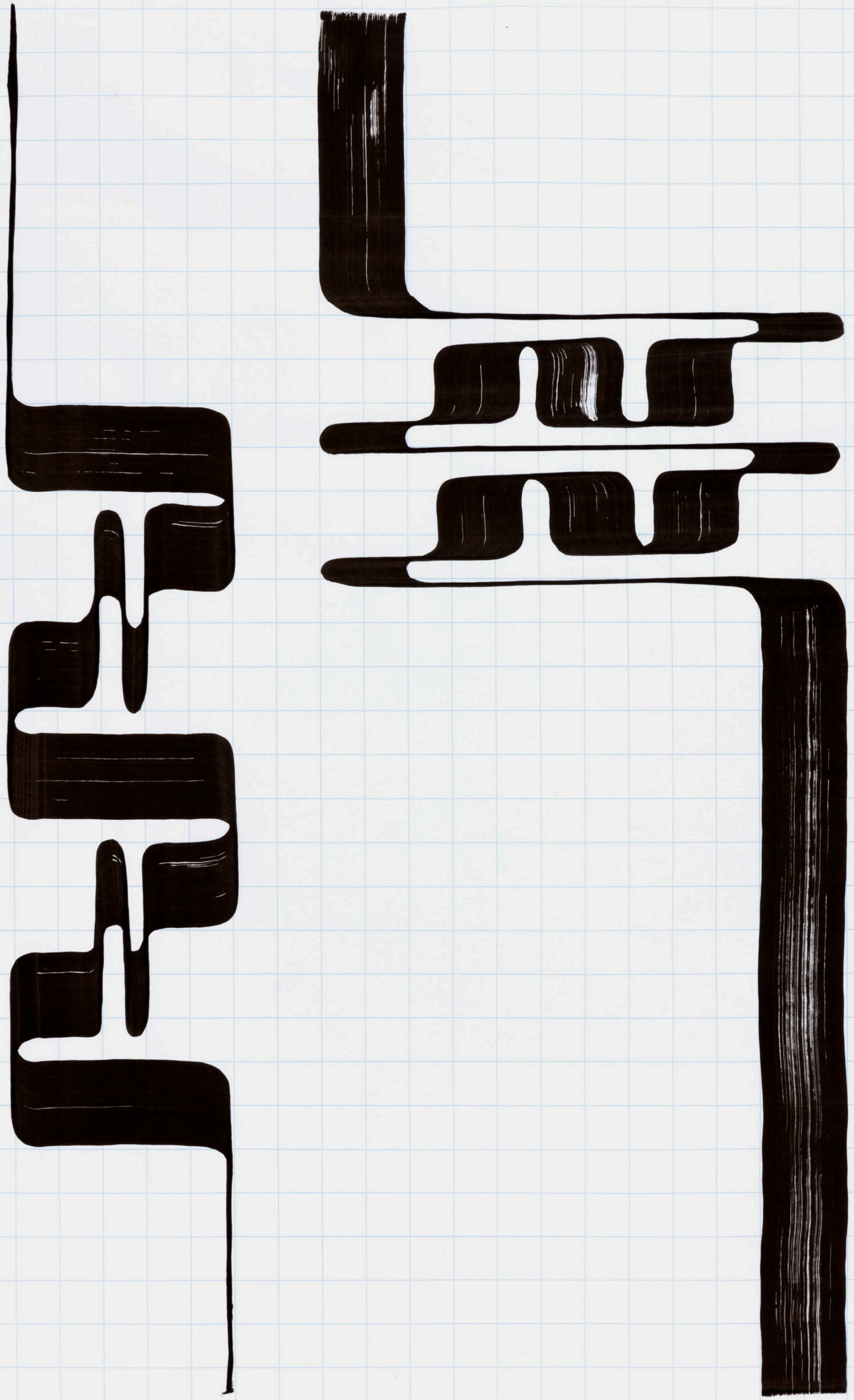


14 DIC. 2018
14 DIC. 2018

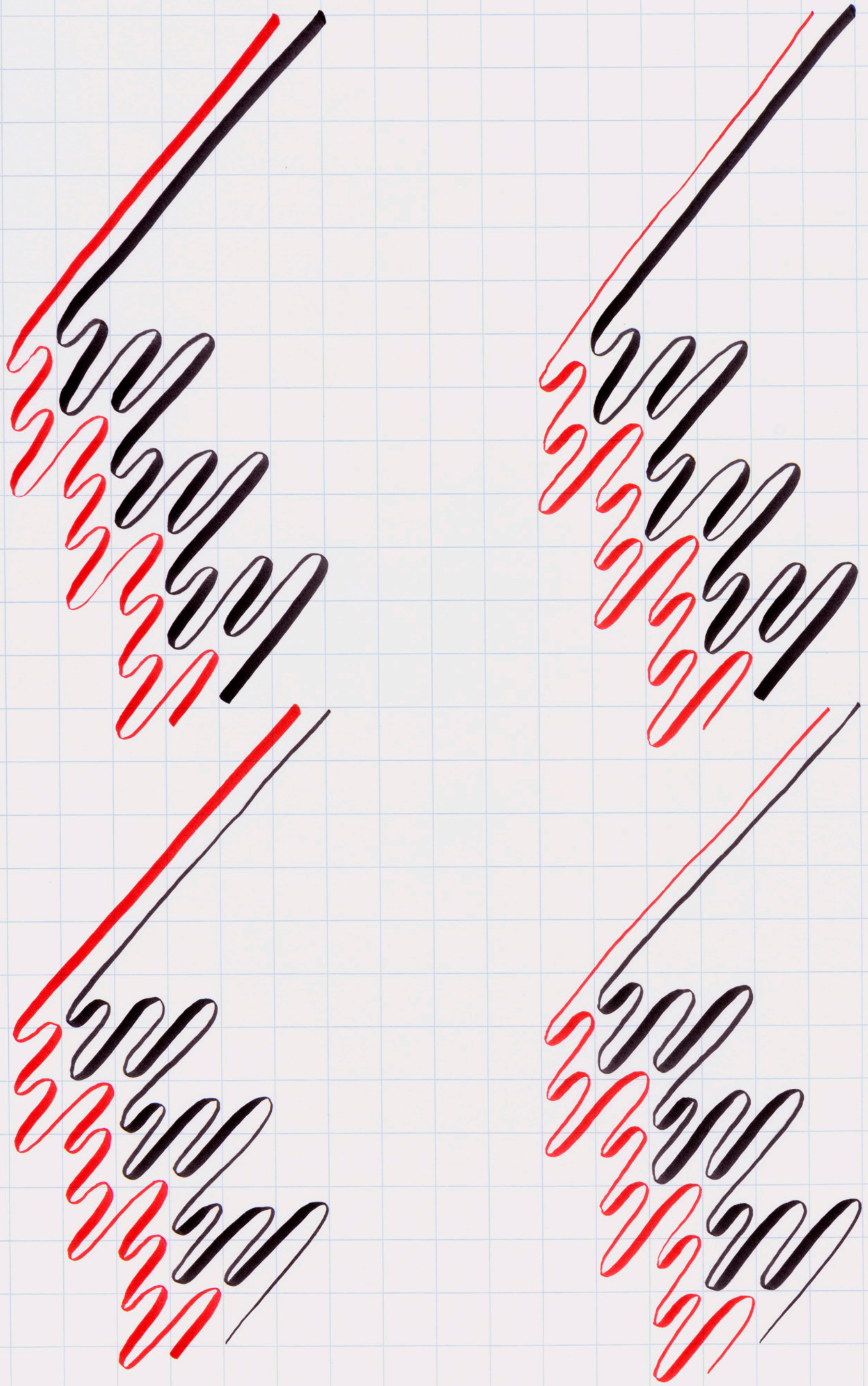

01 FEB. 2019


01 FEB. 2019



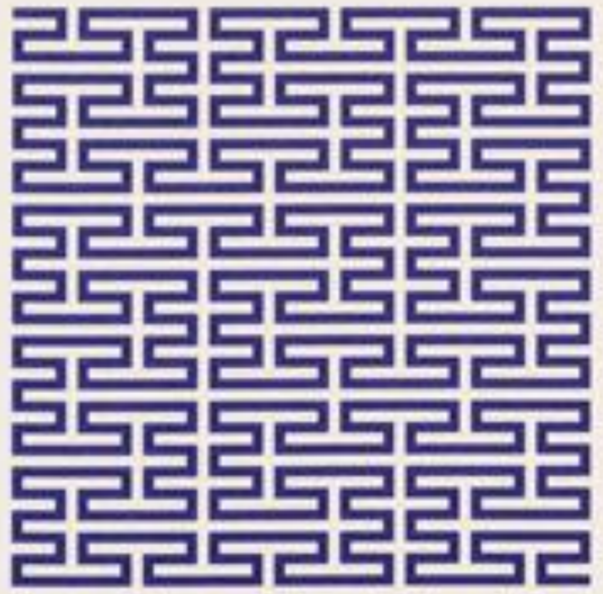


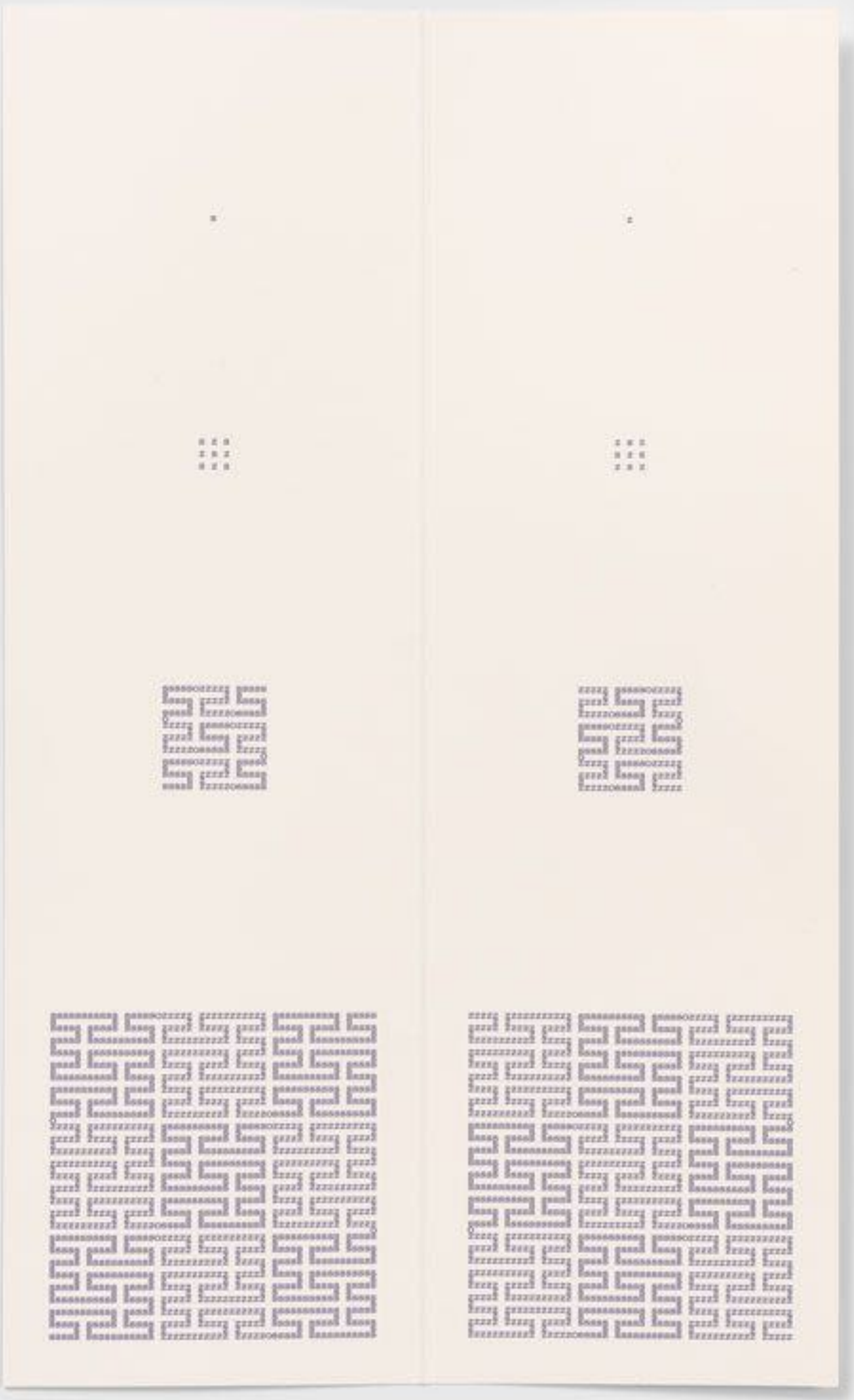
AUG 2 8 2020

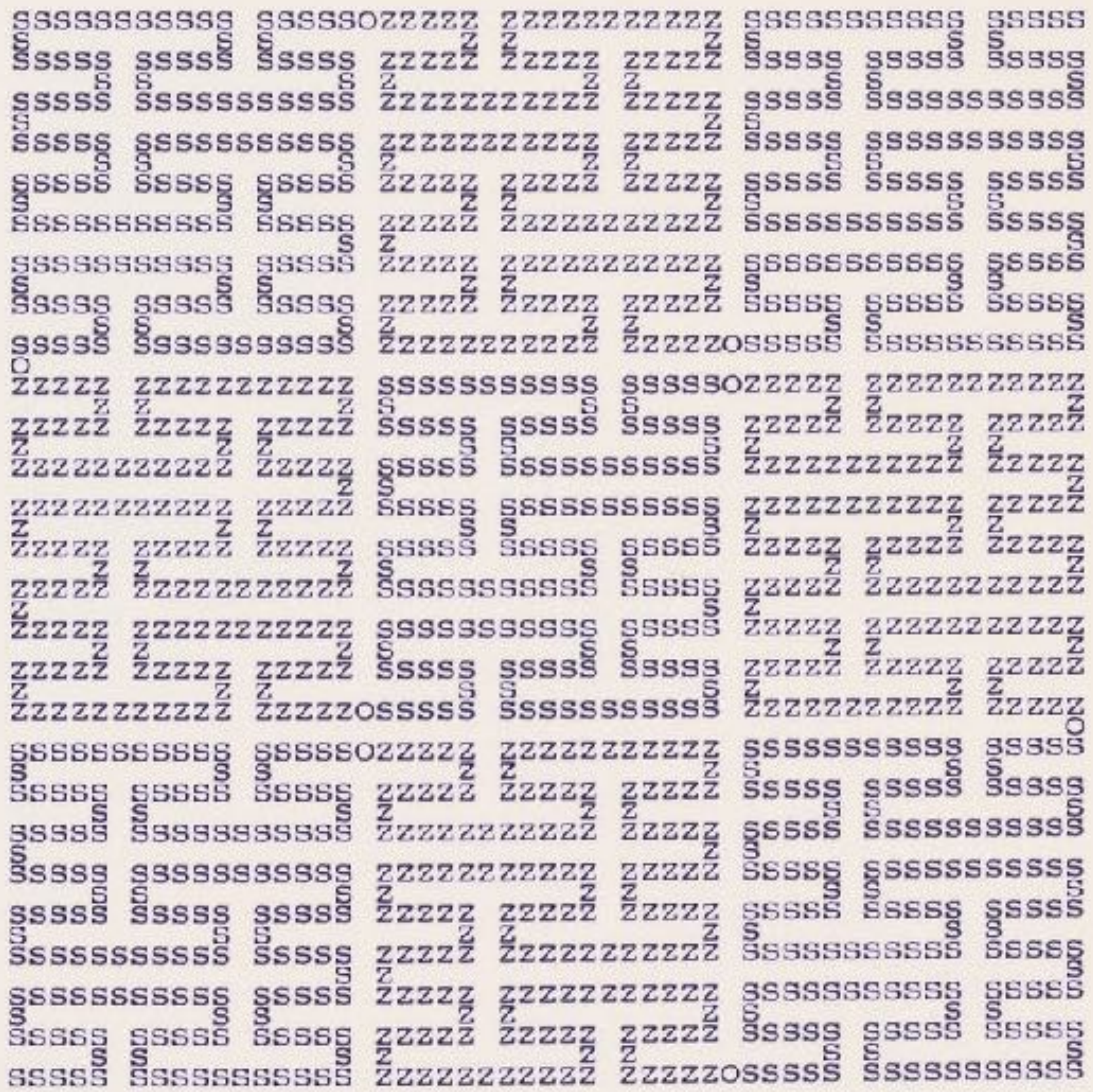


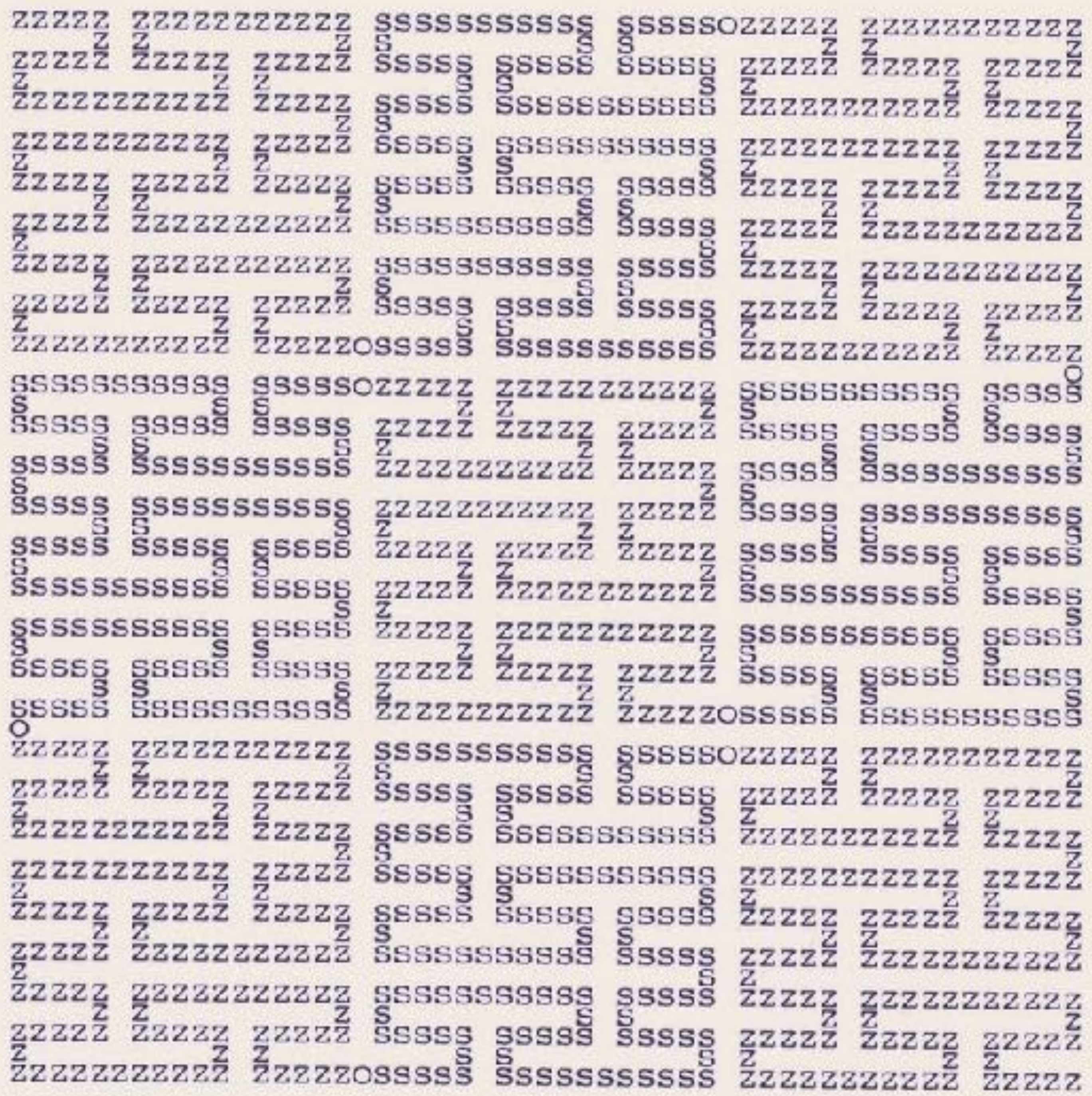

TAUBA AUERBACH
DIAGONAL PRESS

PEANOPOEM
I









```
SSS SSSSSOZZZZZ ZZZZZZZZZZZ SSSSSSSSS
  S S         Z Z         Z S
SSS SSSSS ZZZZZ ZZZZZ ZZZZZ SSSSS SSS
        S Z         Z Z         S S
SSSSSSSSS ZZZZZZZZZZZ ZZZZZ SSSSS SSS
                          Z S
SSSSSSSSS ZZZZZZZZZZZ ZZZZZ SSSSS SSS
        S Z         Z Z         S S
SSS SSSSS ZZZZZ ZZZZZ ZZZZZ SSSSS SSS
  S S         Z Z         Z S
SSS SSSSS ZZZZZ ZZZZZZZZZZZ SSSSSSSSS
        S Z
SSS SSSSS ZZZZZ ZZZZZZZZZZZ SSSSSSSSS
  S S         Z Z         Z S
SSS SSSSS ZZZZZ ZZZZZ ZZZZZ SSSSS SSS
        S Z         Z Z         S S
SSSSSSSSS ZZZZZZZZZZZ ZZZZZOSSSSS SSS

ZZZZZZZZZ SSSSSSSSSSS SSSSSOZZZZZ ZZZ
        Z S         S S         Z Z
ZZZ ZZZZZ SSSSS SSSSS SSSSS ZZZZZ ZZZ
  Z Z         S S         S Z
ZZZ ZZZZZ SSSSS SSSSSSSSSSS ZZZZZZZZZ
        Z S
ZZZ ZZZZZ SSSSS SSSSSSSSSSS ZZZZZZZZZ
  Z Z         S S         S Z
ZZZ ZZZZZ SSSSS SSSSS SSSSS ZZZZZ ZZZ
        Z S         S S         Z Z
ZZZZZZZZZ SSSSSSSSSSS SSSSS ZZZZZ ZZZ
                          S Z
ZZZZZZZZZ SSSSSSSSSSS SSSSS ZZZZZ ZZZ
        Z S         S S         Z Z
```


a
partial
taxonomy of
periodic linear ornament
- both established and original -
...
arranged
by shape, symmetry
dimension, projection and iteration
..
containing
extrude the extrusion
and
ornament as entheogen
.
tauba auerbach
diagonal press

first
edition
-
two thousand
seventeen

subject
to
-likely-
revision

second
edition
-
two thousand
seventeen

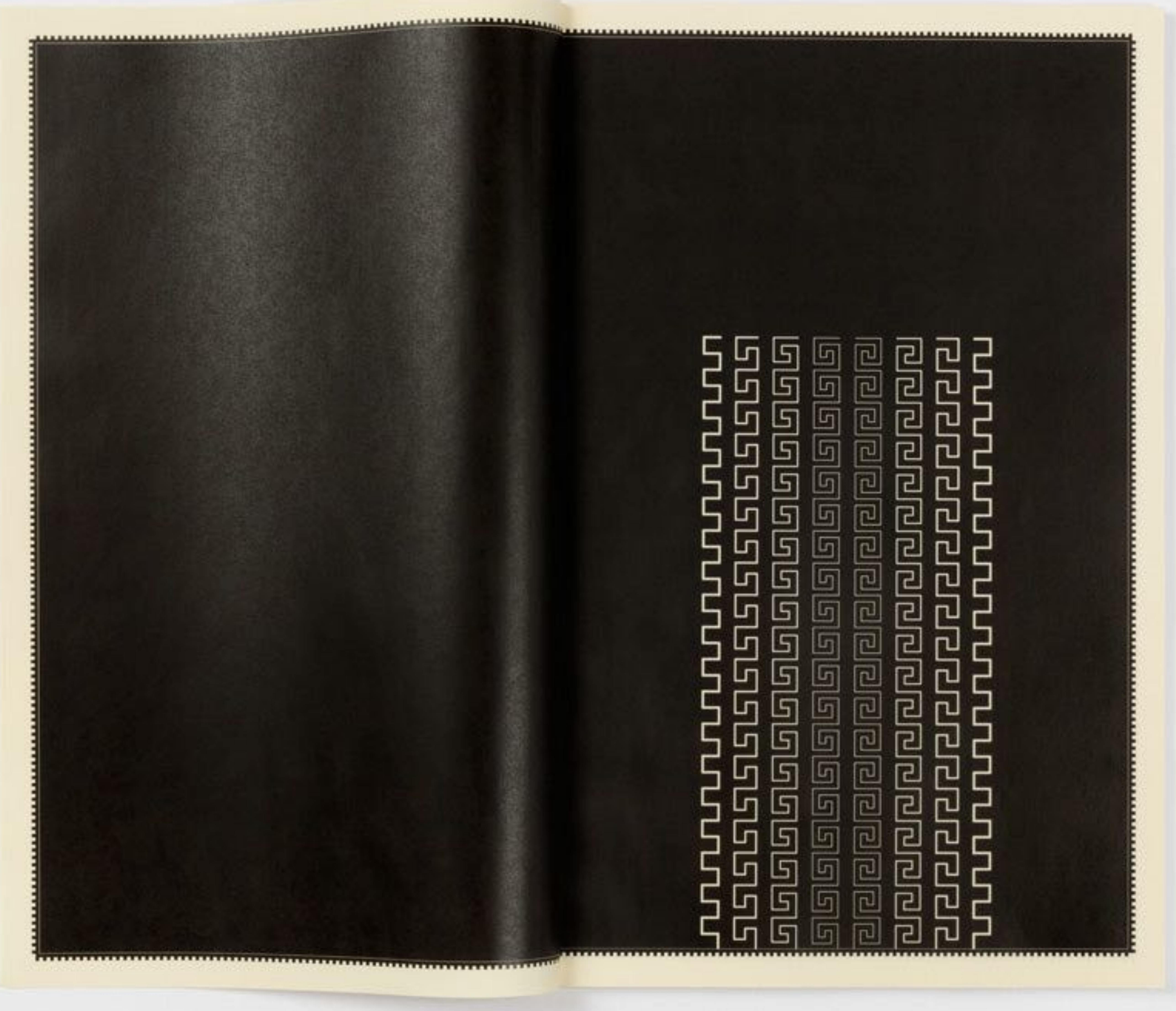
subject
to
-likely-
revision

third
edition
-
two thousand
seventeen

subject
to
-likely-
revision

fourth
edition
-
two thousand
eighteen

subject
to
-likely-
revision



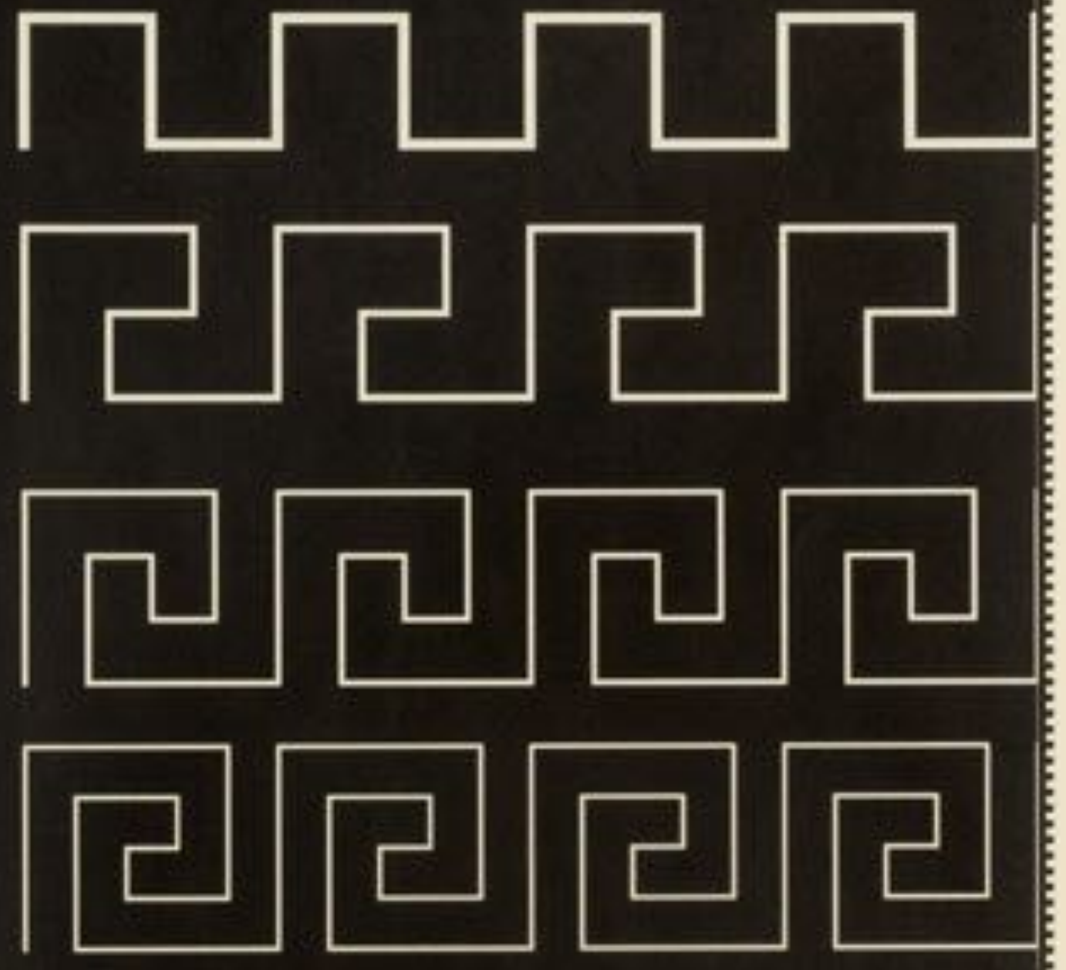
iteration

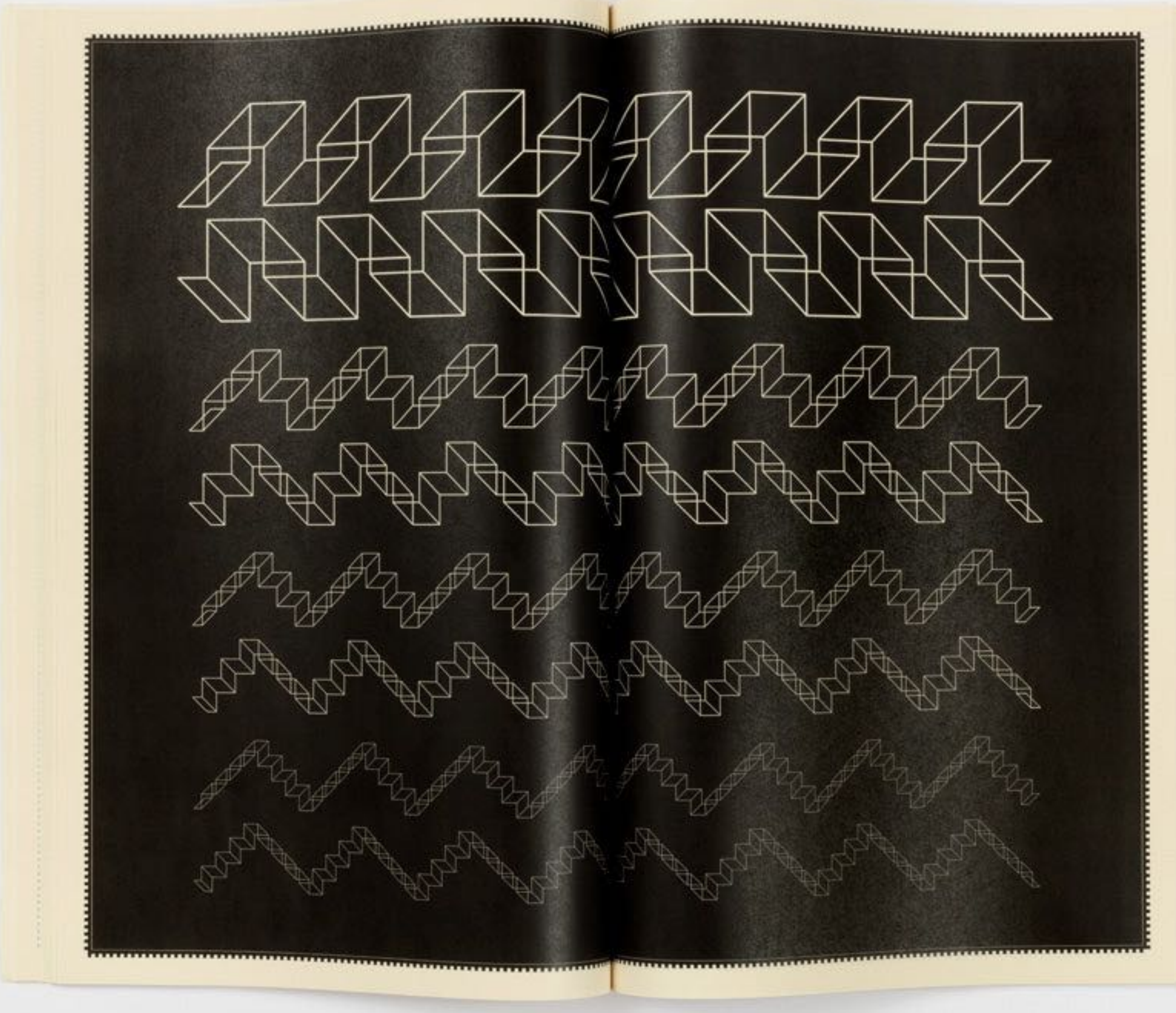
0

1

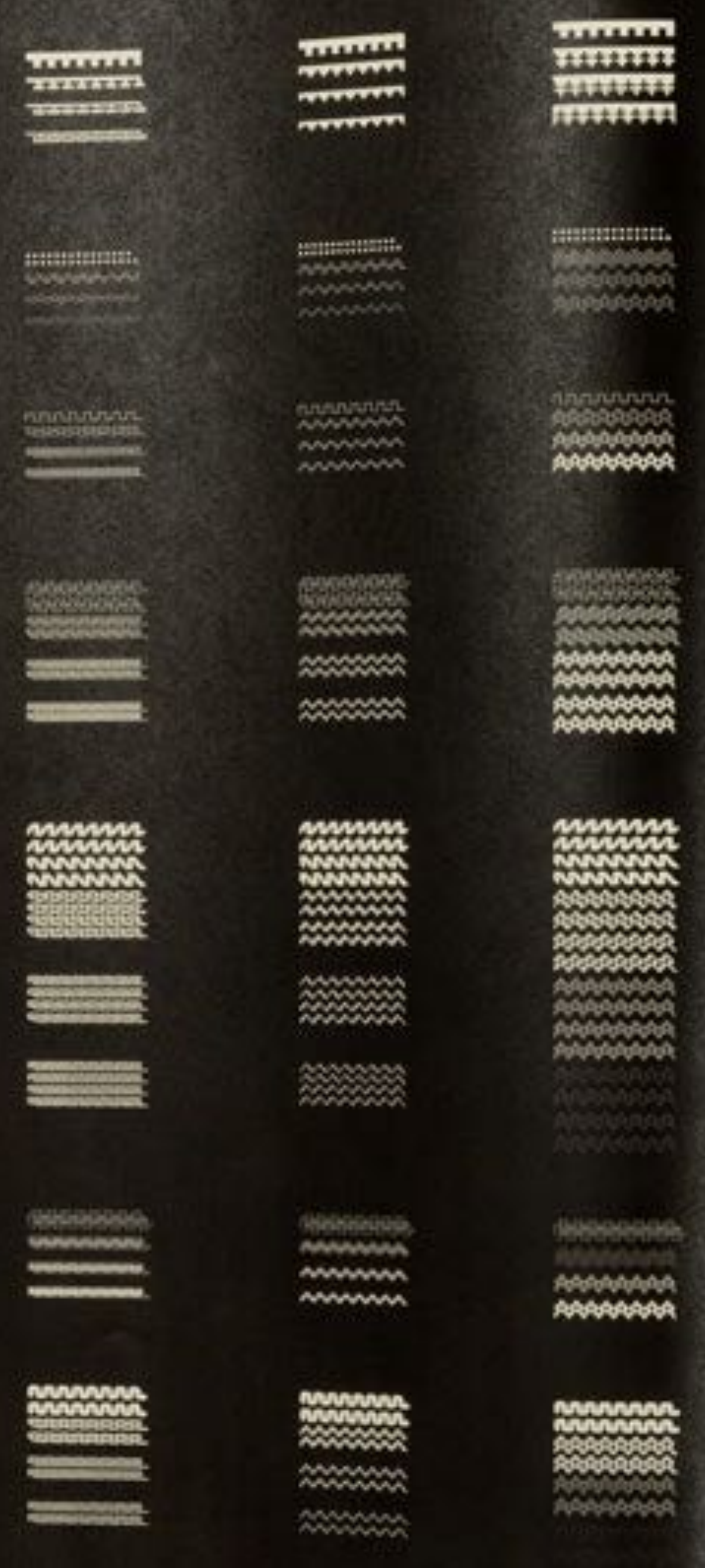
2

3

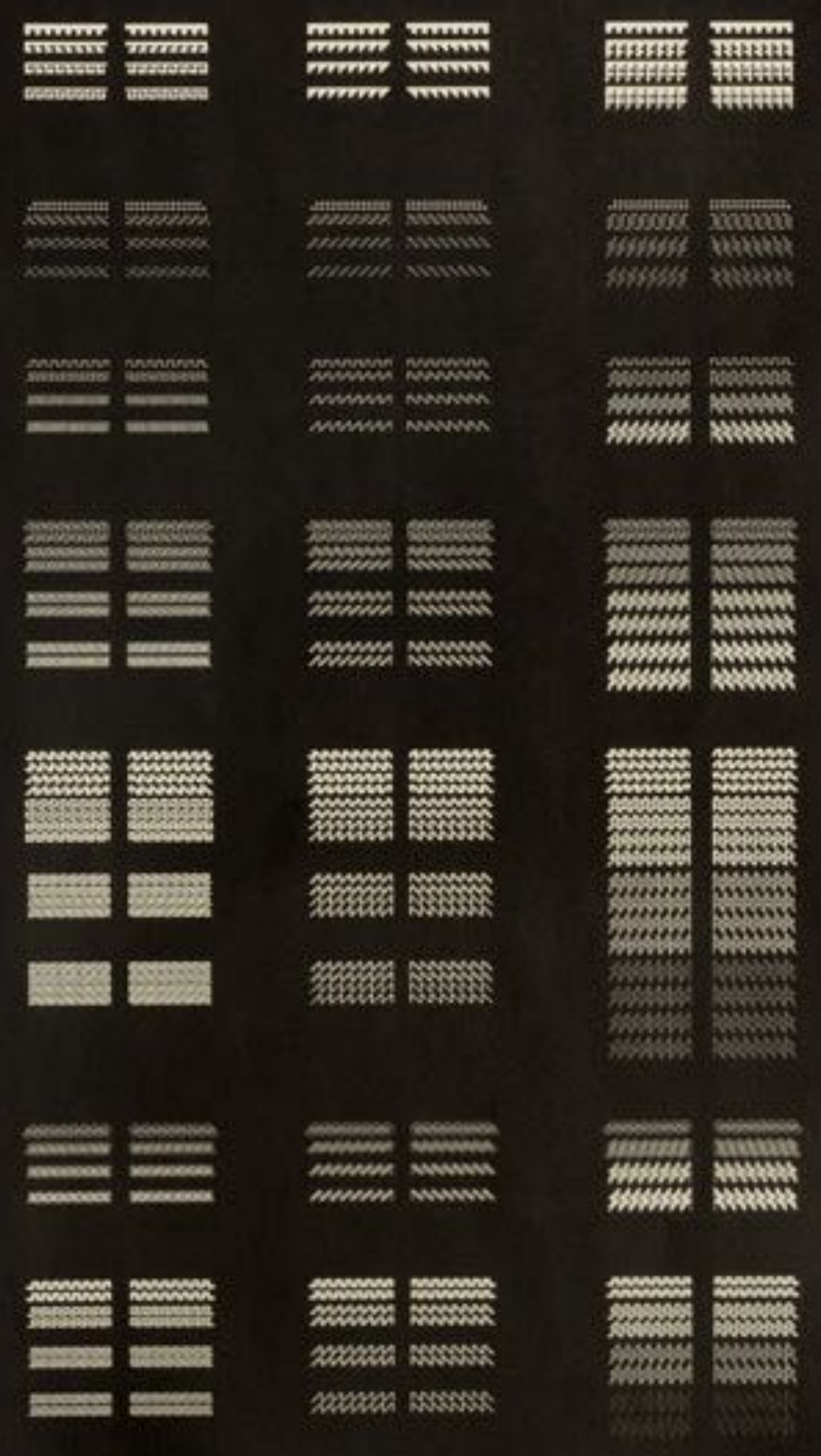




ACHIRAL



CHIRAL



shape symmetry

ROTATIONALLY SYMMETRIC (S)
ROTATIONALLY SYMMETRIC (S)

MIRROR SYMMETRIC

MIRROR SYMMETRIC

ROTATIONALLY SYMMETRIC (Z)
ROTATIONALLY SYMMETRIC (Z)

CHIRAL | ACHIRAL | ACHIRAL | CHIRAL

S and Z designations conform to helix naming conventions (see following pages) and can be "read" in the areas highlighted in grey.

"Square Wave"

Everything is a wave. Depending on where you slice it, it can either have mirror or rotational symmetry.

"Meander"
"Greek Key"
"Chinese Key"
"Running Dog"

This is perhaps the most recognizable of all shapes in the book. I believe it is familiar not only because we've seen it many times,

"T Fret"

but because we recognize it as a diagram of the most fundamental gesture— spin.

This ornament is prehistoric, used by ancient civilizations in many parts of the world.

The form goes by many names. I like to call the rotationally-symmetric variations "Chiral Fret" because they distill the notion of chirality while still being highly symmetric in other ways.

"Stairs"

This simple set of shapes was also used by ancient civilizations all over the world, from the Mayans to the ancient Egyptians.

"Mountain"

Cut the ornament along the black lines and the period is mirror-symmetric, cut along the grey lines and its rotationally-symmetric.

"Cut Cross"

The only seed shape in the book I consider original. I've never seen it, but it is the clear chiral cousin of the mirror-symmetric "Cross" fret. This shape is very dear to me, and now frames my front door.

"Cross"

One of the few places I've seen this mirror-symmetric version is on the end pages of Claude Bragdon's "Frozen Fountain".


MEANDER
Tauba Auerbach (b. 1981, San Francisco, lives and works in New York) studies patterns present at all scales in the universe in a practice that blends mathematics and science with art, design, and craft. In particular, Auerbach has focused on meandering lines and moved fluidly between different media to interrogate their properties. As the artist has noted, these lines wind their way through human history and the natural world: known as meanders, frets, or keys, they appear in diverse ornamental traditions (ancient Mediterranean, Mesoamerican, and East Asian among them), but also as waveforms in physics, space-filling curves in geometry, and the helices of our DNA. Auerbach traces and transforms these lines in multiple dimensions; if they resonate with us, the artist believes, it might be at a fundamental, even cellular level.


IT'S OUR PLEASURE TO SERVE YOU
The Serving Library Annual 2022/23
GALLERY COPY

The year following the publication of "Sur une courbe qui remplit toute une aire plane," Peano purchased a small house in Cavoretto, where he had a copy of Figure 2 made on the balcony with small bits of cement, so that it showed up in black on the white tiles. (He could have had a copy of the *completed* curve by using only one black tile, since the completed curve looks like this: ■)

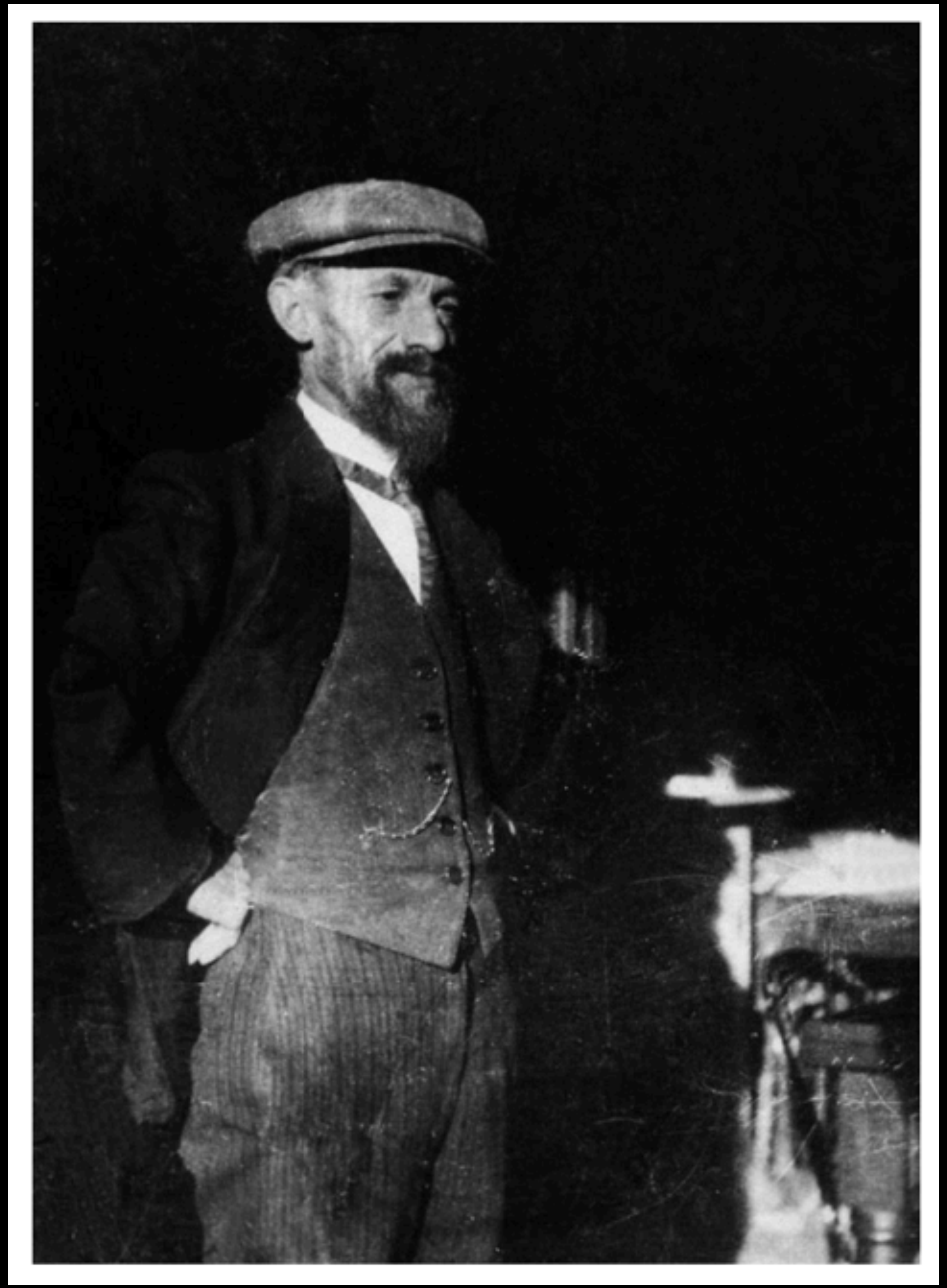
Il 4 Luglio Peano acquista una villa a Cavoretto, in Strada Val Pattonera n. 566, al prezzo di L. 7500, insieme a 8820 mq di terreno circostante. Qui amerà d'ora in avanti trascorrere i mesi estivi. Sulla terrazza fa realizzare, in piastrelle bianche e nere, la riproduzione di un'approssimazione della sua celebre curva che riempie un quadrato, contornata da una greca (v. il grafico sul retro di copertina), di cui oggi resta purtroppo solo una fotografia.

La terrazza con la curva, nella villa a Cavoretto

Strada Comunale Antica di Val Pattonera 253 - San Salvario-Cavoretto-Borgo Po-Nizza Millefonti...


The Clark


The Clark



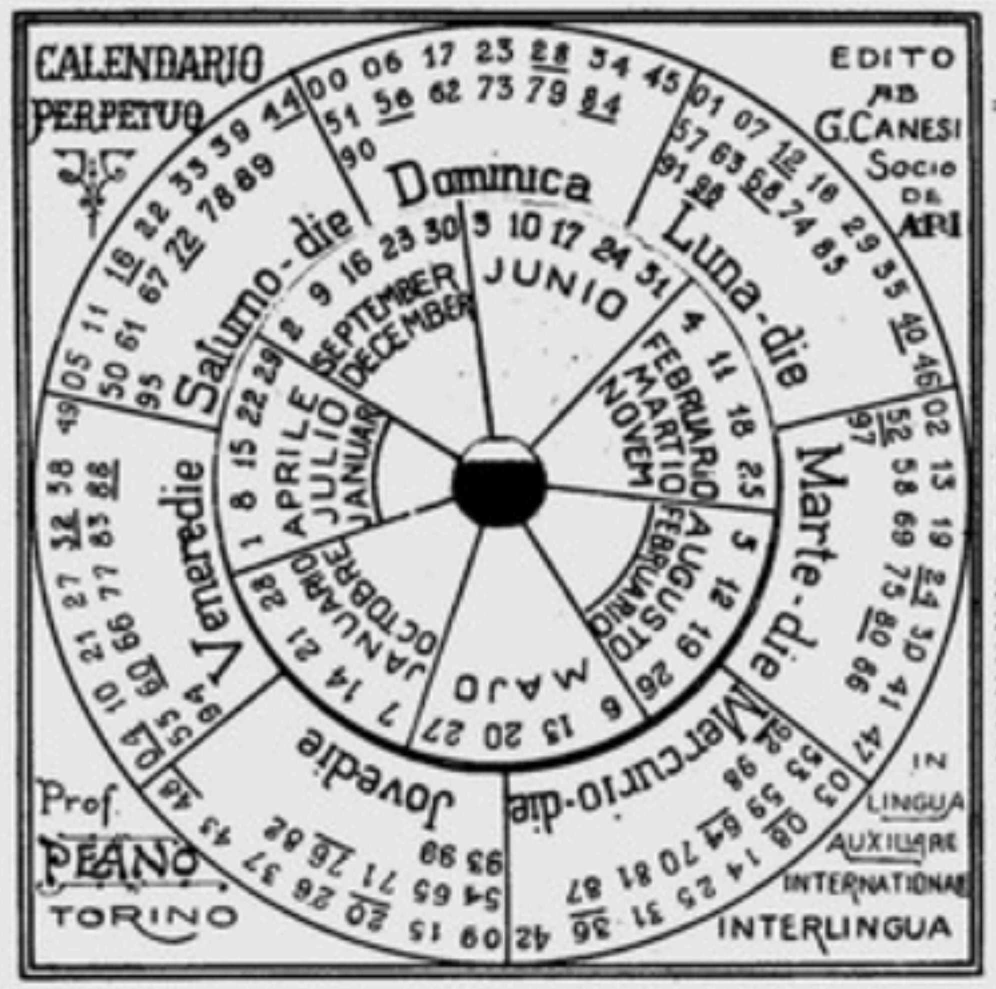
VOS CUM FACILITATE SUMMO
POTE DETERMINA DIE DE SEPTIMANA CORRESPONDENTE AD UNO DATA

Seculo JULIANO

Anno	+
0 99	4
100 199	3
200 299	2
300 399	1
400 499	0
500 599	6
600 699	5
700 799	4
800 899	3
900 999	2
1000-1099	1
1100-1199	0
1200-1299	6
1300-1399	5
1400-1499	4
1500-1582	3

Fine de Calendario Juliano

4 Ottobre 1582



Seculo GREGORIANO

Initio de Calendario Gregoriano

15 Ottobre 1582

1583-1599	0
1600-1699	6
1700-1799	4
1800-1899	2
1900-1999	0
2000-2099	6
2100-2199	4
2200-2299	2
2300-2399	0
2400-2499	6
2500-2599	4

Nos distingue zonas de corona circulare cum 1 et 2, et zonas de disco mobile cum 3 et 4.

Zona 1 contine annos de seculo; zona 2 dies de septimana; zona 3 dies de mense; zona 4 menses.

Anno sublineato in zona 1 responde ad anno bissextile et *januario* et *februario* sublineato in zona 4 es relativo ad anno bissextile.

Omne zona es diviso in 7 parte æquale.

EXEMPLO. - Me vol determina die de septimana correspondente ad *27 augusto 1928.* Me identifica in quale parte de zona 1 es incluso anno 28, et in quale parte de zona 4 es incluso mense augusto; cum motu de disco me fac coincide isto 2 parte; post in zona 3 me quære die 27; tunc in parte correspondente de zona 2 me lege: luna-die.

Figura servi per annos incluso in seculo 20 (1900-1999). Tabula ad latere de figura permitte suo usu pro omne seculo; es sufficiente adjunge ad die determinato cum figura, numero 0, 1, 2, 3... controsignato in tabula ad seculo considerato.

Inde per exemplo pro cognosce in que die de septimana Cristoforo Colombo perveni in America (12 octobre 1492), cum figura nos habe luna-die; nos adiunge 4 (extracto ab tabula) et resulta venere-die.

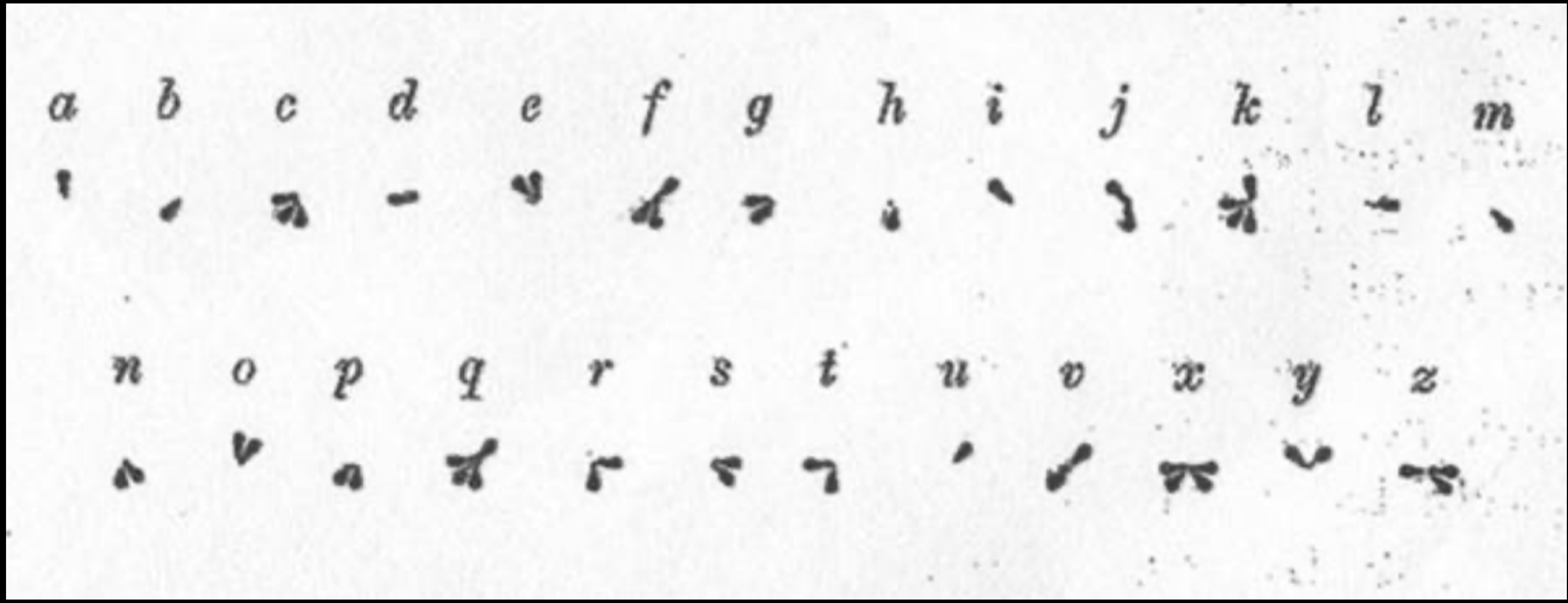
ACADEMIA PRO INTERLINGUA - A.P.I.

Præsidente: G. Peano, prof. Università Cavoretto-Torino

Thesaurario: Ing. G. Canesi Via Costigliole 1, Torino (105)

PRETIO: LIRA ITAL. 3

Gratis ad consocio que inscribe novo socio.


a b c d e f g h i j k l m
n o p q r s t u v x y z

SCHOLA ET VITA - Revista in lingua auxiliare international e
Directore Prof. N. MASTROPAOLO Via E. Pagliano, 46 - MILANO 137

GRAMMATICA COMPLETO

de lingua auxiliare internationale

INTERLINGUA = IL

(ex labores de ACADEMIA PRO INTERLINGUA = API)

Interlingua adopta omne vocabulo latino cum derivatos anglo. Omne vocabulo que existe in latino habe forma de thema (radice) latino. In generale thema de nomine et de adjectivo latino es ablativo. Ex: **rosa, anno, dente, die, novo, me, te...**

Latino habe plure vocabulo sine flexione, invariabile: **ad, in, et, non, semper, heri, quatuor,** etc; illos es interlingua.

Grammatica tormento de pueritia et de... es quasi semper inutile; qui scribe in IL supprime omne elemento de grammatica non necessario.

Alphabeto latino-anglo: a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z.

Orthographia latino: *ae*terno, *h*abile, *ph*ilosop*h*ia, t*h*eatro.. Si vos adopta orthographia non latino, vocabulario latino in plure casu non es apto pro consultatione.

Accentu - Super vocale que praecede ultimo consonante, non finale.

Casu - Non existe in IL: casu latino resulta ex positione, aut per praepositiones **de, ad, ab, in...;** ex: hodie ad me, cras ad te.

Plurale - Es indicato per suffixo - **s**: singulare **monte, flore**; plurale **montes, flores.** Suffixo - s pro plurale es internationale.

Genere - Genere artificiale es complicatione inutile. **Mas** - masculo et **femina** indica genere naturale: cane - mas, cane - femina. Concordantia de adjectivo cum substantivo non existe in anglo et es inutile.

Articulo - Non existe in latino et in russo et es inutile. Ex.: Vos da ad me libro, illo libro, meo libro, illo meo libro, uno meo libro.

Gradu de adjectivo es indicato cum voces disjuncto; Ex.: breve, plus breve, magis breve, multo breve, trans breve, ultra breve, extra breve.

Adverbio ex adjectivo - Plure vice es indicato etiam cum voces disjuncto: Petro stude cum mente diligente, scribe in forma elegante loque in modo claro, canta forte, curre veloce, dormi per longo tempore,

Numeros - uno, duo, tres, quatuor, quinque, sex, septem, octo, novem, decem, centum, mille, milione; **decem et uno = 11**; sex decem et quinque = 65.

Verbo - Non habe suffixo de persona; me habe, te habe, illo habe, nos habe, vos, habe, illos habe.

Tempore de verbo - Si phrasi jam indica tempore, suffixo es inutile; Ex.: heri nos scribe, cras nos lege. Tempores pote es indicato cum: heri, jam, in praeterito, hodie, cras, in proximo tempore, in futuro. Me vol scribe, me debe scribe, me i scribe.

Modo de verbo - Conjunctivo es indicato cum: **si, que, ut...**

Infinitivo - Non habe suffixo; ex.: stude, excelle, adopta...

Flexione *-nte* de participio praesente latino (ama)nte habe internationalitate maximo; pote es eliminato per relatione: studente = que stude, mittente = que mitte.

Participio passivo latino regulare aut irregulare, plure vice vive in derivatos anglo et habe amplo internationalitate. Nos pote elimina illo per inversione: «filio es amato ab matre» = «matre ama filio ».

Publicatione in IL. es multo praeferibile quam publicatione in aliquo lingua nationale; non es necessario versione de tale operes in altero linguas, per causa quod lectore comprehende originale scripto ab auctore.

ACADEMIA PRO INTERLINGUA - API - Praesidente: G. PEANO, Prof. in R. Universitate de Torino - Cavoretto (Torino)
Thesaurario: Ing. G. CANESI - Via Costigliole, 1 bis - Torino 105.

In an 1890 paper, Italian mathematician Giuseppe Peano first described a plane-filling curve:

There exists a continuous curve which goes through every point of a plane ... This result is of interest in the study of the foundations of geometry, for there does not exist a specific character which distinguishes a curve from a surface.

This curve was a radical proposal. Peano asserted that a continuous one-dimensional line could be constructed in a specific way so that it would wind its way through every point of a two-dimensional square. His curve suggested a mapping from the single dimension of points on a line to the paired dimensions of coordinates in a plane. Set theorist Georg Cantor had already devised in 1874 a one-to-one correspondence between the points on a line and the points on a surface, but the correspondence dissolved the continuity of the line. Peano built the argument in his 1890 paper strictly analytically using only the properties of the numbers between zero and one. He included no drawing, diagram, or even a visual description of his curve. One year later, German mathematician David Hilbert offered a drawing based on the curves Peano had conjured. It is a recursive form beginning with a bent line where each segment is replaced by a smaller, self-similar bend, becoming denser and denser, continuing infinitely to completely fill a plane on its theoretical last iteration. The global trajectory of the path (in this case up, to the right, and down) remains unchanged throughout the iterations. Here is Hilbert's drawing of the first three steps in the process:

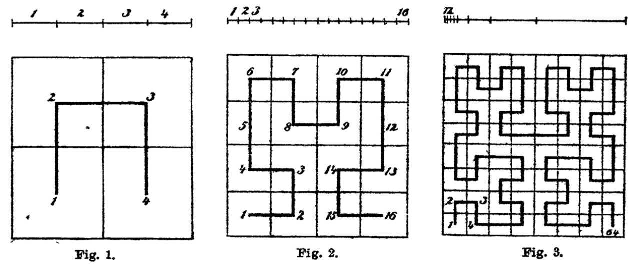
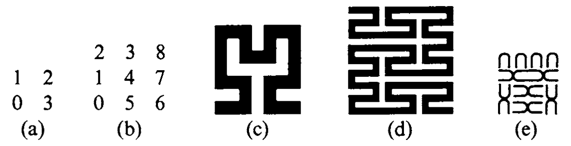


Fig. 1. Fig. 2. Fig. 3.

Peano seems not to have drawn the curve himself, even in his own notes, until the fifth and final edition of his *Formulario Mathematico*, published almost two decades later, in 1908. Here are Peano's drawings from the *Formulario*, which include Hilbert's example using base two (c) and his own example using base three (d):



This drawing is from a family of curves most often referred to as a "Peano curve," with which artist Tauba Auerbach has long been fascinated, incorporating them into their paintings, drawings, sculptures, and the occasional concrete poem. Baffled that Peano wasn't inclined to draw it, the artist went digging. Research on Peano uncovered that he had tiled Hilbert's example into his terrace in Cavoretto, just outside of Turin.

The Serving Library dispatched Tauba Auerbach to Italy to find what remains of Peano's tiled curve:

After a short correspondence, I traveled to meet Professor Clara Silvia Roero, a mathematical historian at the University in Turin, where Peano studied and taught. Roero is an expert on Peano's life, the steward of several of his archives, and the organizer of events and classes on his contributions to mathematics, logic, and international languages. Roero was kind enough to spend a few days showing me around the university, rifling through Peano's archives, and venturing with me to the house on the hill outside of town — to which Peano used to either walk (often picking flowers en route) or ride a horse — and where we were lucky enough to talk our way in to see the terrace where the tiled curve had once been. It was no longer. It was now terra cotta. Roero was already certain this would be the case, because she had spoken to the stone workers tasked with remodeling it years ago, but we both wanted to be in this room for just a moment. I wanted to stand where Peano stood, to look at what he looked at — to get closer to this shape and the person who thought it up so theoretically that it wasn't necessary to draw it, yet he wanted to live with it in visual, concrete form. Since the interior of the room was so transformed, I found myself looking outward, at the landscape in his view and the garden where he once hosted striking sewing factory workers.

I learned many enjoyable facts about Peano from Roero. He was a vegetarian. He was a serious feminist — more than ⅓ of his top protégés were women. He designed a perpetual calendar. He published books and did a lot of his own typesetting. He invented a binary shorthand machine, based on an eight-pointed asterisk, and built it in 1898. He advocated for the adoption of a universal language, "Latino sine flexione," a Latin without inflection, conjugation, or gender, for academic publishing, and headed up an organization devoted to international auxiliary languages. As one of the inventors of mathematical logic, Peano had a big impact on the philosophers of the time, but refused to take a position on philosophical questions because he judged himself not expert enough. He took notes on the cuff of his sleeve. He frequently lost things, so took up a habit of piling his essentials in a cloth napkin and tying the corners together in a bundle. The day he died he taught a class, went to a funny movie (*L'allegro Tenente*), told his wife about it, then had a heart attack in his sleep. He was an anti-fascist from the start, even when most Italians were not. Consequently he asked for a very modest burial, because any puffy ceremony would have unavoidably been fascist at the time (1932). It also just wasn't his style. He had friendships with people of all classes, and when his neighboring farmers attended his funeral, they brought the only flowers.

What follows in this bulletin is a selection from a much larger set of drawings that Auerbach routinely produces. Their *Ligature Drawings* are something close to a writing practice, using markers on oversized graph paper to produce bent lines which develop as they are drawn. The lines often return on themselves, often repeat, and inevitably connect. Most are stamped with the date and they pile up. Their abundance, the programmatic movement of the pen, and the meandering compactness of the lines recalls something from Peano's own description of his curve as

a point in n-dimensional space, which is a continuous function of a real variable, or of time, such that the trajectory of the moving point fills the whole space.

Cover: Marco Bernardi, nephew of Giuseppe Peano, standing on his uncle's tiled terrace in Cavoretto. Photo courtesy Vittorio Massimo Bernardi and Marco Bernardi via Clara Silvia Roero, University of Turin. All following pages: Tauba Auerbach, *Ligature Drawings* (various dates), ink on paper, approx. 32 x 27 in.

BoTSL#17 2022 AUG 26 1:45 PM

MAY

X

A space-filling curve (1890, 1908)*

The following selection contains what is probably, after the postulates for the natural numbers, Peano's best-known discovery. The 'curve which completely fills a planar region' was a spectacular counterexample to the commonly accepted notion that an arc of a curve given by continuous parametric functions could be enclosed in an arbitrarily small region. Indeed, here is a curve given by continuous parametric functions, $x = x(t)$ and $y = y(t)$, such that as t varies throughout the unit interval, the graph of the curve includes every point in the unit square.

We couple this selection with an excerpt from the Formulario *of 1908 in which Peano gives a graphical representation of one 'approach' to such a curve. Hilbert had, after the publication of Peano's original paper, given the first such graphical representation, but Peano was probably led to his discovery by just such a representation. He published his result without diagrams because, it would seem, he wanted no one to think that a false proof lurked in a forced interpretation of a diagram. His proof is purely analytic.*

Peano's curve is a mapping of the unit interval onto its Cartesian product. In the development of topology, this gave rise to the study of Peano spaces. *(A Peano space is a Hausdorff space which is an image of the unit interval under a continuous mapping.) It also raised the question: Which spaces can be mapped continuously onto their Cartesian product? It is of interest that Peano's example is unique, in the sense that: 'The only non-degenerate ordered continuum C which admits a mapping $f : C \to C \times C$ onto its square $C^2 = C \times C$ is the real line segment I.'* (S. Mardešić, 'Mapping Ordered Continua onto Product Spaces,' *Glasnik mat.-fiz. astr. drustvo mat. fiz. hrvatske*, (2) 15 (1960), 85–9; p. 88, Theorem 4.)

1 ON A CURVE WHICH COMPLETELY FILLS A PLANAR REGION

In this note we determine two single-valued and continuous functions

PROVEN AS AFTERTHOUGHT IN PAPER

* 'Sur une courbe, qui remplit toute une aire plane,' *Mathematische Annalen*, 36 (1890), 157–60 [24], and excerpt from *Formulario mathematico*, vol. 5 (Turin: Bocca, 1905–8), pp. 239, 240 [138].

WELL-DEFINED FUNCTION CAN ALSO BE MULTI-VALUED

x and y of a (real) variable t which, as t varies throughout the interval (0, 1), take on all pairs of values such that $0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1$. If, according to common usage, we consider the locus of points whose coordinates are continuous functions of a variable to be a continuous curve, then we have an arc of a curve which goes through every point of a square. Thus, being given an arc of a continuous curve, with no other hypothesis, it is not always possible to enclose it in an arbitrarily small region.

We shall use the number 3 as a base of numeration and refer to each of the numerals 0, 1, 2 as a digit. We now consider the infinite sequence of digits $a_1, a_2, a_3, \ldots$, which we write

SEQUENCE OF DIGITS VS. VALUE

$$T = 0 \,.\, a_1 a_2 a_3 \ldots$$

(for the moment, T is merely a sequence of digits).

If a is a digit, we designate by $\mathbf{k}a$ the digit $2 - a$, the *complement* of a; i.e., we let

$$\mathbf{k}0 = 2, \qquad \mathbf{k}1 = 1, \qquad \mathbf{k}2 = 0.$$

If $b = \mathbf{k}a$, we deduce that $a = \mathbf{k}b$. We also have $\mathbf{k}a \equiv a \pmod 2$. We designate by $\mathbf{k}^n a$ the result of the operation $\mathbf{k}$ repeated n times on a. If n is even, we have $\mathbf{k}^n a = a$; if n is odd, $\mathbf{k}^n a = \mathbf{k}a$. If $m \equiv n \pmod 2$, we have $\mathbf{k}^m a = \mathbf{k}^n a$.

CONGRUENCE

EVEN | ODD

We let correspond to the sequence T the two sequences

$$X = 0 \,.\, b_1 b_2 b_3 \ldots, \qquad Y = 0 \,.\, c_1 c_2 c_3 \ldots,$$

P(T) = (X, Y)

where the digits b and c are given by the relations

$$b_1 = a_1, \quad c_1 = \mathbf{k}^{a_1} a_2, \quad b_2 = \mathbf{k}^{a_2} a_3, \quad c_2 = \mathbf{k}^{a_1 + a_3} a_4,$$
$$b_3 = \mathbf{k}^{a_2 + a_4} a_5, \ldots,$$
$$b_n = \mathbf{k}^{a_2 + a_4 + \ldots + a_{2n-2}} a_{2n-1}, \quad c_n = \mathbf{k}^{a_1 + a_3 + \ldots + a_{2n-1}} a_{2n}.$$

Thus, b_n, the nth digit of X, is equal to a_{2n-1}, the nth digit of uneven rank in T, or to its complement, according as the sum $a_2 + \ldots + a_{2n-2}$ of digits of even rank, which precede it, is even or odd, and analogously for Y. We may thus write these relations in the form:

$$a_1 = b_1, \quad a_2 = \mathbf{k}^{b_1} c_1, \quad a_3 = \mathbf{k}^{c_1} b_2, \quad a_4 = \mathbf{k}^{b_1 + b_2} c_2, \ldots,$$
$$a_{2n-1} = \mathbf{k}^{c_1 + c_2 + \ldots + c_{n-1}} b_n, \quad a_{2n} = \mathbf{k}^{b_1 + b_2 + \ldots + b_n} c_n.$$

.0110 → (X, Y)

If the sequence T is given, then X and Y are determined, and if X and Y are given, then T is determined.

RECURRENCE FUNCTION

$.\bar{9} = 1$ $\quad$ $.\bar{2} = 10$

SEQUENCE vs. VALUE

"VALUE" "REPRESENTATION"?

We give the name *value* of the sequence T to the quantity (analogous to a decimal number having the same notation)

P(t) = (x, y)

$$t = \text{val } T = \frac{a_1}{3} + \frac{a_2}{3^2} + \ldots + \frac{a_n}{3^n} + \ldots$$

$.22\bar{2} = 1$

To each sequence T corresponds a number t such that $0 \leqslant t \leqslant 1$. For the converse, the numbers t in the interval (0, 1) divide into two classes:

TO DO WITH WELL-DEFINEDNESS OF FUNCTION

(α) The numbers, different from 0 and 1, which give an integer when multiplied by a power of 3. They are represented by two sequences, the one

$$T = 0 \,.\, a_1 a_2 \ldots a_{n-1} a_n 222 \ldots,$$

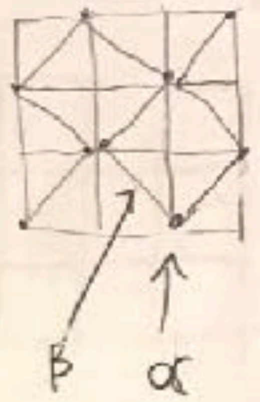
where a_n is equal to 0 or 1, and the other

$$T' = 0 \,.\, a_1 a_2 \ldots a_{n-1} a_n' 000 \ldots,$$

where $a_n' = a_n + 1$. $\quad$ $.1\bar{1}11$ OR $.1021\bar{1}$ OR …

α → 2
β → 1

(β) The other numbers. These are represented by only one sequence T.



Now, the correspondence established between T and (X, Y) is such that if T and T' are two sequences of different form, but $\text{val } T = \text{val } T'$, and if X, Y are the sequences corresponding to T, and X', Y' those corresponding to T', we have

P(t)

$$\text{val } X = \text{val } X', \qquad \text{val } Y = \text{val } Y'.$$

WELL-DEFINED FUNCTION?

Indeed, consider the sequence

$$T = 0 \,.\, a_1 a_2 \ldots a_{2n-3} a_{2n-2} a_{2n-1} a_{2n} 2222 \ldots,$$

PAIRS 2N

where a_{2n-1} and a_{2n} are not both equal to 2. This sequence can represent every number of class α. Letting

$$X = 0 \,.\, b_1 b_2 \ldots b_{n-1} b_n b_{n+1} \ldots$$

we have

$$b_n = \mathbf{k}^{a_2 + \ldots + a_{2n-2}} a_{2n-1}, \qquad b_{n+1} = b_{n+2} = \ldots = \mathbf{k}^{a_2 + \ldots + a_{2n-2} + a_{2n}} 2.$$

MAP vs. CONTINUOUS PATH

Letting T' be the other sequence whose value coincides with val T, we have

$$T' = 0 \,.\, a_1 a_2 \ldots a_{2n-3} a_{2n-2} a'_{2n-1} a'_{2n} 0000 \ldots$$

a' MEANS a IS AUGMENTED AS A RESULT OF $.\bar{2}$ (α)

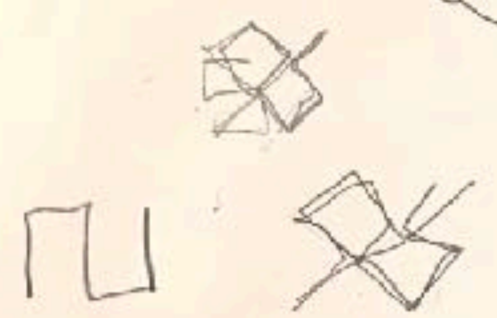
FUNCTION = ONE-TO-ONE CORRESPONDENCE

and

$$X' = 0 \,.\, b_1 \ldots b_{n-1} b_n' b_{n+1}' \ldots$$

CONTINUING 1/2/22

$.\bar{9} = 1$ $\qquad$ $.\bar{2} = 10$



SEQUENCE vs. VALUE

"VALUE" "REPRESENTATION"?

We give the name *value* of the sequence T to the quantity (analogous to a decimal number having the same notation)

$P(t) = (x, y)$
$$t = \text{val } T = \frac{a_1}{3} + \frac{a_2}{3^2} + \dots + \frac{a_n}{3^n} + \dots$$

To each sequence T corresponds a number t such that $0 \leqslant t \leqslant 1$. For the converse, the numbers t in the interval $(0, 1)$ divide into two classes:

$.22\bar{2} = 1$

$\frac{1}{3}\ \frac{2}{9}\ \frac{5}{27}$

$\frac{1}{3} \times 3 = 1$

$\frac{2}{9} \times 9 = \frac{18}{9} = 2$

(α) The numbers, different from 0 and 1, which give an integer when multiplied by a power of 3. They are represented by two sequences, the one

$$T = 0\,.\,a_1 a_2 \dots a_{n-1} a_n 222 \dots,$$

where a_n is equal to 0 or 1, and the other

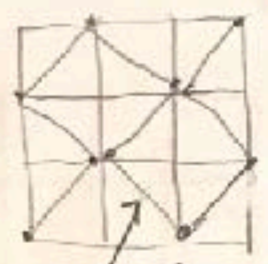
$$T' = 0\,.\,a_1 a_2 \dots a_{n-1} a_n' 000 \dots,$$

where $a_n' = a_n + 1$. $\quad .111\bar{1}$ OR $.10211$ OR ...

(β) The other numbers. These are represented by only one sequence T.

TO DO WITH WELL-DEFINEDNESS OF FUNCTION

$\alpha \to 2$
$\beta \to 1$



$P(t)$

$\beta \quad \alpha$

Now, the correspondence established between T and (X, Y) is such that if T and T' are two sequences of different form, but val T = val T', and if X, Y are the sequences corresponding to T, and X', Y' those corresponding to T', we have

$$\text{val } X = \text{val } X', \qquad \text{val } Y = \text{val } Y'.$$

Indeed, consider the sequence

$$T = 0\,.\,\widehat{a_1 a_2} \dots \widehat{a_{2n-3} a_{2n-2}} \widehat{a_{2n-1} a_{2n}} 2222 \dots,$$

PAIRS 2N

where a_{2n-1} and a_{2n} are not both equal to 2. This sequence can represent every number of class α. Letting

$$X = 0\,.\,b_1 b_2 \dots b_{n-1} b_n b_{n+1} \dots$$

we have

$$b_n = \mathbf{k}^{a_2 + \dots + a_{2n-2}} a_{2n-1}, \qquad b_{n+1} = b_{n+2} = \dots = \mathbf{k}^{a_2 + \dots + a_{2n-2} + a_{2n}} 2.$$

Letting T' be the other sequence whose value coincides with val T, we have

$$T' = 0\,.\,a_1 a_2 \dots a_{2n-3} a_{2n-2} a'_{2n-1} a'_{2n} 0000 \dots$$

a' MEANS A IS AUGMENTED AS A RESULT OF $.\bar{2}$ (α)

and

$$X' = 0\,.\,b_1 \dots b_{n-1} b_n' b_{n+1}' \dots$$

WELL-DEFINED FUNCTION?

MAP vs. CONTINUOUS PATH

FUNCTION = ONE-TO-ONE CORRESPONDENCE

~~STOPPED FOR NOW~~

CONTINUE 1/21/22

$.22\bar{2} = 1$

To each se... the converse, ... (α) The nu... multiplied by ...

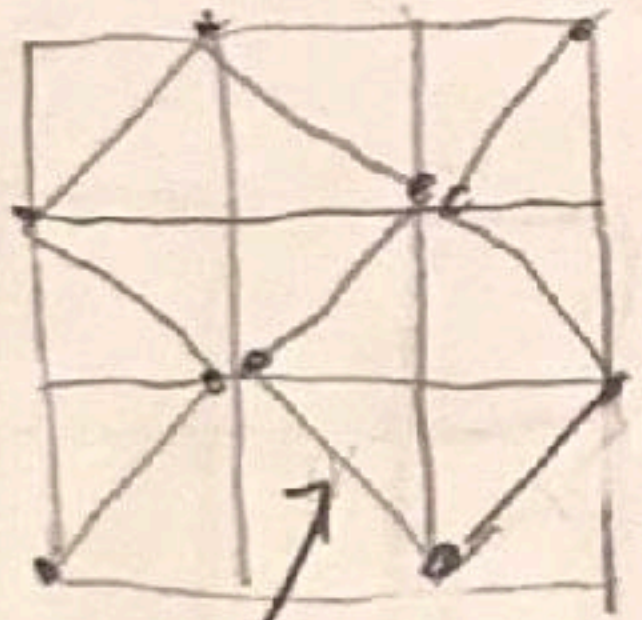
$\frac{1}{3}\ \frac{2}{9}\ \frac{5}{27}$

$\frac{1}{3} \times 3 = 1$

$T = 0\,.\,a$ ... where a_n is eq... $T' = 0\,.$ ...

$\frac{2}{9} \times 9 = \frac{18}{9} = 2$

where $a_n' = a_n$ ... (β) The oth... Now, the co... if T and T' are ... X, Y are the se... to T', we have



$P(t)$

$\beta \quad \alpha$

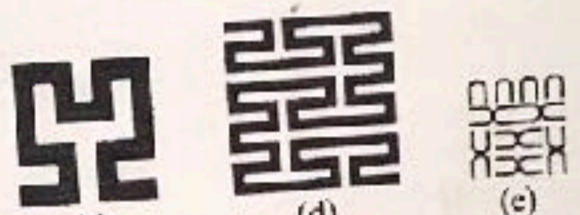
val X = v... Indeed, cons... $T = 0\,.\,\widehat{a_1}$... where a_{2n-1} an... every number o... $X = 0\,.\,b_1$... we have $b_n = \mathbf{k}^{a_2+\dots}$

infinitum. Figure (c) represents the succession of 16 squares in base 2; figure (d) the succession of 81 squares in base 3.

If we represent by the sign ∩ the succession $\begin{matrix}1 & 2\\ 0 & 3\end{matrix}$, or figure (a), then figure (e) represents the succession of 64 squares in base 2.

$$\begin{matrix}1 & 2\\ 0 & 3\end{matrix} \qquad \begin{matrix}2 & 3 & 8\\ 1 & 4 & 7\\ 0 & 5 & 6\end{matrix}$$

(a) (b)



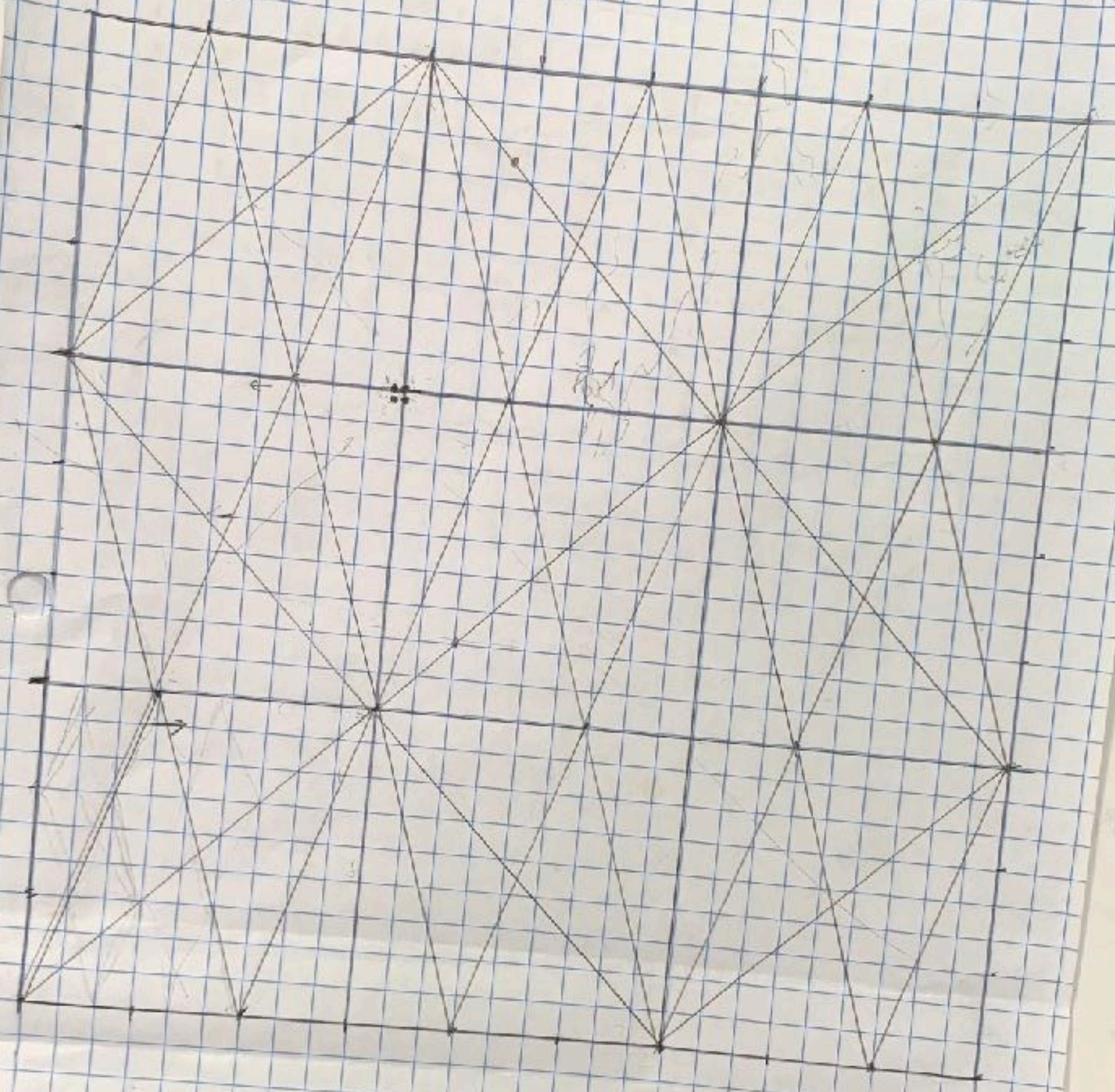
(c) (d) (e)

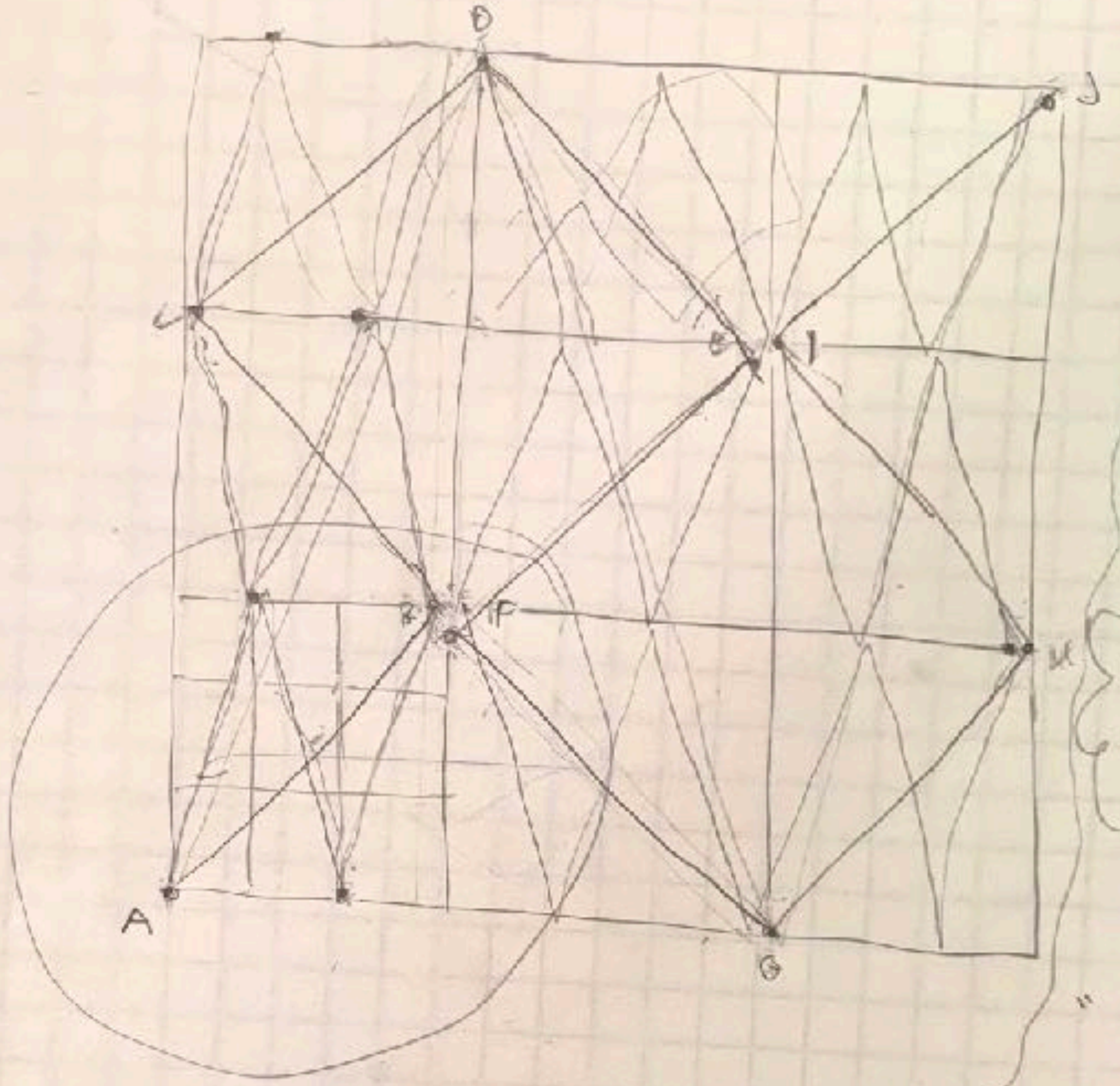
In my article 'Sur une courbe qui remplit toute une aire plane,' I gave the analytic expression for the continuous correspondence between the real number t and the complex number $(x; y)$.

See also Hilbert, *Math. Ann.*, 38 (1891), 459; Cesaro, *Bull. sci. math.*, 21 (1897), 257; Moore, *Trans. Amer. Math. Soc.* (1900), 72; Lebesgue, *Leçons sur l'intégration* (Paris, 1904), p. 45.

MAY

3^0 3^1 3^2 3^3





UPDATE DYNAMICALLY SO ERASE PREVIOUS DRAW AS ITERATES

INCLUDE HELP

Points (3^n):
Order (length n)
Display: (px)

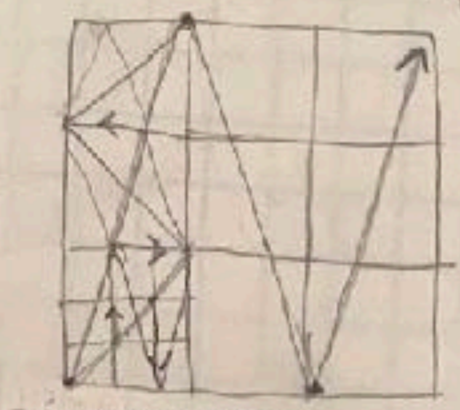
"ORDER"
NUMBER OF DIGITS READ IN T

① HOMEWORK

A. CALCULATE (x, y) FOR THESE

0.001 →	.01, .02
0.002	.02, .00
0.011	$.01\bar{2}$, $.1\bar{2}$
0.012	$.00\bar{2}$, 0.1
0.021	.01, $.\bar{2}$
0.022	.02, $.2\bar{0}$

② READ CANTOR'S DIAGONAL ARGUMENT



$3 \to 9 \to 27 \to 81 \to 243$ C.

ORDER PADDING

$T = .\underline{001}000$

$3^3 = 27$

ORDER OF 3 MAKES 27 POINTS

ORDER OF 4 MAKES 81 POINTS

$3^4 = 81$

$(3^{2^2} = 81)$

$(9^2 = 81)$

B. WHAT HAPPENS WHEN FIRST DIGIT IS "1"; WHAT ABOUT "2"

0.101	0.201
0.102	0.202
0.111	0.211
0.112	0.212
0.121	0.221
0.122	0.222

MAY

n	=	1	2	3	4	5	6		
n	=	0	1	2	3	4	5		
b	=	0	.	1	0	0	0	0	0
b'	=	0	.	0	2	2	2	2	2
c	=	0	.	2	0	0	0	0	0
c'	=	0	.	1	2	2	2	2	2

T = 0 .

1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9

$\boxed{n = 5}$

ODD

$a_{2n-1} = k^{c_1 + c_2 \dots c_{n-1}} b_n$

$a_9 = k^{2+0+0} 0$

$a_9 = 0$

EVEN

$a_{2n} = k^{b_1 + b_2 + b_{n-1}} c_n$

$a_{10} = k^{1+0+0} 0$

$a_{10} = 2$

$\boxed{n = 4}$

EVEN

$a_{2n} = k^{c_0 + c_1 + c_{n-1}} b_n$ ✓

$a_8 = k^{2+0+0} 0$

$a_8 = 0$

ODD

$a_{2n+1} = k^{b_0 + b_1 + b_{n-1}} c_n$ ✓

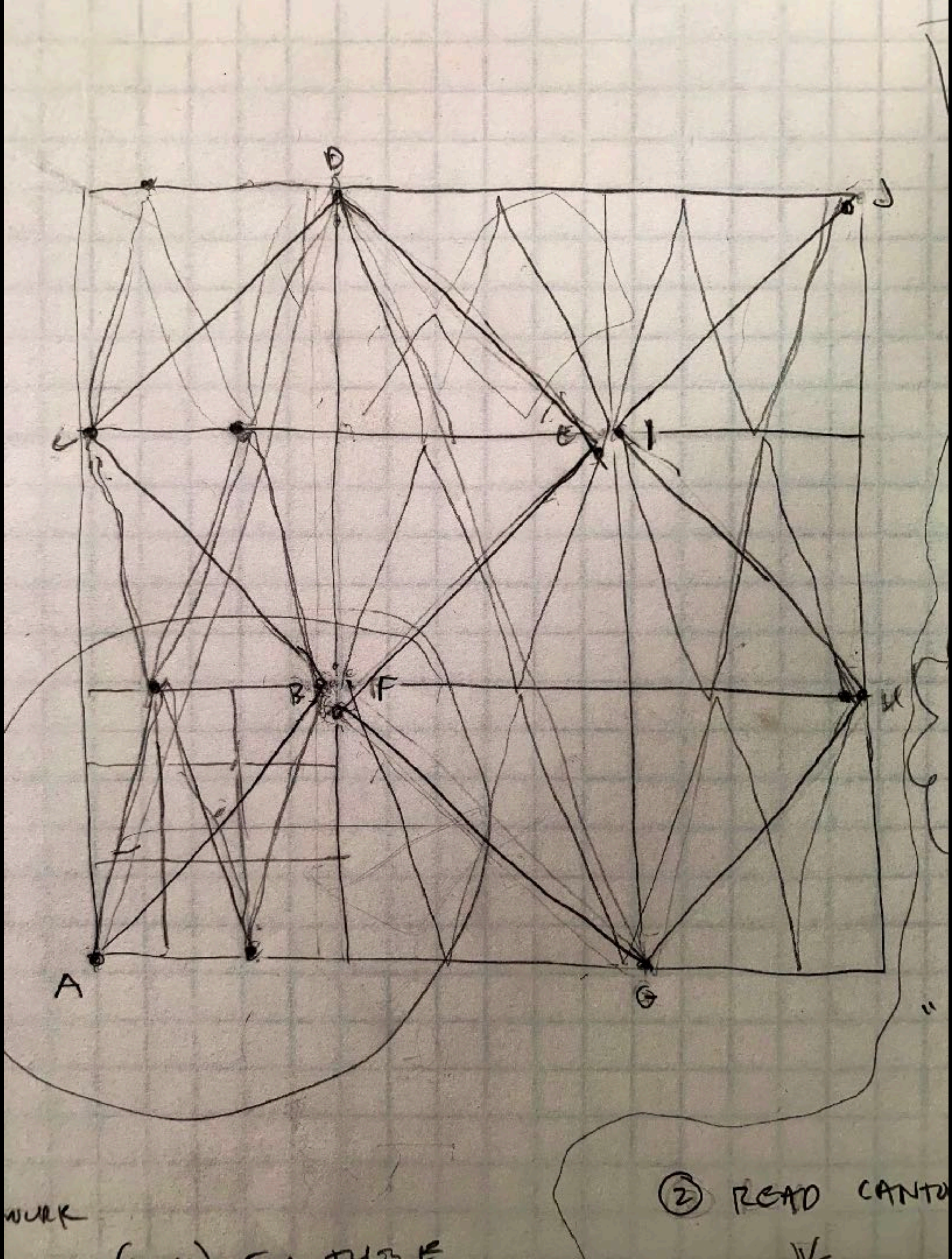
$a_9 = k^1 0$

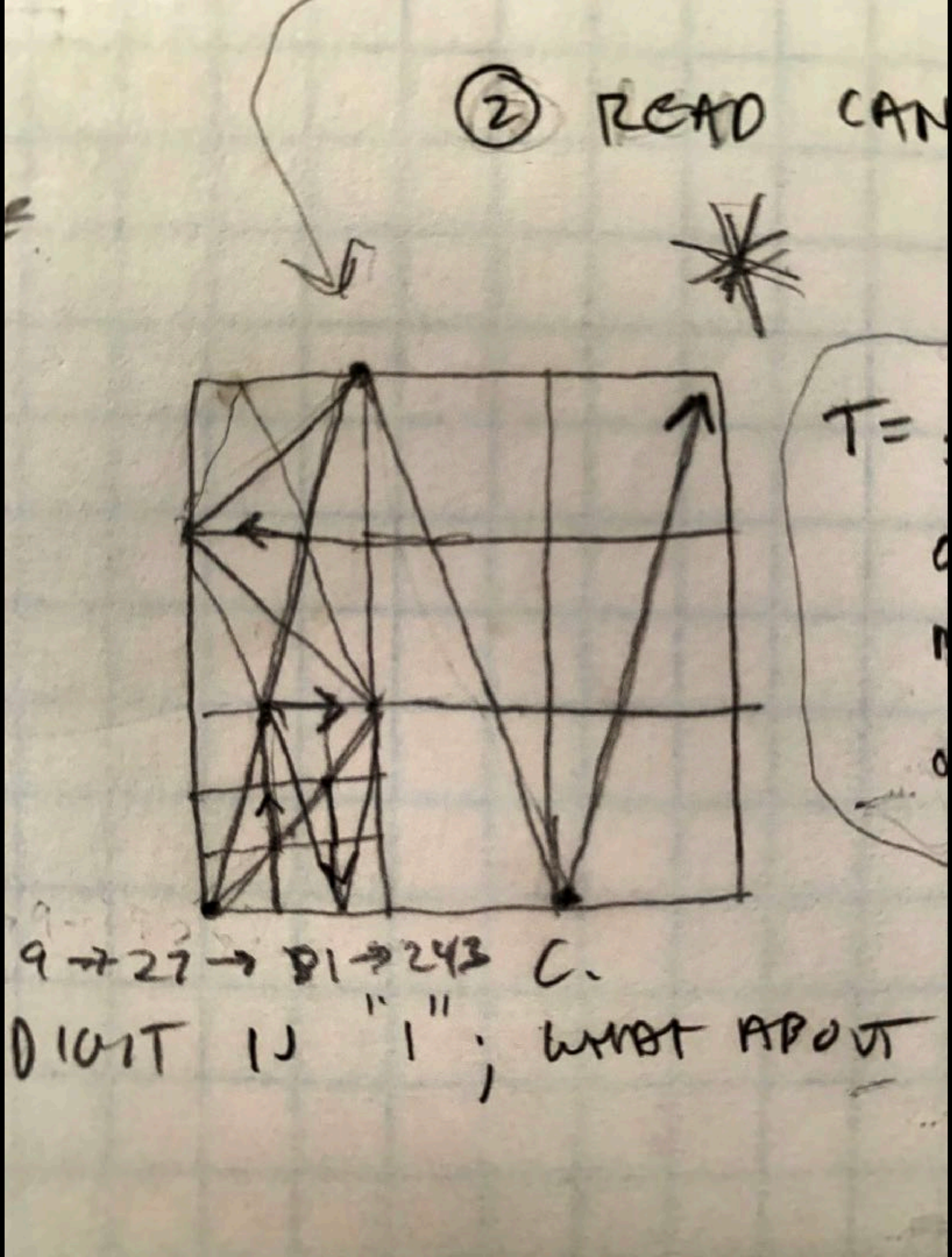
$a_9 = 2$

X = 0 . 1 0
X'' = 0 . 0 2
Y = 0 . 2 0
Y' = 0 . 1 2

T(X, Y) = 0 . 1 0 0 2 ✓
T(X'', Y) = 0 . 0 2 2 0 ✓
T(X, Y') = 0 . 1 1 2 0 ✓
T(X'', Y') = 0 . 0 1 0 2 ✓

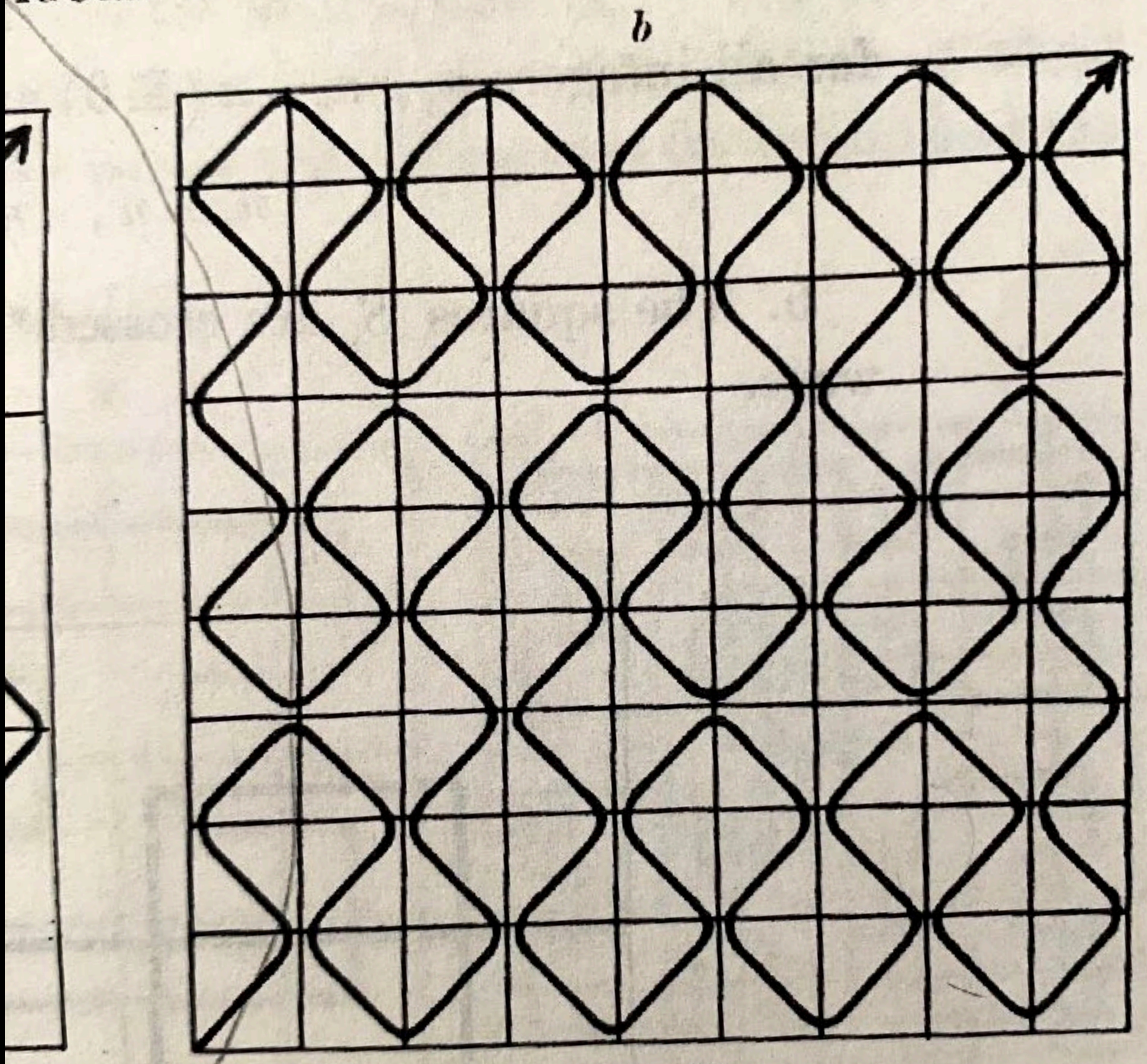
T(X, Y) = 0 . 1 0 0 2 → X = 0 . 1 0 ✓ Y = 0 . 2 0 ✓
T(X', Y) = 0 . 0 2 2 0 → X = 0 . 0 2 ✓ Y = 0 . 2 0 ✓
T(X, Y') = 0 . 1 1 2 0 → X = 0 . 1 0 ✓ Y = 0 . 1 2 ✓
T(X', Y') = 0 . 0 1 0 2 → X = 0 . 0 2 ✓ Y = 0 . 1 2 ✓


D
J
E
I
B
F
A
G
② READ CANTO
WORK


② READ CAN
T=
9 → 27 → 81 → 243 C.
DIGIT IS "1"; WHAT ABOUT

ch it in the ratio $\omega:1$ and break and locate
r diagonals the ω^2 squares S_1 of S_0 in such

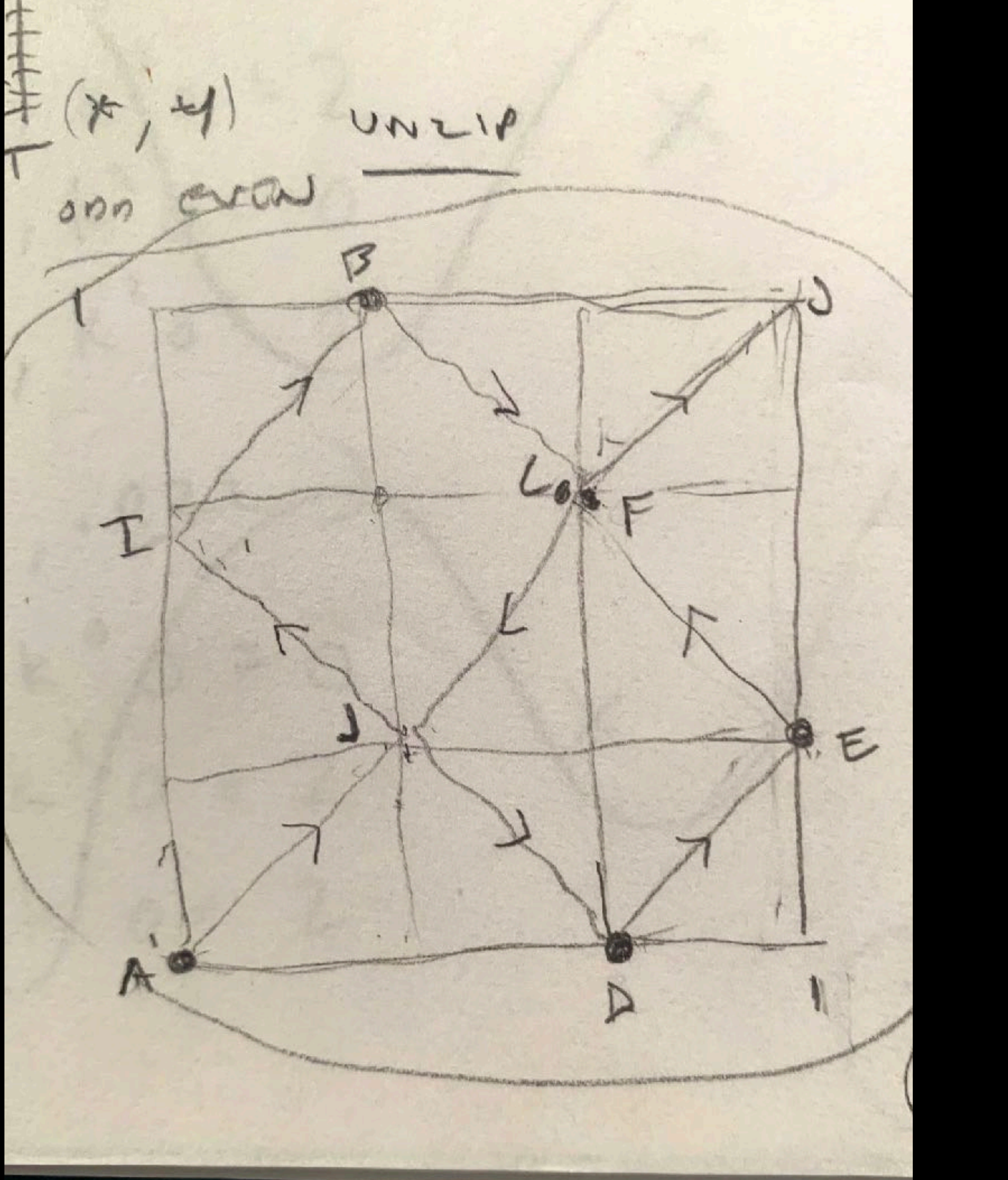
IGURE 3



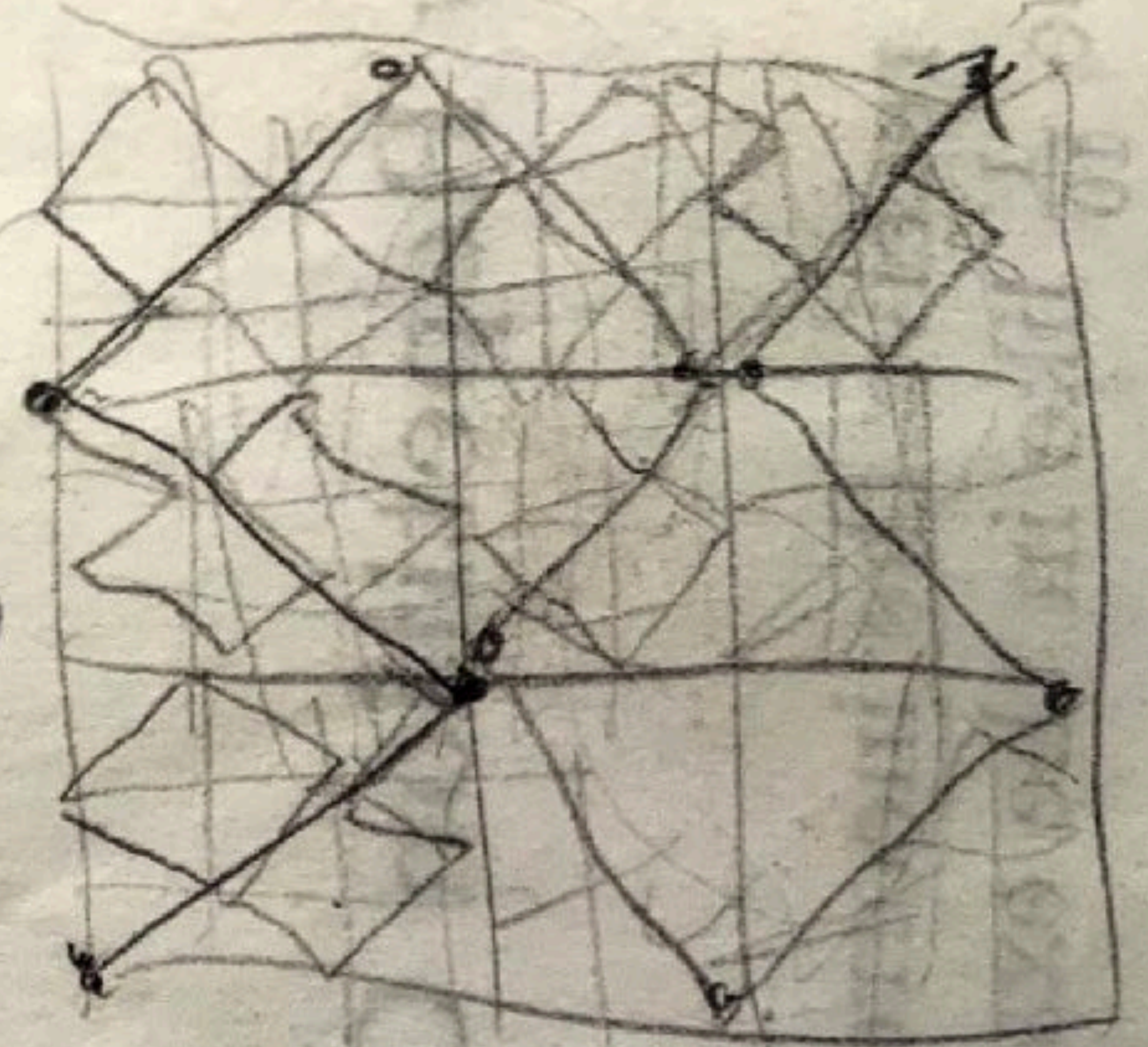
Line $C_2^{(3)}$

l column are traversed sequentially. Then
nals of the ω^2 S_1 changes C_1 into C_2. And

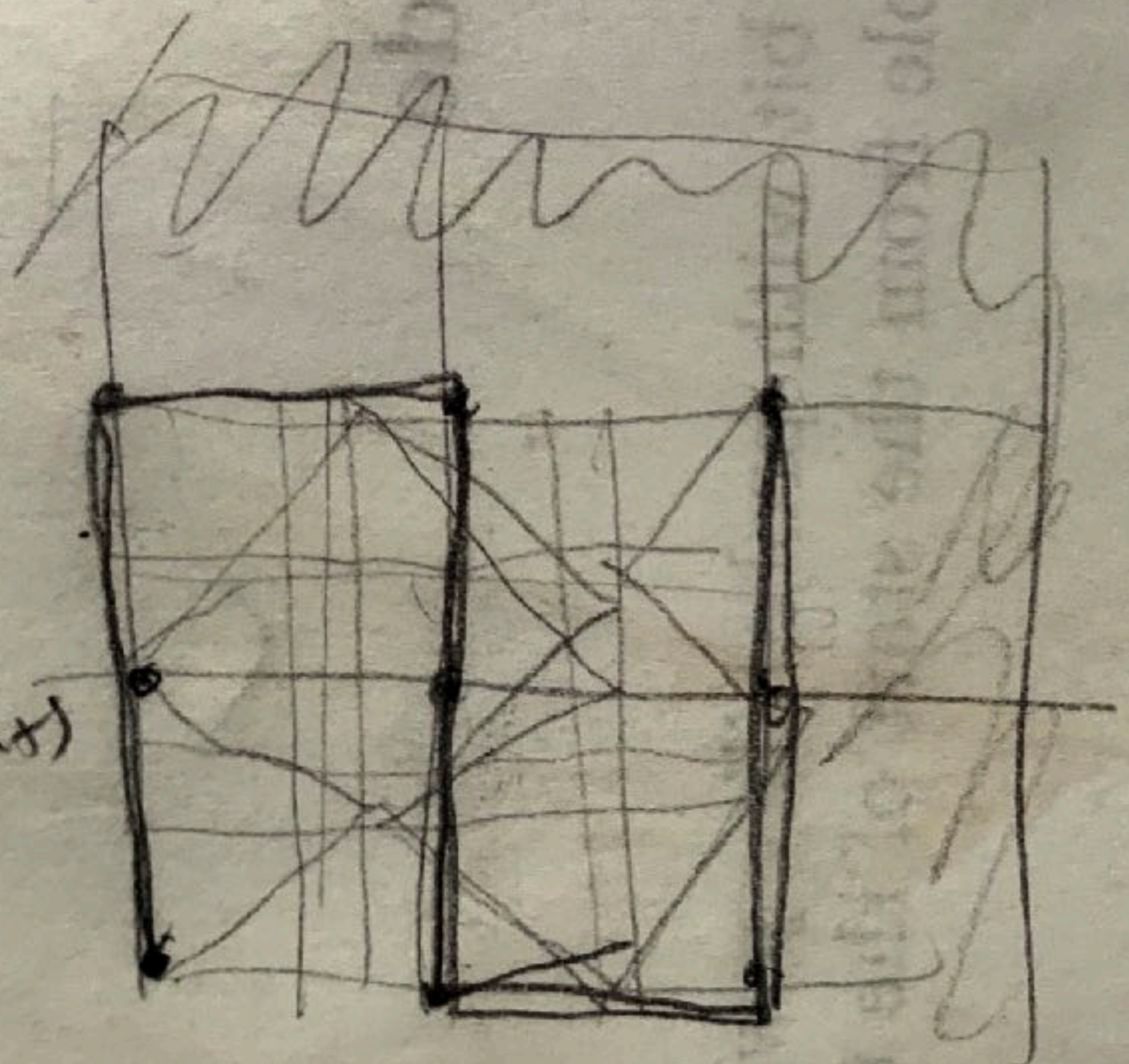
onals of the C_n links (or n-links) and their



3
O(n+1)



2
O(n+1)



.000
.0001
.0002
.0010
.0011
.0012
.0020

✓ A .000
.001
.002
B .010
.011
.012
C .020
.021
.022
✓ D .100
.101
E .102
.110
.111
F .112
.120
.121
.122
✓ G .200
.201
.202
H .210
.211
.212
I .220
.221
✓ J .222

* T

~~READ~~ ~~HY~~

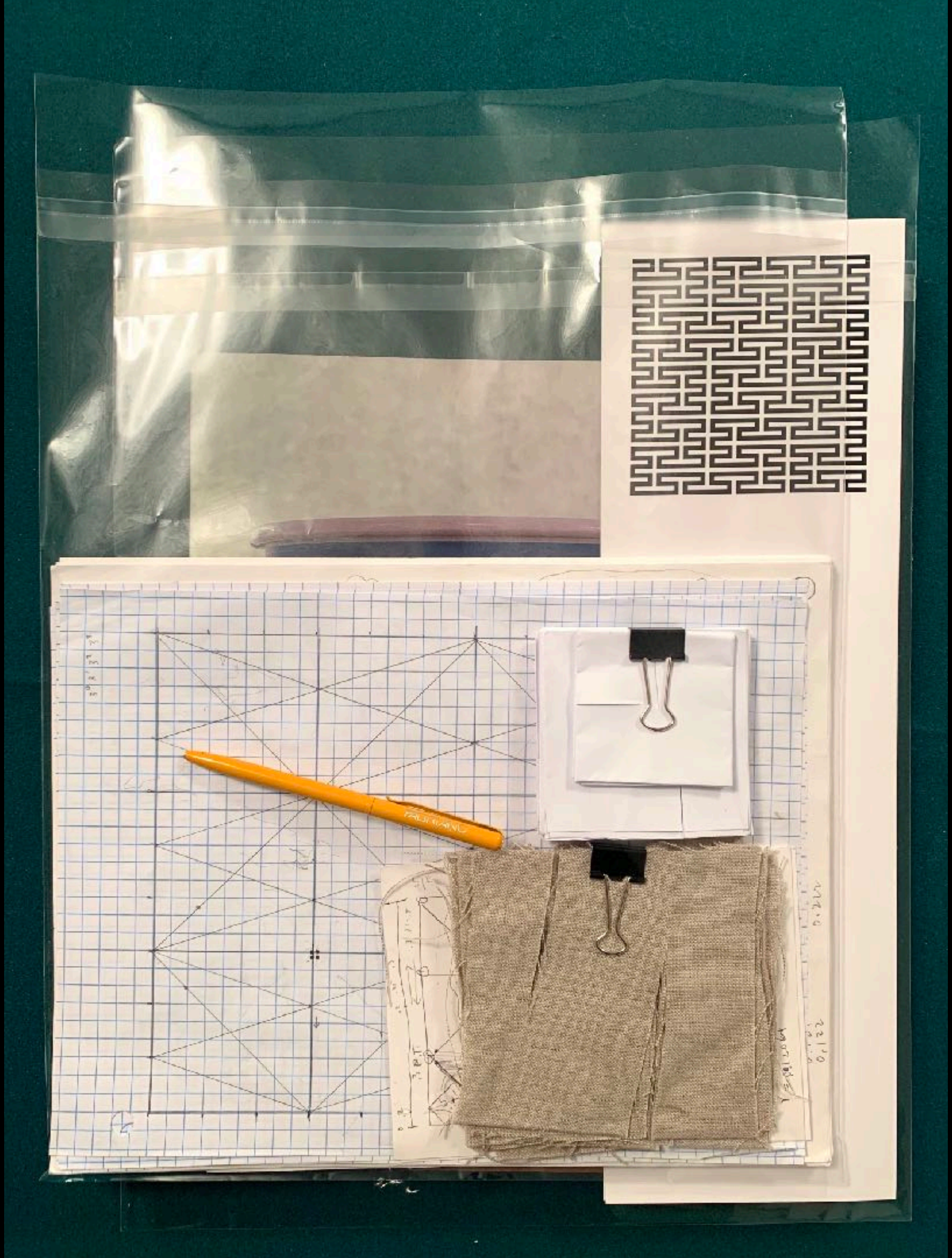
$a_1 = b_1$

$a_2 = k^{b_1} c_1$

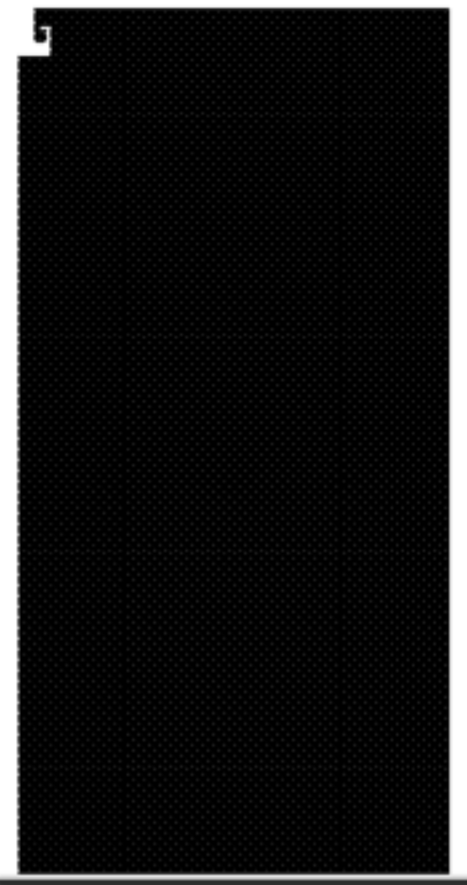
$a_3 = k^{c_1} b_2$

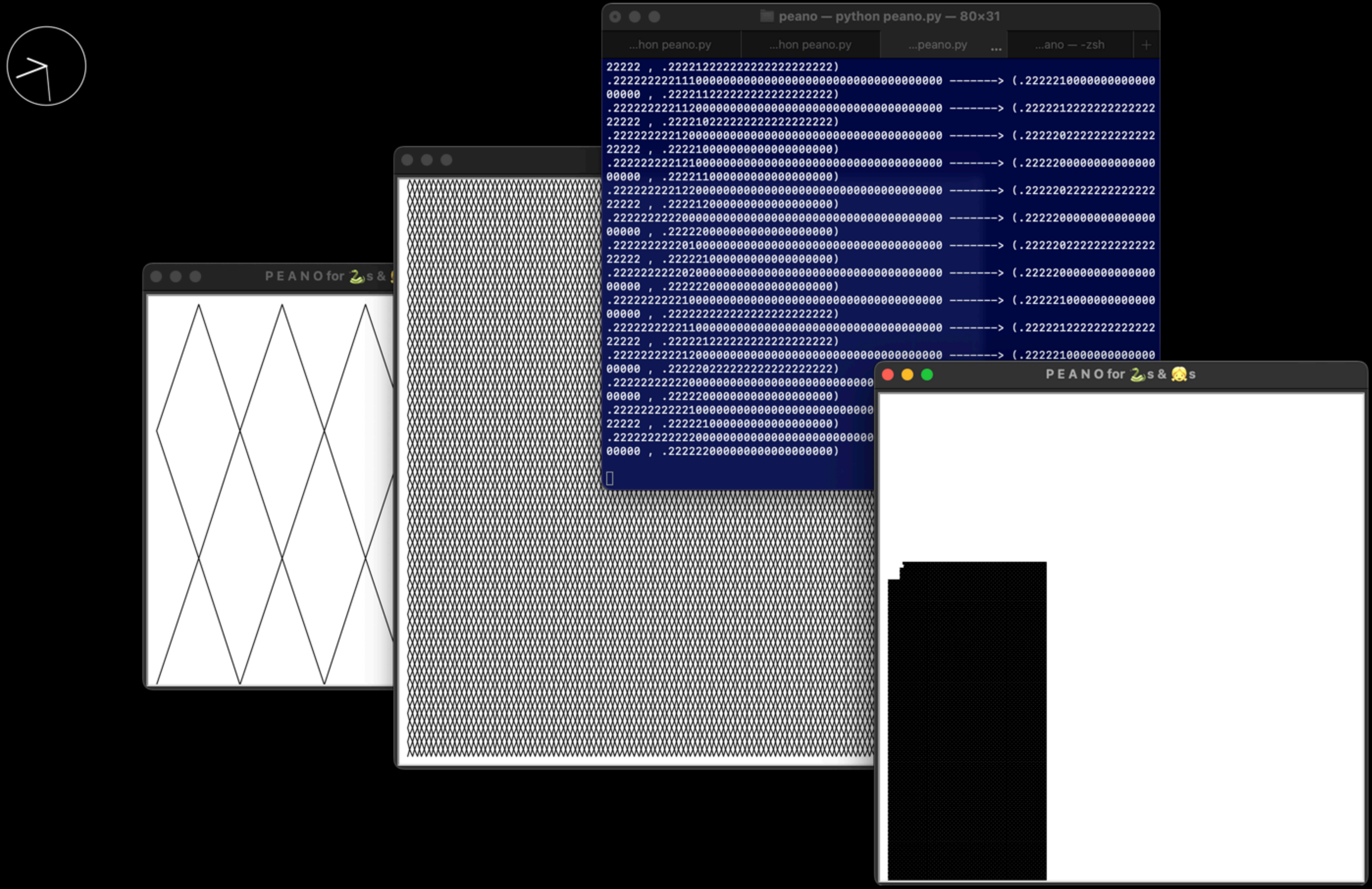
x	y
0	0
1	1
0	2
1	0
2	2
1	1
2	0
0	1
2	2

x x



P E A N O for 🐍s & 👧s





peano — python peano.py — 80×31
...hon peano.py
...hon peano.py
...peano.py
...ano — -zsh
22222 , .22221222222222222222222)
.222222222110000000000000000000000000000000 -------> (.22222100000000000000
00000 , .22221122222222222222222)
.222222222112000000000000000000000000000000 -------> (.22222122222222222222
22222 , .22221022222222222222222)
.222222222120000000000000000000000000000000 -------> (.22222022222222222222
22222 , .22221000000000000000000)
.222222222121000000000000000000000000000000 -------> (.22222000000000000000
00000 , .22221100000000000000000)
.222222222122000000000000000000000000000000 -------> (.22222022222222222222
22222 , .22221200000000000000000)
.222222222200000000000000000000000000000000 -------> (.22222000000000000000
00000 , .22222000000000000000000)
.222222222201000000000000000000000000000000 -------> (.22222022222222222222
22222 , .22222100000000000000000)
.222222222202000000000000000000000000000000 -------> (.22222000000000000000
00000 , .22222200000000000000000)
.222222222210000000000000000000000000000000 -------> (.22222100000000000000
00000 , .22222222222222222222222)
.222222222211000000000000000000000000000000 -------> (.22222122222222222222
22222 , .22222122222222222222222)
.222222222212000000000000000000000000000000 -------> (.22222100000000000000
00000 , .22222022222222222222222)
.2222222222220000000000000000000
00000 , .22222000000000000000000)
.2222222222221000000000000000000
22222 , .22222100000000000000000)
.2222222222222000000000000000000
00000 , .22222200000000000000000)
P E A N O for 🐍s &
P E A N O for 🐍s & 👧s

DEMO I

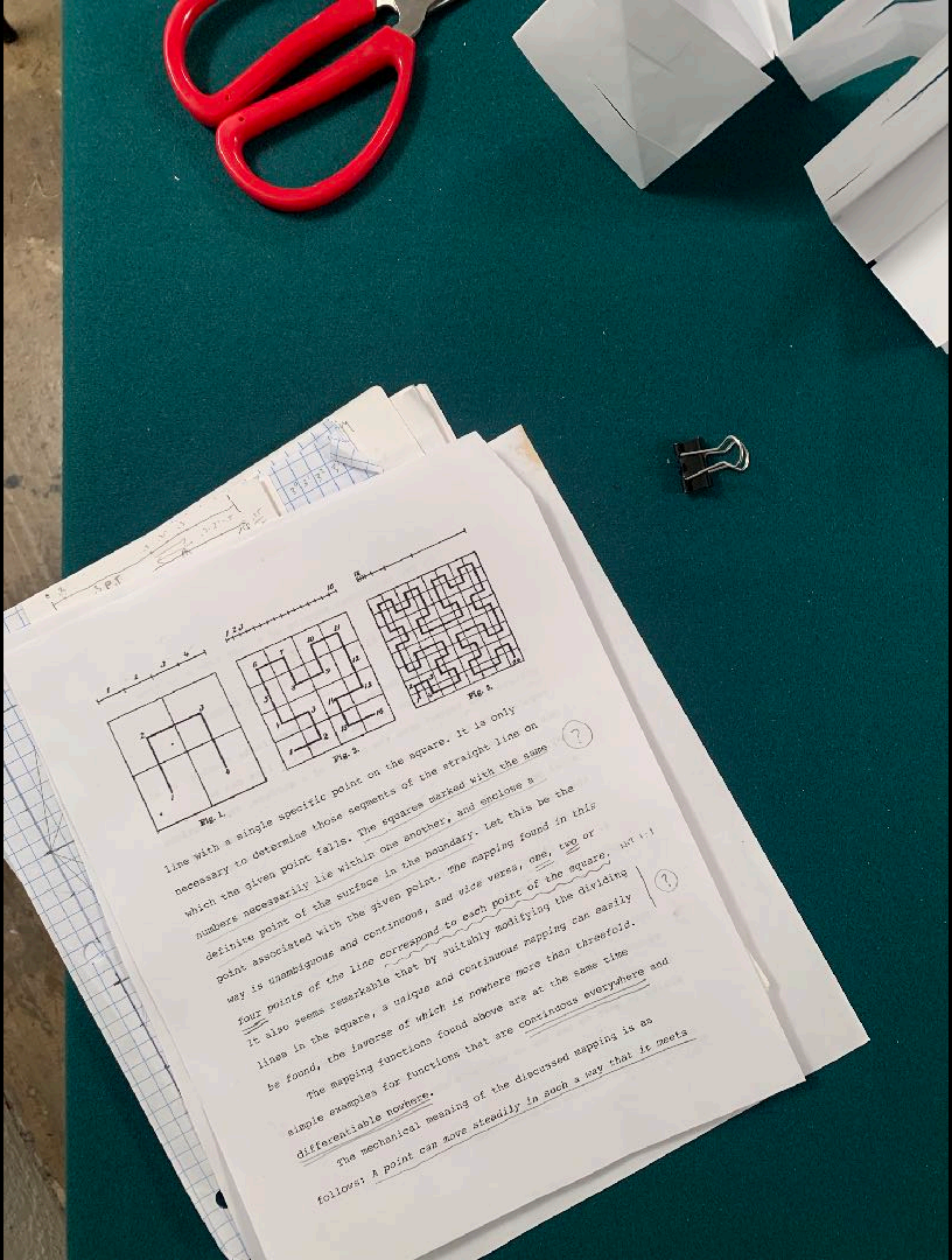
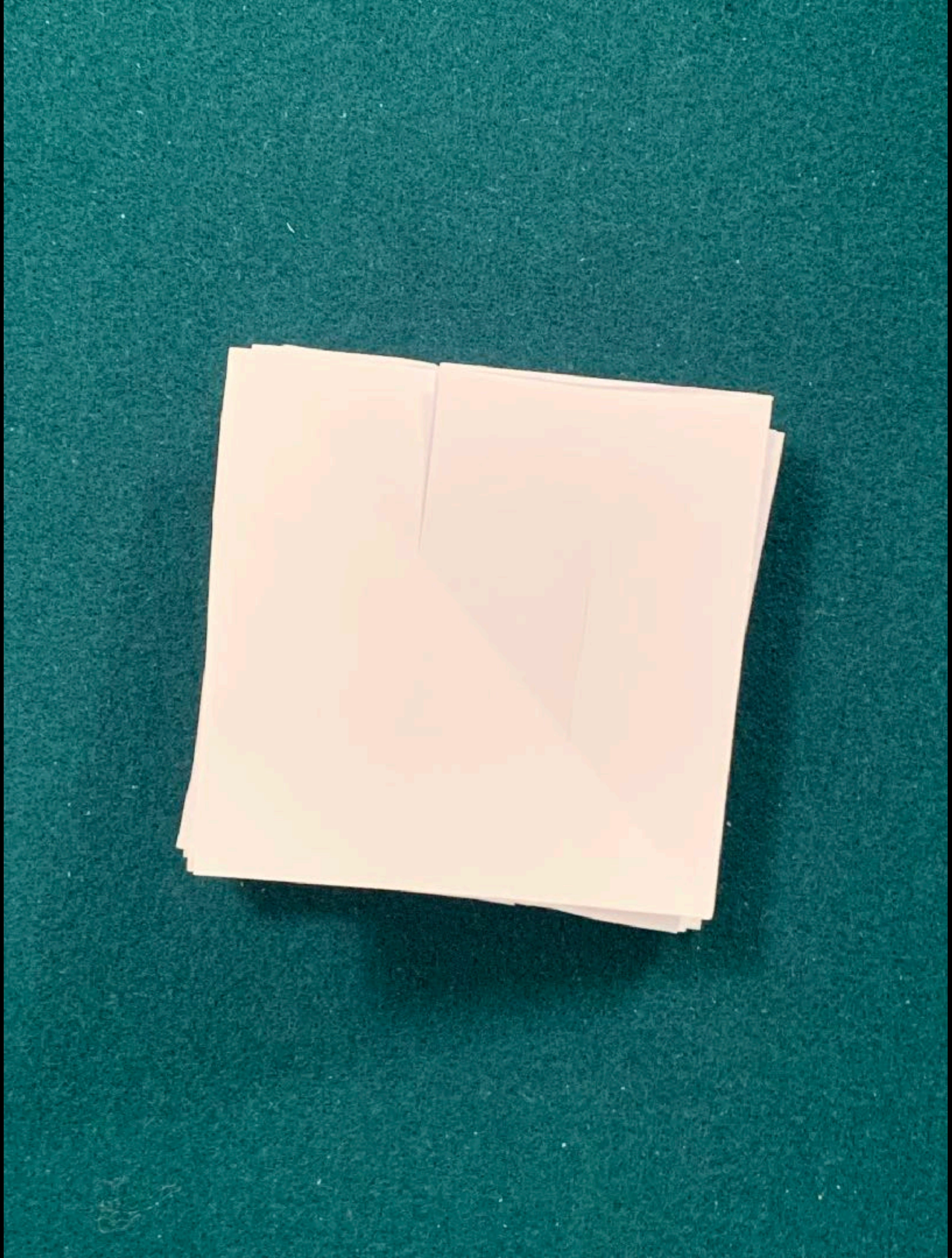
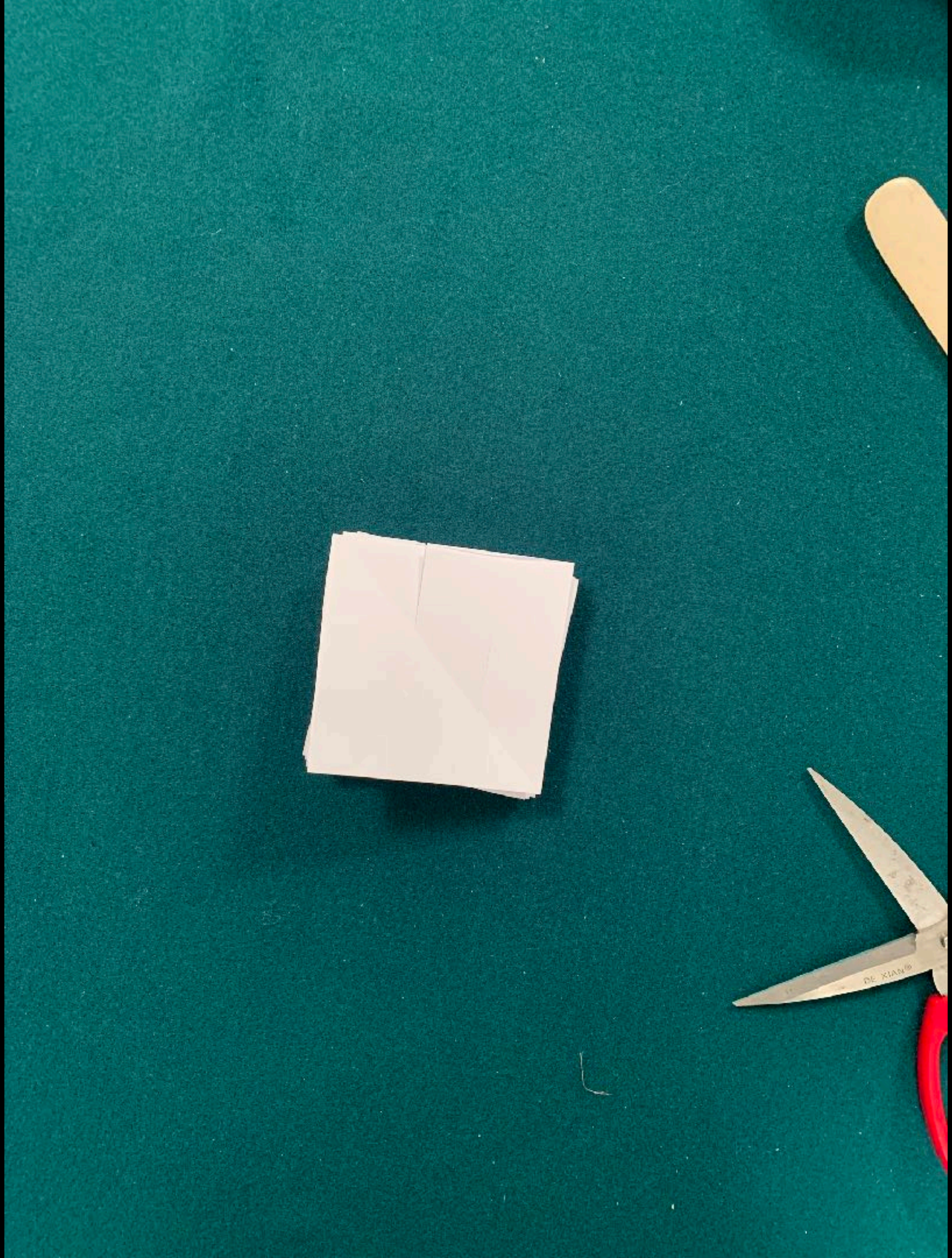
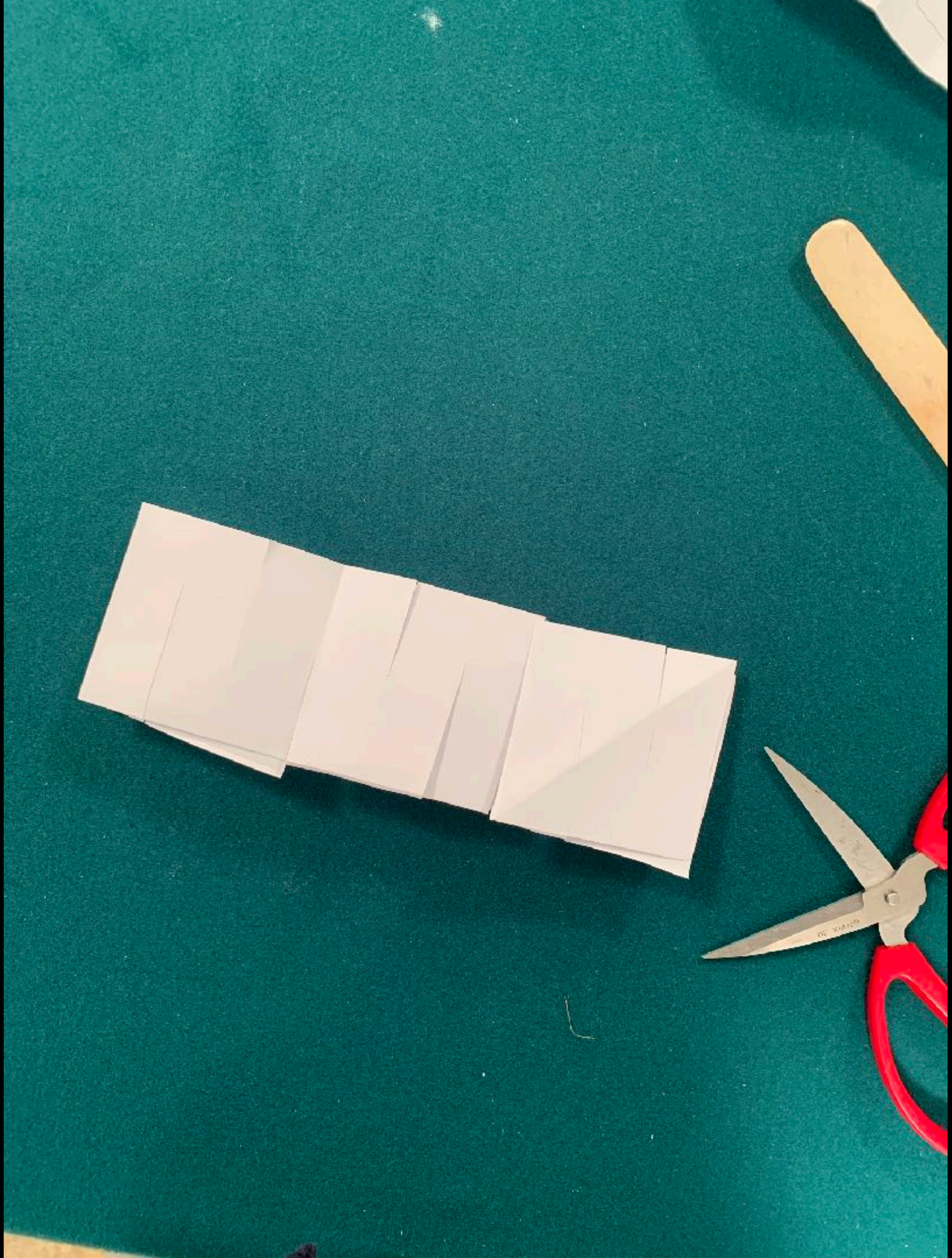
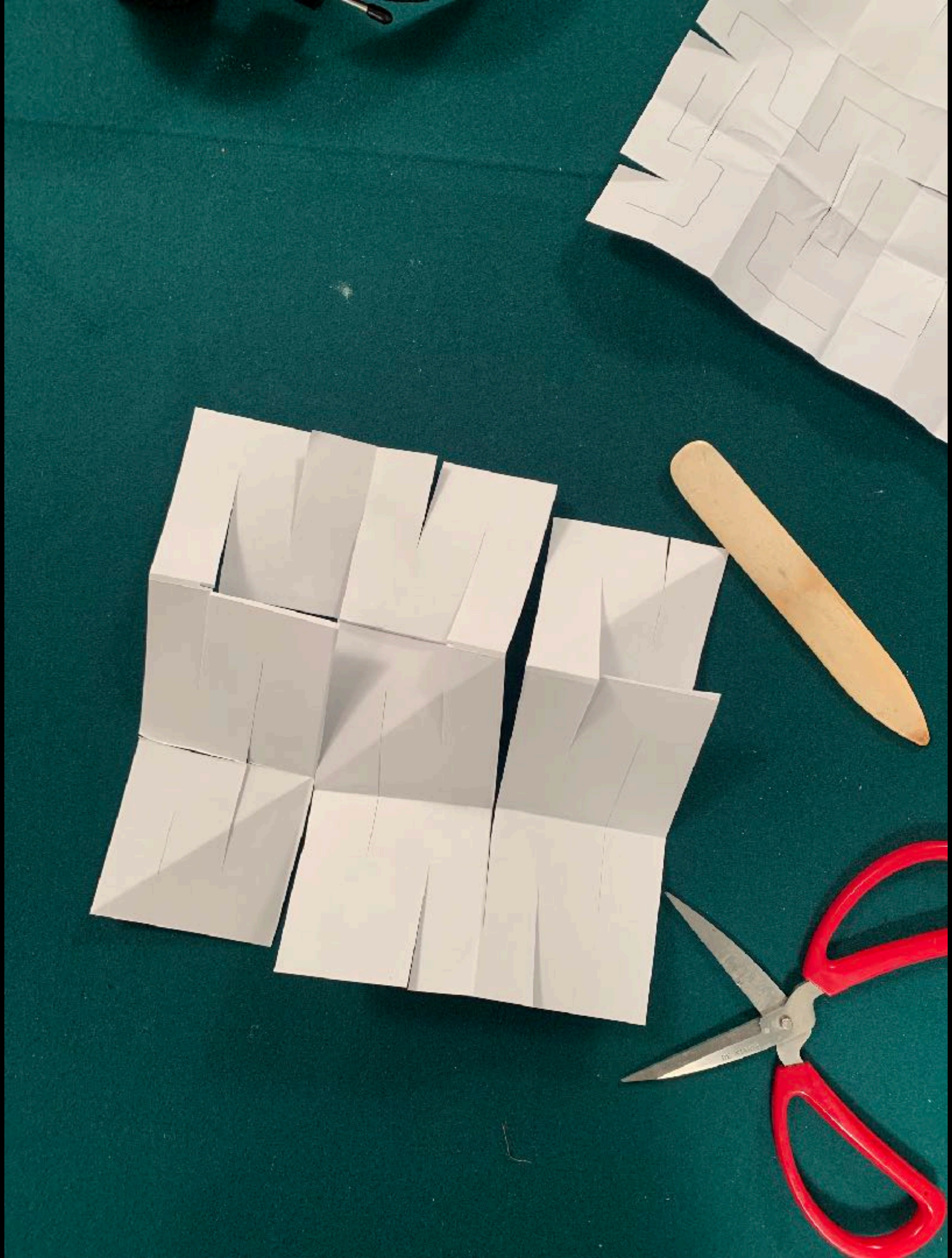
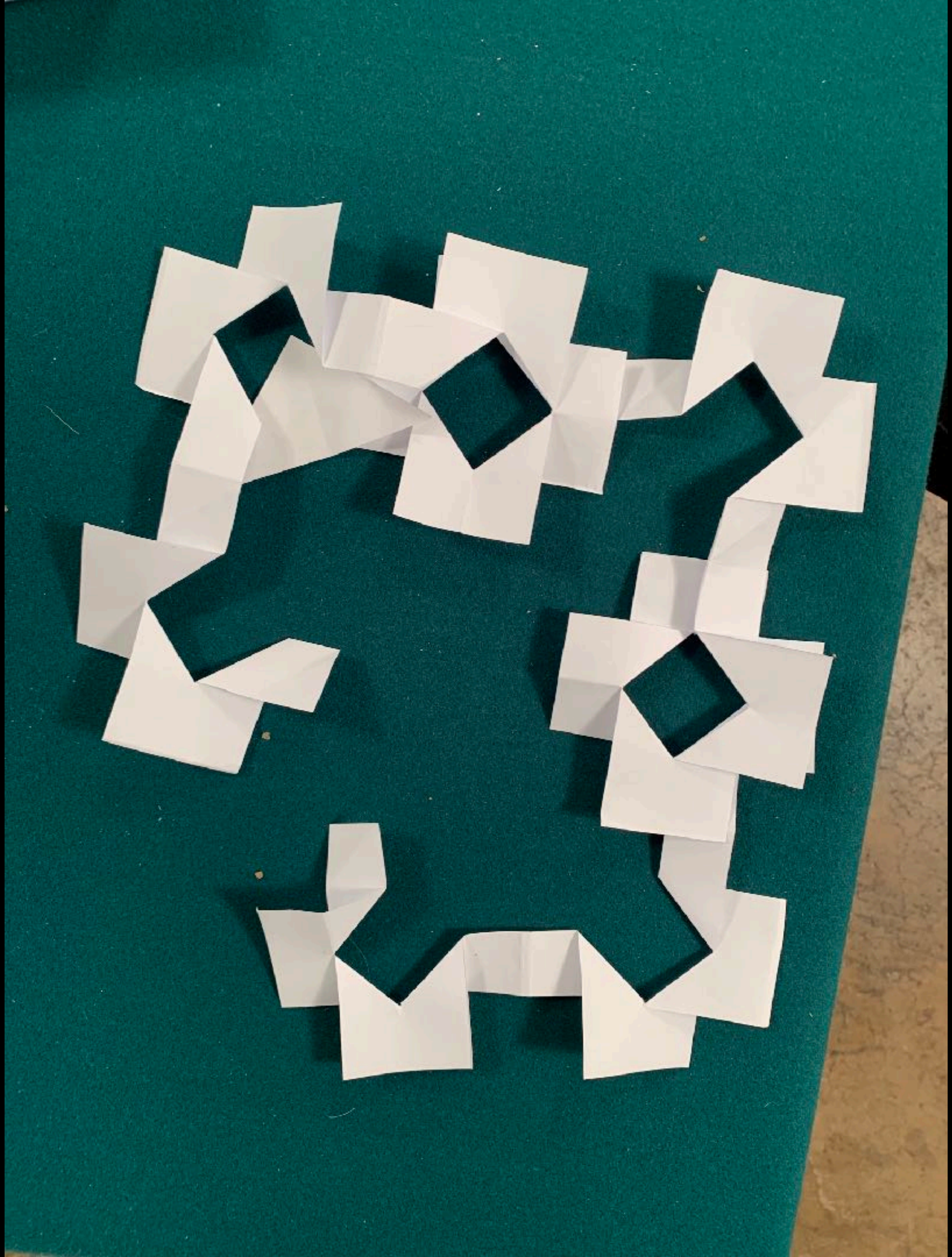


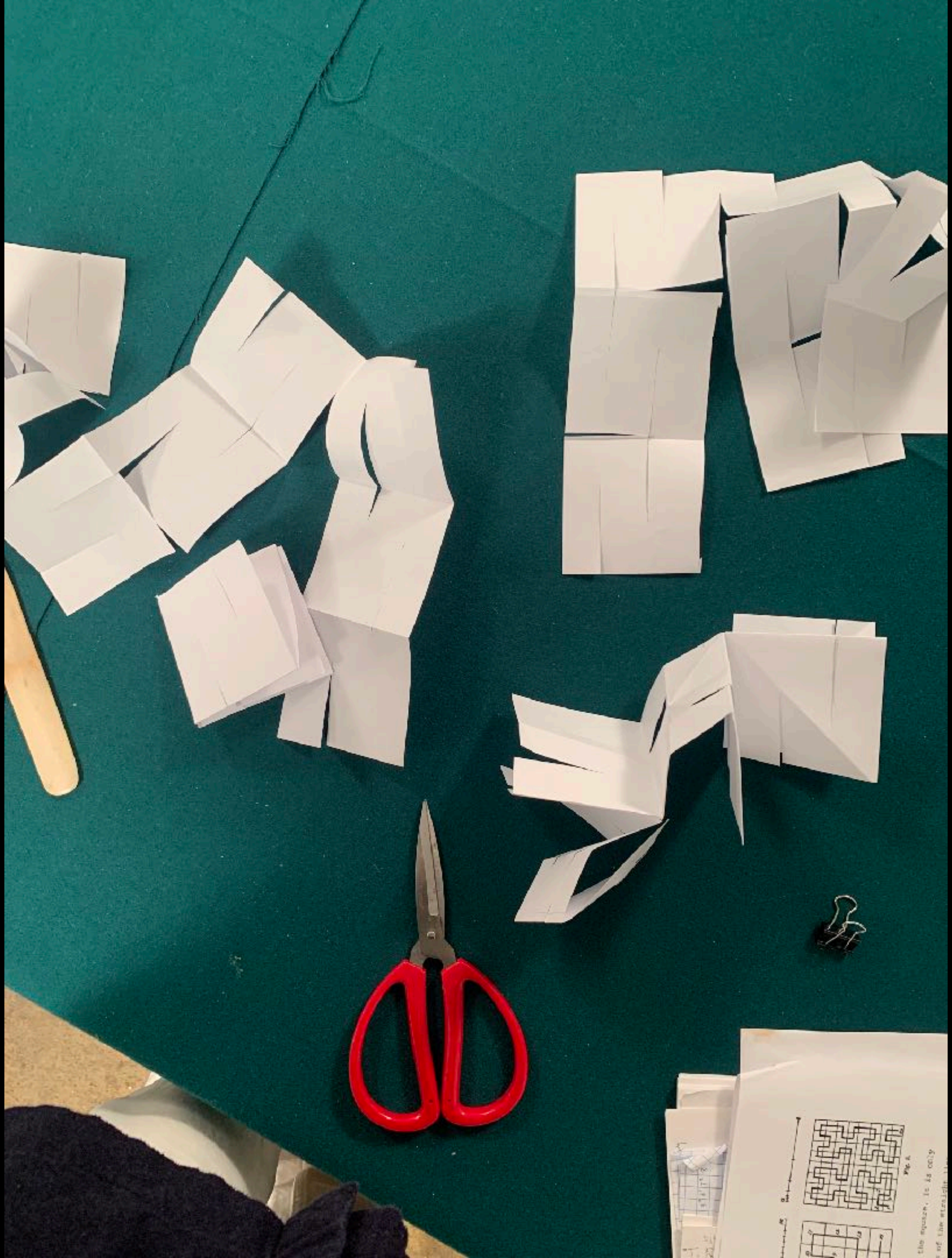
Fig. 1.
Fig. 2.
Fig. 3.
line with a single specific point on the square. It is only
necessary to determine those segments of the straight line on
which the given point falls. The squares marked with the same
numbers necessarily lie within one another, and enclose a
definite point of the surface in the boundary. Let this be the
point associated with the given point. The mapping found in this
way is unambiguous and continuous, and vice versa, one, two or
four points of the line correspond to each point of the square.
It also seems remarkable that by suitably modifying the dividing
lines in the square, a unique and continuous mapping can easily
be found, the inverse of which is nowhere more than threefold.
The mapping functions found above are at the same time
simple examples for functions that are continuous everywhere and
differentiable nowhere.
The mechanical meaning of the discussed mapping is as
follows: A point can move steadily in such a way that it meets

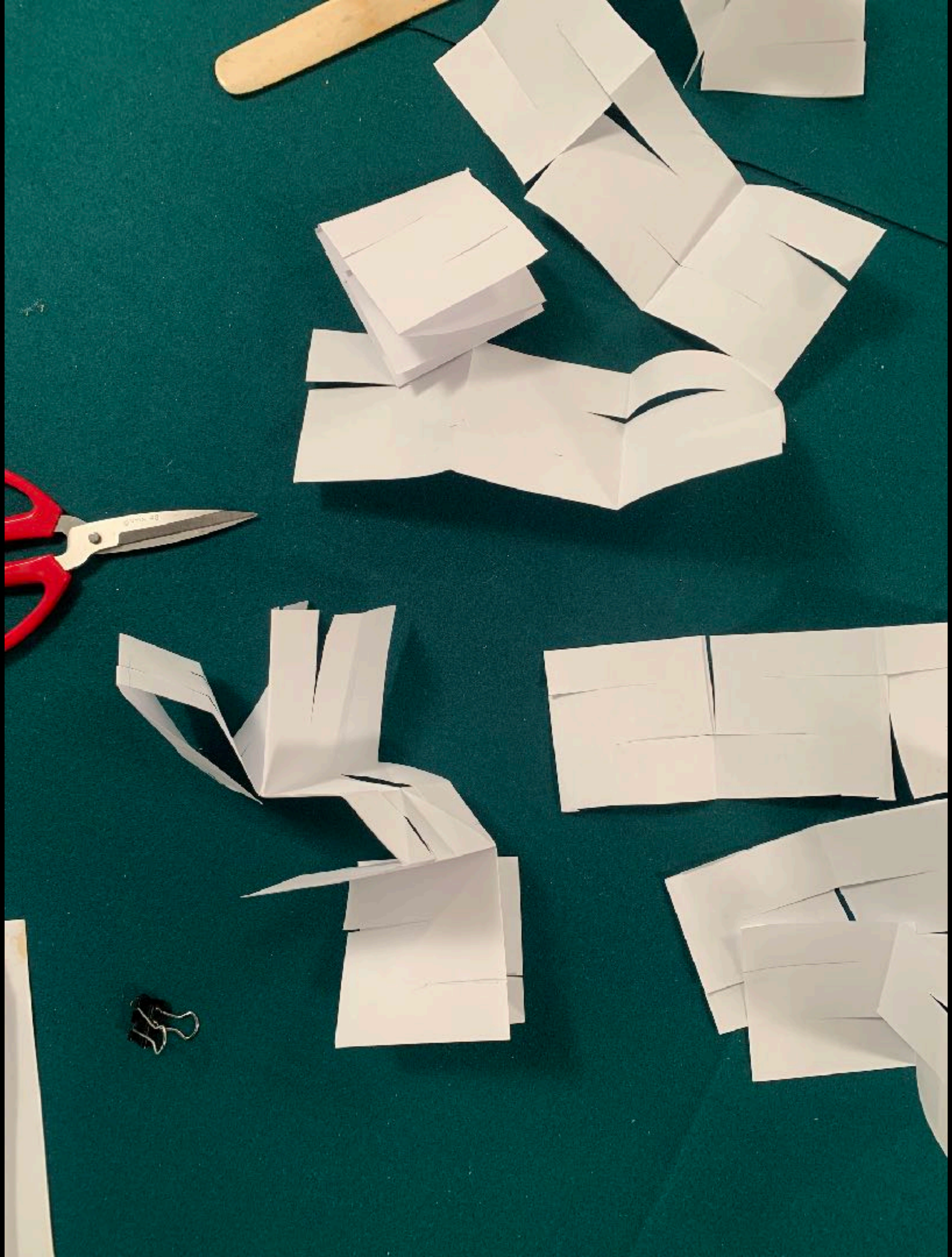




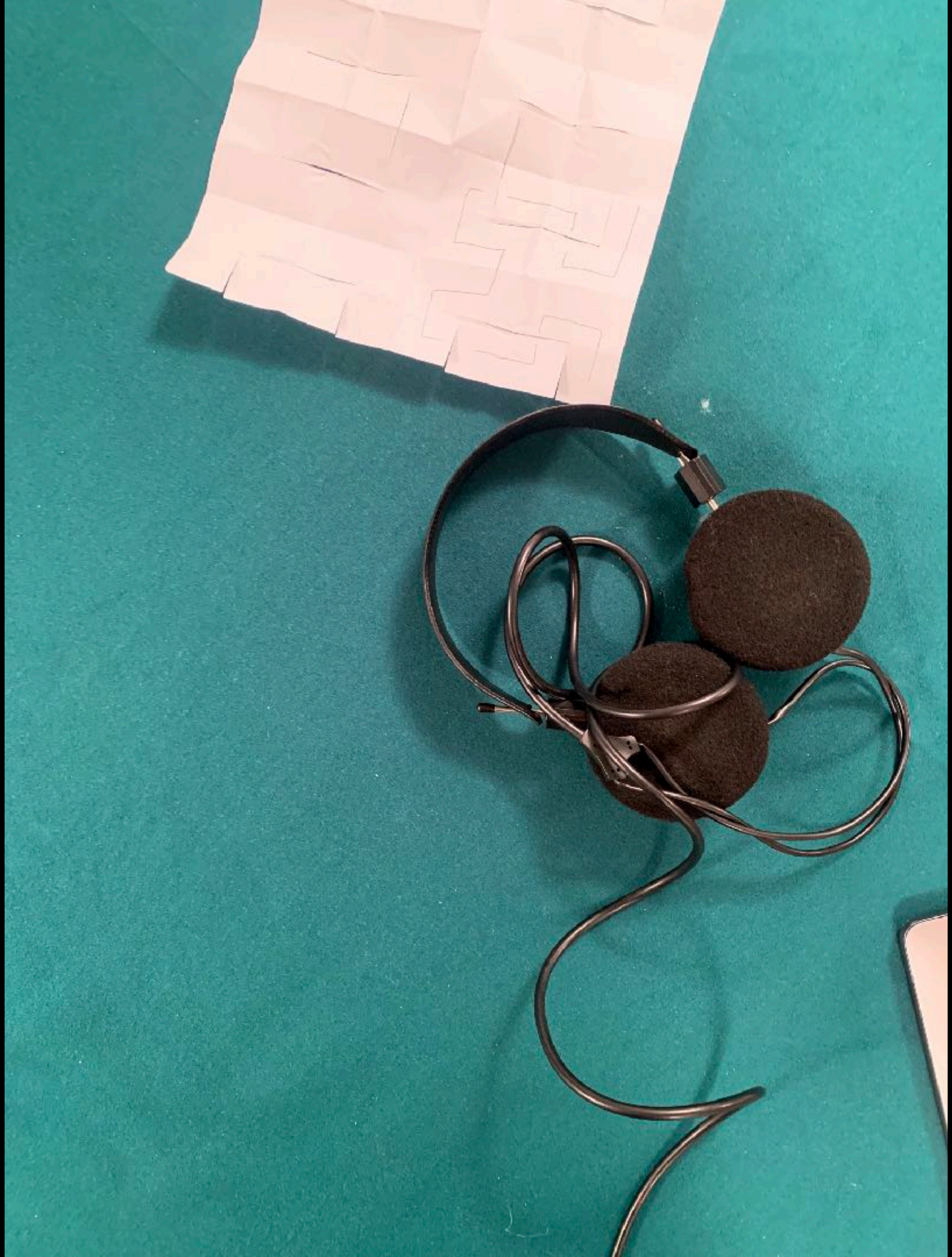


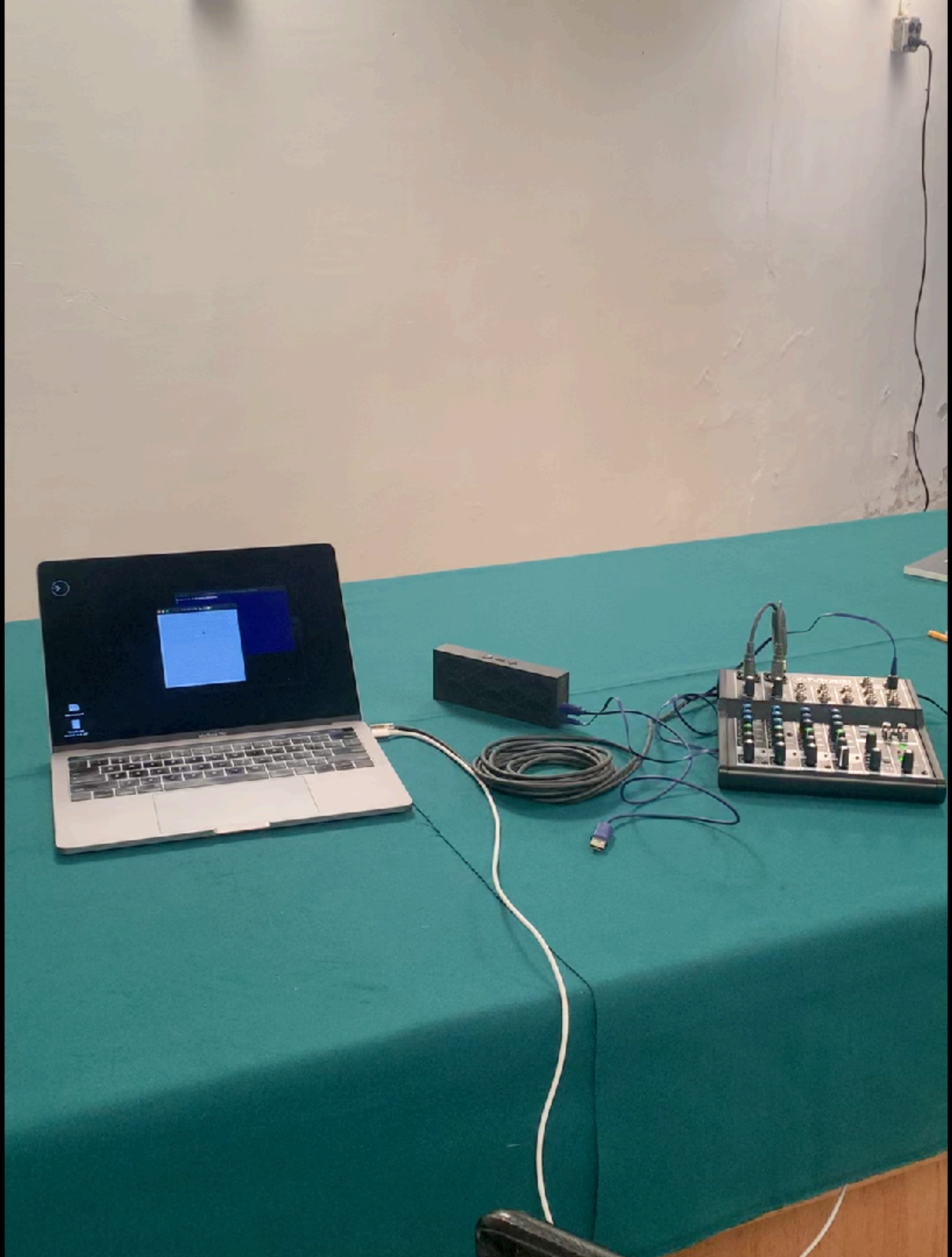



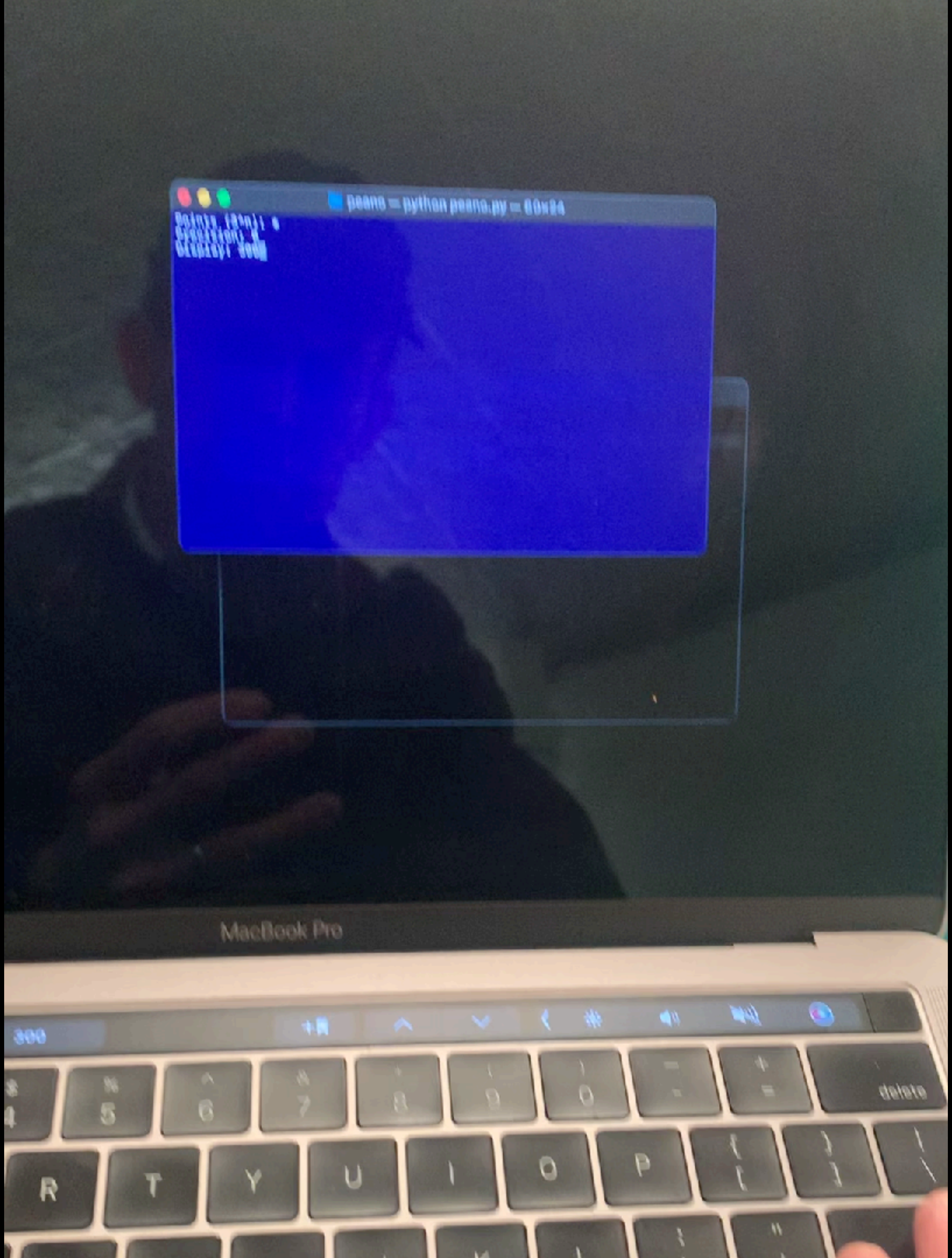


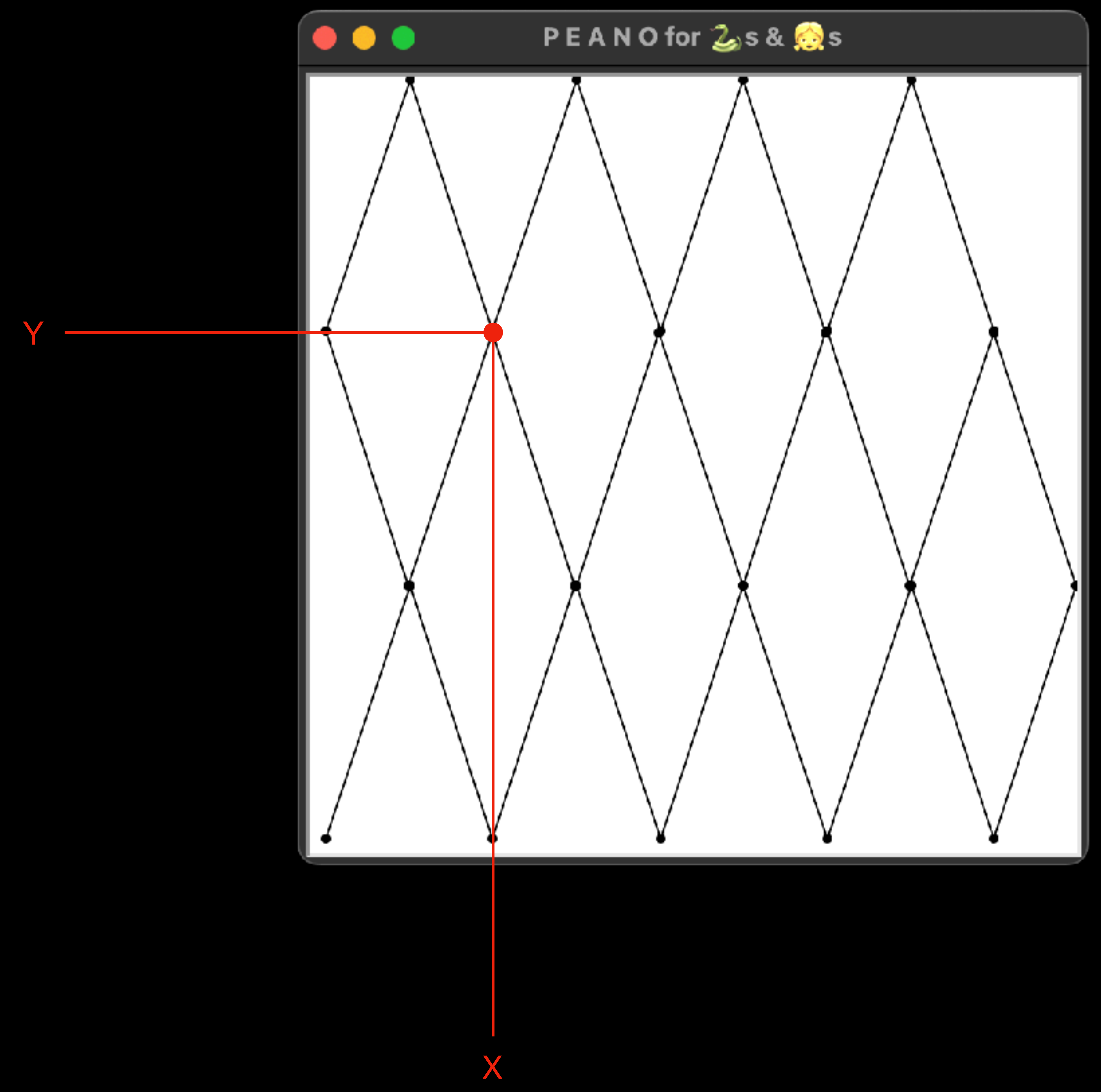


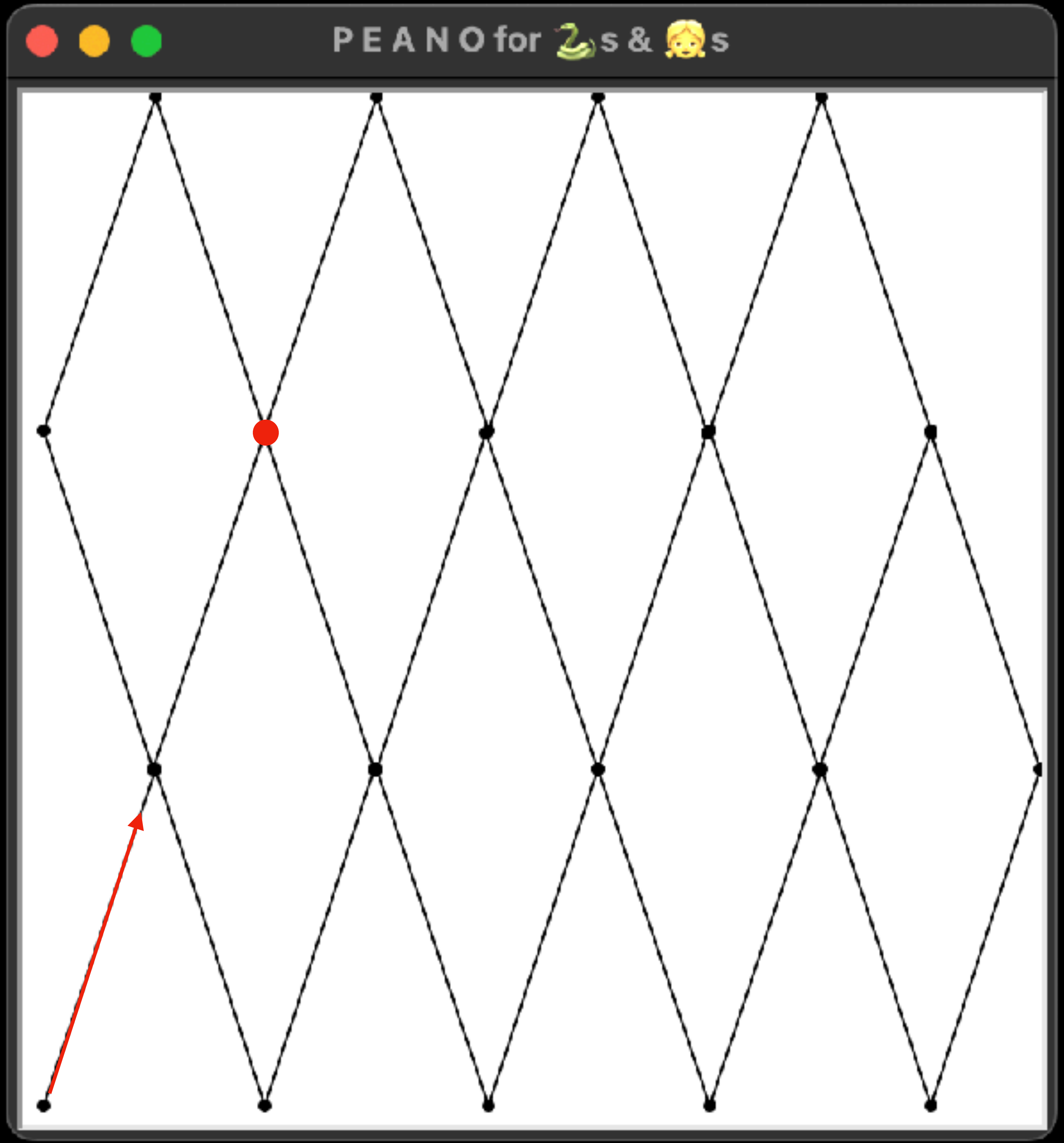

DOVER · THRIFT · EDITIONS
Edwin A. Abbott
FLATLAND
A Romance of Many Dimensions

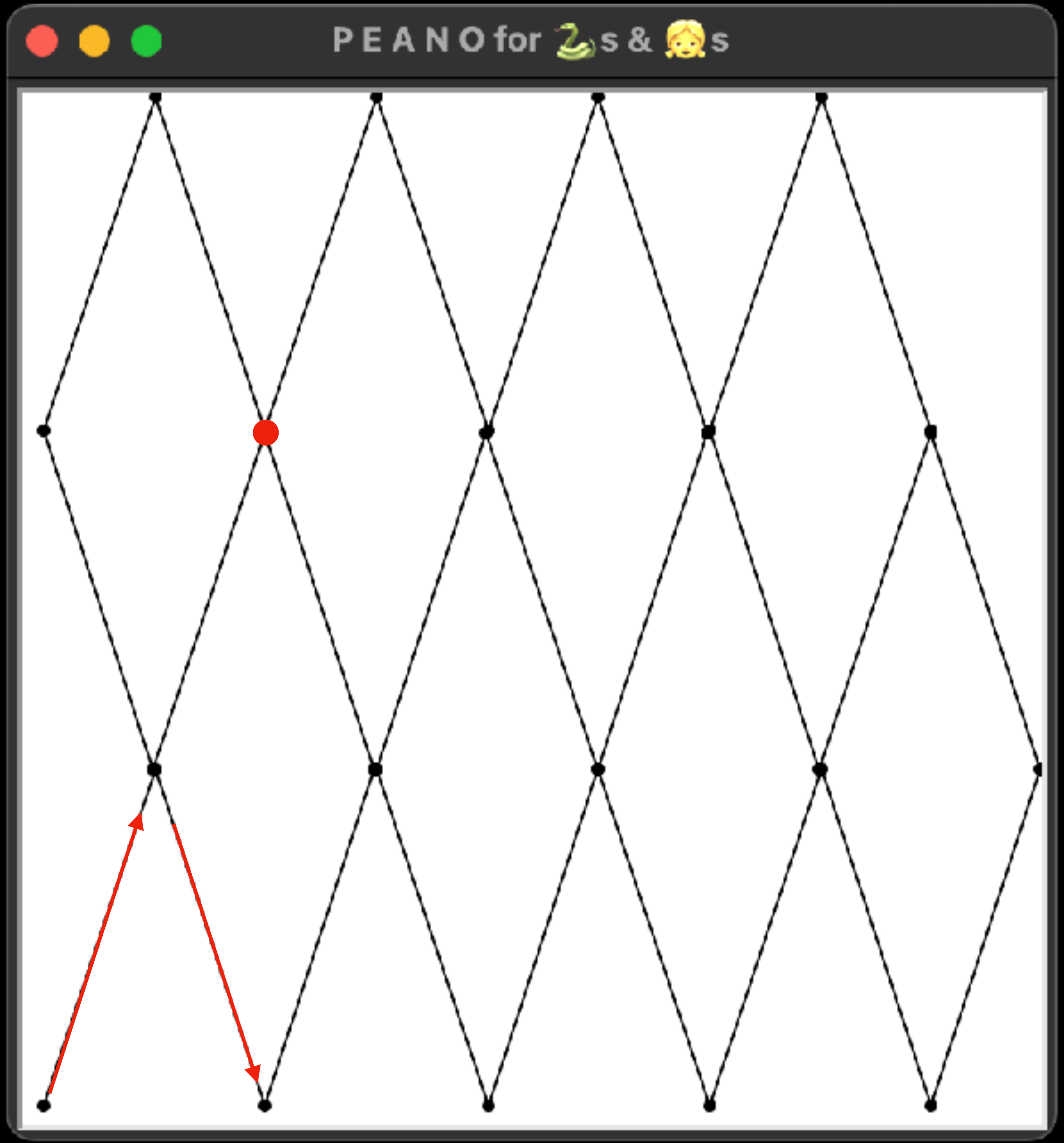


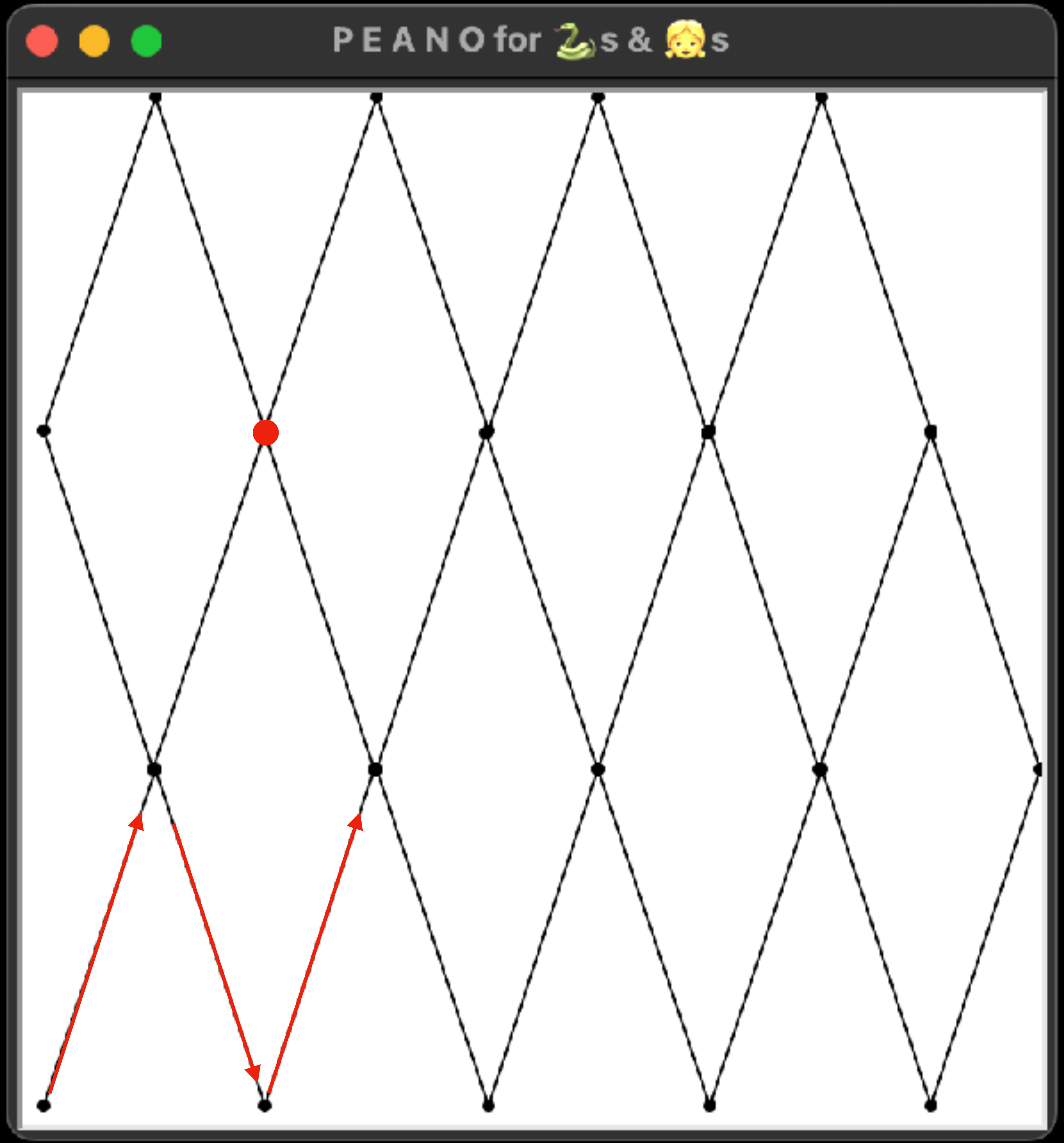


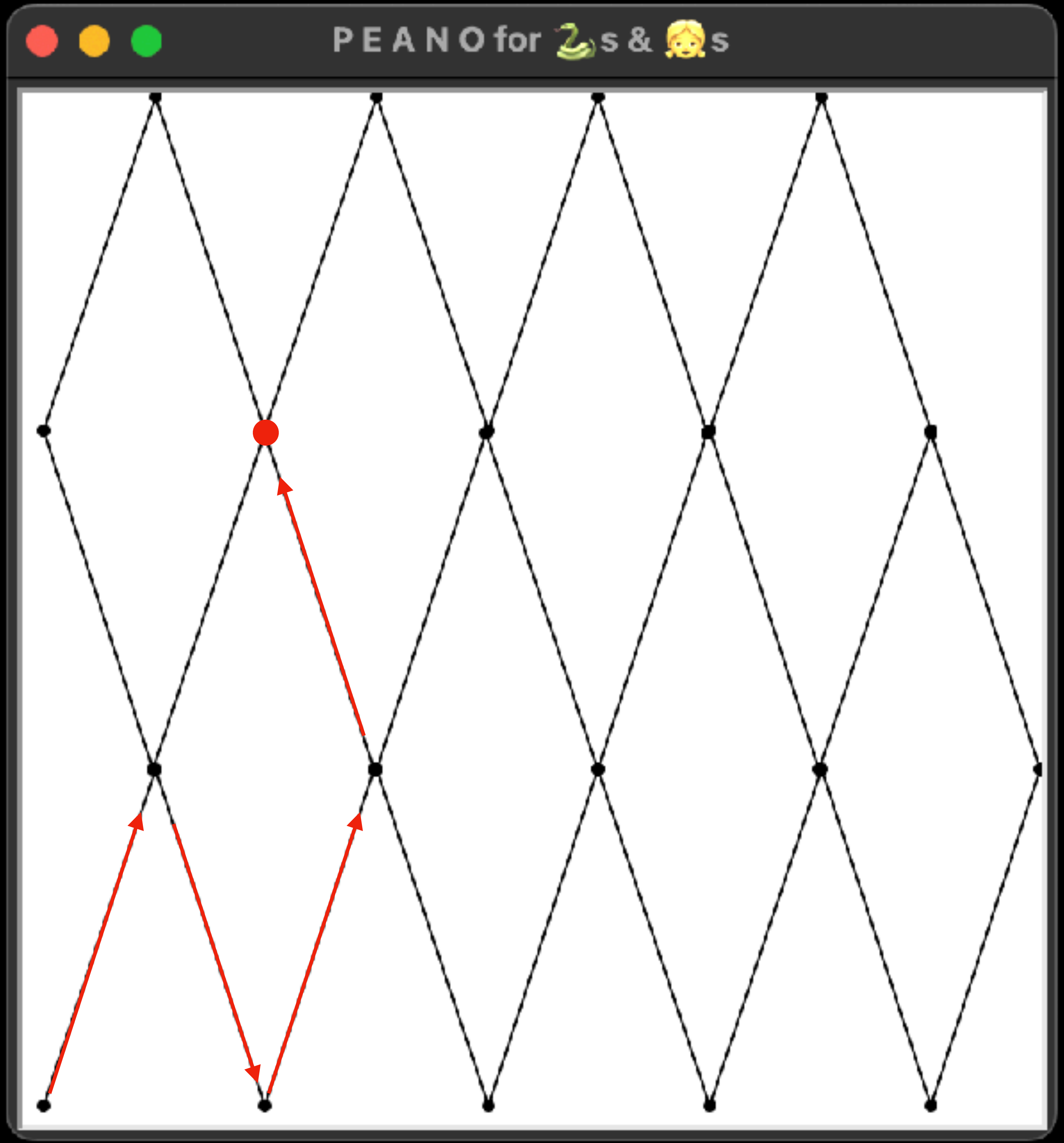

MacBook Pro

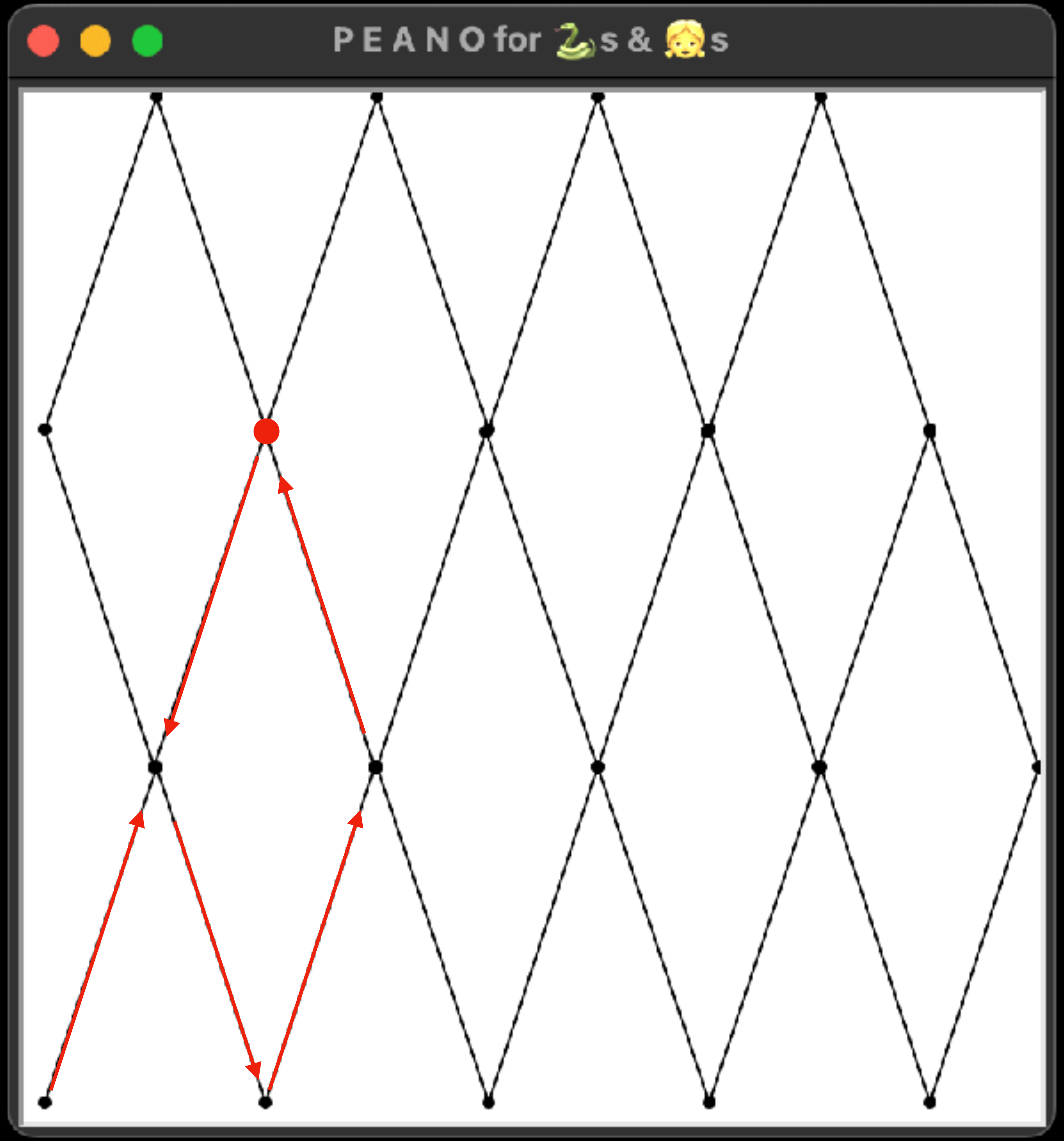

PEANO for 🐍s & 👧s
Y
X

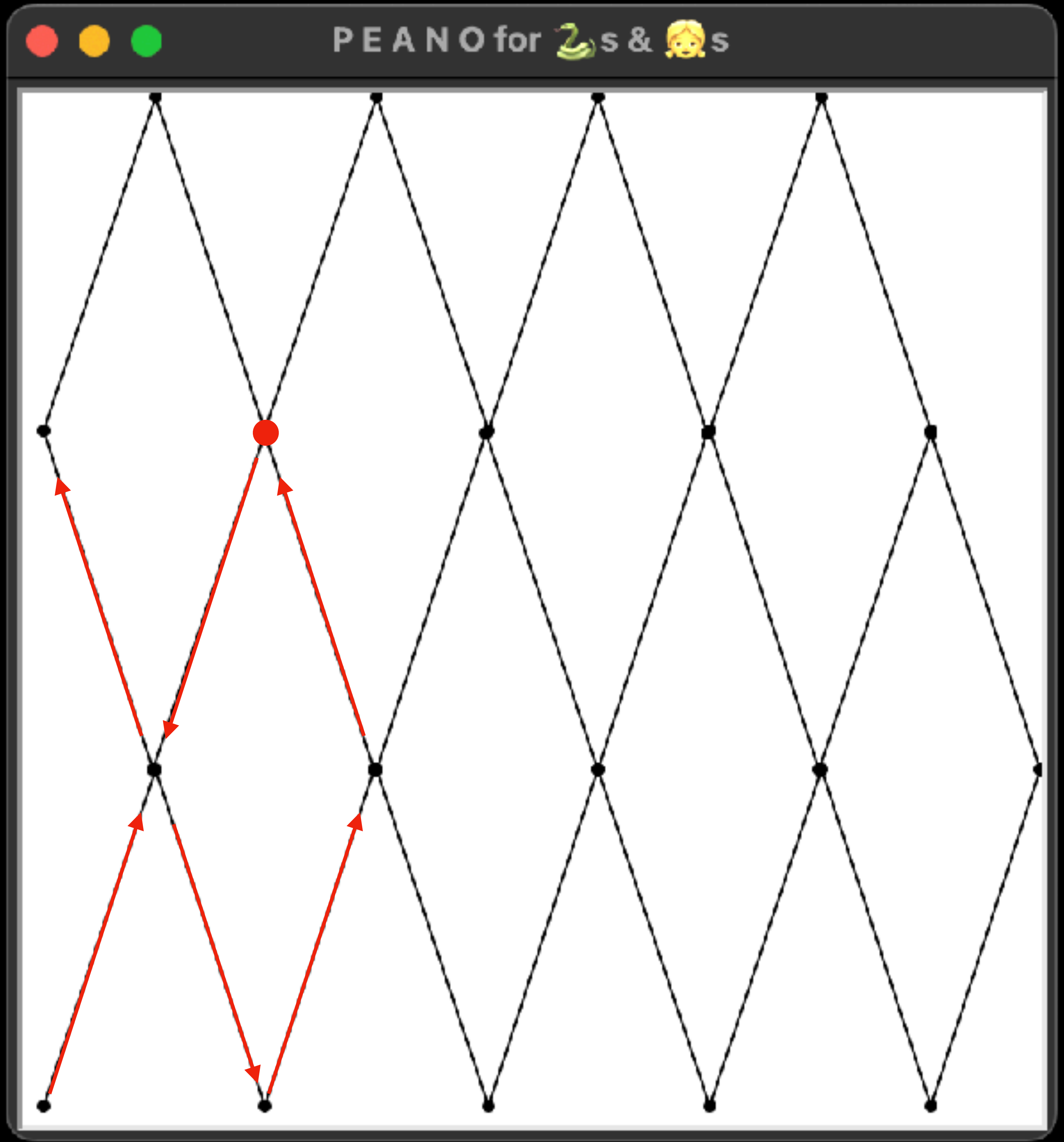

PEANO for 🐍s & 👱s

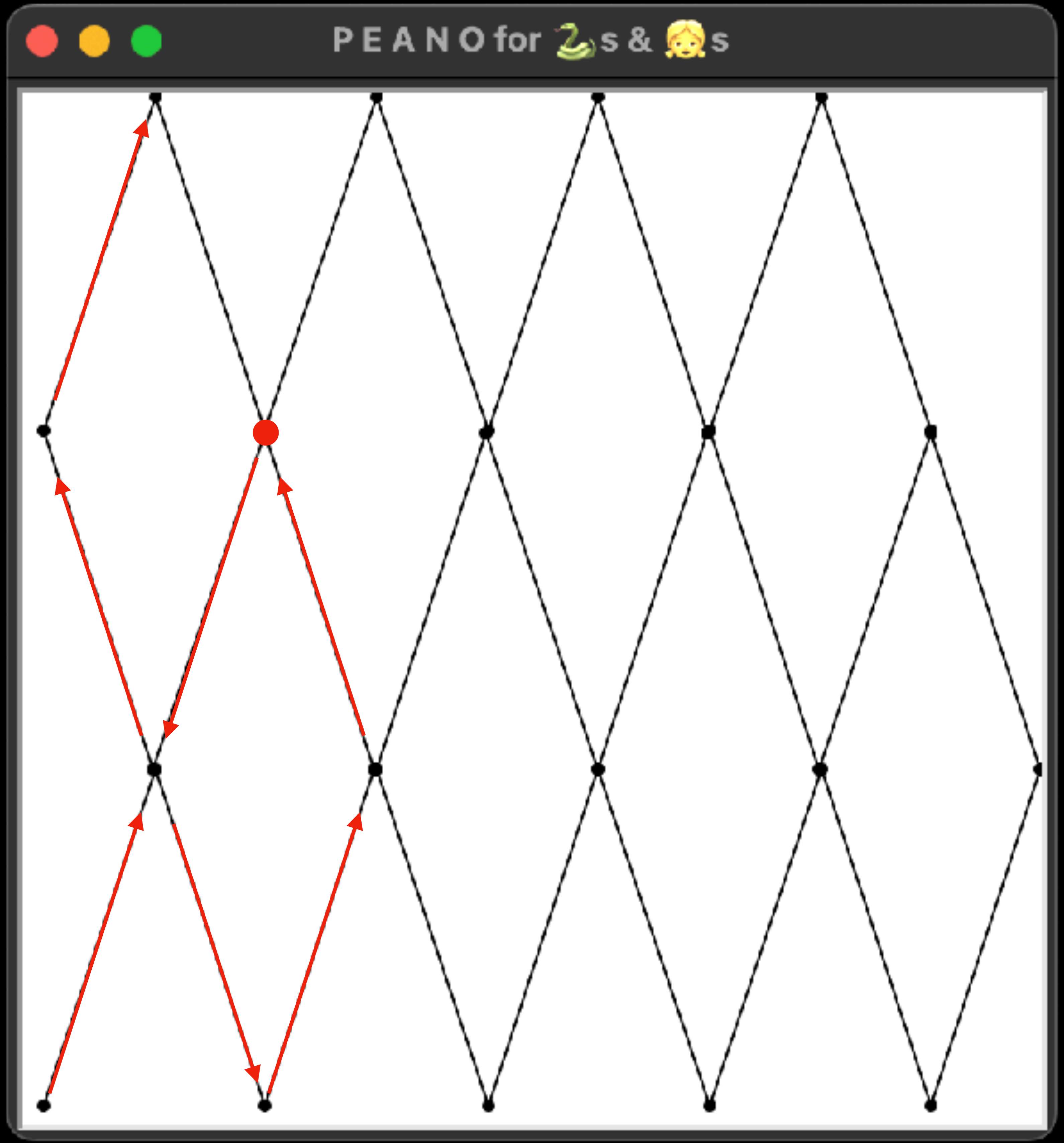

P E A N O for 🐍s & 👩s

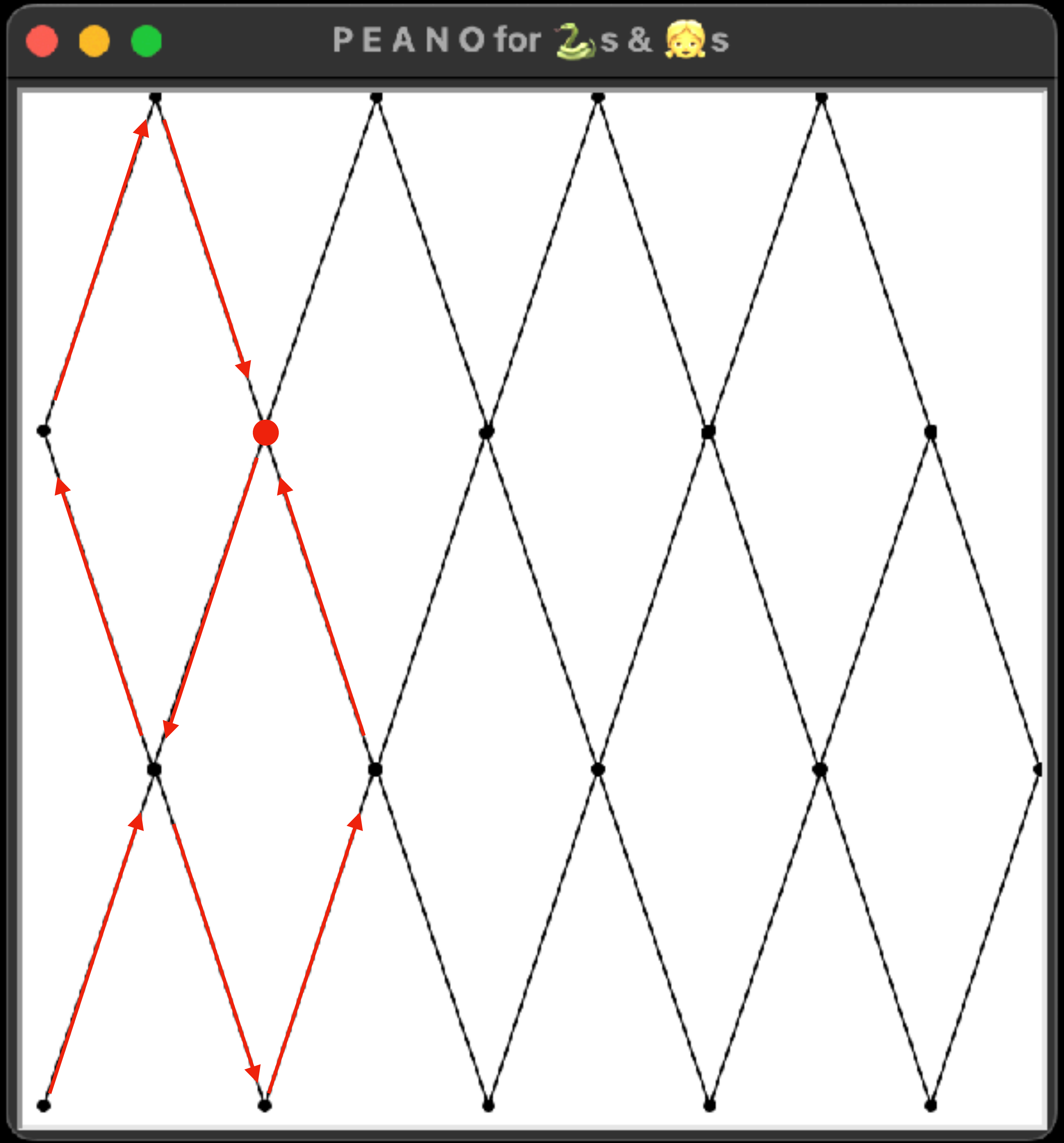

PEANO for 🐍s & 👱s


P E A N O for 🐍s & 👩s

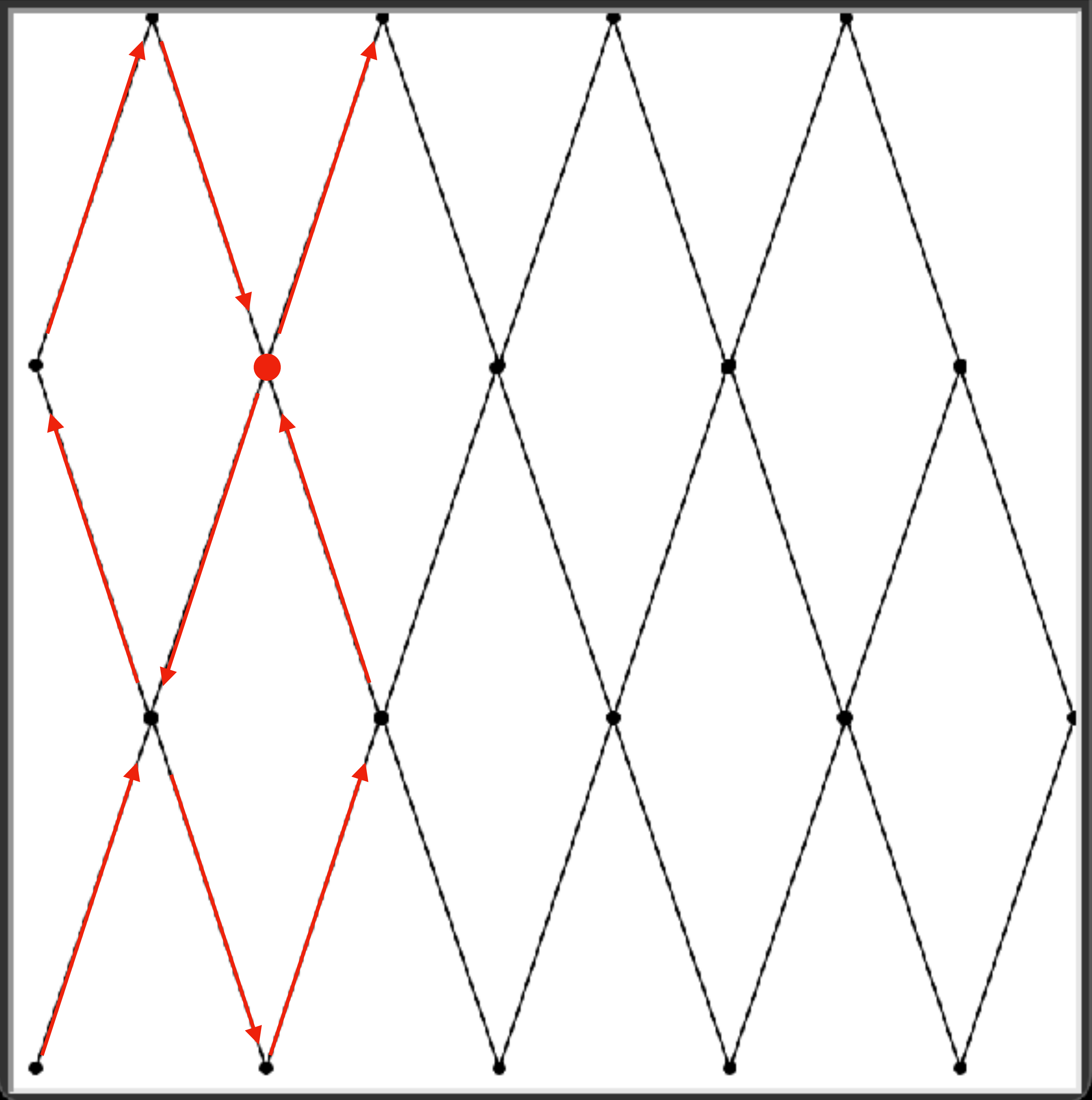

PEANO for 🐍s & 👱s




P E A N O for 🐍s & 👧s


P E A N O for 🐍s & 👧s

PEANO for 🐍s & 👩s



peano — -zsh — 80×21

```
reinfurt@AdrianoOlivetti peano % python peano.py
```

peano — python peano.py — 80×21

```
Points (3^n): 
```

peano — python peano.py — 80×21

```
T -------> 0.9629629629629629)
^C%
reinfurt@AdrianoOlivetti peano % python peano.py
```

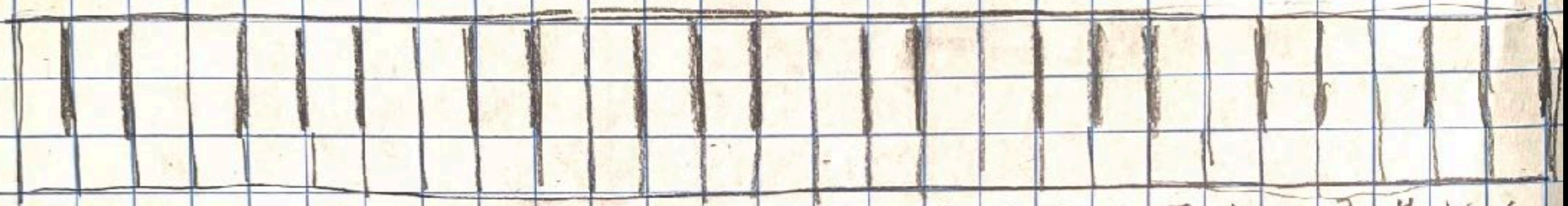
DEMO II (sinewave)

{ 3]

BASED ON # POINTS, SEC A*

LIKE ...

0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2

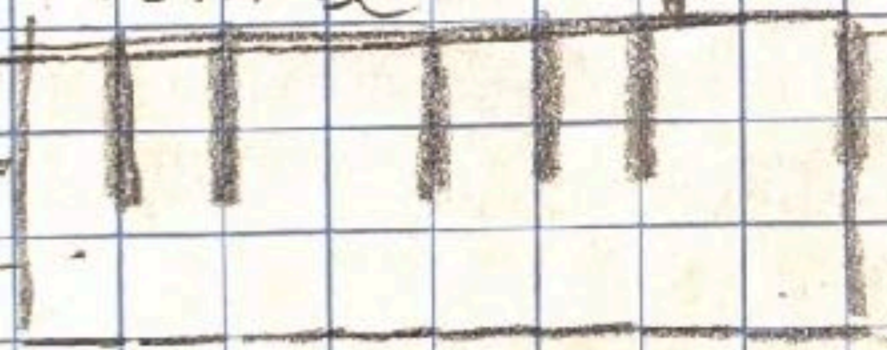


1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	2	3	4	5	6	7	1	2	3	4	5	6
0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8	0	1	2	3	4	5	6	7	8
60	62	64	65	67	69	71	72	74	76	77	79	81	83	84	86	88	89	91	93	94	96	98	100	101	103	105

*

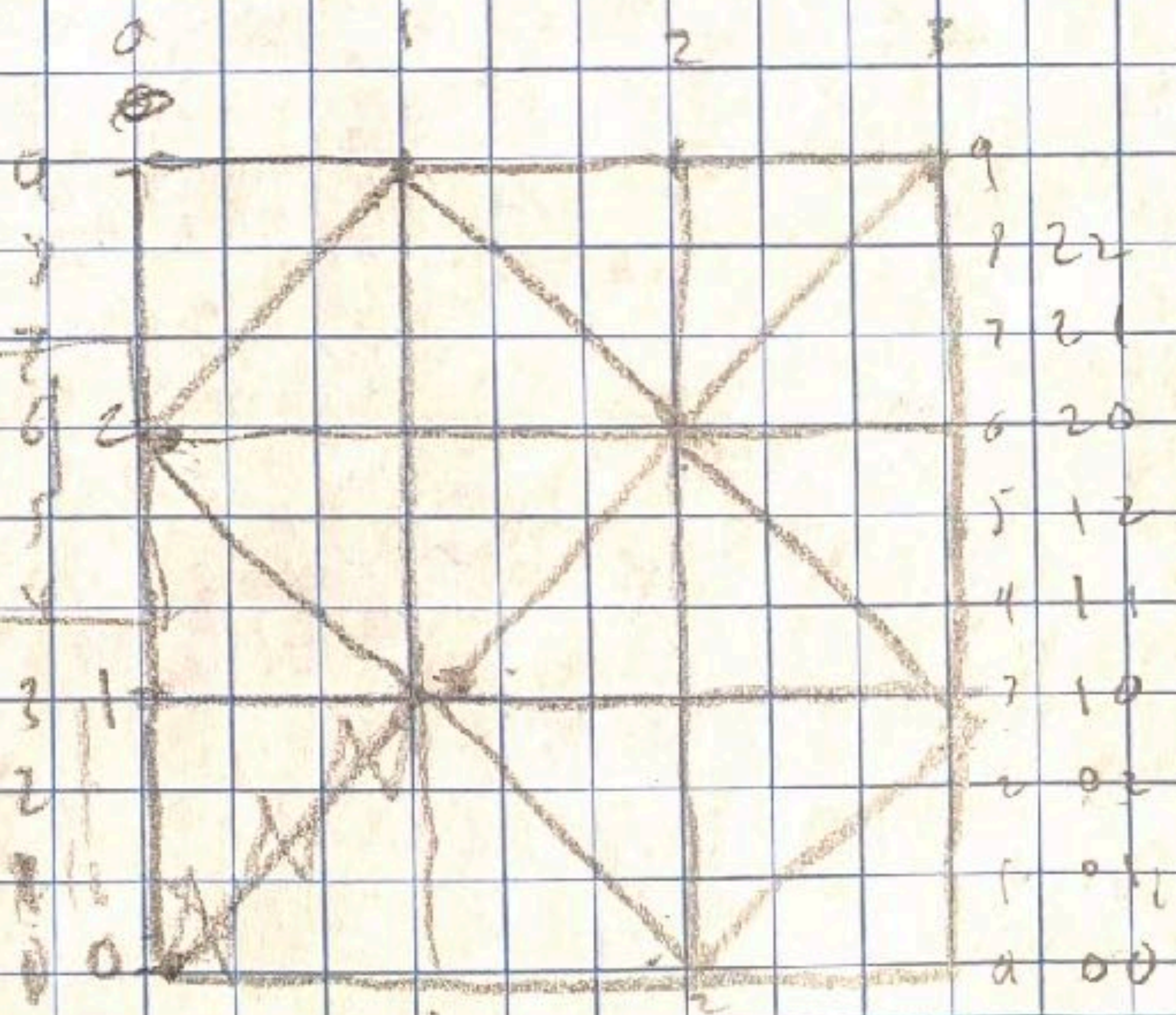
3^1 POINTS = 1 OCTAVE
3^2 POINTS = 2 OCTAVES
3^3 POINTS = 4 OCTAVES
3^4 POINTS = 1 OCTAVES (EXACTLY)

3^5 = 9 OCTAVES



1 2 3 4 5 6 7 8

~~3^n POINTS = $3^{(2n-1)}$~~ OCTAVES REQUIRED



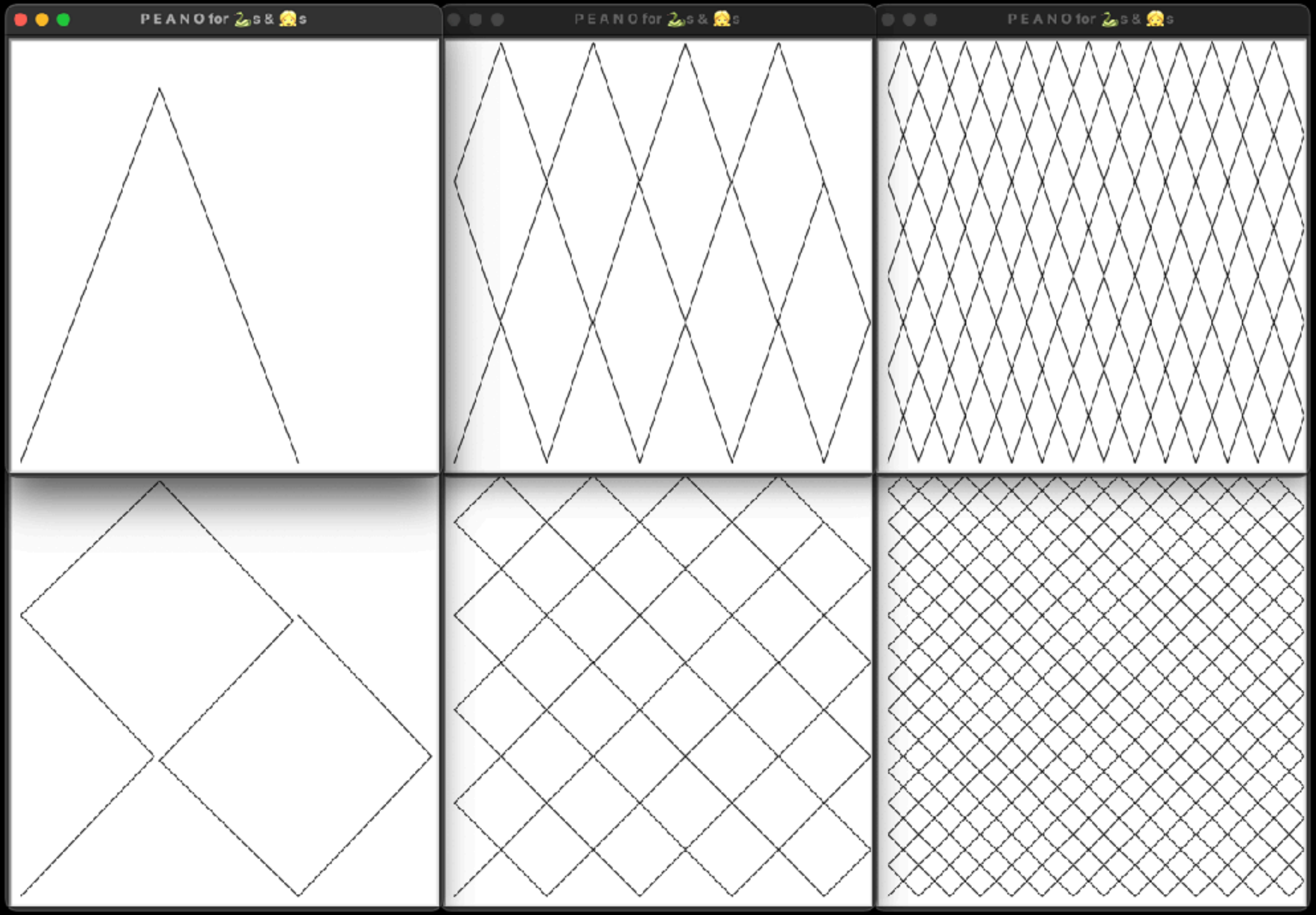
peano — -zsh — 80×18

```
reinfurt@AdrianoOlivetti peano % python peano.py 3 4 200 --screen_x 50 --screen_
y 50 --midi
```

peano — python peano.py 4 4 200 --screen_x 50 --screen_y 50 --midi —...

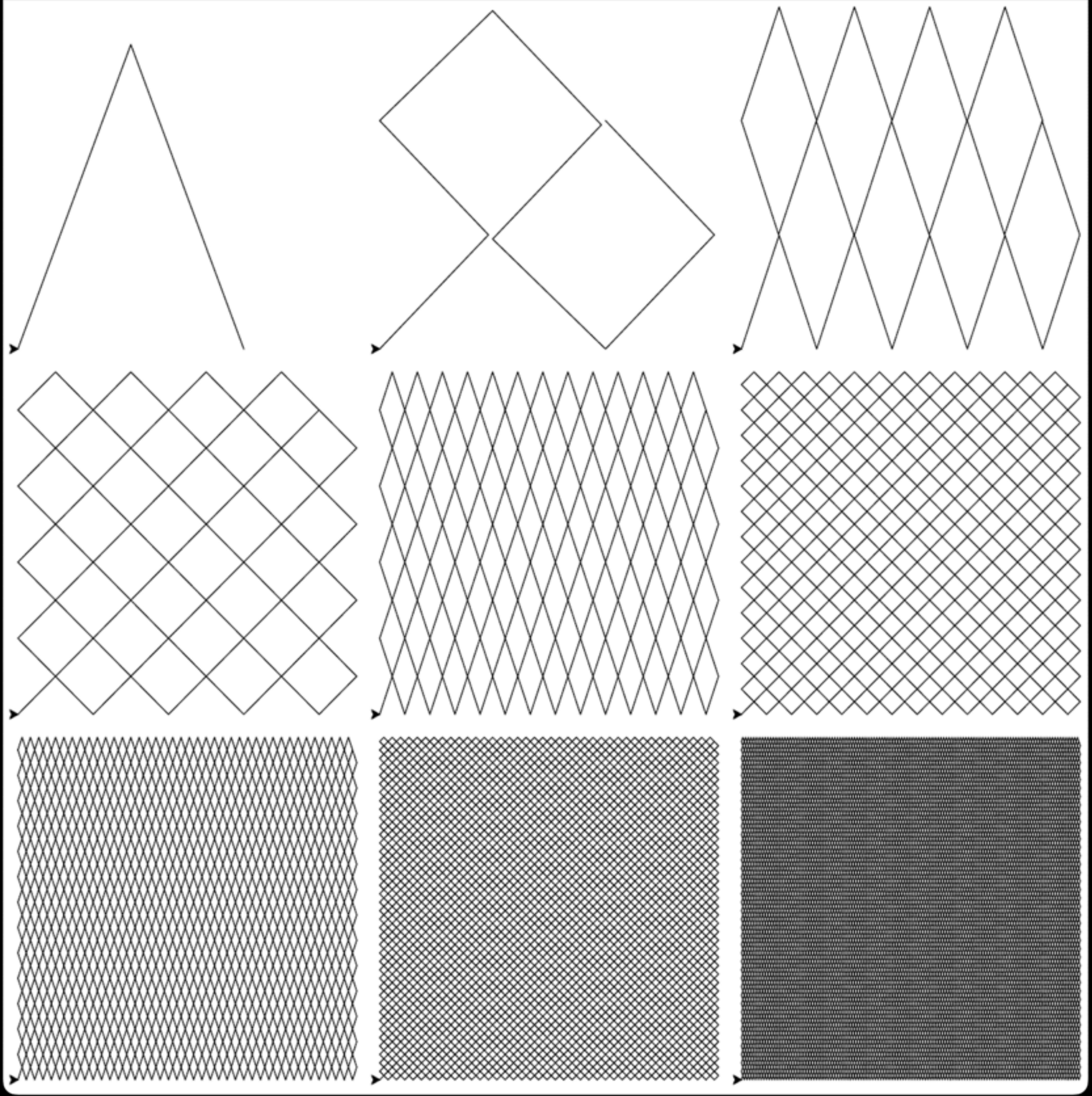
```
P E A N O for 🐍s & 👸s

                    . O .
```

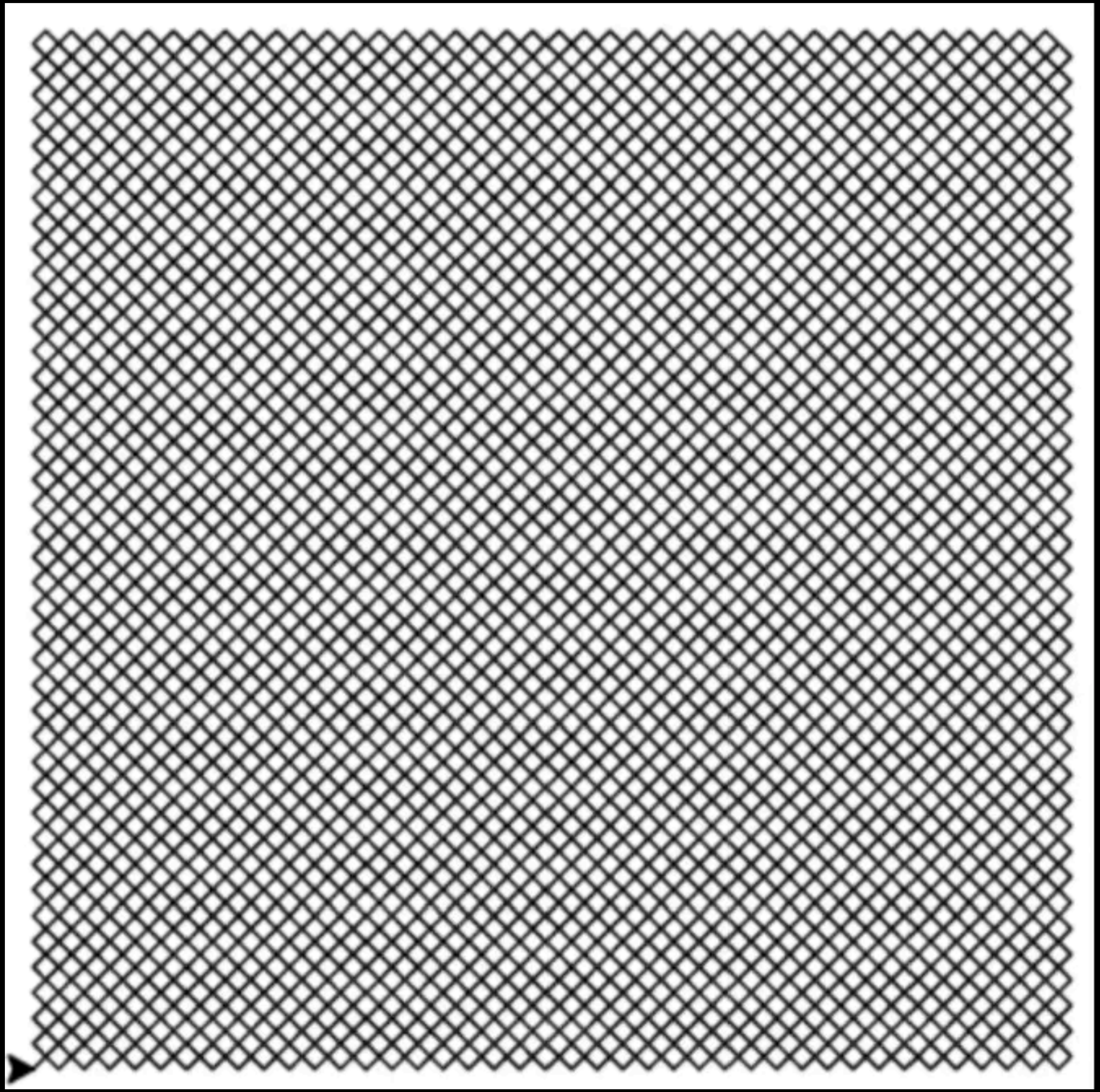
DEMO III (midi)


PEANO for 🐍s & 👱‍♀️s
PEANO for 🐍s & 👱‍♀️s
PEANO for 🐍s & 👱‍♀️s

DEMO IV (_make)

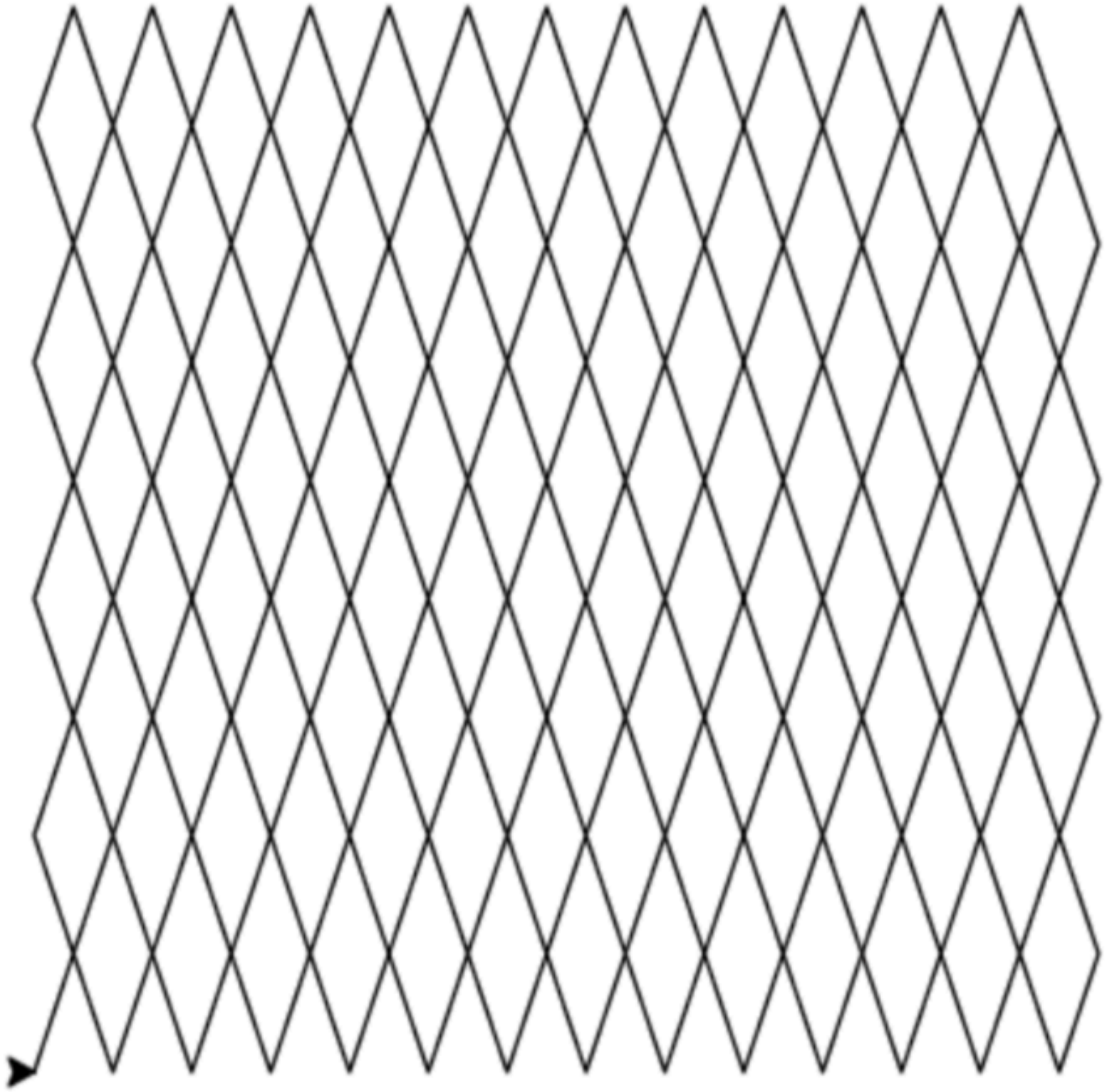
sprite.png





r-e-s-e-a-r-c-h.org

VIS 415, R-e-s-e-a-r-c-h *



VIS 415

February 6

In an 1890 paper, Italian mathematician Giuseppe Peano first described a plane-filling curve:

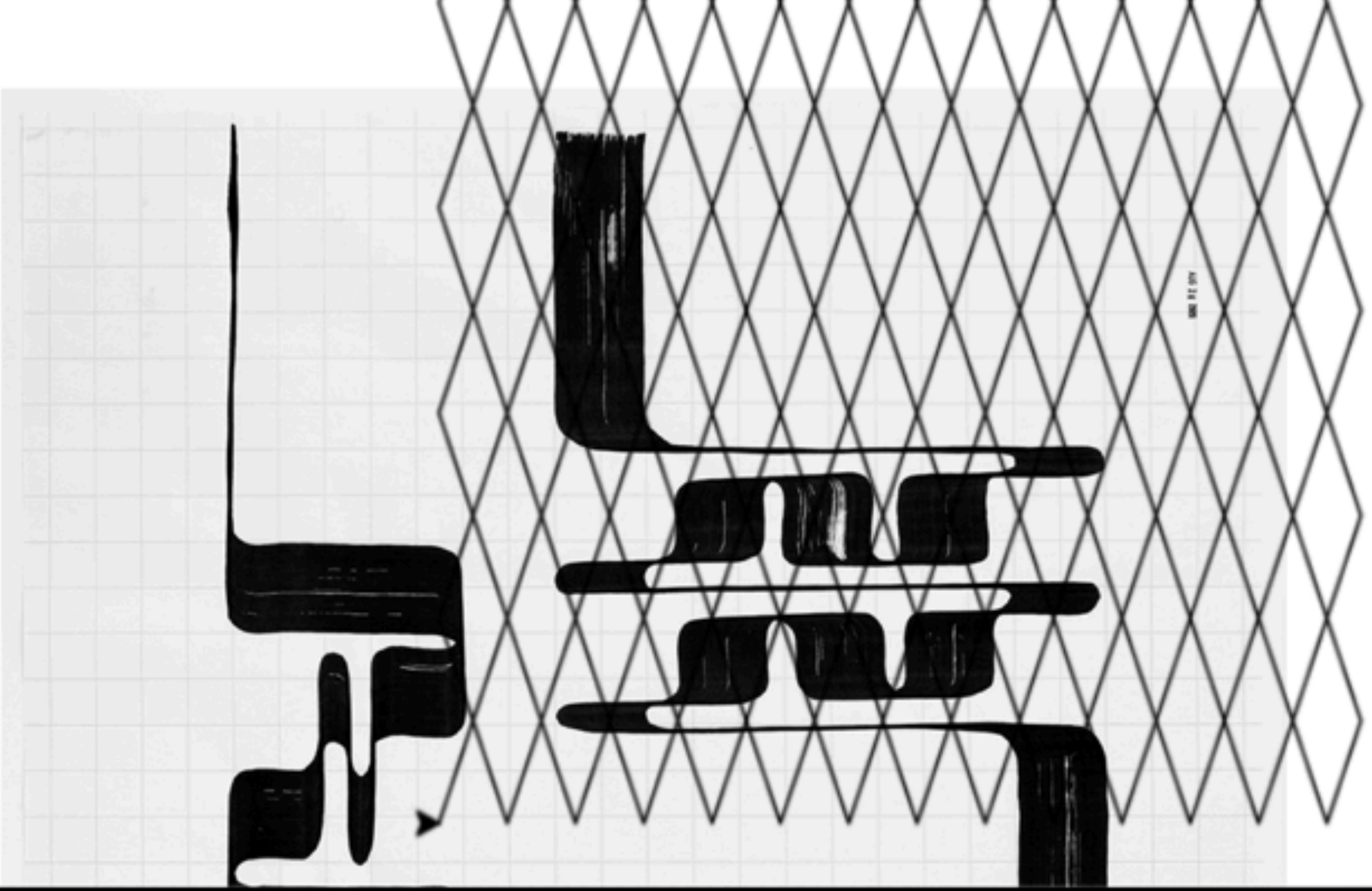
> There exists a continuous curve which goes through every point of a plane ... This result is of interest in the study of the foundations of geometry, for there does not exist a specific character which distinguishes a curve from a surface.

This curve was a radical proposal. Peano asserted that a continuous one-dimensional line could be constructed in a specific way so that it would wind its way through every point of a two-dimensional square.

This drawing is from a family of curves most often referred to as a "Peano curve," with which artist Tauba Auerbach has long been fascinated, incorporating them into their paintings, drawings, sculptures, and the occasional concrete poem.

Auerbach's Ligature Drawings are something close to a writing practice, using markers on oversized graph paper to produce bent lines which develop as they are drawn. The lines often return on themselves, often repeat, and inevitably connect. Most are stamped with the date and they pile up. Their abundance, the programmatic movement of the pen, and the meandering compactness of the lines recalls something from Peano's own description of his curve as

> a point in n-dimensional space, which is a continuous function of a real variable, or of time, such that the trajectory of the moving point fills the whole space.



EXERCISE

The Curve of Peano has been treated by various authors. We give it according to the geometrical construction of the fifth volume of the Formulaire de Mathematiques. Divide the interval [0, 1] into nine equal parts, and number them in order as 1,2,. . .,9. Then divide the square into nine equal parts as in Figure 1, and number them 1,2,. . .,9 to correspond with the segments of the linear interval. Next divide each segment of the straight line into nine equal parts and each of the nine squares into equal parts as in Figure 2. The 81 squares so formed are then numbered in order, so that each square has one side

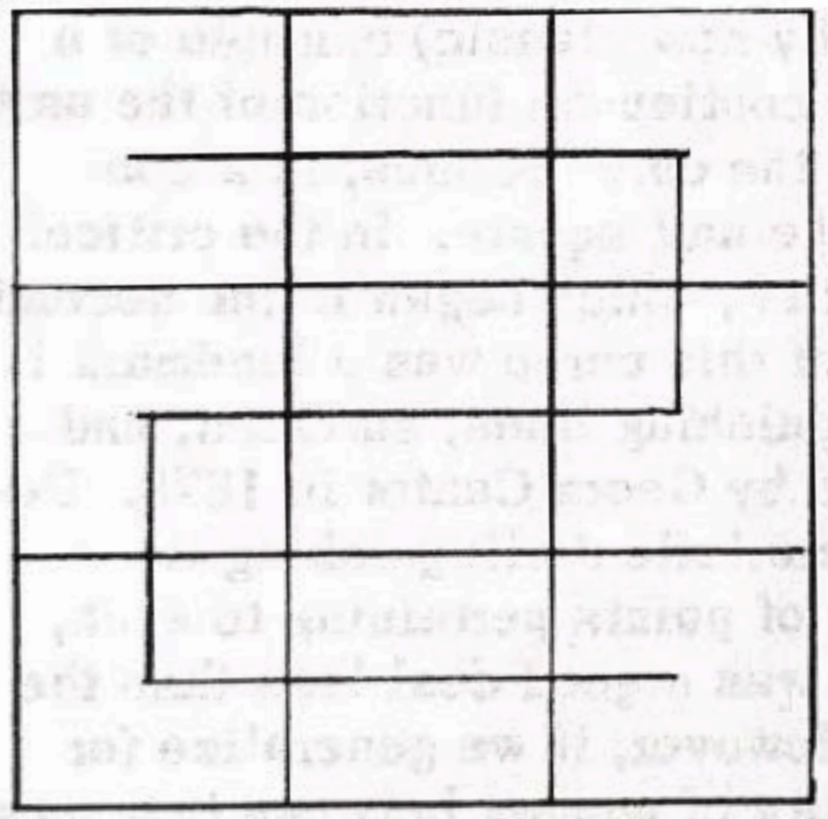


Fig. 1

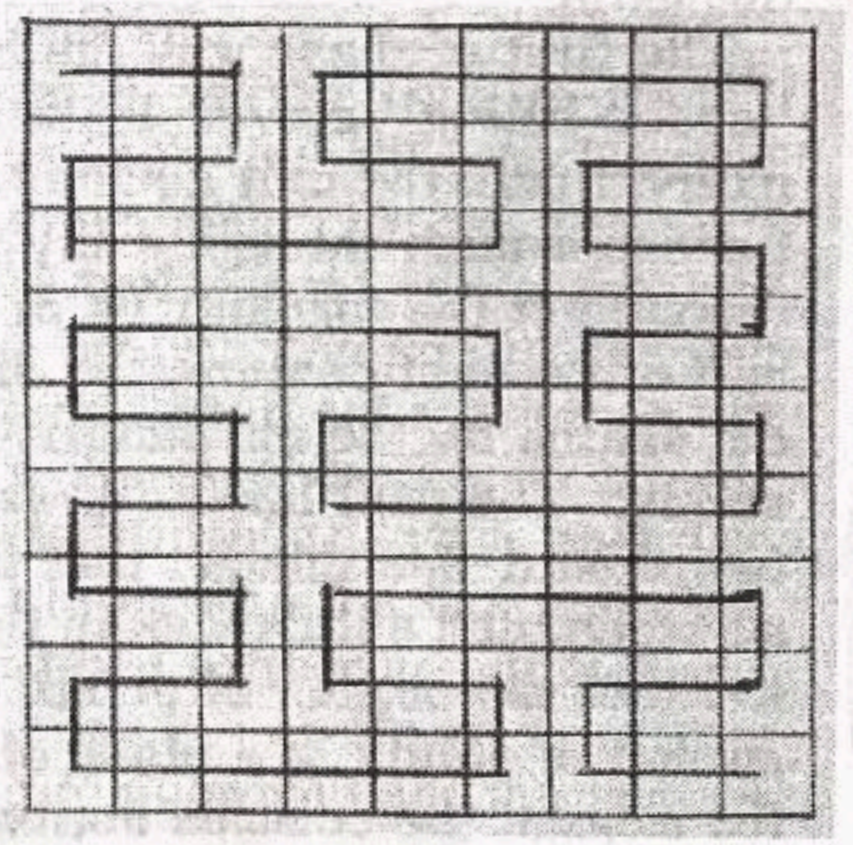


Fig. 2

The line to be mapped --- say, a straight line of length 1 --- we first divide into 4 equal parts 1, 2, 3, 4, and the area, which we assume to be in the form of a square with side length 1, we divide with two straight lines perpendicular to each other into 4 equal squares 1, 2, 3, 4; and each in turn [we divide again] into 4 equal parts, so that we get the 16 sections 1, 2, 3, …, 16 of the straight line; at the same time, each of the 4 squares is divided into 1, 2, 3, 4 equal squares and the resulting 16 squares are then ordered in such a way that each subsequent square is adjacent to the previous one. Let us imagine this procedure continued --- Fig. 3 illustrates the next step --- making it easy to see how to associate any given point on the

!

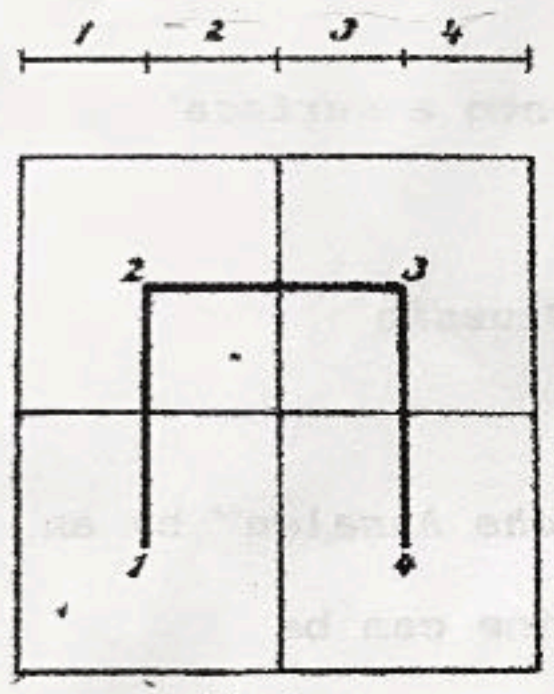


Fig. 1.

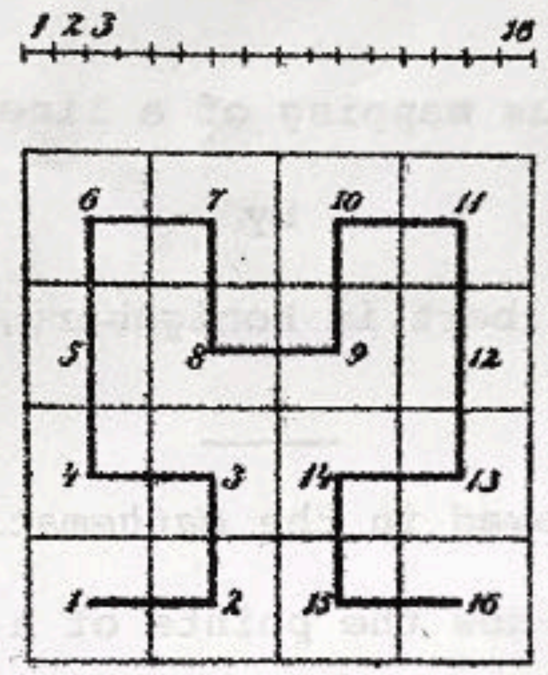


Fig. 2.

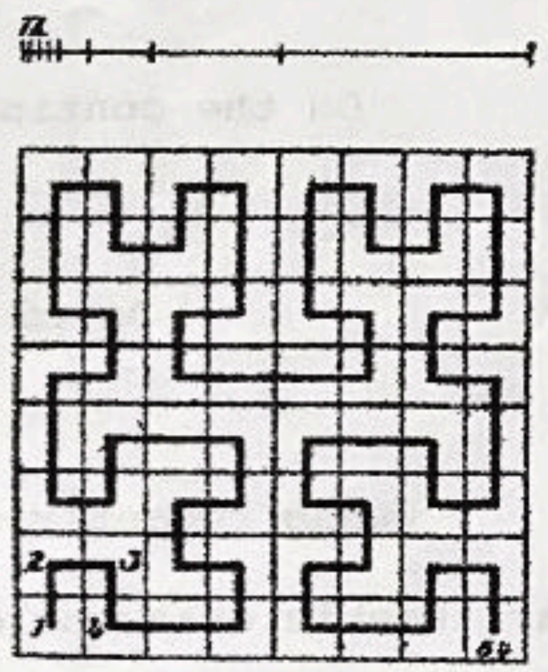
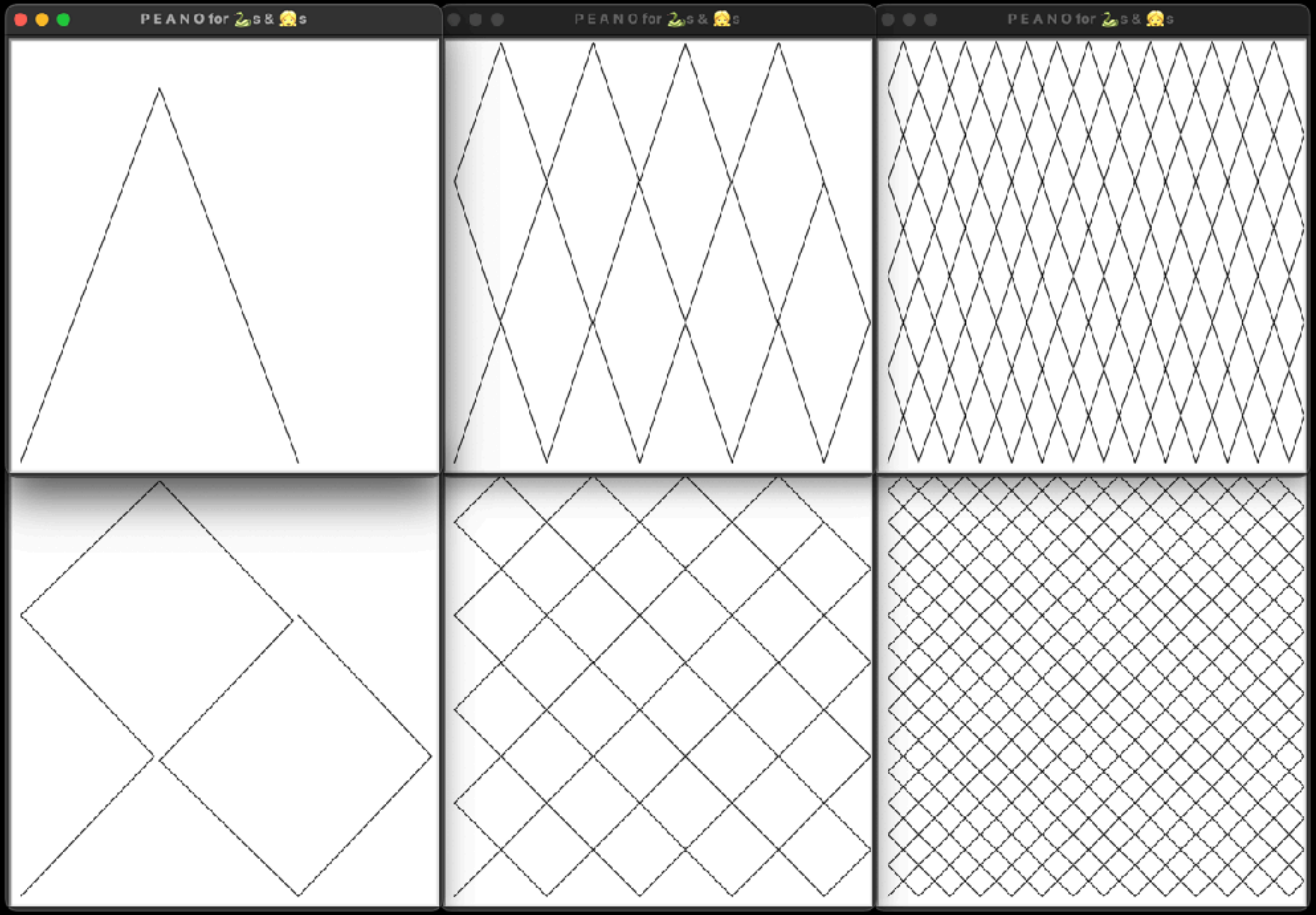
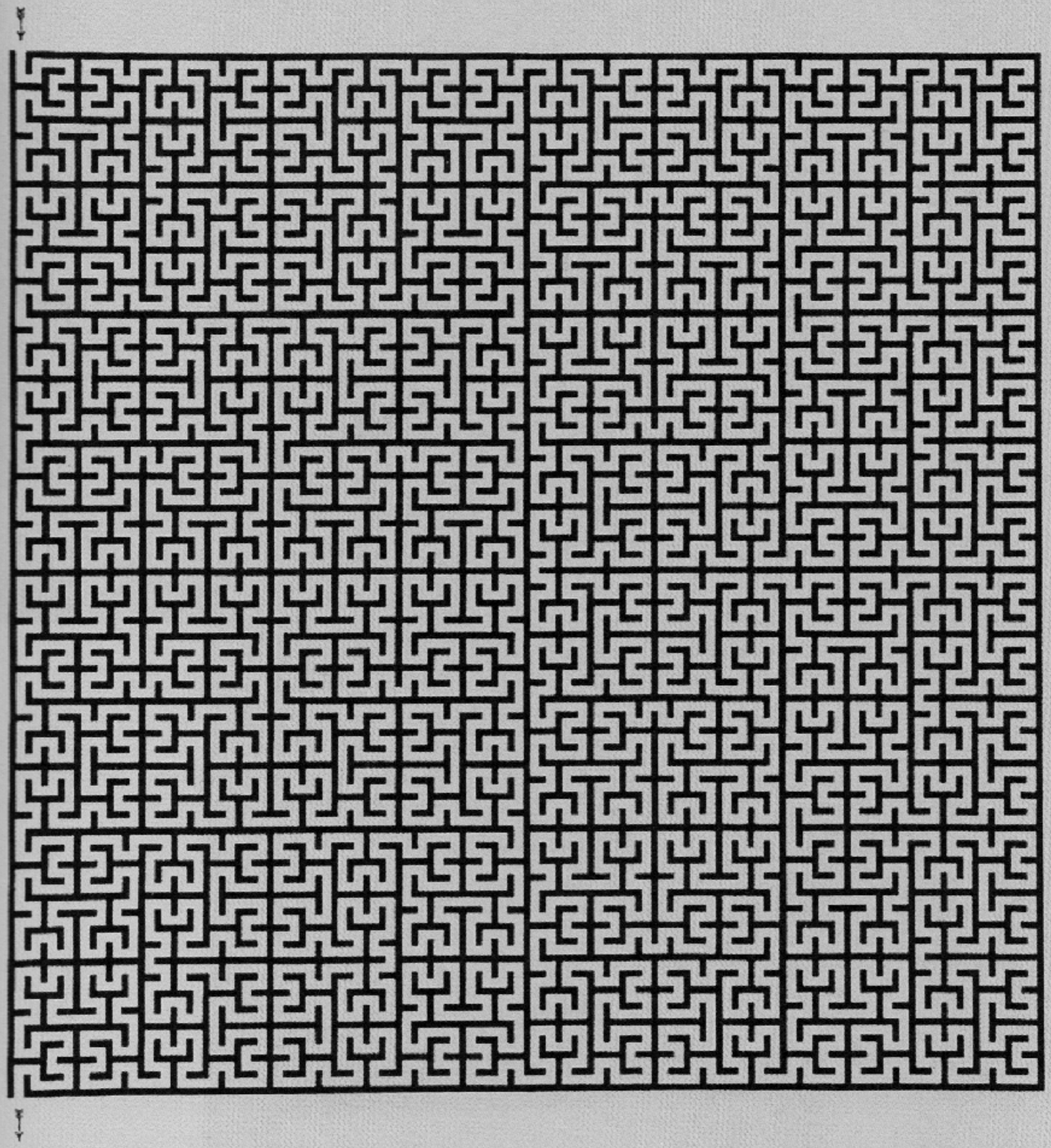


Fig. 3.

line with a single specific point on the square.


PEANO for 🐍s & 👱‍♀️s
PEANO for 🐍s & 👱‍♀️s
PEANO for 🐍s & 👱‍♀️s



ASSIGNMENT

Assignment 1,
Research. Reconsider. Repeat.

Choose a design subject. Make a collection of animated, printed, and physical graphics which help you understand it.

This is a design research project. It is open-ended, self-directed, and semester-length. The goal is to understand the process of design research by doing it. As it is a considerably open assignment, then steps and check-ins are provided en route.

Week 1: Collect nine design research subjects for a semester-length investigation. Post each to your are.na channel with visual references. Come to class prepared to discuss. Together, we will help you select one to work with.

Week 2: *Make something.* This could be as simple as translating the work from one media to another — copy it in pencil at 1:1 size; make a short video of the work moving between different scales; describe the work in your own words recorded as a short audio file; print it in black and white; print it at six different sizes; hang it upside down and make notes on how this changes the work; etc., etc., etc.

The most effective way to learn about a piece of design is by making something new. This is directly opposed to the research process that you might be familiar with which relies on previously written and recorded accounts. In this case, you want to know not so much *what is it?* but rather, *how does it work?* And so, an empirical approach will be best. Bring what you've made to class.

Weeks 3–4: Repeat Week 2 (*Make something.*) three times. In other words, make three new things in relation to your original. Each must be in a different media.

Using the first practical response as a guide, refine your inquiry. For example, try to understand color in the work. What happens if it is monochromatic? Or shifted to warm colors? Experiment with typography — copy the work but change the type. How does this alter the original? What do you know about when and how your subject was made? What were the tools of the time and how did they shape the work? Bring what you've made to class.

Weeks 5–7: Repeat Week 2 (*Make something.*) nine times. In other words, make nine new things in relation to your original. Each should be in a distinct media, at a distinct scale, or with a distinct approach.

All of your design research involves making new work. You will continue to do so. This is a design research project where thinking and making are collapsed into one fluid activity. As before, you should use what you made in Weeks 3–4 to reconsider your subject. Bring what you've made to class.

Weeks 8–10: At this point, you should be on design research autopilot, each work leading you to the next. You will continue to make new things. These can be quick and easy, or complex and involved. At this point, the requirements are very open. We will have individual and/or small group meetings to review your work each week.

It is also time to bring your design research together into a coherent form. You will prepare a 16-minute presentation to be delivered over Zoom to assembled visiting critics. This constitutes the documentation of your design research project. It will be recorded. Therefore, this presentation should be very carefully scripted, choreographed, rehearsed, and performed. You might present from a specific space in the building. You might be wearing specific clothing. You might enlist others to help in the presentation. You might consider lighting and sound. You will present the works you have made, which will necessarily be in different media, and so you should consider how to do this gracefully.

These videos conclude your investigation. They will be archived back onto this website for the future to reconsider.

The working method for this project should more accurately reflect the lateral thinking and meandering processes that are the stock and trade of a lively design practice. In your final video presentations, please speak about everything or most everything that you have made. This will be a lot, but it will also be important to convey the breadth and depth of your investigation.

February 6, 2023
Research. Reconsider. Repeat.

Resources
The Function of Esthetic Research (Enzo Mari)
Traffic: No way out (Enzo Mari)
"...meet the Tetracono"
M-u-l-t-i-p-l-c-i-t-y class videos
