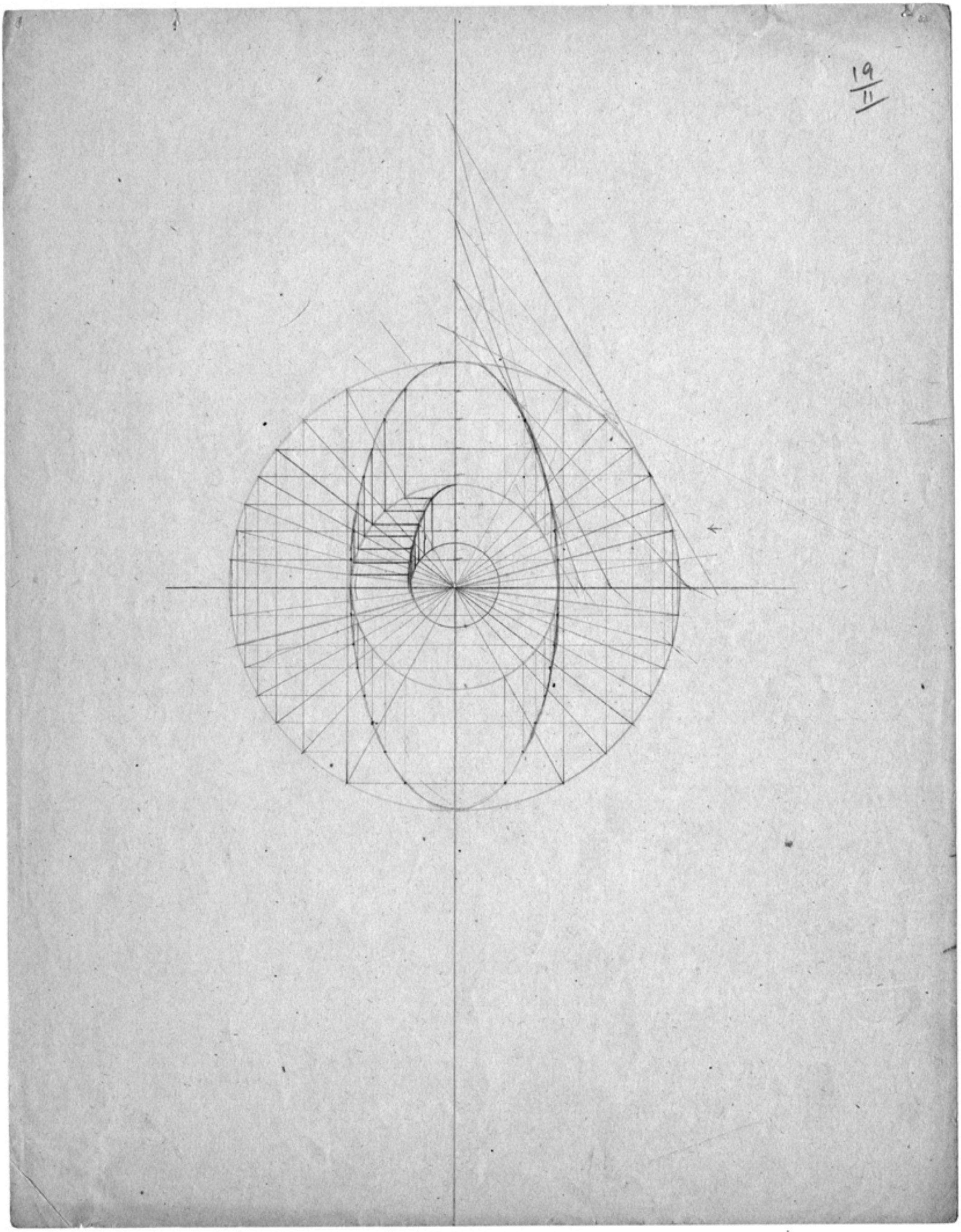
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There is a folder in the Max Dehn Papers at the University of Texas at Austin called "Drawings for Artists" containing mathematical figures drawn by Dehn and his students at Black Mountain College during the spring of 1948. There are conic sections, projective constructions, and double spirals called loxodromes. What's remarkable about the drawings is that they are constructed using incidence relationships rather than metric ones; there is no unit of measurement and no algebra. The points of the beautifully penned ellipse on the cover of this bulletin emerge not by coordinates determined by an equation, but rather by the intersections of the construction lines produced by the interaction of a pair of concentric circles.

Photographs on p.38 of Max Dehn by Eva Dehn, and of William Thurston by Paul Halmos

Cover image: Ellipse from "Drawings for Artists," 1948.
Courtesy Max Dehn Papers, Archives of American Mathematics, The Dolph Briscoe Center for American History, The University of Texas at Austin

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Recently someone told me that there is an island on a lake in an island on a lake in an island in Canada. I can't quite picture this; the triple island remains foggy in my mind's eye, surrounded by five (or are there six?) receding concentric shores. If I imagine it in reverse — in Canada there is an island with a lake that contains an island with a lake that contains an island — the arrangement is slightly more stable. The picture's resolution increases dramatically, however, if I attach artificial details to the triple island construction. For example, I have no trouble picturing a turtle sunning itself on a rock in the middle of a rain puddle on Turtle Island on Turtle Pond in Central Park, Manhattan, despite the fact that the configuration is identical to the unnamed Canadian island.

Mathematics is full of nested spaces like these, chains of abstraction, and seemingly unfathomable limiting situations. Nevertheless, mathematicians often find ways to inhabit the abstractions they construct. This orienteering is usually omitted from the published accounts of mathematical work, but on occasion one is lucky to find a sentence that begins like this:

In order to have a definite intuition as a basis for the following discussion we think of...

A "definite intuition" is one of the unusual by-products of doing mathematics. It's not an oxymoron. People conceive mental models for thinking about problems outside of mathematics, of course. What distinguishes the internal maps of mathematicians may be their specificity, logic, and detail, but also the fact that they have no referent in the world of sense experience. The quote comes from a 1910 paper "On the topology of three-dimensional space" by the German mathematician Max Dehn. Dehn was exceptionally effective at both devising and communicating visual intuitions for mathematics, and his writings offer an unusually vivid depiction of a mathematician's definite intuition. This is due in part to the kinds of problems he chose to work on, many, but not all, of which were geometric in nature. It is also due to his personality. Dehn was, by all accounts, a reflective person, and he studied the history, philosophy, and psychology of mathematics throughout his life with almost the same seriousness as he did mathematics proper. His characteristic humility notwithstanding, Dehn acknowledged the fundamental role of geometric thinking for mathematics:

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The true foundations of mathematics is not only the object of pure thought in Plato's beautiful belief but also the ordinary, if somewhat refined, visualization that the creator of the realm of ideas was somewhat contemptuous of.

Dehn was born in Hamburg in 1878. His mathematical career began at a critical period in the development of mathematics and, in particular, geometry. The concept of symmetry, central to geometry and to mathematics as a whole, evolved during the 18th century, and is often equated with the emergence of modern mathematics. In common usage, symmetry refers to a correspondence between the parts of a figure or pattern. The four sides of a square are equal in length, all points of a circle are at the same distance from the circle center, the left hand mirrors the right hand, etc. A major development in our understanding of symmetry occurred when the correspondence was conceived of as an OPERATION on the figure, rather than a PROPERTY of it. For example, the equality of the lengths of the sides of a square can be expressed as the invariance of the square under a quarter turn. This represents a crucial shift away from a static, figurative conception of symmetry and towards a dynamic, operative one.

With the notion of a symmetry operation two natural possibilities arise. First, there is the possibility of composing two symmetry operations, one after another, and, conversely, decomposing a symmetry operation into simpler components. This gives rise to the concept of a "group," a major focus of Dehn's work. Second, there is the possibility of interpreting any operation on a space as a formal symmetry, be it pure, pleasing, or otherwise. For instance, you might see what happens by extending the symmetries of a square to include arbitrary changes of scale in each coordinate direction. This abstract description becomes more familiar if you read "arbitrary changes in scale in each coordinate direction" as "foreshortening." These symmetries constitute none other than the linear perspective by which, for example, a small square and a large rectangle correspond to identical shapes, merely viewed from distinct vantage points.

What geometric properties, if any, are preserved under this expanded notion of symmetry? Perspective transformations, for one, do not preserve distance nor angle, as we've already seen. What could possibly

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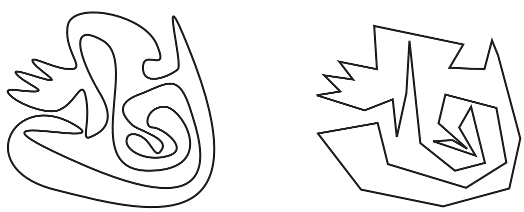
remain of a geometry without distance and angle measure? One answer is incidence. If two lines intersect, then they will still intersect no matter from where we view them. Questions about configurations of incident lines date back to the Greeks, but it wasn't until much later that the mathematical study of perspectival symmetry, projective geometry, took form. By the time Dehn began his doctoral studies in 1899 at Göttingen University, projective geometry had emerged to take a central and unifying position among the bewildering variety of non-Euclidean geometries appearing in the 19th century. For Dehn, these new geometries brought an intelligence to mathematics implicit in sensory experience.

It has become more and more apparent that different systems can be visualized, that, say, different types of spaces are compatible with experience. This is not to imply that the mathematician can now choose his assumptions at will. Not only is such arbitrariness likely to result in developments without beauty, but it is also likely to lead to contradictions that make all of the work an illusion.

Dehn's first mathematical accomplishment was to demonstrate something that would seem to be compatible with experience from kindergarten. If you doodle on a piece of paper in such a way that you never pick up the crayon and you never cross over a mark you've already made except to join the end to the start, then the drawing will separate the paper into two regions: inside the doodle and outside the doodle. A continuous closed curve in the plane that satisfies this no-self-intersection condition is called a Jordan curve. The curve's namesake, French mathematician Camille Jordan, was the first to publish a proof that this curve separates the plane into two regions, but he left the case of polygonal curves (formed by connecting lines) as an exercise for someone else.

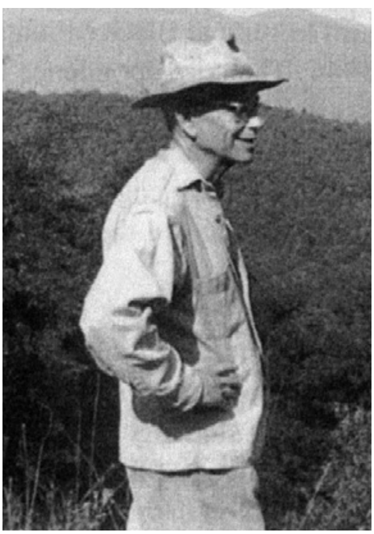


At the suggestion of his advisor, David Hilbert, Dehn proved the Jordan curve theorem in this remaining case; that is, he showed that non-selfintersecting polygons separate the plane in two, too. Given the variety of figures among the infinity of Jordan curves, it should not come as a surprise that Jordan's proof is long and difficult, despite its obvious truth. What's remarkable about Dehn's proof is that it is a fraction of the length of Jordan's, and that he found it using just a few of the most basic axioms of planar geometry, the incidence relations.

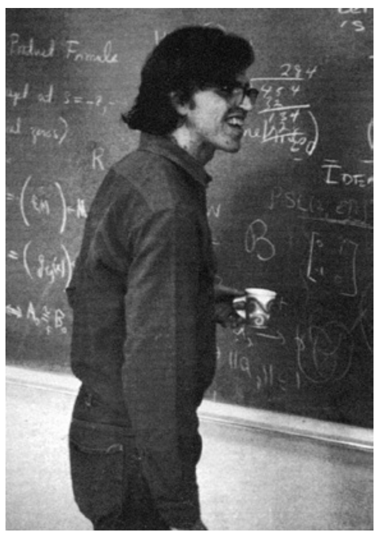
Both Max Dehn's mathematical work and attitudes about mathematics anticipate the influential contemporary geometer William Thurston. Thurston was born in 1946 in Washington D.C., studied at the University of California, Berkeley, and held positions as professor of mathematics at Princeton, MIT, UC Davis and most recently at Cornell, until his untimely death due to cancer last year. Thurston's extraordinary visual intuition allowed him to visualize complex shapes and problems.

"People don't understand how I can visualize four or five dimensions," Thurston told *The Wall Street Journal* in 1983, "Five-dimensional shapes are hard to visualize—but it doesn't mean you can't think about them. Thinking is really the same as seeing." Elsewhere he wrote:

Personally, I put a lot of effort into "listening" to my intuitions and associations, and building them into metaphors and connections. This involves



Max Dehn



William Thurston

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a kind of simultaneous quieting and focusing of my mind. Words, logic, and detailed pictures rattling around can inhibit intuitions and associations.

Both Dehn and Thurston received a similar mixture of praise and criticism for their intuitive approach to mathematics. The mathematician and English translator of Dehn's work, John Stillwell writes, "Dehn's intuition was his greatest strength, and on occasion it exceeded his ability to give rigorous proofs." Compare that to this description of one of Thurston's major theorems: "A grand insight delivered with beautiful but insufficient hints, the proof was never fully published. For many investigators this unredeemed claim became a roadblock rather than an inspiration."

Thurston addressed his critics head on in an article called "Proof and Progress in Mathematics," which argues that this critique misrepresents all mathematical activity as a one-dimensional march from definition to theorem to proof. Mathematics is more of a mental wrestling match and proceeds from fits and starts of speculation. It also comprises the work of communicating the understanding of the objects of mathematic study. Like Dehn, Thurston refuses a strictly Platonic view of mathematics as pure, idealized logic and instead points to its knottier and more quotidian thought patterns.

Dehn's remark that "different systems can be visualized" reflects a prominent value in German mathematics at the time, expressed by the word *Anschauung*. According to a contemporary encyclopedia, the proper word refers to "the perception by the sense of sight and simultaneously the mental conception of an object obtained in this way; in the wider sense generally the immediate cognition of an object in opposition to cognition by thought or that mediated by concepts; the former is also called intuitive, the latter discursive cognition." Mathematicians, like Dehn, who made appeals to their reader's visual intuition, tended to be critical of techniques overly reliant on symbolic computation. Mathematics is more than a game of manipulating meaningless symbols according to arbitrary rules, as proponents of logic and formalism would have it.

Ironically, the greatest proponent of formalism in the strict sense was none other than Dehn's doctoral advisor, David Hilbert. Dehn himself rose to prominence when he succeeded in solving a problem related to

Hilbert's program of formalizing the axioms of Euclidean geometry. This was the third among 23 problems that Hilbert posed for the International Congress of Mathematicians in Paris in 1900. The problems are marked both by their range and also their directness. They touch on number theory, geometry, mathematical physics, topology, algebra, analysis, and foundations, and yet several problems can be stated in terms of high school mathematics. Hilbert's emphasis on clarity, simplicity, and intuition no doubt had a lasting influence on his student, Dehn:

The taste for generalization, for ever more comprehensive concepts, has made us construct things that are very far removed from common intuition. [...] It turned out that, while it is possible to obtain beautiful results in this new realm of pure thought, these constructions lead to contradictions. A complete resolution of the issue is very unlikely for a variety of reasons. I myself think it may be that the reason for this difficult state of affairs is that in the case of very general conceptual constructs there arise contradictions between geometry — the world of extensive magnitudes — and the world of counting, and that analysis — the bridge between the two — will surely remain completely free of contradictions as a long as it is intuitive, that is, as its concepts are at the same time intuitive geometric concepts.

To Dehn, visualization is not merely a "separate, powerful [mental] facility important for mathematical thinking," as Thurston would have it. Instead, visually intuitive arguments are an essential condition for mathematical progress. The role of visualization and spatial intuition in mathematics is, according to him, critical. Geometric intuitiveness is a hallmark of good, lasting mathematics — the kind of mathematical product that will stand up to scrutiny.

At this point, you might be under the impression that a definite mathematical intuition pertains to strictly geometric problems expressed in spatial terms. This was not the case for Dehn, who counted algebra among the "different systems that can be visualized." Unlike geometry, which relies on definite spatial relations, algebra concerns generic abstract structures. Its deductions rely on logical chains of symbolic reasoning according to a fixed set of rules of deduction: one thing stands for something in a definite relation to something else and so on and so on.

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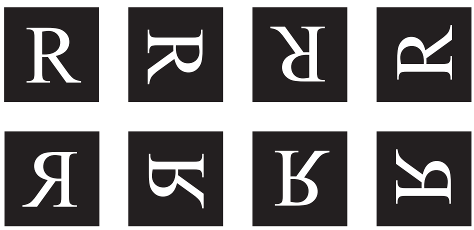
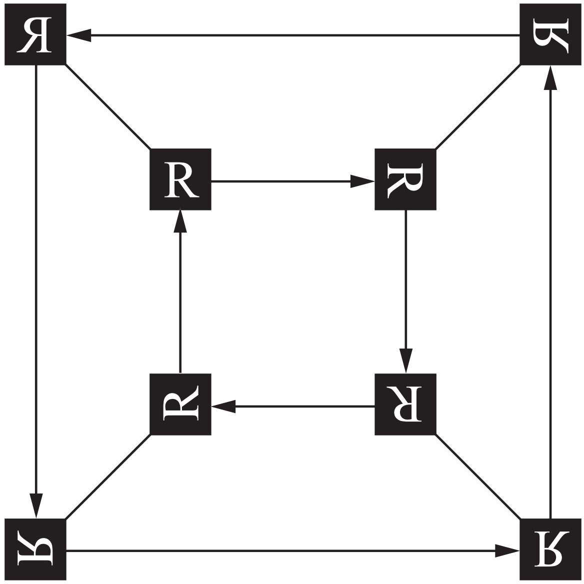
For example, a group is a basic object of modern algebra, and Dehn made significant contributions to the theory of groups. The early 19th-century French mathematician Évariste Galois first conceived of the group in terms rather independent from geometry: A group constitutes the algebra of symmetry operations. There certainly are groups of symmetries of geometric figures, but the structure of the operations, not the figure itself, is the object of study. For example, the group of symmetries of the square consists of four rotations (90, 180, 270, and 360 degrees) and four reflections (about the two diagonals, a horizontal axis, and a vertical axis). A typical question for a group is to find a subset of its elements that, by composition, reproduce or generate the entire group. One answer is that the symmetries of a square may be generated by a 90 degree rotation and a reflection in the vertical axis, like this:



Dehn developed a very effective way of working with groups in visually intuitive terms, using what he called *Gruppenbilder* (group diagrams). A group diagram is a network of points and connecting lines that graphically depicts the elements of a group and the relations between them. To construct a group diagram, one needs a generating set of elements for the whole group. If two elements differ by a single generator, then the points representing them in the group diagram are connected by a line which marks the transformation. For example, consider the symmetry group of the square from above with a generating set given by the 90 degree rotation and the reflection across a vertical axis. In this case, the two points in the group diagram that represent rotation by 180 degrees and rotation by 270 degrees would be connected, since the two rotations differ by 90 degrees, and that operation is a group generator. Two points which represent mirror images of each other under reflection in a vertical line would likewise be joined in the diagram. An arrow represents the rotation and a line marks reflection — the entire group diagram looks like this:



Dehn was not the first mathematician to devise a graphical representation of a group, and in fact, today we call such diagrams Cayley graphs, after the 19th-century English mathematician Arthur Cayley. But Dehn clarified and extended the conception of group diagrams so that they became a point of view in and of themselves. Dehn's student Wilhelm Magnus remembers that when, after six months, Magnus solved a problem Dehn had given him, Dehn said, "Ah, then you could use the Gruppenbild." When told that the proof was purely algebraic, Dehn said: "Da sind Sie also blind." ("Well then, you are blind.")

Dehn used the Gruppenbild to solve an important problem in the study of mathematical knots, a problem that, like the Jordan curve theorem, has a childlike appeal. Take a pipe cleaner, a long bright blue one from a craft store. Tie it loosely into an overhand knot, as if you were starting to tie a shoelace. You could use a shoelace, but I want something that holds its shape in your head. Now bring the free ends together and twist together a little bit of each wire end so the whole pipe cleaner makes a continuous knotted loop. What you have is a model of the simplest non-trivial mathematical knot—a trefoil.

Like a Jordan curve, a trefoil can take infinitely many different shapes. Bending the pipe cleaner this way and that, using a longer or smaller pipe

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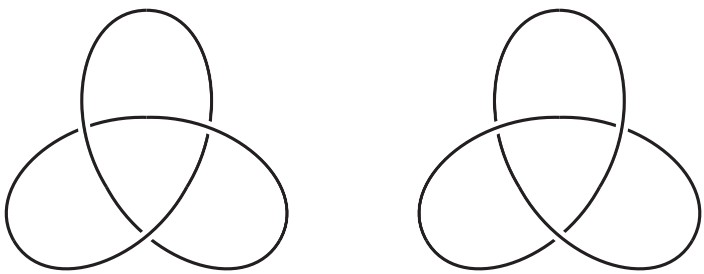
cleaner, etc., result in different-looking trefoils. But these are only changes of appearance; as long as you don't break and rejoin the loop, it's still the same trefoil. Topologists prior to Dehn had proved that a trefoil could never be flattened out into a circle. Even if you had an infinite amount of time and infinite ingenuity, you will never be able to rearrange a trefoil into a circle without cutting it. This is just to say that a trefoil really IS knotted.

What mathematicians were not able to resolve until Dehn, was whether or not there were actually two trefoils. If you hold the blue pipe cleaner trefoil up to a mirror, you see another trefoil. Is the mirror image an intrinsically different knot? Or is the difference merely one of appearance? I don't know if early knot theorists cleaned pipes, but somehow they convinced themselves that no manner of deformation could ever bring a trefoil into the shape of its mirror image. Indeed there ARE two trefoils — a right-handed trefoil and a left-handed trefoil — and Dehn was the first to rigorously prove it.



What he did was first to translate the geometric question into an equivalent algebraic question, in terms of a group naturally associated with a knot. Dehn then translated the question from the knot group back into a visual form by studying its Gruppenbild. Finally, by analyzing the symmetries of this Gruppenbild, he was able to rule out the possibility of a continuous deformation that would transform a trefoil to its mirror image.

. . .

In 1935 Max Dehn was forced from his professorship in Frankfurt into early retirement by the Nazis. Dehn was a Jew, but due to his prior service in WWI he was allowed at first to remain in Germany and receive his pension. Of course, the situation deteriorated, and, as described in

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John W. Dawson Jr.'s fascinating account, "Max Dehn, Kurt Gödel, and the Trans-Siberian Escape Route," Dehn narrowly escaped by the train to China before arriving in the United States. After briefly holding positions at Idaho Southern University, Illinois Institute of Technology, and St. John's College, Dehn landed in 1944 at Black Mountain College, North Carolina. The progressive, experimental liberal arts school was founded in 1933, and its faculty and students included some of the most eminent American artists and writers of the 20th century. Dehn was the only mathematician on the faculty, and he taught Greek, Latin, and philosophy in addition to mathematics. Dawson writes that, "among the documents preserved in the archives of the college is a letter Dehn wrote from Chicago on July 13, 1946, thanking the board of BMC for granting his upcoming leave, expressed the hope that when he returned there later that summer there would be 'some nice work' for him to do, such as 'geometry for artists or hoeing potatoes.'"

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SOURCES

— John W. Dawson, Jr., "Max Dehn, Kurt Gödel, and the Trans-Siberian Escape Route," *NOTICES of the American Mathematical Society*, vol. 49, no. 9 (2002), pp. 1068–1075
— Max Dehn, "Über die Topologie des dreidimensionalen Raumes" ("On the topology of three-dimensional space"), *Mathematische Annalen* 69 (1910), pp. 137–168
— Max Dehn, "Die beiden Kleeblattschlingen" ("The two trefoil knots"), *Mathematische Annalen* 75 (1914), pp. 402–413
— Max Dehn, "Über die geistige eigenart des Mathematikers" (reprinted as "The Mentality of the Mathematician. A Characterization"), Frankfurter Universitätsreden, XXVIII; *The Mathematical Intelligencer*, 1928, 1983
— Arthur Jaffe and Frank Quinn, "'Theoretical mathematics': Toward a cultural synthesis of mathematics and theoretical physics" *Bulletin of the American Mathematical Society*, vol. 29, no. 1 (January 1993), pp. 1–13
— Leslie Kaufman, "William P. Thurston, 65, Math Innovator, Is Dead," *New York Times* (August 23, 2012), p. B15
— Herbert Mehrtens, "Mathematical Models," in *Models: The Third Dimension of Science* (Stanford: Stanford University, 2004)
— R. B. Sher, "Max Dehn and Black Mountain College," *The Mathematical Intelligencer*, vol. 16, no. 1 (1994), pp. 54–55
— John Stillwell, "Max Dehn" (unpublished biography held in the Max Dehn archives at the University of Texas, Austin)
— William P. Thurston, "On Proof and Progress in Mathematics" *Bulletin of the American Mathematical Society*, vol. 30, no. 2 (April 1994), pp. 161–177
— Benjamin Yandell, *The Honors Class: Hilbert's Problems and Their Solvers* (Natick, MA: AK Peters, 2001)

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