Université de Montréal

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IFT-6155 Quantum Computing

Assignment 4

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Presented to
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Question 1

Exercise 3.4.3 Find a classical input to the above circuit that produces $|\Phi^+\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ as output. Determine which quantum states are produced when the other two pairs of classical bits are used as input.

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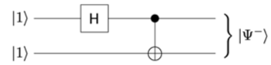
To prove that the quantum exclusive-or gate cannot be factored into a tensor product of two one-qubit transformations, it suffices to show that it can be used to create entanglement out of product states. For this, start with state $|11\rangle$, apply the Hadamard transform to the first qubit, and then apply the quantum exclusive-or to both qubits.



The joint state of both qubits after the Hadamard transform is

$$(\mathsf{H}|1\rangle) \otimes |1\rangle = \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes |1\rangle = \tfrac{1}{\sqrt{2}}|01\rangle - \tfrac{1}{\sqrt{2}}|11\rangle\,.$$

The effect of the quantum exclusive-or is to leave $\frac{1}{\sqrt{2}}|01\rangle$ undisturbed but map $-\frac{1}{\sqrt{2}}|11\rangle$ to $-\frac{1}{\sqrt{2}}|10\rangle$. Therefore this circuit produces entangled state

$$\tfrac{1}{\sqrt{2}}|01\rangle - \tfrac{1}{\sqrt{2}}|10\rangle = |\Psi^-\rangle\,.$$

A) Classical Input to the above circuit that produces $|\Phi^+\rangle = (1/\sqrt{2})\,|00\rangle + (1/\sqrt{2})\,|11\rangle$ as output.

- Classical Input: **$|0\rangle$ & $|0\rangle$**

- Start with state $|00\rangle$
- Apply the Hadamard transform to the first qubit
- Then apply the quantum exclusive-or to both qubits.

$(H|0\rangle) \otimes |0\rangle = ((1/\sqrt{2})\,|0\rangle + (1/\sqrt{2})\,|1\rangle) \otimes |0\rangle = (1/\sqrt{2})\,|00\rangle + (1/\sqrt{2})\,|10\rangle$

- Then, with CNOT, we leave $(1/\sqrt{2})\,|00\rangle$ undisturbed but map $(1/\sqrt{2})\,|10\rangle$ to $(1/\sqrt{2})\,|11\rangle$.
- We therefore get $(1/\sqrt{2})\,|00\rangle + (1/\sqrt{2})\,|11\rangle$, which is $|\Phi^+\rangle$.

B) Which states are produced with $|01\rangle$ and $|10\rangle$.

- For the state $|01\rangle$:

$(H|0\rangle) \otimes |1\rangle = ((1/\sqrt{2})\,|0\rangle + (1/\sqrt{2})\,|1\rangle) \otimes |1\rangle = (1/\sqrt{2})\,|01\rangle + (1/\sqrt{2})\,|11\rangle$

With CNOT, we get: **$(1/\sqrt{2})\,|01\rangle + (1/\sqrt{2})\,|10\rangle$**... which is $|\Psi^+\rangle$.

- For the state $|10\rangle$:

$(H|1\rangle) \otimes |0\rangle = ((1/\sqrt{2})\,|0\rangle - (1/\sqrt{2})\,|1\rangle) \otimes |0\rangle = (1/\sqrt{2})\,|00\rangle - (1/\sqrt{2})\,|10\rangle$

With CNOT, we get: **$(1/\sqrt{2})\,|00\rangle - (1/\sqrt{2})\,|11\rangle$** ... which is $|\Phi^-\rangle$.

- For the state $|11\rangle$:

$(H|1\rangle) \otimes |1\rangle = ((1/\sqrt{2})\,|0\rangle - (1/\sqrt{2})\,|1\rangle) \otimes |1\rangle = (1/\sqrt{2})\,|01\rangle - (1/\sqrt{2})\,|11\rangle$
With CNOT, we get: **$(1/\sqrt{2})\,|01\rangle - (1/\sqrt{2})\,|10\rangle$**... which is $|\Psi^-\rangle$.

Question 2

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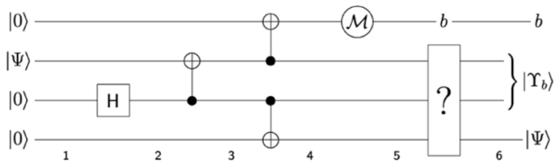
Question 2. Soit $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ un état quelconque d'un qubit, avec $|\alpha|^2 + |\beta|^2 = 1$ bien sûr. Considérez le circuit suivant.



a) Donnez l'état $|\Psi_i\rangle$ des 4 qubits en chacun des points i du circuit, pour $1 \leq i \leq 4$ (nous passerons aux points 5 et 6 par la suite). Bien que $|\Psi_i\rangle$ soit séparable (totalement ou partiellement) en certains de ces points, exprimez-le sous forme explicite de superposition des 16 états de base ($|0000\rangle$, $|0001\rangle$, $|0010\rangle, \ldots, |1111\rangle$), en omettant bien entendu les états d'amplitude zéro.

b) Récrivez explicitement votre état $|\Psi_4\rangle$ sous la forme $\gamma|0\rangle|\Lambda\rangle + \delta|1\rangle|\Gamma\rangle$, où $|\gamma|^2 + |\delta|^2 = 1$ et $|\Lambda\rangle$ et $|\Gamma\rangle$ sont des états légitimes sur trois qubits.

c) Pour passer au point 5 du circuit, mesurez le premier qubit de $|\Psi_4\rangle$ (il s'agit donc une mesure partielle). Donnez les probabilités p_0 et p_1 d'obtenir $b = 0$ ou $b = 1$ comme résultat classique de la mesure, respectivement. Pour chacun des deux résultats classiques possibles b, donnez l'état quantique résiduel $|\Phi_b\rangle$ des trois qubits non mesurés.

d) Donnez explicitement un sous-circuit mystère $\boxed{?}$, fonctionnant sur les trois qubits non mesurés, dont l'effet est de restaurer le $|\Psi\rangle$ de départ dans le dernier qubit. Votre circuit doit fonctionner indépendamment du fait qu'il travaille sur $|\Phi_0\rangle$ ou sur $|\Phi_1\rangle$, *sans connaître au préalable le résultat b de la mesure.*

Note : Ne cherchez rien de trop compliqué : le circuit demandé peut se réduire à deux portes ou-exclusifs (alias portes de négation contrôlée) insérées dans le diagramme ci-dessous.

Que vaut $|\Upsilon_b\rangle$ en fonction de b ? Faites le calcul explicite qui mène de $|\Phi_b\rangle$ à $|\Upsilon_b\rangle|\Psi\rangle$ pour chacune des deux valeurs possibles de b.

Attention : Le dernier qubit ne doit **pas** finir intriqué avec les autres.

e) (optionnel) À quoi ce circuit pourrait-il bien servir ?
Indice: Mettez Alice à gauche du circuit et Bob à droite. . .

- Let $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ a state of a certain qubit, with $|\alpha|^2 + |\beta|^2 = 1$.

a)

- Give the state $|\Psi_i\rangle$ of the 4 qubits at each point i of the circuit, for $1 \leq i \leq 4$.

- For i = 1:

$|\Psi_1\rangle = |0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) \otimes |0\rangle \otimes |0\rangle$
$|\Psi_1\rangle = |0\rangle \otimes (\alpha|000\rangle + \beta|100\rangle)$
$|\Psi_1\rangle = \alpha|0000\rangle + \beta|0100\rangle$

- For i = 2:

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$|\Psi_2\rangle = |0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) \otimes \mathbf{H|0\rangle} \otimes |0\rangle$
$|\Psi_2\rangle = |0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle) \otimes \mathbf{((1/\sqrt{2})\ |0\rangle + (1/\sqrt{2})\ |1\rangle)} \otimes |0\rangle$
$|\Psi_2\rangle = (\alpha|00\rangle + \beta|01\rangle) \otimes ((1/\sqrt{2})\ |00\rangle + (1/\sqrt{2})\ |10\rangle)$
$|\Psi_2\rangle = (\alpha)\ (1/\sqrt{2})\ |0000\rangle + (\beta)\ (1/\sqrt{2})\ |0100\rangle + (\alpha)\ (1/\sqrt{2})\ |0010\rangle + (\beta)\ (1/\sqrt{2})\ |0110\rangle$

- For i = 3:

$|\Psi_3\rangle = |0\rangle \otimes (\oplus\ ((\alpha)\ (1/\sqrt{2}\ |00\rangle + (\beta)\ (1/\sqrt{2})\ |10\rangle + \mathbf{(\alpha)\ (1/\sqrt{2})\ |01\rangle + (\beta)\ (1/\sqrt{2})\ |11\rangle))} \otimes |0\rangle$
$|\Psi_3\rangle = |0\rangle \otimes ((\alpha)\ (1/\sqrt{2}\ |00\rangle + (\beta)\ (1/\sqrt{2})\ |10\rangle + \mathbf{(\beta)\ (1/\sqrt{2})\ |01\rangle + (\alpha)\ (1/\sqrt{2})\ |11\rangle))} \otimes |0\rangle$
$|\Psi_3\rangle = (\alpha)\ (1/\sqrt{2}\ |0000\rangle + (\beta)\ (1/\sqrt{2})\ |0100\rangle + (\beta)\ (1/\sqrt{2})\ |0010\rangle + (\alpha)\ (1/\sqrt{2})\ |0110\rangle$

- For i = 4:

$|\Psi_3\rangle = (\alpha/\sqrt{2})\ |0000\rangle + (\alpha/\sqrt{2})\ |0110\rangle + (\beta/\sqrt{2})\ |0100\rangle + (\beta/\sqrt{2})\ |0010\rangle$
$|\Psi_3\rangle = (\alpha/\sqrt{2})\ |\mathbf{0000}\rangle + (\alpha/\sqrt{2})\ |0\mathbf{110}\rangle + (\beta/\sqrt{2})\ |0\mathbf{100}\rangle + (\beta/\sqrt{2})\ |0\mathbf{010}\rangle$
$|\Psi_4\rangle = (\alpha/\sqrt{2})\ |\mathbf{0000}\rangle + (\alpha/\sqrt{2})\ |\mathbf{1111}\rangle + (\beta/\sqrt{2})\ |\mathbf{1100}\rangle + (\beta/\sqrt{2})\ |\mathbf{0011}\rangle$

b)

- Rewrite explicitly your state $|\Psi_4\rangle$ under the form: $\gamma|0\rangle\,|\Lambda\rangle + \delta|1\rangle\,|\Gamma\rangle$
- Where $|\gamma|^2 + |\delta|^2 = 1$
- And $|\Lambda\rangle$ & $|\Gamma\rangle$ are legitimate 3-qubits states.

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$|\Psi_4\rangle = \gamma|0\rangle\,|\Lambda\rangle + \delta|1\rangle\,|\Gamma\rangle$
$|\Psi_4\rangle = (\alpha/\sqrt{2})\,|0\mathbf{000}\rangle + (\alpha/\sqrt{2})\,|1\mathbf{111}\rangle + (\beta/\sqrt{2})\,|1\mathbf{100}\rangle + (\beta/\sqrt{2})\,|0\mathbf{011}\rangle$
$|\Psi_4\rangle = (1/\sqrt{2})\,|0\rangle\,\boldsymbol{(\alpha|000\rangle + \beta|011\rangle)} + (1/\sqrt{2})\,|1\rangle\,\boldsymbol{(\alpha|111\rangle + \beta|100\rangle)}$

- Verifications:

- $|\gamma|^2 + |\delta|^2 = 1 \quad \rightarrow \quad |(1/\sqrt{2})|^2 + |(1/\sqrt{2})|^2 = 1$
- And $|\Lambda\rangle$ & $|\Gamma\rangle$ are legitimate 3-qubits states as according to the statement: $|\alpha|^2 + |\beta|^2 = 1$.

c)

- For getting to the point 5 of the circuit, measure the 1st qubit of $|\Psi_4\rangle$ (Partial Measurement)
- Give the probability p0 and p1 of getting b=0 and b=1 respectively as a classical result of the measurement.
- For each of the classical results b, give the resulting quantum state $|\Phi_b\rangle$ of the 3 unmeasured qubits

- $|\Psi_4\rangle = (\alpha/\sqrt{2})\,|\mathbf{0}000\rangle + (\alpha/\sqrt{2})\,|\mathbf{1}111\rangle + (\beta/\sqrt{2})\,|\mathbf{1}100\rangle + (\beta/\sqrt{2})\,|\mathbf{0}011\rangle$

- Probability of obtaining b = 0 as a classical result of the measurement:

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p0 = $|(1/\sqrt{2})|^2 = 0.5$

$|\Phi_0\rangle = (\alpha|000\rangle + \beta|011\rangle)$

- Probability of obtaining b = 1 as a classical result of the measurement:

p1 = $|(1/\sqrt{2})|^2 = 0.5$

$\Phi_1\rangle = (\alpha|111\rangle + \beta|100\rangle)$

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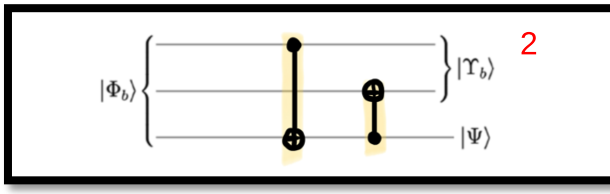
- Provide a circuit "?" working on the 3 unmeasured qubits which has the effect of restauring the initial $|\Psi\rangle$ in the last qubit.
- The circuit need to be independant of $|\Phi_0\rangle$ or $|\Phi_1\rangle$, without knowing in advance the result b of the measurement.
- Do not look for anything too complicated: the **requested circuit can be reduced to two** controlled negation gates.

- Mysterious Circuit:



<ins>- Goal:</ins> Restaure $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\Psi_4\rangle = (1/\sqrt{2})\,|0\rangle\,\mathbf{(\alpha|000\rangle + \beta|011\rangle)} + (1/\sqrt{2})\,|1\rangle\,\mathbf{(\alpha|111\rangle + \beta|100\rangle)}$

<ins>- Effect:</ins>

$|\Phi_0\rangle = (\alpha|000\rangle + \beta|011\rangle)$ becomes $|\Phi_0\rangle = \mathbf{(\alpha|000\rangle + \beta|001\rangle)}$

$|\Phi_1\rangle = (\alpha|111\rangle + \beta|100\rangle)$ becomes $|\Phi_1\rangle = \mathbf{(\alpha|110\rangle + \beta|111\rangle)}$

- Que vaut $|Y_b\rangle$ en fonction de b ?

- Faites le calcul explicite qui mene de $|\Phi_b\rangle$ a $|Y_b\rangle|\Psi\rangle$ pour chacune des deux valeurs possibles de b.
- Attention : **Le dernier qubit ne doit <ins>pas</ins> finir intrique avec les autres**.

<ins>- For b=0:</ins>

$|\Psi_4\rangle = \mathbf{(1/\sqrt{2})\,|0\rangle\,(\alpha|000\rangle + \beta|011\rangle)} + (1/\sqrt{2})\,|1\rangle\,(\alpha|111\rangle + \beta|100\rangle)$
$|\Phi_0\rangle = \alpha|\mathbf{000}\rangle + \beta|\mathbf{011}\rangle$

$|\Phi_0{}^1\rangle = \alpha|\mathbf{000}\rangle + \beta|0\mathbf{11}\rangle$
$|\Phi_0{}^2\rangle = (\alpha|000\rangle + \beta|001\rangle) = |00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$

$\mathbf{|Y_0\rangle|\Psi\rangle = |00\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)}$
$\mathbf{|Y_0\rangle = |00\rangle}$

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<ins>- For b=1:</ins>

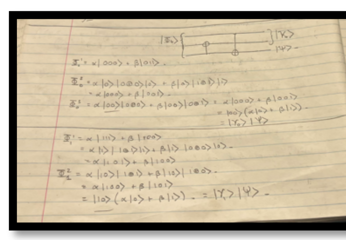
$|\Psi_4\rangle = (1/\sqrt{2})\,|0\rangle\,(\alpha|000\rangle + \beta|011\rangle) + \mathbf{(1/\sqrt{2})\,|1\rangle\,(\alpha|111\rangle + \beta|100\rangle)}$
$|\Phi_1\rangle = (\alpha|\mathbf{111}\rangle + \beta|100\rangle)$

$|\Phi_1{}^1\rangle = (\alpha|1\mathbf{10}\rangle + \beta|10\mathbf{1}\rangle)$
$|\Phi_1{}^2\rangle = (\alpha|110\rangle + \beta|111\rangle) = |11\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$

$\mathbf{|Y_1\rangle|\Psi\rangle = |11\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)}$
$\mathbf{|Y_1\rangle = |11\rangle}$

- Hence: $\mathbf{|Y_b\rangle = |bb\rangle}$

We figured out that it seems that an other circuit (to the right) may be also working... giving $|Y_0\rangle = |11\rangle$ and $|Y_1\rangle = |10\rangle$



<ins>e)</ins>

- (Optionnal): What could be the purpose of this circuit?
***Indice: Mettez Alice à gauche du circuit et Bob à droite...*

~~**Quantum teleportation**~~

~~**(Transmission of information via a classical communication line)**~~
~~**(Alice transmits $|\Psi\rangle$ to Bob)**~~