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IFT-6155 Quantum Computing

Assignment X

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Presented to
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- Considered Scenario:

- We say that a **quantum gate Q** is **universal** if any unitary operation (on any number of qubits) can be obtained by a circuit that uses only arbitrary Q gates and unit gates acting on only one qubit at a time (called unary gates).

- You can assume that the CNOT gate is universal.

- We saw that Controlled-Z gate is universal, where $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ sends $\alpha|0\rangle + \beta|1\rangle$ on $\alpha|0\rangle - \beta|1\rangle$, because one can construct the controlled negation from a Controlled-Z flanked by a Hadamard door H on each side.

- Let **Swap** be the unitary transformation on two qubits that sends $|\Psi\rangle|\Phi\rangle$ to $|\Phi\rangle|\Psi\rangle$

- Let **SS** a square root of Swap, which means that the composition of SS with itself gives Swap.

(I say one and not the square root of Swap because it is not unique, just as the square root of the negation is not unique either)

- The miracle is that **every square root of Swap is universal** (Q4) despite the fact that **Swap itself is not universal** (Q1).

Question 1

- Give a simple proof that the **Swap door is <u>not</u> universal.**

$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle$

$|1\rangle|0\rangle \rightarrow |0\rangle|1\rangle$

$|0\rangle|1\rangle \rightarrow |1\rangle|0\rangle$

$|1\rangle|1\rangle \rightarrow |1\rangle|1\rangle$

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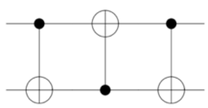
- Swap gate **can not produce entanglement like a CNOT could** (With unary gates)
- The operation is **only able to interchanges the state of two qubits (no interaction)**
- And knowing that single-qubit operations are not able to produce entanglement also
- Set of SWAP/Unary gates can not produce entanglement → **Can not be universal**

- *Interesting: SWAP gate can be obtained from a sequence of three CNOT gates*



Question 2

- Explicitly give the 4×4 matrix which defines **Swap**.

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$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle$ (First Column)

$|1\rangle|0\rangle \rightarrow |0\rangle|1\rangle$

$|0\rangle|1\rangle \rightarrow |1\rangle|0\rangle$

$|1\rangle|1\rangle \rightarrow |1\rangle|1\rangle$ (Last Column)

$$Swap = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 3

- Explicitly give a 4 × 4 matrix that defines SS.

- *Tip: Think about the square root of the negation and use this version:* $S = \begin{pmatrix} \frac{1+i}{2} & \frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}$, where i = $\sqrt{-1}$

- We remember the S Transformation from TP2 #1:

- Transformation S whose composition with itself provides true logical negation

- We will use it here

$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle$

$\mathbf{|1\rangle|0\rangle \rightarrow |0\rangle|1\rangle}$

$\mathbf{|0\rangle|1\rangle \rightarrow |1\rangle|0\rangle}$

$|1\rangle|1\rangle \rightarrow |1\rangle|1\rangle$

- Hence we will put the S matrix in the middle:

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$$\sqrt{Swap} = SS = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1+i) & \frac{1}{2}(1-i) & 0 \\ 0 & \frac{1}{2}(1-i) & \frac{1}{2}(1+i) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 4

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- Prove that the **SS gate is universal** by showing how to achieve the effect of the Controlled-Z or **Controlled-Negation (CNOT)** (at your choice) using only SS gates and unary gates of your choice.

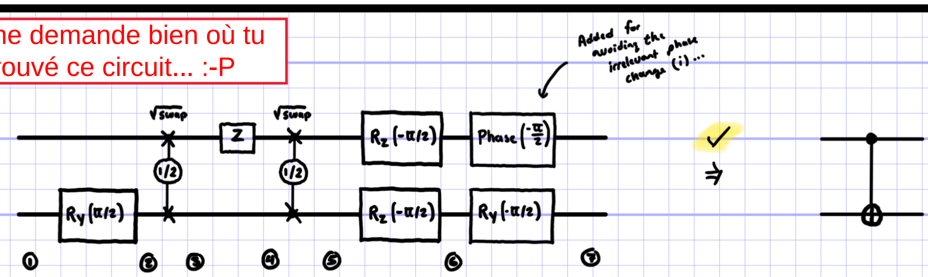
- Need an **explicit quantum circuit** that shows this phenomena.

- Note: The first three questions are very simple (provided you think about them in the right way) but the last one might require you to think about it. Don't underestimate the difficulty!

- **Goal**: Obtain a CNOT gate given the ability to use 1-qubit gates and √SWAP:

Je me demande bien où tu as trouvé ce circuit... :-P

→ Circuit :



$|00\rangle$:

$\psi_1 = |00\rangle$

$\psi_2 = 1/\sqrt{2}\left(|00\rangle + |01\rangle\right)$

$\psi_3 = 1/\sqrt{2}\ |00\rangle + \frac{\sqrt{2}(1+i)}{4}|01\rangle + \frac{\sqrt{2}(1-i)}{4}|10\rangle$

$\psi_4 = 1/\sqrt{2}\ |00\rangle + \frac{\sqrt{2}(1+i)}{4}|01\rangle + \frac{\sqrt{2}(i-1)}{4}|10\rangle$

$\psi_5 = 1/\sqrt{2}|00\rangle + \frac{i\sqrt{2}}{2}|01\rangle$

$\psi_6 = \frac{i\sqrt{2}}{2}\left(|00\rangle + |01\rangle\right)$

$\psi_7 = |00\rangle$

$|10\rangle$:

$\psi_1 = |10\rangle$

$\psi_2 = 1/\sqrt{2}\left(|10\rangle + |11\rangle\right)$

$\psi_3 = \frac{\sqrt{2}(1-i)}{4}|01\rangle + \frac{\sqrt{2}(1+i)}{4}|10\rangle + \frac{\sqrt{2}}{2}|11\rangle$

$\psi_4 = \frac{\sqrt{2}(1-i)}{4}|01\rangle - \frac{\sqrt{2}(1+i)}{4}|10\rangle - \frac{\sqrt{2}}{2}|11\rangle$

$\psi_5 = \frac{-i\sqrt{2}}{2}|10\rangle - \frac{\sqrt{2}}{2}|11\rangle$

$\psi_6 = \frac{-i\sqrt{2}}{2}|10\rangle + \frac{i\sqrt{2}}{2}|11\rangle$

$\psi_7 = |11\rangle$

$|01\rangle$:

$\psi_1 = |01\rangle$

$\psi_2 = 1/\sqrt{2}\left(-|00\rangle + |01\rangle\right)$

$\psi_3 = -\frac{\sqrt{2}}{2}|00\rangle + \frac{\sqrt{2}(1+i)}{4}|01\rangle + \frac{\sqrt{2}(1-i)}{4}|10\rangle$

$\psi_4 = -\frac{\sqrt{2}}{2}|00\rangle + \frac{\sqrt{2}(1+i)}{4}|01\rangle - \frac{\sqrt{2}(1-i)}{4}|10\rangle$

$\psi_5 = \frac{-\sqrt{2}}{2}|00\rangle + \frac{i\sqrt{2}}{2}|01\rangle$

$\psi_6 = \frac{-i\sqrt{2}}{2}|00\rangle + \frac{i\sqrt{2}}{2}|01\rangle$

$\psi_7 = |01\rangle$

$|11\rangle$:

$\psi_1 = |11\rangle$

$\psi_2 = 1/\sqrt{2}\left(-|10\rangle + |11\rangle\right)$

$\psi_3 = \frac{\sqrt{2}(i-1)}{4}|01\rangle - \frac{\sqrt{2}(1+i)}{4}|10\rangle + \frac{\sqrt{2}}{2}|11\rangle$

$\psi_4 = \frac{\sqrt{2}(i-1)}{4}|01\rangle + \frac{\sqrt{2}(1+i)}{4}|10\rangle - \frac{\sqrt{2}}{2}|11\rangle$

$\psi_5 = \frac{i\sqrt{2}}{2}|10\rangle - \frac{\sqrt{2}}{2}|11\rangle$

$\psi_6 = \frac{i\sqrt{2}}{2}|10\rangle + \frac{i\sqrt{2}}{2}|11\rangle$

$\psi_7 = |10\rangle$