Université de Montréal

23/28

IFT-6155 Quantum Computing

Assignment 2

Ayoub Echchahed *Nassim El Massaudi*

Presented to
Mr. Gilles Brassard

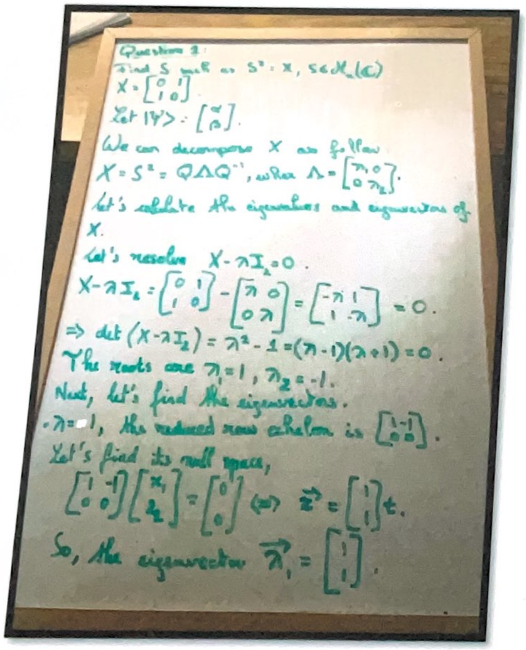
Department of Computer Science and Operations Research
University of Montreal
Winter 2023

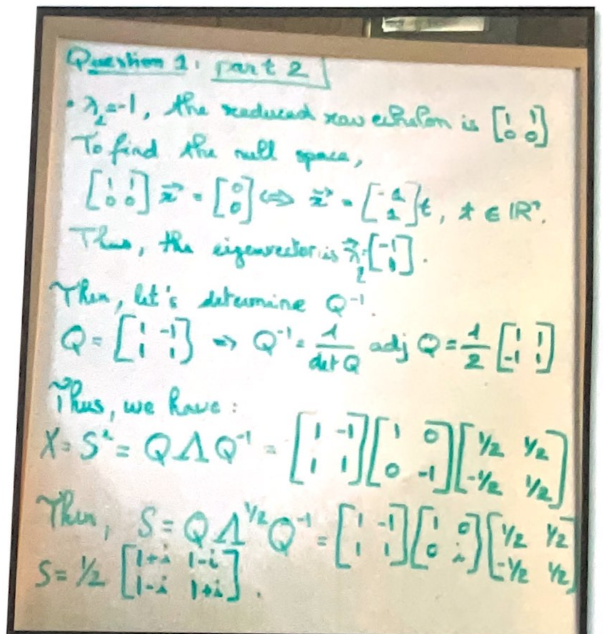
Question 1

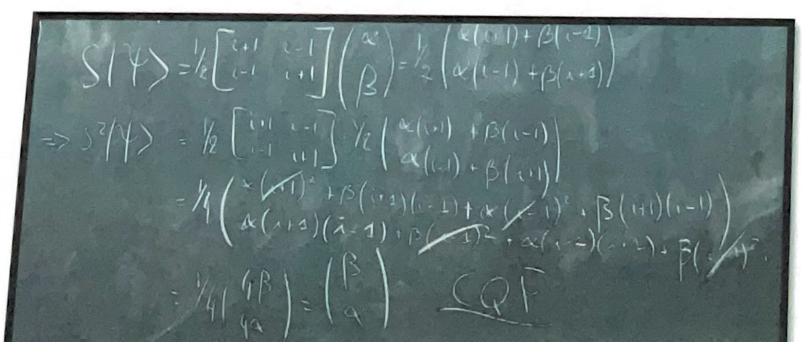
5/5

- Find a transformation S whose composition with itself provides true logical negation.
- In other words, it must be that $\mathbf{S^2 = X}$, which means that $S^2\,|\Psi\rangle = X|\Psi\rangle$ for all $|\Psi\rangle$

- Steps:
1) *Decomposition of the transformation*
2) *Computations for finding the eigenvalues*
3) *Computations for finding the eigenvectors*
4) *Finding Q^{-1} from Q*
5) *Determining S*
6) *Demonstrating $S^2 = X$*







418

Question 2

- **abcd Theorem:** Prove that any two-qubit state $|\Gamma\rangle$ as given in Equation 2.10 is separable if and only if ad = bc.

- The "only if" part has already been established above; no need to repeat the argument. For the "if" part, exhibit explicit values for complex numbers α, β, γ and δ (as functions of a, b, c and d) such that if we define one-qubit states $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\Phi\rangle = \gamma|0\rangle + \delta|1\rangle$, then $|\Gamma\rangle = |\Psi\rangle \otimes |\Phi\rangle$.

$$|\Gamma\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle \qquad (2.10)$$

- Of course, both $|\Psi\rangle$ and $|\Phi\rangle$ must be legitimate states, meaning that $|\alpha|^2 + |\beta|^2 = 1$ and $|\gamma|^2 + |\delta|^2 = 1$.

- *Note that if either one of $|\Psi i$ or $|\Phi i$ is proven to be normalized, then the normalization of the other follows from the fact that $|\Gamma\rangle = |\Psi\rangle \quad |\Phi\rangle$ and of course that $|\Gamma\rangle$ itself is normalized since it is given as a quatum state.*

- *Hint: To prove the "if" part, you may need to consider as a special (easy) case the situation in which abcd = 0, which means that at least one of a or d and at least one of b or c equals 0 since ad = bc.*

Question 2: (ad - bc = 0 ⇒ separable).

Let's consider $|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ such that $ad - bc = 0$. Note that $\|\Psi\|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$ which means that $|\Psi\rangle$ is normalized.

wlog, let's assume at least one of $a, b, c, d \in \mathbb{C}$ is not null.

So, let's assume $a \neq 0$, and thus, $d = \frac{bc}{a}$.

et si $a=0$?

By definition (Ψ is separable), we can assume that $\exists v, w \in K$ such as

$$|\Psi\rangle = |v\rangle \otimes |w\rangle = (a|0\rangle + c|1\rangle) \otimes (|0\rangle + \frac{b}{a}) = a|00\rangle + b|01\rangle + c|10\rangle + \frac{bc}{a}|11\rangle.$$

In this decomposition, we have $\|w\|^2 = 1 + \left|\frac{b}{a}\right|^2 > 1$. So $|w\rangle$ is not normalized, let's consider $\lambda \in \mathbb{R}_+^*$ such that:

$\lambda = \frac{1}{\|v\|} = \frac{1}{\sqrt{|a|^2 + |c|^2}}$, thus we have, $\|\lambda v\| = 1$ and

$$\left\|\frac{w}{\lambda}\right\| = |a|^2 + |c|^2 + \frac{|a|^2 + |c|^2}{|a|^2} \overset{??}{=} |a|^2 + |b|^2 + |c|^2 + \left|\frac{bc}{a}\right|^2 = \|\Psi\|^2 = 1.$$

Also, $$\|\lambda v\|^2 \left\|\frac{w}{\lambda}\right\| = \|v\|\,\|w\| = (|a|^2 + |c|^2)\left(1 + \left|\frac{b}{a}\right|^2\right) = |a|^2 + |b|^2 + |c|^2 + \left|\frac{bc}{a}\right|^2 = |a|^2 + |b|^2 + |c|^2 + |d|^2 = \|\Psi\|^2 = 1.$$

Ceci sert à rien?

We can conclude that $|\Psi\rangle$ is separable if $ad = bc$. QED.

Je présume qu'ici, vous voulez dire $\left|\frac{|w\rangle}{\lambda}\right|^2$.

Ce qui serait: $(|a|^2 + |c|^2)(1 + |b/a|^2)$

et qui n'a aucun rapport avec ce qui est écrit.

Question 3 4/5

A)

Exercise 2.5.2 Consider an arbitrary one-qubit unitary transformation U and apply it independently to each qubit of a $|\Psi^-\rangle$ pair. Prove that the resulting state is again $|\Psi^-\rangle$, possibly up to an irrelevant phase factor. In other words, prove that there exists a complex number η of unit norm, which depends only on U, such that

$$[\mathsf{U}\otimes\mathsf{U}]|\Psi^-\rangle = \tfrac{1}{\sqrt{2}}(\mathsf{U}|0\rangle\otimes\mathsf{U}|1\rangle) - \tfrac{1}{\sqrt{2}}(\mathsf{U}|1\rangle\otimes\mathsf{U}|0\rangle) = \eta|\Psi^-\rangle .$$

Give explicitly the value of η as a function of the paremeters α, β, γ and δ so that $\mathsf{U}|0\rangle = \alpha|0\rangle + \beta|1\rangle$ and $\mathsf{U}|1\rangle = \gamma|0\rangle + \delta|1\rangle$. □

2.5.2 :

$$[U\otimes U]|\Psi^-\rangle = \tfrac{1}{\sqrt{2}}(U|0\rangle\otimes U|1\rangle) - \tfrac{1}{\sqrt{2}}(U|1\rangle\otimes U|0\rangle)$$

$$\tfrac{1}{\sqrt{2}}\left[(\alpha|0\rangle+\beta|1\rangle)\otimes(\gamma|0\rangle+\delta|1\rangle) - (\gamma|0\rangle+\delta|1\rangle)\otimes(\alpha|0\rangle+\beta|1\rangle)\right].$$

$$= \tfrac{1}{\sqrt{2}}\left[(\alpha\gamma|00\rangle+\alpha\delta|01\rangle+\beta\gamma|10\rangle+\cancel{\beta\delta|11\rangle}) - (\gamma\alpha|00\rangle+\gamma\beta|01\rangle+\delta\alpha|10\rangle+\cancel{\delta\beta|11\rangle})\right].$$

$$= \tfrac{1}{\sqrt{2}}\left[(\beta\gamma-\delta\alpha)|10\rangle + (\alpha\delta-\gamma\beta)|01\rangle\right].$$

$$= \frac{\alpha\delta-\gamma\beta}{\sqrt{2}}|01\rangle - \frac{\delta\alpha-\beta\gamma}{\sqrt{2}}|10\rangle = (\delta\alpha-\beta\gamma)|\Psi^-\rangle.$$

On a donc $\eta = (\delta\alpha - \beta\gamma)$ où $\alpha,\beta,\gamma,\delta \in \mathbb{C}$.

Est-ce que $|\eta| = 1$?

5/5

B)

Exercise 2.5.3 First prove that $|\Phi^+\rangle$ is invariant under bilateral application of the Hadamard transform: $[\mathsf{H} \otimes \mathsf{H}]|\Phi^+\rangle = |\Phi^+\rangle$. In contrast to Exercise 2.5.2, however, find explicitly a one-qubit unitary transformation U such that $[\mathsf{U} \otimes \mathsf{U}]|\Phi^+\rangle \neq |\Phi^+\rangle$, even up to an irrelevant phase factor. (*Hint:* Use complex numbers in your definition of U.) Your solution must be such that if U is applied bilaterally to each qubit forming $|\Phi^+\rangle$, and if the resulting state is observed, two opposite bits will be obtained with certainty (this cannot be farther away from a $|\Phi^+\rangle$). Optional: Generalize this result by proving that no two-qubit state $|\Lambda\rangle$ can exist with the property that the amplitude of both $|01\rangle$ and $|10\rangle$ is zero in $[\mathsf{U} \otimes \mathsf{U}]|\Lambda\rangle$ for every one-qubit unitary transformation U. □

2.5.3:

$$[H \otimes H]|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}\left[(H|0\rangle \otimes H|0\rangle) + (H|1\rangle \otimes H|1\rangle)\right].$$

$$= \tfrac{1}{\sqrt{2}}\left[\left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle\right) + \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle\right)\right]$$

$$= \frac{1}{2\sqrt{2}}\left[|00\rangle + |01\rangle + |10\rangle + |11\rangle + |00\rangle - |01\rangle - |10\rangle + |11\rangle\right].$$

$$= \tfrac{1}{2\sqrt{2}}\left[2|00\rangle + 2|11\rangle\right] = \frac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle = |\Phi^+\rangle.$$

Part 2 : 2.5.3.

$$(U \otimes U)|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}\left[(U|0\rangle \otimes U|0\rangle) + (U|1\rangle \otimes U|1\rangle)\right].$$

$$= \tfrac{1}{\sqrt{2}}\left[(\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle) + (\gamma|0\rangle + \delta|1\rangle) \otimes (\gamma|0\rangle + \delta|1\rangle)\right].$$

$$= \tfrac{1}{\sqrt{2}}\left[\alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle + \gamma^2|00\rangle + \gamma\delta|01\rangle + \delta\gamma|10\rangle + \delta^2|11\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[(\alpha^2+\gamma^2)|00\rangle + (\beta^2+\delta^2)|11\rangle\right] + \tfrac{1}{\sqrt{2}}\left[(\alpha\beta+\gamma\delta)|01\rangle + (\alpha\beta+\delta\gamma)|10\rangle\right].$$

$$= \frac{1}{\sqrt{2}}\left[(\alpha^2+\gamma^2)|00\rangle + (\beta^2+\delta^2)|11\rangle\right] + (\alpha\beta+\gamma\delta)|\Psi^+\rangle$$

En prenant $\alpha = \tfrac{1}{\sqrt{2}}$, $\beta = \tfrac{i}{\sqrt{2}}$, $\gamma = \frac{-i}{\sqrt{2}}$ et $\delta = \frac{-1}{\sqrt{2}}$

On a

$$\begin{cases} |\alpha|^2 + |\gamma|^2 = \frac{1}{2} - \frac{1}{2} = 0. \\ |\beta|^2 + |\delta|^2 = \left|\frac{i}{\sqrt{2}}\right|^2 + \left|\frac{-1}{\sqrt{2}}\right|^2 = 0 \\ \alpha\gamma^* + \beta\delta^* = \frac{1}{\sqrt{2}} \times \frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} = 0 \\ |\alpha|^2 + |\beta|^2 = 1 \\ |\gamma|^2 + |\delta|^2 = 1 \end{cases}$$

$$U = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ -i/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Question 4

5/5

Question 4. Soit R_θ l'opération de rotation de polarisation de photon par un angle θ dans le sens antihoraire, qui envoie $|0\rangle = |0^\circ\rangle$ et $|1\rangle = |90^\circ\rangle$ sur

$$|\theta\rangle = (\cos\theta)|0\rangle + (\sin\theta)|1\rangle$$

et

$$|90^\circ + \theta\rangle = (\cos(90^\circ + \theta))|0\rangle + (\sin(90^\circ + \theta))|1\rangle = -(\sin\theta)|0\rangle + (\cos\theta)|1\rangle\,,$$

respectivement. En d'autres termes.

$$\mathsf{R}_\theta \;:\; \begin{cases} |0\rangle \mapsto (\cos\theta)|0\rangle + (\sin\theta)|1\rangle \\ |1\rangle \mapsto -(\sin\theta)|0\rangle + (\cos\theta)|1\rangle\,. \end{cases}$$

Considérons également les quatre états suivants sur deux qubits :

$$|\Phi^+\rangle = \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle \quad |\Phi^-\rangle = \tfrac{1}{\sqrt{2}}|00\rangle - \tfrac{1}{\sqrt{2}}|11\rangle$$
$$|\Psi^+\rangle = \tfrac{1}{\sqrt{2}}|01\rangle + \tfrac{1}{\sqrt{2}}|10\rangle \quad |\Psi^-\rangle = \tfrac{1}{\sqrt{2}}|01\rangle - \tfrac{1}{\sqrt{2}}|10\rangle\,,$$

qui portent le nom d'« états de Bell ».

Calculez l'effet d'une rotation par un angle θ du premier qubit et par un angle ϕ du second qubit de $|\Phi^+\rangle$, où θ et ϕ sont deux angles arbitraires. En d'autres termes, calculez

$$(\mathsf{R}_\theta \otimes \mathsf{R}_\phi)\,|\Phi^+\rangle\,.$$

Exprimez votre réponse le plus simplement possible *en termes d'états de Bell.*

Rappel : $\sin(a-b) = \sin a\cos b - \cos a \sin b$; $\cos(a-b) = \sin a \sin b + \cos a \cos b$.

$$(R_\theta \otimes R_\phi)\,|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(R_\theta|0\rangle \otimes R_\phi|0\rangle\right) + \frac{1}{\sqrt{2}}\left(R_\theta|1\rangle \otimes R_\phi|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\Big(\big(\cos\theta\,|0\rangle + \sin\theta\,|1\rangle\big)\otimes\big(\cos\phi\,|0\rangle + \sin\phi\,|1\rangle\big)\Big)$$
$$+ \frac{1}{\sqrt{2}}\Big(\big(-(\sin\theta)|0\rangle + (\cos\theta)|1\rangle\big)\otimes\big(-(\sin\phi)|0\rangle + (\cos\phi)|1\rangle\big)\Big)$$

$$= \frac{\cos\theta\cdot\cos\phi}{\sqrt{2}}|00\rangle + \frac{\sin\theta\cdot\sin\phi}{\sqrt{2}}|11\rangle + \frac{\cos\theta\cdot\sin\phi}{\sqrt{2}}|01\rangle$$
$$+ \frac{\sin\theta\cdot\cos\phi}{\sqrt{2}}|10\rangle$$

$$+$$

$$\frac{-(\sin\theta)\cdot-(\sin\phi)}{\sqrt{2}}|00\rangle + \frac{-(\sin\theta)\cdot(\cos\phi)}{\sqrt{2}}|01\rangle + \frac{\cos\theta\cdot-(\sin\phi)}{\sqrt{2}}|10\rangle$$
$$+ \frac{(\cos\theta)\cdot(\cos\phi)}{\sqrt{2}}|11\rangle$$

$$= \frac{1}{\sqrt{2}}\Big(\Big[\big(\cos\theta\cdot\cos\phi\big) + \big(-(\sin\theta)\cdot-(\sin\phi)\big)|00\rangle\Big]$$
$$+ \Big[\big(\cos\theta\cdot\sin\phi\big) + \big(-(\sin\theta)\cdot(\cos\phi)\big)|01\rangle\Big]$$
$$+ \Big[\big(\sin\theta\cdot\cos\phi\big) + \big(\cos\theta\cdot-(\sin\phi)\big)|10\rangle\Big]$$
$$+ \Big[\big(\sin\theta\cdot\sin\phi\big) + \big(\cos\theta\cdot\cos\phi\big)|11\rangle\Big]\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(\cos(\theta-\phi)|00\rangle + \sin(\phi-\theta)|01\rangle - \sin(\phi-\theta)|10\rangle$$
$$+ \cos(\theta-\phi)|11\rangle\Big)$$

$$= \cos(\theta-\phi)|\Phi^+\rangle + \sin(\phi-\theta)|\Psi^-\rangle.$$