Université de Montréal

7/20

IFT-6155 Quantum Computing

Assignment 1

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Presented to
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Question 1

We have seen that two states are orthogonal if and only if they can be distinguished at all times and with certainty. The funny thing is that when two 'states' are distinct without being orthogonal, we can either distinguish them 'every time but without certainty or with certainty but not every time! We will explore this idea in this exercise and the next, doing our best to distinguish between states.

$$\boxed{|\psi\rangle = |0\rangle \quad \text{et} \quad |\phi\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle}$$

In this first exercise, I ask you **to invent a device A that takes a physical qubit |Γ⟩ as input and returns a binary answer A(|Γ⟩). Your device tries to return 0 on |ψ⟩ and 1 on |φ⟩ whenever possible.** It will not be able to be sure since these two states are **not orthogonal**. Let p0 and p1 be the probabilities of error on |ψ⟩ and |φ⟩, respectively, i.e. p0 = Prob[A(|ψ⟩) = 1] and p1 = Prob[A(|φ⟩) = 0]. Your task is to **minimize the probability of error of your device on the worst possible entry, which is defined by p_{error} = max(p0, p1).**

- Goal: Determine Angle θ that minimise p_{error} and compute the p_{error} value for the proposed system:

- Intuition for the possible inputs:

|ψ⟩ -- Horizontal (0◦) Polarized Photon
|φ⟩ -- (45◦) Polarized Photon

- Input: **Physical qubit |Γ⟩**
- Output: **Binary Response A(|Γ⟩)**

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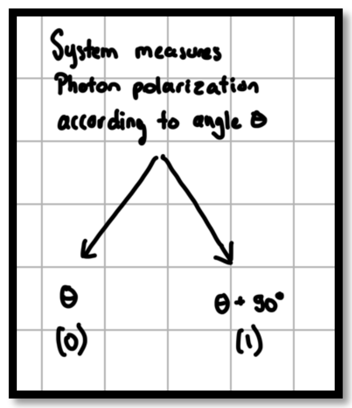

1) if Angle θ = 0◦

- Deviation according to:
  - 0◦ (System return 0)
  - 90◦ (System return 1)

P_{error} = max (p0, p1) as worst possible entry

p0 = Prob [A(|ψ⟩) = 1] = 0
p1 = Prob [A(|φ⟩) = 0] = 0.5

P_{error} = 0.5
Let's do better

2) System:

- Angle θ = 22.5◦ (π / 8)
- Deviation according to:
  - 22.5◦ (System return 0)
  - 90+22.5 = 112.5◦ (System return 1)

P_{error} = max (p0, p1) as worst possible entry

un filtre à pi/8 est le pire cas! Tu maximises l'erreur sur phi.

p0 = Prob [A(|ψ⟩) = 1]
= $\sin^2(\pi / 8) = (2 - \sqrt{2}) / 4$ = **0.1465**

p1 = Prob [A(|φ⟩) = 0]
~~= $\sin^2(\pi / 8) - (\pi / 4)$ = 0.1465~~ p1=cos^2(pi/8)=0.85

P_{error} = max (p0, p1) = 0.1465

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This exercise asks you to demonstrate that the |ψ⟩ and |φ⟩ states of the preceding exercise can be distinguished with certainty, although not every time since they are not orthogonal. This is called a ***conclusive measure.*** *More specifically, you* ***must invent another device B that takes a physical qubit |Γ⟩ as input and returns a ternary answer B(|Γ⟩) chosen from the set {0, 1, ?}.***

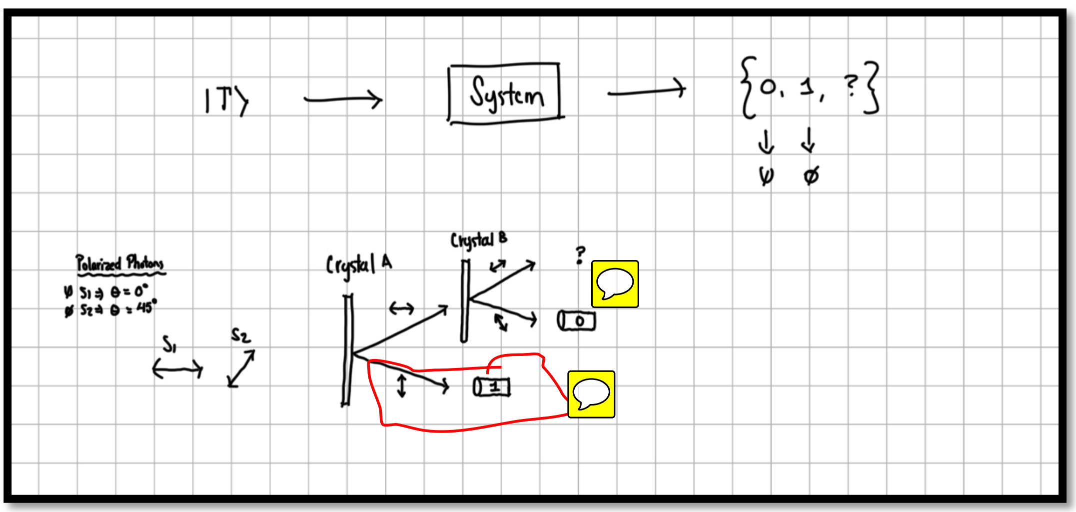
$$\boxed{|\psi\rangle = |0\rangle \quad \text{et} \quad |\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle}$$

Answer 0 means that |Γ⟩ = |ψ⟩ with certainty and answer 1 means that |Γ⟩ = |φ⟩, also with certainty. No error is tolerated in these two cases of repetition, as long as the State |Γ⟩ entering is either |ψ⟩ or |φ⟩. On the other hand, your device has the right to respond "?", which means that it irreversibly lost the qubit when entering without having been able to identify it positively.

B(|ψ⟩) ∈ {0, ?}
B(|φ⟩) ∈ {1, ?}

- Goal*: Maximise Pconclusive = min (q0, q1), which is the probability of your device answering conclusively on the worst possible entry.*

- System:



$P_{conclusive}$ = min (q0, q1)

~~q0 = Prob[B(|ψ⟩) = 0] = 1 x 0.5 = 0.5~~ q0=0

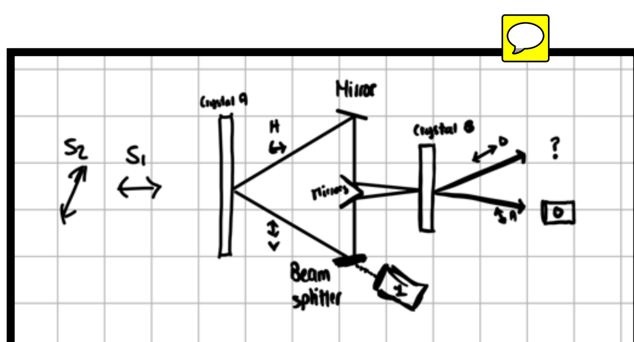
q1 = Prob[B(|φ⟩) = 1] = 0.5

$P_{conclusive}$ = 0.5

Hesitations as Crystal B inputs may be 2 horizontally polarized photons (information loss)

 # Hence my other system using a beam splitter and mirrors:

(Figure to the right)



- Side Research on the optimal response

- Did a small literature review in order to potentially find the optimal solution to the situation above:
- **Problem**: Discriminating among given nonorthogonal quantum states
- **Setting**: Unambiguous state discrimination
- Minimum-Error State Discrimination VS Error minimizing discrimination strategy

- Von Neumann projective measurement (2 outcomes) is not what we are looking for
- **Optimal Positive Operator-Valued Measure (POVM)** is I think the optimal strategy (3 outcomes) (Ivanovic-Dieks-Peres (IDP) measurement)

- The 3 papers below are detailing setups for this strategy:

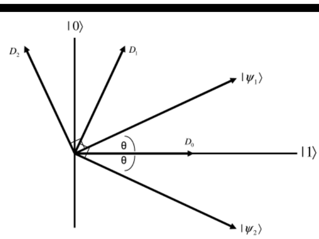


Fig. 11.3. Optimal POVM that discriminates $|\psi_1\rangle$ and $|\psi_2\rangle$ unambiguously. The detector $D_0 = \Pi_0$ is set up symmetrically between $|\psi_1\rangle$ and $|\psi_2\rangle$, the detector $D_1 = \Pi_1$ is set up along $\bar{P}_2$ and the detector $D_2 = \Pi_2$ is set up along $\bar{P}_1$. When a click in the D_i detector occurs we know for certain that $|\psi_i\rangle$ was prepared $(i = 1, 2)$ as the input state since it is the only one that has a component along this direction. When a click in the detector D_0 occurs we learn nothing about which state was prepared as the input since both have a component along this direction. This measurement outcome then corresponds to the inconclusive result so D_0 is the failure detector and the probability that it clicks is Q^{POVM}. (The figure is for illustrative purposes only. Arrows representing POVM detection operators *do not correspond* to simple projections along their respective directions. They are drawn shorter than arrows representing state vectors which, in turn, have unit length. For an implementation of the POVM, see Sect. 11.2.3.)

$|\psi_1\rangle$ (and correspondingly for $p_2 = 0$). The set-up of the detection operators, yielding Q^{POVM} for the failure probability, is depicted in Fig. 11.3.

These findings can be summarized as follows. The optimal failure probability, Q^{opt}, is given as

$$Q^{opt} = \begin{cases} Q^{POVM} & \text{if } \frac{\cos^2\Theta}{1+\cos^2\Theta} \leq \eta_1 \leq \frac{1}{1+\cos^2\Theta} \ , \\ Q_1 & \text{if } \eta_1 < \frac{\cos^2\Theta}{1+\cos^2\Theta} \ , \\ Q_2 & \text{if } \frac{1}{1+\cos^2\Theta} < \eta_1 \ . \end{cases} \tag{11.13}$$

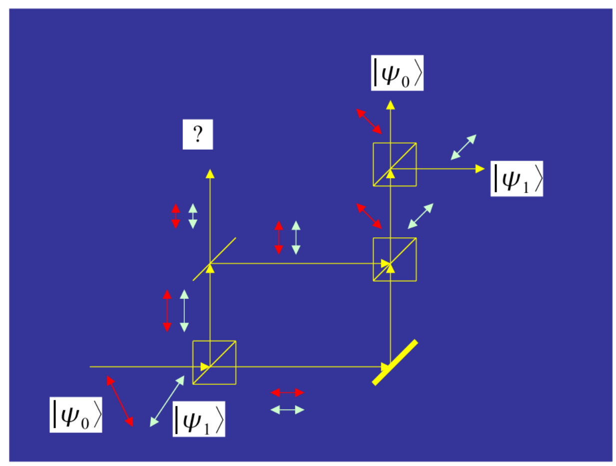


Fig. 9. Schematic of the Clarke *et al.* experimental realisation of unambiguous discrimination between two non-orthogonal polarisation states.

- **References**:

[1] János A. Bergou (2010) Discrimination of quantum states, Journal of Modern Optics, 57:3, 160-180, DOI: 10.1080/09500340903477756
https://www.hunter.cuny.edu/physics/faculty/bergou/repository/files/publications/06490417.pdf

[2] Stephen M. Barnett and Sarah Croke, "Quantum state discrimination," Adv. Opt. Photon. 1, 238-278 (2009)
https://opg.optica.org/aop/fulltext.cfm?uri=aop-1-2-238&id=176580

[3] Roger B. M. Clarke, Anthony Chefles, Stephen M. Barnett, and Erling Riis. Experimental Demonstration of Optimal Unambiguous State Discrimination. Phys. Rev. A 63, 040305(R) – Published 21 March 2001. *https://arxiv.org/pdf/quant-ph/0007063.pdf*