The Life Cycle of Plants in India and Mexico

Details of Derivation and Simulation

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Coverpage

1. The first section is math derivation of Section IV. Some very complex coefficients are not computed. The average $\bar{\tau}$ can be normalized in some form rather that the footnote's expression(This only pins down the term in (1.17)).
2. The strategic interaction in aggregate output is absorbed in Π_d, (2.3). So we do not need to do Krusell-Smith here.
3. In the deterministic model, I think it is okay to simulate without generated firms like Monte Carlo, it is used in determine the cutoff rule and entrance log-normal distribution.
4. The original derivation omits the partial derivative on Y, I guess the authors use the condition that $\sum_a N_a$ is large (1.4).

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Monopolistic Competition with Two Factors I

Profit maximization of final goods:

$$\max_{Y_{a,i}} \pi = PY - \sum_a \sum_i^{N_a} P_{a,i} Y_{a,i} \tag{1.1}$$

$$Y = \left[\sum_a \sum_i^{N_a} Y_{a,i}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \tag{1.2}$$

(given price) FOC yields:

$$PY^{1/\sigma} = P_{a,i} Y_{a,i}^{1/\sigma} \to P = \left[\sum_a \sum_i^{N_a} P_{a,i}^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \tag{1.3}$$

$$\frac{P_{a,i} Y_{a,i}}{PY} = \left(\frac{P}{P_{a,i}} \right)^{\sigma-1}$$

Monopolistic Competition with Two Factors II

This is the aggregation rule of price. By Euler Theorem:

$$\sum_{a,i} \frac{P_{a,i} Y_{a,i}}{PY} = 1$$

Plug in (1.3), the static profit maximization is:

$$\max_{L_{a,i}, K_{a,i}} (1-\tau_{a,i}) PY^{1/\sigma} [A_{a,i} K_{a,i}^{\alpha} L_{a,i}^{1-\alpha}]^{1-1/\sigma} - (1+\tau_{L_{a,i}}) w L_{a,i} - (1+\tau_{K_{a,i}}) R K_{a,i} \tag{1.4}$$

cost minimization yields:

$$K_{a,i}(1-\alpha) R (1+\tau_{K_{a,i}}) = \alpha w (1+\tau_{L_{a,i}}) L_{a,i} \equiv E_{a,i} \tag{1.5}$$

Omit strategic interaction terms on Y, as N_a is very large. We can convert it to AK model, the cost function:

$$(1+\tau_{L_{a,i}}) w L_{a,i} + (1+\tau_{K_{a,i}}) R \frac{\alpha}{1-\alpha} \frac{w}{R} \frac{1+\tau_{L_{a,i}}}{1+\tau_{K_{a,i}}} L_{a,i} = \frac{1+\tau_{L_{a,i}}}{1-\alpha} w L_{a,i} = \frac{E_{a,i}}{\alpha(1-\alpha)} \tag{1.6}$$

Monopolistic Competition with Two Factors III

Therefore:

$$\max_{E_{a,i}} = (1-\tau_{Y_{a,i}})PY^{1/\sigma}\left[A_{a,i}\left(\frac{1}{(1-\alpha)R(1+\tau_{K_{a,i}})}\right)^{\alpha}\left(\frac{1}{\alpha w(1+\tau_{L_{a,i}})}\right)^{1-\alpha}\right]^{1-1/\sigma} E_{a,i}^{1-1/\sigma} - \frac{E_{a,i}}{\alpha(1-\alpha)} \tag{1.7}$$

we have:

$$E = \alpha^{\sigma}(1-\alpha)^{\sigma}\left(1-\frac{1}{\sigma}\right)^{\sigma} P^{\sigma}Y\left[A_{a,i}\frac{1}{(1-\alpha)^{\alpha}R^{\alpha}(1+\tau_{K_{a,i}})^{\alpha}}\frac{1}{\alpha^{1-\alpha}w^{1-\alpha}(1+\tau_{L_{a,i}})^{1-\alpha}}\right]^{\sigma-1}(1-\tau_{Y_{a,i}})^{\sigma} \tag{1.8}$$

That is:

$$L_{a,i} = \text{coefficient1} * \frac{1-\tau_{Y_{a,i}}}{1+\tau_{L_{a,i}}} * (\tau_{a,i}/A_{a,i})^{1-\sigma} \tag{1.9}$$

$$K_{a,i} = \text{coefficient2} * \frac{1-\tau_{Y_{a,i}}}{1+\tau_{K_{a,i}}} * (\tau_{a,i}/A_{a,i})^{1-\sigma} \tag{1.10}$$

Monopolistic Competition with Two Factors IV

That is:

$$Y_{a,i} = \left[A_{a,i} \left(\frac{\alpha}{R(1+\tau_{K_{a,i}})} \right)^{\alpha} \left(\frac{1-\alpha}{w(1+\tau_{L_{a,i}})} \right)^{1-\alpha} \right]^{\sigma} \left(\frac{\sigma-1}{\sigma} \right)^{\sigma-1} (1-\tau_{Y_{a,i}})^{\sigma} P^{\sigma} Y \tag{1.11}$$

Or in a concise way:

$$Y_{a,i} \propto \left(\frac{A_{a,i}}{\tau_{a,i}} \right) \tag{1.12}$$

We can solve for $P_{a,i}$:

$$P_{a,i} = \frac{1}{A_{a,i} \left(\frac{\alpha}{R(1+\tau_{K_{a,i}})} \right)^{\alpha} \left(\frac{1-\alpha}{w(1+\tau_{L_{a,i}})} \right)^{1-\alpha} \left(\frac{\sigma-1}{\sigma} \right)^{\frac{\sigma-1}{\sigma}} (1-\tau_{Y_{a,i}})} \tag{1.13}$$

According to the aggregation rule of price (1.3):

$$P = \frac{\left(\frac{R}{\alpha}\right)^{\alpha} \left(\frac{w}{1-\alpha}\right)^{1-\alpha}}{\left(\frac{\sigma-1}{\sigma}\right)^{\frac{\sigma-1}{\sigma}}} \left[\sum_{a} \sum_{i}^{N_a} \left(\frac{(1+\tau_{K_{a,i}})^{\alpha} (1+\tau_{L_{a,i}})^{1-\alpha}}{A_{a,i}(1-\tau_{Y_{a,i}})} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}} \tag{1.14}$$

Monopolistic Competition with Two Factors V

In this way, we can solve the "TFP" here as we did in (1.13) in terms of weighted $\tau_{a,i} \to \bar{\tau}$ intuitively. The following is the derivation. To proceed, we first compute market share:

$$\left(\frac{P}{P_{a,i}}\right)^{\sigma-1} = \frac{\left(\frac{(1+\tau_{K_{a,i}})^{\alpha}(1+\tau_{L_{a,i}})^{1-\alpha}}{A_{a,i}(1-\tau_{Y_{a,i}})}\right)^{1-\sigma}}{\sum_a \sum_i^{N_a} \left(\frac{(1+\tau_{K_{a,i}})^{\alpha}(1+\tau_{L_{a,i}})^{1-\alpha}}{A_{a,i}(1-\tau_{Y_{a,i}})}\right)^{1-\sigma}} = \frac{\left(\frac{\tau_{a,i}}{A_{a,i}}\right)^{1-\sigma}}{\sum_a \sum_i^{N_a} \left(\frac{\tau_{a,i}}{A_{a,i}}\right)^{1-\sigma}} \tag{1.15}$$

We need to compute the TFP, according to aggregation rule:

$$TFP = \frac{Y}{(\sum_{a,i} K_{a,i})^{\alpha}(\sum_{a,i} L_{a,i})^{1-\alpha}} \tag{1.16}$$

So the summation of denominator generates a structure of sales weight (from (1.9), (1.15)), i.e.,

$$\text{denominator of } \bar{\tau} \propto \left[\sum\sum\left(\frac{1-\tau_{Y_{a,i}}}{1+\tau_{K_{a,i}}}\right)\cdot\left(\frac{P_{a,i}Y_{a,i}}{PY}\right)\right]^{\alpha}\left[\sum\sum\left(\frac{1-\tau_{Y_{a,i}}}{1+\tau_{L_{a,i}}}\right)\cdot\left(\frac{P_{a,i}Y_{a,i}}{PY}\right)\right]^{1-\alpha} \tag{1.17}$$

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First order conditions

Predetermined case can be done in a similar way

Plug in (1.3), we can rewrite the static profit maximization problem as:

$$\Pi_{a,i} = \max_{Y_{a,i},L_{a,i},P_{a,i}} (1-\tau_{a,i}) P Y^{1/\sigma} Y_{a,i}^{1-1/\sigma} - Y_{a,i}/A_{a,i} - F_{a,i} \tag{2.1}$$

In the supplementary material, the author omits the strategic interactions among firms(*the impact on* $Y_{a,i}$). First order condition yields (reduced form of (1.9)):

$$\begin{aligned} L_{a,i} &= \left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} P^{\sigma} Y (1-\tau_{a,i})^{\sigma} A_{a,i}^{\sigma-1} \\ Y_{a,i} &= \left(\frac{\sigma}{\sigma-1}\right)^{-\sigma} P^{\sigma} Y (1-\tau_{a,i})^{\sigma} A_{a,i}^{\sigma} \\ P_{a,i} &= \left(\frac{\sigma}{\sigma-1}\right) \frac{1}{A_{a,i}} \frac{1}{1-\tau_{a,i}} \end{aligned} \tag{2.2}$$

and the revenue (*definition of* Π_d *and* $\Pi_{a,i}$):

$$P_{a,i} Y_{a,i} = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma-1} (P)^{\sigma} Y \left(1-\tau_{a,i}\right)^{\sigma-1} A_{a,i}^{\sigma-1} \equiv \Pi_d A_{a,i}^{\sigma-1} (1-\tau_{a,i})^{\sigma} \equiv \Pi_{a,i} + F_{a,i} \tag{2.3}$$

Firm Dynamics and Entry

If there's no uncertainty, which means that the firm's innovation channel is shut down[1]:

$$V_{a,i}(A_{a,i}) = \Pi_{a,i} + (1-\delta_{a,i})V_{a+1,i}(A_{a+1,i}) \tag{2.4}$$

If each period, firms have to invest on C to maintain productivity

$$V_{a,i}\left(A_{a,i}\right) = \max\left\{0, V^0_{a,i}\left(A_{a,i}\right)\right\}$$
$$V^0_{a,i}\left(A_{a,i}\right) = \max_{q\in[0,1]} \Pi_{a,i} - C\left(A_{a,i}, q\right) + (1-\delta_{a,i})\left(\frac{1}{R}\right)\left[qV_{a+1,j}\left(sA_{a,i}\right) + (1-q)V_{a+1,j}\left(\frac{A_{a,i}}{s}\right)\right] \tag{2.5}$$

Entry:

$$\text{Free entry:} n_e = \frac{1}{R}\sum_{i=1}^{N_1}\frac{V_{1,i}}{N_1},\ \text{fixed entry:} n_e = \frac{1}{R}V_{1,1} \tag{2.6}$$

[1] This is quite ambiguous, are we assume exogeneous path for each i here?

Market Clearing Condition

1. Consumption goods:
$$C = Y \tag{2.7}$$
2. Labor market clearing:
$$L_Y + L_R = \sum_a \sum_i L_{a,i} + L_R = L \tag{2.8}$$
3. The entry, innovation and overhead labor demand:
$$L_R = \frac{N_1}{N} n_e + \sum_a \sum_i (F_{a,i} + C(A_{a,i}, q_{a,i})) \tag{2.9}$$

Remark: Since the labor's wage has been set to be one. So N_1 can be pinned down endogenously by the entry condition and aggregate output[2].

[2] In a free entry case, maybe iteration is needed here because n_e is determined by N_1

Computational Steps

This is modified from the appendix part, I add some details here.

- Exogenous Productivity
- Endogenous Productivity

Computational Steps I

Exogenous Productivity

1. Create the group specific matrices $[A_{a,i}]_{20\times 2000}$, $[\delta_{a,i}]_{20\times 2000}$, $[\tau_{a,i}]_{20\times 2000}$, $[F_{a,i}]_{20\times 2000}$, the dimension of i labels productivity group .
2. Match the entrants $[A_{1,i}]_{1\times 2000}$ to the data, with linear interpolation.
   - In the endogenous case, assume firm's qualities are drawn from log-normal distribution $\exp(\mu + \sigma Z)$. Iterate as the distribution's SE fits with data and free (fixed) entry condition. In this way, μ and σ can be solved (& cutoff entrance rule).[3]
3. Adjust $[A_{a,i}]_{20\times 2000}$ and $[\delta_{a,i}]_{20\times 2000}$ to match the productivity distribution, exit by age and productivity.[4]

Computational Steps II

Exogenous Productivity

4. Adjust $\tau_{a,i}$ and $F_{a,i}$ in each grid point to match: (τ) the observed elasticity of average revenue products with respect to productivity in the U.S.; (F) linearly increasing in log-productivity, with the maximum slope such that no firms exit endogenously (*which means the productivity level is very low*).
5. Compute value function[5] by backward induction and iterate Π_d until the free entry condition is satisfied.
   Remark: This can be done through Newton Method. For example, Π_d is less than the true level, then the entrance will be low (life time value), so as the labor demand.

Computational Steps III

Exogenous Productivity

1. Create the productivity matrix of Indian and Mexico: $[A_{a,i}]_{20\times 2000}^{\text{Indian, Mexico}}$. When doing counterfactuals, other coefficients(matrices) are kept.
2. Find out the new productivity "cutoff rule" of Indian and Mexico.
3. Find out Π_d in the same way in Indian and Mexico.

[3] I think here we should solve a loop iteration problem, because we have to find out μ of the entrants. However, I bet the authors pinned down μ by $\bar{A}$.

[4] Shall we simulate here? Like a self-consistent loop.

[5] It is relatively easy in this deterministic case.

Computational Steps I

Endogenous Productivity

1. Create the group specific matrices $[A_{a,i}]_{20\times 66}$, $[\delta_{a,i}]_{20\times 66}$, $[\tau_{a,i}]_{20\times 66}$, the dimension of i labels productivity group. s is pinned down from employment growth volatility.
2. Match the probability distribution function of entrants' productivity to the data, with linear interpolation.
3. Adjust $[A_{a,i}]_{20\times 66}$ and $[\delta_{a,i}]_{20\times 66}$ to match the productivity distribution, exit by age and productivity.
4. Adjust tax rate $[\tau_{a,i}]_{20\times 66}$ to match the observed elasticity of average products with respect to productivity in the U.S.
5. Iterate on b and h to match the productivity by age's data, given the endogenous decision of innovation.
6. Iterate to find out Π_d and value function.

Repeat 3. - 6. until it converges.

Computational Steps II

Endogenous Productivity

Roadmap:

- If we set $A_{a,i}$ to be the value of cross-sectional productivity without innovation.
- b, h is set before iteration, then we can get a innovation investment rule and may distort the dynamic innovation rule because of the value function.
- Therefore, we have to adjust b, h, so as Π_d.

Once $A_{a,i}$ is solved, the step for the counterfactuals in India and Mexico is relatively easy.

1. Rematch $\tau_{a,i}$ with the the observed elasticity of average products with respect to productivity in India or Mexico.
2. Find out investment decision rule.
3. Compute value function and iterate Π_d until free entry condition is satisfied.

Repeat 2. - 3. until it converges.

Solving Procedure

- The Key is to solve Π_d, as I have mentioned, Newton method.
- Conditioned on Π_d, we update V and other parameters.
- Then we plug in the updated parameters to find updated Π_d.

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- I studied Saeed Shaker-Akhtekhane's work and Benkamin Moll's related work on it. They use a different formulation on the entrance. Actually, we don't need to add such age related variables here to generate the life cycle problem because once the entrance and exit bar has been set (exogenously or endogenously), the life cycle can be determined by green function.
- I don't know whether someone has worked on impulse control's problem like what we have learnt from Stokey's book, i.e., firm can invest, because the entrance problem quite resembles Sannikov and Demarzo (2006).
- Master equation, i.e., Adrien (2021), is useful to solve the aggregate shock's case (KS), I learned it from statistical mechanic's course and I am quite familiar with it because I have done some related research on quantum many-body system, which also depends on master equation.
- We can adjust η in Ben's code.

Benchmark Model (Exogeneous Entry)

Since Saeed Shaker-Akhtekhane's presentation of model is little wired, I follow many traditional corporate fin's literature and write it down in Ben's way:

$$v(z)=\max_{\{n_t\}_{t\geq 0},\tau}\left\{\mathbb{E}_0\int_0^\tau e^{-\rho t}\left(pf\left(z_t,n_t\right)-wn_t-c_f\right)dt+e^{-\rho\tau}v^*\right\} \tag{3.1}$$

$$dz_t=\mu\left(z_t\right)dt+\sigma\left(z_t\right)dW_t,\quad z_0=z \text{ BM is reflected at the boundary.}$$

We can maximize it for every period. The HJBVI problem here is:

$$\min\left\{\rho v(z)-v'(z)\mu(z)-\frac{1}{2}v''(z)\sigma^2(z)-\pi(z),v(z)-v^*\right\}=0,\quad \text{all } z\in(0,1) \tag{3.2}$$

It is easy to see that the remaining active rule is a single cutoff one: $\mathcal{Z}=[x,1]$, which means once z drops below x, the firm has to exit. The g is normalized to be one: $\int_{\mathcal{Z}}g(z,t)dz=1$.

Entry and Exit

In the baseline model, the entry is used to fill the gap created by exits. So, Kolmogorov Forward Equation:

$$\frac{\partial g(z,t)}{\partial t} = -\frac{\partial(\mu(z)g(z,t))}{\partial z} + \frac{1}{2}\frac{\partial^2}{\partial z^2}(\sigma^2(z)g(z,t)) + m(t)\psi(z) \tag{3.3}$$

where $\psi(z)$ is the initial productivity distribution and satisfies: $\int_x^1 \psi(z)dz = 1$. In the stationary equilibrium:

$$0 = -(\mu(z)g(z))' + \frac{1}{2}\left(\sigma^2(z)g(z)\right)'' + m\psi(z), \qquad \text{all } z \in \mathcal{Z} \tag{3.4}$$

Boundary conditions:

$$\begin{aligned} &m = -\int_{\mathcal{Z}} \left(\mathcal{A}^* g\right)(z)dz,\ g(x,t) = 0 \\ &p = D(Q), \quad w = W(N), \quad Q = \int_z q(z)g(z)dz, \quad N = \int_z n(z)g(z)dz \end{aligned} \tag{3.5}$$

Conversion of LCP

We can get $[0, 1]$'s grid points, i.e., $\frac{i}{N}, 0 \leq i \leq N$, then we can convert the HJBVI problem into a LCP problem:

$$\begin{aligned} Bx + q &\geq 0 \\ x &\geq 0 \\ x^T(Bx + q) &= 0 \end{aligned} \tag{3.6}$$

where:$x = v - v^*$ $(x = x_{N\times 1})$, $B = \rho I_{N\times N} - A_{N\times N}$.

Main Model (Endogenous Entry)

We add the elastic endogenous entry condition here, where η is the elasticity and $\eta \to \infty$ represents the free entry's case:

$$m = \bar{m} \exp\left(\eta(\int_0^1 v(z)\psi(z)dz - c_e)\right)$$

Thus, the main model can be characterized by:

$$\begin{aligned}
0 &= \min\left\{\rho v(z) - v'(z)\mu(z) - \frac{1}{2}v''(z)\sigma^2(z) - \pi(z), v(z) - v^*\right\}, \quad \text{all } z \in (0,1) \\
0 &= -(\mu(z)g(z))' + \frac{1}{2}\left(\sigma^2(z)g(z)\right)'' + m\psi(z), \quad \text{all } z \in \mathcal{Z}, \\
m &= \bar{m} \exp\left(\eta\left(\int_0^1 v(z)\psi(z)dz - c_e\right)\right) \\
p &= D(Q), \quad w = W(N), \quad Q = \int_{\mathcal{Z}} q(z)g(z)dz, \quad N = \int_{\mathcal{Z}} n(z)g(z)dz
\end{aligned} \tag{3.7}$$

Shaker's Model considers the life cycle distribution problem by adding the entry cost discounted into the future. Ben's model can similarly generate such life cycle's problem.

Main Model (Endogenous Entry)

Generate the Life cycle's problem

We consider the Green's function's evolution (response function of Dirac delta function):

$$\frac{\partial G(z,t;z_0,0)}{\partial t} = -\frac{\partial(\mu(z)G(z,t;z_0,0))}{\partial z} + \frac{1}{2}\frac{\partial^2[\sigma^2(z)G(z,t;z_0,0)]}{\partial z^2}, \forall z \in \mathcal{Z}, \text{aka KF} \tag{3.8}$$

Recall the difference between above eq and (3.3):

$$\frac{\partial g(z,t)}{\partial t} = -\frac{\partial(\mu(z)g(z,t))}{\partial z} + \frac{1}{2}\frac{\partial^2}{\partial z^2}(\sigma^2(z)g(z,t)) + \underbrace{\boxed{m(t)\psi(z)}}_{\text{injected density}}, \forall z \in \mathcal{Z}$$

(3.8) tells us the evolution path's density in the future and can be solved directly because it is an heat equation (Analytically solvable by separating variables). Further, as the exit rule is a single cutoff rule, it is sufficient to consider KF.

Main Model (Endogenous Entry)

Generate the Life cycle's problem

The real life cycle's cross section (by convolution) in a stationary Eq:

$$\rho(z,a) = \int_{z_0 \in \mathcal{Z}} G(z,a;z_0,0)\psi(z_0)dz_0 \tag{3.9}$$

Once the stationary Eq has been obtained (z_{exit}), we can find out the cross section density: $\rho(z,a)$. ρ is decreasing of a because every time there's some firm die due to the idiosyncratic shock.

Remark: Shaker's consideration of rigid entry and exit is not wrong, I guess he forget this method after getting the stationary equilibrium. Considering the age as an variable is meaningful, but it is not necessary here.

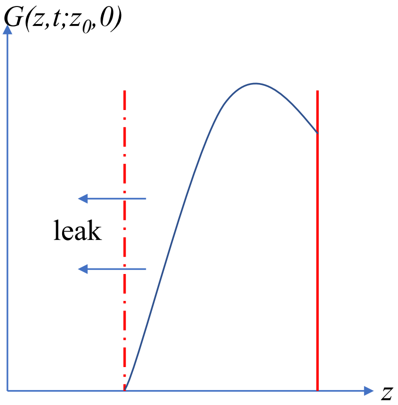

Figure: Exit of a single firm