Upwind Derivative vs Downwind Derivative

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Upwind and downwind method

Let us start from convective equation (square root of wave Eq):

$$\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = 0, \tag{0.1}$$

we assume $a > 0$, it has a solution:

$$u(x, t+\Delta) = u(x - a\Delta, t)$$

Upwind method utilizes upstream's information.

- Upwind $(a > 0)$:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

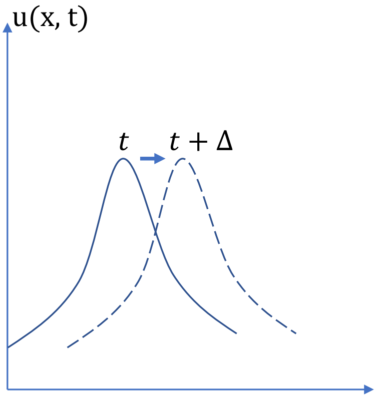
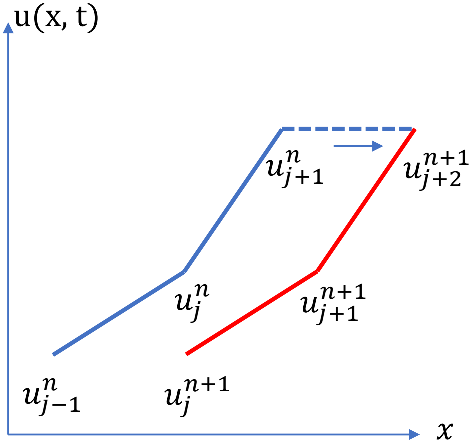
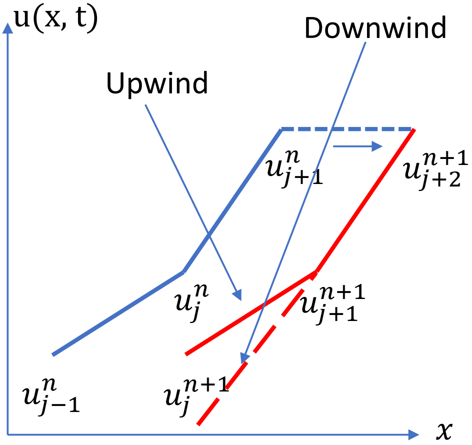


Figure: Convective Equation

Upwind and downwind method Courant et al. (1952)[1]

Why upwind/downwind?





[1] Courant, Isaacson, and Rees proposed the CIR method

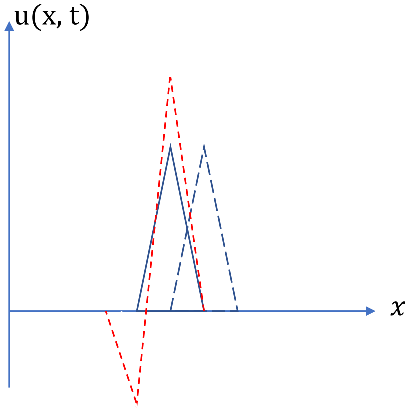
Upwind and downwind method

Why upwind/downwind?

We can also consider an extreme case that $u^0_{n_1} = 1, u^0_n = 0, n \neq n_1$. The wave behave morbidly.

- Intuitively, the information for the part uninfluenced cannot be utilized, they may be wrong.
- Quantitatively, it will lead to an error: $|(u^n_{j+1} - u^n_j) - (u^n_j - u^n_{j-1})|$, for uninfluenced region, first part is generally zero while the second part is a finite one $O(1)$, or at least $O(\Delta x)$.
- **Remark**: there's another error comes from pseudo diffusion in convective Eq, it can be eliminated by choosing $\Delta t = \frac{\Delta x}{a}$.

$$\frac{\partial u(x,t)}{\partial t} + a\frac{\partial u(x,t)}{\partial x} = \frac{1}{2}a\Delta x\left(1 - a\frac{\Delta t}{\Delta x}\right)\frac{\partial^2 u(x,t)}{\partial x^2}$$



Upwind and downwind method

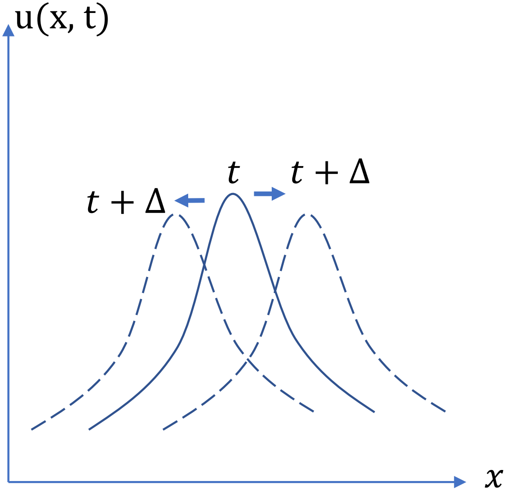
In practice, we consider a PDE:

$$\frac{\partial u}{\partial t} + \frac{f(u)}{\partial x} = 0$$

can be decomposed into:

$$\frac{\partial u}{\partial t} + \frac{f^+}{\partial x} + \frac{f^-}{\partial x} = 0$$

We introduce the upwind to aviod information from upstream getting stucked.



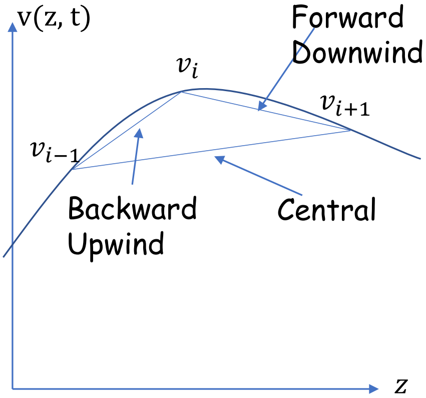
Ben Moll's HJB and KF

Recall Ben's HJB:

$$\rho v(z) - v'(z)\mu(z) - \frac{1}{2}v''(z)\sigma^2(z) - \pi(z)$$

Standard Algorithm with upwind finite difference:

- **Forward approximation** (downwind): $v'(z) \approx \frac{v_{j+1}^n - v_j^n}{\Delta z}$
- **Backward approximation** (upwind): $v'(z) \approx \frac{v_j^n - v_{j-1}^n}{\Delta z}$
- Central difference diverges in convective Eq.



The HJB in difference method

Explicit:

$$\frac{v_j^{n+1} - v_j^n}{\Delta} + \rho v_j^n = \pi(z) + \mu(z) 1_{\mu(z)>0} \partial_F v_j^n + \mu(z) 1_{\mu(z)<0} \partial_B v_j^n + \frac{1}{2}\sigma_j^2 \partial^2 v_j^n$$

Or equivalently:

$$\frac{v_j^{n+1} - v_j^n}{\Delta} + \rho v_j^n = \pi(z) + \mu(z) 1_{\mu(z)>0} \frac{v_{j+1}^n - v_j^n}{\Delta z} + \mu(z) 1_{\mu(z)<0} \frac{v_j^n - v_{j-1}^n}{\Delta z} + \frac{1}{2}\sigma_j^2 \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta z^2}$$

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%Construct matrix
X = -min(mu,0)/dz + sig2/(2*dz2);
Y = -max(mu,0)/dz + min(mu,0)/dz - sig2/dz2;
Z = max(mu,0)/dz + sig2/(2*dz2);
A = spdiags(Y,0,I,I) + spdiags(X(2:I),-1,I,I) +
spdiags([0;Z(1:I-1)],1,I,I);
```

positive derivative/
negative derivative

Matrix Formulation

We reconstruct the matrix formulation of HJB in finite difference:

$$\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^n = \pi_n + \mathcal{A}^n v^n$$

where:

$$\mathcal{A}^n = \begin{bmatrix} y_1 & z_1 & 0 & \cdots & 0 \\ x_2 & y_2 & z_2 & \cdots & 0 \\ 0 & x_3 & y_3 & z_3 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & x_N & y_N \end{bmatrix}$$

and

$$x_j = -\frac{\mu(z) 1_{\mu<0}}{\Delta a}, y_j = -\frac{\sigma_j^2}{\Delta z^2}, z_j = \frac{\mu(z) 1_{\mu>0}}{\Delta a}$$

If we have multi variables to deal with, i.e., z, s, we should consider $\mathcal{Z} \otimes \mathcal{S}$ and $\mathcal{A}_z \otimes \mathcal{A}_s$, it is not a triangular matrix any more.

References

Courant, Richard, Eugene Isaacson, and Mina Rees, "On the solution of nonlinear hyperbolic differential equations by finite differences," *Communications on pure and applied mathematics*, 1952, *5* (3), 243–255.