1 Proof of Theorem 3

Additional Proof of submartingale property in Theorem 3 of [3].

$$G_t^X = \int_0^t (Y_s + \tilde{u}(Z_s))ds - \int_0^t e^{-\rho s} U_d(Y_s) dX_s + e^{-rt} J(Y_t, Z_t). \tag{1.1}$$

the last term is expected J with respect to filtration $\mathcal{F}_t$. Consider a time $v < t$, it suffices to show that:

$$E[G_t^X | \mathcal{F}_v] \geq G_v^X \tag{1.2}$$

The LHS can be written as:

$$\begin{aligned}\int_0^v e^{-rs}(Y_s + \tilde{u}(Z_s))ds - \int_0^v e^{-\rho s} U_d(Y_s) dX_s \\ + E\left[\int_v^t e^{-rs}(Y_s + \tilde{u}(Z_s))ds - \int_v^t e^{-\rho s} U_d(Y_s) dX_s + e^{-rt} J(Y_t, Z_t) | \mathcal{F}_v\right]\end{aligned} \tag{1.3}$$

The first 2 term corresponds to G_v^X's first 2 terms. We add $e^{-rs} J(Y_s, Z_s)$ (*realized at* v), then a "revealed preference"-like algebra manipulation shows the rest is greater than zero. Recall:

$$J(y, z) = \inf_{X \in \mathcal{I}(x)} E\left[\int_0^\infty e^{-rs}(Y_s + \tilde{u}(Z_s))ds - \int_0^\infty e^{-\rho s} U_d(Y_s) dX_s\right] \tag{1.4}$$

The feasible allocation $\mathcal{I}(x_t)$ adapted to filtration $\mathcal{F}_t$, is the subset of $\mathcal{I}(X_s)$ (i.e. realized at t is $X_t \in \mathcal{I}(X_v)$, thus

$$\begin{aligned}e^{-rv} \inf_{X \in \mathcal{I}(x_v)} E\left[\int_0^\infty e^{-r(s-v)}(Y_s + \tilde{u}(Z_s))ds - \int_0^\infty e^{-\rho(s-v)} U_d(Y_s) dX_s | \mathcal{F}_v\right] \leq \\ E\left[\int_v^t e^{-rs}(Y_s + \tilde{u}(Z_s))ds - \int_v^t e^{-\rho s} U_d(Y_s) dX_s | \mathcal{F}_v\right] + \\ e^{-rt} E\left[\inf_{X \in \mathcal{I}(x_t)} E\left[\int_0^\infty e^{-r(s-v)}(Y_s + \tilde{u}(Z_s))ds - \int_0^\infty e^{-\rho(s-v)} U_d(Y_s) dX_s | \mathcal{F}_t\right] | \mathcal{F}_v\right]\end{aligned} \tag{1.5}$$

The rest of the proof agrees with the proof from the article on core priciples. Here, instead, we consider the Doob-Mayer decomposition. Since G_t^X is a submartingale, we can write it into martingale part and non-dicreasing part (*may be with some jumps*). According to above derivation, we know that martingale part is nothing but J, while non-decreasing part is the difference conditional on filtration. The difference part is generally greater than zero and is zero if and only if the optimality is satisfied. It is also okay to "differtiate" G_t^X and cancel the non-decreasing part pointwisely.

References

[1] Karatzas, I. and Shreve, S.: *Brownian Motion and Stochastic Calculus*. Springer Verlag, 2 ed, New York, 1998

[2] Stokey, N. (2008): *The Economics of Inaction: Stochastic Control Models with Fixed Costs.* Princeton University Press, New Jersey.

[3] Miao, Jianjun, and Yuzhe Zhang. A duality approach to continuous-time contracting problems with limited commitment. *Journal of Economic Theory* 159 (2015): 929-988.