1 Newton

(a) We know that

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{1}$$

$$\frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{2}$$

$$p(r) = K\rho(r)^{1+\frac{1}{n}} \tag{3}$$

(2) can be rewritten as:

$$\frac{1}{\rho}\frac{dp}{dr} = -\frac{Gm}{r^2} \tag{4}$$

Taking its derivative with respect to r yields

$$\begin{aligned}\frac{d}{dr}\left(\frac{1}{\rho}\frac{dp}{dr}\right) &= \frac{2Gm}{r^3} - \frac{G}{r^2}\frac{dm}{dr} \\ &= -\frac{2}{r}\left(\frac{1}{\rho}\frac{dp}{dr}\right) - 4\pi G\rho\end{aligned} \tag{5}$$

By multiplying Eq.(5) with r^2, and collecting the r derivatives of p on one side, we get:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G r^2\rho \tag{6}$$

By introducing a new function θ, which satisfies the relation $\rho = \rho_c\theta^n$(ρ_c is a constant), we can rewrite Eq.(3) as:

$$p = K\rho_c^{1+\frac{1}{n}}\theta^{n+1} \tag{7}$$

Inserting Eq.(7) into (6), we get:

$$\frac{d}{dr}\left(\frac{r^2}{\rho_c\theta^n}K\rho_c^{\frac{1}{n}+1}(n+1)\theta^n\frac{d\theta}{dr}\right) = -4G\pi r^2\rho_c\theta^n \tag{8}$$

Simplified,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2K\rho_c^{\frac{1-n}{n}}(n+1)\frac{d\theta}{dr}\right) = -4G\pi\rho_c\theta^n \tag{9}$$

By defining $\alpha \coloneqq \sqrt{K\rho_c^{\frac{1-n}{n}}(n+1)/4\pi G}$ and introducing a new variable ξ which satisfies the relation $r = \alpha\xi$, we can rewrite Eq.(9) as:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{10}$$

which is the Lane-Emden equation.

Analytical solutions of Lane-Emden equation only exist for $n = 0, 1, 5$. The regular solutions near the center, i.e. $\xi \approx 0$ can be approximated as a power series:

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 + \dots \tag{11}$$

This series has an error of order $O(\xi^6)$.

The Mathematica code used to calculate this series expression can be found in the Supplementary Material.

Eq.(1) can be rewritten as

$$dm(r) = 4\pi r^2\rho(r)dr \tag{12}$$

which, after scaling the appropriate variables ($\rho = \rho_c$, $r = \alpha\xi$), becomes

$$dm = 4\pi\rho_c\alpha^3\xi^2\theta^n d\xi \tag{13}$$

Integrating both sides from 0 to ξ_n gives

$$\begin{aligned} m &= 4\pi\rho_c\alpha^3 \int_0^{\xi_n} \xi^2\theta^n d\xi \\ &= 4\pi\rho_c\alpha^3 \int_0^{\xi_n} -\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) d\xi \\ &= 4\pi\rho_c\alpha^3\xi_n^2\left(-\theta'(\xi_n)\right) \end{aligned} \tag{14}$$

Since $r = \alpha\xi$ and ξ_n is the maximum value of ξ where $\theta(\xi) \geq 0$, we can conclude that $R = \alpha\xi_n$ is the radius of the star. Multiplying and dividing Eq.(14) with ξ_n to write it in terms of R, we get:

$$M = 4\pi\rho_c R^3\left(-\frac{\theta'(\xi_n)}{\xi_n}\right) \tag{15}$$

In order to find the total mass of a star in terms of its radius, we need to combine Eq.(14) and $R = \alpha\xi_n$.

We'll get rid of α, and write its true value instead, with the aim to connect the two equations by isolating ρ_c in each one. Eq.(14), with this prescription, can be written as:

$$M = 4\pi\left(\frac{K(n+1)}{4\pi G}\right)^{\frac{3}{2}}\left(-\xi_n^2\theta'(\xi_n)\right)\rho_c^{\frac{3-n}{2n}} \tag{16}$$

Similarly,

$$R = \alpha\xi_n = \left(\frac{K(n+1)}{4\pi G}\right)^{\frac{1}{2}}\xi_n\rho_c^{\frac{1-n}{2n}} \tag{17}$$

Isolating ρ_c form both equations, we get:

$$\begin{aligned} \rho_c &= \left(\frac{M}{4\pi\left(\frac{K(n+1)}{4\pi G}\right)^{\frac{3}{2}}\left(-\xi_n^2\theta'(\xi_n)\right)}\right)^{\frac{2n}{3-n}} \\ &= \left(\frac{R}{\left(\frac{K(n+1)}{4\pi G}\right)^{\frac{1}{2}}\xi_n}\right)^{\frac{2n}{1-n}} \end{aligned} \tag{18}$$

which results in the relation:

$$M = (4\pi)^{\frac{1}{1-n}}\left(\frac{K(n+1)}{G}\right)^{\frac{n}{n-1}}\xi_n^{\frac{n+1}{n-1}}(-\theta'(\xi_n))R^{\frac{3-n}{1-n}} \tag{19}$$

(b) The Python code for extracting the *.csv* file can be found in the Supplementary Material. The M vs R plot of the white dwarfs is included in Figure 1.

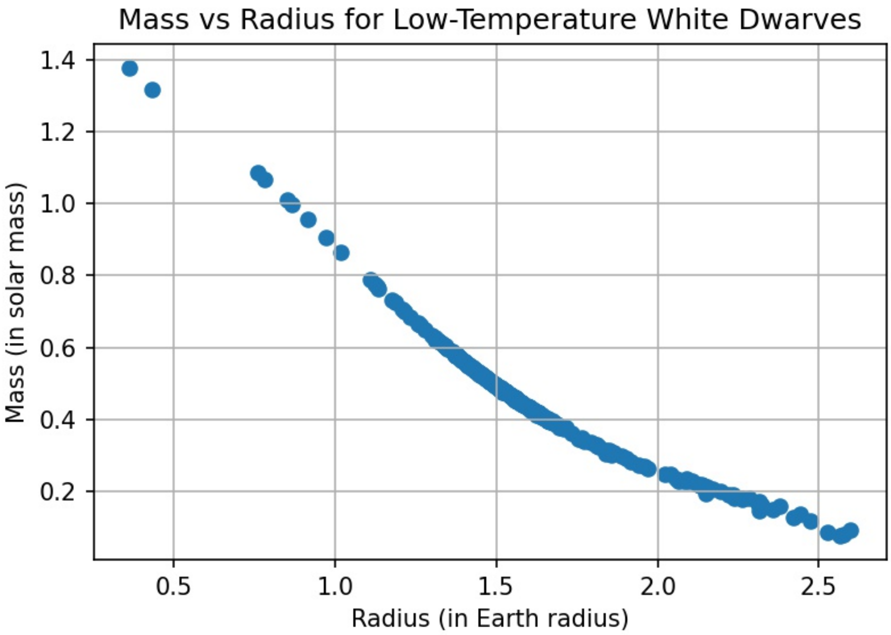

Figure 1: Mass v Radius plot of low-temperature White Dwarves.

(c) The series expansion (obtained by Mathematica) for the polytropic approximation of pressure is:

$$\begin{aligned} P &= \frac{Cx^5}{5} + \mathcal{O}(x^6) \\ &\simeq \frac{8C}{5D^{5/q}} \rho^{1+\frac{1}{q/(5-q)}} \end{aligned} \tag{20}$$

which yields the constants K_* and n_*.

After making the appropriate fit, q seems to be fluctuating around 3, as seen in Figure 2. Since we know from theory that q is an integer, we can deduce that q is exactly equal to 3.

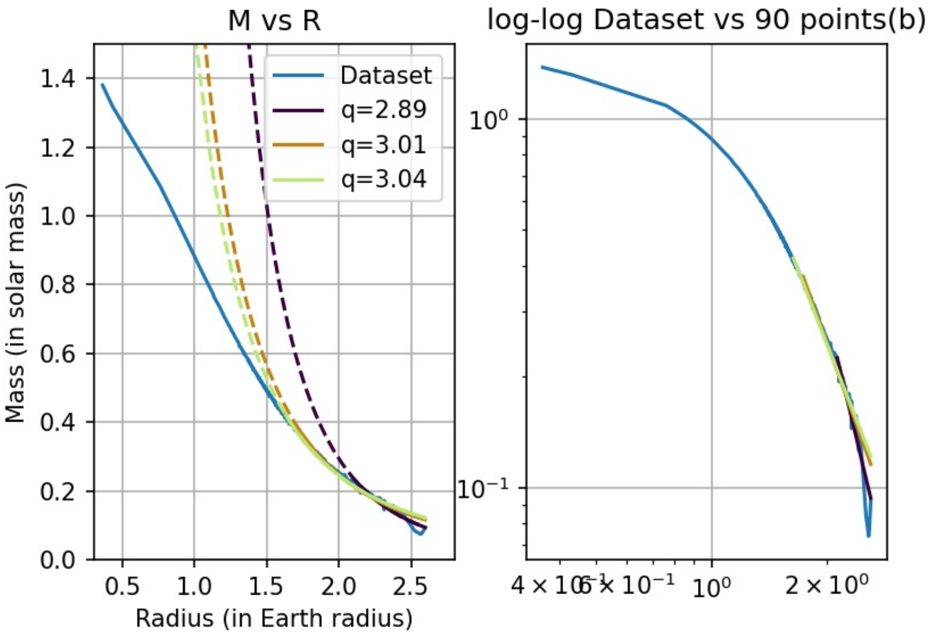

Figure 2: a) M-R plot, low-mass fit. b) log-log plots.

After obtaining the specific value of q, and subsequently n_*, another fitting reveals the value of K_*, which turns out to be $\approx 2.83 \times 10^{12}$ cm^4 g$^{-\frac{2}{3}}$ s^{-2}.

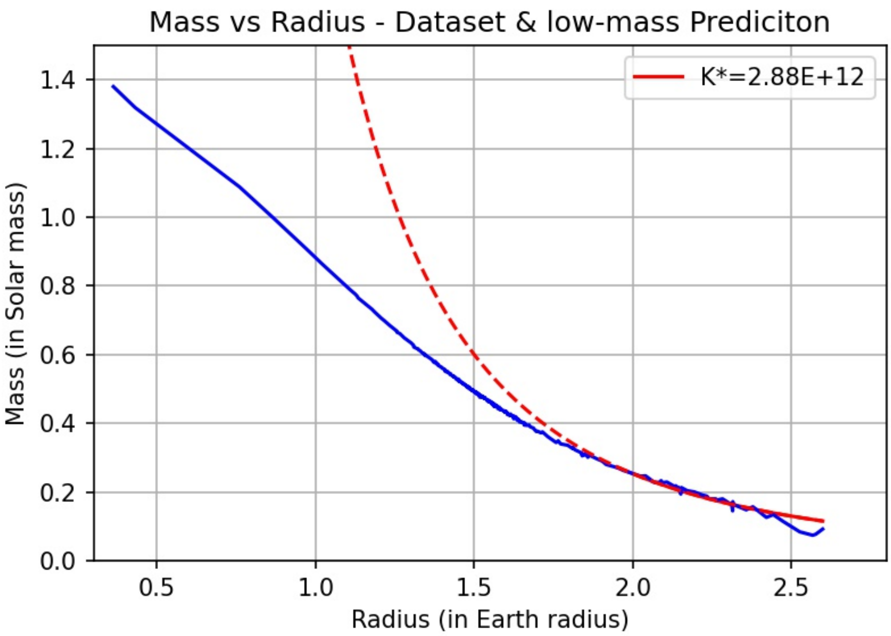

Figure 3: M-R plot and K_* fit

Central density ρ_c has the formula:

$$\rho_c = \frac{M}{4\pi R^3} \frac{\xi_n^3}{\left(-\xi^2 \theta'(\xi)\right)_{\xi=\xi_n}} \tag{21}$$

ξ_n and $\left(-\xi^2\theta'(\xi)\right)_{\xi=\xi_n}$ can be obtained by solving the Lane-Emden Equation. After substituting the appropriate values, we find ρ_c to be proportional to M^2, which can be seen in Figure 4.

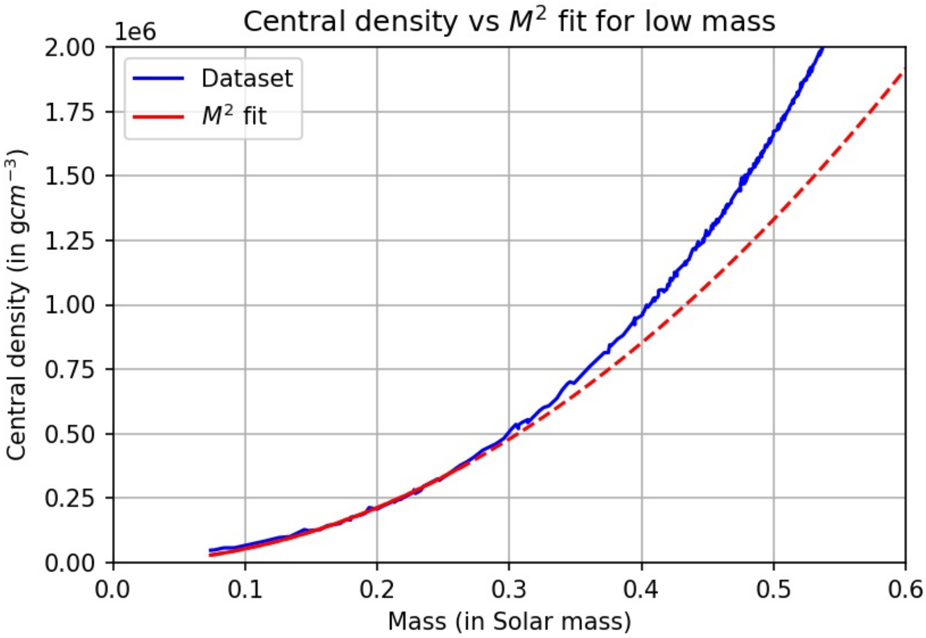

Figure 4: Density vs Mass, M^2 fit (red)

(d) A range of $(10^5, 10^9)$ was used for ρ_c and possible D values between $(10^6, 10^7)$. The D value with lowest RMS error approximated the theoretical value $D = 2.00 \times 10^6$, which was thus used.

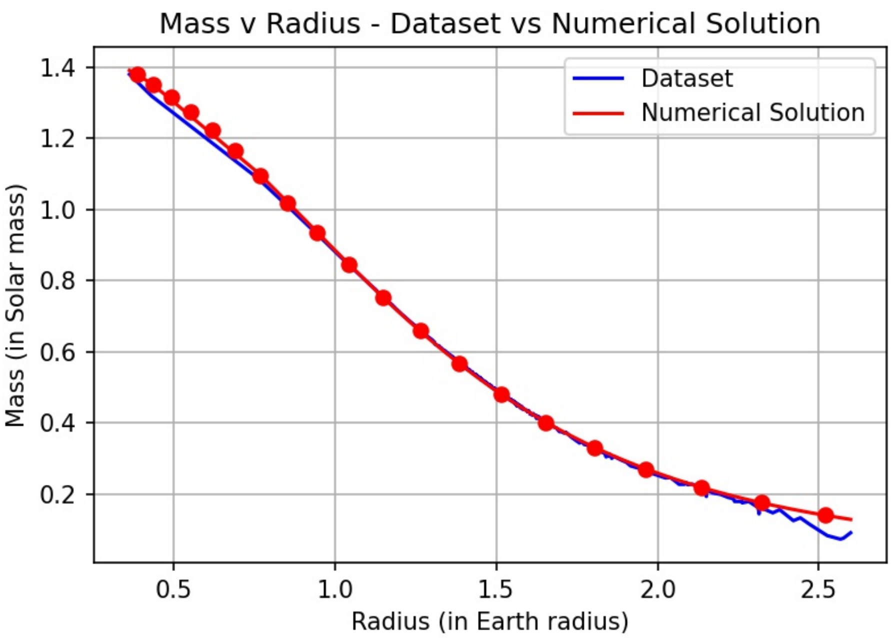

Figure 5: Mass v Radius plot, $D = 2.00 \times 10^6$ fit

(e) After plotting a number of ρ_c values between 10^{10} and 10^{15},the White Dwarf mass for which a solution existed was around $1.39 M_{\odot}$, which is in accordance with the currently accepted Chandrasekhar limit, $1.4 M_{\odot}$. The M-R relationship can be seen in Figure 6.

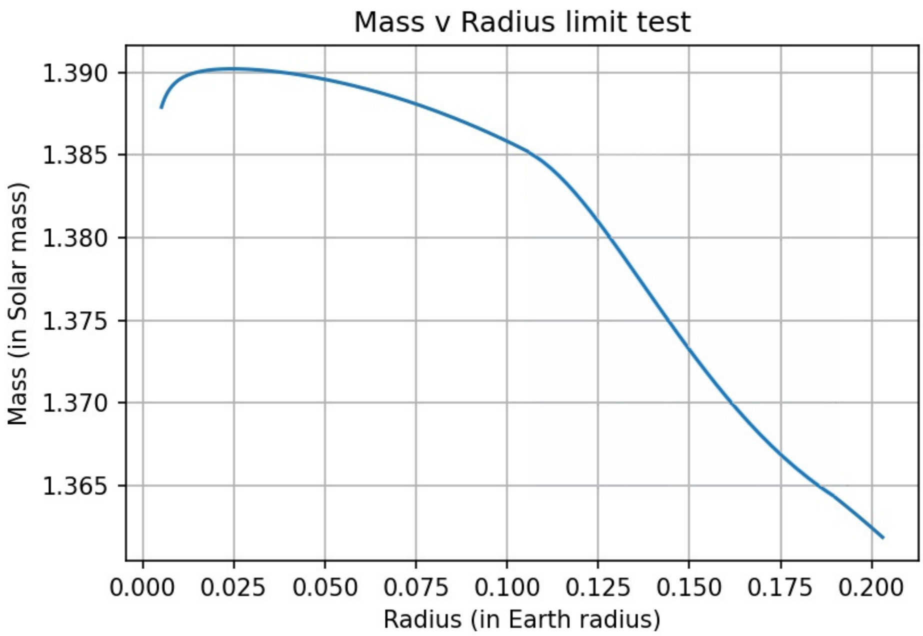

Figure 6: Mass v Radius plot for various ρ_c values.

2 Einstein

(a) The mass-radius curve for NSs is presented in Figure 7.

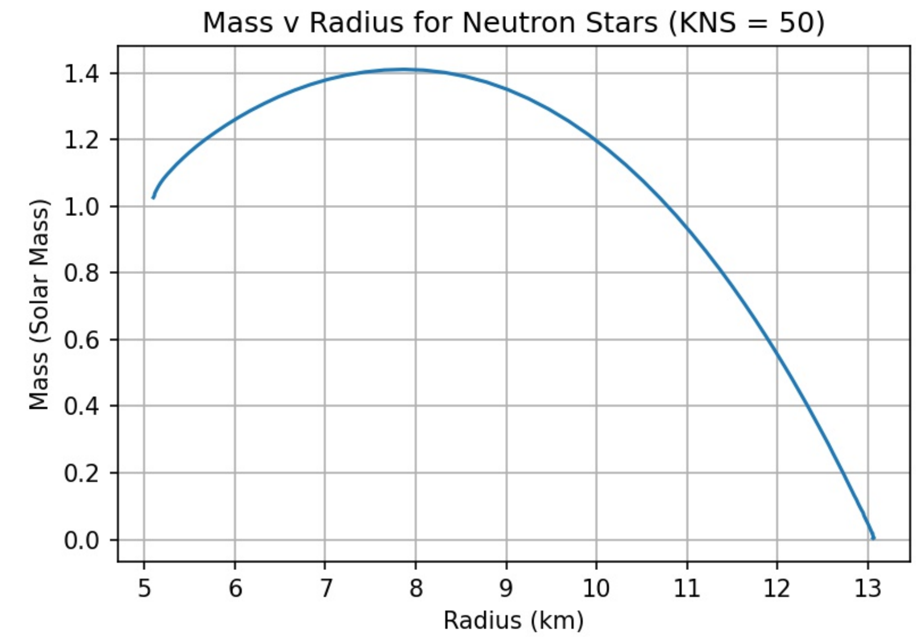

Figure 7: M-R plot for neutron stars with $K_{NS} = 50$.

(b) The fractional binding energy(Δ)-radius curve for NSs is presented in Figure 8.

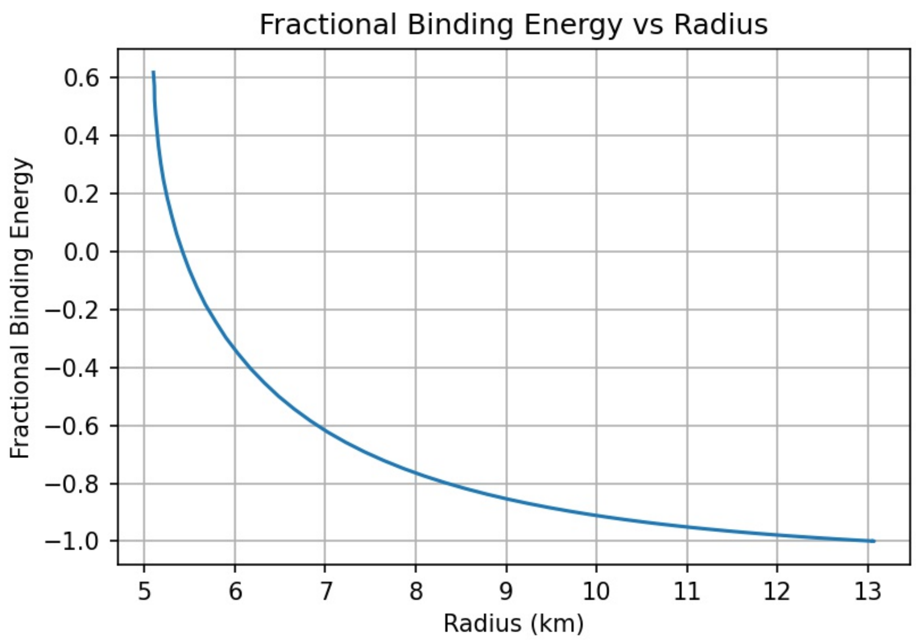

Figure 8: Δ-R plot for neutron stars with $K_{NS} = 50$.

(c) The mass-central density(ρ_c) curve for NSs is presented in Figure 9. The maximum stable mass for which the solution exists is $1.41 M_{\odot}$, coinciding with $K_{NS} = 50$.

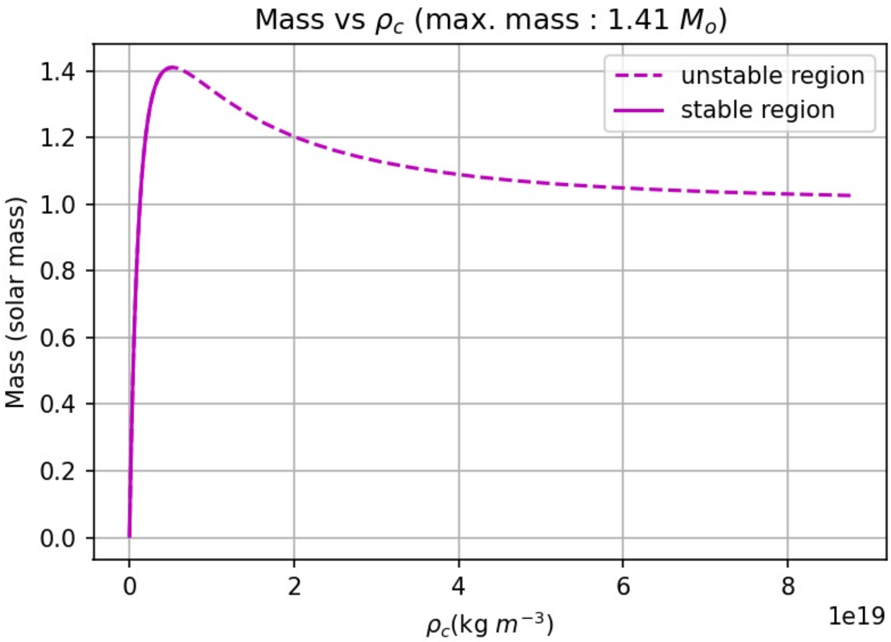

Figure 9: M-ρ_c plot for neutron stars with $K_{NS} = 50$.

(d) The lowest K_{NS} value to have maximum mass greater than $2.14M_\odot$, so any K_{NS} is $K_{NS} = 116$, the M and $\Delta - R$ and M-ρ_c curves for which are presented in Figures 10, 11, 12. To show another possible value, the graphs for $K_{NS} = 125$ were drawn and presented in Figures 13, 14, 15.

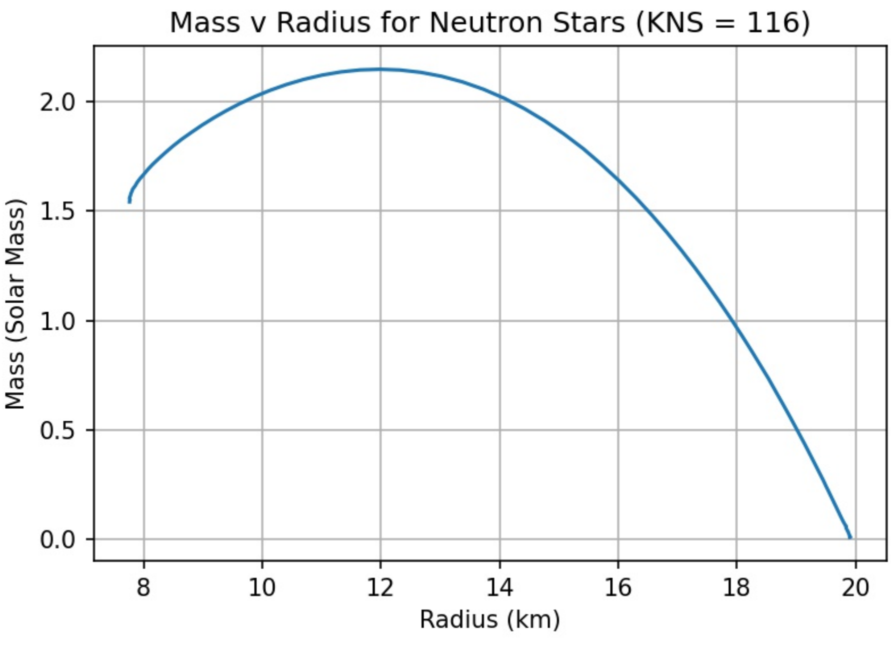

Figure 10: M-R plot for neutron stars with $K_{NS} = 116$.

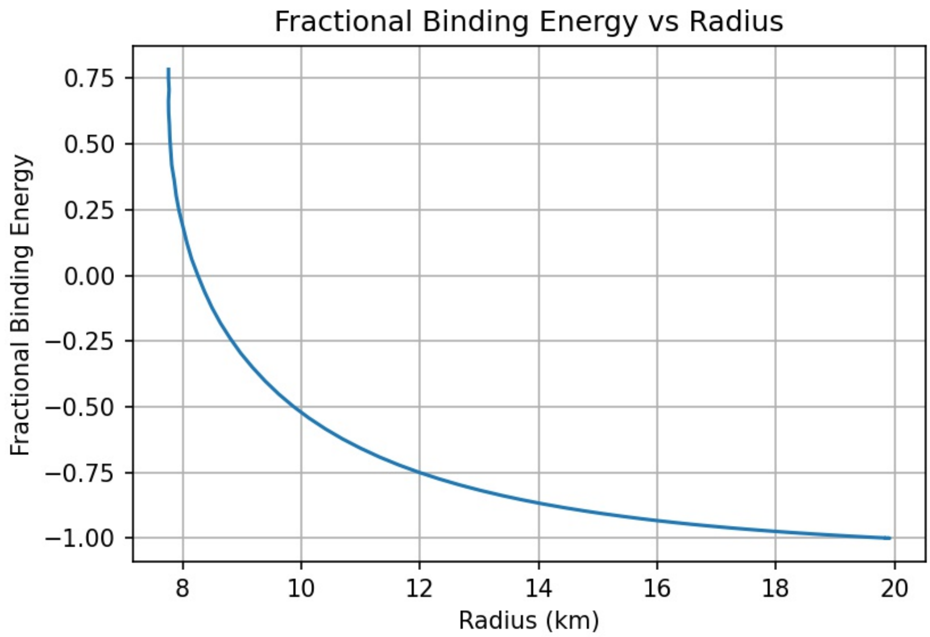

Figure 11: Δ-R plot for neutron stars with $K_{NS} = 116$.

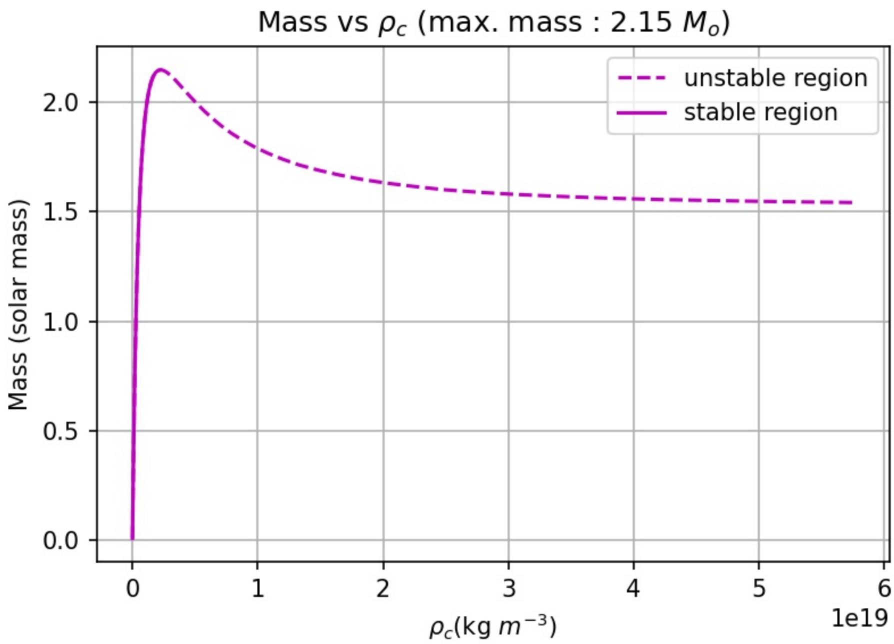

Figure 12: M-ρ_c plot for neutron stars with $K_{NS} = 116$.

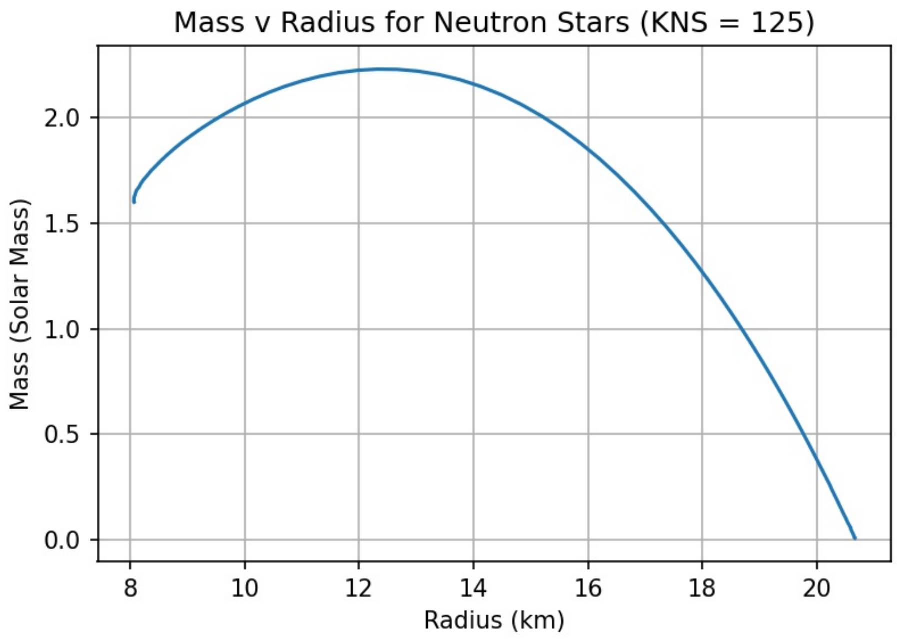

Figure 13: M-R plot for neutron stars with $K_{NS} = 125$.

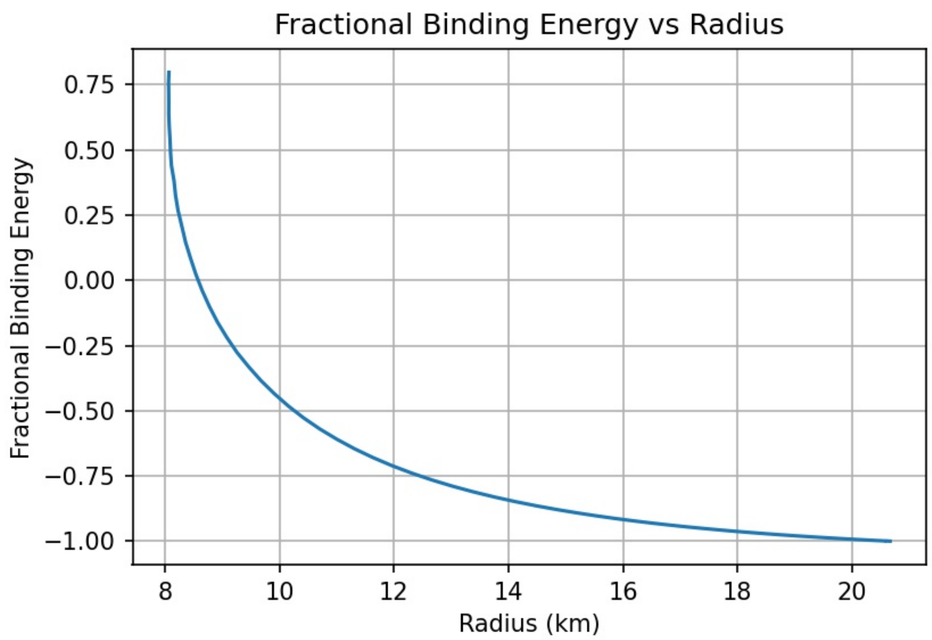

Figure 14: Δ-R plot for neutron stars with $K_{NS} = 125$.

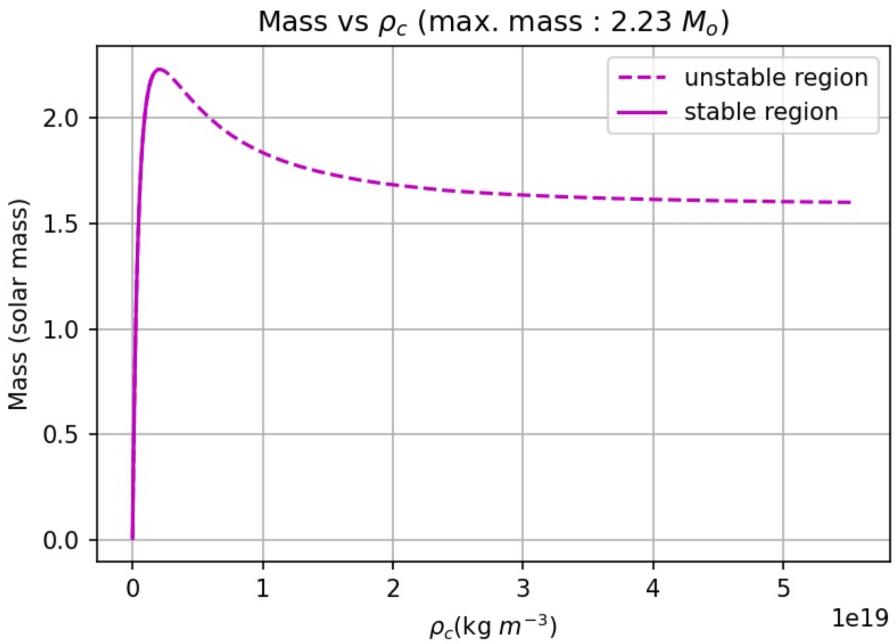

Figure 15: M-ρ_c plot for neutron stars with $K_{NS} = 125$.

(e) When $(r > R)$, $\bar{\nu}(r) = \ln\left(1 - \frac{2M}{r}\right)$. Since $\bar{\nu}(R) = \ln\left(1 - \frac{2M}{R}\right)$, adding $\pm\left(\bar{\nu}(R) - \ln\left(1 - \frac{2M}{R}\right)\right)$ would not change the value.