CMPSC 360 | Discrete Mathematics for Computer Science
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Worksheet 7

1. Determine which functional properties the following functions satisfy and whether they are bijective:

   (a) $f : \mathbb{Z} \to \mathbb{Q}$ and $f(x) = \frac{2x-3}{5}$

   (b) $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = 3x^2 + 11x$

   Answer:

   (a) **Injectivity:**
   Assume $x_1, x_2 \in \mathbb{Z}$ and $f(x_1) = f(x_2)$.
   $\implies \frac{2x_1-3}{5} = \frac{2x_2-3}{5}$
   $\implies x_1 = x_2$
   Thus, by the definition of injectivity, f is injective.
   Surjectivity:
   Suppose that for some $x \in \mathbb{Z}$, $y = f(x)$. Then,
   $y = \frac{2x-3}{5}$
   $\implies x = \frac{5y+3}{2}$
   For $y = 2 \in \mathbb{Q}$, $x = \frac{13}{2} \notin \mathbb{Z}$. Therefore, f is not surjective.

   (b) **Injectivity:**
   Not injective because $f(1) = f(\frac{-14}{3}) = 14$ and $1 \neq \frac{-14}{3}$.
   Surjectivity:
   For some $x \in \mathbb{R}$, let $y = f(x)$
   $\implies y = 3x^2 + 11x$
   $\implies 3x^2 + 11x - y = 0$
   $\implies x = \frac{-11+\sqrt{121+12y}}{6}$
   For $y = -11$, $x \notin \mathbb{R}$
   Therefore, f is not surjective on $\mathbb{R}$

2. Suppose $f(x) = \sqrt{2x-5}$, $g(x) = 5x^2 - 3$. What is the domain of f? What is the domain of g? Find the composite functions below. For each composite function, state the domain.

   (a) $f \circ g(x)$

   (b) $g \circ f(x)$

   (c) $f \circ f(x)$

   (d) $g \circ g(x)$

   Answer:

   (a) $f \circ g(x)$
   $f \circ g(x) = \sqrt{2(5x^2-3)-5} = \sqrt{10x^2-11}$. Domain = $\{x \in \mathbb{R} \mid \ |x| \geq \sqrt{\frac{11}{10}}\}$.

(b) $g \circ f(x)$
$g \circ f(x) = 5(2x-5) - 3 = 10x - 28$. Domain = domain of f = $\{x \in \mathbb{R} | x \geq 5/2\}$.

(c) $f \circ f(x)$
$f \circ f(x) = \sqrt{2\sqrt{2x-5} - 5}$. Domain: we have to make sure $\sqrt{2x-5} \geq 5/2$. Some algebraic work gives $x \geq 45/8$ (which encompasses the domain of $f : x \geq 5/2$.

(d) $g \circ g(x)$
$g \circ g(x) = 5(5x^2 - 3)^2 - 3$. Domain: $\mathbb{R}$.

3. Consider the function $f : \mathbb{R} \setminus \{-4\} \to \mathbb{R} \setminus \{2\}$ defined as $f(x) = \frac{2x-1}{x+4}$.

(a) Determine whether the function f has an inverse, and if so, find the expression for $f^{-1}(x)$.

(b) Would f have an inverse if we change the function's domain and codomain to $f : \mathbb{Z} \setminus \{-4\} \to \mathbb{Z} \setminus \{2\}$? Explain your reasoning.

Answer:

(a) Determining if the Function Has an Inverse:
To determine if the function f has an inverse, we need to check both injectivity (one-to-one) and subjectivity of the function.
Suppose $f(a) = f(b)$ for some a and b in the domain, $\mathbb{R} \setminus \{-4\}$. We have:

$$f(x) = \frac{2x-1}{x+4}$$

So, if $f(a) = f(b)$, then:

$$\frac{2a-1}{a+4} = \frac{2b-1}{b+4}$$

Cross-multiplying:

$$(2a-1)(b+4) = (2b-1)(a+4)$$

Expanding and simplifying:

$$2ab + 8a - b - 4 = 2ab + 8b - a - 4 \implies 8a - b - 4 = 8b - a - 4 \implies 9a = 9b \implies a = b$$

Since $f(a) = f(b)$ implies $a = b$, the function is injective.
Let y be an arbitrary real number in $\mathbb{R} \setminus \{2\}$, we need to find an x such that:

$$\frac{2x-1}{x+4} = y$$
$$2x - 1 = xy + 4y$$
$$(2-y)x = 4y + 1$$

Since $y \neq 2$, then $y - 2 \neq 0$.
So, we have $x = \frac{4y+1}{2-y}$. Let's check whether x could be -4:

If $x = -4$, then $4y + 1 = -8 + 4y$, thus $1 = -8$ but this is not true, so $x \neq 4$. Then we can say the function is surjective.
Finding the Inverse Function $f^{-1}(x)$:
To find the inverse function, we swap x and y in the equation:

$$y = \frac{2x-1}{x+4}$$

and solve for y:

$$x = \frac{2y-1}{y+4} \implies y = \frac{4x+1}{2-x}$$

Now, y is the subject of the formula:

$$f^{-1}(x) = \frac{4x+1}{2-x}$$

(b) Inverse does not exist. For values in the new domain, there are no equivalent values in the codomain. Example: f(1).

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 1$ and let $g : \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = -x$.
Find the composition $g \circ f$ and determine if it is a bijection over the set of integers, $\mathbb{Z}$.
Answer:
Given the functions:

$$f(x) = x + 1$$
$$g(x) = -x$$

To find the composition $g \circ f$, we apply function f first and then apply function g to the result:

$$g \circ f(x) = g(f(x))$$
$$g \circ f(x) = g(x+1)$$
$$g \circ f(x) = -(x+1)$$
$$g \circ f(x) = -x - 1$$

So, the composition $g \circ f$ is $g \circ f(x) = -x - 1$.

Next, we'll determine if $g \circ f$ is a bijection over the set of integers, $\mathbb{Z}$.

For a function to be bijective, it must be both injective (one-to-one) and surjective (onto).

Injective: A function is injective if for every distinct pair of inputs, the outputs are also distinct. Suppose:

$$g \circ f(a) = g \circ f(b)$$

for some integers a and b. This implies:

$$-a - 1 = -b - 1$$

$$-a = -b$$

$$a = b$$

Since a must equal b for the outputs to be the same, $g \circ f$ is injective.

Surjective: A function is surjective if every element in the codomain (in this case, $\mathbb{Z}$) has a preimage in the domain. Let y be an arbitrary integer in $\mathbb{Z}$. We need to find an x such that:

$$g \circ f(x) = y$$

$$-x - 1 = y$$

$$-x = y + 1$$

$$x = -y - 1$$

For any integer y, the value $x = -y - 1$ is also an integer. Hence, $g \circ f$ is surjective.

Since $g \circ f$ is both injective and surjective, it is a bijection over the set of integers, $\mathbb{Z}$.

5. Given the functions $h : M \to N$ and $k : N \to P$ where

$$M = \{a, b, c, d, e\}$$

$$N = \{j, k, l, m, n\}$$

$$P = \{s, t, u, v, w\}$$

and the functions are represented by the rosters:

$$h = \{(a, j), (b, k), (c, l), (d, m), (e, n)\}$$

$$k = \{(j, t), (k, s), (l, u), (m, v), (n, w)\}$$

Determine $k \circ h$ and $h \circ k$.
Answer:
Using the definition of function composition, we can compute $k \circ h$ for each element in M:

$$\begin{aligned}
(k \circ h)(a) &= k(h(a)) = k(j) = t \\
(k \circ h)(b) &= k(h(b)) = k(k) = s \\
(k \circ h)(c) &= k(h(c)) = k(l) = u \\
(k \circ h)(d) &= k(h(d)) = k(m) = v \\
(k \circ h)(e) &= k(h(e)) = k(n) = w
\end{aligned}$$

$$k \circ h = \{(a, t), (b, s), (c, u), (d, v), (e, w)\}$$

$h \circ k$ is not defined because $h : M \to N$ and $k : N \to P$.