Worksheet 5 Solutions

1. Proof or disprove the following statements.

   (a) For $a, b \in \mathbb{R}$, if $a^2 = b^2$, then $a = b$.

   (b) $n^2 + n + 41$ is prime for $n \in \mathbb{Z}$.

   Answer:

   (a) We can find a counterexample for this statement: $a = 1, b = -1$. In this example, $a^2 = b^2 = 1$, but $a \neq b$. $\square$

   (b) We can find a counterexample for this statement.
   Since $n^2 + n + 41 = (n+1)n + 41$, we can try to construct a perfect square, where $n = 40$. Then, $(n+1)n + 41 = (40+1) \times 40 + 41 = 41 \times 40 + 41 = 41 \times 41$, which contradicts the statement that $n^2 + n + 41$ is prime. $\square$

2. Prove by contrapositive that if a number is divisible by 4, then its last two digits in base 10 is divisible by 4.

   Proof:

   The contrapositive statement is: *If a number's last two digits in base 10 is not divisible by 4, then the number itself is not divisible by 4.*

   Assume that we have $n = 100a + b, a, b \in \mathbb{Z} \wedge b \in [0, 100)$, where $b \equiv k \pmod 4, k \neq 0$.
   Since $100a = 25(4a) \equiv 0 \pmod 4$, and by the compatibility of addition in modular arithemetic we would have $100a + b \equiv 0 + k \pmod 4 \equiv k \pmod 4$.
   Since $k \neq 0$, we can conclude that $4 \nmid (100a + b)$, or $4 \nmid n$.

   Thus we proved the contrapositive statement, and hence proving the original statement. $\square$

3. Suppose $q \in \mathbb{Z}$. Prove by contrapositive that if $6q + 7$ is even, then q is odd.

   Proof:

   The contrapositive statement is: *If q is even, then $6q + 7$ is odd.*

   Assume that q is even. By definition, $q = 2n$ where $n \in \mathbb{Z}$.

   Consider the expression $6q + 7$:

   $$6q + 7 = 6(2n) + 7 = 12n + 7 = 12n + 6 + 1 = 2(6n + 3) + 1$$

   Since $n \in \mathbb{Z}$, we have $6n + 3 \in \mathbb{Z}$. Hence, by the definition of odd, we have $6q + 7$ is odd.

   Since we have proven that if q is even, then $6q + 7$ is odd, by contrapositive, it follows that if $6q + 7$ is even, then q must be odd. $\square$

4. For $a,b,c \in \mathbb{R}^+$, prove that if $ab = c$, then $a \leqslant \sqrt{c}$ or $b \leqslant \sqrt{c}$.

   Answer:

   Suppose a, b, and c are positive real numbers such that $ab = c$, and suppose $a > \sqrt{c}$ and $b > \sqrt{c}$. By ordering properties of real numbers:

   - Since $a > 0$, we have $b > \sqrt{c} \leftrightarrow ab > a\sqrt{c}$.
   - Since $\sqrt{c} > 0$, $a > \sqrt{c} \leftrightarrow a\sqrt{c} > \sqrt{c}.\sqrt{c} = c$.

   Thus, $ab > a\sqrt{c} > \sqrt{c}.\sqrt{c} = c$, which implies $ab > c$.

   However, $ab > c$ contradicts the assumption that $ab = c$.
   Therefore, if $ab = c$, then $a \leqslant \sqrt{c}$ or $b \leqslant \sqrt{c}$. $\square$

5. Suppose $n \in \mathbb{Z}$ and $p \in \mathbb{P}$ (i.e., p is a prime number). Prove the statement: if $p \mid n$, then $p \nmid (n+1)$.

   Proof:

   Let's assume there exists some integer n and prime p such that $p \mid n$ and $p \mid (n+1)$.

   By definition of divisibility:

$$p \mid n \rightarrow p \cdot r = n, r \in \mathbb{Z} \tag{1}$$

$$p \mid (n+1) \rightarrow p \cdot s = n+1, s \in \mathbb{Z} \tag{2}$$

   Combining Eq. 1 and Eq. 2, we have $p \cdot s = p \cdot r + 1$, which can be derived as $p \cdot (s - r) = 1$.

   From the definition of divisibility we can say that $p \mid 1$.

   As $p \mid 1$, we have $p \leq 1$. But $p \in \mathbb{P}$ requires $p > 1$ which leads to contradiction.

   Hence, For any integer n and any prime p, if $p \mid n$, then $p \nmid (n+1)$ $\square$