CMPSC 360 Discrete Mathematics for Computer Science
Fall 2024 Mahfuza Farooque

Worksheet 4

1. Simplify the following expressions.

   (a) $\neg\forall x\exists y(P(x,y)\wedge\exists zR(x,y,z))$

   (b) $\neg\forall x(\exists y\forall zP(x,y,z)\wedge\exists z\forall yP(x,y,z))$

   Answer:

   (a)

$$\begin{aligned}
\neg\forall x\exists y(P(x,y)\wedge\exists zR(x,y,z)) &\equiv \exists x\neg\exists y(P(x,y)\wedge\exists zR(x,y,z)) && \text{(De Morgan's Law for Quantifiers)}\\
&\equiv \exists x\forall y\neg(P(x,y)\wedge\exists zR(x,y,z)) && \text{(De Morgan's Law for Quantifiers)}\\
&\equiv \exists x\forall y(\neg P(x,y)\vee\neg\exists zR(x,y,z)) && \text{(De Morgan's Law for Conjunctions)}\\
&\equiv \exists x\forall y(\neg P(x,y)\vee\forall z\neg R(x,y,z)) && \text{(De Morgan's Law for Quantifiers)}
\end{aligned}$$

   (b)

$$\begin{aligned}
\neg\forall x(\exists y\forall zP(x,y,z)\wedge\exists z\forall yP(x,y,z)) &\equiv \exists x\neg(\exists y\forall zP(x,y,z)\wedge\exists z\forall yP(x,y,z)) && \text{(DML for Quantifiers)}\\
&\equiv \exists x(\neg\exists y\forall zP(x,y,z)\vee\neg\exists z\forall yP(x,y,z)) && \text{(DML for Conjunctions)}\\
&\equiv \exists x(\forall y\exists z\neg P(x,y,z)\vee\forall z\exists y\neg P(x,y,z)) && \text{(DML for Quantifiers)}
\end{aligned}$$

2. Use the rules of inference to show that the following arguments are valid:

   (a) $q\leftrightarrow\overline{p},\ \overline{q}\wedge r,\ p\to s,\ s\to t\ \vdash t\vee r$

   (b) $u\to s,\ \overline{p}\wedge(\overline{p}\vee\overline{q})\to\overline{s},\ u\ \vdash p$

   Answer:

   (a) $q\leftrightarrow\overline{p} \implies (q\to\overline{p})\wedge(\overline{p}\to q)$ (Def of biconditional)
   $(q\to\overline{p})\wedge(\overline{p}\to q) \implies (q\to\overline{p}),\ (\overline{p}\to q)$ (Simplification)
   $\overline{q}\wedge r \implies \overline{q},\ r$ (Simplification)
   $\overline{p}\to q,\ \overline{q} \implies p$ (Modus Tollens)
   $p\to s,\ s\to t \implies p\to t$ (Hypothetical Syllogism)
   $p,\ p\to t \implies t$ (Modus Ponens)
   $t,r \implies t\vee r$ (Addition)

   (b) $u\to s,\ u \implies s$ (Modus Ponens)
   $\overline{p}\wedge(\overline{p}\vee\overline{q})\to\overline{s},\ s \implies \overline{\overline{p}\wedge(\overline{p}\vee\overline{q})}$ (Modus Tollens)
   $\overline{\overline{p}\wedge(\overline{p}\vee\overline{q})} \implies p\vee(\overline{\overline{p\wedge q}})$ (DeMorgan's Law and double negation)
   $p\vee(p\wedge q) \implies p$ (Absorption Law)

3. **Inference rules for predicate logics:**
A plant is considered endangered if it has not been found in at least three different locations. The Blueleaf plant has been found in only one location this year. According to environmental regulations, an endangered plant cannot be commercially traded. We have a specific Blueleaf plant in question.

This scenario can be formalized using the following predicates, assuming the universe of discourse is the set of all plants:

- $E(x)$: x is endangered.
- $F(x,n)$: x is found in n different locations.
- $T(x)$: x can be traded.

The term ”Blueleaf” refers to a particular instance of a plant.

Given:

(a) $\forall x(\neg F(x,3) \rightarrow E(x))$ - A plant not found in at least three locations is endangered.
(b) $\forall x(E(x) \rightarrow \neg T(x))$ - An endangered plant cannot be traded.
(c) $F(\text{Blueleaf},1)$ - The Blueleaf plant has been found in only one location.

To Prove:

- $E(\text{Blueleaf}) \wedge \neg T(\text{Blueleaf})$ - The Blueleaf plant is endangered and cannot be traded.

Answer:

Proof:

(a) By Universal Instantiation of assumption (1), we have $\neg F(\text{Blueleaf},3) \rightarrow E(\text{Blueleaf})$.
(b) From assumption (3), $F(\text{Blueleaf},1)$, we infer $\neg F(\text{Blueleaf},3)$.
(c) By Modus Ponens of step (1) due to step (2), we conclude $E(\text{Blueleaf})$, proving the Blueleaf plant is endangered.
(d) By Universal Instantiation of assumption (2), we get $E(\text{Blueleaf}) \rightarrow \neg T(\text{Blueleaf})$.
(e) By Modus Ponens of step (4) due to step (3), we conclude $\neg T(\text{Blueleaf})$, proving the Blueleaf plant cannot be traded.
(f) Therefore, we have shown that $E(\text{Blueleaf}) \wedge \neg T(\text{Blueleaf})$, meaning the Blueleaf plant is both endangered and cannot be traded.

□

4. Let n be a positive integer. Prove that if n is an even, then $4n^2 + 48n + 144$ is a perfect square.

(a) What is the given statement and your assumptions?
(b) What is your conclusion?
(c) Construct the proof:

Answer:

(a) For positive n, we assume n is even.

(b) We want to show $4n^2 + 48n + 144$ is a perfect square

(c) Proof: By definition of being even, there exists some integer k such that $n = 2k$. Plugging it back into the equation: $16k^2 + 96k + 144$. Factorizing and simplifying: $16(k^2 + 6k + 9) = 16(k+3)^2$. Since $16(k+3)^2$ is in perfect square form this proves our statement QED.

5. Using the direct proof method, prove that if n is an odd perfect square, then n is of the form 4k+1. Answer the questions as directed.

   (a) What are the assumptions?

   (b) What are you trying to conclude?

   (c) Show your work.

   Answer: Proof:

   (a)Existential instantiation of assumption (a) for term a: $n = 2a + 1$
   (b)Existential instantiation of assumption (b) for term b: $n = b^2$
   $(a),(b) \implies b^2 = 2a + 1 \implies b^2 - 1 = 2a \implies (b+1)(b-1) = 2a$

   Therefore $b+1$ or $b-1$ should be even: $\exists x(b+1 = 2x \vee b-1 = 2x)$ Case 1: $b+1 = 2x, x \in Z \implies n = b^2 = (2x-1)^2 = 4x^2 - 4x + 1 = 4(x^2 - x) + 1 = 4k + 1$ Case 2: $b - 1 = 2x, x \in Z \implies n = b^2 = (2x+1)^2 = 4x^2 + 4x + 1 = 4(x^2 + x) + 1 = 4k + 1$

   QED

6. Consider the following theorem:
   "For an integer n, if n is odd, then n^3 is odd."

   Verify that the theorem is true using direct proof.
   Answer:
   Direct Proof: Assume arbitrary $n \in \mathbb{Z}$ is odd, then by definition, there exists some $k \in \mathbb{Z}$ such that $n = 2k+1$. Therefore, $n^3 = (2k+1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$. Since $k \in \mathbb{Z}$, $4k^3 + 6k^2 + 3k \in \mathbb{Z}$, then by definition, n^3 is also odd. Q.E.D.