CMPSC 465 Fall 2025	Data Structures & Algorithms Ke Chen and Yana Safonova	Worksheet X

Monday, Sep 8, 2025

1. (pts.) **Compare Growth Rates**. Order the following functions by asymptotic growth:

(i) $f_1(n) = 3^n$

(ii) $f_2(n) = n^{\frac{1}{3}}$

(iii) $f_3(n) = 12$

(iv) $f_4(n) = 2^{\log_2 n}$

(v) $f_5(n) = \sqrt{n}$

(vi) $f_6(n) = 2^n$

(vii) $f_7(n) = \log_2 n$

(viii) $f_8(n) = 2^{\sqrt{n}}$

(ix) $f_9(n) = n^3$

Answer:

- $f_3 = O(f_7)$: By the definition of Big-O, let $c = 12$ and $n_0 = 2$. Then $12 \leq c \log_2 n$ for all $n \geq n_0$.

- $f_7 = O(f_2)$: You could use the fact that any polynomial dominates any logarithm. Without this, you could also prove it by using basic facts about the log function and the definition of Big O as follows. From the definition of O it suffices to show that $\log_2 n \leq 3n^{1/3}$, which is equivalent to $\log_2 n^{1/3} \leq n^{1/3}$, and this follows from the fact that $\log_2 x \leq x$ for all $x > 0$; the latter is true because $x - \log_2 x$ is convex with a derivative of $1 - \frac{1}{x \ln 2}$ which is 0 at $x = \frac{1}{\ln 2}$ which means $\min_x x - \log_2 x = \frac{1}{\ln 2} + \log_2 \ln 2 = 0.9139 \geq 0$

- $f_2 = O(f_5)$: $\sqrt{n} = n^{\frac{1}{2}}$. Then, it suffices to observe that $n^{1/3} \leq n^{1/2}$ since $\frac{1}{2} > \frac{1}{3}$.

- $f_5 = O(f_4)$: $2^{\log_2 n} = n = n^1$. As $1 > \frac{1}{2}$, $f_4 \geq f_5$.

- $f_4 = O(f_9)$: $3 > 1$, so $f_9 \geq f_4$.

- $f_9 = O(f_8)$: You could use the fact that any exponential dominates any polynomial. Alternatively, you could prove this from first principles, similar to the way f_7 was shown to be $O(f_2)$ (check this!).

- $f_8 = O(f_6)$: By definition. Take $c = 1$ and $n_0 = 1$. $2^{\sqrt{n}} \leq 2^n$ for all $n \geq 1$.

- $f_6 = O(f_1)$: By definition. Take $c = 1$ and $n_0 = 1$. $2^n \leq 3^n$ for all $n \geq 1$.

  Note that unlike logarithm bases, which do not affect the growth rate of the logarithm, larger exponent bases grow faster than smaller exponent bases. This can be shown with the following limit:
  $$\lim_{n\to\infty} \frac{3^n}{2^n} = \lim_{n\to\infty} (\tfrac{3}{2})^n = \infty$$

  An aside about the equivalent growth rates of logarithms of different bases - this can be explained by following the change of base formula, which shows that any two logarithms of different bases are separated by a constant multiple.

2. (pts.) **Running Time Analysis**. For each pseudo-code below, give the asymptotic running time in Θ notation.

(a)
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for i := 1 to n do
    j := i;
    while j < n do
        j := j + 5;
    end
end
```

(b)
```
for i := 1 to n do
    for j := 4i to n do
        s := s + 2;
    end
end
```

(c)
```
for i := 1 to n do
    j := 2;
    while j < i do
        j := j^4;
    end
end
```

Answer:

(a) Inside the inner loop, a constant number of basic steps are performed. For any fixed i, the inner loop performs $\lceil \frac{n-i}{5} \rceil$ iterations. A lower bound for the total running time is:

$$\sum_{i=1}^{n} \lceil \tfrac{n-i}{5} \rceil \geq \sum_{i=1}^{n} \frac{n-i}{5} = \frac{1}{5}\sum_{i=1}^{n}(n-i) = \frac{1}{5}\sum_{i=0}^{n-1} i = \frac{1}{5}(\tfrac{(n-1)(n)}{2}) = \Omega(n^2).$$

And an upper bound (using the fact that $\lceil x \rceil \leq x+1$) is:

$$\sum_{i=1}^{n} \lceil \tfrac{n-i}{5} \rceil \leq \sum_{i=1}^{n} (\frac{n-i}{5} + 1) = \frac{1}{5}\sum_{i=1}^{n}(n-i) + \sum_{i=1}^{n} 1 = \frac{1}{5}\sum_{i=0}^{n-1}(i) + n = \frac{1}{5}(\tfrac{(n-1)(n)}{2}) + n = O(n^2).$$

Therefore, the asymptotic running time is $\Theta(n^2)$.

A faster way to arrive at the solution is observing that $\lceil \frac{n-i}{5} \rceil = \Theta(n-i)$. This means that for every fixed i, there is some coefficients a_i and b_i such that $a_i(n-i) \leq \lceil \frac{n-i}{5} \rceil \leq b_i(n-i)$ for all n greater than some n_0. If we instead use the minimum a_{min} and the maximum b_{max} among these coefficients, we can ensure that for every term, $a_{min}(n-i) \leq \lceil \frac{n-i}{5} \rceil \leq b_{max}(n-i)$. Applying this to the entire sum using the distributive property, we get:

$$a_{min} \sum_{i=1}^{n} (n-i) \leq \sum_{i=1}^{n} \lceil \tfrac{n-i}{5} \rceil \leq b_{max} \sum_{i=1}^{n} (n-i)$$

$$a_{min} \sum_{i=0}^{n-1} i \leq \sum_{i=1}^{n} \lceil \tfrac{n-i}{5} \rceil \leq b_{max} \sum_{i=0}^{n-1} i$$

$$a_{min}(\tfrac{(n-1)(n)}{2}) \leq \sum_{i=1}^{n} \lceil \tfrac{n-i}{5} \rceil \leq b_{max}(\tfrac{(n-1)(n)}{2})$$

$$\sum_{i=1}^{n} \lceil \tfrac{n-i}{5} \rceil = \Theta(n^2)$$

Applying the idea of distributing the minimum and maximum coefficients to the general case, we can see that:

$$a_{min} \sum_{i=1}^{n} f(i) \leq \sum_{i=1}^{n} \Theta(f(i)) \leq b_{max} \sum_{i=1}^{n} f(i)$$

$$\sum_{i=1}^{n} \Theta(f(i)) = \Theta(\sum_{i=1}^{n} f(i))$$

(b) For any fixed $i \leq n/4$, the inner loop iterates $n-4i+1$ times, and zero times when $i > n/4$. Then, the running time is

$$\frac{3n}{4} + \sum_{i=1}^{n/4} (n-4i+1) = \frac{3n}{4} + \sum_{i=1}^{n/4} n - 4\sum_{i=1}^{n/4} i + n/4 = \frac{3n}{4} + \frac{n^2}{4} - 4 \cdot \frac{\frac{n}{4}(\frac{n}{4}+1)}{2} + n/4 = \Theta(n^2).$$

Note that a constant amount of work is still done by the outer loop even when the inner loop does not iterate (and the inner loop when considering that it must check that $i > n/4$).

(c) For each fixed $i > 2$, the inner loop iterates at most $\log_4 \log_2 i + 1 = \Theta(\log\log i)$ times, since the value of j in the k-th iteration of the inner loop is $2^{4^{k-1}}$, and it runs while $2^{4^{k-1}} < i$. Hence, the running time is $O(1) + \sum_{i=3}^{n} \Theta(\log\log i) = \Theta(\sum_{i=3}^{n} \log\log i)$.

$$\sum_{i=3}^{n} \log\log i \leq \sum_{i=3}^{n} \log\log n \leq n \log\log n = O(n \log\log n)$$

$$\sum_{i=3}^{n} \log\log i \geq \sum_{i=\lceil n/2 \rceil}^{n} \log\log i \geq \sum_{i=\lceil n/2 \rceil}^{n} \log\log \frac{n}{2} \geq \lfloor \tfrac{n}{2} \rfloor \log\log \tfrac{n}{2} = \Omega(n \log\log n),$$

where the first inequality holds for $n \geq 6$ say. Note that this style of proof can be applied to the summation of any asymptotically increasing function f. The lower bound additionally requires that $f(\frac{n}{2}) = \Theta(f(n))$, which is the case for logarithms and polynomials, but not exponentials.

Bounding the summation of an exponential is included in homework 1. For this upper bound, you replace all the terms with $f(n)$:

$$\sum_{i=1}^{n} f(i) \le \sum_{i=1}^{n} f(n) = nf(n) = O(nf(n))$$

For the lower bound, you take only the upper half of the terms, then replace them all with $f(\frac{n}{2})$:

$$\sum_{i=1}^{n} f(i) \ge \sum_{i=\lceil n/2 \rceil}^{n} f(i) \ge \sum_{i=\lceil n/2 \rceil}^{n} f(\tfrac{n}{2}) \ge \lfloor \tfrac{n}{2} \rfloor f(\tfrac{n}{2}) = \Omega(nf(n))$$

These bounds show that $\sum_{i=1}^{n} f(i) = \Theta(nf(n))$.

Using this and the fact that $\sum_{i=1}^{n} \Theta(f(i)) = \Theta(\sum_{i=1}^{n} f(i))$ established in part (i), we see that for any asymptotically increasing function f that has $f(\frac{n}{2}) = \Theta(f(n))$:

$$\sum_{i=1}^{n} \Theta(f(i)) = \Theta(nf(n))$$

3. (pts.) **Identities**. Show that the following statements hold true.

(a) $\sum_{k=1}^{n} k^j = \Theta(n^{j+1})$ for any constant $j > 0$. Note that $k \le n$ for all terms and $k \ge \frac{n}{2}$ for many terms.

(b) $\sum\limits_{i=1}^{n} \sum\limits_{j=1, j \ne i}^{n} ij = \Theta(n^4)$.

Answer:

1. Since $k \le n$ every term in the sum is at most n, so

$$\sum_{k=1}^{n} k^j = 1^j + \cdots + n^j \le n^j + \cdots + n^j = \sum_{k=1}^{n} n^j = n^{j+1}.$$

We do something similar for the lower bound. The only additional idea is that we only look at the second half of the sum. The smallest element in the second half of the sum corresponds to $k = n/2$ (assuming without loss of generality that n is even). Then,

$$\sum_{k=1}^{n} k^j \ge \sum_{k=n/2}^{n} k^j \ge \sum_{k=n/2}^{n} (\tfrac{n}{2})^j \ge (\tfrac{n}{2})^{j+1} = \frac{1}{2^{j+1}} \cdot n^{j+1}.$$

2. First we show the upper bound. $\sum_{i=1}^{n} \sum\limits_{j \ne i, j=1}^{n} ij \le (\sum_{i=1}^{n} i)^2 = (\frac{n(n+1)}{2})^2 = O(n^4)$. For the lower bound, we can write:

$$\sum_{i=1}^{n} \sum_{j \ne i, j=1}^{n} ij \ge \sum_{i=\lceil \frac{n}{2} \rceil}^{n} i \sum_{j=\lceil \frac{n}{2} \rceil + 1}^{n} j \ge \sum_{i=\lceil \frac{n}{2} \rceil}^{n} \frac{n}{2} \sum_{j=\lceil \frac{n}{2} \rceil + 1}^{n} \left(\frac{n}{2} + 1\right) \quad \ge \tfrac{n}{2}(\tfrac{n}{2} + 1) \cdot (\frac{n}{2})^2 \ge \frac{n(n+2)}{4} \cdot \frac{n^2}{4} = \Omega(n^4).$$

Alternatively, the summation can be expanded into $(\sum_{i=1}^{n} i)^2 - \sum_{i=1}^{n} i^2$, then simplified using the standard summations from 1 to n. This will give a polynomial in n with degree 4, which has been shown to be $\Theta(n^4)$. The intuition for this approach is that the only thing stopping the original summation from being easily decoupled is the condition $j \ne i$, which excludes the same products as the new i^2 summation subtracts.

4. (pts.) **Recurrence Relations** Solve the following recurrence relations and give the tightest correct upper bound for each of them in O notation. Assume that $T(1) = O(1)$. Show all work.

(a) $T(n) = 16 \cdot T(n/2) + 15n^4 + 3n^3$

(b) $T(n) = 49 \cdot T(n/25) + n^{3/2} \log n$

(c) $T(n) = n^{\frac{1}{3}} T(n^{\frac{1}{3}}) + 5n^{\frac{2}{3}}$

Answer:

(a) $T(n) = 16 \cdot T(n/2) + 15n^4 + 3n^3 = 16 \cdot T(n/2) + O(n^4) \implies T(n) = O(n^4 \log n)$

(b) $T(n) = 49T(n/25) + n^{3/2} \log n = O(n^{3/2} \log n)$. By unrolling we have:

$$\begin{aligned}
\sum_{i=0}^{\log_{25} n} 49^i \left(\frac{n}{25^i}\right)^{3/2} \log\left(\frac{n}{25^i}\right) &= \sum_{i=0}^{\log_{25} n} \left(\frac{49}{25^{3/2}}\right)^i n^{3/2} \log\left(\frac{n}{25^i}\right) \\
&< \sum_{i=0}^{\log_{25} n} \left(\frac{49}{125}\right)^i O(n^{3/2} \log n) \\
&= O(n^{3/2} \log n) \sum_{i=0}^{\log_{25} n} \left(\frac{49}{125}\right)^i \\
&= O(n^{3/2} \log n) O(1) \\
&= O(n^{3/2} \log n)
\end{aligned}$$

Note that in the latter equation, we used the fact that the geometric series will be $O(1)$, since $\frac{49}{125} < 1$.

(c) The given recurrence relation is:

$$\begin{aligned}
T(n) &= n^{\frac{1}{3}} T(n^{\frac{1}{3}}) + 5n^{\frac{2}{3}} \\
&= n^{\frac{1}{3}} (n^{\frac{1}{9}} T(n^{\frac{1}{9}}) + 5n^{\frac{1}{3} \cdot \frac{2}{3}}) + 5n^{\frac{2}{3}} \\
&= n^{\frac{1}{3} + \frac{1}{9}} T(n^{\frac{1}{9}}) + 5(n^{\frac{2}{3}} + n^{\frac{5}{9}}) \\
&= n^{\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^k}} T(n^{\frac{1}{3^k}}) + 5(n^{\frac{2}{3}} + n^{\frac{5}{9}} + n^{\frac{14}{27}} + \ldots)
\end{aligned}$$

Note that for $n^{\frac{1}{3^k}}$ to be constant, choose $k = \Theta(\log \log n)$ (hint: set $n^{\frac{1}{3^k}} = c$ for any $c > 1$). Also, $\frac{1}{3} \leq \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^k} \leq \frac{1}{2}$, so, we get $n^{\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^k}} T(n^{\frac{1}{3^k}}) = O(n^{\frac{1}{2}})$. The exponents of the second term converge to $\frac{1}{2}$, so $5(n^{\frac{2}{3}} + n^{\frac{5}{9}} + n^{\frac{14}{27}} + \ldots) = O(n^{\frac{2}{3}})$. Therefore, $T(n) = O(n^{\frac{2}{3}})$.