CMPSC 465: LECTURE VIII

Binary Heaps

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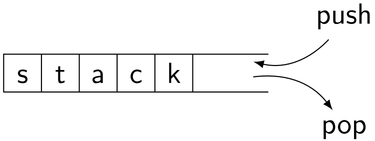
September 15, 2025

Quick review: stack and queue

Stack

- Last In, First Out (LIFO).
- Constant time insert (push) and delete (pop) at the same end.
- Common implementations: array, linked list.



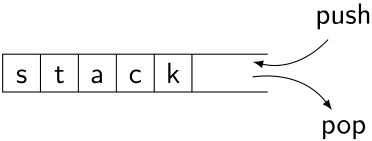
Quick review: stack and queue

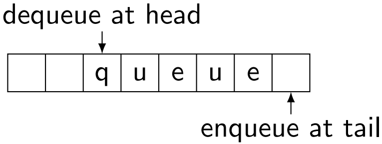
Stack

- Last In, First Out (LIFO).
- Constant time insert (push) and delete (pop) at the same end.
- Common implementations: array, linked list.

Queue

- First In, First Out (FIFO).
- Constant time insert (enqueue) and delete (dequeue) at different ends.
- Common implementations: circular array, linked list with a tail pointer.





Priority queue

Motivation Can we reorder elements in the container so that those with higher priority are handled first?

- $\diamond$ GetMax
- $\diamond$ Insertion
- $\diamond$ Deletion
- $\diamond$ ChangePriority

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Sorted list

- ⋄ GetMax
- ⋄ Insertion
- ⋄ Deletion
- ⋄ ChangePriority

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◇ GetMax	$O(1)$
◇ Insertion	
◇ Deletion	
◇ ChangePriority	

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◇ GetMax	$O(1)$
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Priority queue

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	Sorted list	
◇ GetMax	$O(1)$	
◇ Insertion	$O(n)$	array can do binary search but in worst-case requires linear shift; linked list supports constant-time insertion/deletion but cannot binary search
◇ Deletion	$O(n)$	
◇ ChangePriority	$O(n)$	

Priority queue

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	Sorted list	Array with pointer to max
⋄ GetMax	$O(1)$	
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◇ ChangePriority	$O(n)$		

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◇ GetMax	$O(1)$	$O(1)$	
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◇ Deletion	$O(n)$	$O(n)$	worst-case: max is removed
◇ ChangePriority	$O(n)$	$O(n)$	may need to find new max

Priority queue

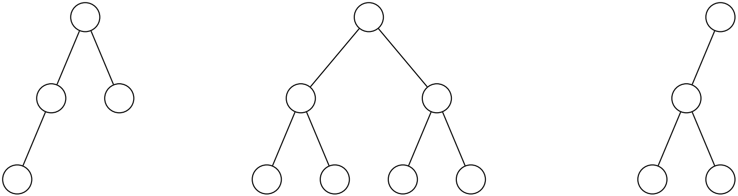
Motivation Can we reorder elements in the container so that those with higher priority are handled first?

	Sorted list	Array with pointer to max	Binary heap
◇ GetMax	$O(1)$	$O(1)$	$O(1)$
◇ Insertion	$O(n)$	$O(1)$	$O(\log n)$
◇ Deletion	$O(n)$	$O(n)$	$O(\log n)$
◇ ChangePriority	$O(n)$	$O(n)$	$O(\log n)$

Complete tree

Definition A binary tree is complete if

- the tree is completely filled in all levels except possibly in the lowest level; and
- all nodes in the last level are as far left as possible.



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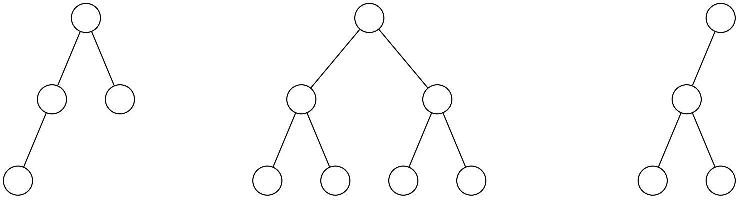

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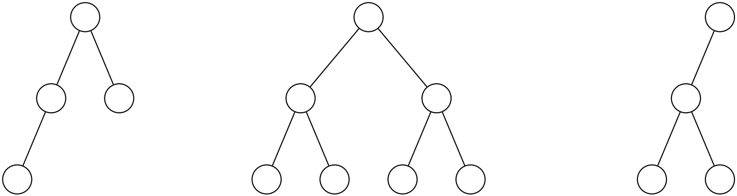

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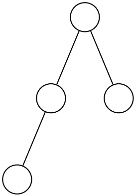

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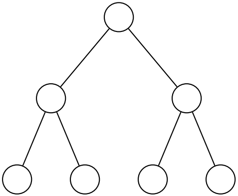

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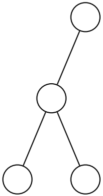
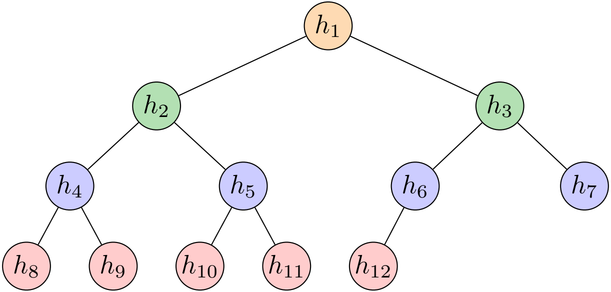


not complete

A complete tree in an array

Observe that a complete tree fits snugly in an array.

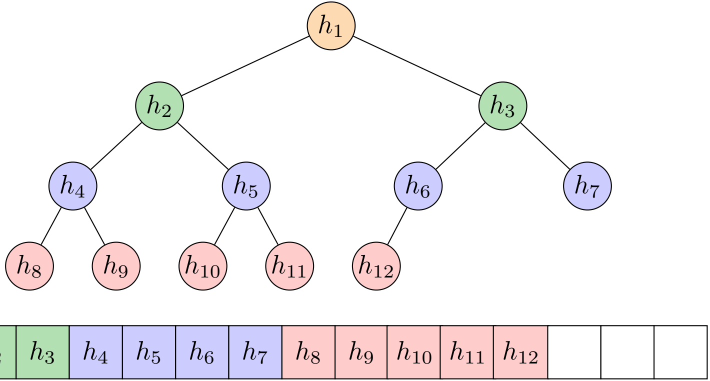


h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9	h_{10}	h_{11}	h_{12}			

A complete tree in an array

Observe that a complete tree fits snugly in an array.

For an index i, $parent(i) = \lfloor i/2 \rfloor$, $left(i) = 2i$, and $right(i) = 2i + 1$.



Binary max-heap

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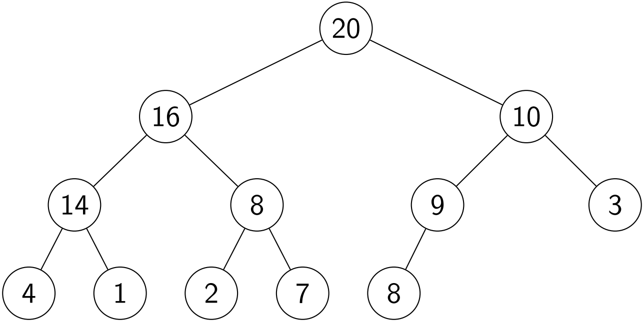

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Exercise: Does an array sorted in non-decreasing order represent a max-heap? E.g., 10, 8, 7, 7, 3, 2, 2, 1.

We will see later that we don't need to sort in order to construct a heap from an array.

Heap operations

Suppose $H[1..n]$ is a binary max-heap.

GetMax Takes $O(1)$, all we have to do is return $H[1]$.

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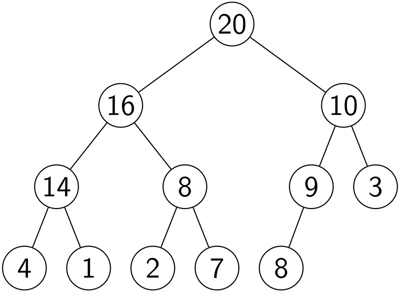
- Insert in the first available position $H[n+1]$.

Example Suppose we want to insert 18 in:



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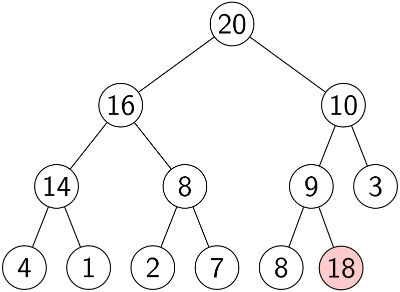

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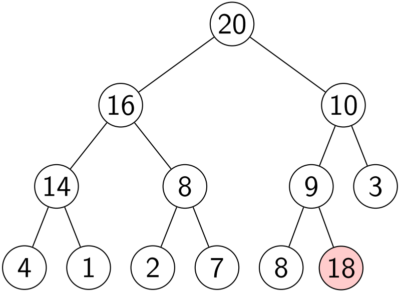

Violate the max-heap property

Heap operations

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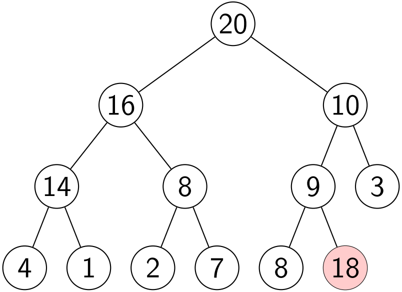
- **Idea** Let the new element "bubble up" by swapping until it finds the right position.



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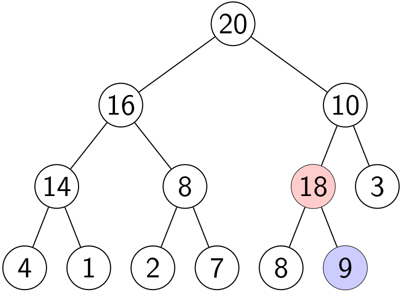

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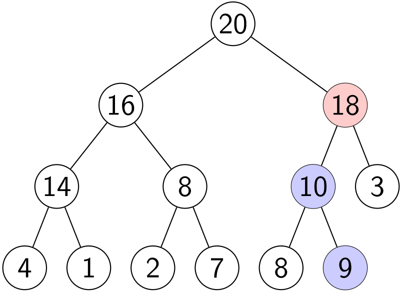

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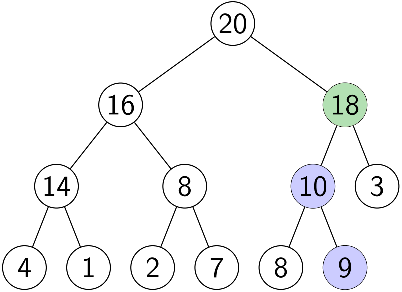

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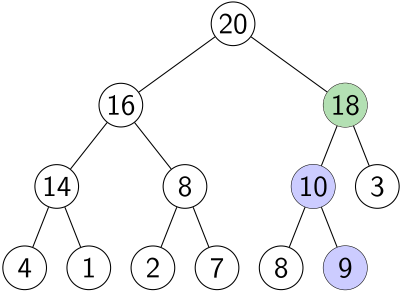

- This process is called **HeapifyUp**.

Heap operations

HeapifyUp(H, i)

```
while i > 1 and H[i] > H[parent(i)] do
    swap H[i] and H[parent(i)]
    i = parent(i)
```

Insert($H[1..n]$, key)

```
// Assume the array H still has available
   space
H[n+1] = key
HeapifyUp(H, n+1)
n = n+1
```

Correctness?

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HeapifyUp takes $O(h)$ time where h is the height of the heap.

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$n = n + 1$

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Time complexity?

HeapifyUp takes $O(h)$ time where h is the height of the heap.

$h = O(\log n)$.