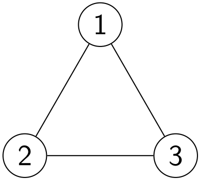
# CMPSC 465: LECTURE X

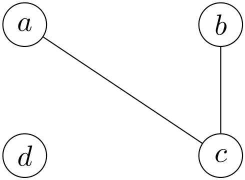
## Graphs and graph algorithms

Ke Chen

September 22, 2025

# Graphs



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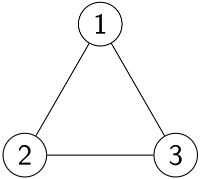


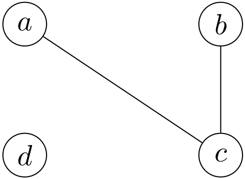


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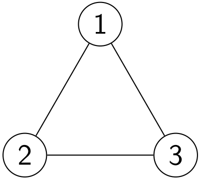


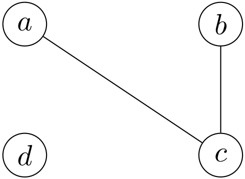


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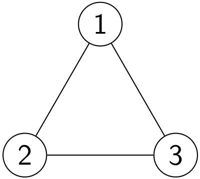


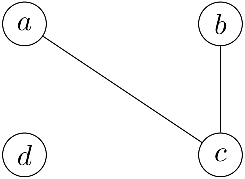


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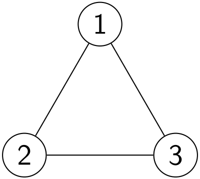


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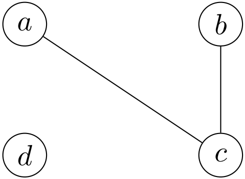


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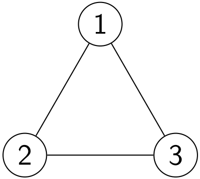


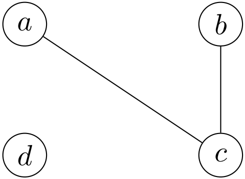


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- A graph is defined by a set of vertices $V$ and a set of edges $E$.
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- We write $G = (V, E)$ for a graph with vertex set $V$ and edge set $E$.

# Why graphs?

By Jakob Emanuel Handmann

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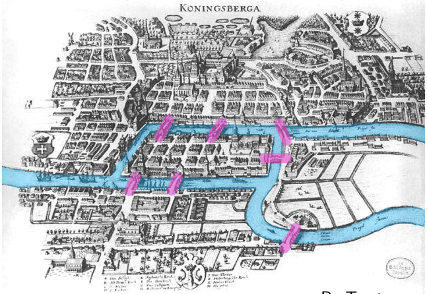


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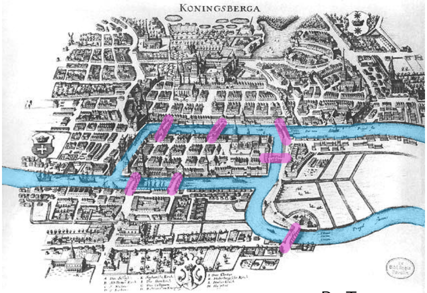


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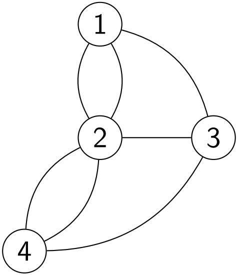

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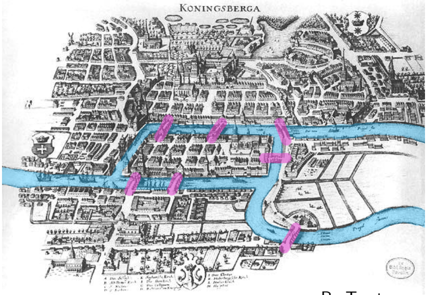


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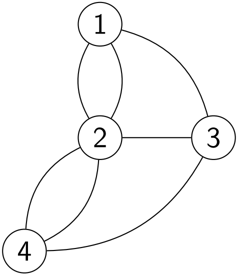


Since then graphs have been used to model a variety of things: maps, relationships, constraints, networks, ...

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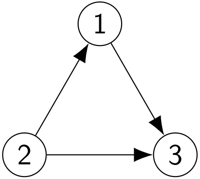


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- $A$ is called the adjacency matrix of $G$.
- If $G$ is undirected, $A$ is symmetric ($A_{ij} = A_{ji}$).
- Diagonal entries are often set to $0$, since we typically work with simple graphs (no self-loop, no multi-edges).

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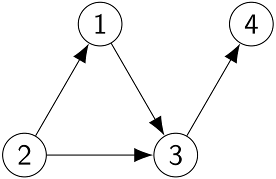


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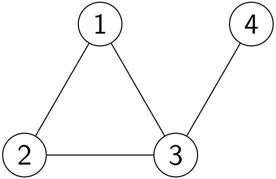


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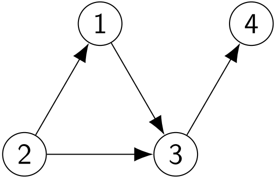


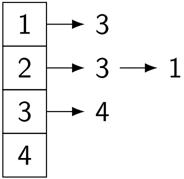

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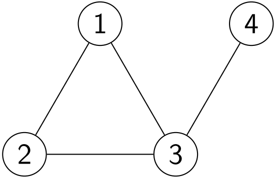


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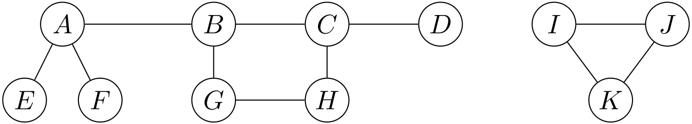

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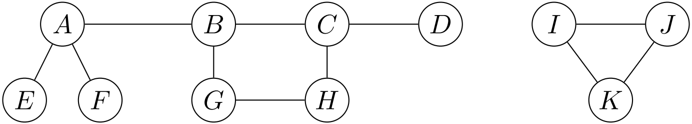


- $A$ is connected to $G$.
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- $A$ is not connected to $I$, $J$, or $K$.

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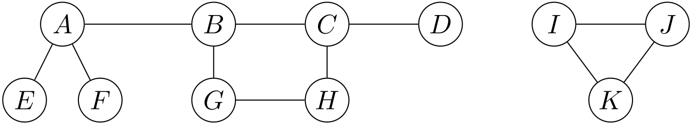


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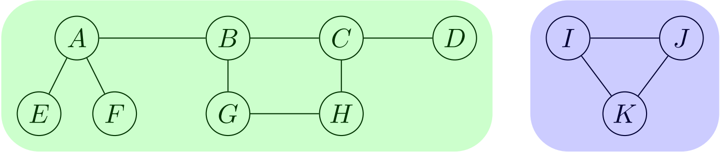


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- DFS and BFS are very powerful algorithms, with interesting properties and applications.