CMPSC 360 Discrete Mathematics for Computer Science
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Worksheet 6 Solutions

1. Determine if the following relations are reflexive, symmetric, antisymmetric, transitive, equivalence relation, or poset (i.e., partially ordered set) on the respective sets:

   (a) $R_0 = \{(a,a),(b,b),(a,c),(c,a)\}$ on $A_0 = \{a,b,c,d\}$

   (b) $R_1 = \{(a,a),(c,c),(b,b),(a,d),(d,b),(a,b),(a,c)\}$ on $A_1 = \{a,b,c,d,e\}$.

   (c) Also, determine the relational properties that R_1 satisfies on A_0 from above.

   (d) $R_2 = \{(x,y) \in P \times P \mid x \subseteq y\}$ defined on P such that P is the power set of an arbitrary set X

   Answer:

   (a) **Reflexive:** Not reflexive since $(d,d),(c,c) \notin R_0$.
   Symmetric: Yes
   Antisymmetric: No, since $(a,c),(c,a) \in R_0$
   Transitive: No, since $(c,a),(a,c)$, but no (c,c)
   R_0 is neither an equivalence relation nor a poset on A_0.

   (b) **Reflexive:** Not reflexive since $(d,d),(e,e) \notin R_1$.
   Symmetric: No, since $(d,a) \notin R_1$
   Antisymmetric: Yes
   Transitive: Yes
   R_1 is neither an equivalence relation nor a poset on A_1.

   (c) **Reflexive:** Not reflexive since $(d,d) \notin R_1$.
   Symmetric: No, since $(d,a) \notin R_1$
   Antisymmetric: Yes
   Transitive: Yes
   R_1 is neither an equivalence relation nor a poset on A_0.

   (d) **Reflexive:** Yes, any set is a subset of itself.
   Symmetric: No
   Antisymmetric: Yes, since equal sets can be interchangeably written as subsets of each other.
   Transitive: Yes
   R_2 is a partial order relation on P.

2. Prove that the following relation R is an equivalence relation on $\mathbb{Z}$:

$$R = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} \mid \frac{x^2 - y^2}{4} \in \mathbb{Z}\}$$

Answer:

Proof

The relation R is an equivalence relation on set $\mathbb{Z}$ if it is reflexive, symmetric and transitive.

The relation R is reflexive on set $\mathbb{Z}$ as $\forall x \in \mathbb{Z}((x,x) \in R)$. This is because for any integer x, $\frac{x^2-x^2}{4} = 0$ and $0 \in \mathbb{Z}$.

To check if it is symmetric, let's assume a pair $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ that satisfies our relation R. If $(a,b) \in R$, $\frac{a^2-b^2}{4} \in \mathbb{Z}$. This means that $a^2 - b^2$ is divisible by 4. If $a^2 - b^2$ is divisible by 4, $b^2 - a^2$ is also divisible by 4. Therefore, $\frac{b^2-a^2}{4} \in \mathbb{Z}$ and therefore, $(b,a) \in R$. Since we have proved $\forall a \forall b((a,b) \in R \to (b,a) \in R)$, the following relation is symmetric.

To check if it is transitive, let's assume two pairs (a,b) and $(b,c) \in \mathbb{Z} \times \mathbb{Z}$ that satisfies our relation R. Therefore, $(a,b) \in R \wedge (b,c) \in R$, which means that $\frac{a^2-b^2}{4} \in \mathbb{Z}$ and $\frac{b^2-c^2}{4} \in \mathbb{Z}$. Since adding two integers results in an integer, we can say that $\frac{a^2-b^2}{4} + \frac{b^2-c^2}{4} \in \mathbb{Z}$. Therefore, $\frac{a^2-c^2}{4} \in \mathbb{Z}$, which means that $(a,c) \in R$. Since we have proved $\forall a \forall b \forall c((a,b) \in R \wedge (b,c) \in R \to (a,c) \in R$, the following relation is transitive.

Therefore the relation above is an equivalence relation on $\mathbb{Z}$. $\square$

3. Let $A = a,b,c,d,e$. Suppose R is an equivalence relation on A. Suppose R has three equivalence classes. Also aRd and bRc. Write out R as a set.

Answer:

Given that R is an equivalence relation on A, we know R is reflexive, symmetric and transitive.

From reflexivity, we know that $(a,a),(b,b),(c,c),(d,d),(e,e) \in R$.

From symmetry, we know that $(a,d),(d,a),(b,c),(c,b) \in R$.

Then, because R has three equivalence classes, and from the existing relations, we have $[a] = [d] = \{a,d\}$, $[b] = [c] = \{b,c\}$, and $[e] = \{e\}$.

Trying to add any other relations would result in a contradiction of the above findings. Try it out yourself. Thus the set R is:

$$R = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,d),(d,a),(b,c),(c,b)\}$$

4. Define a relation R on $\mathbb{Z}$ as xRy if and only if $3x-5y$ is even. Prove R is an equivalence relation. Describe its equivalence classes.

Answer:

Equivalence Relation:

Informally, if $3x-5y$ is even, then x and y must have same parity. *Try to prove it formally by yourself.*

The relation R is an equivalence relation on set $\mathbb{Z}$ if it is reflexive, symmetric and transitive.

The relation R is reflexive on set $\mathbb{Z}$ as $\forall x \in \mathbb{Z}((x,x) \in R)$. This is because for any integer x, $3x-5x = -2x = 2(-x)$, which is even by definition.

To check if it is symmetric, let's assume a pair $(a,b) \in \mathbb{Z} \times \mathbb{Z}$ that satisfies our relation R. If $(a,b) \in R$, $3a-5b = 2k$ where $k \in \mathbb{Z}$. From the same parity property of x and y above, it's not hard to see that, if $a = 2m, b = 2n$ for some $m,n \in \mathbb{Z}$, then $3b-5a = 6n-10m = 2(3n-5m)$. On the other hand, if $a = 2m+1, b = 2n+1$ for some $m,n \in \mathbb{Z}$, then $3b-5a = 6n+3-10m-5 = 2(3n-5m-1)$. Therefore, in both cases, $(b,a) \in R$. Since we have proved $\forall a \forall b((a,b) \in R \to (b,a) \in R)$, the relation is symmetric.

To check if it is transitive, let's assume two pairs (a,b) and $(b,c) \in \mathbb{Z} \times \mathbb{Z}$ that satisfies our relation R. Therefore, $(a,b) \in R \wedge (b,c) \in R$, which means that $3a-5b = 2k_1$ and $3b-5c = 2k_2$ where $k_1, k_2 \in \mathbb{Z}$. Adding the two equations, we get $3a-5c = 2(k_1+k_2+b)$. Therefore, $(a,c) \in R$. Since we have proved $\forall a \forall b \forall c((a,b) \in R \wedge (b,c) \in R \to (a,c) \in R$, the relation is transitive.

Therefore the relation above is an equivalence relation on $\mathbb{Z}$. $\square$

Equivalence Classes:

From the above, we already proved that x and y must have the same parity for xRy. As this is the only constraint on x and y, the equivalence class containing 0 seems like a reasonable place to start.

$$[0] = \{x \in \mathbb{Z} : x = 2k, k \in \mathbb{Z}\} = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$$

Note that $[0] = [2] = [4] = [-2] = [-4] = \ldots$, which forms the first equivalence class. Following the same logic, we can find the other equivalence class:

$$[1] = \{x \in \mathbb{Z} : x = 2k+1, k \in \mathbb{Z}\} = \{\ldots, -3, -1, 1, 3, \ldots\}$$

Note that $[1] = [3] = [-1] = [-3] = \ldots$, which forms the second equivalence class.

5. Determine if the following relations are functions:

   (a) $f_0 : \{1,2,3\} \to \{1,2,3\}$ and $f_0 = \{(1,1),(2,1),(3,2)\}$
   (b) $f_1 : \{1,2,3,4\} \to \{1,2,3\}$ and $f_1 = \{(1,1),(2,1),(3,2)\}$
   (c) $f_2 : \{a,b,c\} \to \{a,b\}$ and $f_2 : \{(a,a),(b,a),(b,c)\}$

   Answer:

   (a) Yes, it is a function. All elements in the domain are mapped to an element in the co-domain and there is no one-to-many mapping.
   (b) No, it is not a function (or an ill-defined function) since f_1 is not everywhere defined. Any valid function needs to have all elements in the domain mapped to an element in the co-domain. Here, 4 is not mapped.
   (c) f_2 is not a function because b is mapped to c and a. A one-to-many relation is not a function.

6. Determine which functional properties the following functions satisfy and whether they are bijective:

   (a) $f : \mathbb{Z} \to \mathbb{Q}$ and $f(x) = \frac{2x-3}{5}$
   (b) $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = 3x^2 + 11x$

   Answer:

   (a) **Injectivity:**
   Assume $x_1, x_2 \in \mathbb{Z}$ and $f(x_1) = f(x_2)$.
   $\implies \frac{2x_1-3}{5} = \frac{2x_2-3}{5}$
   $\implies x_1 = x_2$
   Thus, by the definition of injectivity, f is injective.
   Surjectivity:
   Suppose that for some $x \in \mathbb{Z}$, $y = f(x)$. Then,
   $y = \frac{2x-3}{5}$
   $\implies x = \frac{5y+3}{2}$
   For $y = 2 \in \mathbb{Q}$, $x = \frac{13}{2} \notin \mathbb{Z}$. Therefore, f is not surjective.
   (b) **Injectivity:**
   Not injective because $f(1) = f(\frac{-14}{3}) = 14$ and $1 \neq \frac{-14}{3}$.
   Surjectivity:
   For some $x \in \mathbb{R}$, let $y = f(x)$
   $\implies y = 3x^2 + 11x$
   $\implies 3x^2 + 11x - y = 0$
   $\implies x = \frac{-11+\sqrt{121+12y}}{6}$
   For $y = -11$, $x \notin \mathbb{R}$
   Therefore, f is not surjective on $\mathbb{R}$