CMPSC 465: LECTURE XIV

Breath First Search
and
Shortest Paths

Ke Chen

October 01, 2025

Breadth First Search (BFS)

Input: Graph $G = (V, E)$, a starting vertex s, an integer $color$
Output: All vertices reachable from s marked with $color$

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// visited is an array of length |V|, filled with 0's
BFS-Explore(G, s, color)
    Q.enqueue(s) // Q is a queue
    visited[s] = color
    while Q is not empty do
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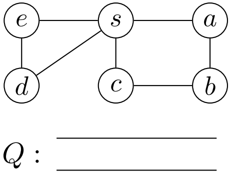

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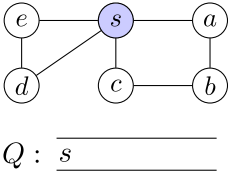

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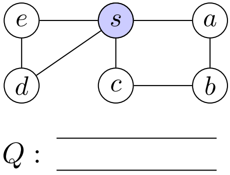

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`//` $visited$ `is an array of length` $|V|$`, filled with 0's`

BFS-Explore(G, s, $color$)

- Q.enqueue(s) `//` Q `is a queue`
- $visited[s] = color$
- **while** Q *is not empty* **do**
  - $v = Q$.dequeue()
  - **foreach** *edge* $\{v, w\} \in E$ **do**
    - **if** $visited[w] == 0$ **then**
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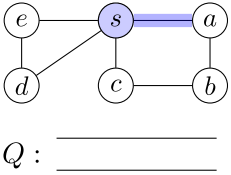

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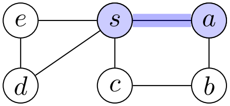

Q : a

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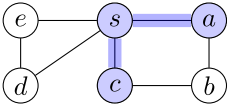

$Q:$ $a \; c$

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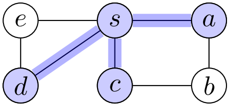

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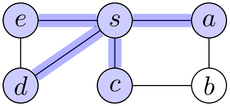

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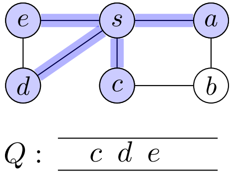

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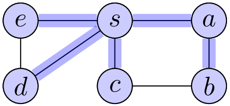

Q : $c \; d \; e \; b$

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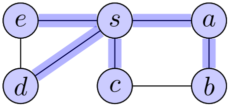

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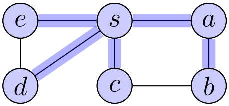

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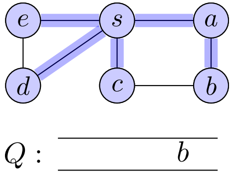

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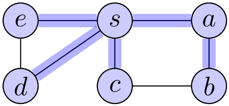

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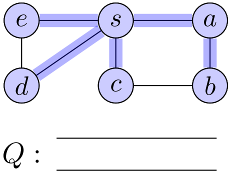

Correctness? Almost the same as Explore for DFS (Exercise).

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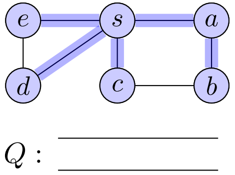

Time complexity?

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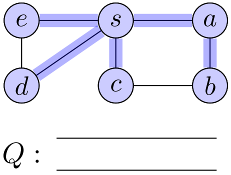

Time complexity? Each vertex is enqueued exactly once.
Each edge is considered twice.

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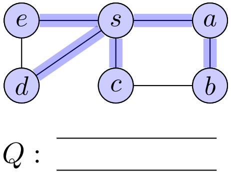

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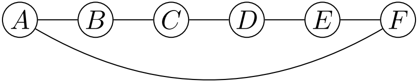
Adjacency list: $O(|V| + |E|)$ Adjacency matrix: $O(|V|^2)$

BFS vs DFS



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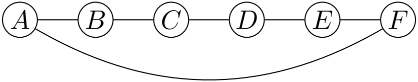
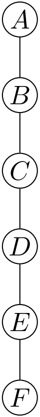


DFS tree:



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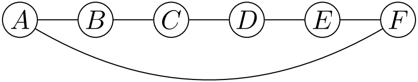

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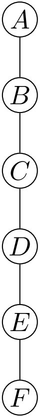

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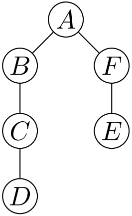

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- DFS finds some path, BFS finds the shortest path

BFS vs DFS

- Just like DFS, the same BFS algorithm works for both directed and undirected graphs.
- Same complexity (how about MicroMouse?)
- Both can answer queries "Can I go from A to B?"
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The shortest path problem

Given a graph G, what is the shortest path between two vertices?

First scenario G is unweighted, i.e., all edges have distance one.

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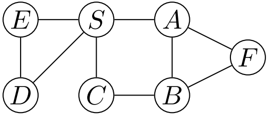
Example



- Shortest path between D and F?

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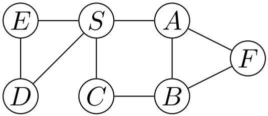

- Shortest path between D and F? $D \rightarrow S \rightarrow A \rightarrow F$, distance=3

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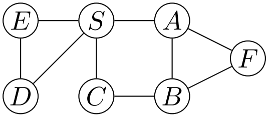

- Shortest path between D and F? $D \to S \to A \to F$, distance=3
- Shortest path between C and A?

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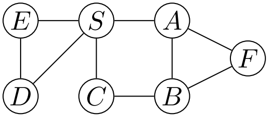

- Shortest path between D and F? $D \to S \to A \to F$, distance=3
- Shortest path between C and A? $C \to S(\text{or } B) \to A$, distance=2

Unweighted shortest path with BFS

Idea We can modify BFS to keep track of distances.

Unweighted shortest path with BFS

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Input: Graph $G = (V, E)$, a starting vertex s
Output: Shortest distances to all vertices reachable from s

```
// dist is an array of length |V|, filled with ∞'s
BFS-ShortestPath(G, s)
    Q.enqueue(s) // Q is a queue
    dist[s] = 0
    while Q is not empty do
        v = Q.dequeue()
        foreach edge {v, w} ∈ E do
            if dist[w] == ∞ then
                Q.enqueue(w)
                dist[w] = dist[v] + 1
```

Unweighted shortest path with BFS

Time complexity? Same as BFS.

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Correctness? Prove by induction that BFS explores "layer by layer": for $d = 0, 1, 2, \ldots$, there is a moment at which

- All nodes with distance $\leq d$ from s have the correct distances.
- All other nodes have distances ∞.
- The queue contains exactly all nodes at distance d.

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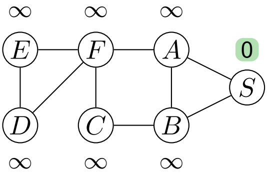

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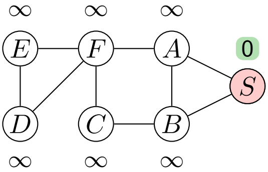

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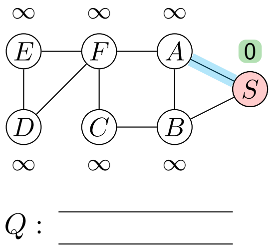

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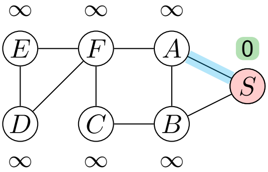
Example



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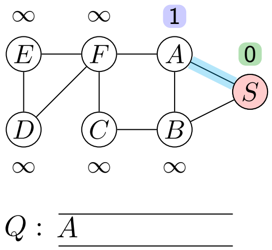

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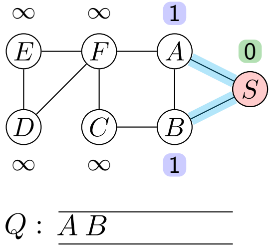

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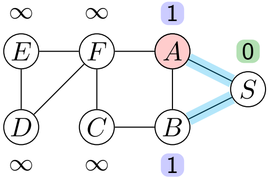

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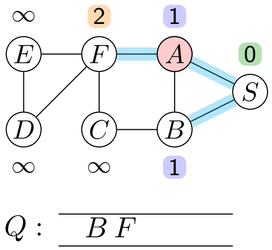

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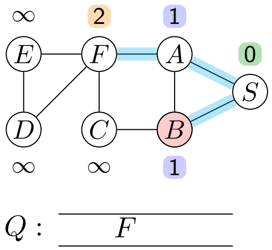

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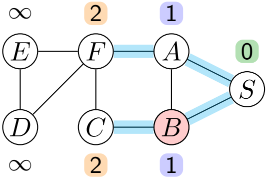

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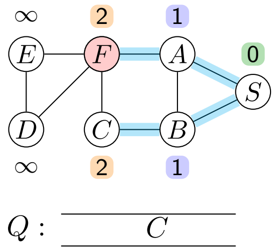

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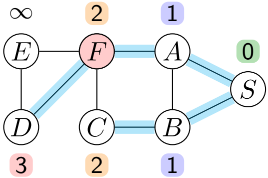

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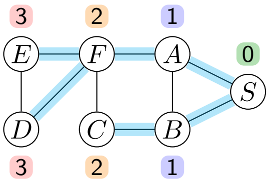

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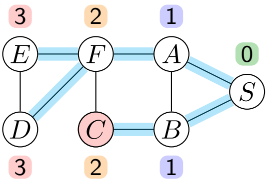

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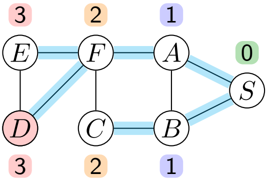

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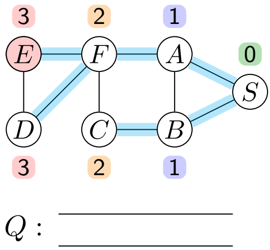

Unweighted shortest path with BFS

Remarks We find all shortest distances from s, this is called the single-source shortest path problem.

- No need to call Explore on unvisited vertices.
- Have to make a fresh start if want shortest distances from a different node.

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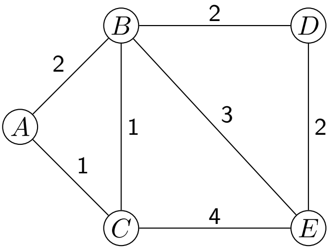

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