CMPSC 465 | Data Structures & Algorithms
Fall 2025 | Ke Chen and Yana Safonova

HW 1

Due September 15, 10:00 pm

Instructions: You are encouraged to solve the problem sets on your own, or in groups of three to five people, but you must write your solutions strictly by yourself. You must explicitly acknowledge in your write-up all your collaborators, as well as any books, papers, web pages, etc. you got ideas from.

Formatting: Each problem should begin on a new page. Each page should be clearly labeled with the problem number. The pages of your homework submissions must be in order. You risk receiving no credit for it if you do not adhere to these guidelines.

Late homework will not be accepted. Please, do not ask for extensions since we will provide solutions shortly after the due date. Remember that we will drop your lowest three scores.

This homework is due Monday, September 15, at 10:00 pm electronically. You need to submit it via Gradescope (Course ID: 1087979). Please ask on Campuswire about any details concerning Gradescope and formatting.

1. (5 pts.) **Getting started**. Please read the course policies on the syllabus, especially the course policies on collaboration. If you have any questions, contact the instructors. Once you have done this, please write "I understand the course policies." on your homework to get credit for this problem

2. (36 pts.) **Comparing growth rates.** In each of the following situations, indicate whether $f = O(g)$, or $f = \Omega(g)$, or both (in which case $f = \Theta(g)$). Justify your answers.

	$f(n)$	$g(n)$
a)	$4n \cdot 3^n + n^{50}$	4^n
b)	$\log(5n)$	$\log(4n)$
c)	$e^n + n^{10}$	2^n
d)	7^n	8^n
e)	$\log(n^8 + n^2)$	$\log(6n)$
f)	$2^n + n^{100}$	$3 \cdot 2^n + \log n$
g)	$\sqrt{n} + \log n$	$4\sqrt{n} + n^{1/3}$
h)	$(\log_3 n)^{\log_3 n}$	$3^{(\log_3 n)^2}$
i)	$\frac{\log n}{n}$	$n^{\frac{1}{n}}$

Answer:

(a) $f(n) = O(g(n))$: $\lim_{n\to\infty} \frac{4n\cdot 3^n + n^{50}}{4^n} = \lim_{n\to\infty} 4n\left(\frac{3}{4}\right)^n + \frac{n^{50}}{4^n} = 0$, as exponentials with base less than 1 decay and polynomials grow slower than exponentials, so $f = O(g)$.

(b) $f(n) = \Theta(g(n))$: $\lim_{n\to\infty} \frac{\log(5n)}{\log(4n)} = \lim_{n\to\infty} \frac{\log 5 + \log n}{\log 4 + \log n} = 1$, since $\log n$ dominates constants, implying $f = \Theta(g)$.

(c) $f(n) = \Omega(g(n))$: $\lim_{n\to\infty} \frac{e^n + n^{10}}{2^n} = \lim_{n\to\infty} \left(\frac{e}{2}\right)^n + \frac{n^{10}}{2^n} = \infty$, since $e/2 > 1$ and polynomials are dominated by exponentials, so $f = \Omega(g)$.

(d) $f(n) = O(g(n))$: $\lim_{n\to\infty} \frac{7^n}{8^n} = \left(\frac{7}{8}\right)^n \to 0$, as the base $7/8 < 1$ implies exponential decay, so $f = O(g)$.

(e) $f(n) = \Theta(g(n))$: Since $n^8 \le n^8 + n^2 \le 2n^8$, we have $8\log n \le \log(n^8 + n^2) \le \log 2 + 8\log n$, and $g(n) = \log 6 + \log n$, so both are $\Theta(\log n)$, implying $f = \Theta(g)$.

(f) $f(n) = \Theta(g(n))$: Both are dominated by 2^n (faster than n^{100} or $\log n$), so $f(n) = \Theta(2^n)$, $g(n) = \Theta(2^n)$, and thus $f = \Theta(g)$.

(g) $f(n) = \Theta(g(n))$: Both are dominated by $\sqrt{n}$ (faster than $\log n$ or $n^{1/3}$), so $f(n) = \Theta(\sqrt{n})$, $g(n) = \Theta(\sqrt{n})$, implying $f = \Theta(g)$.

(h) $f(n) = O(g(n))$: Let $k = \log_3 n$, so $f(n) = k^k$, $g(n) = 3^{k^2} = (3^k)^k$; since $3^k \ge k$ for all $k \ge 1$, $g(n) \ge f(n)$, so $f = O(g)$.

(i) $f(n) = O(g(n))$: Since $\log x \le x$ for all $x \ge 1$, we have $f(n) = \log n^{1/n} < n^{1/n} = g(n)$. By the definition of big-O with $c = 1$ and $n_0 = 1$, $f(n) = O(g(n))$.

3. (24 pts.) **Proofs.**

(a) Prove that $3n^2 + 4n + 5 = \Theta(n^2)$.

(b) Prove that if $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, then $f_1(n) + f_2(n) = O(\max(g_1(n), g_2(n)))$.

(c) Prove that if $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, then $f(n) = \Theta(h(n))$ i.e., transitivity of theta notation.

Answer:

(a) *Proof.* To prove that $3n^2 + 4n + 5 = \Theta(n^2)$, we need to show that there exist positive constants c_1, c_2, and n_0 such that $c_1 n^2 \le 3n^2 + 4n + 5 \le c_2 n^2$ for all $n \ge n_0$.

For the lower bound: $3n^2 + 4n + 5 \ge 3n^2$ for all $n \ge 0$, since $4n + 5 \ge 0$. Thus, with $c_1 = 3$ and $n_0 = 1$, the inequality holds.

For the upper bound: For $n \ge 1$, $4n \le 4n^2$ (since $n \le n^2$) and $5 \le 5n^2$ (since $1 \le n^2$). Therefore, $3n^2 + 4n + 5 \le 3n^2 + 4n^2 + 5n^2 = 12n^2$. Thus, with $c_2 = 12$ and $n_0 = 1$, the inequality holds.

Combining both, $3n^2 \le 3n^2 + 4n + 5 \le 12n^2$ for all $n \ge 1$, so $3n^2 + 4n + 5 = \Theta(n^2)$. $\square$

(b) *Proof.* Assume $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$, with asymptotically non-negative functions. By definition, there exist constants $c_1 > 0$, n_1 such that $0 \le f_1(n) \le c_1 g_1(n)$ for $n \ge n_1$, and $c_2 > 0$, n_2 such that $0 \le f_2(n) \le c_2 g_2(n)$ for $n \ge n_2$.

Let $m(n) = \max(g_1(n), g_2(n))$. Then, $g_1(n) \le m(n)$ and $g_2(n) \le m(n)$, so $f_1(n) + f_2(n) \le c_1 g_1(n) + c_2 g_2(n) \le c_1 m(n) + c_2 m(n) = (c_1 + c_2) m(n)$ for $n \ge \max(n_1, n_2)$.

Since $c_1 + c_2 > 0$ and $f_1(n) + f_2(n) \ge 0$, we have $0 \le f_1(n) + f_2(n) \le (c_1 + c_2) m(n)$ for $n \ge \max(n_1, n_2)$, so $f_1(n) + f_2(n) = O(m(n)) = O(\max(g_1(n), g_2(n)))$. $\square$

(c) *Proof.* Assume $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$, with asymptotically non-negative functions. By definition, there exist $c_1, c_2 > 0$ and n_1 such that $c_1 g(n) \le f(n) \le c_2 g(n)$ for $n \ge n_1$, and $d_1, d_2 > 0$ and n_2 such that $d_1 h(n) \le g(n) \le d_2 h(n)$ for $n \ge n_2$.

For the lower bound: $f(n) \ge c_1 g(n) \ge c_1 d_1 h(n)$. For the upper bound: $f(n) \le c_2 g(n) \le c_2 d_2 h(n)$.

Thus, $c_1 d_1 h(n) \le f(n) \le c_2 d_2 h(n)$ for $n \ge \max(n_1, n_2)$, where $c_1 d_1 > 0$ and $c_2 d_2 > 0$. Therefore, $f(n) = \Theta(h(n))$. $\square$

4. (14 pts.) **Useful Identities.** Show the following statements hold true.

(a) Show that

$$\log(n!) = \Theta(n \log n).$$

(b) Show that

$$\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n).$$

(*Hint:* To show an upper bound, compare $\frac{1}{i}$ with $\frac{1}{t}$, where t is a power of two just smaller than i. To show a lower bound, compare $\frac{1}{i}$ with $\frac{1}{t}$, where t is a power of two just greater than i.)

Answer:

(a) Observe that

$$n! = 1 * 2 * 3 \cdots * n \leq n * n * n \cdots * n = n^n$$

and

$$n! = 1 * 2 * 3 \cdots * n \geq n * (n-1) * (n-2) \cdots * (n - \lfloor \tfrac{n}{2} \rfloor) \geq \left(\frac{n}{2}\right)^{\lfloor \frac{n}{2} \rfloor} \geq \left(\frac{n}{2}\right)^{\frac{n}{2}-1}$$

Hence $\left(\frac{n}{2}\right)^{\frac{n}{2}-1} \leq n! \leq n^n$. Then by taking the logarithm:

$$\left(\frac{n}{2} - 1\right) \log\left(\frac{n}{2}\right) \leq \log(n!) \leq n \log n.$$

Finally note that

$$\left(\frac{n}{2} - 1\right) \log\left(\frac{n}{2}\right) \geq \frac{1}{2}\left(\frac{n}{2} - 1\right) \log(n) \geq \left(\frac{n}{8}\right) \log(n).$$

Where both of these inequalities hold for all $n \geq 4$.

$$\frac{1}{8} n \log(n) \leq \log(n!) \leq n \log n.$$

for all $n \geq 4$.

(b) The main idea for both upper and lower bound is that for any k there exist a unique value of i such that $\frac{1}{2^{i+1}} \leq \frac{1}{k} \leq \frac{1}{2^i}$. Note that for several k's the value of i is the same, so we can replace several $1/k$'s by the same inverse power of 2 to obtain a more manageable sum. More precisely, for the upper bound we replace each $1/k$ with $1/2^i$, where 2^i is the largest power of 2 less than or equal to k. Let $\ell = \lceil \log_2 n \rceil$.

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{k} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \\
&\leq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots \\
&\leq 1 + 2 \cdot \tfrac{1}{2} + 4 \cdot \tfrac{1}{4} + \cdots + 2^{\ell} \cdot \frac{1}{2^{\ell}}. \\
&= 1 + 1 + 1 + \cdots + 1 = \ell + 1 = O(\log n)
\end{aligned}$$

There are $\ell + 1$ terms in the final sum because there is a single 1 for each power of 2 from 0 to ℓ. This is always an upper bound, regardless of whether or not n is a power of 2, due to including all 2^{ℓ} of the $\frac{1}{2^{\ell}}$ terms. At most one of these terms is present in the original sum, since $n \leq 2^{\ell}$. We can do something similar for the lower bound, except we replace $1/k$ with $1/2^i$, where 2^i is the smallest power of 2 greater than k. We can guarantee this sum is a lower bound by only including terms up to $\frac{1}{2^{\ell-1}}$.

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \geq \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots \geq \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \frac{\ell - 1}{2} = \Omega(\log n)$$

Note. This problem can also be solved by integrating: $\sum_{k=1}^{n} \frac{1}{k} = \Theta(\int_1^n \frac{1}{x} dx) = \Theta(\log n)$.

5. (21 pts.) **Recurrence Relations**. Solve the following recurrence relations and give the tightest correct upper bound for each of them in O notation. Assume that $T(O(1)) = O(1)$. Show all work.

(a) $T(n) = 4T(n/2) + 5n^3$

(b) $T(n) = 3T(n/4) + n^{0.6}$

(c) $T(n) = 7T(n/3) + n^2$

(d) $T(n) = 8T(n/2) + 3n^3$

(e) $T(n) = T(n-1) + d^n$, where $d \geq 1$ is a constant

(f) $T(n) = T(n/3) + 0.75^n$

(g) $T(n) = T(4n/7) + T(3n/7) + \Theta(n)$. (*Hint: Assume $T(k) \leq ck \log k$ for some constant $c > 0$ for all $k < n$. Prove that $T(n) \leq c_1 n \log n$ for some constant $c_1 > 0$ by induction.*)

Answer:

(a) $a = 4$, $b = 2$, $d = 3$; $\log_2 4 = 2 < 3$, so we get $O(n^3)$ using Master Theorem.

(b) $a = 3$, $b = 4$, $d = 0.6$; $\log_4 3 \approx 0.792 > 0.6$, so we get $O(n^{0.792})$ using Master Theorem.

(c) $a = 7$, $b = 3$, $d = 2$; $\log_3 7 \approx 1.771 < 2$, so we get $O(n^2)$ using Master Theorem.

(d) $a = 8$, $b = 2$, $d = 3$; $\log_2 8 = 3 = d$, so we get $O(n^3 \log n)$ using Master Theorem.

(e) If $d = 1$, unrolling gives us $T(n) = T(n-i) + i$, with the base case $T(1) = O(1)$, we conclude $T(n) = O(n)$. If $d > 1$, after unrolling, $T(n) = T(0) + d\frac{d^n - 1}{d-1} = O(d^n)$.

(f) $T(n) = T(n/3) + 0.75^n = (T(n/9) + 0.75^{n/3}) + 0.75^n = ((T(n/27) + 0.75^{n/9}) + 0.75^{n/3}) + 0.75^n = \ldots = T(1) + \sum_{i=0}^{\log_3 n - 1} 0.75^{n/3^i}$
Now, $T(1)$ is a constant, say, c. Then, we may rewrite the series as follows:
$T(n) = c + \sum_{i=0}^{m-1} (0.75)^{3^{m-i}}$ where $m = log_3 n$
$\implies T(n) = c + \sum_{i=1}^{m} (0.75)^{3^i}$
We also know $\sum_{i=1}^{m} (0.75)^{3^i} \leq \sum_{i=1}^{\infty} (0.75)^{3^i} = C$ (some constant)
Thus, $T(n) = O(1)$

(g) *Proof.* Define $P(n) : T(n) = O(n \log n)$. We proceed using induction.
Base case. ($n = 1$): $T(1) = O(1)$, so, $P(1)$ holds true.
Inductive step. Assume that for all $k < n$, we have $T(k) \leq c_1 k \log k$. We want to show that there exists c_1 such that for all large enough n, $T(n) \leq c_1 n \log n$. Since both $\frac{4n}{7}$, and $\frac{3n}{7}$ are less than n according to inductive hypothesis, we have: $T(\frac{4n}{7}) \leq c_1 \frac{4n}{7} \log \frac{4n}{7}$, and $T(\frac{3n}{7}) \leq c_1 \frac{3n}{7} \log \frac{3n}{7}$. Now using recurrence

relation for $T(n)$ we can write:

$$\begin{aligned} T(n) \le T(\frac{3n}{7}) + T(\frac{4n}{7}) + cn \le c_1 \frac{3n}{7}\log\frac{3n}{7} + c_1\frac{4n}{7}\log\frac{4n}{7} + cn \\ = c_1\frac{3n}{7}\log\frac{3}{7} + c_1\frac{3n}{7}\log n + c_1\frac{4n}{7}\log\frac{4}{7} + c_1\frac{4n}{7}\log n + cn \\ = c_1 n\log n + c_1 n(\frac{3}{7}\log\frac{3}{7} + \frac{4}{7}\log\frac{4}{7}) + cn \\ = c_1 n\log n + cn - c_1 n(\frac{3}{7}\log\frac{7}{3} + \frac{4}{7}\log\frac{7}{4}) \end{aligned} \tag{1}$$

Now, note that if we let $c_1 = \frac{c}{\frac{3}{7}\log\frac{7}{3} + \frac{7}{4}\log\frac{4}{7}}$, and replace it in the latter equation, we get $cn - c_1 n(\frac{3}{7}\log\frac{7}{3} + \frac{7}{4}\log\frac{4}{7}) = 0$, which means that $T(n) \le c_1 n\log n$. So, induction step is complete, and we proved that $T(n) = O(n\log n)$ $\square$

Rubric:

Problem 1, ? pts

?

Problem 2, ? pts

?

Problem 3, ? pts

?

Problem 4, ? pts

?

Problem 5, ? pts

?