CMPSC 465: LECTURE V

Information Theory Lower Bound
QuickSort

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September 08, 2025

Recall from last week...

The time complexity of MergeSort satisfies
$T(n) = 2T(n/2) + \Theta(n)$.

We can solve this recurrence relation by

- substitution (guess and induction),
- unrolling,
- recursion tree,
- master theorem,

to get $T(n) = \Theta(n \log n)$.

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Can we do better?

No. Any comparison-based sorting algorithm needs $\Omega(n \log n)$ comparisons in the worst case.

Decision-tree algorithms

Consider sorting three distinct numbers a, b, and c:

$$a < b?$$

Decision-tree algorithms

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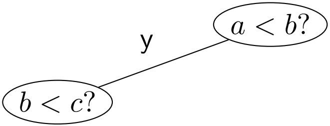

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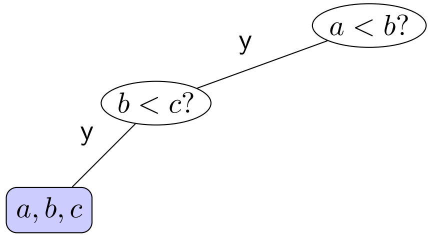

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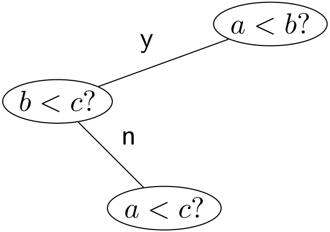

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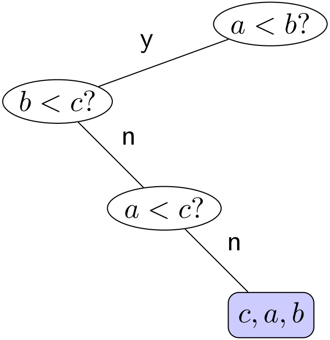

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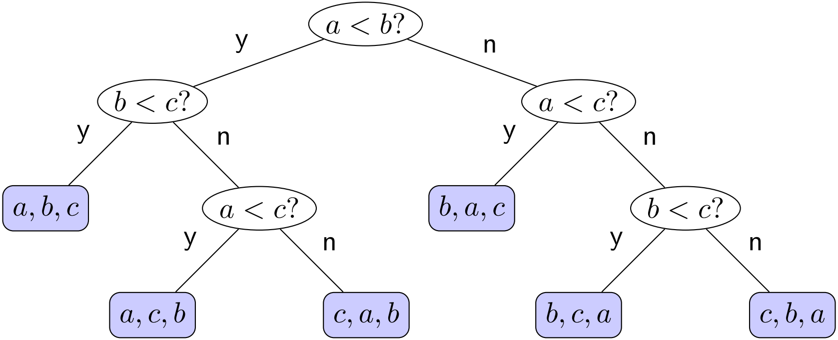

The information theory lower bound (ITLB)

Observations:

- Comparison-based sorting algorithms can be modeled by decision trees.
- Each leaf node corresponds to a possible output.
- The execution of the algorithm corresponds to a path from root down to a leaf.
- A longest path (i.e., height of the tree) gives the worst-case number of comparisons needed.

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A leaf-counting argument:

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- A binary tree of height h can have at most 2^h leaves.
- So $2^h \geq n!$, or $h \geq \log(n!) = \Omega(n \log n)$.

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Theorem: Any comparison-based sorting algorithm requires $\Omega(n \log n)$ comparisons in the worst case.

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Notes:

- Not all sorting algorithms are comparison-based: e.g., CountingSort, RadixSort, and BucketSort run in $O(n)$ time (with constraints on inputs).
- ITLB can also be used to obtain lower bounds for other comparison-based problems.

ITLB for insertion

Insertion problem: Given a sorted set of n distinct numbers, insert a new number into the proper position.

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- Note that ITLB only considers comparisons.
  How many other operations are needed for insertion?
  Can insertion be done in $O(\log n)$ time?

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Morale: The ITLB does not always give a tight bound, it can be very far off.

Back to MergeSort

Time complexity?

- $\Theta(n \log n)$, asymptotic optimal.

Space complexity?

- $O(n)$ extra space for merging.
- Can be reduced to $O(1)$, but is complicated and not practical.

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Can we do better?

QuickSort

Input: 8,1,9,2,8,4,6,5

Idea: divide and conquer

1. Split input by a pivot $\quad$ $\overset{<}{1,2,4}$ $\overset{\text{pivot}}{5}$ $\overset{>}{8,9,6,8}$

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1	Split input by a pivot	1,2,4	5	8,9,6,8
2	Sort each part	1,2,4	5	6,8,8,9

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3	TADA!	1,2,4,5,6,8,8,9		

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QuickSort(A, st, ed)

```
if st < ed then
    p = Partition(A, st, ed)
    QuickSort(A, st, p − 1)
    QuickSort(A, p + 1, ed)
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How to do Partition?

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How to do Partition? Easy!

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How to do Partition in place, namely, with $O(1)$ extra space?

Partition around pivot

Idea: check elements one by one against the pivot and maintain a $<$ pivot region and a $\geq$ pivot region

$$8 \quad 1 \quad 9 \quad 2 \quad 8 \quad 4 \quad 6 \quad 5$$

Partition around pivot

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last element in the $<$ pivot region

8	1	9	2	8	4	6	5
st							ed

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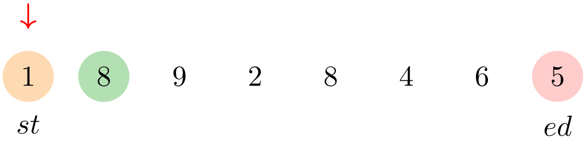

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st							ed
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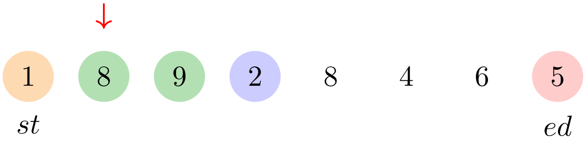

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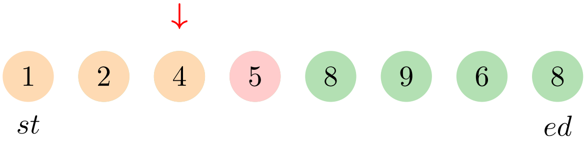

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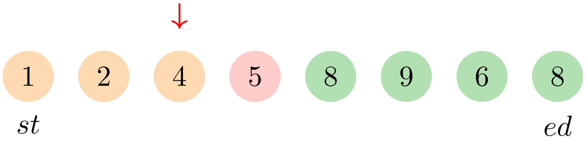

Partition(A, st, ed)

$pivot = A[ed]$
$bd = st - 1$
for $cur = st$ **to** $ed - 1$ **do**
  if $A[cur] < pivot$ **then**
    $bd = bd + 1$
    swap $A[bd]$ with $A[cur]$
swap $A[bd + 1]$ with $A[ed]$
return *bd+1*

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Space complexity? $\Theta(1)$

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Time complexity?

- Worst-case is when each partition results in sizes $(0, 1, n-1)$.

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- Can you find an input that achieves this worst-case performance?
- Why is it called QuickSort then?