CMPSC 465: LECTURE XII

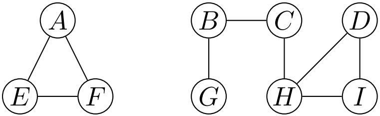
DFS on Directed Graphs

Ke Chen

September 26, 2025

Recall from the previous lecture ...

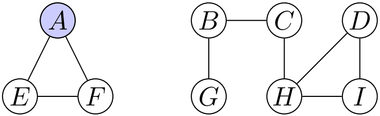
On an undirected graph:



Recall from the previous lecture ...

On an undirected graph:

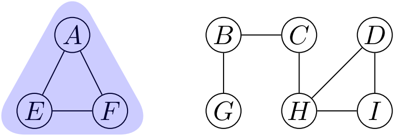
- Explore a node:



Recall from the previous lecture ...

On an undirected graph:

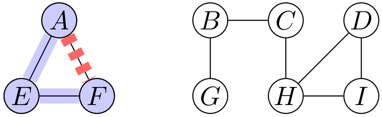
- Explore a node:
  - reveals one connected component



Recall from the previous lecture ...

On an undirected graph:

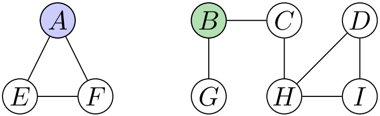
- Explore a node:
  - reveals one connected component
  - produces a DFS tree $\rightsquigarrow$ tree edges, back edges



Recall from the previous lecture ...

On an undirected graph:

- Explore a node:
  - reveals one connected component
  - produces a DFS tree $\rightsquigarrow$ tree edges, back edges
- DFS repeatedly calls Explore:



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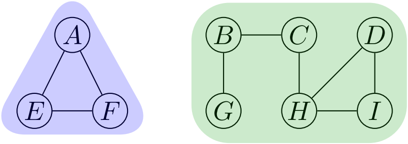
On an undirected graph:

- Explore a node:
  - reveals one connected component
  - produces a DFS tree $\rightsquigarrow$ tree edges, back edges
- DFS repeatedly calls Explore:
  - finds all connected components



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On an undirected graph:

- Explore a node:
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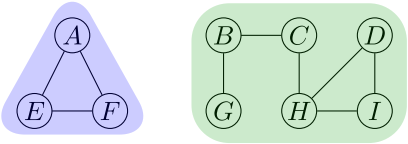
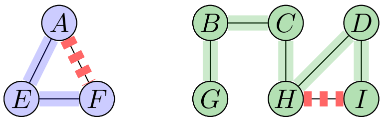
  - produces a DFS tree $\rightsquigarrow$ tree edges, back edges
- DFS repeatedly calls Explore:
  - finds all connected components $\rightsquigarrow$ "Can I go from A to B?"



Recall from the previous lecture ...

On an undirected graph:

- Explore a node:
  - reveals one connected component
  - produces a DFS tree ⇝ tree edges, back edges
- DFS repeatedly calls Explore:
  - finds all connected components ⇝ "Can I go from A to B?"
  - yields a DFS forest



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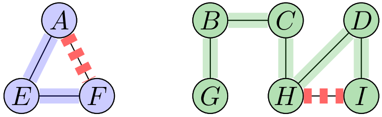
- Explore a node:
  - reveals one connected component
  - produces a DFS tree $\rightsquigarrow$ tree edges, back edges
- DFS repeatedly calls Explore:
  - finds all connected components $\rightsquigarrow$ "Can I go from A to B?"
  - yields a DFS forest $\rightsquigarrow$ cycle detection



Pre- and post-visit timestamps (undirected)

We can collect more information during Explore by keeping a global clock that ticks every time we:

- visit a node for the first time, and
- leave a node for good.

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```
// pre and post are integer arrays of size |V|
// clock is a global integer counter starting at 1
Explore(G, s, color)
    visited[s] = color
    pre[s] = clock
    clock = clock + 1
    foreach edge {s, v} ∈ E do
        if visited[v] == 0 then
            Explore(G, v, color)
    post[s] = clock
    clock = clock + 1
```

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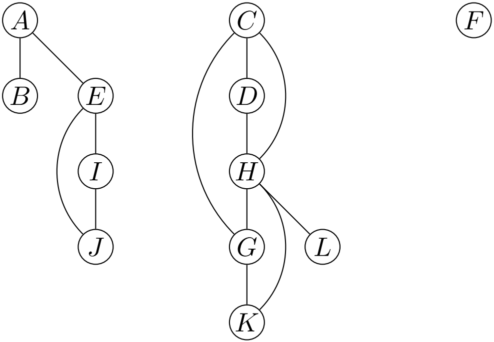

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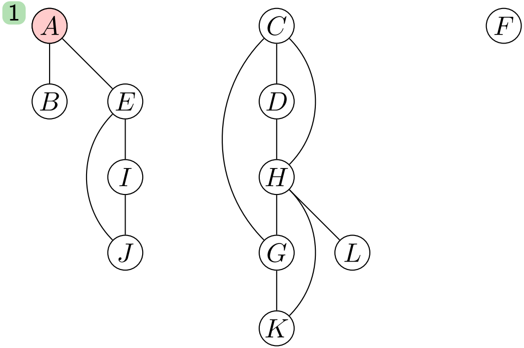

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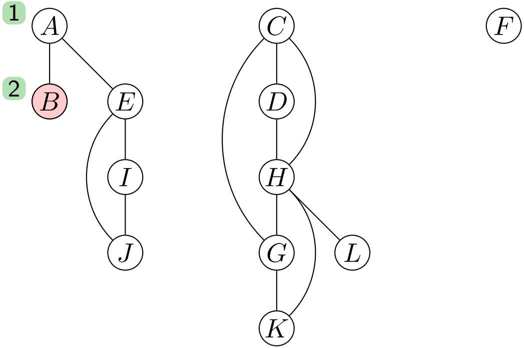

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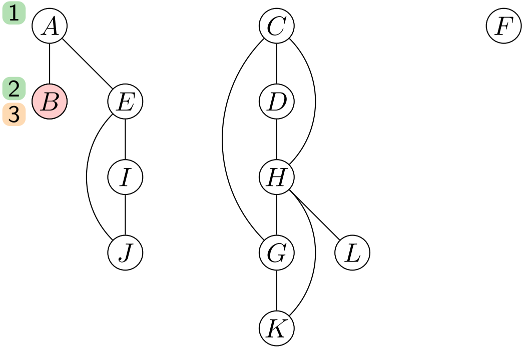

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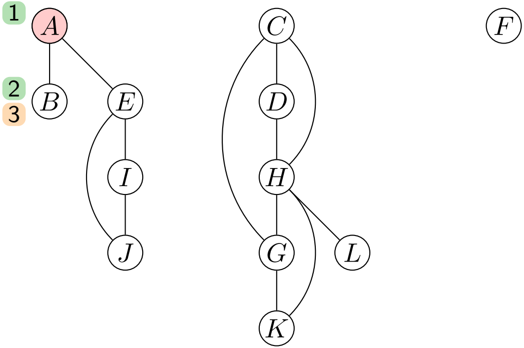

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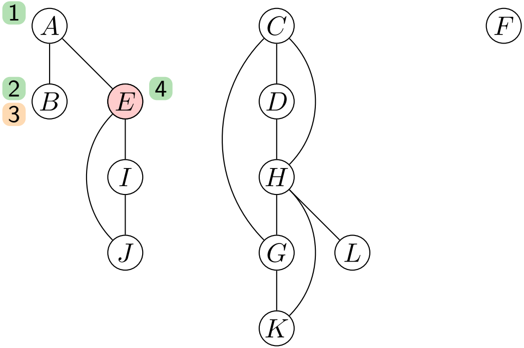

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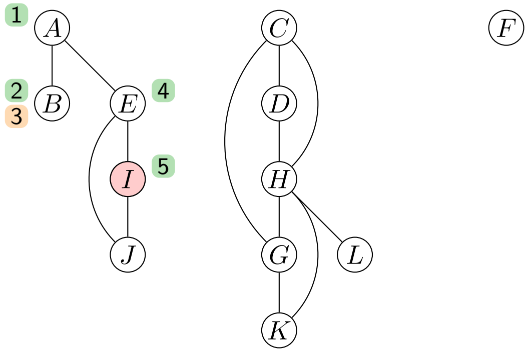

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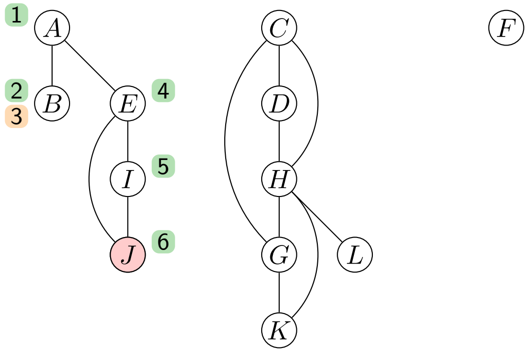

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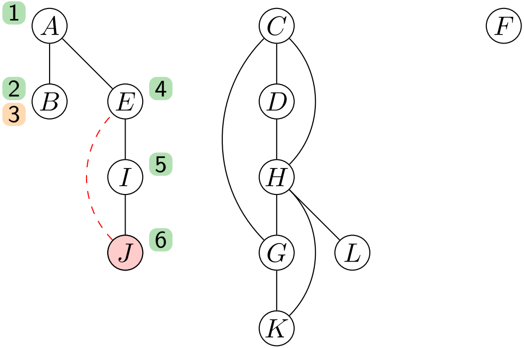

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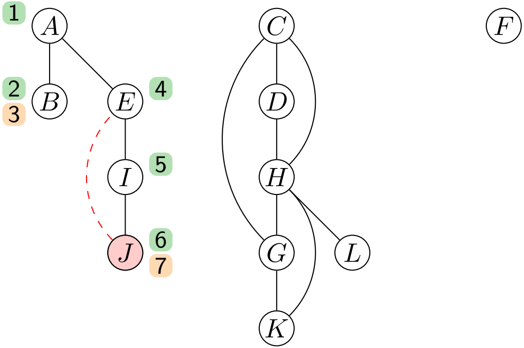

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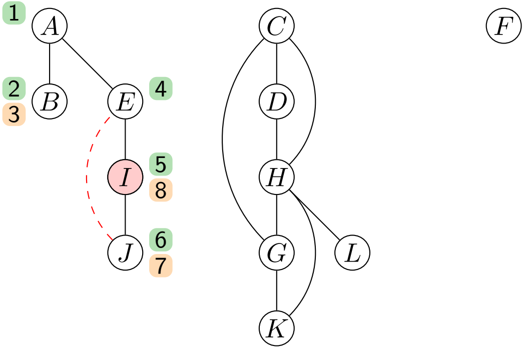

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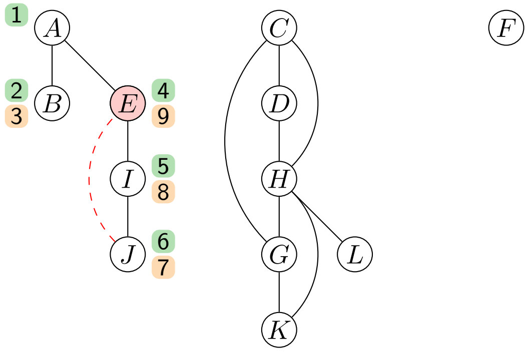

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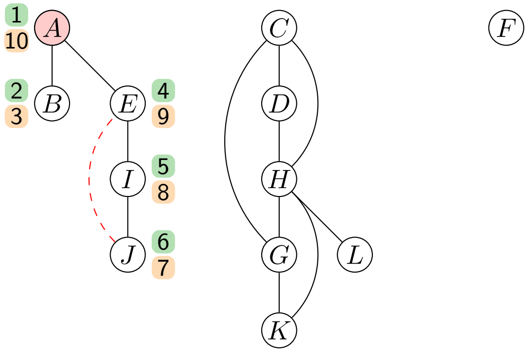

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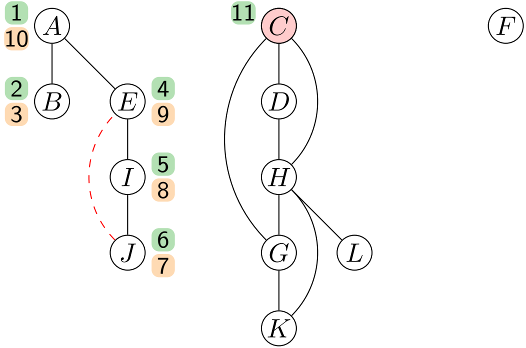

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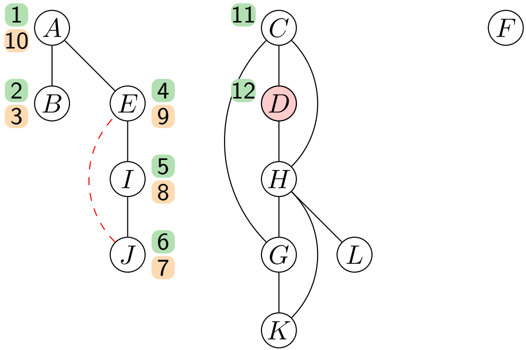

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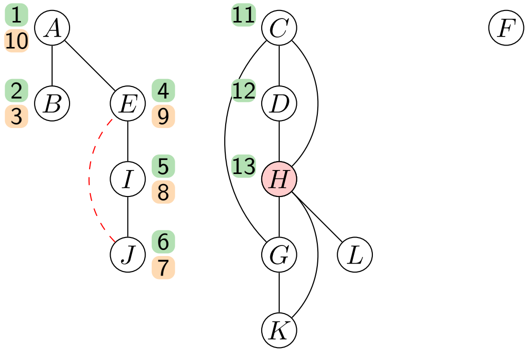

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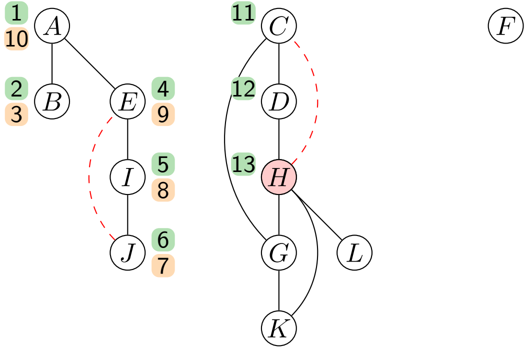

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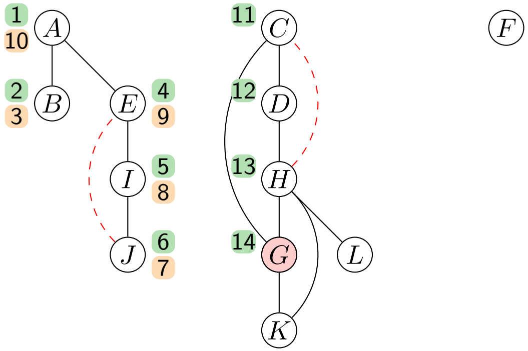

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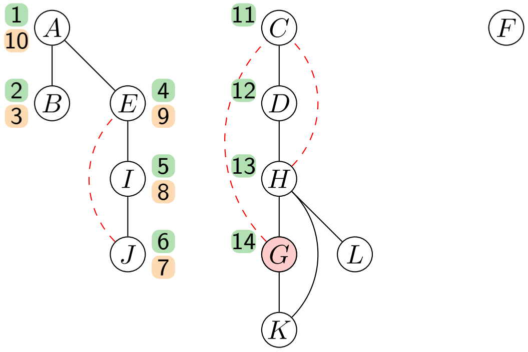

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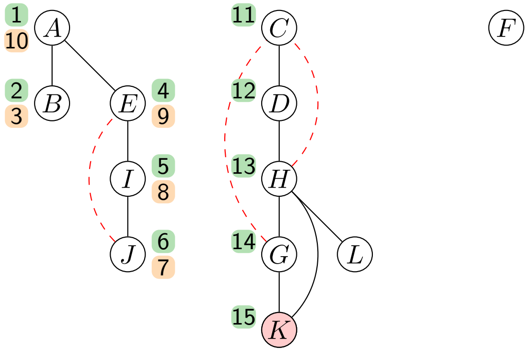

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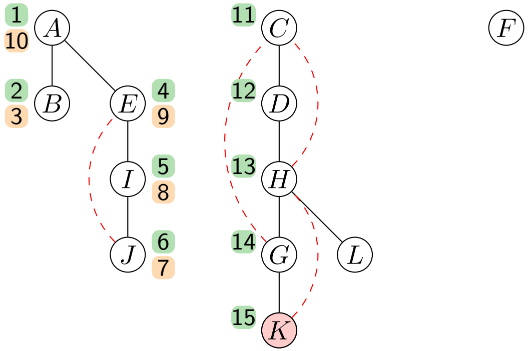

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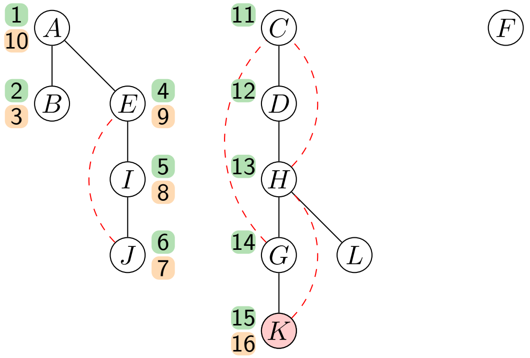

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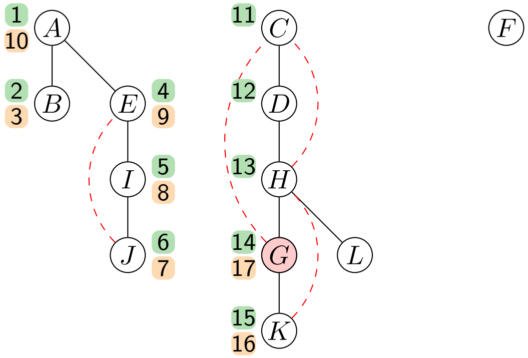

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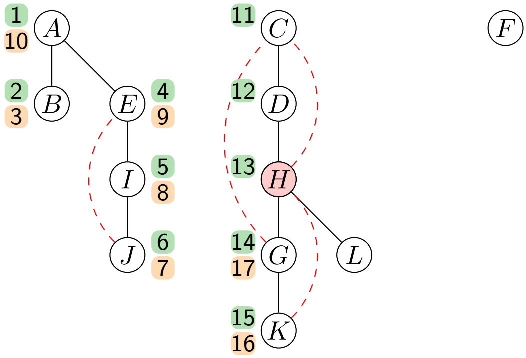

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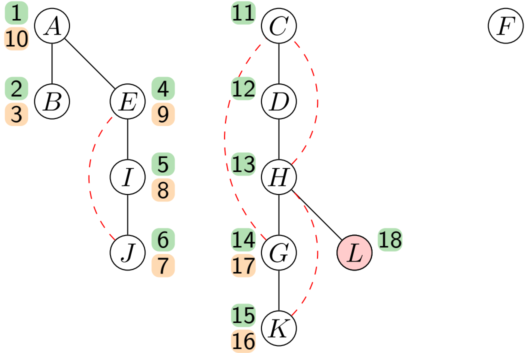

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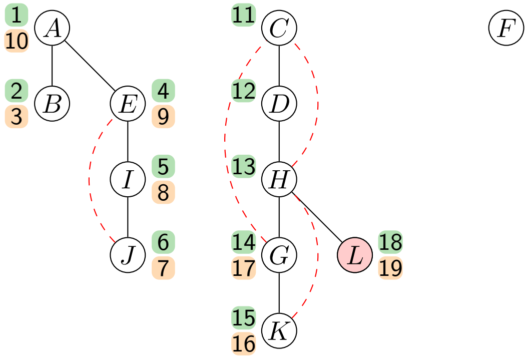

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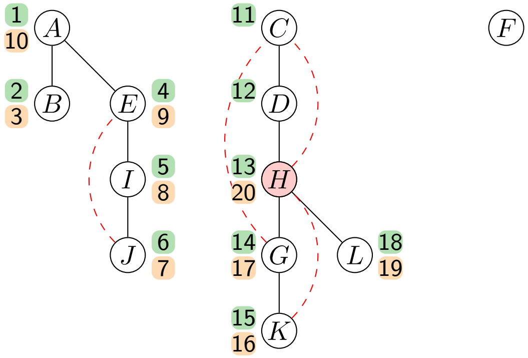

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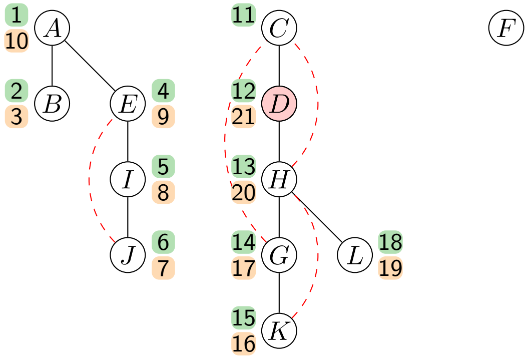

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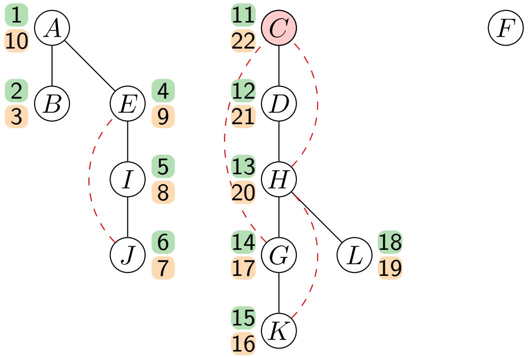

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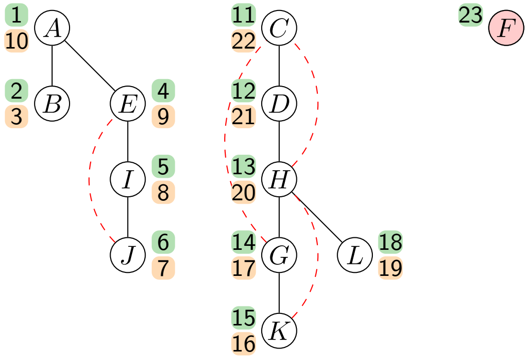

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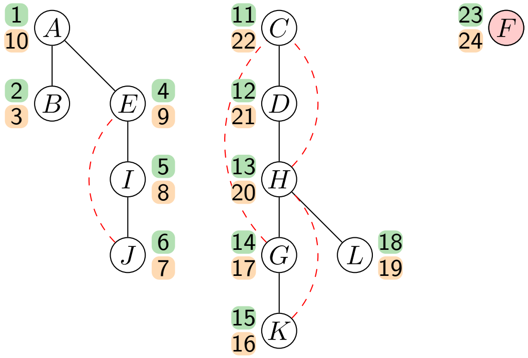

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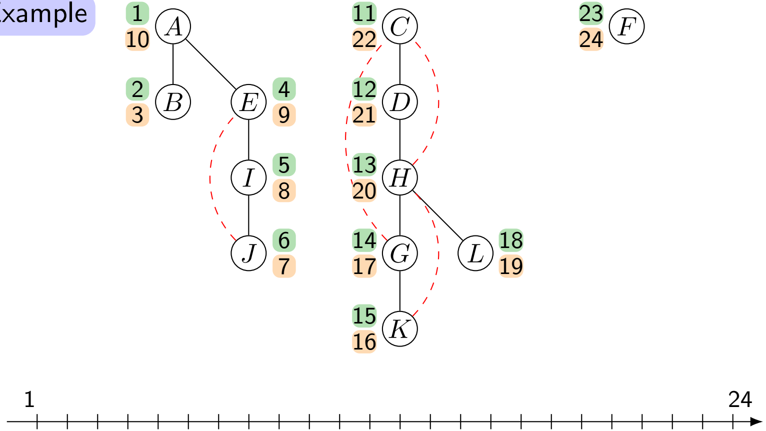

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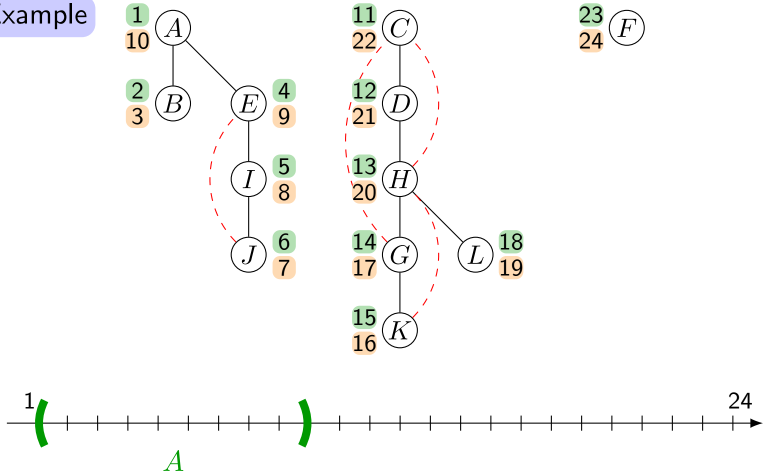

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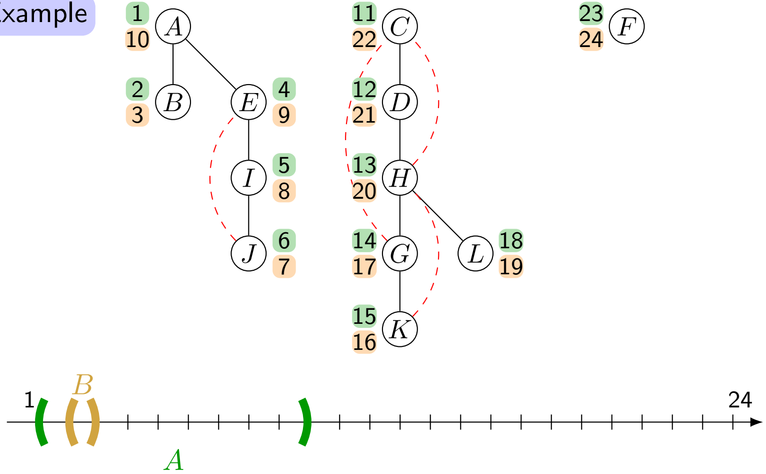

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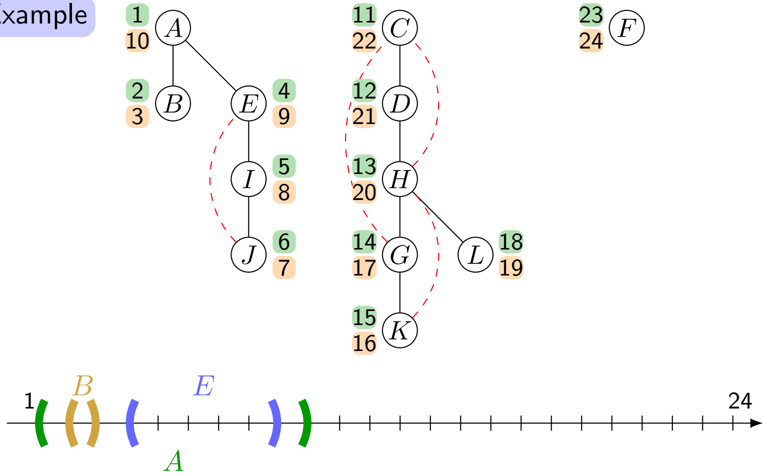

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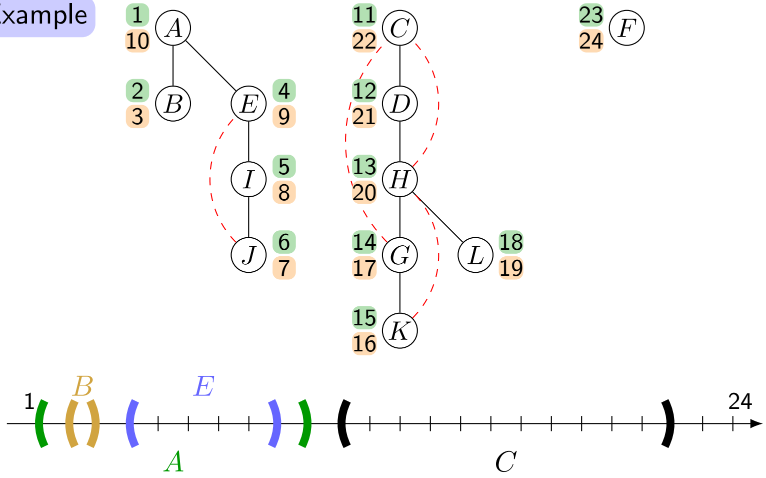

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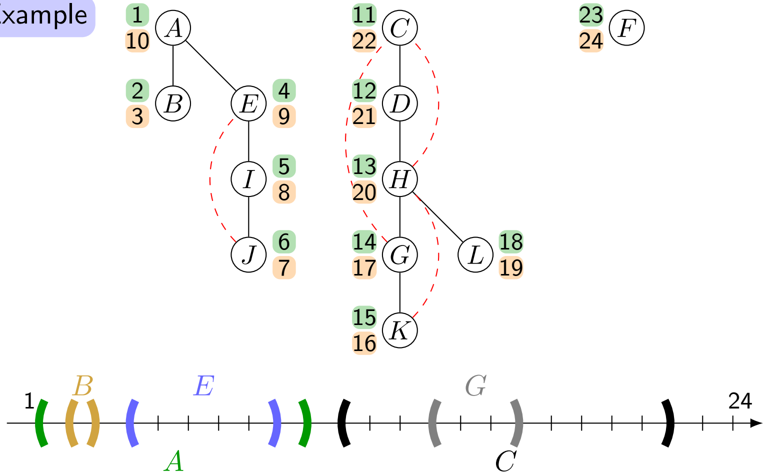

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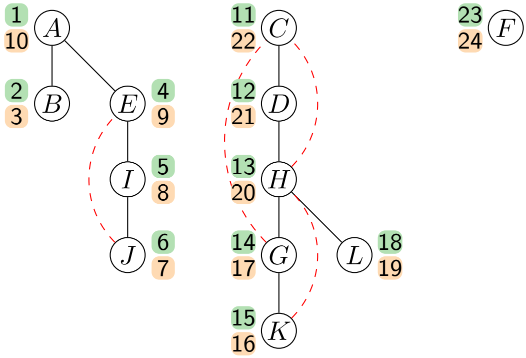
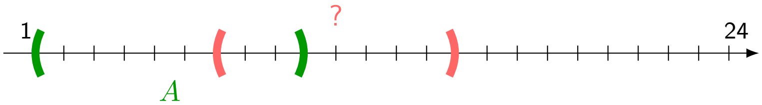
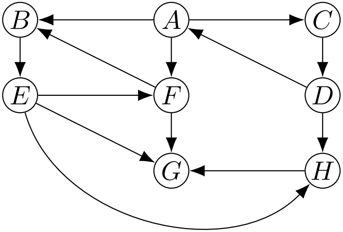
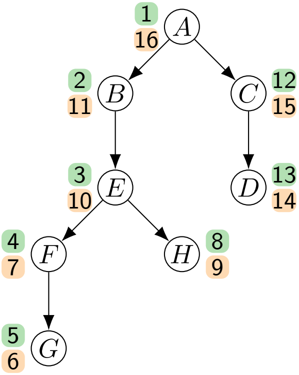




DFS in directed graphs





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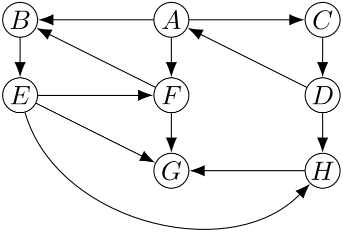

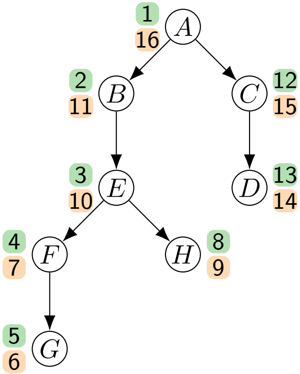

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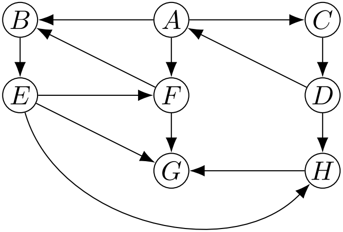

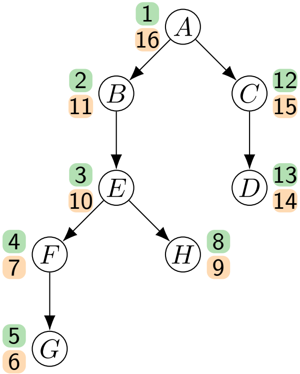

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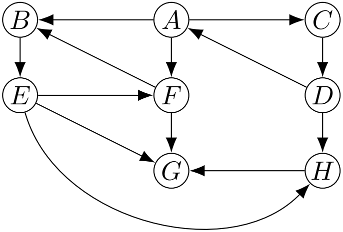

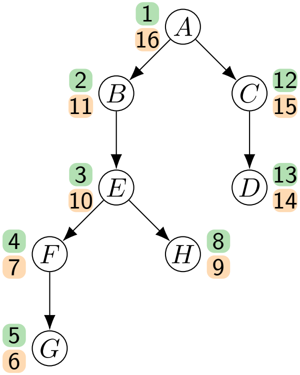

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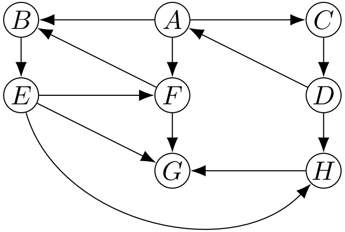

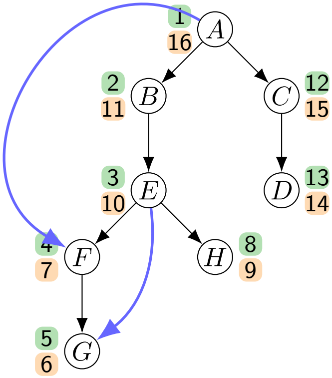

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But we have more edge types:

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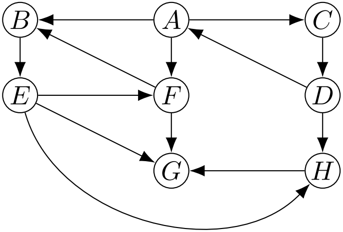

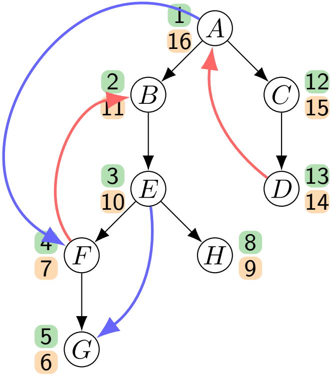

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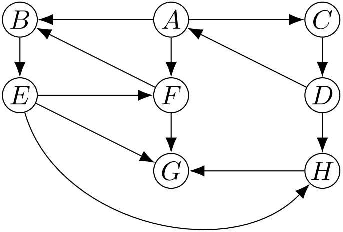
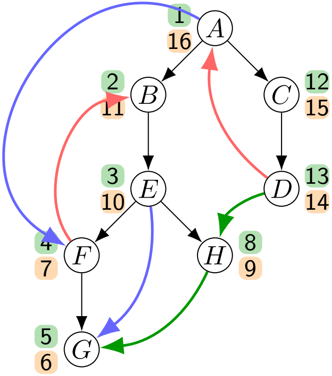
- forward edges lead to a non-child descendant.
- back edges lead to an ancestor.
- cross edges lead to neither an ancestor nor a descendant.

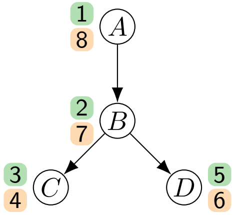




Pre-, post-visit numbers and edge types

Fact If vertex w is an ancestor of vertex v in the DFS tree, then

$$pre[w] < pre[v] < post[v] < post[w]$$



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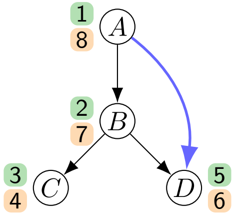

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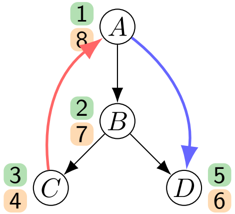

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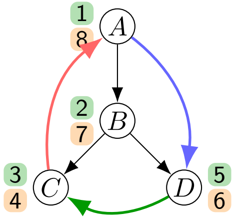

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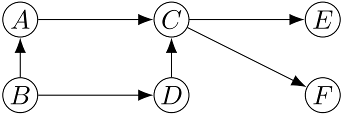

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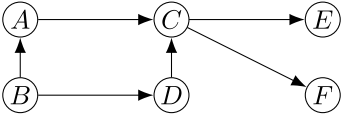

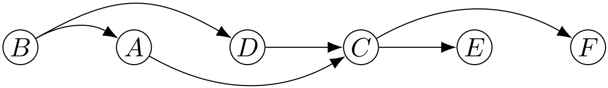

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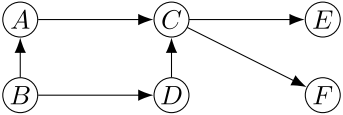

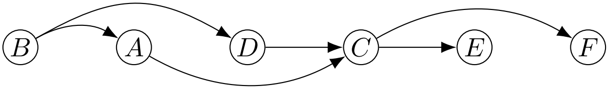

Such an ordering is called a topological order or a linearization of the DAG.

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Can we do better?

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// top_order will contain the ordered vertices
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Explore(G, s, color)
    visited[s] = color
    pre[s] = clock,    clock = clock + 1
    foreach edge {s, v} ∈ E do
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