CMPSC 465: LECTURE XV

Dijkstra's Algorithm

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October 03, 2025

Shortest path on weighted graphs

In many applications, having weights on edges is useful.

The edge weights could represent distances, cost, time, etc.

Second scenario Find shortest paths on weighted graphs.

Idea Suppose all the weights are positive integers, we can add dummy nodes to represent edge weights.

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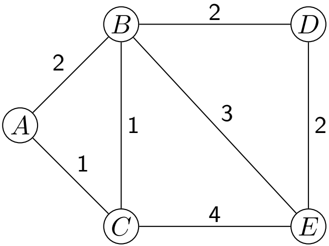
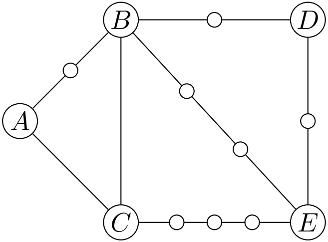
Example





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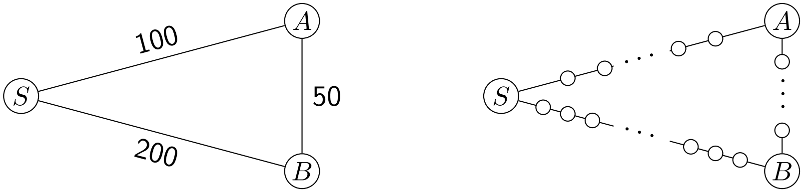

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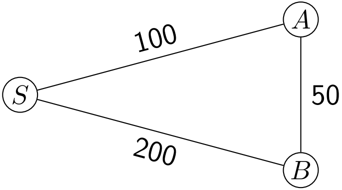

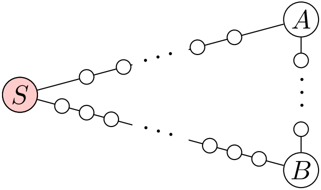

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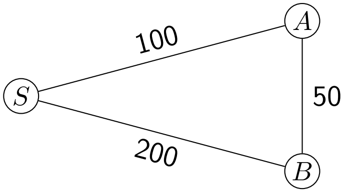

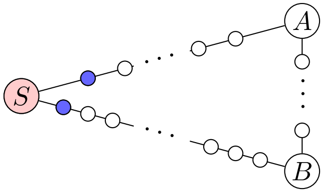

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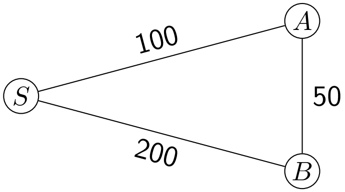

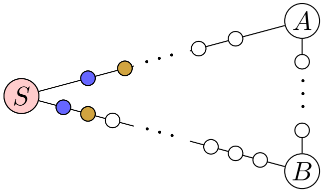

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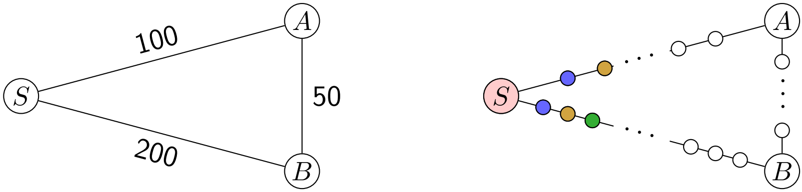

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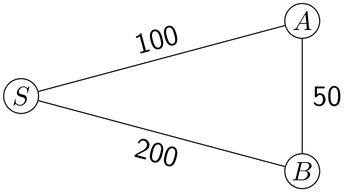

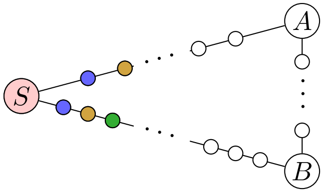

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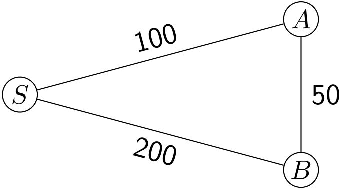

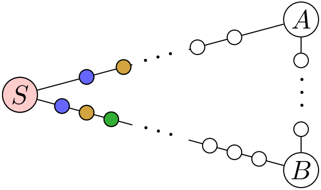

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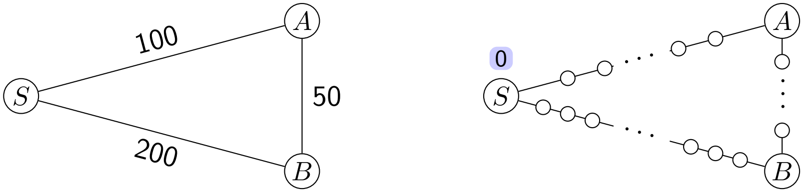

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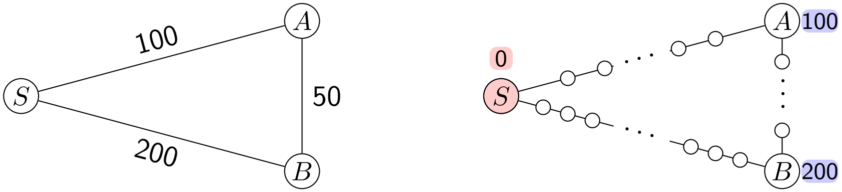

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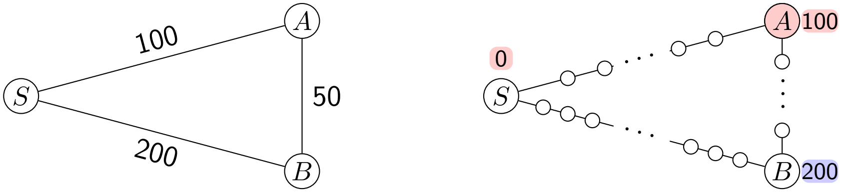

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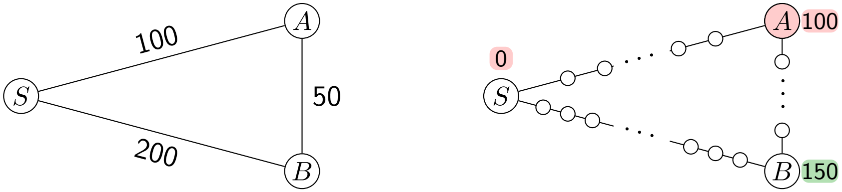

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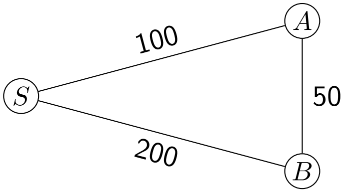

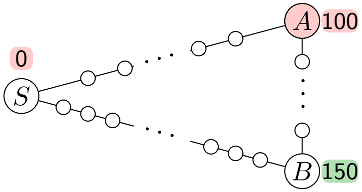

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- For the first 99 seconds, we are waiting for the dummy nodes we don't care about.
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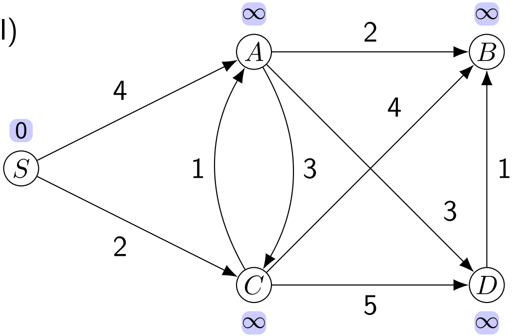
- These are the estimated upper bounds for arrival time.

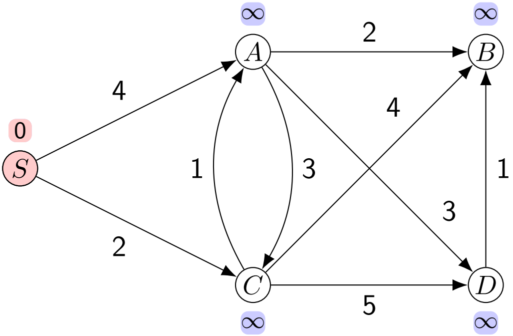
Another example

Time: (initial)



Another example

Time: 0



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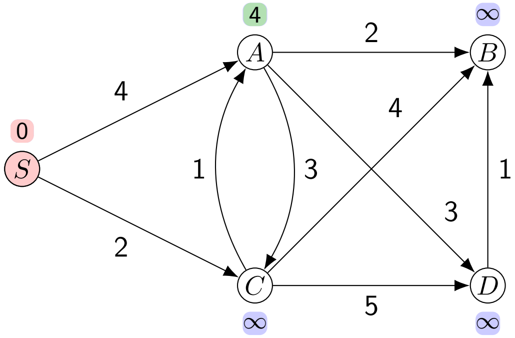

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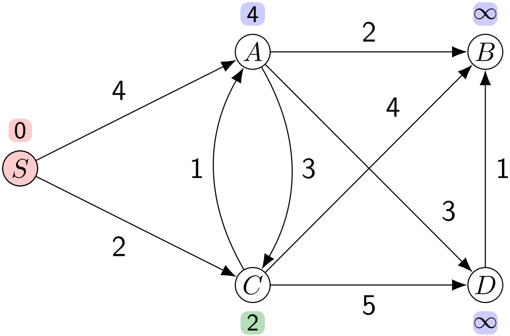

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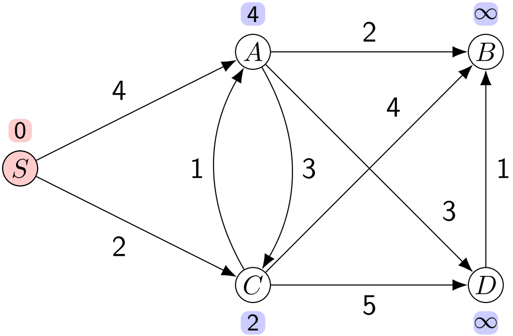
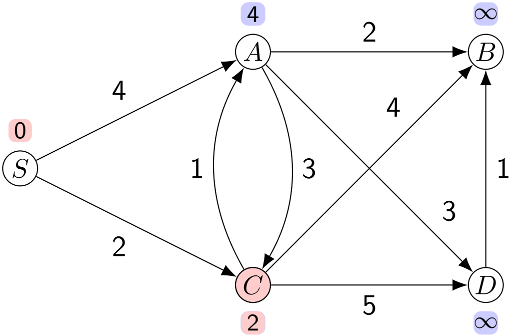


Another example

Time: 2



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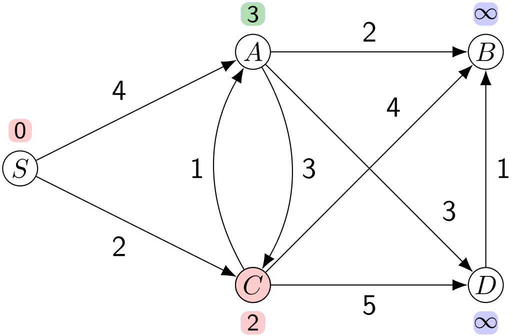

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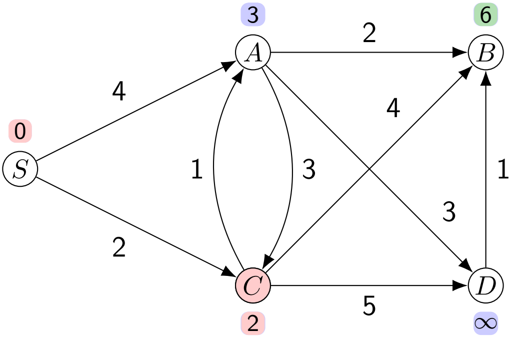

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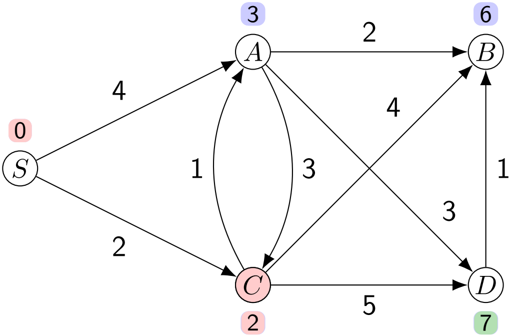

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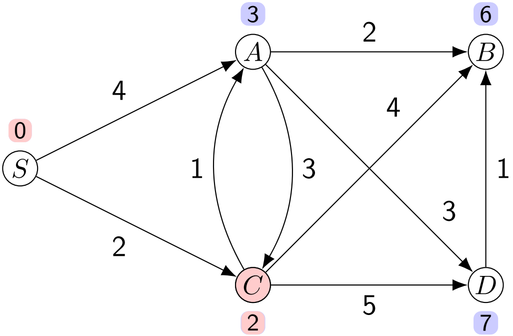

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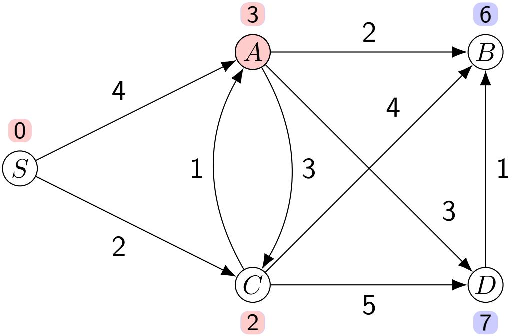

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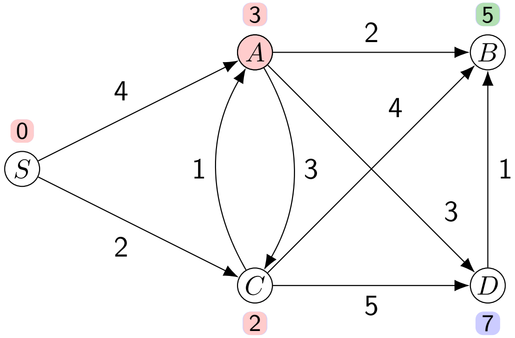

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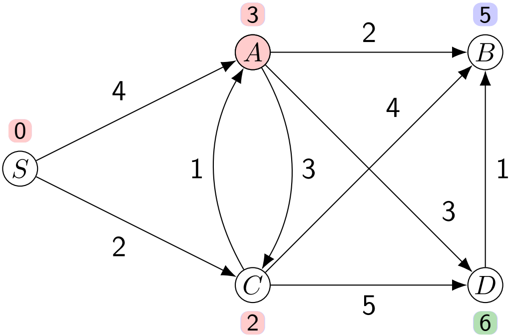

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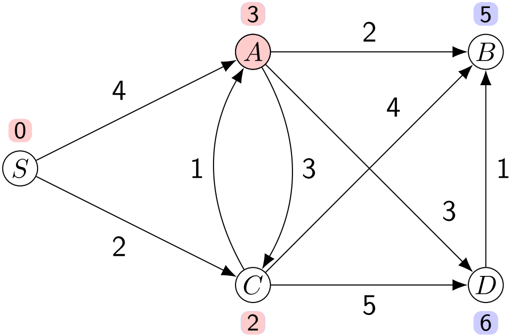

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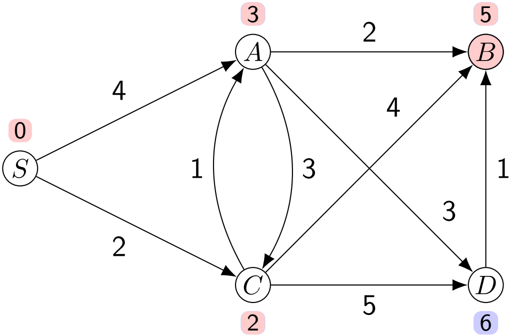

Another example

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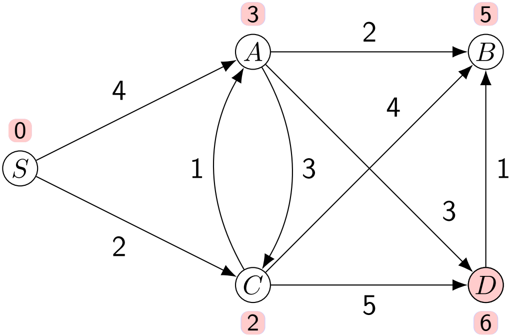

Dijkstra's algorithm

Input: A graph $G = (V, E, \ell)$ where $\ell : E \to \mathbb{N}$ maps edges to weights, a starting vertex s

Output: Shortest paths from s to any other vertex

Dijkstra(G, s)

```
// dist has length |V|, for the alarm clocks
foreach v ∈ V do
    dist[v] = ∞
dist[s] = 0

repeat |V| times do
    Find the vertex v with the smallest time among those
      whose alarms have not rung yet

    foreach (v, w) ∈ E do
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$O(|V|^2)$ with naive implementation, we'll see how to improve later.

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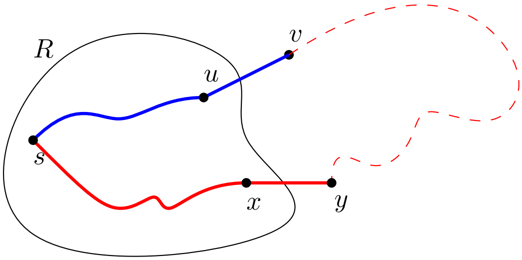

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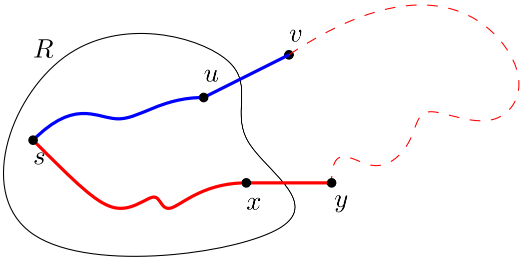

$len(\text{red path}) \geq dist[x] + \ell(x, y) \geq dist[y] \geq dist[v] = len(\text{blue path})$.

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- Each vertex also needs to know its index in the heap.