CMPSC 360 Discrete Mathematics for Computer Science
Fall 2024 Mahfuza Farooque **Worksheet 9 Solutions**

1. **Induction:** Using induction, prove that $\forall n \in \mathbb{N}$, $x^{2n} - y^{2n}$ is divisible by $x+y$.

   Answer:
   $P(n) = x^{2n} - y^{2n}$ is divisible by $x+y$
   Base case: P(1) should be true.
   $x^2 - y^2 = (x+y)(x-y)$
   This is clearly divisible by $x+y$.
   Hence, $P(1)$ is true.

   We assume $P(k)$ is true.
   $P(k) = x^{2k} - y^{2k}$ is divisible by $x+y$.
   Let $x^{2k} - y^{2k} = t(x+y)$ for some $t \in \mathbb{Z}$.
   We have to show that $P(k+1)$ is true when $P(k)$ is true.
   $P(k+1) = x^{2(k+1)} - y^{2(k+1)}$ is divisible by $x+y$.

   Consider from $P(k+1)$
   $x^{2(k+1)} - y^{2(k+1)} = x^{2k+2} - y^{2k+2}$
   $= x^{2k}x^2 - y^{2k}y^2$
   $= x^{2k}x^2 - x^{2k}y^2 + x^{2k}y^2 - y^{2k}y^2$ (Adding and subtracting $x^{2k}y^2$)
   $= x^{2k}(x^2 - y^2) + y^2(x^{2k} - y^{2k})$
   $= x^{2k}(x+y)(x-y) + y^2t(x+y)$ (Using the assumption $P(k)$)
   $= (x+y)[x^{2k}(x-y) + y^2t]$
   This is clearly divisible by $x+y$.
   We showed that $P(k+1)$ is true when $P(k)$ is true.
   Hence, the statement $P(n)$ is proved by induction.

2. **Strong Induction:** Use strong induction to prove the following proposition:

   $$\text{If } n \in \mathbb{N}, \text{ then } 12 | (n^4 - n^2).$$

   Answer:
   Strong induction involves assuming each of statements $P_1, P_2, \ldots, P_k$ is true, and showing that this forces P_{k+1} to be true. In particular, if P_1 through P_k are true, then certainly P_{k-5} is true, provided that $1 \leq k-5 < k$. The idea is then to show $P_{k-5} \Rightarrow P_{k+1}$ instead of $P_k \Rightarrow P_{k+1}$. For this to make sense, our basis step must involve checking that $P_1, P_2, \ldots, P_6$ are all true. Once this is established, $P_{k-5} \Rightarrow P_{k+1}$ will imply that the other P_k are all true. For example, if k = 6, then $P_{k-5} \Rightarrow P_{k+1}$ is $P_1 \Rightarrow P_7$, so P_7 is true; for k = 7, then $P_{k-5} \Rightarrow P_{k+1}$ is $P_2 \Rightarrow P_8$, so P_8 is true.
   Proceed by strong induction on n. First, note that the statement is true for the first six positive integers:
   For $n = 1$, 12 divides $1^4 - 1^2 = 0$.

For $n = 2$, 12 divides $2^4 - 2^2 = 12$.
For $n = 3$, 12 divides $3^4 - 3^2 = 72$.
For $n = 4$, 12 divides $4^4 - 4^2 = 240$.
For $n = 5$, 12 divides $5^4 - 5^2 = 600$.
For $n = 6$, 12 divides $6^4 - 6^2 = 1260$.
Next, for $k \geq 6$, assume $12|(n^4 - n^2)$ for all n such that $1 \leq n \leq k$.

We must show that $P(k+1)$ is true, that is, $12|((k+1)^4 - (k+1)^2)$. Now $P(k-5)$ being true means $12|(k-5)^4 - (k-5)^2$. To simplify, define $l := k-5$, so $12|l^4 - l^2$, meaning $l^4 - l^2 = 12a$, for some $a \in \mathbb{Z}$, and $k+1 = l+6$. Then:

$(k+1)^4 - (k+1)^2 = (l+6)^4 - (l+6)^2$
$= l^4 + 24l^3 + 216l^2 + 864l + 1296 - (l^2 + 12l + 36)$
$= (l^4 - l^2) + 24l^3 + 216l^2 + 852l + 1260$
$= 12a + 24l^3 + 216l^2 + 852l + 1260$
$= 12(a + 2l^3 + 18l^2 + 71l + 105)$

Because $a + 2l^3 + 18l^2 + 71l + 105 \in \mathbb{Z}$, we get $12|(k+1)^4 - (k+1)^2$.

3. **Strong Induction:** Using induction, prove that a rectangular chocolate bar composed of n pieces can be split into individual pieces using at most $n-1$ breaks for all $n \in \mathbb{N}$.

Answer:
Suppose that the given theorem is $P(n)$: A chocolate bar composed of n pieces requires at most $n-1$ breaks to be split into individual pieces.

Base Case($n = 1$):
A chocolate bar composed of 1 piece cannot be split any further and requires $1 - 1 = 0$ breaks. Thus, $P(1)$ is true.

Inductive Hypothesis:
Suppose that $P(1) \wedge P(2) \wedge \ldots \wedge P(k)$ for some $k \geq 1$.

Inductive Step:
Consider a chocolate bar with $k+1$ pieces. Now consider a subdivision such that the chocolate bar is broken into two bars of sizes p and q such that $p + q = k + 1$.

According to the inductive hypothesis, these smaller bars require $p-1$ and $q-1$ breaks. Thus, the total number of breaks is given as:
No. of breaks = $(p-1) + (q-1) + 1$ (The extra 1 is because of the initial break to divide the bar into smaller bars of sizes p and q)
No. of breaks = $p + q - 1 = k + 1 - 1 = k$
Thus, a bar of size $k+1$ requires at most k breaks, which completes the inductive step. Therefore $\forall n \in \mathbb{N},\ P(n)$ is true using strong induction.

4. Number Theory: Solve the following:

(a) $18^{10} \bmod 39$

(b) $8^{176} \bmod 11$

Answer:

(a) $18^{10} \bmod 39 = 2^{10} \bmod 39 \times 9^{10} \bmod 39 = 12$

(b) $8^{176} \bmod 11 = 9^{88} \bmod 11 = 4^{44} \bmod 11 = 3^{11} \bmod 11 = 3$

5. Number Theory: Solve the following:

(a) Convert $(354)_{10}$ to base 8.

(b) Convert $(542)_8$ to base 10.

Answer:

(a) $354 = 44 * 8 + 2, \quad 44 = 5 * 8 + 4, \quad 5 = 0 * 8 + 5$
$\therefore (354)_{10} = (542)_8$

(b) $(542)_8 = (5 \times 8^2) + (4 \times 8^1) + (2 \times 8^0) = (354)_{10}$