CMPSC 465: LECTURE XVII

Shortest Path with Negative Weights

Ke Chen

October 08, 2025

Negative edge weights

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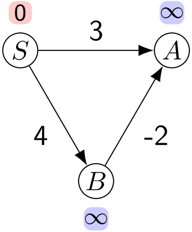

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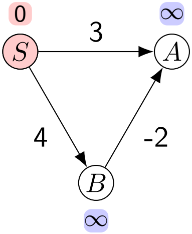

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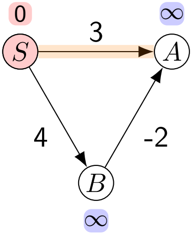

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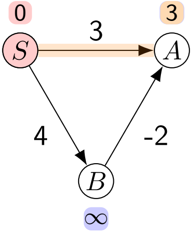

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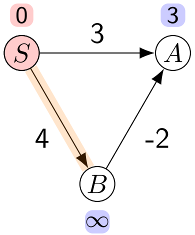

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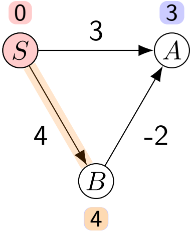

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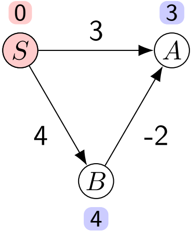

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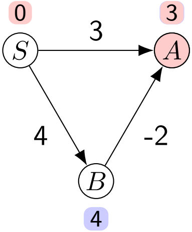

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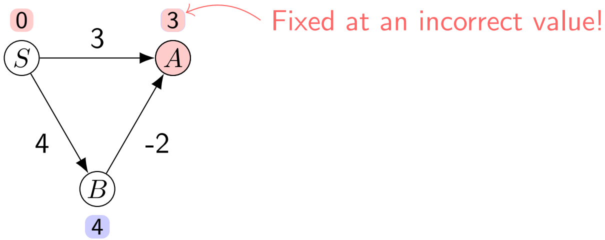

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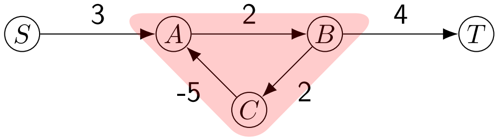
- What happens if there are negative edges?
- Even worse, if there are negative cycles, the "shortest distance" is not well-defined.



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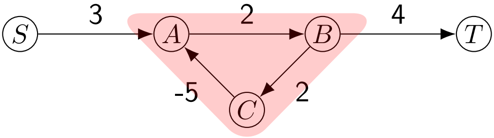

- Note that in directed graphs, negative edges $\neq$ negative cycles; however, in undirected graphs, a negative edge $=$ a negative cycle. (Why?)

Update is fine with negative edges

Recall the Update operation:

Update($(v, w) \in E$)

if $dist[w] > dist[v] + \ell(v, w)$ **then**

$\quad dist[w] = dist[v] + \ell(v, w)$

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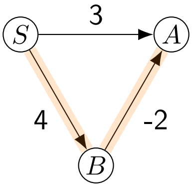

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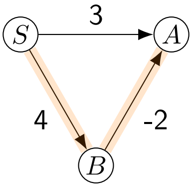

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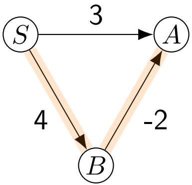

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- Dijkstra applies a smart sequence of only $O(|E|)$ Update calls that's guaranteed to include the required sequence for each node if no negative weights.

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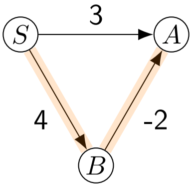

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- Having additional Update calls doesn't hurt.
- Dijkstra applies a smart sequence of only $O(|E|)$ Update calls that's guaranteed to include the required sequence for each node if no negative weights. It may fail with negative edges.

One sequence to rule them all

Can we find a sequence of Update calls that always work?

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Yes! The following sequence contains ALL possible sequences of Update calls of length at most $|V| - 1$:

$$\left.\begin{array}{l}
\mathrm{edge}_1 \longrightarrow \mathrm{edge}_2 \longrightarrow \mathrm{edge}_3 \longrightarrow \cdots \longrightarrow \mathrm{edge}_{|E|} \\
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\end{array}\right\} |V| - 1 \text{ times}$$

One sequence to rule them all

Idea Call Update on each edge, and repeat $|V| - 1$ times.

- Why $|V| - 1$?

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- The order of edges doesn't matter.

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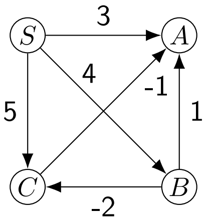
Example



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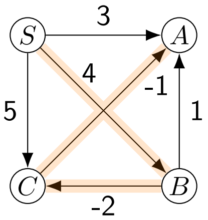
One sequence to rule them all

Idea Call Update on each edge, and repeat $|V|-1$ times.

- Why $|V|-1$? A shortest path contains at most $|V|-1$ edges.
- The order of edges doesn't matter.

Example



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- Why $|V| - 1$? A shortest path contains at most $|V| - 1$ edges.
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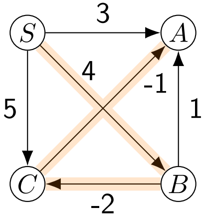

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Bellman-Ford algorithm

Input: Graph $G = (V, E, \ell)$, starting vertex s
Output: Shortest path from s to any other vertex

```
Bellman-Ford(G, s)
    // dist stores distances from s
    foreach v ∈ V do
        dist[v] = ∞
    dist[s] = 0

    repeat |V| − 1 times do
        foreach e ∈ E do
            Update(e)
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Time complexity?

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Can we do better?

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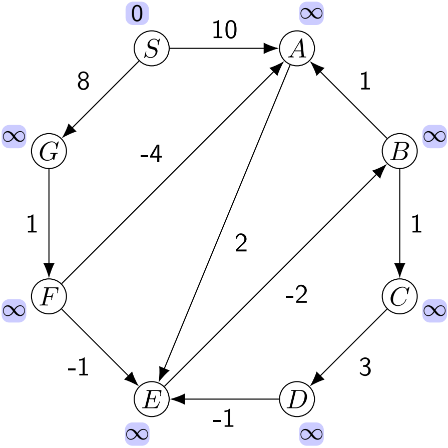
Round:



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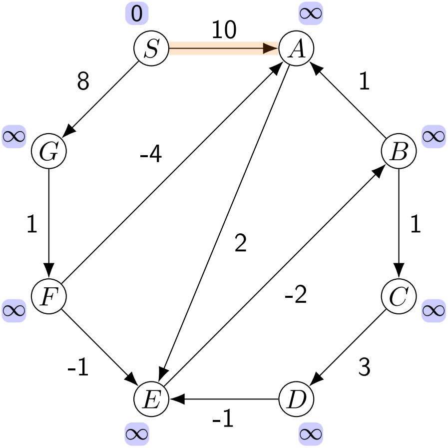

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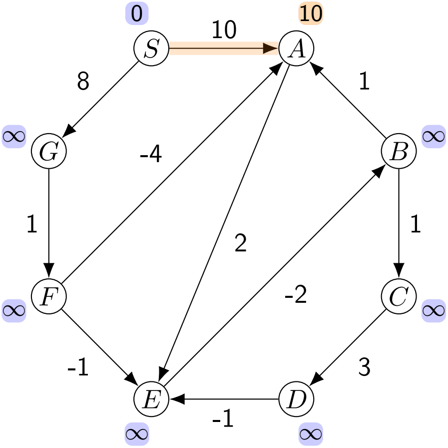

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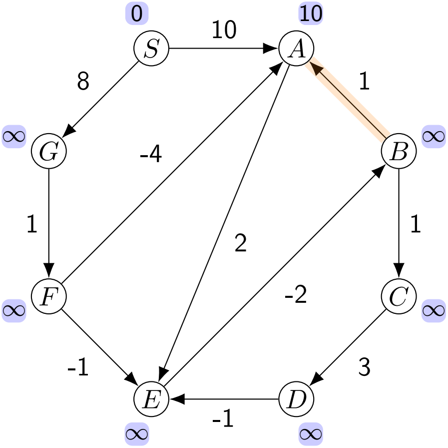

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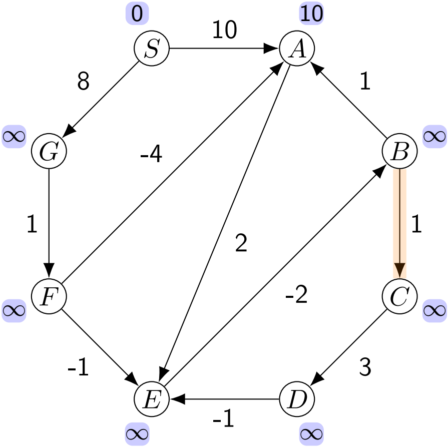

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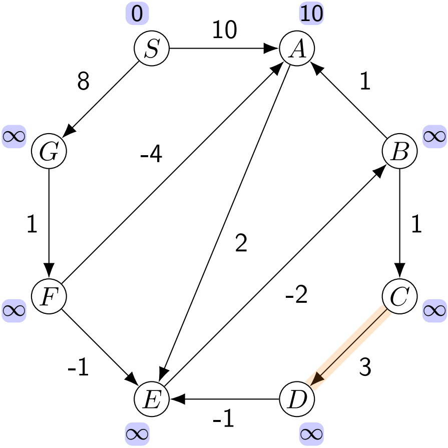

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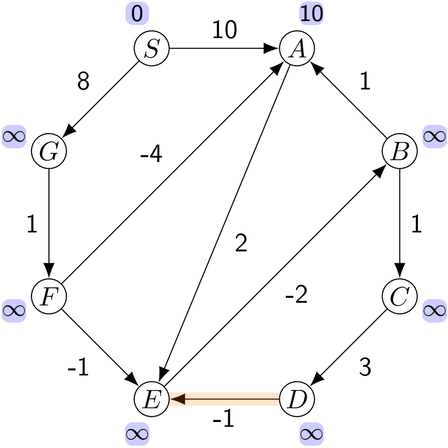

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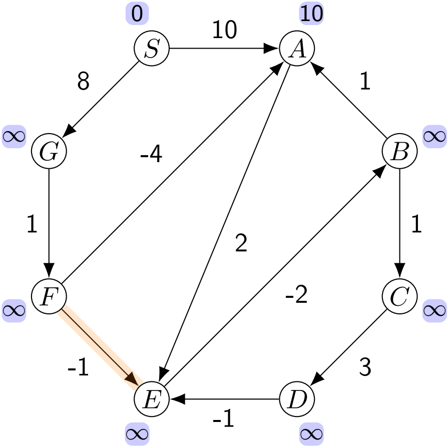

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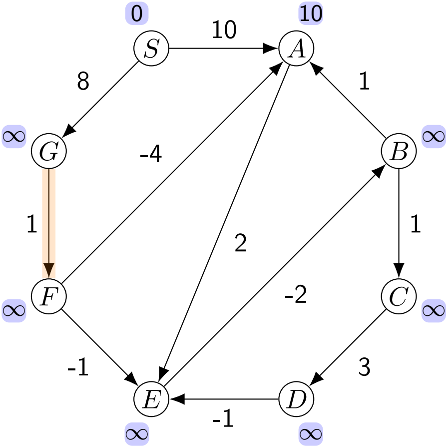

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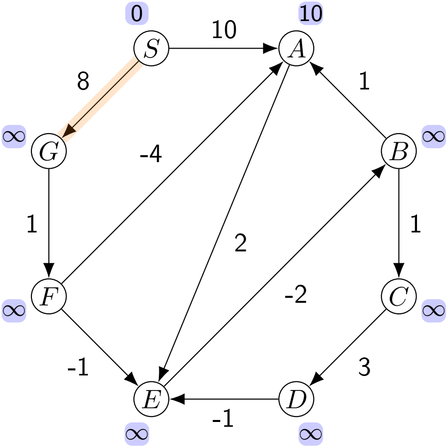

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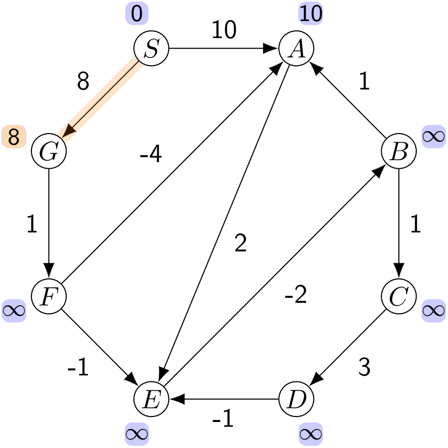

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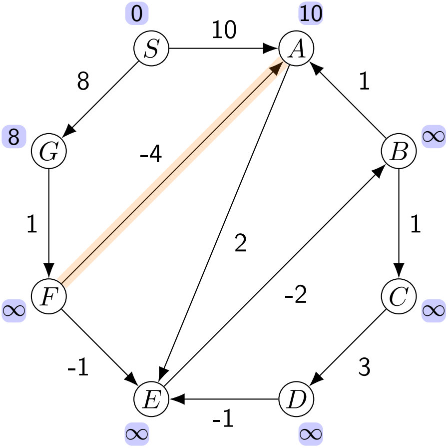

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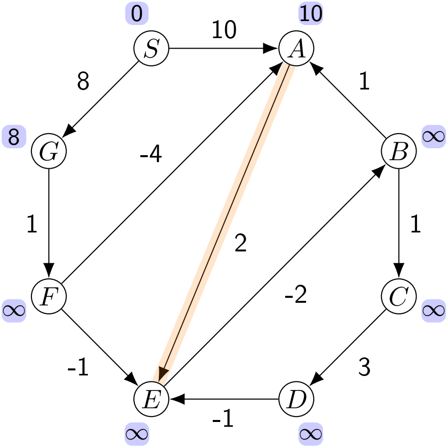

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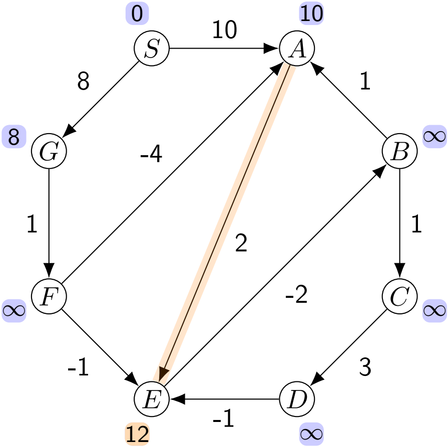

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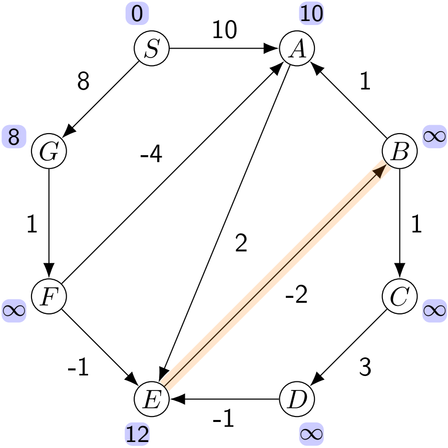

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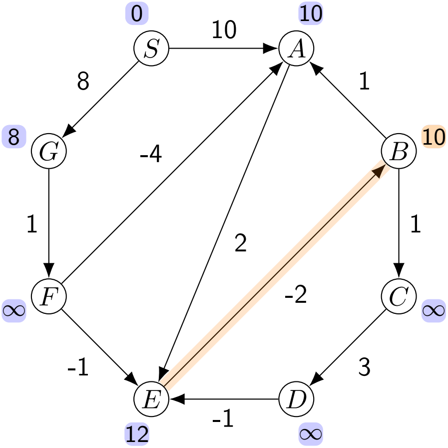

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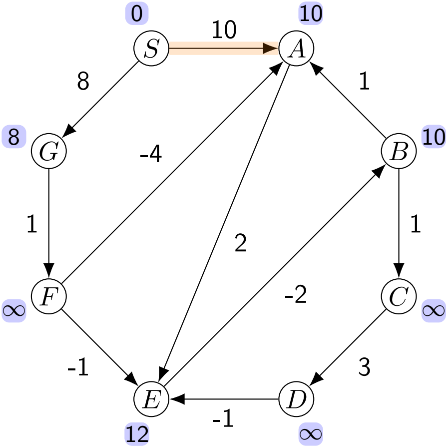
Round: 2



Bellman-Ford algorithm

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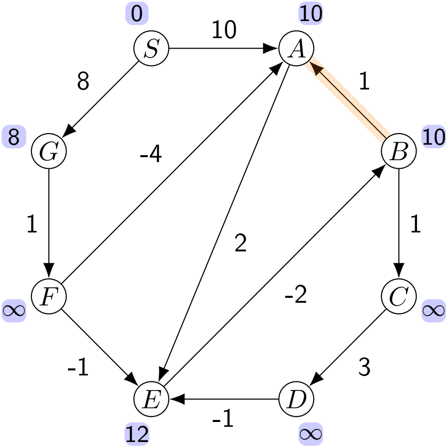
Assume we Update edges in the order (S, A), (B, A), (B, C), (C, D), (D, E), (F, E), (G, F), (S, G), (F, A), (A, E), (E, B)

Round: 2



Bellman-Ford algorithm

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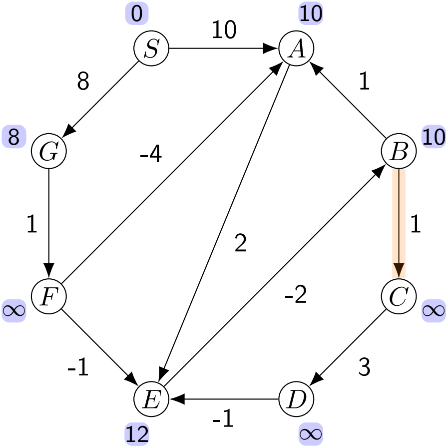
Round: 2



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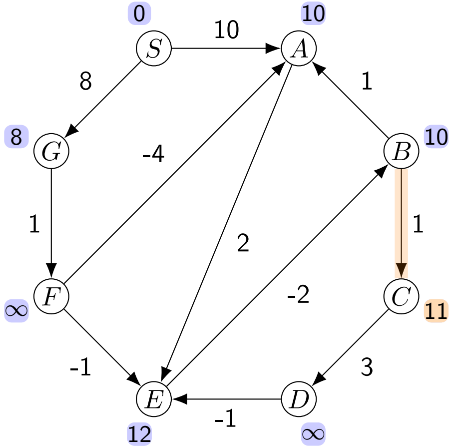
Round: 2



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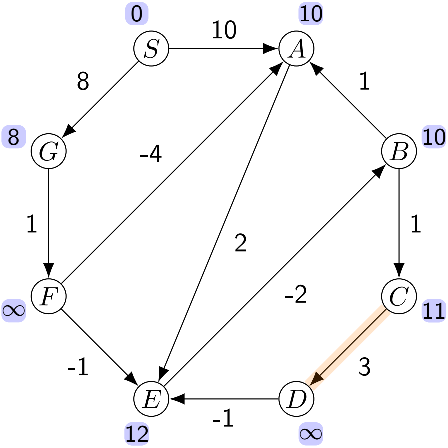
Round: 2



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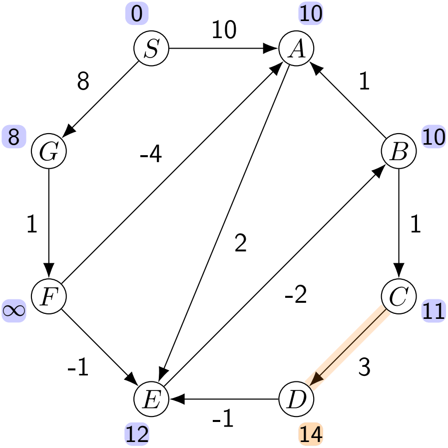

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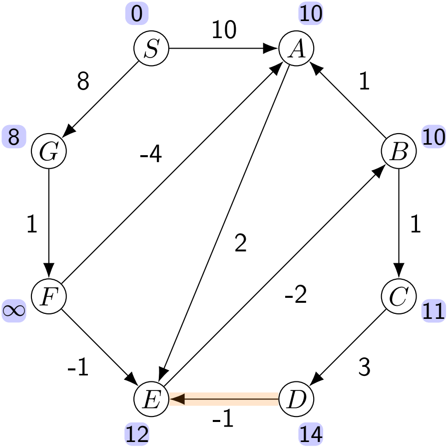

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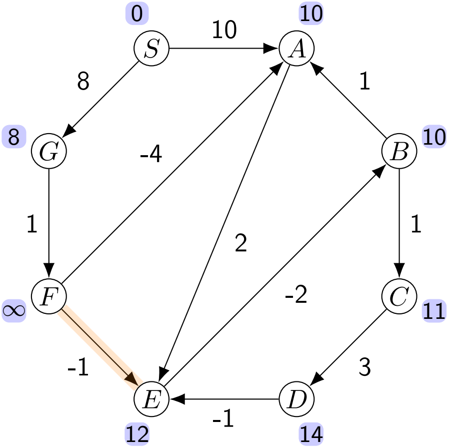

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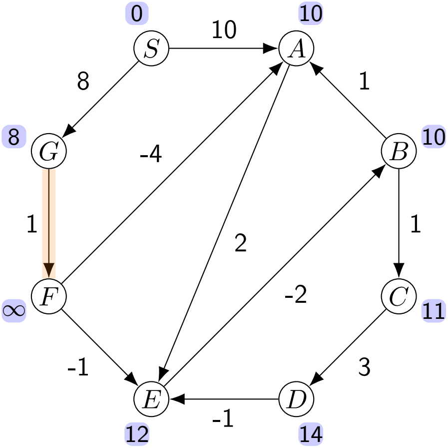

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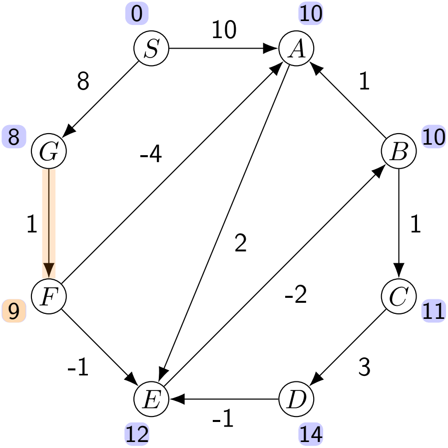

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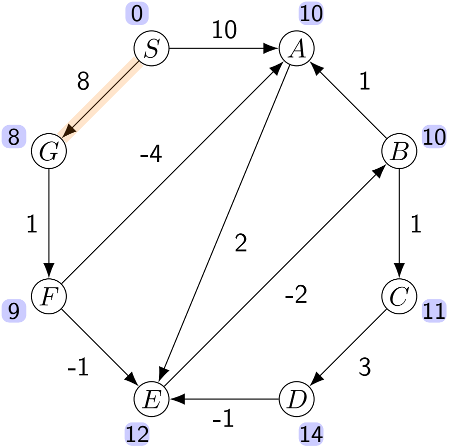

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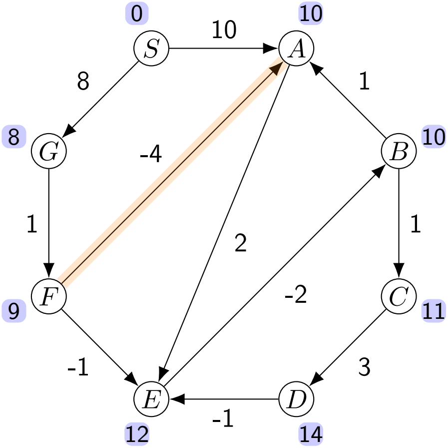

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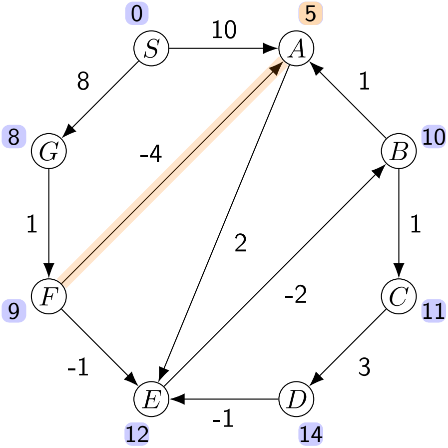

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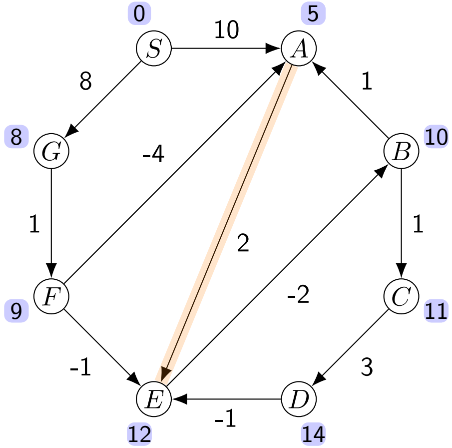

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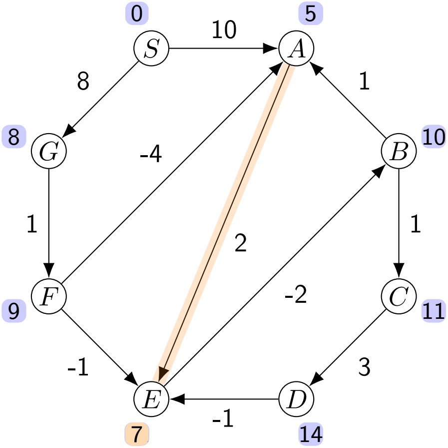

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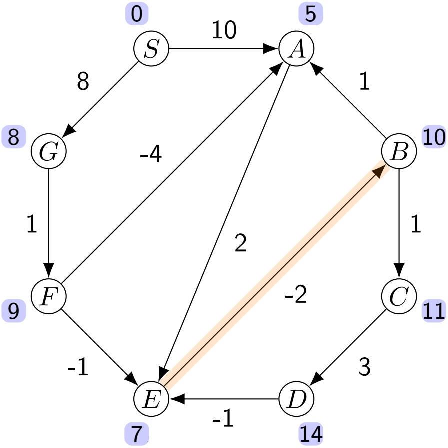

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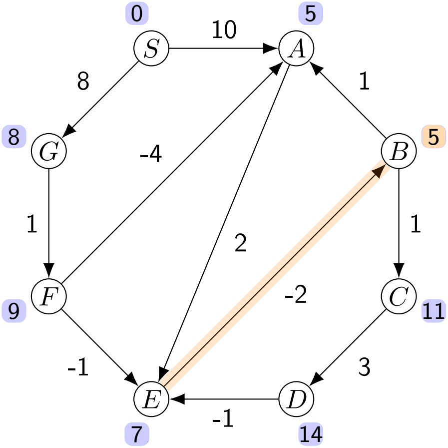
Round: 2



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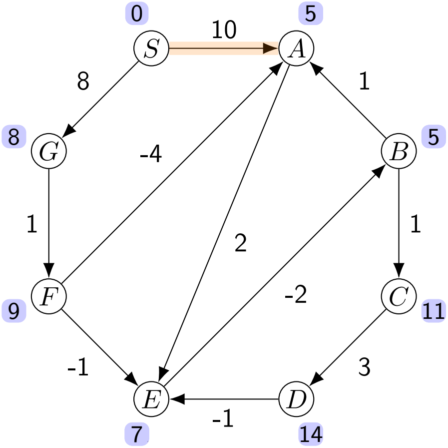
Consider the following example:

Assume we Update edges in the order (S, A), (B, A), (B, C), (C, D), (D, E), (F, E), (G, F), (S, G), (F, A), (A, E), (E, B)

Round: 3



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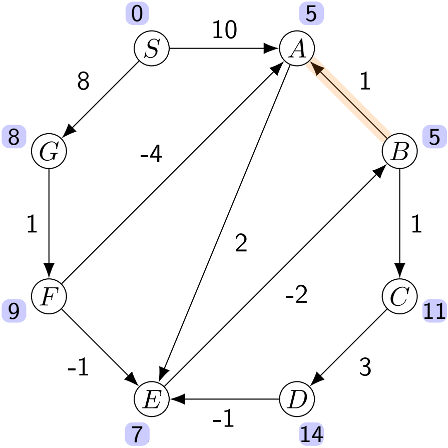
Round: 3



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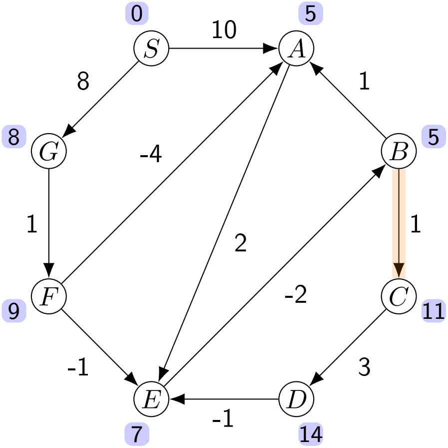

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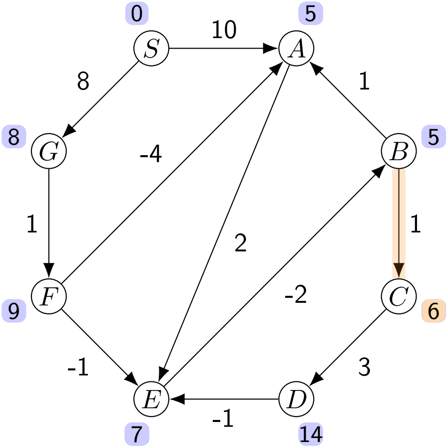

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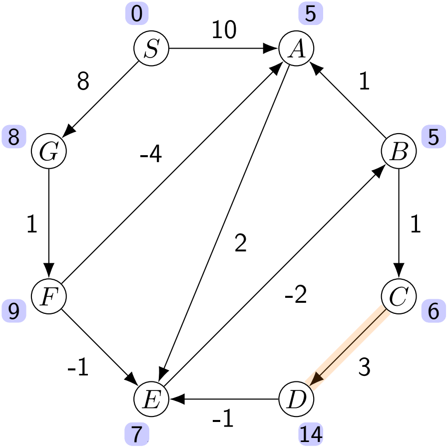

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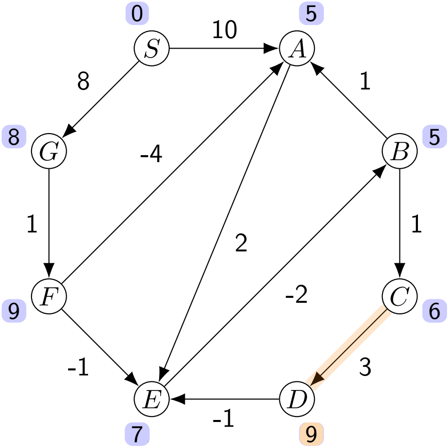

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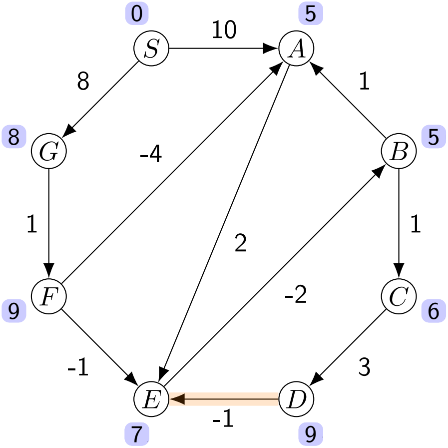

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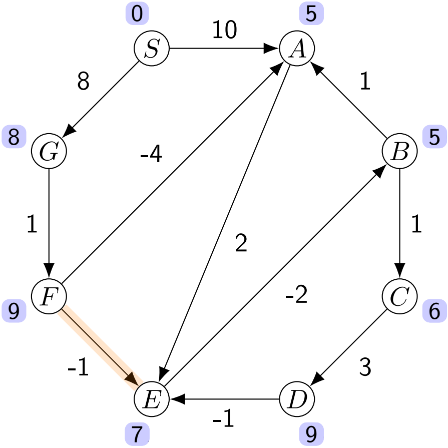

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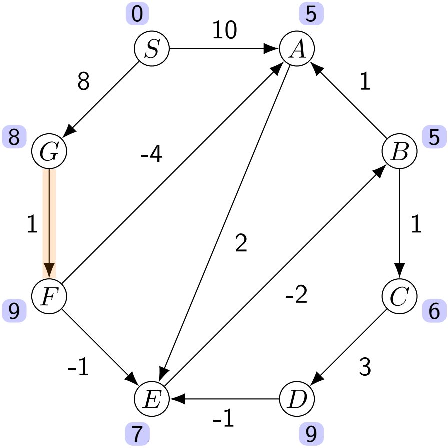

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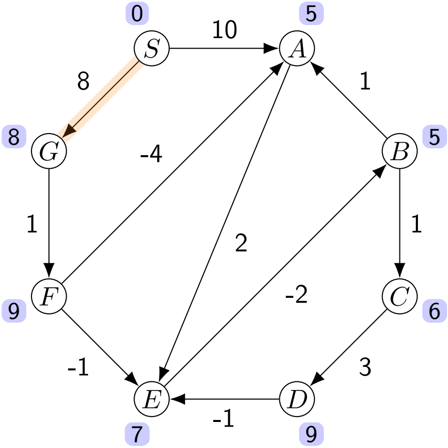

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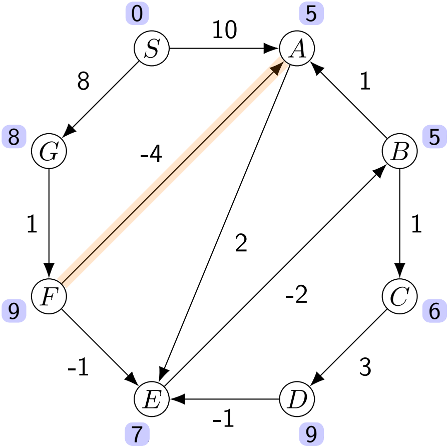

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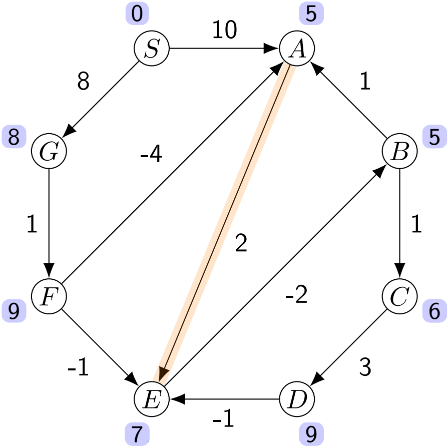

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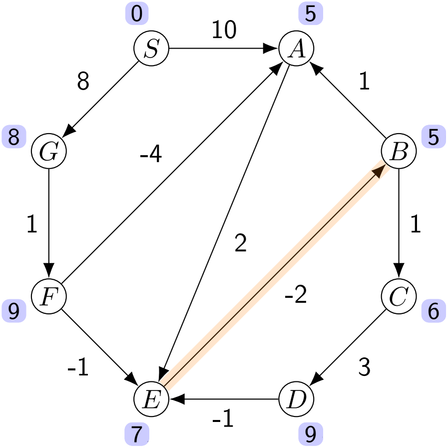

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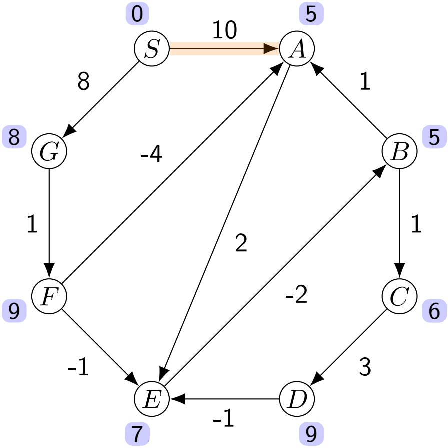
Round: 4



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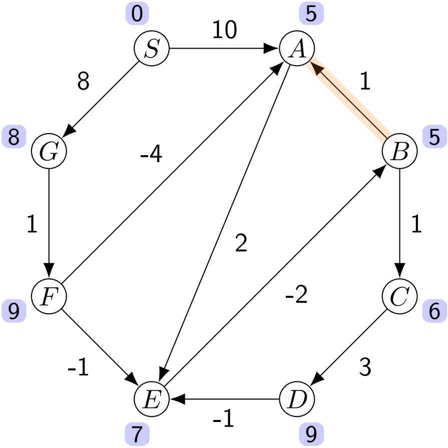

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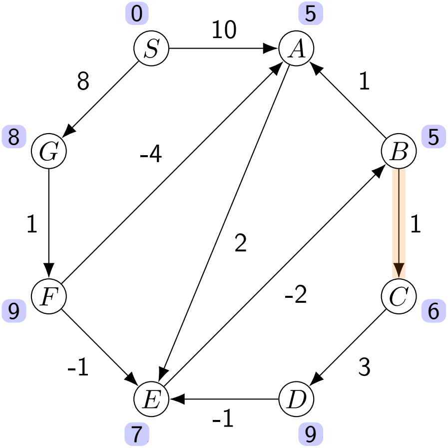

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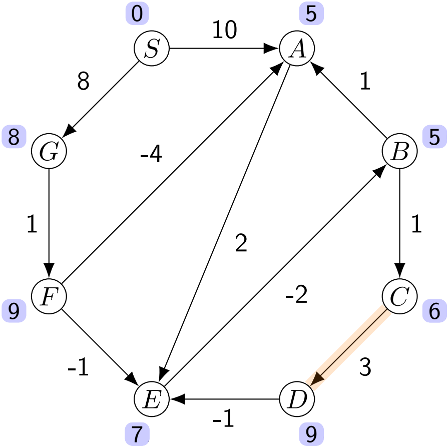

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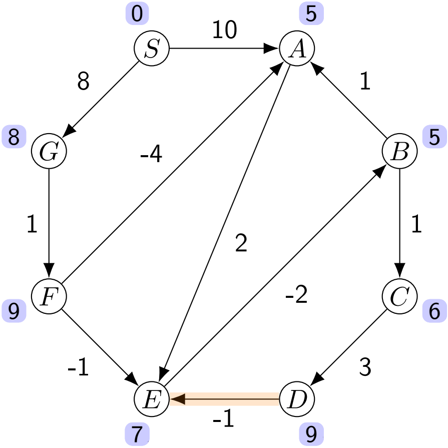

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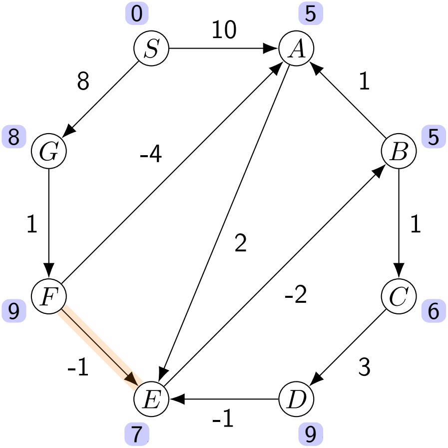

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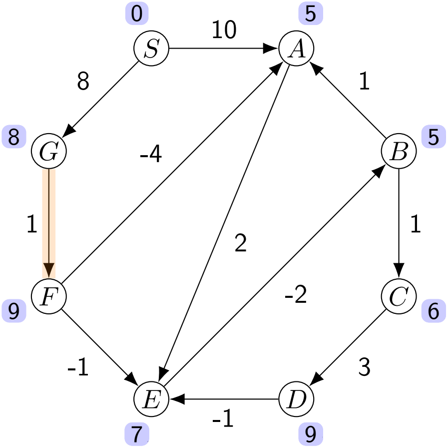

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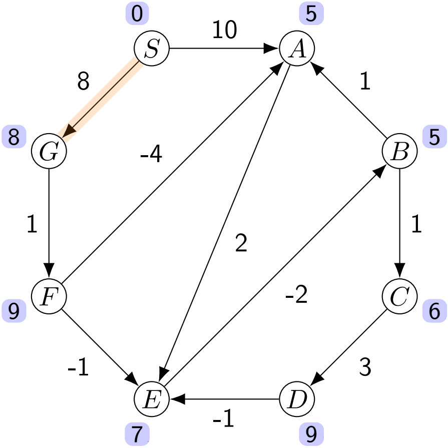

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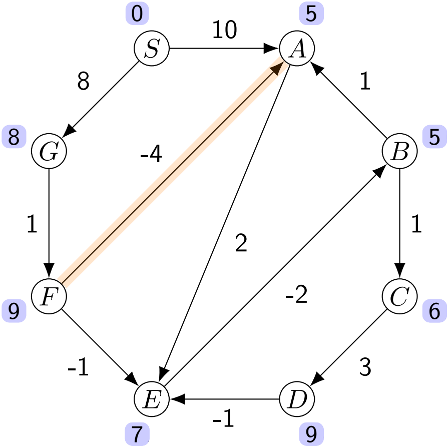

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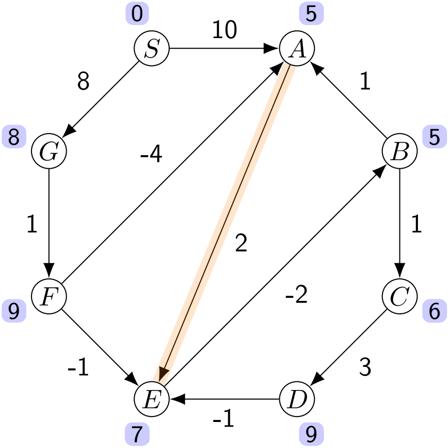

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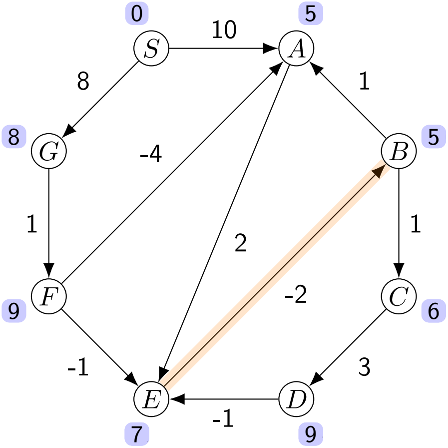

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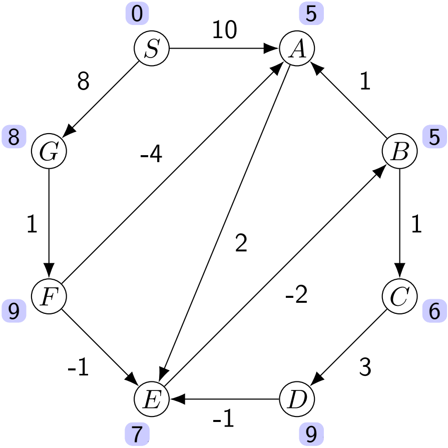
Round: 4



Bellman-Ford algorithm

Can we do better?

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Bellman-Ford algorithm

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Bellman-Ford algorithm

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Bellman-Ford algorithm

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What happens if we run $|V|$ iterations?

Bellman-Ford algorithm

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- This can be used to detect negative cycles in a graph.

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- Can we use information from other rows to speed things up?