# CMPSC 465: LECTURE XVIII

## All Pairs Shortest Paths
## Maximum Flow

Ke Chen

October 10, 2025

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- Repeating Bellman-Ford fills $dist$ row by row. $O\left(|V|^2 \cdot |E|\right)$.
- Can we use information from other rows to speed things up?

# A recursive formulation

**Initially:** $$dist_0[i,j] = \begin{cases} 0 & \text{if } i = j \\ \ell(i,j) & \text{if } (i,j) \in E \text{ is an edge} \\ \infty & \text{otherwise} \end{cases}$$

This is the **adjacency matrix** of the weighted graph $G$.

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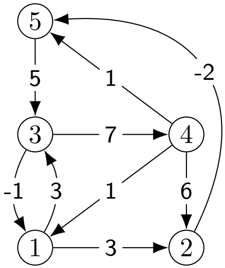


$dist_0$ :

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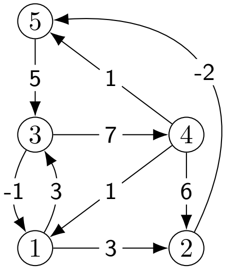


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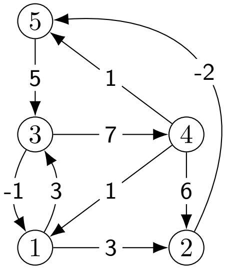


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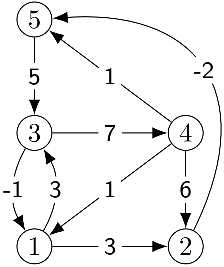


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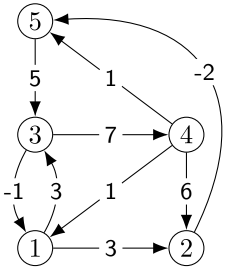


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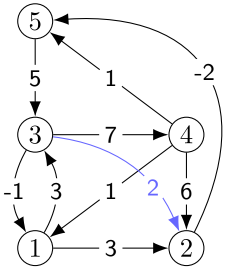


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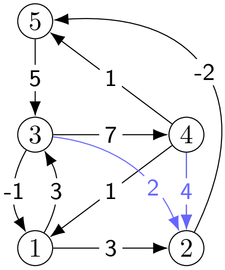


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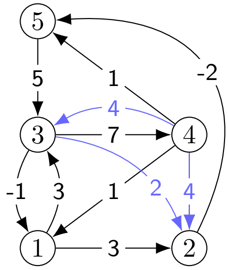


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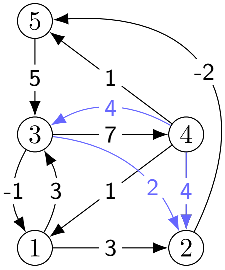


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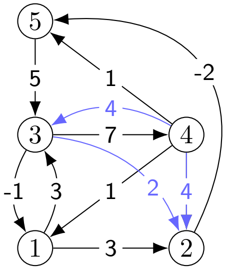


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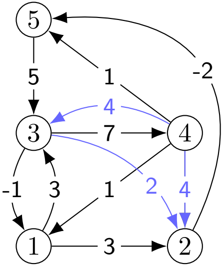


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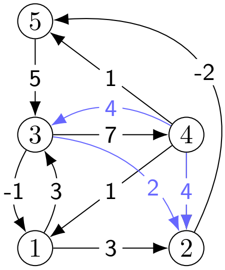


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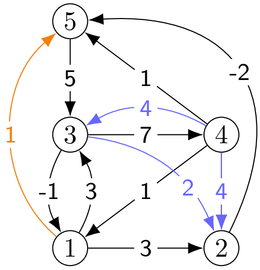


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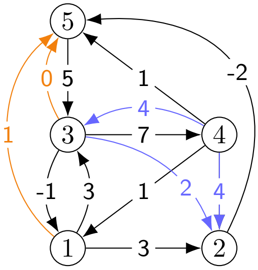


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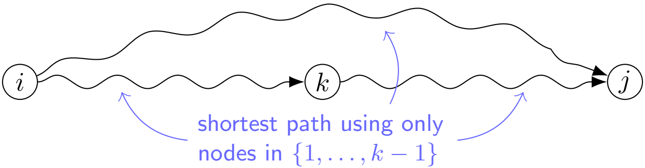

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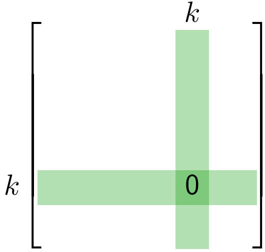


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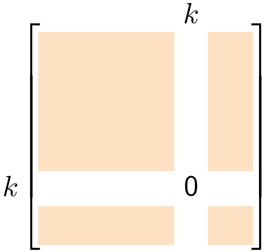

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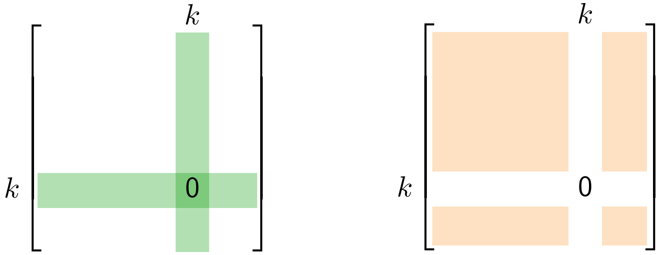


The values we need from $dist_{k-1}$ will not change when computing $dist_k$, so we can simply do updates in place.

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**Floyd-Warshall**($G = (V, E, \ell)$)

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foreach (i, j) ∈ {1, ..., n}^2 do
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| nonnegative weights | Dijkstra | $O((\|V\| + \|E\|) \log \|V\|)$ |
| no negative cycles | Bellman-Ford | $O(\|V\| \cdot \|E\|)$ |
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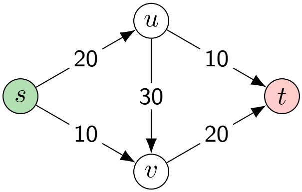
Definition A Flow Network is a directed graph $G = (V, E)$ s.t.:

1. Each edge $e \in E$ has a nonnegative capacity $c(e)$.
2. There is a unique source node $s \in V$.
3. There is a unique sink node $t \in V$.

# Flow

Example A flow network $G$:



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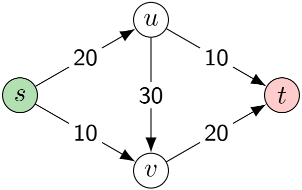


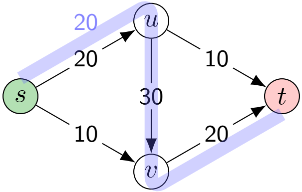

Definition An $s-t$ flow in a flow network is a function $f$ that maps each edge to a nonnegative real number ($f : E \to \mathbb{R}_{\geq 0}$) satisfying:

1. [Capacity Condition] $0 \leq f(e) \leq c(e) \quad \forall e \in E$
2. [Flow Conservation] For all $v \in V$, $v \notin \{s, t\}$:

$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e).$$

# Flow

Example A flow network $G$:

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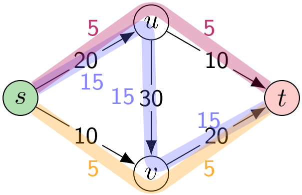


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Max Flow Problem Given a flow network, find the flow of maximum value.