

CMPSC 465: LECTURE XIII

Strongly Connected Components

Ke Chen

September 29, 2025

Recall from last week ...

On a directed graph:

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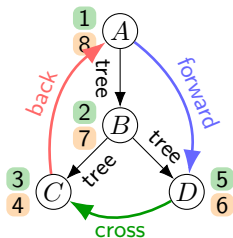
On a directed graph:

- ▶ The same DFS works.

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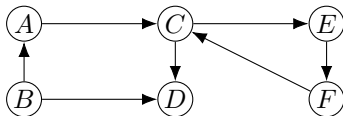
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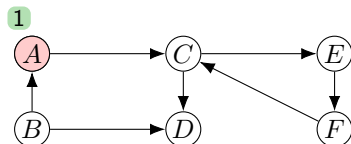
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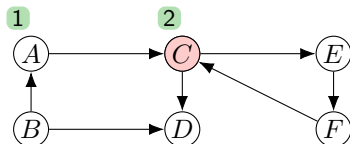
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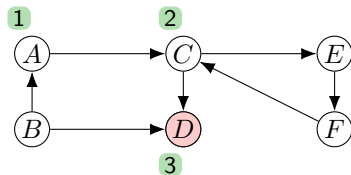
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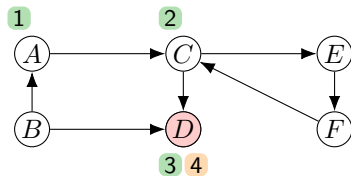
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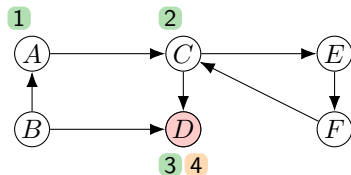
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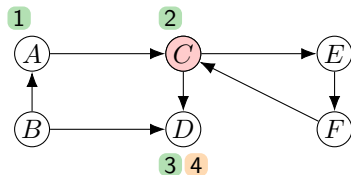


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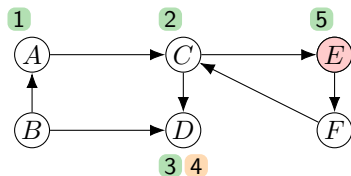


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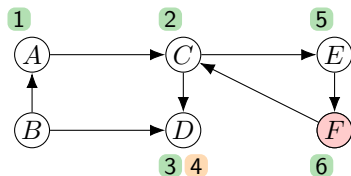


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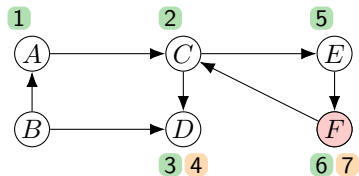


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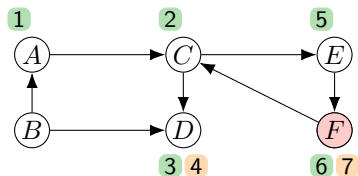


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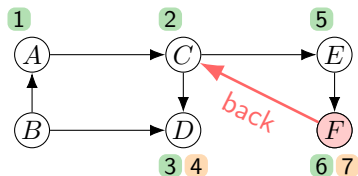
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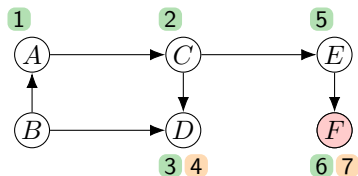
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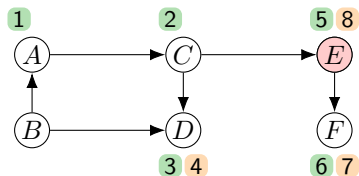
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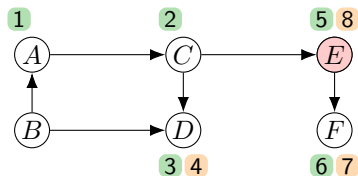
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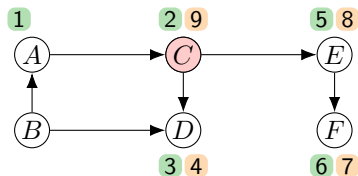
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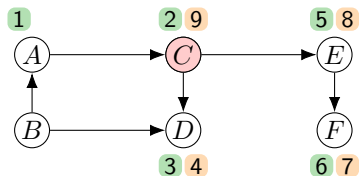
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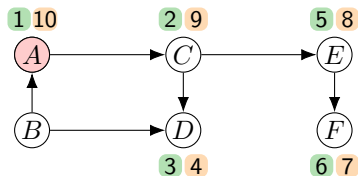
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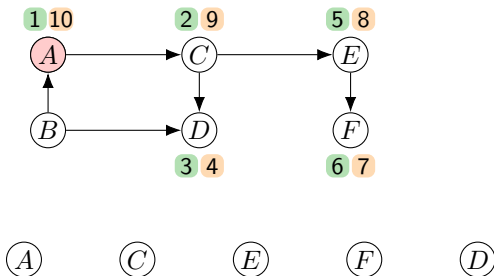
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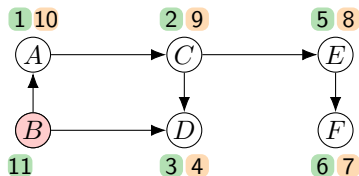
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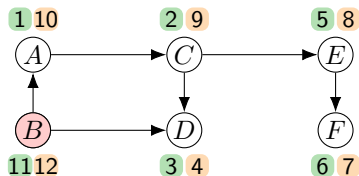
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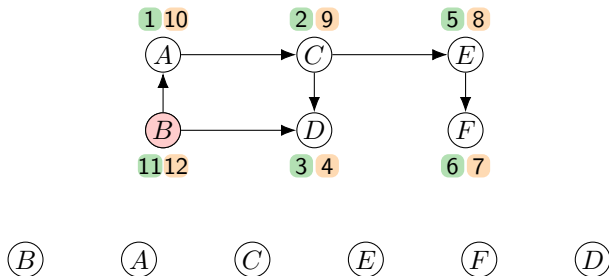
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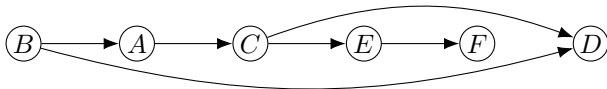
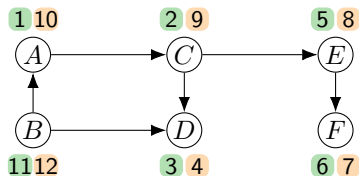
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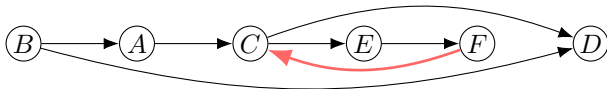
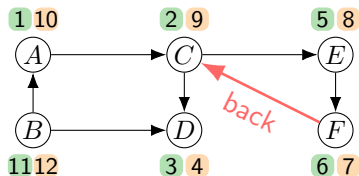
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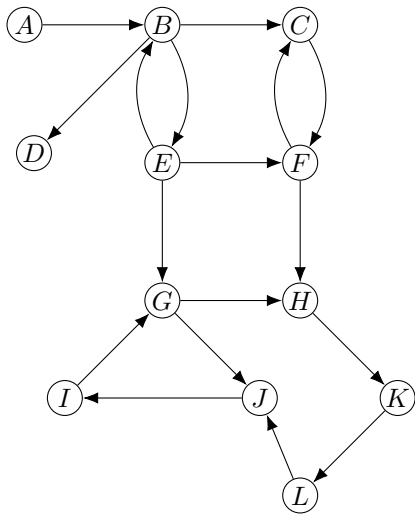
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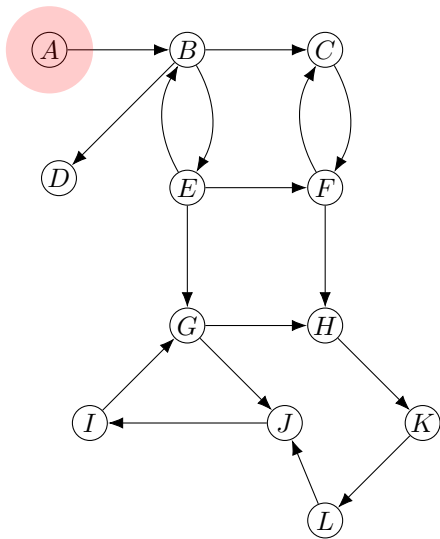
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- ▶ Each subset is called a **strongly connected component (SCC)**.

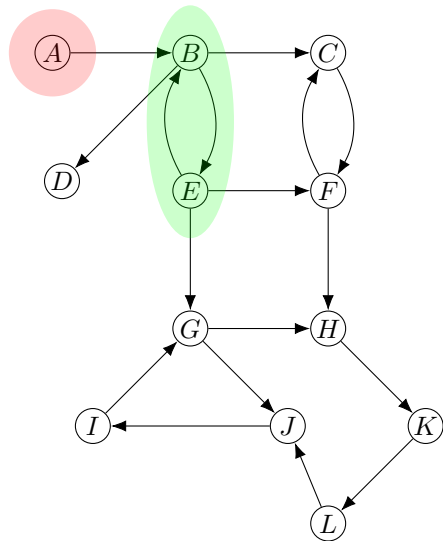
Strongly connected components (SCC's)



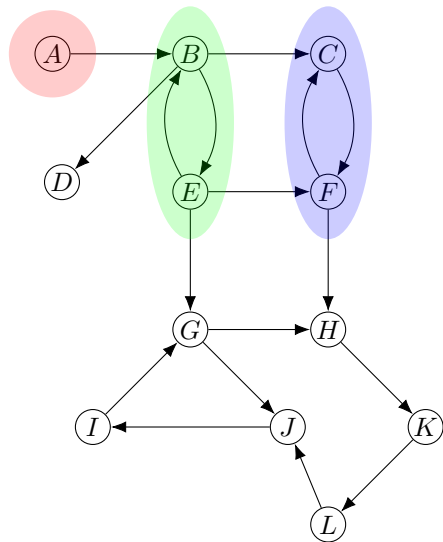
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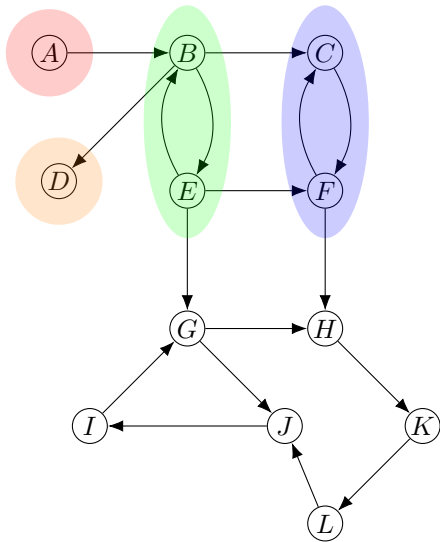
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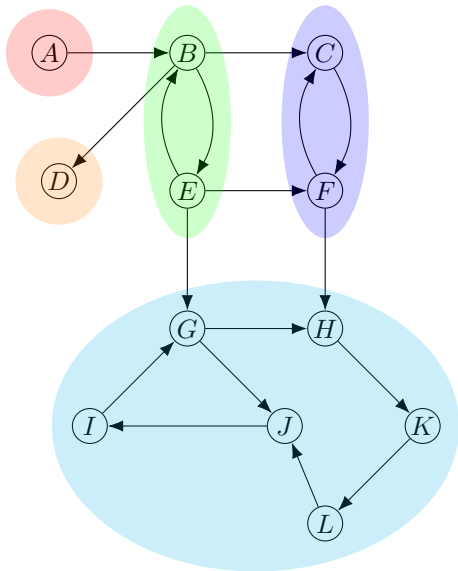
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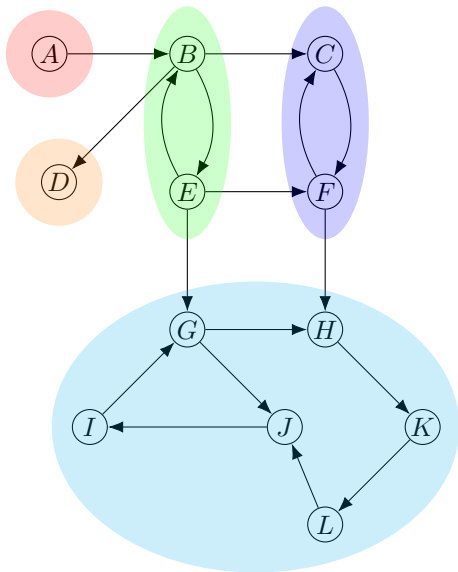


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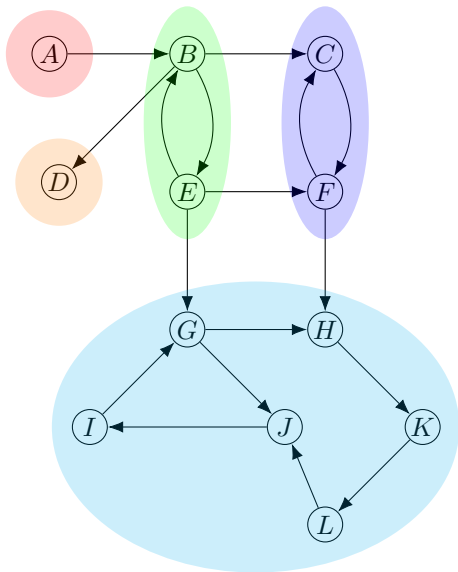
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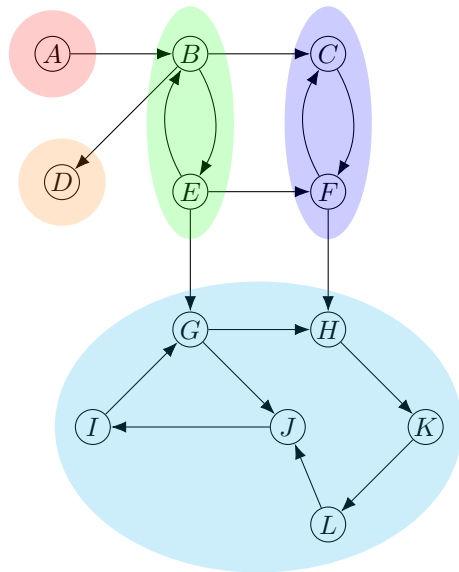
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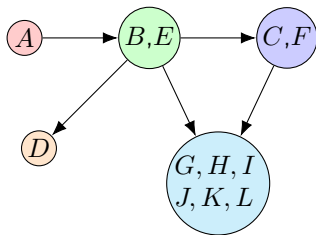


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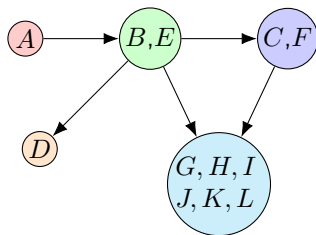


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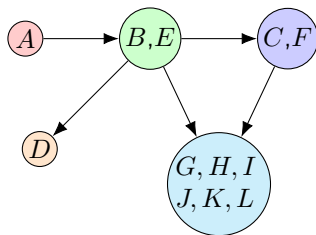
Meta graph



Each directed graph has a corresponding meta graph

- ▶ Vertices correspond to SCC's.
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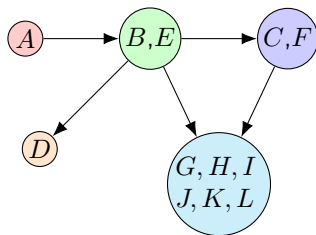


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Fact The meta graph is a DAG.

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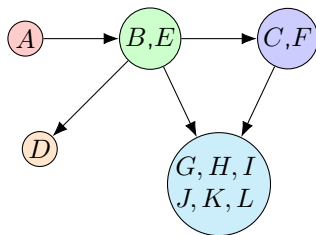
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Fact The meta graph is a DAG.

Proof. If there was a cycle in the meta graph, then the SCC's in the cycle would be merged together.

Meta graph

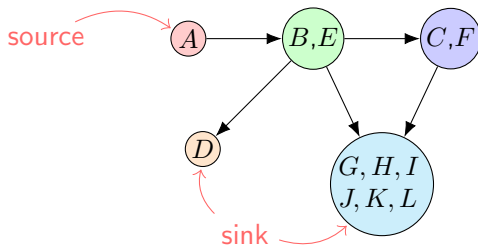


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Definition In a directed graph, a **source** is a vertex with no incoming edges; a **sink** is a vertex with no outgoing edges.

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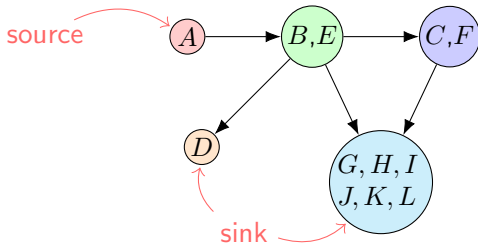


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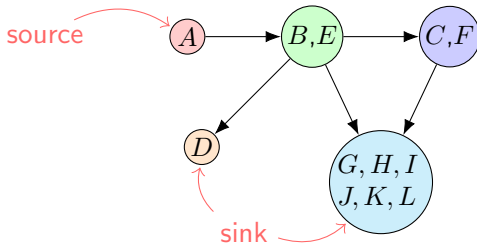


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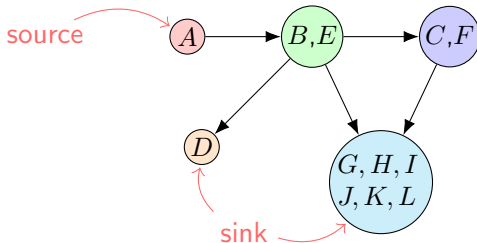
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1. Find a vertex in a sink SCC.
2. Call Explore on it.
3. Remove the found SCC and repeat.


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1. Find a vertex in a sink SCC.  But how?
2. Call Explore on it.
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Turns out to be not easy ...

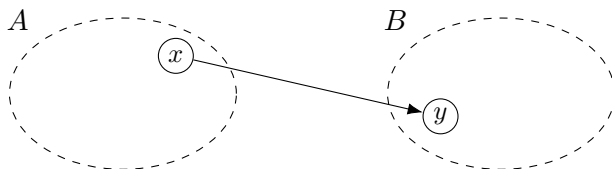
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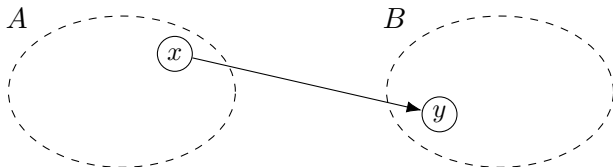
Fact Suppose A and B are two SCC's and there is an edge from a vertex in A to a vertex in B . Then, the vertex with the **largest post number** must be in **?**.



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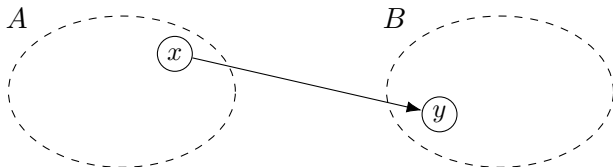
Proof. Among all vertices in $A \cup B$, if DFS first visits

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- ▶ a node v in B .

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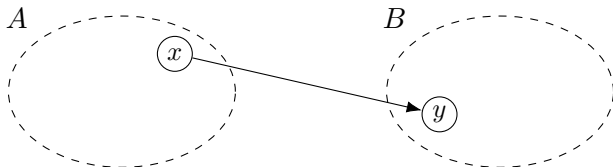
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- ▶ a node v in B .

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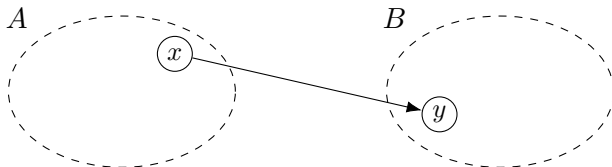
Proof. Among all vertices in $A \cup B$, if DFS first visits

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- ▶ a node v in B . will finish all B without visiting anything in A .

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Turns out to be not easy ... How about a vertex in a **source** SCC?

Fact Suppose A and B are two SCC's and there is an edge from a vertex in A to a vertex in B . Then, the vertex with the **largest post number** must be in A .



Proof. Among all vertices in $A \cup B$, if DFS first visits

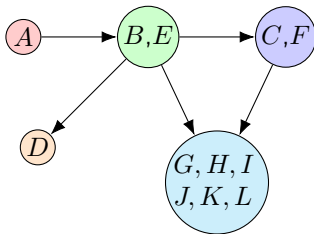
- ▶ a node v in A , will finish all B before going back to v .
- ▶ a node v in B . will finish all B without visiting anything in A .

Find a vertex in a sink SCC

Turns out to be not easy ... How about a vertex in a **source** SCC?

Fact Suppose A and B are two SCC's and there is an edge from a vertex in A to a vertex in B . Then, the vertex with the **largest post number** must be in A .

A key consequence of this fact is that the vertex with the **largest post number** must be in a **source SCC**.



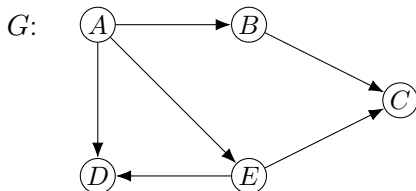
Find a vertex in a sink SCC

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Find a vertex in a sink SCC

But we need a vertex from a **sink** SCC, not a **source** SCC.

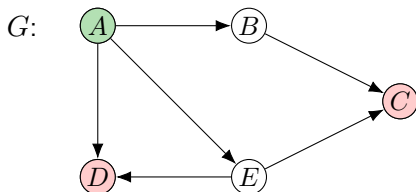
Idea Consider the reverse graph G^R .



Find a vertex in a sink SCC

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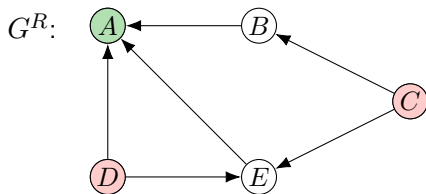
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Find a vertex in a sink SCC

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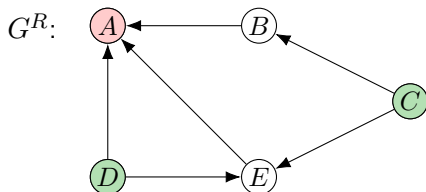
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Find a vertex in a sink SCC

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Facts about G^R

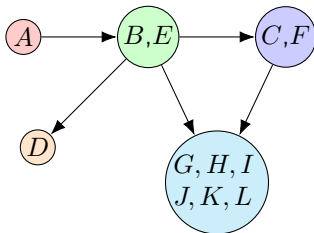
- ▶ G and G^R have the same SCC's.

Find a vertex in a sink SCC

But we need a vertex from a **sink** SCC, not a **source** SCC.

Facts about G^R

- ▶ G and G^R have the same SCC's.
- ▶ In the meta graph of G^R , every source SCC corresponds to a sink SCC in G .

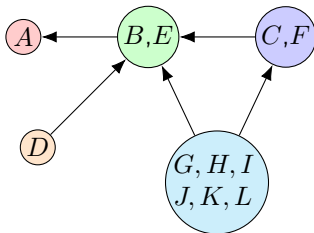


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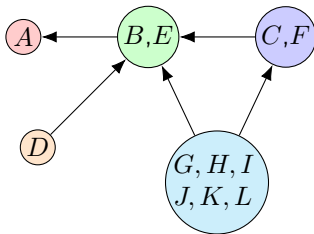


Find a vertex in a sink SCC

But we need a vertex from a **sink** SCC, not a **source** SCC.

Facts about G^R

- ▶ G and G^R have the same SCC's.
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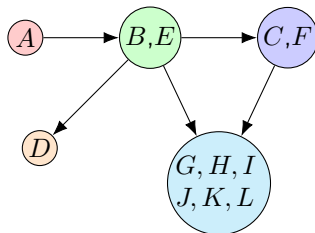


Therefore, if we run DFS on G^R and choose the node with the highest post number, it must be in a **sink** SCC of G !

Find SCC's


Recall our plan:

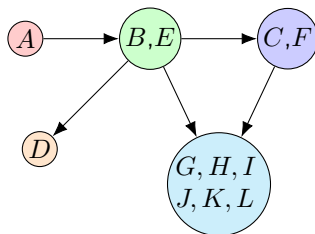
1. Find a vertex in a sink SCC.
2. Call Explore on it.
3. Remove the found SCC and repeat.



Find SCC's

Recall our plan:

1. Find a vertex in a sink SCC. 
2. Call Explore on it.
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Find SCC's

Input: Graph $G = (V, E)$ in adjacency list.

1. Build adjacency list for G^R .
2. Run DFS on G^R , assign pre/post numbers and output nodes sorted in **descending order of post number**. Let v_1, v_2, \dots, v_n be the ordering of the vertices.
3. Run DFS on G using the this ordering:

// visited is an array of size n filled with 0's
color = 1

for $i = 1$ **to** n **do**

```
| if  $visited[i] == 0$  then  
| | Explore( $G, v_i, color$ )  
| |  $color = color + 1$ 
```

Find SCC's

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Time complexity?

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Time complexity? Steps 1, 2, and 3 each takes $O(|V| + |E|)$,
so overall **$O(|V| + |E|)$** .

Find SCC's

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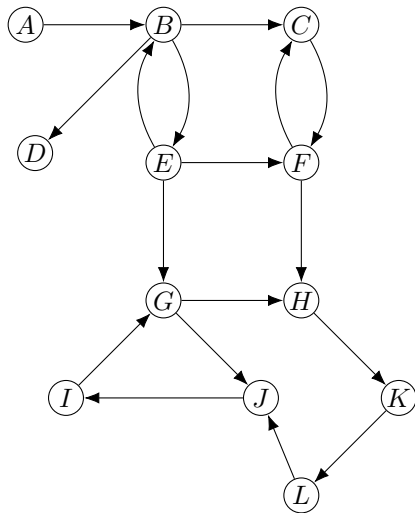
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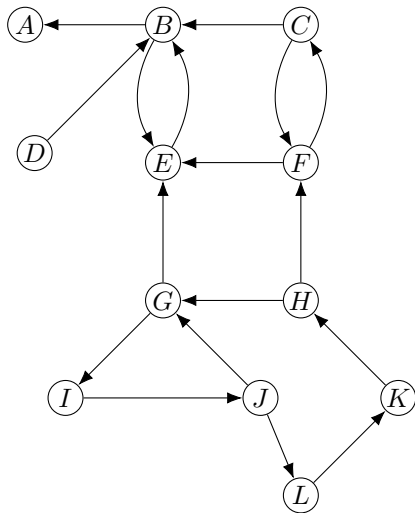
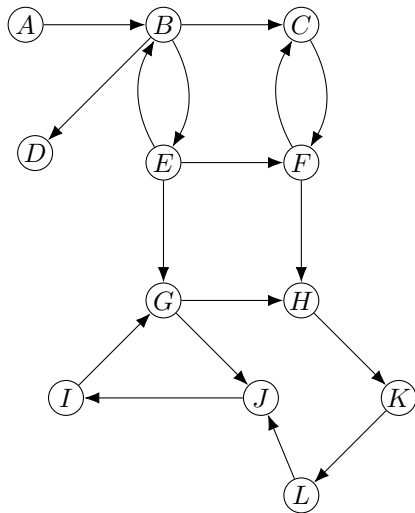
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        Explore( $G, v_i, color$ )  
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```

Final remark Note that we don't need to actually "remove" any SCC. Any node with a nonzero entry in *visited* is no longer part of the graph for DFS.

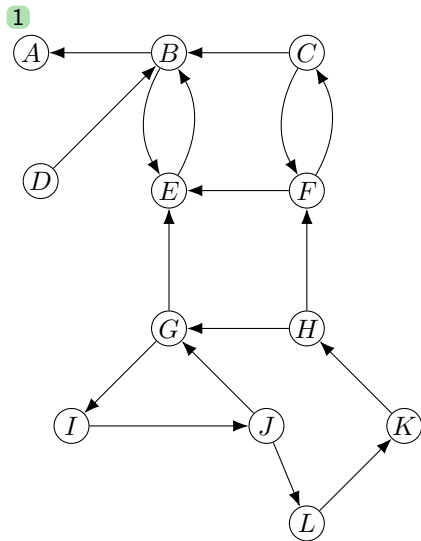
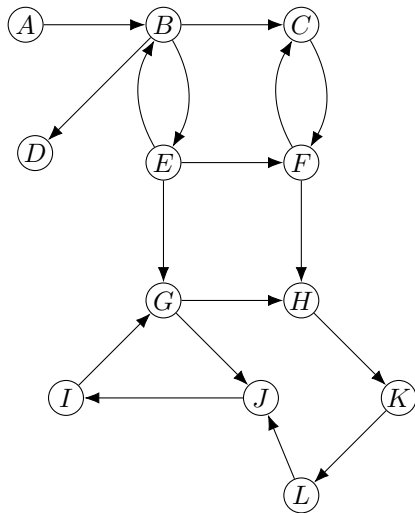
Find SCC's



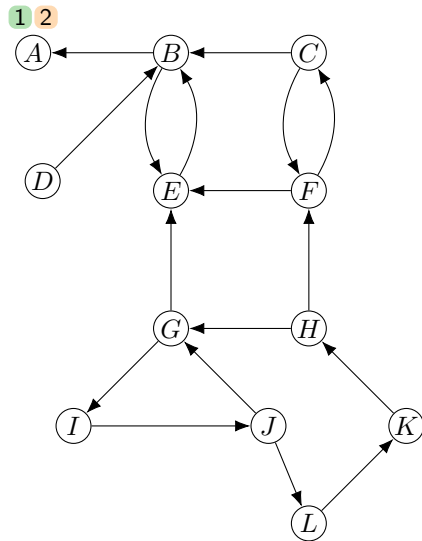
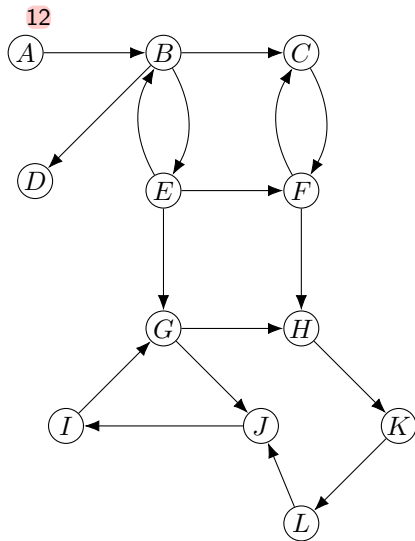
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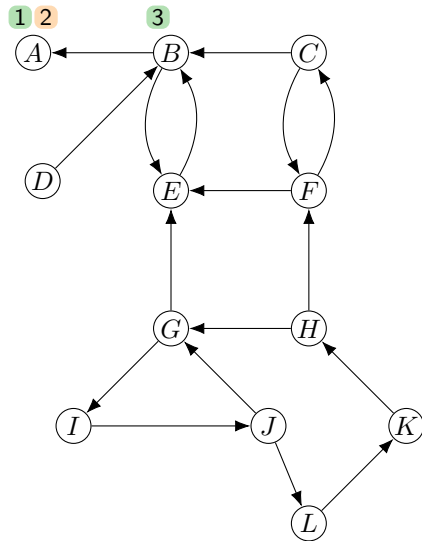
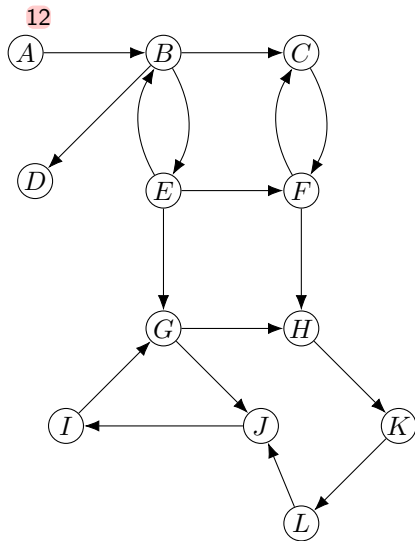
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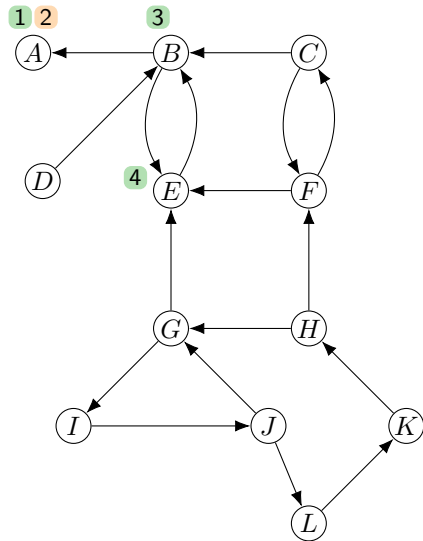
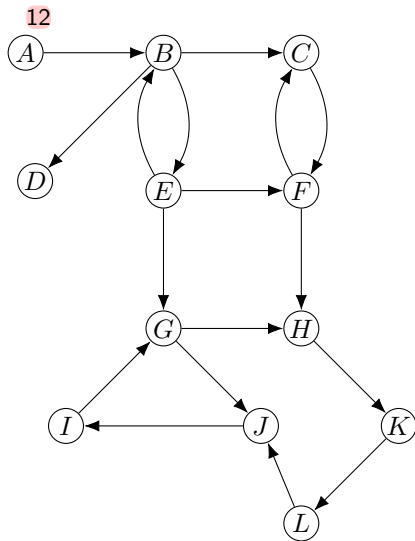
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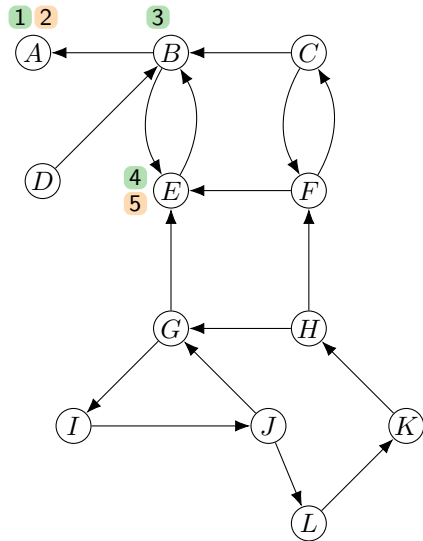
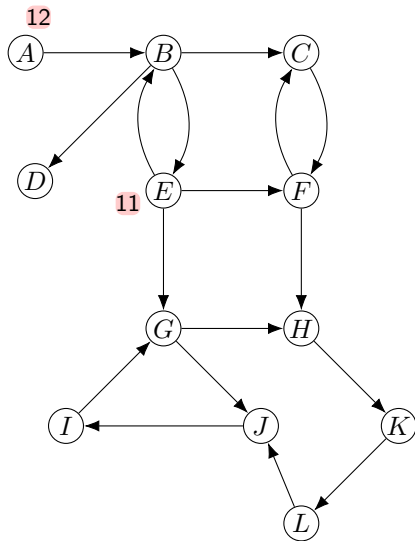
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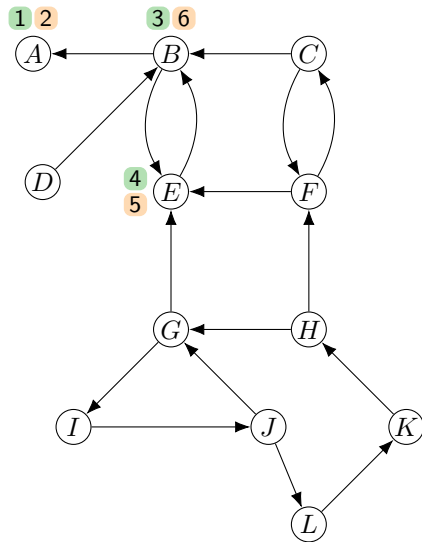
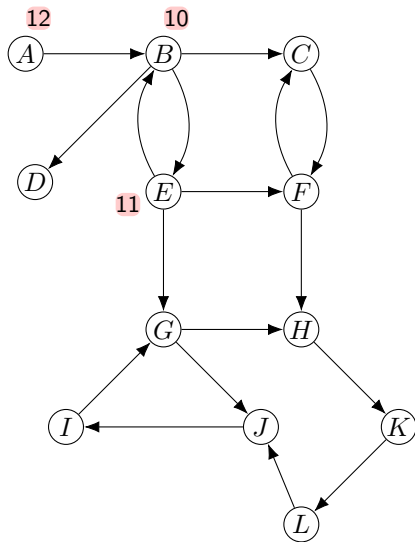
Find SCC's



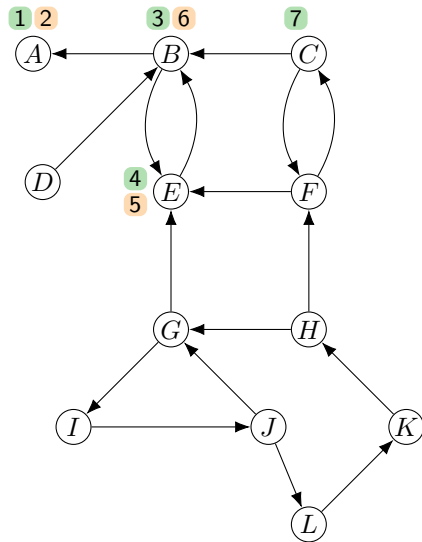
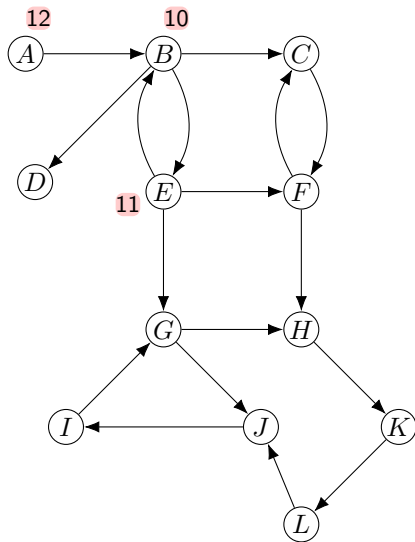
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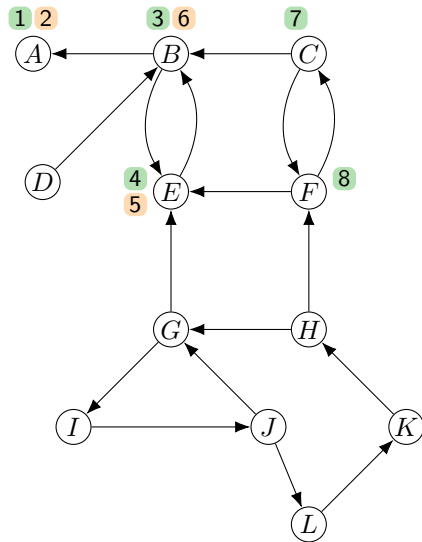
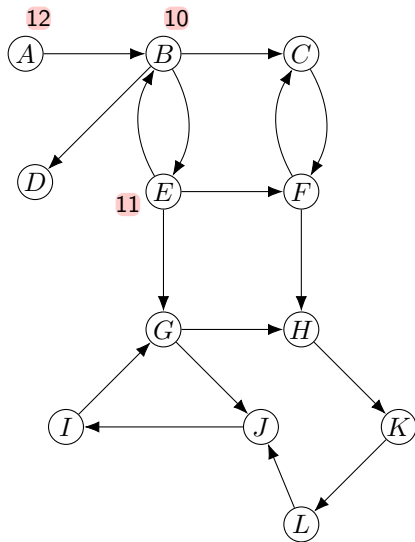
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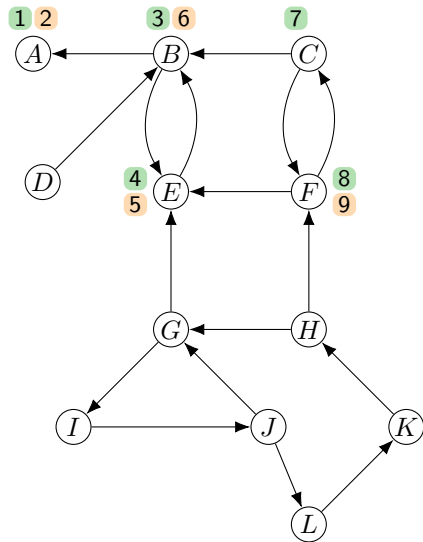
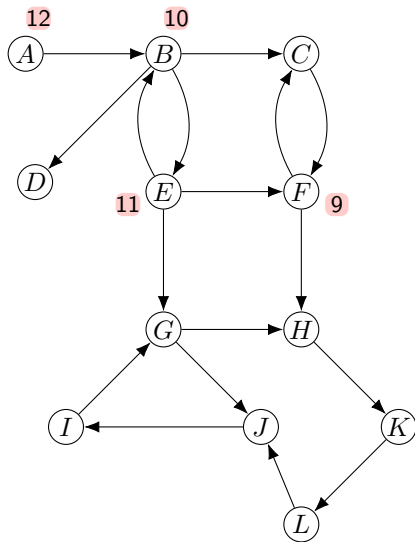
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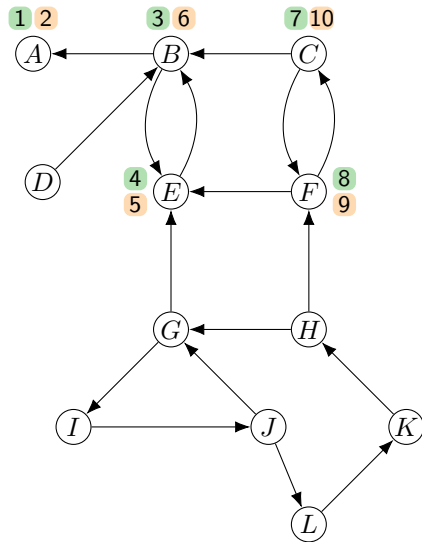
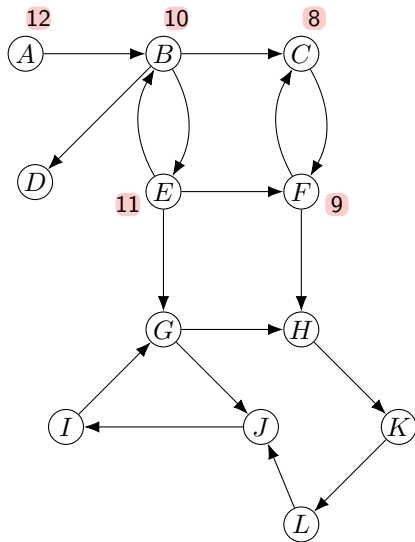
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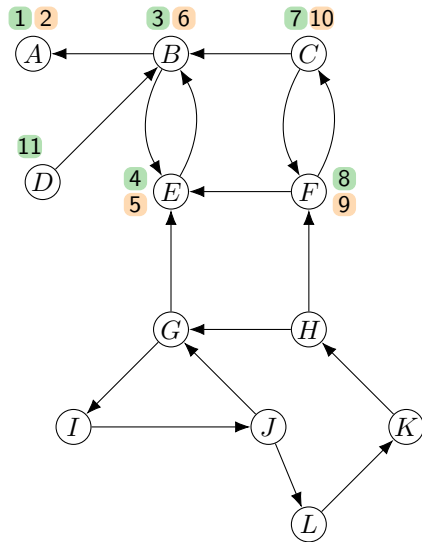
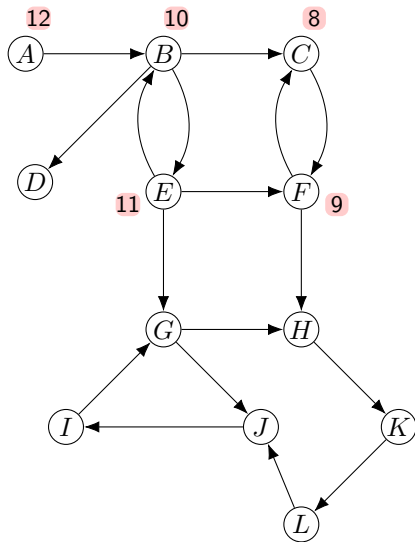
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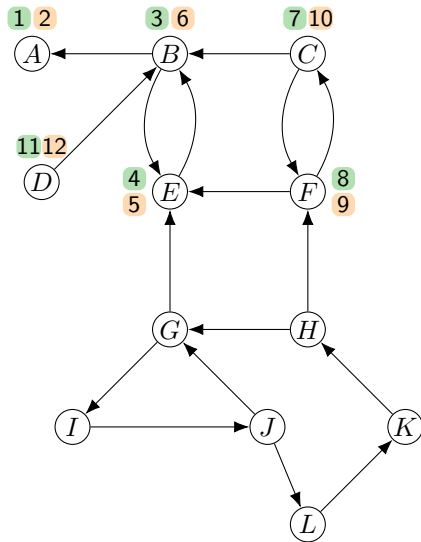
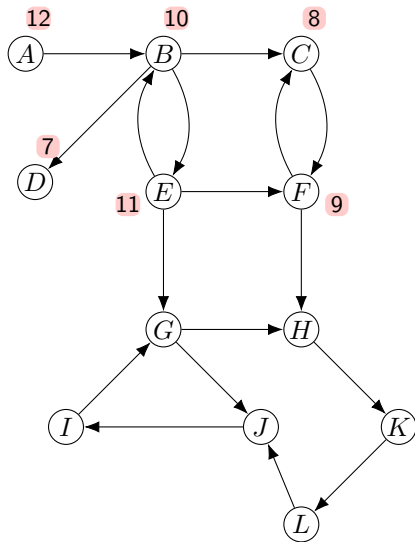
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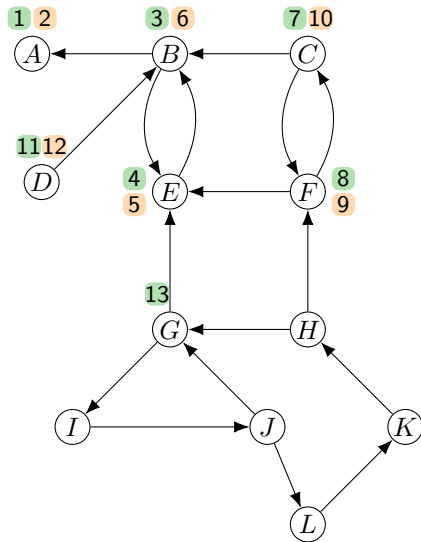
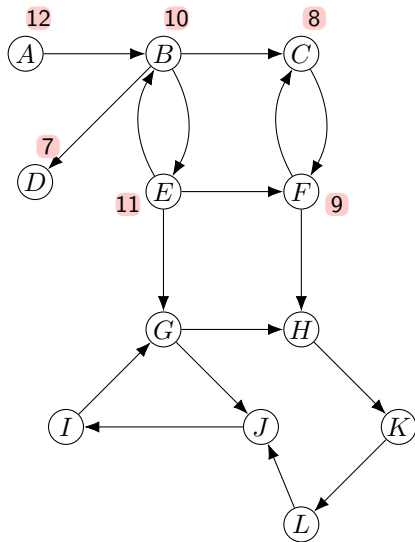
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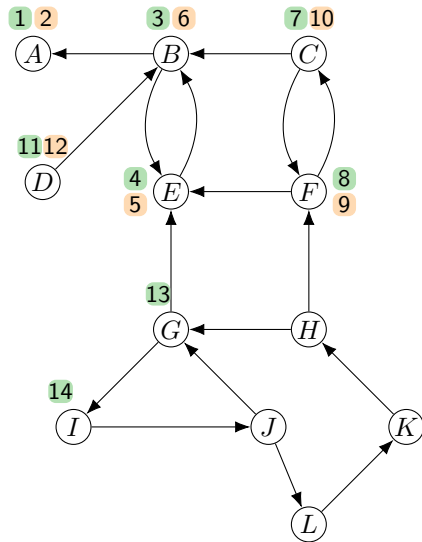
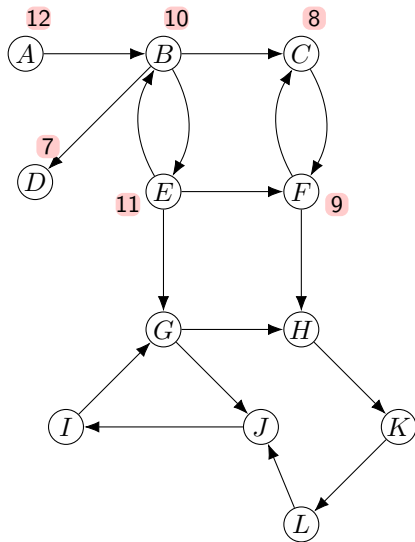
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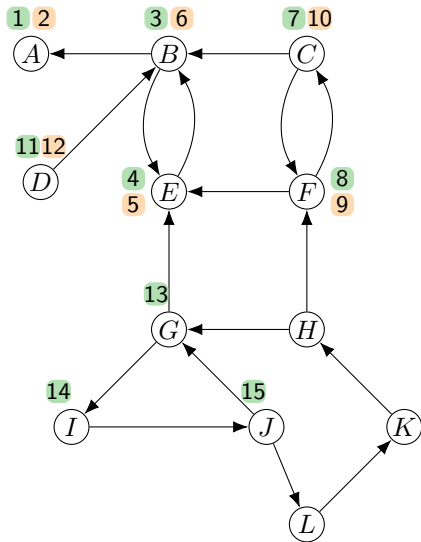
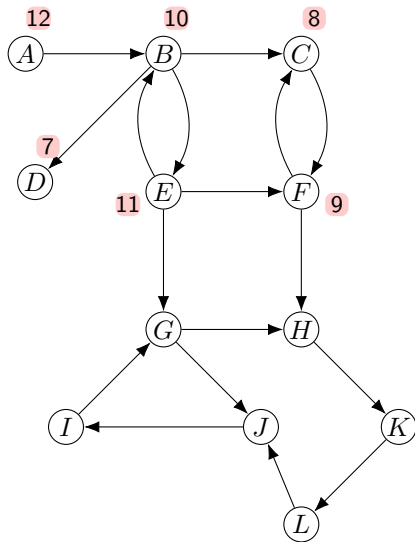
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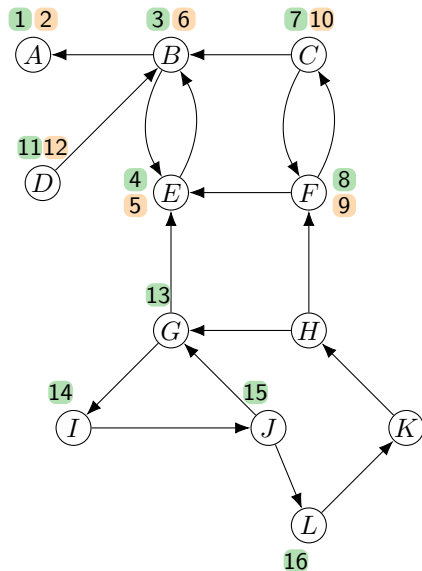
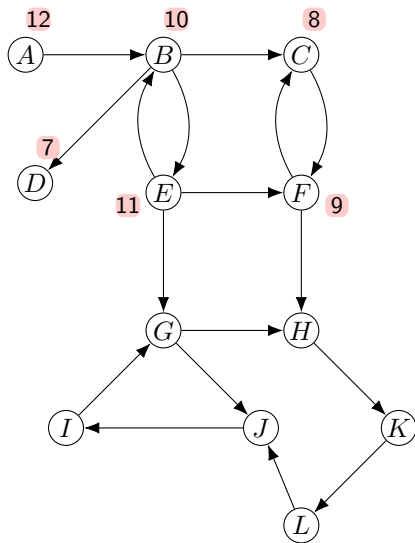
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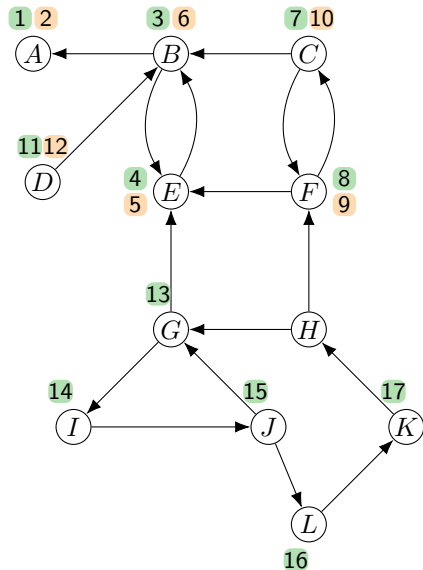
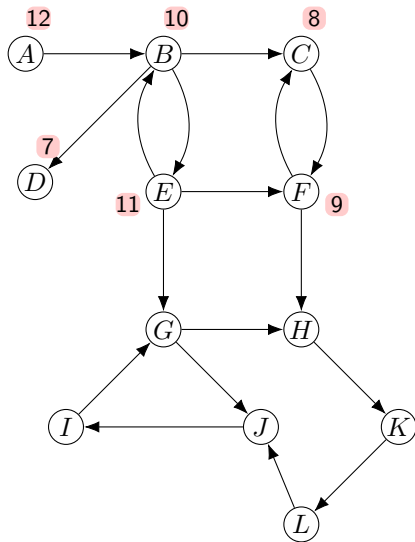
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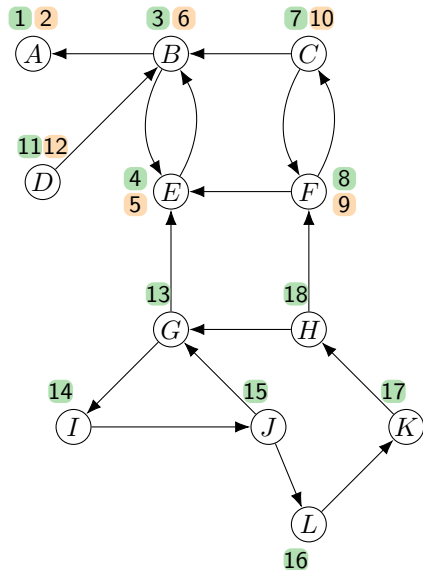
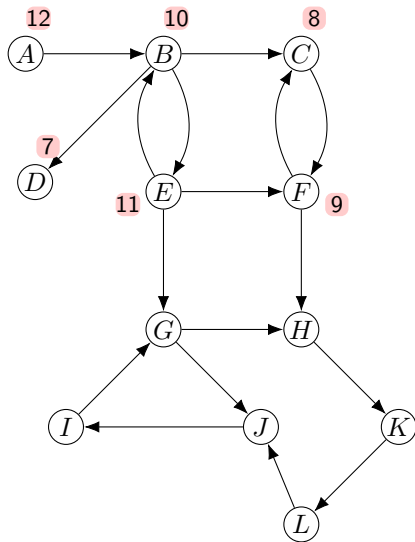
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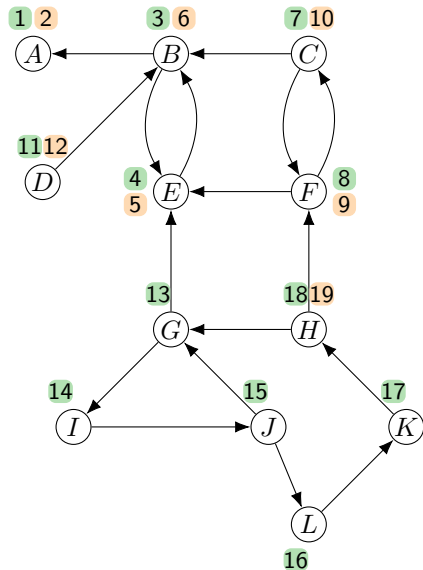
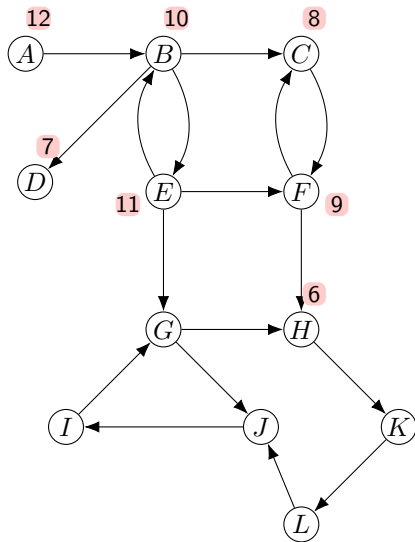
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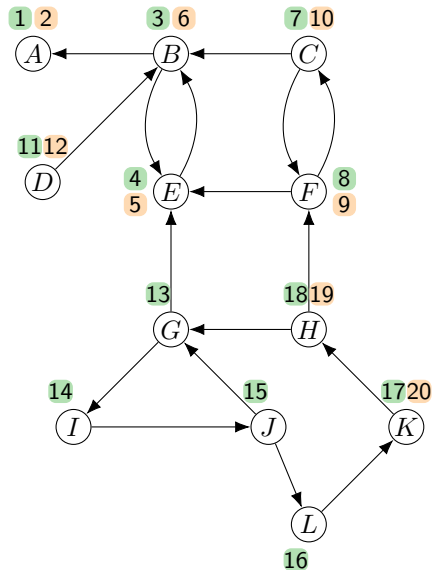
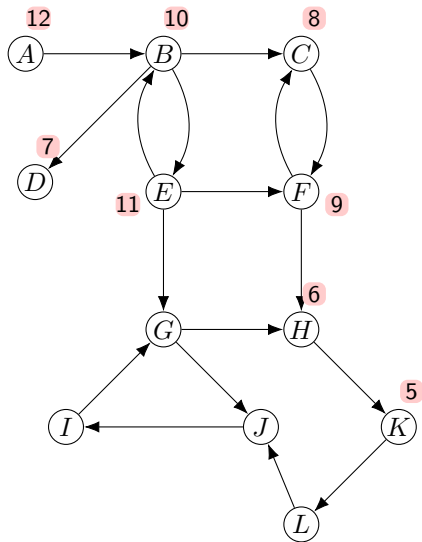
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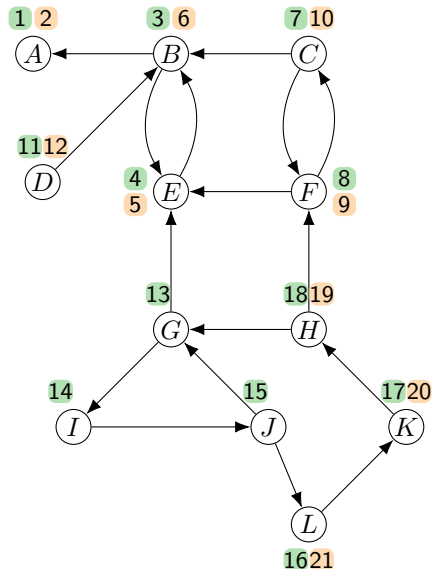
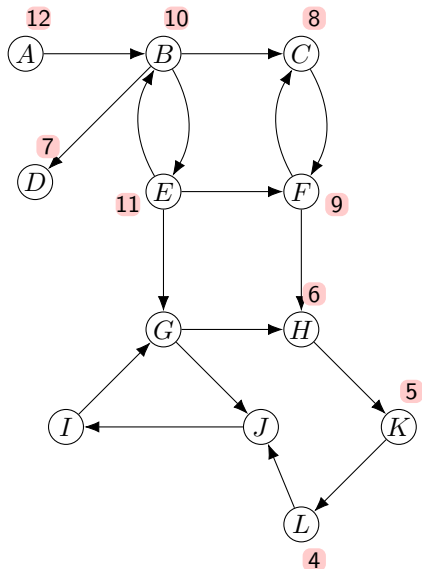
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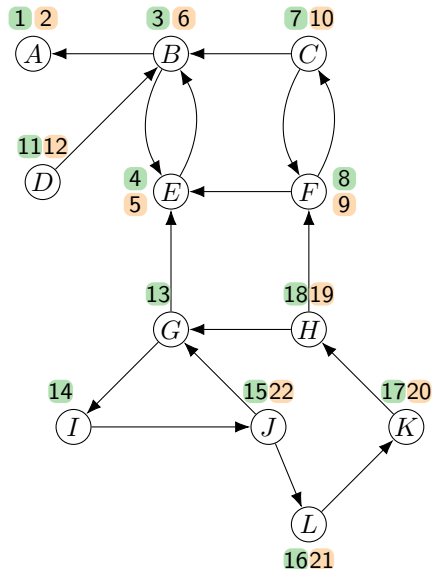
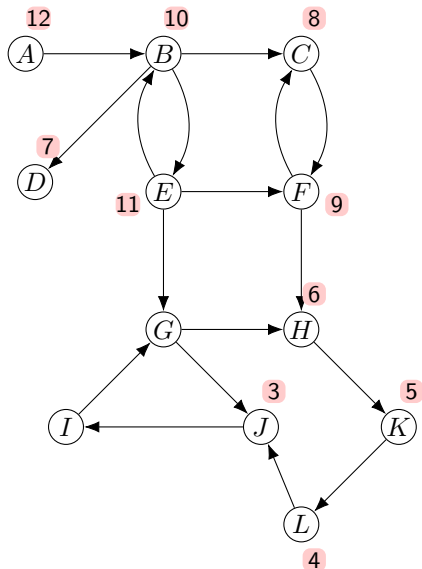
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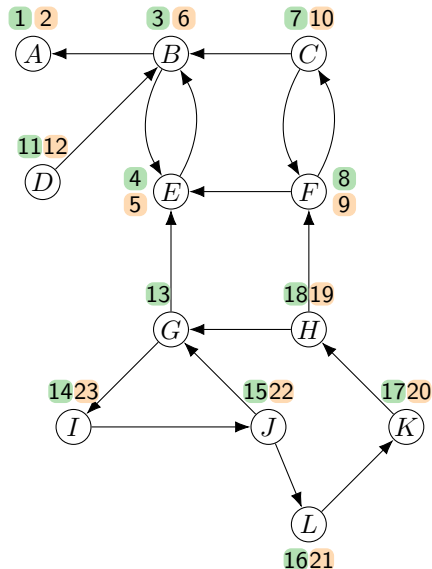
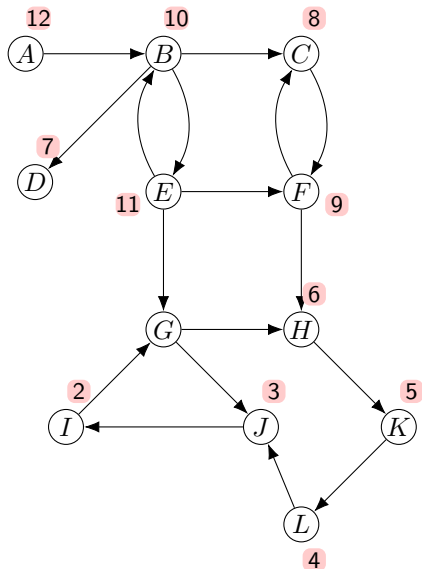
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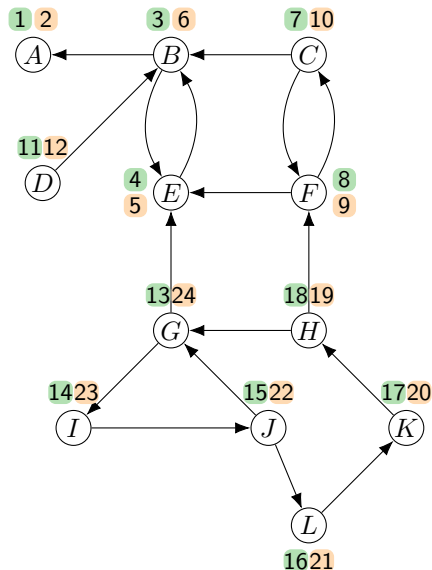
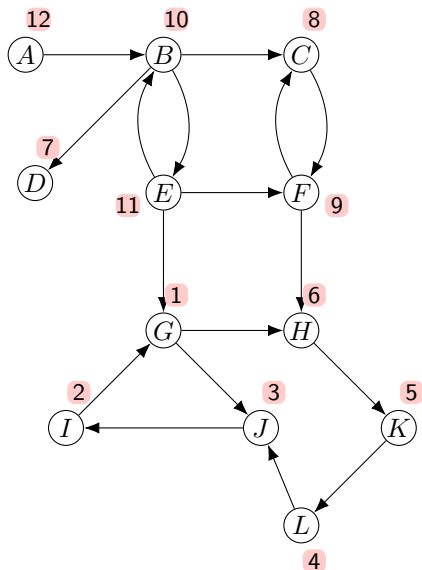
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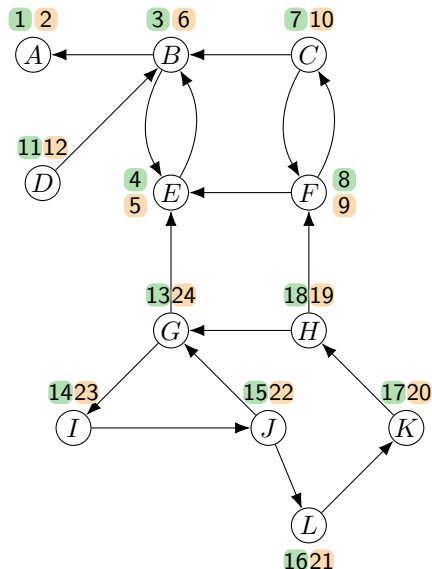
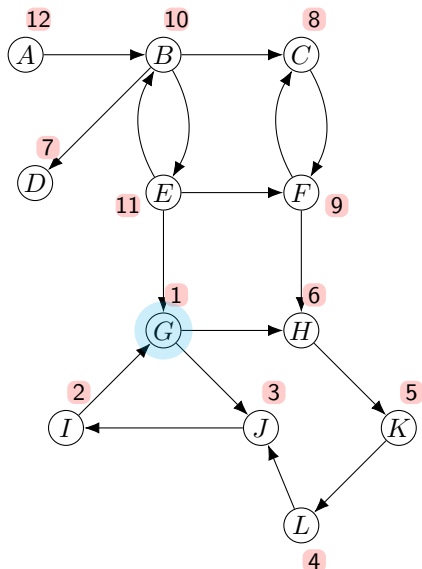
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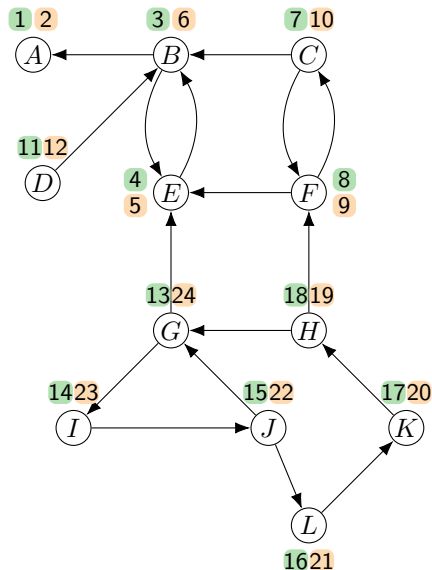
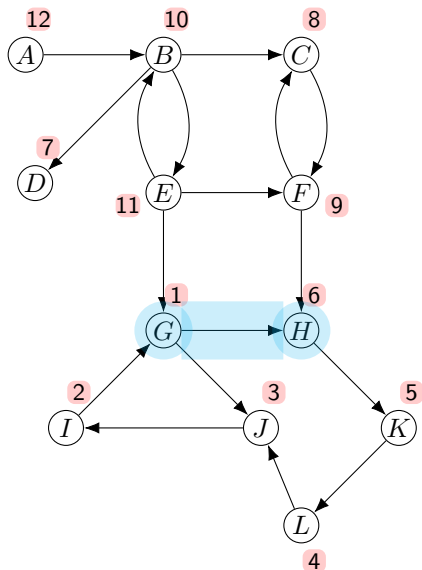
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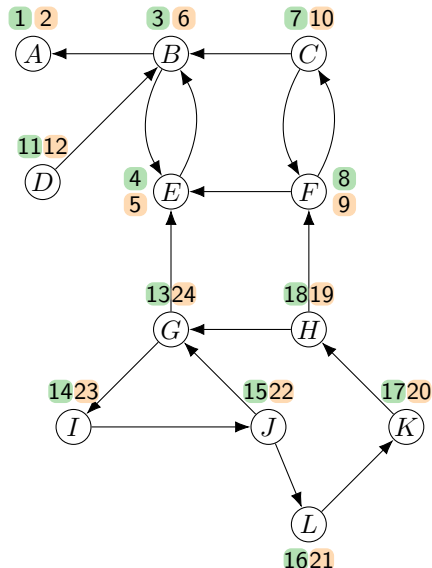
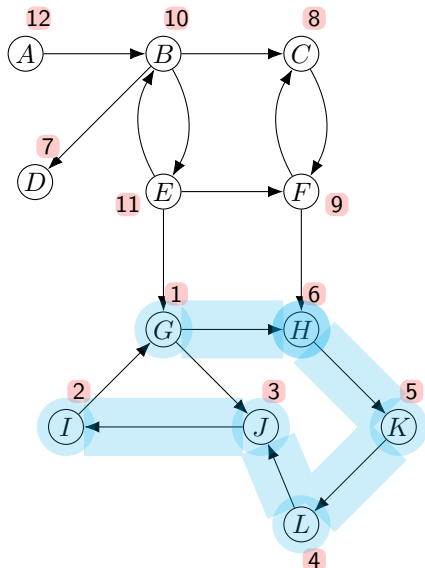
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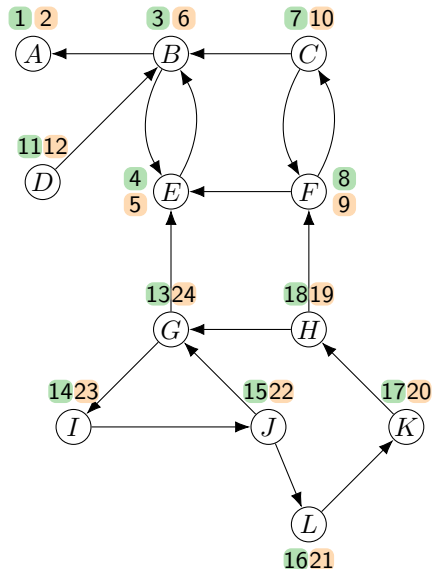
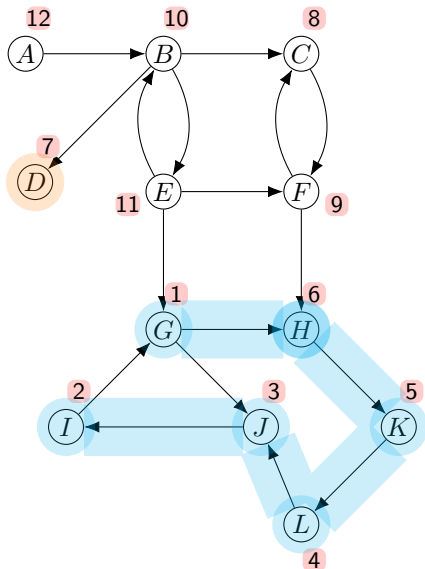
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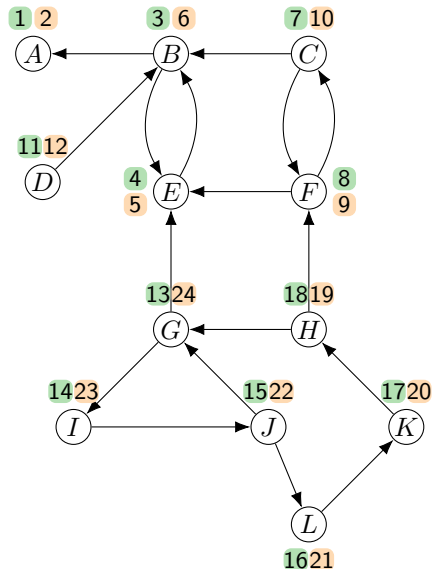
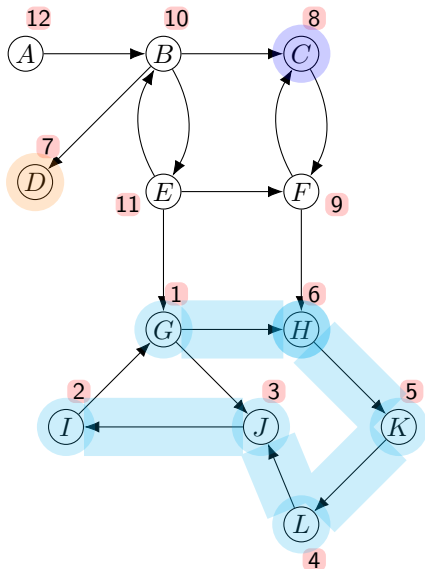
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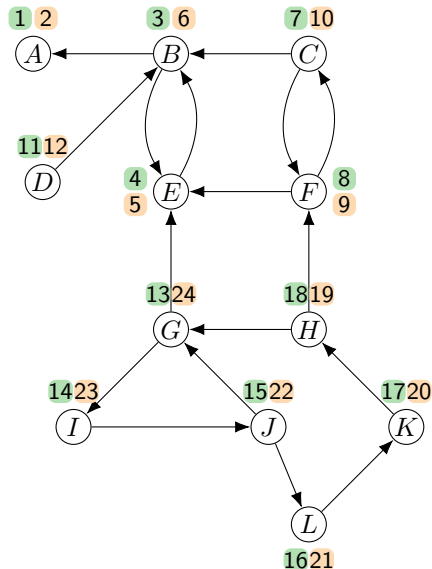
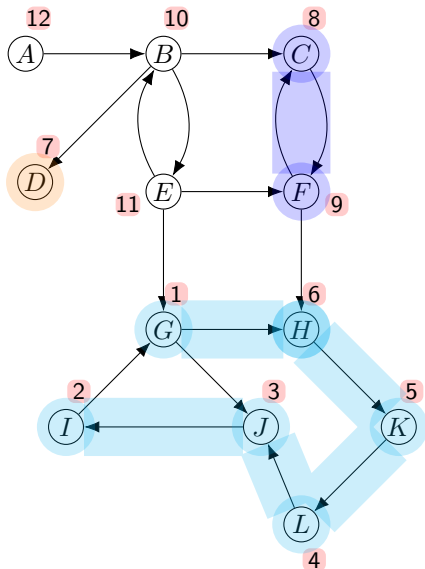
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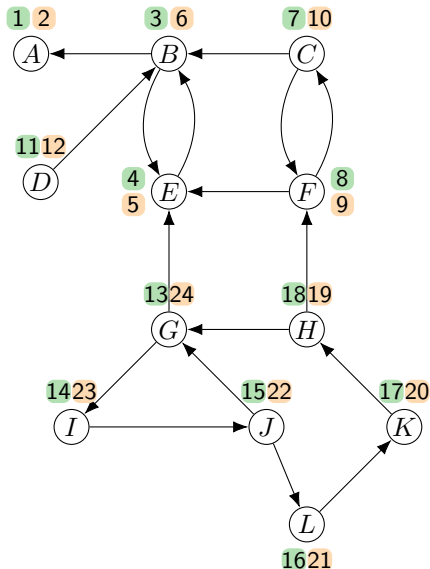
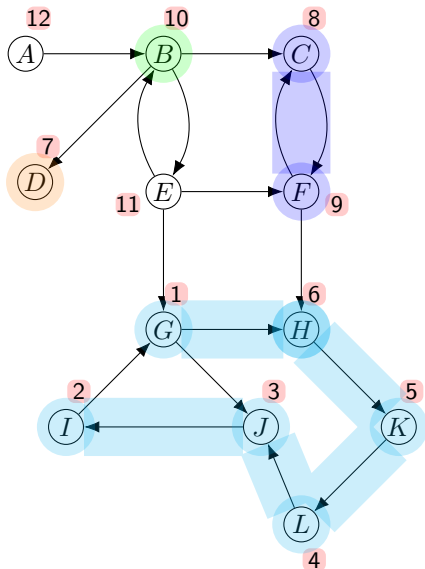
Find SCC's



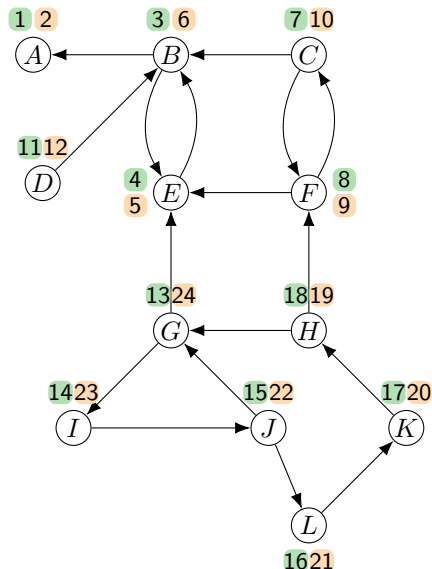
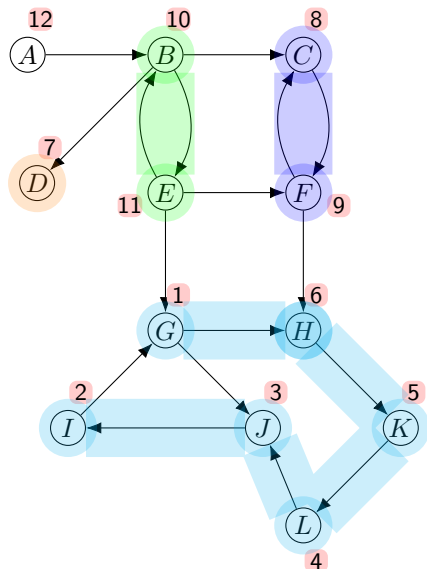
Find SCC's



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