CMPSC 465: LECTURE XIX

Maximum Flow Problem

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October 13, 2025

Flow network

Recall:

Definition A Flow Network is a directed graph $G = (V, E)$ s.t.:

1. Each edge $e \in E$ has a nonnegative capacity $c(e)$.
2. There is a unique source node $s \in V$.
3. There is a unique sink node $t \in V$.

Definition An $s - t$ flow in a flow network is a function f that maps each edge to a nonnegative real number ($f : E \to \mathbb{R}_{\geq 0}$) satisfying:

1. [Capacity Condition] $0 \leq f(e) \leq c(e) \quad \forall e \in E$
2. [Flow Conservation] For all $v \in V$, $v \notin \{s, t\}$:

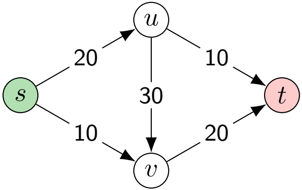
$$\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e).$$

The maximum flow problem

The value of a flow f is $v(f) = \sum_{e \text{ out of } s} f(e)$.

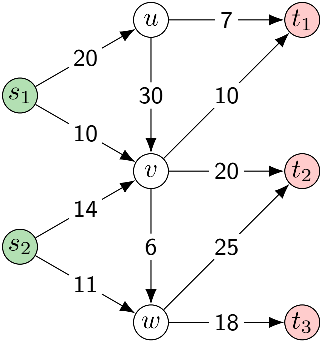
Note that $v(f) = \sum_{e \text{ into } t} f(e)$ (why?).

Max Flow Problem Given a flow network, find the flow of maximum value.



Simplifying assumptions

1. Single source, single sink.



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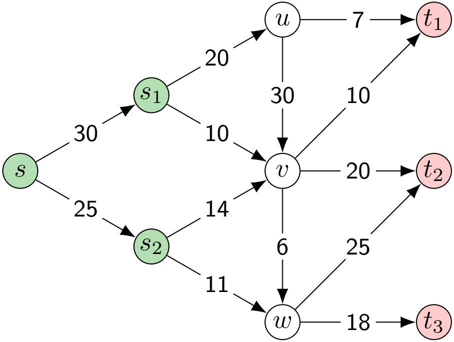

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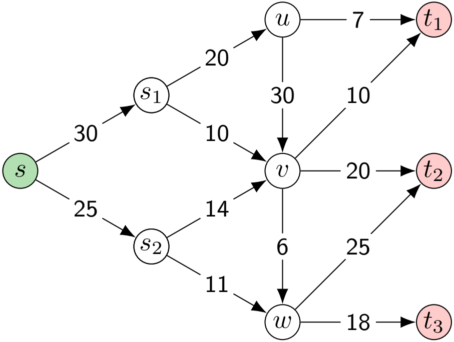

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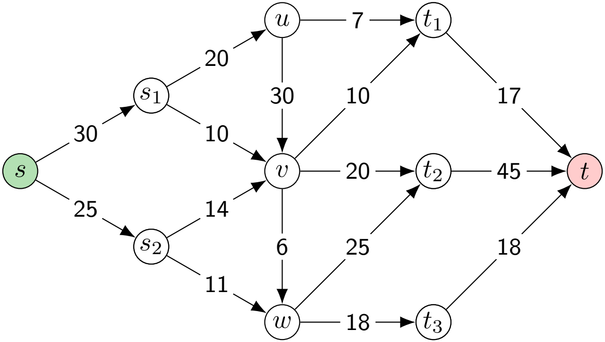

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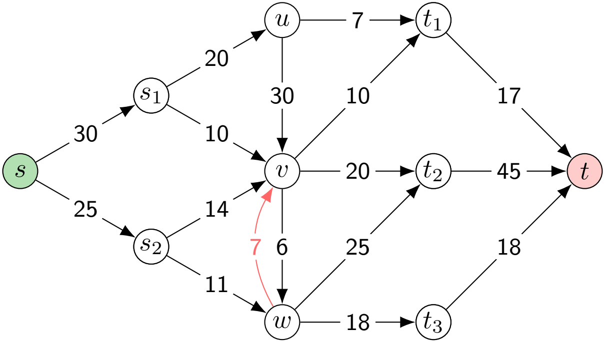
1. Single source, single sink.
2. No antiparallel edges.



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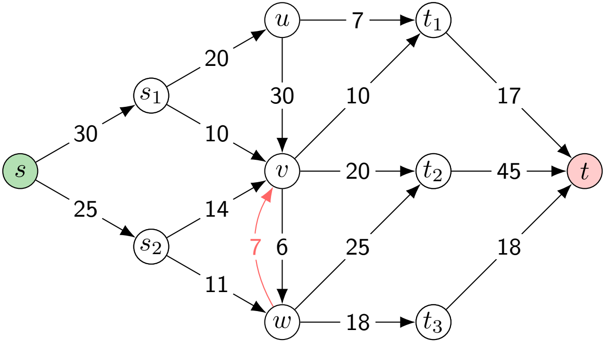

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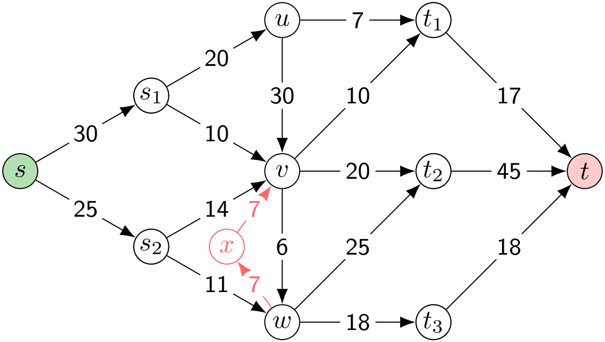

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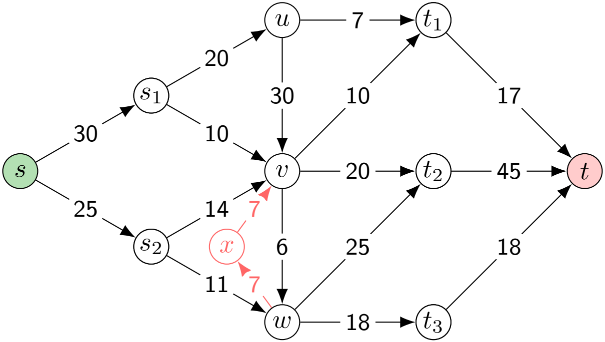
3. Integer flow capacity.



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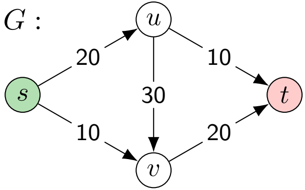

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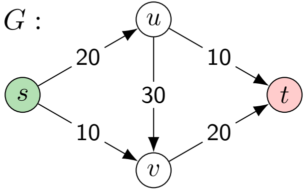

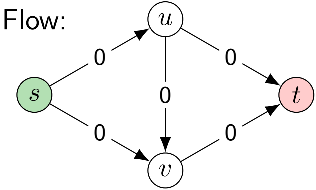

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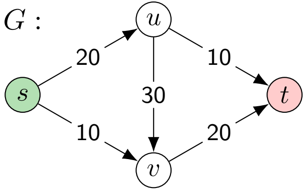

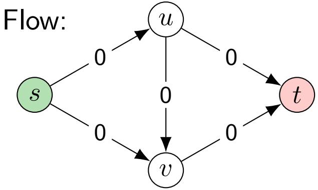

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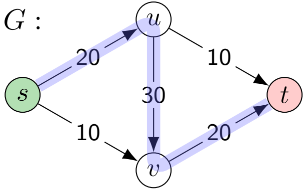

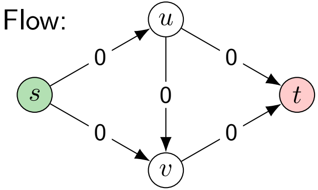

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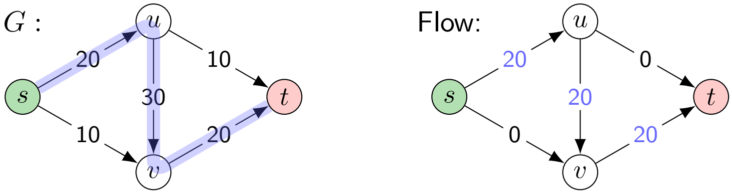

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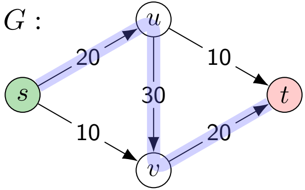

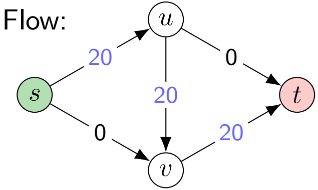

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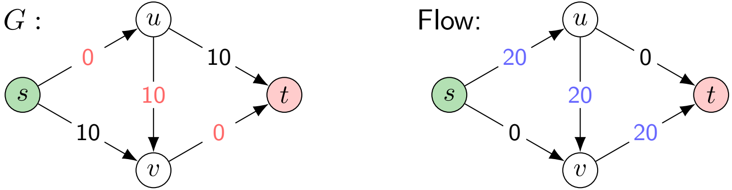

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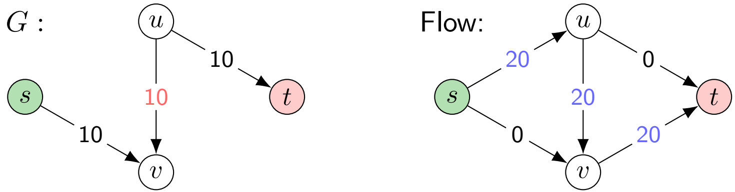

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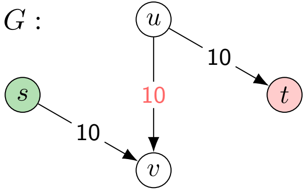

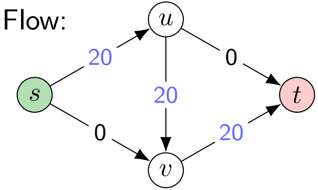

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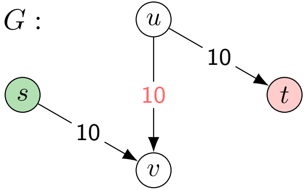

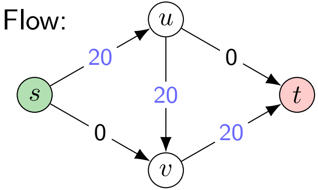

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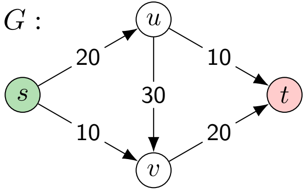

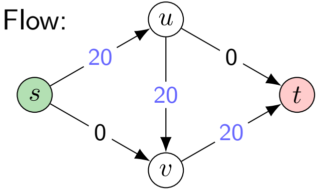

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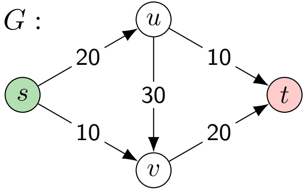

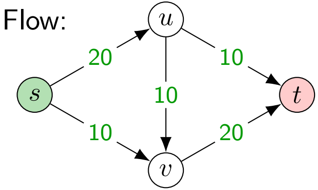

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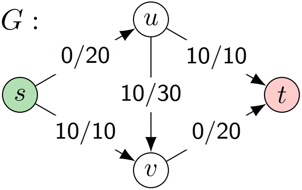
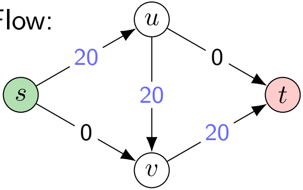


Flow:



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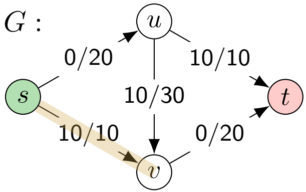

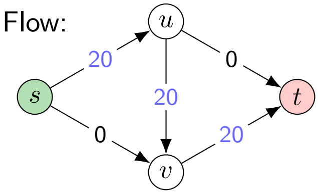

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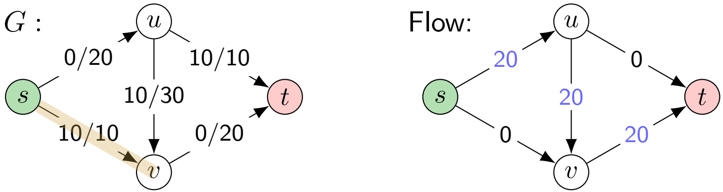

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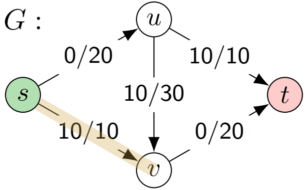

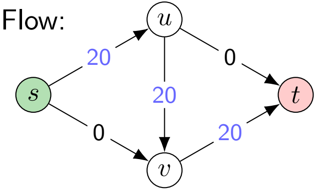

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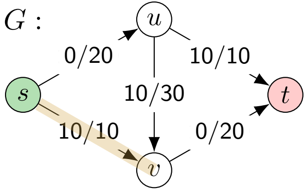

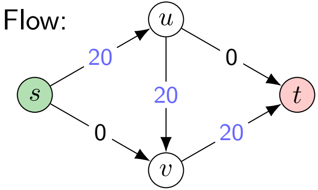

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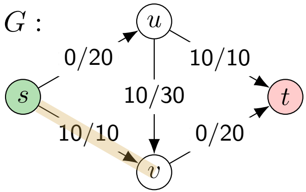

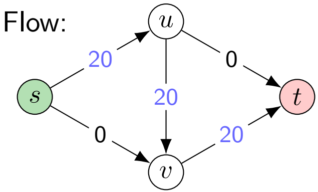

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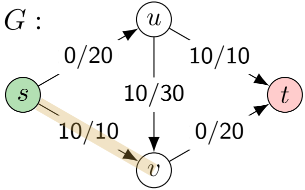

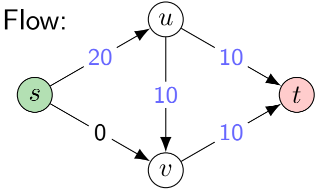

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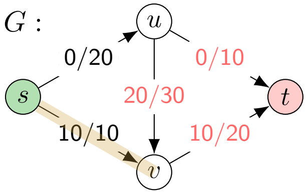

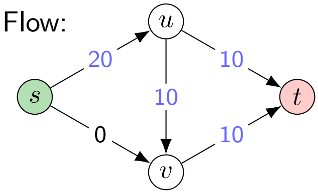

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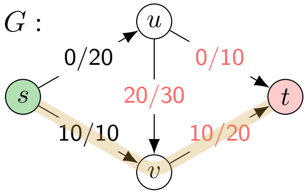

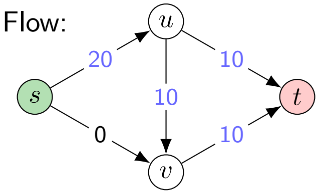

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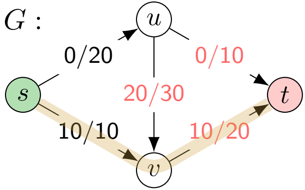

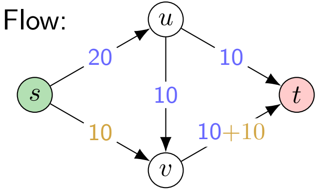

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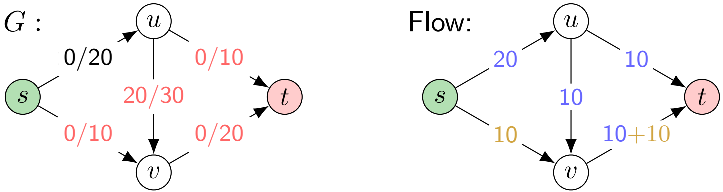

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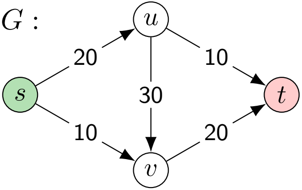
- For an edge $e = (u, v) \in E$ with $f(e) > 0$, we want to be able to "send back" the flow, so we include the reverse edge (v, u) in G_f with capacity $f(e)$ [backward edge].

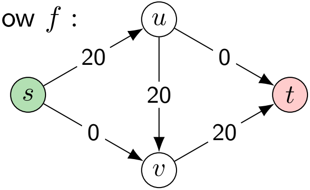
Residual graph

Example

G :



Flow f :



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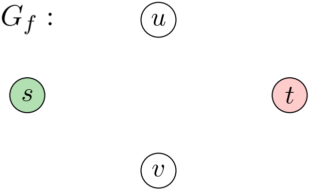

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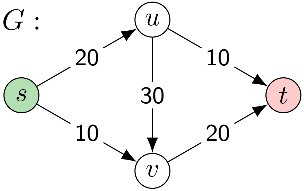
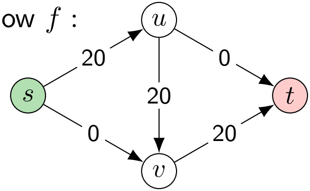
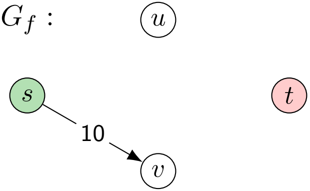


Flow f :



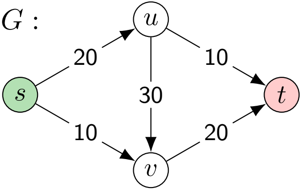
G_f :



Residual graph

Example

G :



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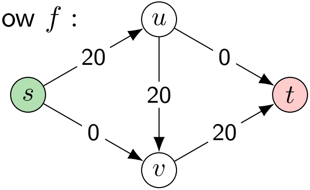

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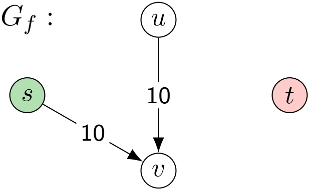

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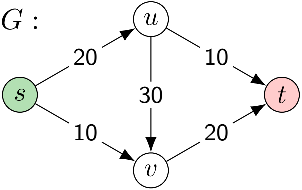

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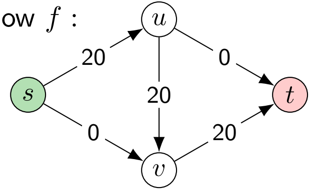

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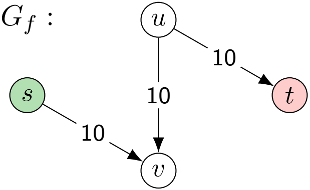

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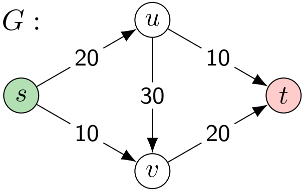

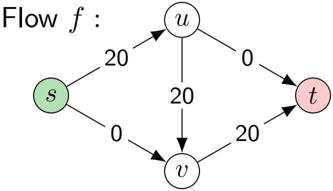

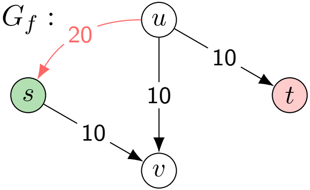

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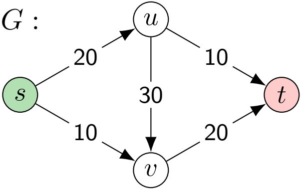

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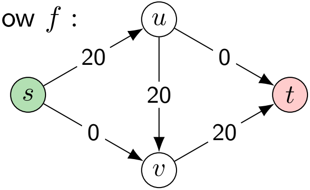

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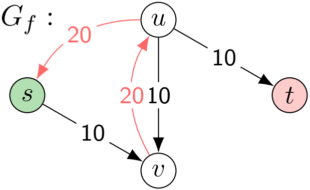
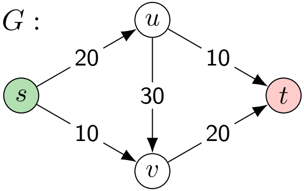
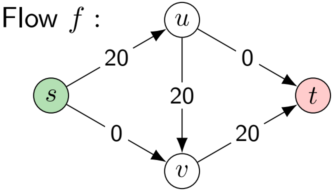
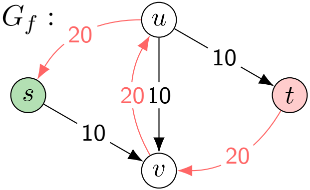


Residual graph

Example







Residual graph

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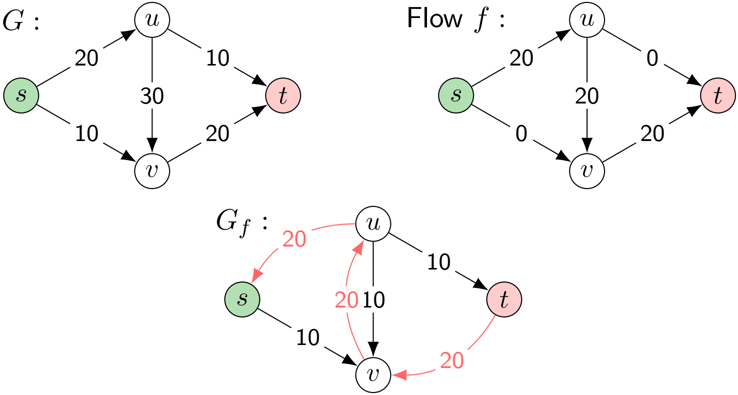

Key Insight An $s - t$ path in G_f can be used to improve the value of the flow.

Augmenting paths on residual graph

- Let P be an $s - t$ path in G_f.

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Augment(P, f)

$b = bottleneck(P, f)$
foreach *edge* $(x, y) \in P$ **do**
  if (x, y) *is a forward edge* **then**
    $f(x, y) = f(x, y) + b$
  if (x, y) *is a backward edge* **then**
    $f(x, y) = f(x, y) - b$
return *f*

The Ford-Fulkerson algorithm

Input: Flow network $G = (V, E, c)$
Output: Maximum flow f
Ford-Fulkerson(G)

- Initialize $f(e) = 0$ for all $e \in E$
- $G_f = G$
- **while** *there is an* $s - t$ *path* P *in* G_f **do**
  - $f =$ Augment(P, f)
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Time complexity?