CMPSC 465: LECTURE XX

Max-flow Min-cut Theorem

Ke Chen

October 15, 2025

The Ford-Fulkerson algorithm

Input: Flow network $G = (V, E, c)$
Output: Maximum flow f

Augment(P, f)

- $b = bottleneck(P, f)$
- **foreach** *edge* $(x, y) \in P$ **do**
  - **if** (x, y) *is a forward edge* **then** $f(x, y) = f(x, y) + b$
  - **if** (x, y) *is a backward edge* **then** $f(x, y) = f(x, y) - b$
- **return** f

Ford-Fulkerson(G)

- Initialize $f(e) = 0$ for all $e \in E$
- $G_f = G$
- **while** *there is an* $s - t$ *path* P *in* G_f **do**
  - $f =$ Augment(P, f)
  - Build new residual graph G_f
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Correctness? Time complexity?

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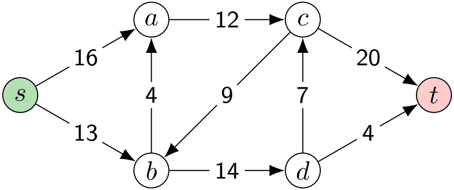

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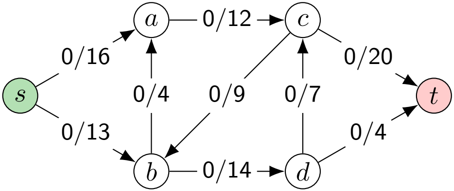

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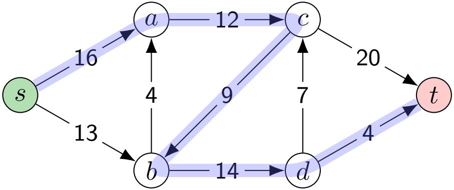

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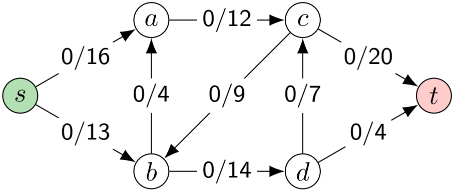

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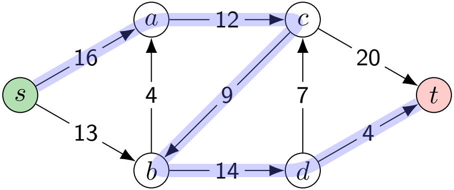

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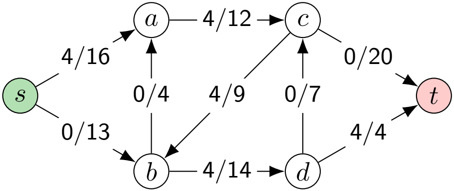

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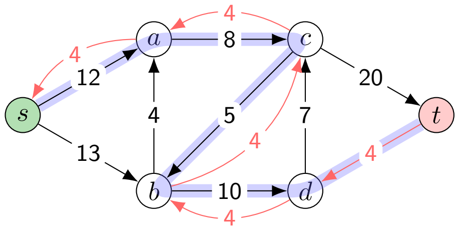

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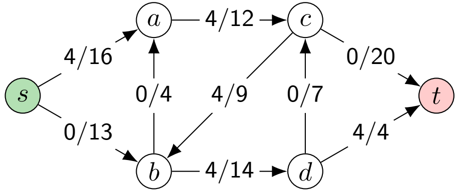

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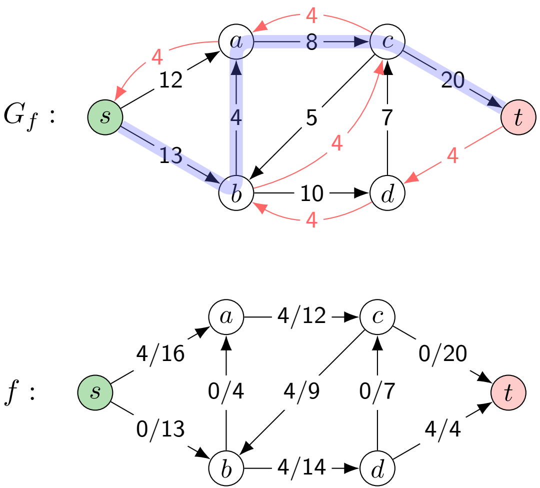

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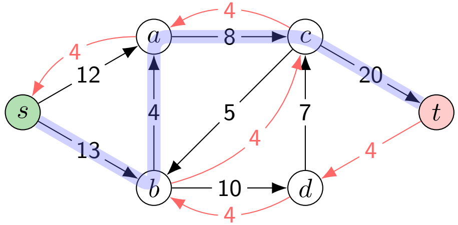

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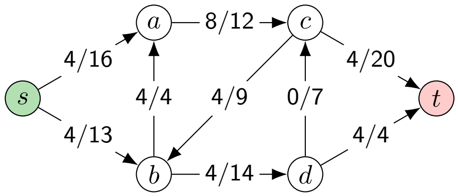

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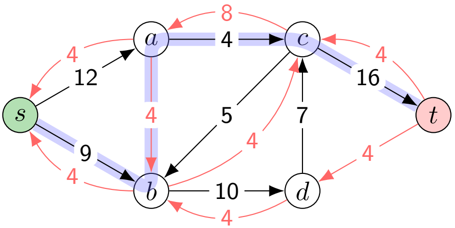

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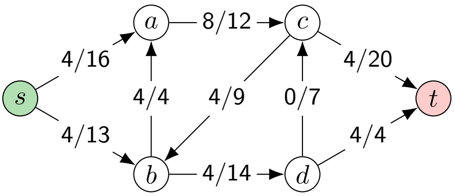

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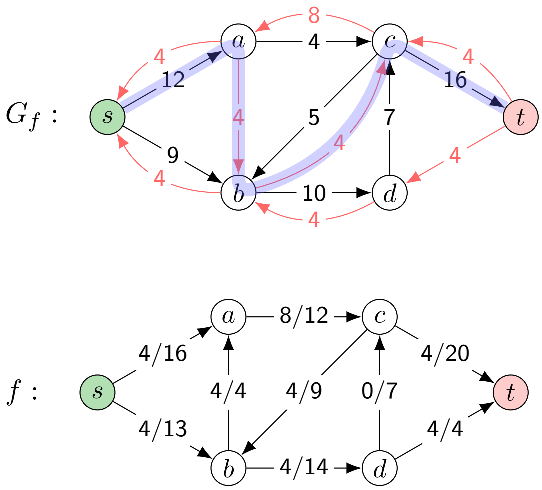

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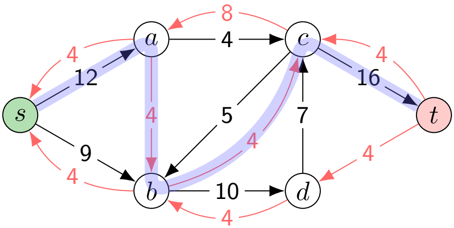

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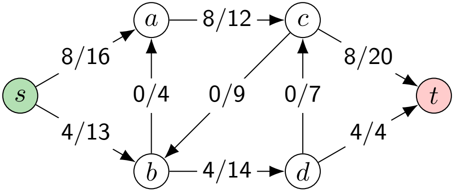

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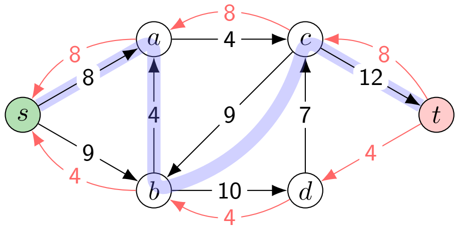

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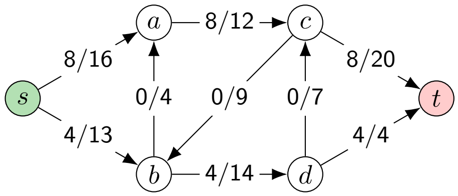

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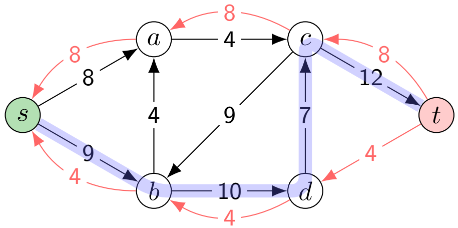

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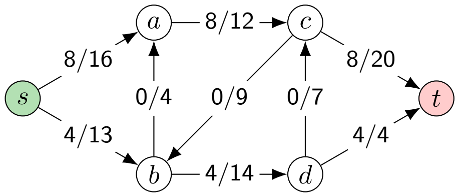

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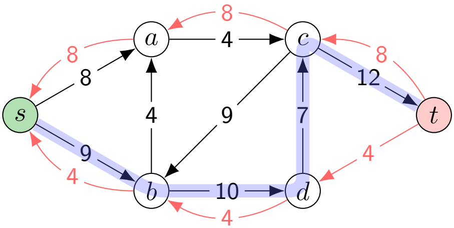

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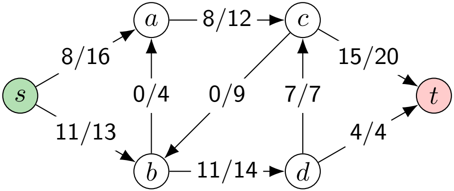

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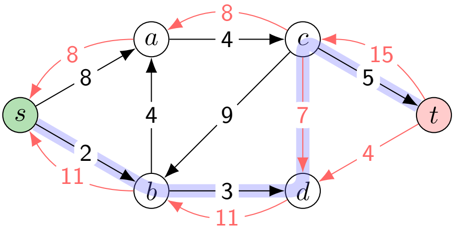

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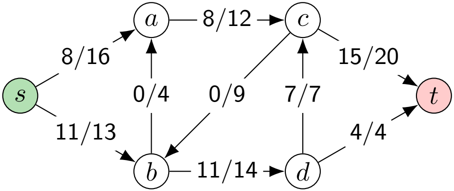

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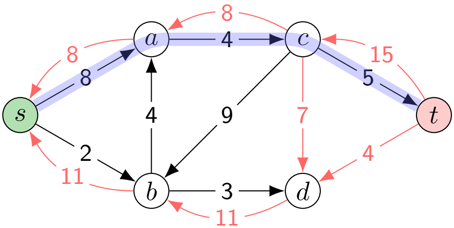

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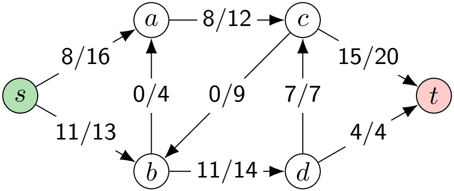

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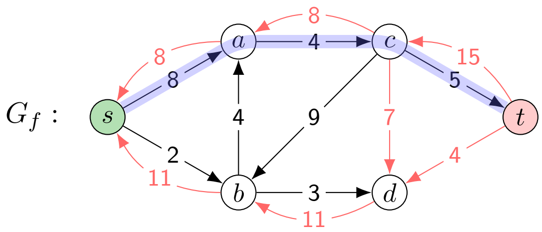

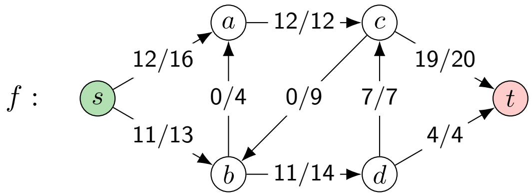

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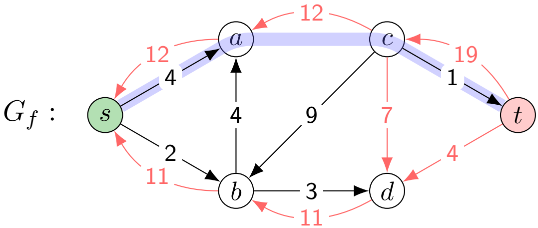

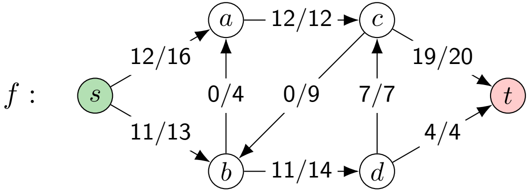

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Capacity condition holds by the construction of the residual graph G_f.

- $e \notin P$, then the flow on e does not change, $f'(e) = f'(e) \leq c(e)$.
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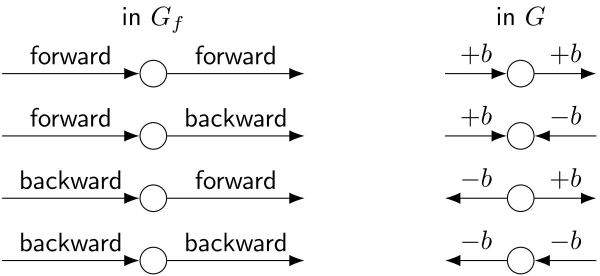
Flow conservation has four possible cases for vertices in P:



□

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Not polynomial in the input size!

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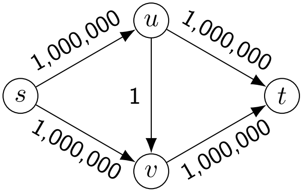

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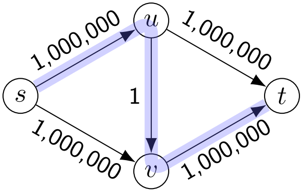

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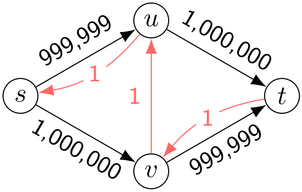

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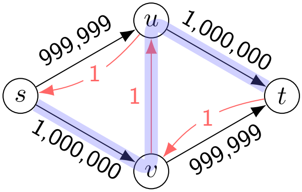

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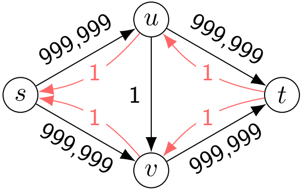

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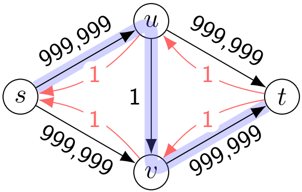

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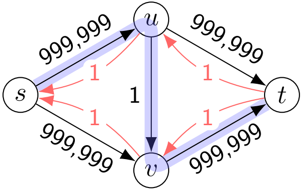

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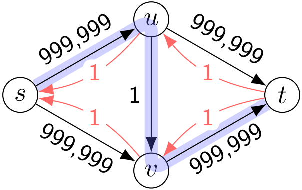

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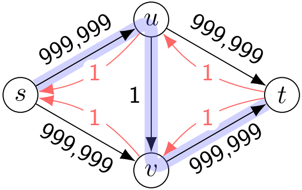

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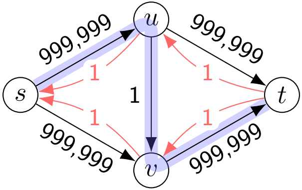

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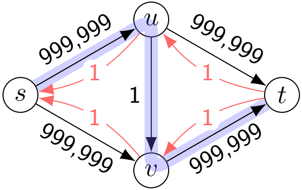

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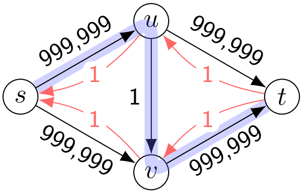

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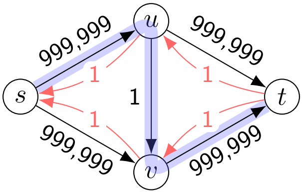

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- And many more...

Cuts in graph

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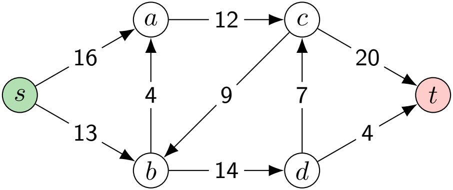

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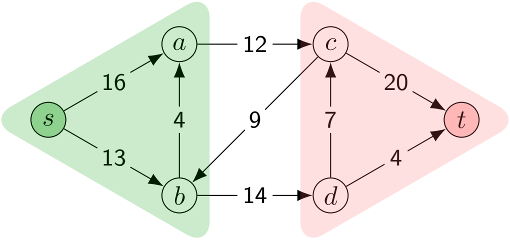

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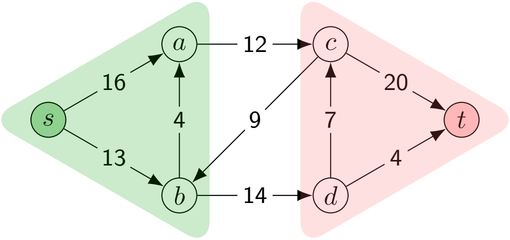

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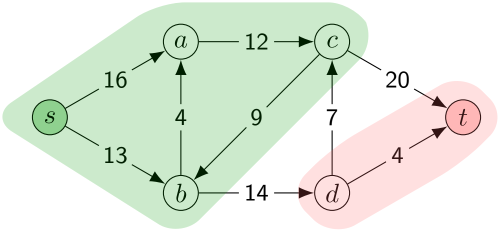

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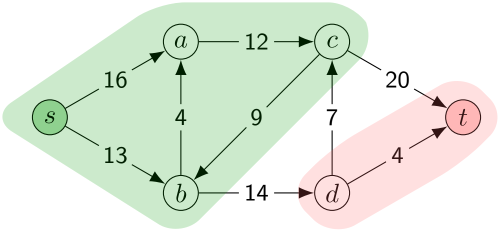

$$c(S, T) = 34$$

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Therefore,

$$c(S, T) = \sum_{e \text{ out of } S} c(e) = \sum_{e \text{ out of } S} f(e) - \sum_{e \text{ into } S} f(e) = v(f).$$

$\square$

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