CMPSC 360	Discrete Mathematics for Computer Science	
Fall 2024	Mahfuza Farooque	Worksheet 7

1. a) Given two functions f and g such that $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{a}{x^4+2}$ and $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = bx - 1$. Determine the values of constants a and b such that:
$\begin{cases} g \circ Id_{\mathbb{R}}(0) = f \circ Id_{\mathbb{R}}(0) \\ g \circ Id_{\mathbb{R}}(1) = f \circ Id_{\mathbb{R}}(1) \end{cases}$, where $Id_{\mathbb{R}}(x) = x$ for all $x \in \mathbb{R}$.

   b) Given two functions f and g such that $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = ae^x + b$ and $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = cx^2 + |x| + 1$. Determine the values of constants a, b and c such that:
$\begin{cases} g \circ Id_{\mathbb{R}}(0) = f \circ Id_{\mathbb{R}}(0) \\ g \circ Id_{\mathbb{R}}(1) = f \circ Id_{\mathbb{R}}(1) \\ g \circ Id_{\mathbb{R}}(2) = f \circ Id_{\mathbb{R}}(2) \end{cases}$, where $Id_{\mathbb{R}}(x) = x$ for all $x \in \mathbb{R}$.

Answer:

(a) $g(0) = f(0) \implies a = -2, \quad g(1) = f(1) \implies \frac{a}{3} = b - 1 \implies \frac{-2}{3} + 1 = b = \frac{1}{3}$

(b) $g(0) = f(0) \implies a + b = 1$
$g(1) = f(1) \implies ae + b = c + 2$
$g(2) = f(2) \implies ae^2 + b = 4c + 3$
$\implies a = \frac{6}{e^2-4e+3}, b = 1 - \frac{6}{e^2-4e+3}, c = \frac{6e-6}{e^2-4e+3} + 1$

2. Evaluate the following:

(a) $(\sum_{x=1}^{12} \frac{1}{x+6})(\prod_{y=1}^{17} -4y + y^2 - 21)$

(b) $\left(\prod_{m=1}^{5} m^8\right)^{\frac{1}{4}}$

(c) $\sum_{x=1}^{17}(x+3) - \sum_{x=1}^{19}(x+9)$

Answer:

(a) $(\sum_{x=1}^{12} \frac{1}{x+6})(\prod_{y=1}^{17}(y-7)(y+3))$
$(\sum_{x=1}^{12} \frac{1}{x+6})((1-7)(1+3)\dots\mathbf{(7\text{-}7)}(7+3)\dots(17-7)(17+3))$
$= 0$

(b) $(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5)^2 = 14400$

(c) $\sum_{x=1}^{17} -6 - \sum_{x=18}^{19} x + 9$
$-102 - 27 - 28 = -157$

3. Use $\sum$ notation and/or $\prod$ notation to rewrite the following sums and/or products.

(a) $x_1y_1^4 + x_2(y_1^4 - y_2^4) + x_3(y_1^4 - y_2^4 + y_3^4)$

(b) $\frac{2}{2} + \frac{2^2}{2(2+1)} + \frac{2^3}{2(2+1)(2+2)} + \dots + \frac{2^n}{2(2+1)(2+2)\dots(2+(n-1))}$

(c) $\left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right)\left(1 + \frac{1}{3^2}\right)\dots\left(1 + \frac{1}{n^2}\right)$

(d) $1-\left(\frac{1}{2}\cdot\frac{3}{2}\right)+\left(\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\right)-\ldots+(-1)^{n+1}\left(\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\ldots\frac{(2n-1)}{2}\right)$

(e) $1-\frac{2}{1!}+\frac{3^2}{2!}-\frac{4^3}{3!}+\cdots+(-1)^{n+1}\frac{n^{n-1}}{(n-1)!}$

Answer:

(a) $\sum_{i=1}^{3} x_i(\sum_{j=1}^{i}(-1)^{j+1}y_j^4)$

(b) $\sum_{k=1}^{n}\frac{2^k}{\prod_{j=0}^{k-1}(2+j)}$

(c) $\prod_{k=1}^{n}\left(1+\frac{1}{k^2}\right)$

(d) $0.5+\sum_{k=0}^{n}(-1)^k\prod_{j=0}^{k}\frac{2j+1}{2}$

(e) $\sum_{k=1}^{n}(-1)^{k+1}\frac{k^{k-1}}{(k-1)!}$

4. Prove using induction that $n! > 3^n$ for all natural numbers $n \geq 7$.

Answer:

We Proceed by induction on n.
Let $P(n)$ be $n! > 3^n, n \geq 7$
Base Case: $n = 7$
LHS of $P(7) = n! = 7! = 5040$
RHS of $P(7) = 3^7 = 2187$
Therefore, P(7) is true as $5040 > 2187$.

Inductive Hypothesis:(n=k)
Assume that $P(n)$ is true for some $n = k$ such that $k \geq 7$.
i.e., $P(k) = k! > 3^k$

Inductive Step:(n=k+1)
We need to show that $P(k+1)$ is true, which means $(k+1)! > 3^{(k+1)}$
We know that,

$$\begin{aligned} 3^{(k+1)} &= 3^k \times 3 \\ &< 3 \times k! && \text{(By Induction Hypothesis)} \\ &< (k+1) \times k! && (\text{since } n \geq 7, k \geq 7) \\ &= (k+1)! \end{aligned}$$

Therefore, we can say that $(k+1)! > 3^{(k+1)}$.

Hence $P(k+1)$ is true.

Therefore, P(n) is true for all $n \in \mathbb{N}$, $n \geq 7$ by Induction.

5. Using induction, prove that for any $n \in \mathbb{N}$, $1^3+2^3+3^3+\cdots+n^3 = \frac{1}{4}n^2(n+1)^2$

Answer:

Let $P(n) : 1^3+2^3+3^3+\cdots+n^3 = \frac{1}{4}n^2(n+1)^2$
Base Case is $P(1)$
LHS of $P(1) = 1^3 = 1$
RHS of $P(1) = \frac{1}{4}1^2(1+1)^2 = 1$

$\therefore P(1)$ is true.

Inductive Hypothesis:
Now, assume $P(k)$ is true for some natural number k, i.e.
$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{1}{4}k^2(k+1)^2$

Inductive Step:
Need to show $P(k+1)$ is true. First, let us define $P(k+1)$:
$P(k+1) : 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 = \frac{1}{4}(k+1)^2(k+1+1)^2$
LHS of $P(k+1)$ = $1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$
Applying the Inductive hypothesis to the above equation, we get
$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$
$= \frac{1}{4}(k+1)^2(k^2 + 4(k+1))$
$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$
$= \frac{1}{4}(k+1)^2(k+2)^2$
$= \frac{1}{4}(k+1)^2(k+1+1)^2 =$ RHS of $P(k+1)$
$\therefore P(k+1)$ is true if $P(k)$ is true.
$\therefore P(n)$ is true for all natural numbers n by Principle of Mathematical Induction.
Q.E.D.

6. Prove by induction that $6^n + 4$ is divisible by 5 for all $n \in N$.

Answer:

We proceed by induction on n.
Let $P(n)$ be $6^n + 4$ is divisible by 5, $n \in N$
Base Case: $(n = 1)$
$P(1) = 6^1 + 4 = 10$ which is divisible by 5
Therefore, $P(1)$ is true.

Inductive Hypothesis:(n=k)
Assume that $P(n)$ is true for any arbitrary $n = k$ such that $k \in \mathbb{N}$.
i.e., $6^k + 4$ is divisible by 5. By the definition of divides $6^k + 4 = 5m$, $m \in N$
So, $6^k = 5m - 4$.

Inductive Step:(For n=k+1)
We need to show that $P(k+1)$ is true. That means $5|6^{k+1} + 4$ using the definition of P(k).
To do that,
$6^{k+1} + 4 = 6 \times 6^k + 4$
$=6 \times (5m-4) + 4$
$=30m + 20 = 5(6m+4)$
$= 5p$ where $p = 6m+4, p \in N$

Hence, P(k+1) is divisible by 5.
Therefore, $P(n)$ is true for all $n \in \mathbb{N}$ by Induction.
Q.E.D.