CMPSC 465: LECTURE XVI

Revisit Dijkstra's Algorithm

Ke Chen

October 06, 2025

Dijkstra is weighted BFS

	BFS	Dijkstra
What		
Why		
How		

Dijkstra is weighted BFS

	BFS	Dijkstra
What	Visit the unvisited node with the smallest number of links from s	
Why		
How		

Dijkstra is weighted BFS

	BFS	Dijkstra
What	Visit the unvisited node with the smallest number of links from s	Add the unvisited node with the shortest distance from s to R
Why		
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Dijkstra is weighted BFS

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What	Visit the unvisited node with the smallest number of links from s	Add the unvisited node with the shortest distance from s to R
Why	With more iterations, the number of links from s can only increase	
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How	Use queue (FIFO, earlier in queue, fewer links needed)	

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What	Visit the unvisited node with the smallest number of links from s	Add the unvisited node with the shortest distance from s to R
Why	With more iterations, the number of links from s can only increase	With more iterations, the shortest distance from s can only increase (positive edge weights!)
How	Use queue (FIFO, earlier in queue, fewer links needed)	Use priority queue

Dijkstra with priority queue

```
Dijkstra(G = (V, E, ℓ), s)
    // dist stores distances from s
    // prev can be used to reconstruct paths
    foreach v ∈ V do  dist[v] = ∞, prev[v] = null
    dist[s] = 0, Insert(Q, {dist[s], s}) // Q is a priority queue

    while Q is not empty do
        v = GetMin(Q), Delete(Q, 0)
        foreach (v, w) ∈ E do
            if dist[w] > dist[v] + ℓ(v, w) then
                dist[w] = dist[v] + ℓ(v, w)
                prev[w] = v

                // pos[w] returns the index of w in Q
                if w in Q then  DecreaseKey(Q, pos[w], dist[w])
                else  Insert(Q, {dist[w], w})
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Dijkstra with priority queue

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Time complexity?

Dijkstra with priority queue

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```
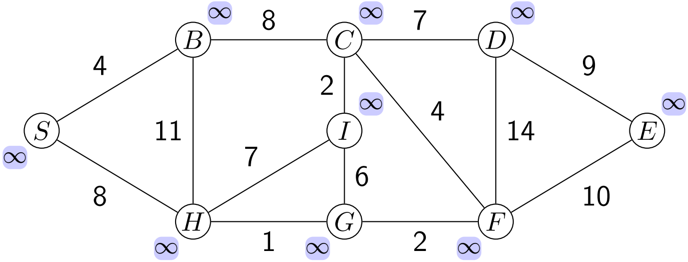

Time complexity? Calls at most $|V|$ Insert/Delete, and at most $|E|$ DecreaseKey, with a binary min-heap, $O\left((|V| + |E|)\log|V|\right)$.

One more example

G :



Q :

[empty]

One more example

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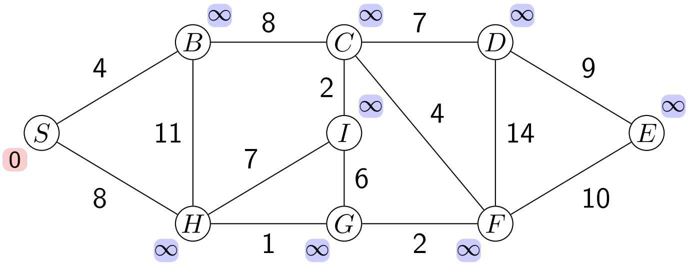

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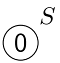

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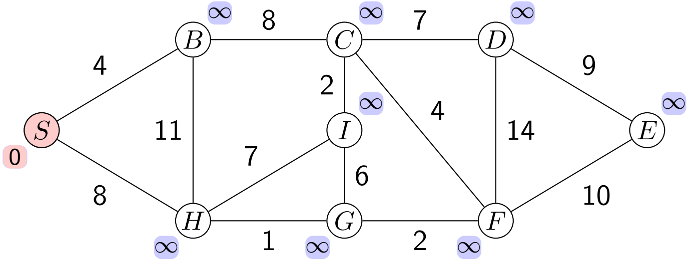

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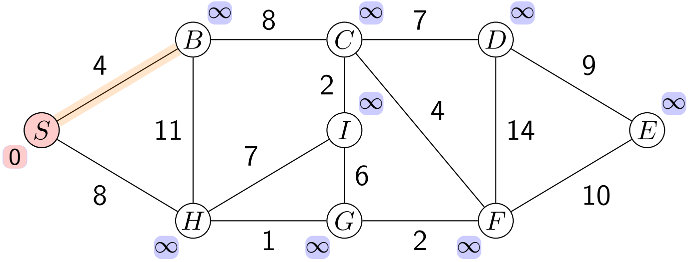

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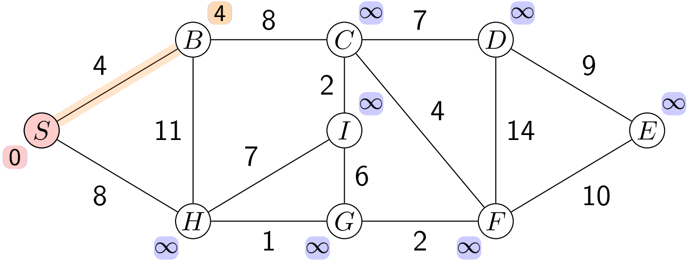

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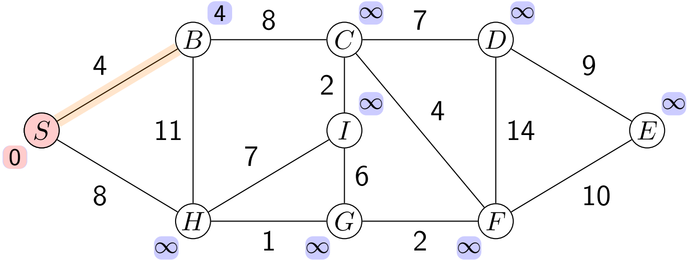

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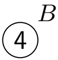

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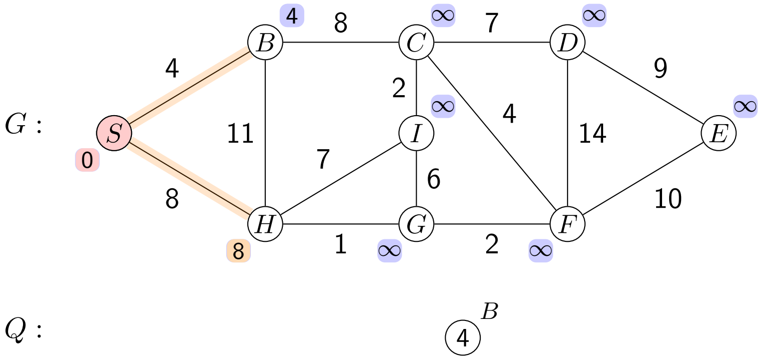

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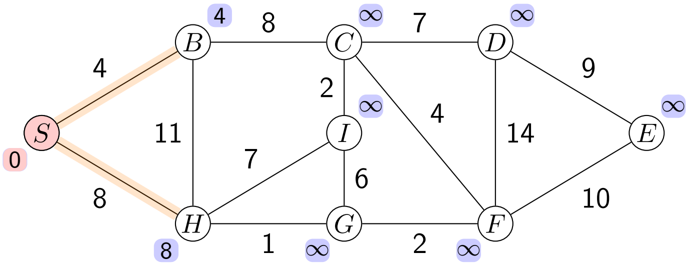

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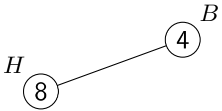

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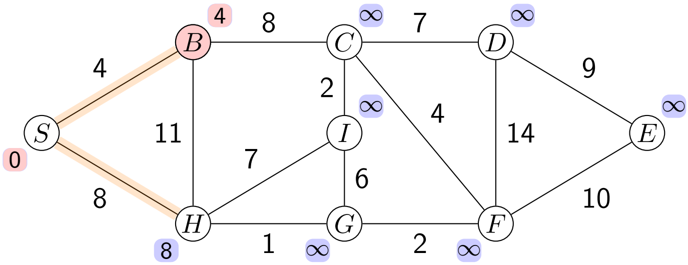

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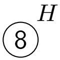

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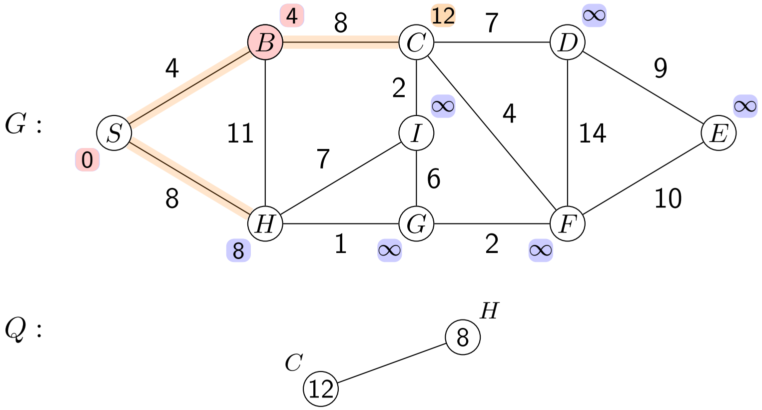

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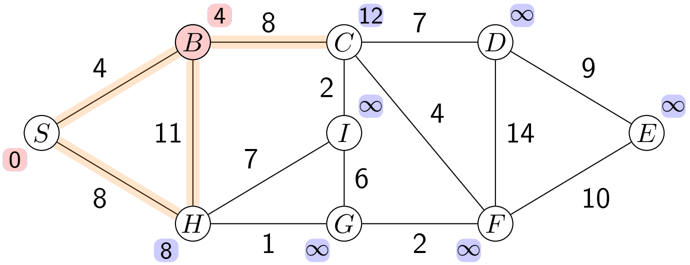

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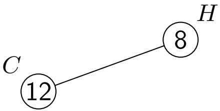

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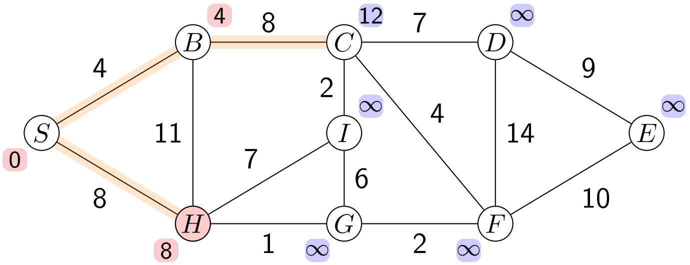

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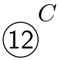

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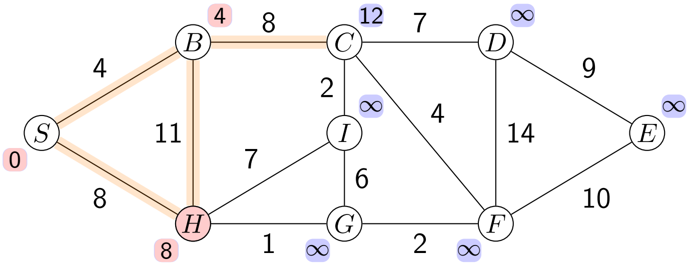

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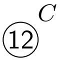

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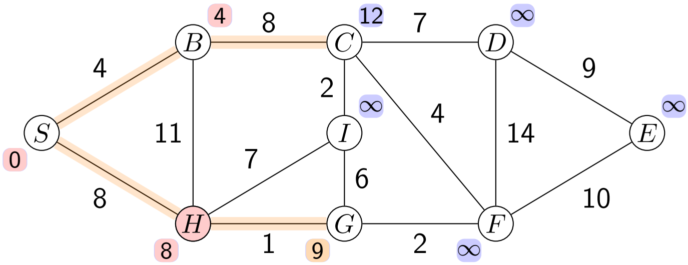

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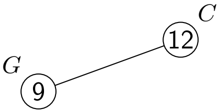

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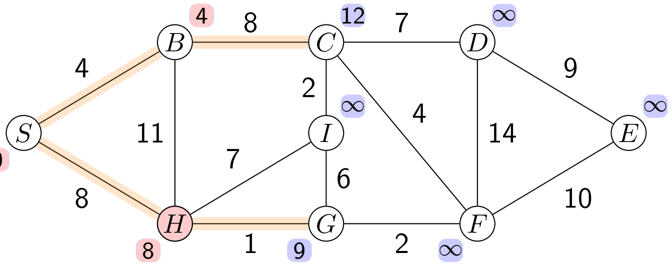

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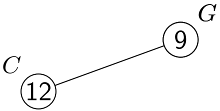

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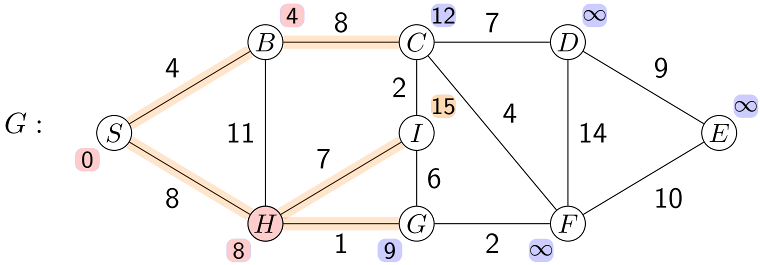

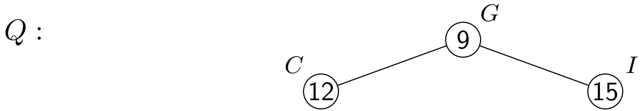

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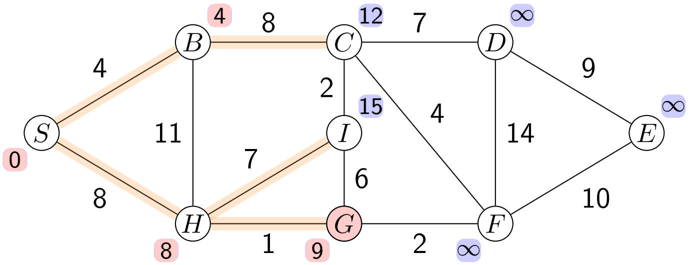

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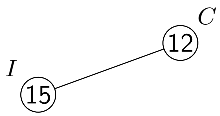

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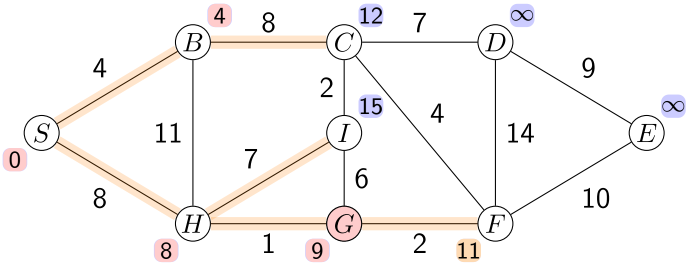

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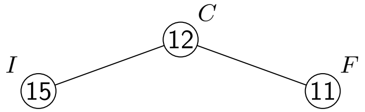

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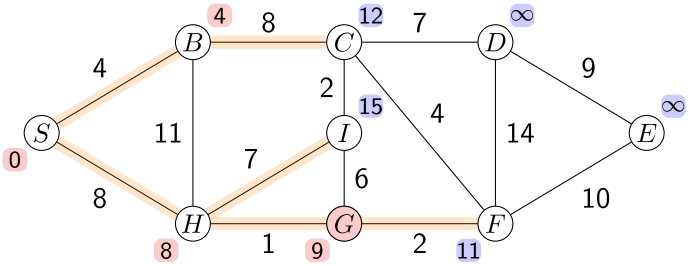

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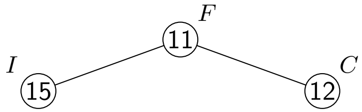

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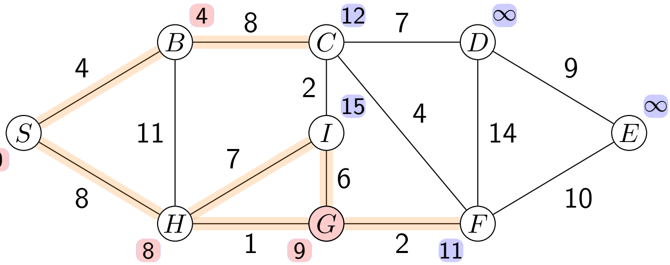

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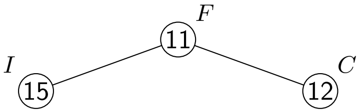

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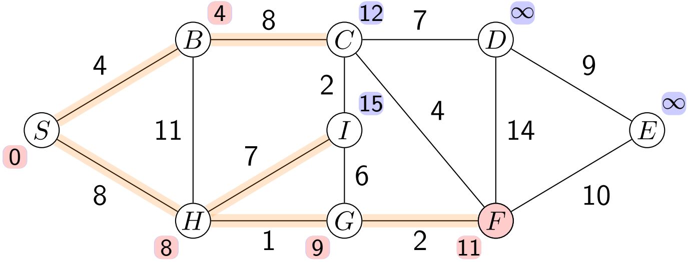

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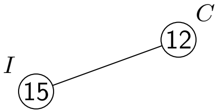

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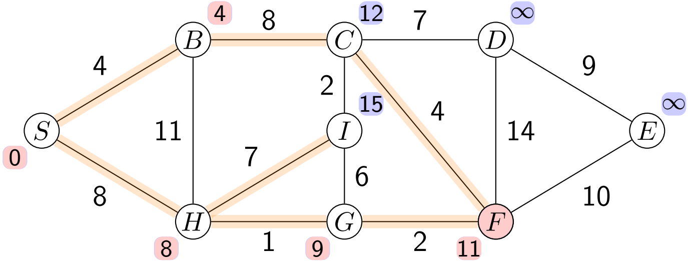

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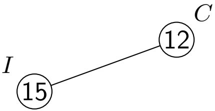

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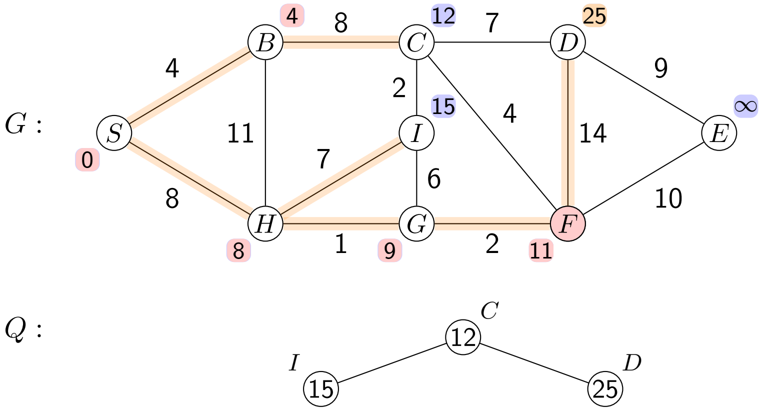

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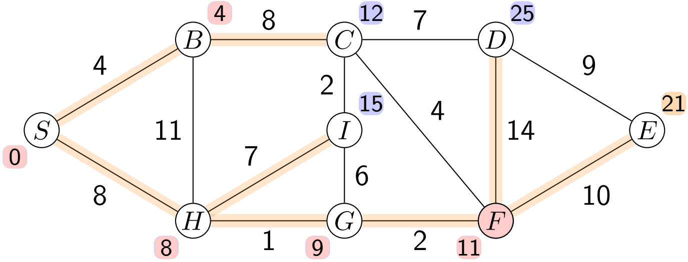

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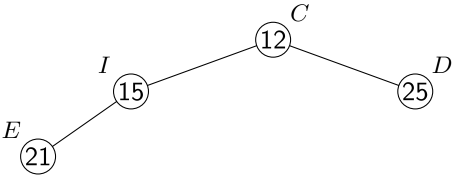

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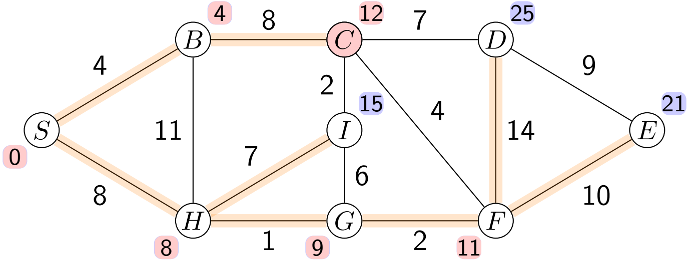

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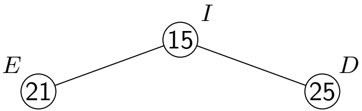

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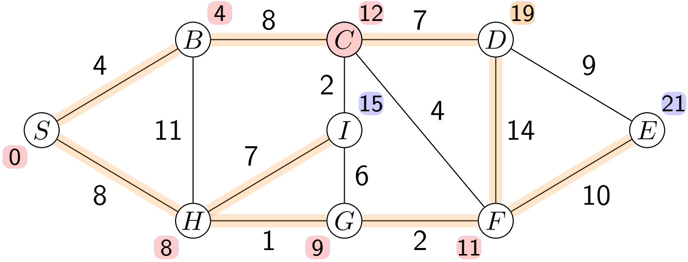

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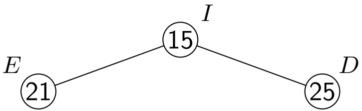

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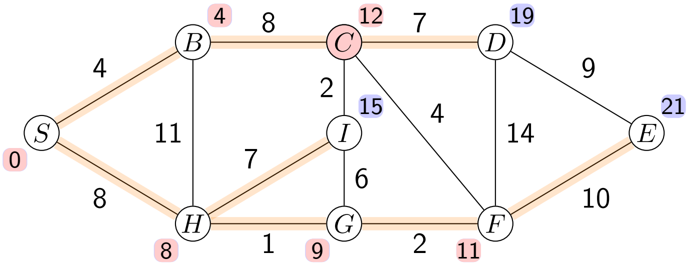

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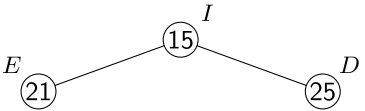

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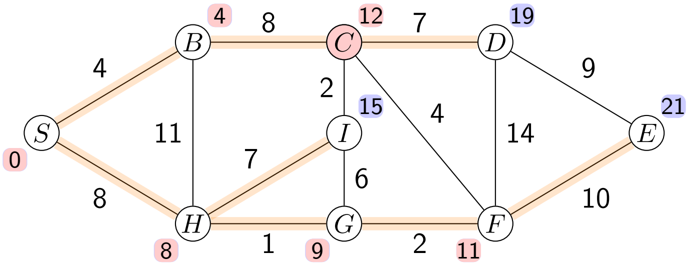

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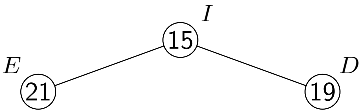

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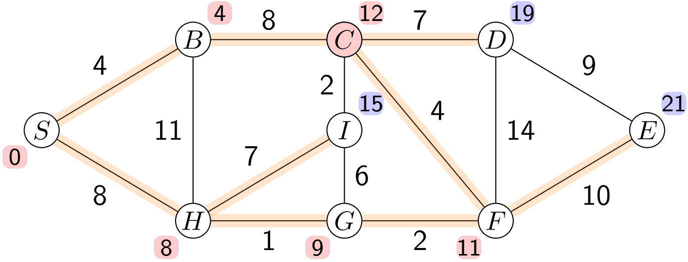

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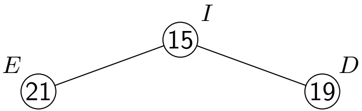

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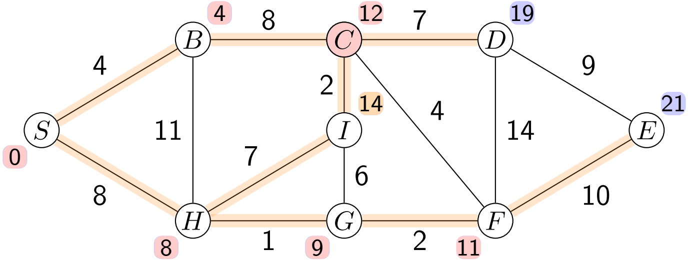

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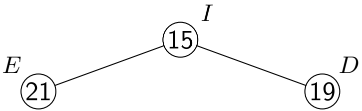

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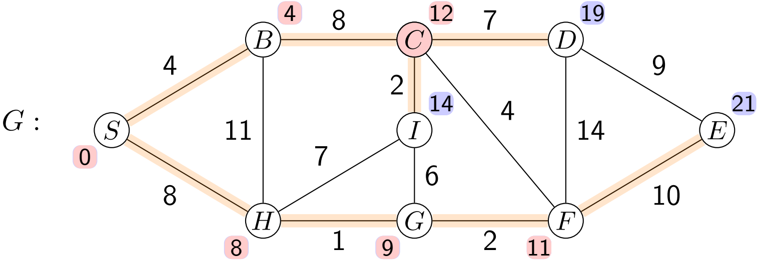

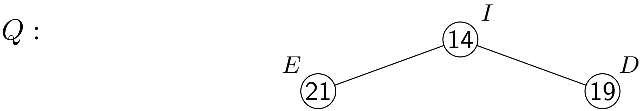

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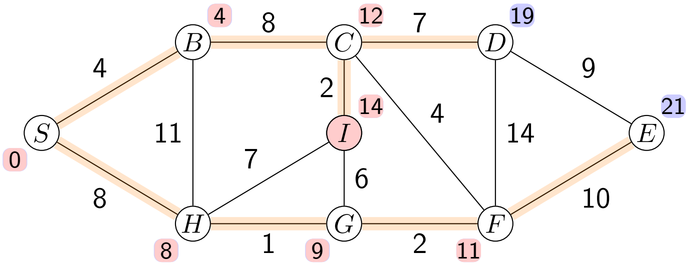

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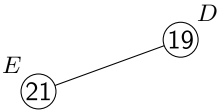

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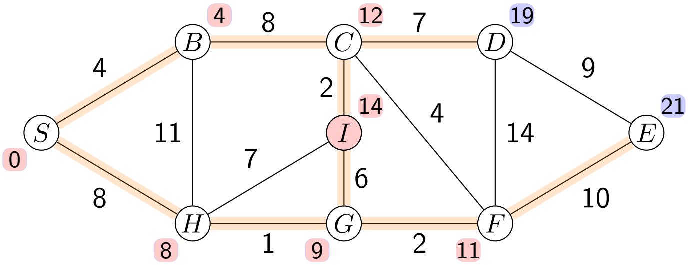

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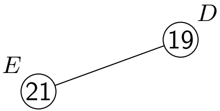

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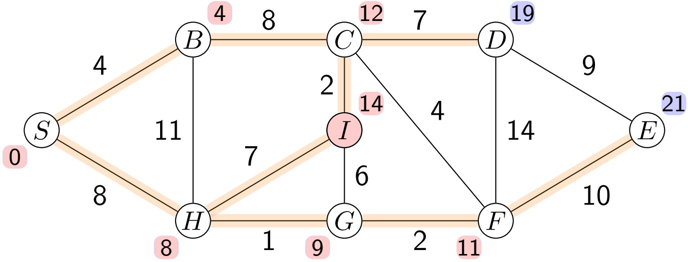

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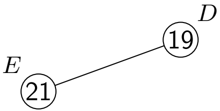
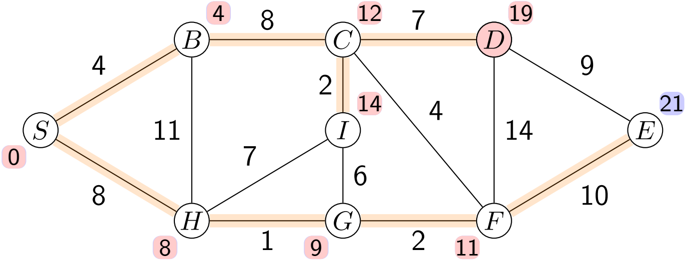


One more example

G :



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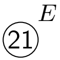

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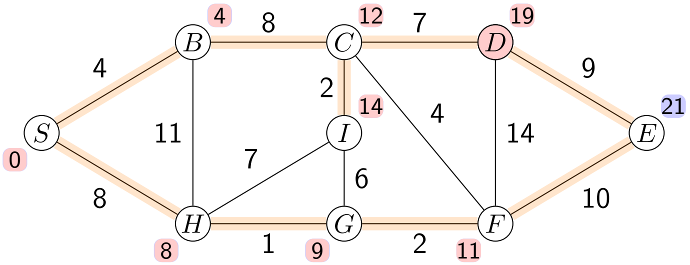

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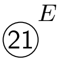

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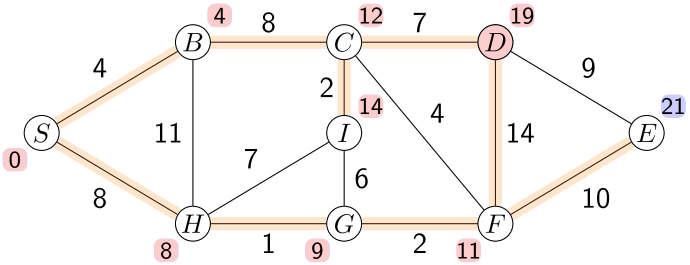

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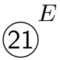

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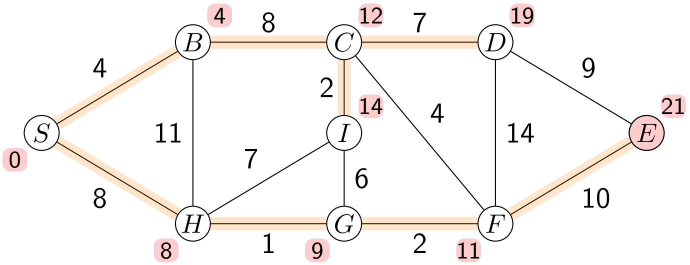

Q :

[empty]

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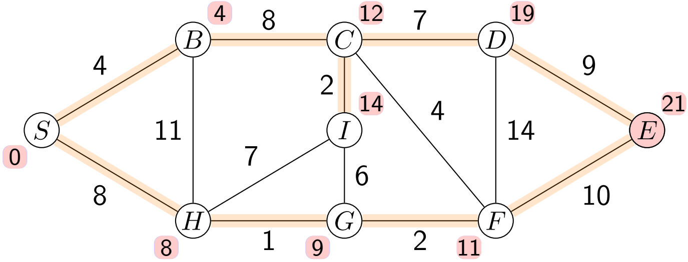

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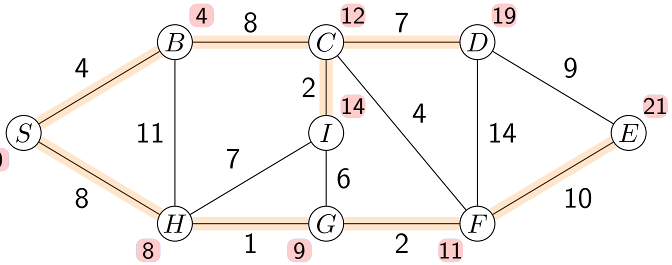

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[empty]

Dijkstra with priority queue

Time complexity?

- Calls at most $|V|$ Insert and Delete, and at most $|E|$ DecreaseKey, with a binary min-heap, $O\left((|V| + |E|) \log |V|\right)$.

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- A breakthrough this year by Duan et al. won the best paper award at STOC 2025.

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- Dijkstra's algorithm can be viewed as a clever sequence of Update operations.

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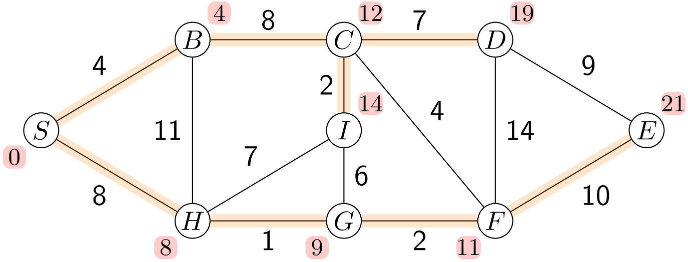

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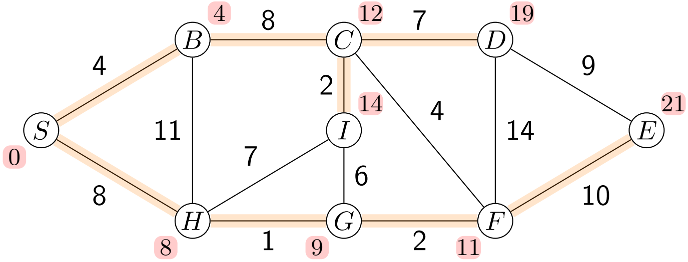

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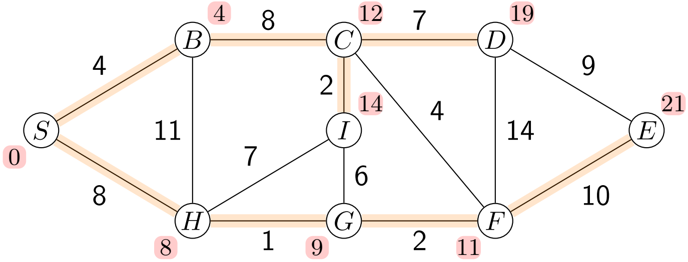

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- Note that having additional Update calls doesn't hurt – Update is safe.