

# Stability criterion for a two-neuron reciprocally coupled network based on the phase and burst resetting curves

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## Abstract

Experimental and theoretical studies have shown that the amount of phase resetting produced in a neural oscillator by a synaptic input depends not only on the stimulus timing (phase), but also on its intensity and duration. To eliminate the influence of the stimulus intensity on the amount of the phase resetting we studied only strong synaptic couplings. We defined the burst resetting curve (BRC) as the change in burst duration in response to an input at various phases in the open loop condition. We combined the newly defined BRC with the phase resetting curve (PRC) to derive a stability criterion for a two-neuron network of bursting neurons.

**Keywords:** Phase resetting curve, Bursting, Phase-locking, Stability

## 1. Introduction

We previously developed analytical and numerical methods to analyze ring networks using the PRC method [1,2,3,7]. In an open loop experiment, a single stimulus is applied at a given time ( $t_s$ ), or phase  $\varphi = t_s/P_0$ , during the ongoing rhythm and the effect on the cycle period is tabulated as a phase resetting curve. The stimulus is a free-running burst in the presynaptic neuron. The beginning of each cycle is measured from the burst onset (phase zero). The intrinsic period ( $P_0$ ), and the intrinsic burst duration ( $b_0$ ) can be transiently lengthened or shortened by a hyperpolarizing perturbation to the values  $P_1$  and  $b_1$ , and the duration of the second cycle may be perturbed as well, often due largely to the change in burst duration (Fig. 1A). We define the first order effects of the perturbation by the first order PRC  $F_1(\varphi) = P_1/P_0 - 1$  [5,10], and the first order BRC  $G_1(\varphi) = b_1/b_0 - 1$ ; the second order PRC  $F_2(\varphi) = P_2/P_0 - 1$  (Fig. 1 B and C). If a significant change occurred in the burst duration of the first burst after the stimulus was applied ( $b_1$ ) compared to its

intrinsic value ( $b_0$ ) then the new burst duration is assumed to be the stimulus duration for the postsynaptic neuron in the closed loop condition.

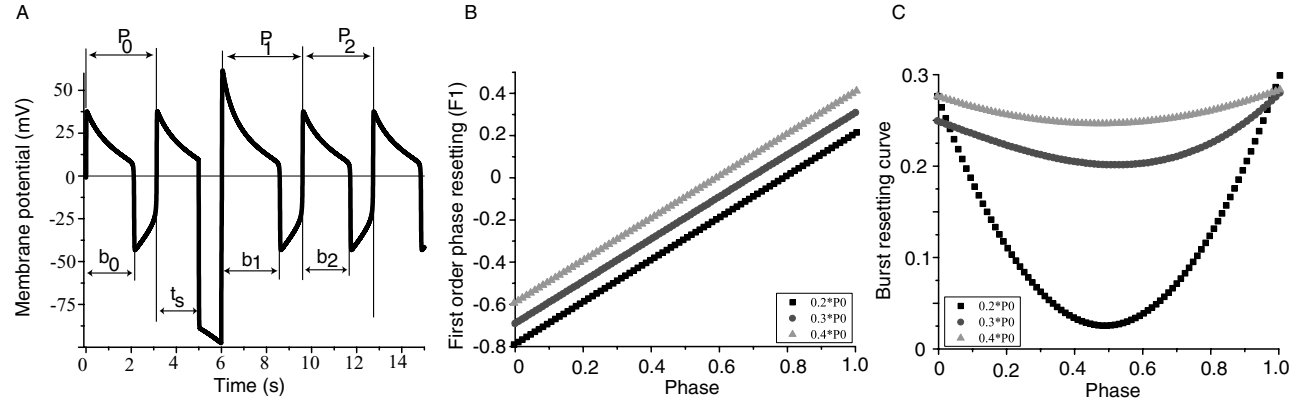


Fig. 1 Perturbed membrane potential trace for the Type II Morris-Lecar (ML) model neuron in an open loop experiment (A). The effect of the perturbation on the current cycle (B), and the subsequent burst duration (C) are shown for perturbation durations from 0.2 to 0.4 of the intrinsic period.

Our goal is to derive a stability criterion for the phase-locked mode(s) that includes the effect of the change in burst duration. For this purpose, we considered the simplest possible network: a two-neuron network with strong reciprocally inhibitory synapses (Fig. 2). In a two-neuron network, the stimulus time ( $t_s$ ) of one neuron equals the recovery time ( $t_r$ ) of the other. The stimulus time ( $t_s$ ) is defined as the time elapsed between the start of the burst in the reference neuron and the beginning of the input stimulus. The recovery time ( $t_r$ ) is the time elapsed between the beginning of the stimulus and the subsequent burst of the reference neuron. The type II regime of the ML model neuron is characterized by an extended spike duration that mimics the voltage envelope of a bursting neuron [4].

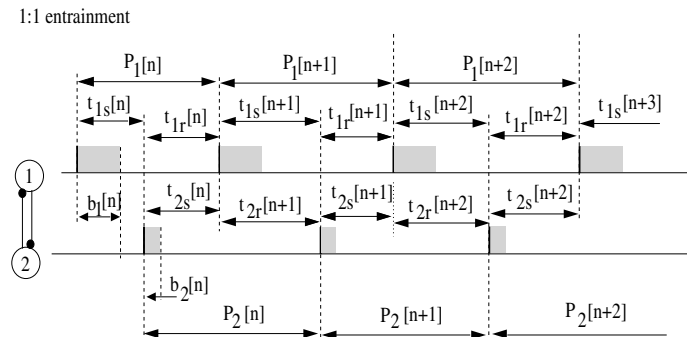


Fig. 2 Schematic representation of a two-neuron ring network (left) developing a 1:1 firing pattern (right). The burst onset is marked with a vertical thick black line and the burst duration for each neuron is indicated by a filled rectangle.

Experiments on bursting neurons have shown that under certain conditions the amount of the phase resetting depends on the stimulus phase and its duration [9]. Fig. 1C

shows that the burst duration can change significantly compared to its intrinsic value. Therefore, in order to predict the phase-locked mode(s), the effect of the change in the duration of the burst in the presynaptic neuron on the total phase resetting must be considered.

## 2. Method

Based on Fig. 2, the relationship between the stimulus and the recovery times of the two neurons in the  $n$ -th cycle is given by

$$\begin{aligned} t_{1s}[n] + t_{1r}[n] = P_1[n] &= P_{10} \left( 1 + F_{11} \left( \frac{t_{1s}[n]}{P_{10}}, \frac{b_2[n]}{P_{10}} \right) + F_{12} \left( \frac{t_{1s}[n-1]}{P_{10}}, \frac{b_2[n-1]}{P_{10}} \right) \right), \\ t_{2r}[n+1] + t_{2s}[n] = P_2[n] &= P_{20} \left( 1 + F_{21} \left( \frac{t_{2s}[n]}{P_{20}}, \frac{b_1[n]}{P_{20}} \right) + F_{22} \left( \frac{t_{2s}[n-1]}{P_{20}}, \frac{b_1[n-1]}{P_{20}} \right) \right), \end{aligned} \quad (1)$$

where  $\phi_i[n] = t_{is}[n]/P_{i0}$  is the normalized stimulus phase, and  $\tau_i[n] = b_i[n]/P_{i-10}$  is the normalized burst duration. Let  $(\phi_1^*, \tau_1^*, \phi_2^*, \tau_2^*)$  be the phase-locked mode for the discrete map (1) in the infinity limit  $n \rightarrow \infty$ . By assuming that the  $n$ -th cycle values are close to the steady solution we linearize the map (1). The linearized version of (1) is a first order recursion written in terms of the infinitesimal displacements in phase  $\Delta\phi_i[n]$  from the steady state in the  $n$ -th and  $(n-1)$ -th cycle. The characteristic equation of this recursion has the characteristic roots  $\lambda$ , which measure the speed of growth of small perturbations. A steady phase-locked mode  $(\phi_1^*, \tau_1^*, \phi_2^*, \tau_2^*)$  is stable if all characteristic roots are inside the unit circle ( $|\lambda| < 1$ ).

## 3. Results

In order to determine the stability of the steady solution (phase-locked mode), if it exists, we linearize the maps (1) around the steady state  $(\phi_1^*, \tau_1^*, \phi_2^*, \tau_2^*)$ . Let  $\Delta\phi_i[n]$  be the temporal phase difference, presumably infinitesimal, between the steady solution  $\phi_i^*$  of (1) and the actual stimulus phase  $\phi_i[n]$  in the  $n$ -th cycle. In a similar way we define the deviation of the normalized current burst duration  $\tau_i[n]$  from its steady value  $\tau_i^*$  by  $\Delta\tau_i[n]$ . A direct substitution of  $\phi_i[n] = \Delta\phi_i[n] + \phi_i^*$  and  $\tau_i[n] = \Delta\tau_i[n] + \tau_i^*$  into (1), and linearization around the steady state, leads to a system of two equations with four unknowns

$$\begin{aligned}
& P_{10} (-\Delta\phi_1 [n] + m_{11} \Delta\phi_1 [n] + h_{11}\Delta\tau_1 [n]) + P_{20} (-\Delta\phi_2 [n-1] + m_{21} \Delta\phi_2 [n-1] + h_{21}\Delta\tau_2[n-1]) \\
& \equiv P_{10} (m_{11} \Delta\phi_1 [n] + h_{11}\Delta\tau_1[n] + m_{12} \Delta\phi_1 [n-1] + h_{12}\Delta\tau_1 [n-1]), \\
& P_{10} (-\Delta\phi_1 [n] + m_{11} \Delta\phi_1 [n] + h_{11}\Delta\tau_1 [n]) + P_{20} (-\Delta\phi_2 [n] + m_{21} \Delta\phi_2 [n] + h_{21}\Delta\tau_2 [n]) \\
& \equiv P_{20} (m_{21} \Delta\phi_2 [n] + h_{21}\Delta\tau_2 [n] + m_{22} \Delta\phi_2 [n-1] + h_{22}\Delta\tau_2 [n-1]), \quad (2)
\end{aligned}$$

where  $m_{ij} = (\partial F_{ij} / \partial \phi)_\tau$  is the slope of the  $j$ -th order PRC for neuron  $i$  in the network when the phase is changing at a constant burst duration, and  $h_{ij} = (\partial F_{ij} / \partial \tau)_\phi$  is the slope of the  $j$ -th order PRC for neuron  $i$  in the network when the burst duration is changing at a given phase. To determine the slopes  $h$  we need the PRCs for different burst durations. The system (2) cannot be solved unless two more equations are added or additional assumptions are made. The two additional “equations” are the burst resetting curves (Fig. 1C). The only difference between the PRC and BRC is that the burst resetting curves measure the relative change in the burst duration as a result of the synaptic input

$$\begin{aligned}
b_1[n] &= b_{10} \left( 1 + G_{11} \left( \phi_1[n]; \frac{b_2[n]}{P_{10}} \right) + G_{12} \left( \phi_1[n-1]; \frac{b_2[n-1]}{P_{10}} \right) \right), \\
b_2[n] &= b_{20} \left( 1 + G_{21} \left( \phi_2[n]; \frac{b_1[n]}{P_{20}} \right) + G_{22} \left( \phi_2[n-1]; \frac{b_1[n-1]}{P_{20}} \right) \right), \quad (3)
\end{aligned}$$

where  $G_k = b/b_{k0} - 1$  is the relative burst duration change for the neuron labeled  $k$ . By linearizing the additional set of equations (3) we get

$$\begin{aligned}
\Delta\tau_2 [n] &\equiv (p_{11}\Delta\phi_1 [n] + q_{11} \Delta\tau_1 [n] + p_{12}\Delta\phi_1 [n-1] + q_{12}\Delta\tau_1 [n-1] + \dots ) b_{10}/P_{20}, \\
\Delta\tau_2 [n] &\equiv (p_{21}\Delta\phi_2 [n] + q_{21} \Delta\tau_2 [n] + p_{22}\Delta\phi_2 [n-1] + q_{22}\Delta\tau_2 [n-1] + \dots ) b_{20}/P_{10}, \quad (4)
\end{aligned}$$

where  $p_{ij} = (\partial G_{ij} / \partial \phi)_\tau$ ,  $q_{ij} = (\partial G_{ij} / \partial \tau)_\phi$ , and the normalized burst durations are  $\tau_2[n] = b_1[n]/P_{20}$ , respectively,  $\tau_1[n] = b_2[n]/P_{10}$ . The characteristic equation associated with the linearized equations (2) and (4) was obtained using the Z-transform method:

$$\begin{pmatrix} P_{i1} & 0 & 0 & 0 \\ P_{10}(1-m_{11}) & P_{20} & -P_{10}h_{11} & 0 \\ p_{11}\frac{b_{10}}{P_{20}} & 0 & q_{11}\frac{b_{10}}{P_{20}} & -1 \\ 0 & p_{21}\frac{b_{20}}{P_{10}} & -1 & q_{21}\frac{b_{20}}{P_{10}} \end{pmatrix} \Delta[n] + \begin{pmatrix} P_{10}m_{12} & P_{20}(1-m_{21}) & P_{10}h_{12} & -P_{20}h_{21} \\ 0 & P_{20}m_{22} & 0 & P_{20}h_{22} \\ p_{12}\frac{b_{10}}{P_{20}} & q_{12}\frac{b_{10}}{P_{20}} & 0 & 0 \\ 0 & 0 & p_{22}\frac{b_{20}}{P_{10}} & q_{22}\frac{b_{20}}{P_{10}} \end{pmatrix} \Delta[n-1] = 0$$

where  $\Delta[n] = [\Delta\phi_1[n], \Delta\phi_2[n], \Delta\tau_1[n], \Delta\tau_2[n]]^T$ . Although the characteristic polynomial of the above first order recursion is of fourth degree (not shown) it can be reduced to a second

degree polynomial if all the contributions of the BRC are neglected ( $p_{ij} = 0$  and  $q_{ij} = 0$ , for neuron  $i$  and all orders  $j = 1, 2, \dots$ ). The characteristic equation becomes [7]

$$\lambda^2 - \lambda((1-m_{11})(1-m_{21}) - m_{12} - m_{22}) + m_{12}m_{22} = 0, \quad (6)$$

and it is valid only if the closed loop stimulus to every neuron in the network is identical to its open loop stimulus. Particularly, it means that the burst duration of each neuron during the phase-locked mode exactly matches the corresponding intrinsic burst duration. By neglecting the second order PRC contributions to the total phase resetting ( $m_{j2} = 0$ ,  $j = 1, 2$ ) we recover a previously derived stability criterion [3].

$$\lambda = (1-m_{11})(1-m_{21}), \quad (7)$$

which is valid only if every neuron receives identical stimuli both in open and closed loop, and the neurons return to their unperturbed activity during one cycle (strongly attracting limit cycle). On the other hand, the unreduced fourth order characteristic polynomial applies even if the burst durations in the open and closed loop conditions are not equal, but the assumption of a return near the limit during the cycle between perturbations still applies.

#### 4. Discussion and conclusions

Much theoretical work on the stability of synchrony and phase locked modes has focused on weakly coupled, simple integrate and fire neurons or Type I oscillators [4,6], which enables the use of phase models. But there can be no generalization to the relaxation oscillator models that characterize bursting neurons, at least not for strong inputs that saturate as a function of synaptic strength and exert their effects largely by virtue of terminating a burst due to their strength, or by delaying a postsynaptic burst due to their long duration. The major result of this study is the stability criterion for a two-neuron network reciprocally coupled through strong inhibitory synapses when the open and closed loop stimuli significantly differ from each other. We restricted this study to strong inhibitory couplings to ensure that the first order PRCs are nearly independent of the stimuli intensity [9], because a sufficiently strong perturbation is assumed to consistently cause a switch from one branch of the limit cycle to another, with no other effect. This allowed the simplification of neglecting the dependence of resetting on stimulus intensity and considering only the dependence on stimulus phase and duration.

The definition of the burst resetting curve and its contribution to the characteristic equation are novel results.

## 5. Acknowledgments

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