Localized Activations in a Simple Neural Field Model

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Abstract

A quarter of a century ago Amari [1] has presented a comprehensive and very elegant solution of the one-dimensional neural field equation. In the two-dimensional case analytical results on localized solutions are available under the assumption of rotational invariance, as numerical evidence indicates that no other stable solutions exist. We present analytic results for a special case of a "tophat" interaction function, which partially justifies the implicit assumption of circular solutions and allows us to discuss the possibility of non-generic deviations from circularity.

Key words: neural fields, Amari model, local excitation, bubbles, circularity
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1 Introduction

Neural fields describe the dynamics of macroscopic activity distributions on neural layers. The one-dimensional version of the model has been proposed and comprehensively studied in [1,3,7]. These results on existence and stability of various classes of stationary solutions have been adapted to two-dimensional neural fields under the somewhat artificial assumption of radial symmetry [8], where the results from the one-dimensional case seem to carry over. Yet, even under this restriction two-dimensional neural fields may have stationary configurations, cf. Fig. 2 (right), which are not present in the one-dimensional case [9]. Noteworthy, there is a lack of formal justification for the restriction to radial symmetry, more complex patterns are observable in real neural tissues, and there are no analytic results (except for [8,9]) on two-dimensional neural

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fields, cf. e.g. [4], and again emphasized in [5], such that the consideration of another special case seems justified.

A neural field describes the activation of a sheet of neural tissue which is considered at a sufficiently large length scale such that individual neurons are not resolved and the neural activity is a function of the continuous position $x \in \mathbb{R}^n$. We further assume that connections are isotropic and temporally constant. More specifically, a neuron at position x receives external inputs $I(x)$ and inputs from neurons at locations x' in the same layer via synaptic interactions $w(x, x')$ which are assumed to depend only on the distance $||x - x'||$. The local field $u(x, t)$ of neurons at position x and time t evolves according to

$$\tau \frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{R[u]} w(|x - x'|) dx' + h, \tag{1}$$

where h is a constant input or a neural threshold. The activation is assumed to be either unity or zero in dependence on whether or not the local field u is positive. The integration in (1) is taken only over the activated area

$$R\left[u\right] = \left\{x \in \mathbb{R}^n | u\left(x\right) > 0\right\}. \tag{2}$$

In general $R\left[u\right]$ can form complex patterns (cf. e.g. [2]), but we will restrict ourselves to bounded simply connected areas. Only equilibrium solutions

$$u(x,t) = \int_{R[u]} w(|x - x'|) dx' + h \tag{3}$$

are considered. Conditions on w and h that govern the qualitative properties of R have been given in [8,9]. Here only the case I of their classification is relevant and h is fixed to a value slightly below zero.

Since in numerical simulations only radially symmetric solutions have been found, the existence of more complex stable solutions is questionable, although unstable stationary states with more complex activated regions have not been searched for systematically. We will present here a simplified interaction function w which allows to derive analytical constraints for solutions of a simplified two dimensional neural field equation by geometrical considerations.

2 A special interaction function

We will assume in the following the activated area $R\left[u\right]$ is simply connected and has a smooth boundary ∂R. Further, by the "tophat" interaction function

$$w\left(\|x\|\right) = \begin{cases} g_+ & \text{if } \|x\| < w_0 \\ g_- & \text{otherwise} \end{cases} \tag{4}$$

the stationarity condition (3) is reduced to the areal balance of activated regions that contribute positively or negatively to the local field $u(x)$ at a point x. Splitting the activated area R according to (4) into an excitatory and an inhibitory part A_+ and A_-, resp., cf. Fig. 1 (left), we have for a point c at the boundary ∂R of R

$$u(c) = g_+ |A_+(c)| + g_- |A_-(c)| = 0. \tag{5}$$

If R is a circle of radius r, the weighting factors g_+ and g_-, which are required in order to obtain a solution with given $q = w_0/r$ can be calculated numerically from (5) and e.g. for $q = 1$, i.e. $w_0 = r$, the weights are given by $g = (4\pi - 3\sqrt{3})/(2\pi + 3\sqrt{3}) \approx 0.642$, where $g = g_-/g_+$. Thus, within the limits of the analysis in [8] circular regions of any size (relative to w_0) can be stabilized by an appropriate choice of g. In the following we will fix the length scale of the neural field such that $w_0 = 1$, but keep r as a parameter which can be adjusted by an appropriate choice of g. Note that r is required to be larger or equal to $w_0/2$ since otherwise no point receives any inhibition.

3 Two-dimensional bubbles

From a boundary point $c \in \partial R$ two intersection points are defined by the crossing of the excitatory range of the kernel and ∂R, c.f. Fig. 1. We assume that these are the only points of intersection, which is implied by transversality of such intersections for all $c \in \partial R$. From Eq. (5) we derive a condition for these intersection points which allows us to show that for the present case only circular solutions exist.

Consider two nearby points c_1 and c_2 on ∂R, cf. Fig. 1 (right). Because ∂R is assumed to be smooth it can be locally approximated by an osculating circle of radius r_c (measured in units of w_0) passing through c_1 and c_2 around a center M_c, which depends on c_1 and c_2 and which in the case of a circular solution coincides with the center of the activated area. Without loss of generality we set $M_c = (0,0)$ and $c_1 = (0, -r_c)$. The distance between c_1 and c_2 is denoted by Δ_c. The circle of radius $w_0 = 1$ around c_i intersects the boundary of R in two points a_i and b_i, $i = 1, 2$. By definition of a_i and b_i it holds that $\|a_i - c_i\| = w_0$ and $\|b_i - c_i\| = w_0$, where $\|\cdot\|$ is the Euclidean norm. Because both c_1 and c_2 are boundary points, the areas A_a and A_b, by which their excitatory and inhibitory areas differ, must be equal independently on the weights g_+, g_-, cf. (5). The relation between these areas will yield a condition for the distances of the points a_i and b_i from the center M_c.

The assumptions of smoothness and limited curvature of ∂R allow us to conclude that the equality of arc lengths follows from the equality of areas.

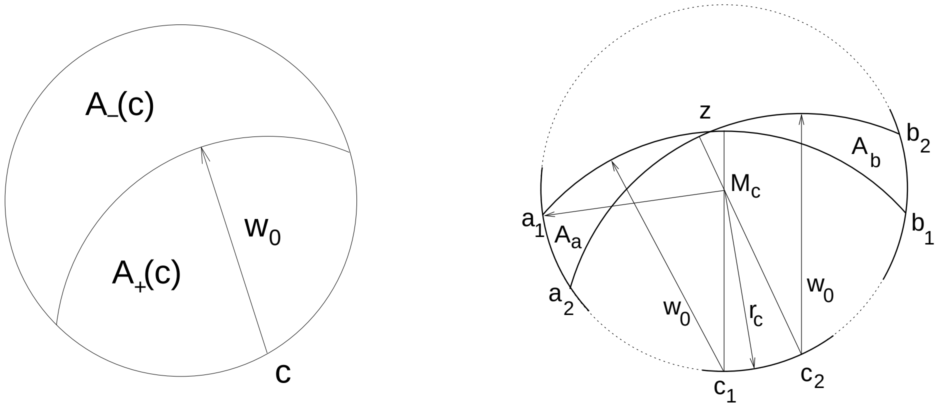

Figure 1. Left: Inhibitory and excitatory areas in an activated region of circular shape. Right: A configuration of points on the boundary of the activated area. Although a circular region is drawn, this property is not used in the analysis.

Namely, if c_2 approaches c_1 then the condition $|A_a| = |A_b|$ requires a_2 (b_2) to approach a_1 (b_1) such that the arcs from z to a_1 and from z to b_1 are equal. The points a_1 and b_1 are thus symmetric with respect to the line connecting c_1 and M_c. The same holds analogously for a_2, b_2, and c_2. Therefore choosing M_c as the origin we can represent the points a_i and b_i in terms of unit vectors $\tilde{c}_i$, $\tilde{c}_i^\perp$ parallel to, resp., c_i and its orthogonal complement $c_i^\perp$:

$$a_i = \alpha_i \tilde{c}_i - \beta_i \tilde{c}_i^\perp, b_i = \alpha_i \tilde{c}_i + \beta_i \tilde{c}_i^\perp, \quad i = 1, 2 \tag{6}$$

with $\alpha_i = a_i \cdot \tilde{c}_i$ and $\beta_i = a_i \cdot \tilde{c}_i^\perp$. Expanding to second order in $\Delta_c = \|c_1 - c_2\|$ we find from $|A_a| = |A_b|$

$$(\alpha_1\beta_2 - \alpha_2\beta_1)\left(\Delta_c^2 - 2r_c^2\right) w_0 - (\beta_1 - \beta_2)\left(\Delta_c^2 (r_c - w_0)^2 - 2r_c^2 w_0 (2r_c - 4w_0)\right) =$$
$$2r_c^2 w_0^2 \left(\arcsin\left(\frac{r_c - \alpha_1}{\sqrt{\beta_1^2 + (r_c - \alpha_1)^2}}\right) - \arcsin\left(\frac{r_c - \alpha_2}{\sqrt{\beta_2^2 + (r_c - \alpha_2)^2}}\right)\right) \tag{7}$$

While β_i can be obtained from the relation $\beta_i^2 + (r_c - \alpha_i)^2 = w_0^2$, α_i in (7) is expressed by a variable $\Delta(a_i)$ defined by $\|a_i\| = r_c + \Delta(a_i)$, $i = 1, 2$, noting that a_i is given relative to M_c. The assumption that ∂R is nearly circular allows us to expand with respect to $\Delta(a_i)$. In second order, we find two solutions of (7) one of which implies directly $\Delta(a_1) = \Delta(a_2)$. Since in the limit $\Delta_c \to 0$ we must have $a_1 = a_2$, the second solution $\Delta(a_2) = F(\Delta(a_1))$ is not valid unless a specific relation between r_c and $\Delta(a_1)$ holds, which is given (for $w_0 = 1$) by

$$\Delta(a_1) = \frac{1 - 4r_c^2}{24r_c^3(2r_c^2 - 1)}\left(1 - 4r_c^2 + 8r_c^4 \pm \sqrt{1 + 8\left(-1 + 2r_c^2\right)\left(r_c + 2r_c^3\right)^2}\right) \tag{8}$$

Even for this special relation, $\Delta(a_1)$ and $\Delta(a_2)$ do not differ in first order, i.e. the normal near a_1 is directed towards the center of the osculating circle at c.

Thus for both solutions of (7) the osculating circles at c_1 and a_1 have the same center. If we start now, instead from c_1, from a_1 and continue iteratively, either all points of ∂R are reached and have thus a common osculating circle or we return to c_1 after finitely many, say k, steps. In the latter case the curve formed by the centers of the osculating circles, also called the evolute of ∂R, must be k-periodic. It is easy to show [6] that the evolute at the center of the osculating circle at a point c is tangent to the normal to ∂R at c. If ∂R is nearly circular and the evolute has well defined tangents, then the evolute cannot be tangent to the normal at a_1 as well. Since the evolute of a smooth curve is piecewise smooth, we conclude that the evolute of ∂R consists only of a single isolated point. Evolutes consisting of several isolated points are excluded by smoothness and simple connectedness of R. Therefore R is necessarily circular, and we have $\Delta(a_1) = \Delta(a_2) = 0$.

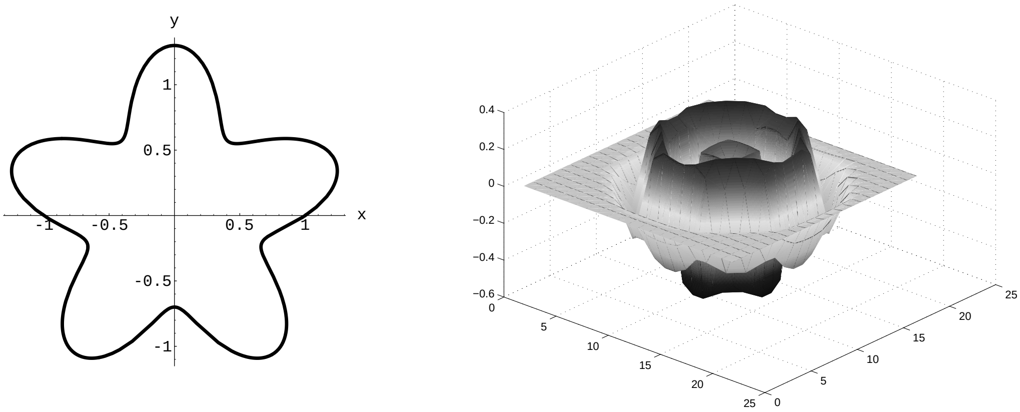

Figure 2. Left: Hypothetical activated area. Right: Local field profile of a radially symmetric ring-shaped solution obtained using a smooth interaction function.

4 Discussion

In this paper we have presented an interesting special case of a two-dimensional neural field equation which allows to treat certain properties of the stationary states analytically. In principle it should possible to directly calculate the evolute of the boundary of R, but the complexity of the expressions led us to the present approach. Since stationary solutions of the neural field equation with a "tophat" interaction kernel are circular the assumption of circularity in [8,9] seem justified, although in this way no direct conclusions on stability can be drawn. While we can exclude continuous bifurcations from the circular solution, more complicated activated areas which considerably deviate from a circle (such as Fig. 2, left), discontinuous bifurcation of solutions or solutions which are not simply connected (cf. [5]) may still exist. In addition to the present result is possible to derive limits to the maximal curvature of the activated region in terms of w_0 which allow to rule out some of the stationary configurations with multiple symmetry axes.

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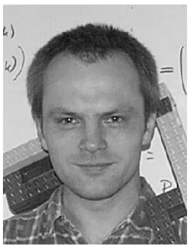

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