Localized Activations in a Simple Neural Field Model

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Abstract

A quarter of a century ago Amari [1] has presented a comprehensive and very elegant solution of the one-dimensional neural field equation. In the two-dimensional case analytical results on localized solutions are available under the assumption of rotational invariance, although numerical evidence indicates that no other stable solution exist. We present analytic results for a special case of an interaction function, which partially justifies the implicit assumption of circular solutions and allows to discuss the possibility of non-generic deviations from circularity.

1 Introduction

The one-dimensional version of the neural field model has been proposed and comprehensively studied in [1, 4, 5]. These results have been adapted to two-dimensional neural fields under the assumption of radially symmetric activated regions ('bubbles') [2], where most of the results from the one-dimensional case carry over. Yet, even under this restriction two-dimensional neural fields may have stationary configurations, cf. Figure 2 (right), which are not present in the one-dimensional case as pointed out in [3]. The lack of formal justification for the restriction to radial symmetry, the more complex pattern that are observable in real neural tissues, and the fact that (except for [2, 3]) there are no analytic results on two-dimensional neural fields, cf. e.g. [6], might suffice to justify the consideration of another special case, namely a particularly simple interaction function, which is constant, resp., in the excitatory region and the inhibitory region. In this way, the study of stationary solutions of the field equations can be reduced to an essentially geometric problem and a more detailled analysis becomes possible.

A neural field describes the activation of a sheet of neural tissue which is considered at a sufficiently large length scale such that individual neurons are not resolved and the neural activity is merely a function of the continuous position $x \in \mathbb{R}^n$ in the field. We further simplify the model by stating that neurons and connections are characterized by real variables and that connections are isotropic and temporally constant. More specifically, we assume that a neuron at position x receives external inputs $I(x)$ and inputs from neurons at other locations x' in the same layer via synaptic interactions $w(x, x')$ which are assumed to depend only on the distance $||x - x'||$. The local field $u(x, t)$ of neurons at position x and time t evolves according to

$$\tau \frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{R[u]} w(|x - x'|) dx' + h \tag{1}$$

where h is a constant input or a neural threshold. The integration in (1) is taken only over the activated area

$$R\left[u\right] = \left\{ x \in \mathbb{R}^n \middle| u\left(x\right) \geq 0 \right\}, \tag{2}$$

and the activation is assumed to be either unity or zero in dependence on whether or not the local field u is positive. In general $R\left[u\right]$ can form complex pattern (cf. e.g. [7]), but we will restrict ourselves on the study of bounded simply connected areas. Moreover only equilibrium solutions

$$u(x,t) = \int_{R[u]} w(|x - x'|)dx' + h \tag{3}$$

are of interest. Conditions on w and h that govern the qualitative properties of R have been given in [2, 3]. Here only the case I of their classification is relevant and h is fixed to a value slightly below zero.

Since by means of extensive numerical simulations only radially symmetric solutions have been found the existence of more complex stable solutions is questionable, although unstable stationary states with activated regions of more complex shapes have not been searched for systematically. It will become clear that these are difficult to analyze even in the present simplified setting such that a complete analytic treatment of the two-dimensional neural field equation seems presently infeasible or will at least not be possible as elegantly as in the one-dimensional case [1].

2 A special interaction function

We will assume in the following the activated area $R\,[u]$ is simply connected and has a smooth boundary ∂R. Further, by the interaction function

$$w\left(\|x\|\right) = \begin{cases} g_+ & \text{if } \|x\| < w_0 \\ g_- & \text{otherwise} \end{cases} \tag{4}$$

the stationarity condition (3) is reduced to the areal balance of activated areas that contribute positively or negatively to the local field $u\,(x)$ of a point x. Splitting the activated area R according to (4) into an excitatory and an inhibitory part A_+ and A_-, resp., cf. Figure 1 (left), we have for a point c at the boundary of R

$$u(c) = g_+ |A_+(c)| + g_- |A_-(c)| = 0. \tag{5}$$

If R is a circle of radius r, the weighting factors g_+ and g_-, which are required in order to obtain a solution with given $q = w_0/r$ can be calculated from

$$-2\pi q^2 g_- + (g_+ + g_-)\left(q^2 \arccos\left(\frac{1}{2q^4} - \frac{2}{q^2} + 1\right) + \arccos\left(\frac{1}{2q^2} - 1\right) - \sqrt{4q^2 - 1}\right) = 0 \tag{6}$$

e.g. for $q = 1$, i.e. $w_0 = r$, the weights are given by $g = (4\pi - 3\sqrt{3})/(2\pi + 3\sqrt{3}) \approx 0.642$, where $g = g_-/g_+$. Thus, within the limits of the analysis in [2] circular regions of any size (relative to the length scale of the interaction function) can be stabilized by an appropriate choice of g. In the following we will fix the length scale of the neural field such that $w_0 = 1$, but keep r as a parameter which can be adjusted by an appropriate choice of g. Note that r is required to be larger or equal to $w_0/2$ since otherwise boundary points do not receive any inhibition.

3 Two-dimensional bubbles

Consider two nearby points c_1 and c_2 on ∂R, cf. Figure 1 (right). Because ∂R is assumed to be smooth it can be locally approximated by an osculating circle of radius r_c (measured in units of w_0) passing through c_1 and c_2 around a center M, which depends on c_1 and c_2 and which only in the case of a circular solution coincides with the center of the activated area. Without loss of generality we set $M = (0, 0)$ and $c_1 = (0, -r_c)$. The circle of radius $w = 1$ around c_i intersects the boundary of R in two points a_i and b_i, $i = 1, 2$, and we assume that these are the only points of intersection. The latter assumption is weaker than convexity. Instead we require R to be star-shaped with respect to M for any pair of close points $c_i \in \partial R$ and that the curvature of ∂R is lower than the curvature of a circle with radius w. By definition of a_i and b_i it holds that $\|a_i - c_i\| = w_0$ and $\|b_i - c_i\| = w_0$, where $\|\cdot\|$ denotes the Euclidean norm.

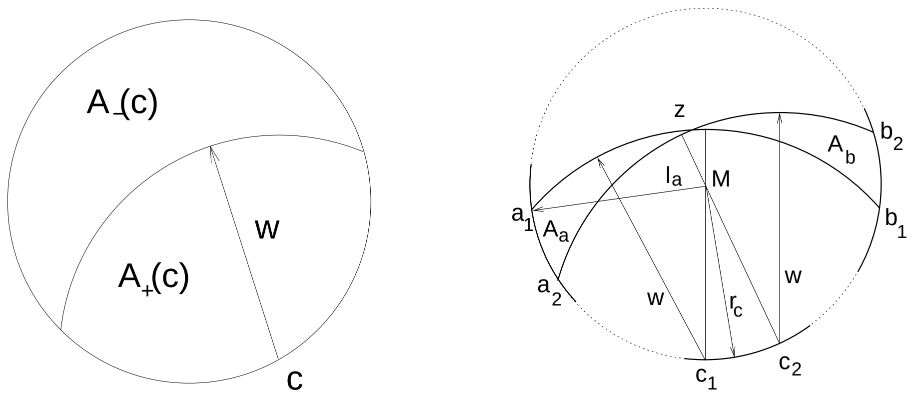

Figure 1: Left: Inhibitory and excitatory areas in an activated region of circular shape. Right: A configuration points on the of boundary of the activated area. Although a circular regions are drawn, this property is not used in the analysis.

Because both c_1 and c_2 are boundary points, the areas A_a and A_b, by which their excitatory and inhibitory areas differ, must be equal independently on the weights g_+, g_-, cf. (5). We will determine these areas in order to derive conditions for the distances of the points a_i and b_i from the center M. From $|A_a| = |A_b|$ we find

$$\begin{aligned} & a_2^x a_1^y + b_1^x b_2^y + c_2^x\left(a_2^y + b_2^y\right) + c_1^y\left(a_1^x + b_1^x\right) + 2z^x\left(b_1^y + c_2^y\right) + 2z^y\left(b_2^x - c_1^x\right) \\ & +r_c\left(\arcsin\left((a_1 - c_1)\cdot(z - c_1)\, r_c^{-2}\right) + \arcsin\left((b_2 - c_2)\cdot(z - c_2)\, r_c^{-2}\right)\right) \\ = \; & a_1^x a_2^y + b_2^x b_1^y + c_2^y\left(a_2^x + b_2^x\right) + c_1^x\left(a_1^y + b_1^y\right) + 2z^x\left(b_2^y + c_1^y\right) + 2z^y\left(b_1^x - c_2^x\right) \\ & +r_c\left(\arcsin\left((a_2 - c_2)\cdot(z - c_2)\, r_c^{-2}\right) + \arcsin\left((b_1 - c_1)\cdot(z - c_1)\, r_c^{-2}\right)\right) \end{aligned} \tag{7}$$

where z denotes the intersection of the two circles of radius w_0 around c_1 and c_2 which is inside R. When deriving (7) we have neglected the curvature of the boundary near the points a_i and b_i because it effects the equality only in third order in $\Delta = \|c_1 - c_2\|$. The assumptions of smoothness and limited curvature of ∂R allow to conclude (in first order in Δ) that the equality of arc length follows from the equality of areas. Namely, if c_2 approaches c_1 then the condition $|A_a| = |A_b|$ requires a_2 (b_2) to approach a_1 (b_1) such that the arcs from z to a_1 and from z to b_1 are equal. The points a_1 and b_1 are thus symmetric with respect to the straight line connecting c_1 and M. The same holds analogously for a_2, b_2, and c_2. Therefore it is possible to represent the points a_i and b_i in terms of the vector c_i and its orthogonal complement $c_i^\perp$ in the following way.

$$\begin{array}{rclcrcl} a_1 & = & \alpha\frac{c_1}{r_c} - \beta\frac{c_1^\perp}{r_c} & , & b_1 & = & \alpha\frac{c_1}{r_c} + \beta\frac{c_1^\perp}{r_c} \\ a_2 & = & (\alpha+\gamma)\frac{c_2}{r_c} - (\beta+\delta)\frac{c_2^\perp}{r_c} & , & b_2 & = & (\alpha+\gamma)\frac{c_2}{r_c} + (\beta+\delta)\frac{c_2^\perp}{r_c} \end{array} \tag{8}$$

with $\alpha = a_1 \cdot c_1 / r_c$ and $\beta = a_1 \cdot c_1^\perp / r_c$, because $\|c_i\| = \left\|c_i^\perp\right\| = r_c$. Inserting (8) into (7) and expanding to second order in $\Delta = \|c_1 - c_2\|$ we find

$$\begin{array}{rl} 8r_c^2\left(\alpha\delta r_c - \beta\gamma r_c - \delta\left(1+r_c\right)\right) + 4\Delta r_c\left(1+r_c\right)\left(\left(2\alpha+\gamma-2\right)r_c - 2\right) & \\ +\Delta^2\left(4\beta\gamma r_c + \delta\left(1+2r_c-4\alpha r_c + r_c^2\right)\right) & = \ 0 \end{array} \tag{9}$$

The increments γ and δ are determined by (9) and by $\|a_2 - c_2\| = w_0$. In the case of a circular solution with center M both increments must vanish, otherwise γ and δ describe the deviations from circularity. Geometrical consideration imply the relations $\alpha = \left(r_c^2 + l_a^2 - w_0^2\right)/(2r_c)$ and $\beta = \sqrt{l_a^2 - \left(r_c^2 + l_a^2 - w_0^2\right)^2/(2r_c)^2}$, where $l_a = \|a_1\| = \|b_1\|$ is the distance between a_1 or b_1 and $M = (0,0)$, which allows to express γ and δ, and thus the distance from a_2 (and b_2) to M in terms of w_0, r_c, and l_a. This relation $\|a_2 - M\| = F\left(w_0, r_c, l_a\right)$ has been obtained by MATHEMATICA and is too complex to be reproduced here.

Checking for consistency with the circular case, i.e. equating $r_c = \|c_1\| = \|c_2\| = \|a_1\| = \|b_1\| = l_a$, we find that the map $\|a_2\| = F_\Delta\left(w, \|c_1\| = \|c_2\|, \|a_1\|\right)$ has in the limit $\Delta \to 0$ (up to second-order deviations) a fixed point $\|a_2\| = \|a_1\| = r_c$. This fixed point is present for any $r_c \geq \frac{1}{2}$, cf. Section 2. By exchanging the indices in a_i, b_i, and c_i we may conclude

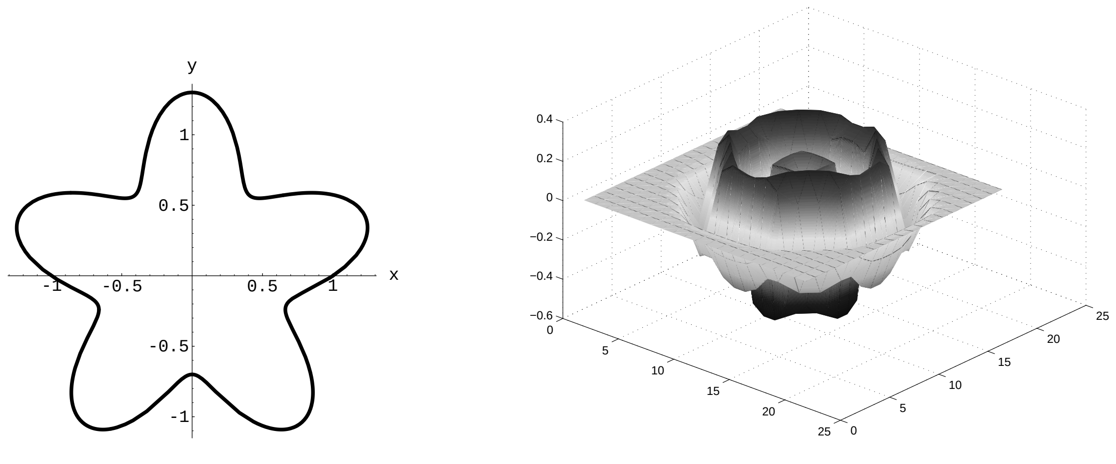

Figure 2: Left: Hypothetical activated area. Right: Local field profile of a radially symmetric ring-shaped solution obtained using a smooth interaction function.

that for boundary points on the same circle also nearby points will be on this circle. If the curvature of ∂R at c_i is minimal it will be minimal also at a_i and b_i. Replacing b_i by c_i we can construct a sequence of points of minimal curvature along ∂R which generically reaches every point such that ∂R is of constant curvature, i.e. R is a circle.

Only for special values of g and thus of r_c the sequence will be periodic and other configurations are allowed in principle. Let us assume that R is of non-circular shape, e.g. as shown in Figure 2 (left). Then there are points a_i, b_i, c_i which do not share the same curvature circle. Therefore the map F has to be applied to r_c and distances $l_a = l_b \neq r_c$ of $a_1(b_1)$ from M. Except for two special values of r_c the value $\|a_2\|$ strongly deviates from l_a even for arbitrarily small Δ such that the smoothness conditions on ∂R are violated. For $r_c = 1$ and $r_c = \sqrt{5}$ the boundary changes smoothly, but in a way that contradicts, resp., the assumption of a simply connected activated region or the curvature condition for ∂R. In this way, it is evident that under the present assumptions only circular activated regions can be stationary.

4 Discussion

Since only circular regions are allowed as stationary configuration, the results on stability for most of the cases considered in the radially symmetric case seem to carry over to the present case, although in [2, 3] only circularly symmetric perturbations have been considered. It is possible although beyond the scope of the present paper to derive limits to the maximal curvature of the activated region in terms of w_0 which allow to rule out stationary configurations with multiple symmetry axes. In this way the presence of a stable solution in between two unstable solutions can be conjectured. However, the conditions on the activated regions which we have imposed here, may be violated transiently when bubbles escape the region between unstable states. In this way the adoption of the stability criteria from [2, 3] requires a deeper understanding of the general dynamics of neural fields.

In this paper we have presented an interesting special case of a two-dimensional neural field equation which allows to treat certain properties of the stationary states analytically. A few further results within the present setting are clearly possible and will be presented in a forthcoming version of this paper. Concerning the more challenging questions arising in the two-dimensional case in connection with more general interaction functions (not to speak about transient dynamical behavior, more complex stationary behaviors, or a more realistic setting of the model itself), the relevance of the present approach can hardly be estimated at present, although it has an interest in itself and may thus stimulate further work on this topic. The interaction function w (4) permits to reduce the problem of the existence of localized solutions to the consideration of infinitesimal regions near the boundary. In order to have self-consistent conditions for boundary points in more general cases, contributions along the whole boundary have to integrated, which is difficult since the boundary cannot be assumed to be known.

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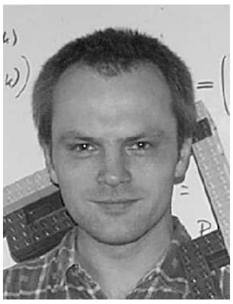

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