CHAPTER 7

PROPERTIES OF FUNCTIONS

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7.2 One-to-One, Onto, and Inverse Functions

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One-to-One Functions

One-to-One Functions

Definition

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently, $$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

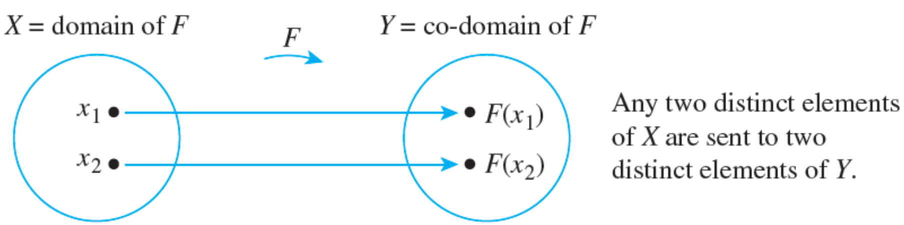
Symbolically:

$$F\colon X \to Y \text{ is one-to-one} \Leftrightarrow \forall x_1, x_2 \in X, \text{ if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

Thus,

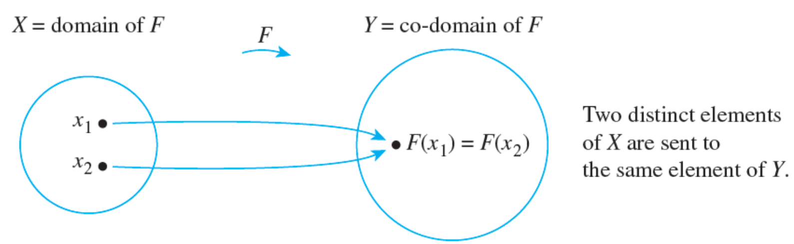
A function $F\colon X \to Y$ is *not* one-to-one $\Leftrightarrow$ $\exists$ elements x_1 and x_2 in X with $F(x_1) = F(x_2)$ and $x_1 \neq x_2$.

One-to-One Functions



A One-to-One Function Separates Points

Figures 7.2.1(a)

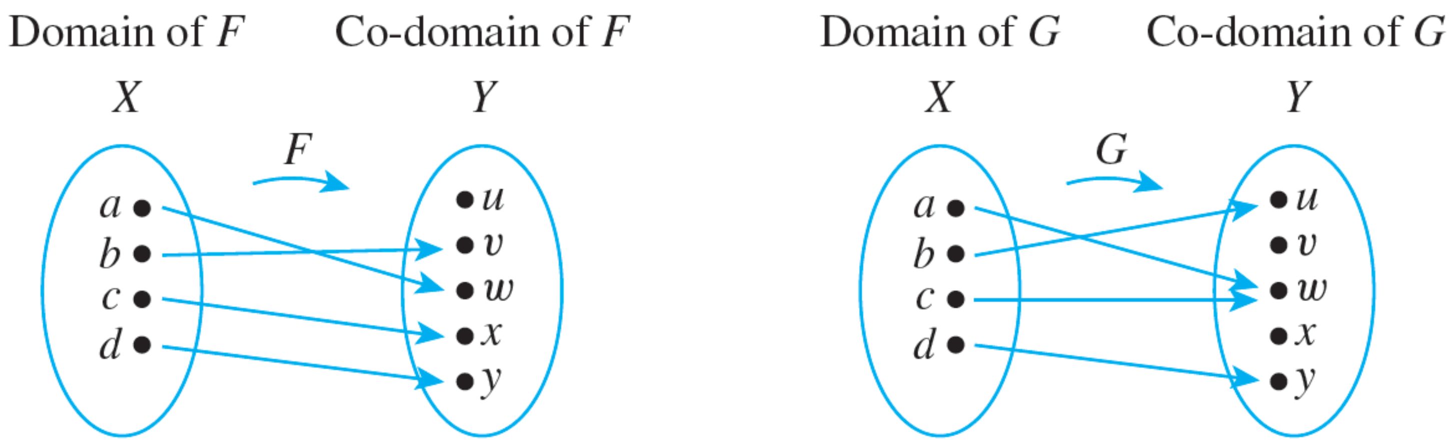


A Function That Is Not One-to-One Collapses Points Together

Figures 7.2.1(b)

Example 7.2.1 – *Identifying One-to-One Functions Defined on Finite Sets*

a. Do either of the arrow diagrams in Figure 7.2.2 define one-to-one functions?



Figures 7.2.2

Example 7.2.1 – *Identifying One-to-One Functions Defined on Finite Sets*

continued

b. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, and $H(3) = d$.

Define $K: X \to Y$ as follows: $K(1) = d$, $K(2) = b$, and $K(3) = d$. Is either H or K one-to-one?

Example 7.2.1 – *Solution*

a. F is one-to-one, but G is not. F is one-to-one because no two different elements of X are sent by F to the same element of Y.

G is not one-to-one because the elements a and c are both sent by G to the same element of Y: $G(a) = G(c) = w$ but $a \neq c$.

Example 7.2.1 – *Solution*

continued

b. H is one-to-one, but K is not. H is one-to-one because each of the three elements of the domain of H is sent by H to a different element of the co-domain: $H(1) \neq H(2)$, $H(1) \neq H(3)$, and $H(2) \neq H(3)$.

K, however, is not one-to-one because $K(1) = K(3) = d$ but $1 \neq 3$.

One-to-One Functions on Infinite Sets

One-to-One Functions on Infinite Sets

Now suppose f is a function defined on an infinite set X. By definition, f is one-to-one if, and only if, the following universal statement is true:

$$\forall x_1, x_2 \in X, \text{ if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

Thus, to prove f is one-to-one, you will generally use the method of direct proof:

suppose x_1 and x_2 are elements of X such that $f(x_1) = f(x_2)$

and **show** that $x_1 = x_2$.

One-to-One Functions on Infinite Sets

To show that f is *not* one-to-one, you will ordinarily

> **find** elements x_1 and x_2 in X so that $f(x_1) = f(x_2)$ but $x_1 \neq x_2$.

Example 7.2.2 – *Proving or Disproving That Functions Are One-to-One*

Define $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{Z} \to \mathbf{Z}$ by the rules

$$f(x) = 4x - 1 \text{ for all } x \in \mathbf{R}$$

and

$$g(n) = n^2 \quad \text{for all } n \in \mathbf{Z}.$$

a. Is f one-to-one? Prove or give a counterexample.

b. Is g one-to-one? Prove or give a counterexample.

Example 7.2.2 – *Solution*

continued

Answer to (a):

If the function $f: \mathbf{R} \to \mathbf{R}$ is defined by the rule $f(x) = 4x - 1$, for each real number x, then f is one-to-one.

Example 7.2.2 – *Solution*

continued

Proof: Suppose x_1 and x_2 are real numbers such that $f(x_1) = f(x_2)$. *[We must show that* $x_1 = x_2$*.]* By definition of f,

$$4x_1 - 1 = 4x_2 - 1.$$

Adding 1 to both sides gives

$$4x_1 = 4x_2,$$

and dividing both sides by 4 gives

$$x_1 = x_2,$$

[as was to be shown].

Example 7.2.2 – *Solution*

continued

Answer to (b):

If the function $g: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $g(n) = n^2$, for all $n \in \mathbf{Z}$, then g is not one-to-one.

Counterexample:

Let $n_1 = 2$ and $n_2 = -2$. Then by definition of g,

$$g(n_1) = g(2) = 2^2 = 4 \text{ and also}$$
$$g(n_2) = g(-2) = (-2)^2 = 4.$$

Hence $g(n_1) = g(n_2)$ but $n_1 \neq n_2$,

and so, g is not one-to-one.

Application: Hash Functions

Skip

Onto Functions

Onto Functions

Definition

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F: X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Onto Functions

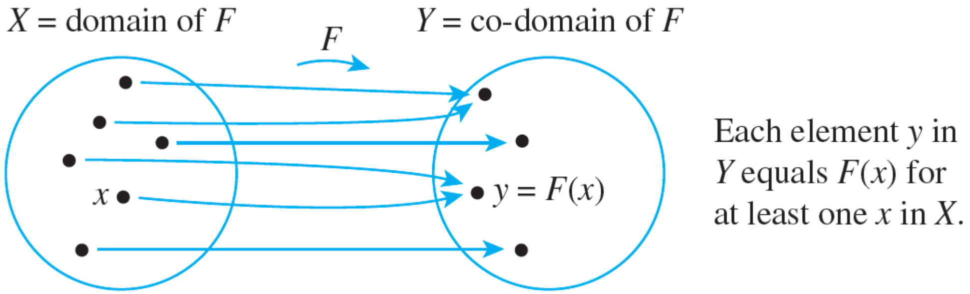
To obtain a precise statement of what it means for a function *not* to be onto, take the negation of the definition of onto:

$$F: X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X,\ F(x) \neq y.$$

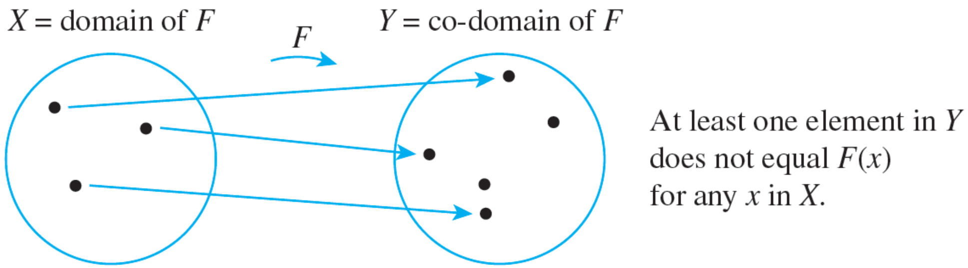
That is, there is some element in *Y* that is *not* the image of *any* element in *X*.

Onto Functions

This is illustrated in Figures 7.2.3(a) and 7.2.3(b).



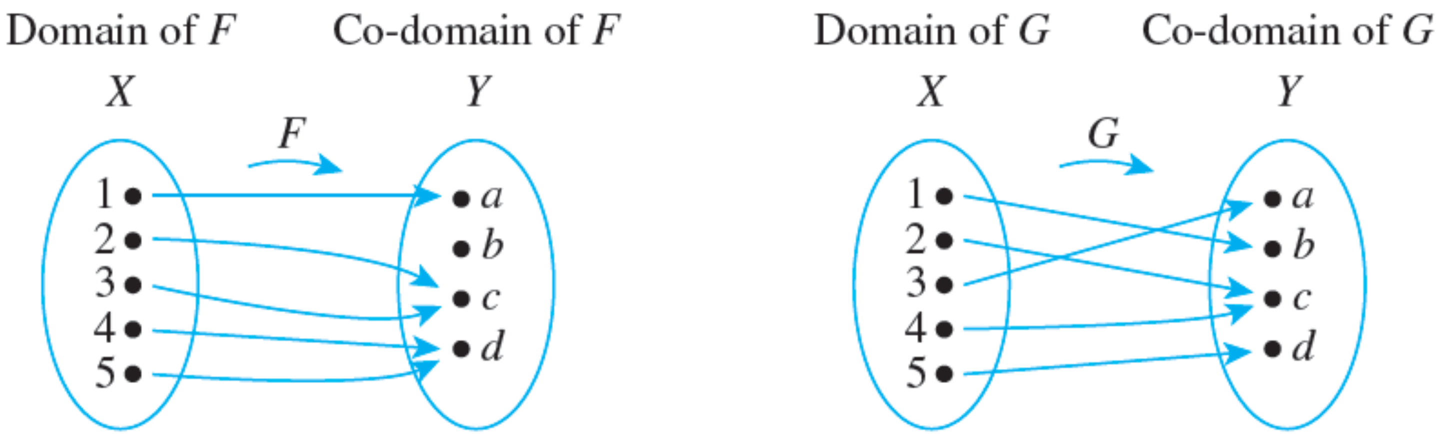
A Function That Is Onto
Figures 7.2.3(a)



A Function That Is Not Onto
Figures 7.2.3(b)

Example 7.2.4 – *Identifying Onto Functions Defined on Finite Sets*

a. Do either of the arrow diagrams in Figure 7.2.4 define onto functions?



Figures 7.2.4

Example 7.2.4 – *Identifying Onto Functions Defined on Finite Sets*

continued

b. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Define $H : X \to Y$ as follows: $H(1) = c$, $H(2) = a$, $H(3) = c$, $H(4) = b$.

Define $K : X \to Y$ as follows: $K(1) = c$, $K(2) = b$, $K(3) = b$, and $K(4) = c$. Is either H or K onto?

Example 7.2.4 – *Solution*

a. F is not onto because $b \neq F(x)$ for any x in X. G is onto because each element of Y equals $G(x)$ for some x in X : $a = G(3)$, $b = G(1)$, $c = G(2) = G(4)$, and $d = G(5)$.

b. H is onto, but K is not. H is onto because each of the three elements of the co-domain of H is the image of some element of the domain of H : $a = H(2)$, $b = H(4)$, and $c = H(1) = H(3)$. K, however, is not onto because $a \neq K(x)$ for any x in $\{1, 2, 3, 4\}$.

Onto Functions on Infinite Sets

Onto Functions on Infinite Sets

Now suppose F is a function from a set X to a set Y, and suppose Y is infinite. By definition, F is onto if, and only if, the following universal statement is true:

$$\forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

Thus, to prove F is onto, you will ordinarily use the method of generalizing from the generic particular:

suppose that y is any element of Y

and **show** that there is an element x in X with $F(x) = y$.

Onto Functions on Infinite Sets

To prove F is *not* onto, you will usually

find an element y of Y such that $y \neq F(x)$ for *any* x in X.

Example 7.2.5 – *Proving or Disproving That Functions Are Onto*

Define $f : \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{Z} \to \mathbf{Z}$ by the rules

$$f(x) = 4x - 1 \text{ for each } x \in \mathbf{R}$$

and

$$h(n) = 4n - 1 \text{ for each } n \in \mathbf{Z}.$$

a. Is f onto? Prove or give a counterexample.

b. Is h onto? Prove or give a counterexample.

Example 7.2.5 – *Solution*

a. The best approach is to start trying to prove that *f* is onto and be alert for difficulties that might indicate that it is not.

Now $f : \mathbf{R} \to \mathbf{R}$ is the function defined by the rule

$$\boxed{f(x) = 4x - 1 \quad \text{for each real number } x.}$$

Example 7.2.5 – *Solution*

continued

To prove that f is onto, you must prove

$$\forall y \in Y, \exists x \in X \text{ such that } f(x) = y.$$

Substituting the definition of f into the outline of a proof by the method of generalizing from the generic particular, you

suppose y is a real number

and **show** that there exists a real number x such that $y = 4x - 1$.

Example 7.2.5 – *Solution*

continued

Scratch Work: **If** such a real number *x* exists, then

$$
\begin{aligned}
4x - 1 &= y \\
4x &= y + 1 && \text{by adding 1 to both sides} \\
x &= \frac{y+1}{4} && \text{by dividing both sides by 4.}
\end{aligned}
$$

Thus, *if* such a number *x* exists, it must equal (y + 1)/4.
Does such a number exist? Yes.

Example 7.2.5 – *Solution*

continued

To show this, let $x = (y + 1)/4$, and then make sure that (1) x is a real number and that (2) f really does send x to y.

The following formal answer summarizes this process.

Answer to (a):

If $f: \mathbf{R} \to \mathbf{R}$ is the function defined by the rule $f(x) = 4x - 1$ for each real number x, then f is onto.

Example 7.2.5 – *Solution*

continued

Proof: Let $y \in \mathbf{R}$. *[We must show that* $\exists x$ *in* **R** *such that* $f(x) = y$.*]* Let $x = (y + 1)/4$.

Then x is a real number since sums and quotients (other than by 0) of real numbers are real numbers. It follows that

$$f(x) = f\left(\frac{y+1}{4}\right) \qquad \text{by substitution}$$

$$= 4\cdot\left(\frac{y+1}{4}\right) - 1 \qquad \text{by definition of } f$$

Example 7.2.5 – *Solution*

continued

$= (y + 1) - 1 = y$ by basic algebra,

[as was to be shown].

b. The function $h : \mathbf{Z} \to \mathbf{Z}$ is defined by the rule

$$\boxed{h(n) = 4n - 1 \quad \text{for each integer } n.}$$

To prove that h is onto, you must prove that

$\forall$ integer m, $\exists$ an integer n such that $h(n) = m$.

Example 7.2.5 – *Solution*

continued

Substituting the definition of *h* into the outline of a proof by the method of generalizing from the generic particular shows that you need to

suppose *m* is any integer

and **show** that there is an integer *n* with $4n - 1 = m$.

Can you reach what is to be shown from the supposition? No! If $4n - 1 = m$,

$$n = \frac{m+1}{4} \quad \text{by adding 1 and dividing by 4.}$$

Example 7.2.5 – *Solution*

continued

But n must be an integer. And when, for example, $m = 0$,

$$n = \frac{0+1}{4} = \frac{1}{4},$$

which is *not* an integer.

Thus, in trying to prove that h is onto, you run into difficulty, and this difficulty reveals a counterexample that shows h is not onto.

Example 7.2.5 – *Solution*

continued

This discussion is summarized in the following formal answer.

Answer to (b):

If the function $h: \mathbf{Z} \to \mathbf{Z}$ is defined by the rule $h(n) = 4n - 1$ for each integer n, then h is not onto.

Counterexample: The co-domain of h is $\mathbf{Z}$ and $0 \in \mathbf{Z}$.

But $h(n) \neq 0$ for any integer n.

Example 7.2.5 – *Solution*

continued

For if $h(n) = 0$, then

$$4n - 1 = 0 \qquad \text{by definition of } h$$

which implies that

$$4n = 1 \qquad \text{by adding 1 to both sides}$$

and so

$$n = \frac{1}{4} \qquad \text{by dividing both sides by 4.}$$

But $\frac{1}{4}$ is not an integer. Hence there is no integer n for which $f(n) = 0$, and thus f is not onto.

Relations between Exponential and Logarithmic Functions

Relations between Exponential and Logarithmic Functions

For positive numbers $b \neq 1$, the **exponential function with base *b***, denoted $\exp_b$, is the function from **R** to $\mathbf{R}^+$ defined as follows: For each real number x,

$$\exp_b(x) = b^x,$$

where $b^0 = 1$ and $b^{-x} = 1/b^x$.

When working with the exponential function, it is useful to recall the laws of exponents from elementary algebra.

Relations between Exponential and Logarithmic Functions

Laws of Exponents

If b and c are any positive real numbers and u and v are any real numbers, the following laws of exponents hold true:

$$b^u\, b^v = b^{u+v} \tag{7.2.1}$$

$$(b^u)^v = b^{uv} \tag{7.2.2}$$

$$\frac{b^u}{b^u} = b^{u-v} \tag{7.2.3}$$

$$(bc)^u = b^u c^u \tag{7.2.4}$$

It can be shown using calculus that both the exponential and logarithmic functions are one-to-one and onto.

Relations between Exponential and Logarithmic Functions

Therefore, by definition of one-to-one, the following properties hold true:

For any positive real number b with $b \neq 1$,

$$\text{if } b^u = b^v \text{ then } u = v \text{ for all real numbers } u \text{ and } v, \qquad 7.2.5$$

and

$$\text{if } \log_b u = \log_b v \text{ then } u = v \text{ for all positive real numbers } u \text{ and } v. \qquad 7.2.6$$

These properties are used to derive many additional facts about exponents and logarithms.

Relations between Exponential and Logarithmic Functions

In particular we have the following properties of logarithms.

Theorem 7.2.1 Properties of Logarithms

For any positive real numbers b, c, x, and y with $b \neq 1$ and $c \neq 1$ and for every real number a:

a. $\log_b(xy) = \log_b x + \log_b y$

b. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

c. $\log_b(x^a) = a \log_b x$

d. $\log_c x = \frac{\log_b x}{\log_b c}$

Example 7.2.7 – *Solution*

By Theorem 7.2.1(d),

$$\log_2 5 = \frac{\ln 5}{\ln 2} \cong \frac{1.609437912}{0.6931471806}$$

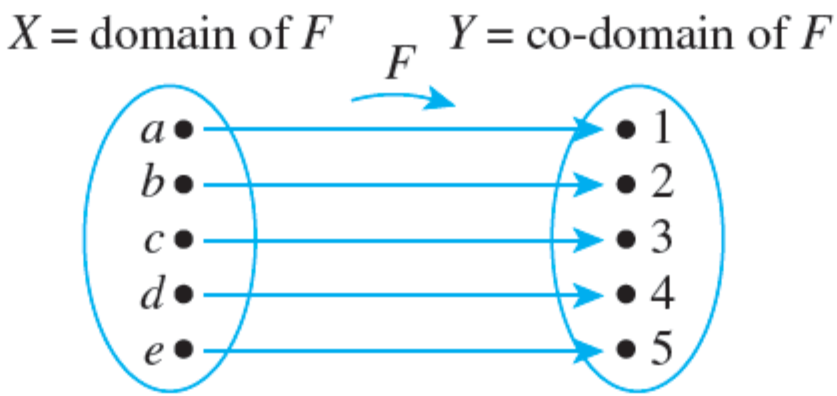
$$\cong 2.321928095.$$

One-to-One Correspondences

One-to-One Correspondences

Definition

A **one-to-one correspondence** (or **bijection**) from a set X to a set Y is a function $F: X \to Y$ that is both one-to-one and onto.



Example 7.2.8 – *A Function from a Power Set to a Set of Strings*

Let $\mathscr{P}(\{a, b\})$ be the set of all subsets of $\{a, b\}$ and let S be the set of all strings of length 2 made up of 0's and 1's. Then $\mathscr{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $S = \{00, 01, 10, 11\}$.

Define a function h from $\mathscr{P}(\{a, b\})$ to S as follows: Given any subset A of $\{a, b\}$, a is either in A or not in A, and b is either in A or not in A.

If a is in A, write a 1 in the first position of the string $h(A)$; otherwise write a 0 there.

Example 7.2.8 – *A Function from a Power Set to a Set of Strings*

continued

Similarly, if b is in A, write a 1 in the second position of the string $h(A)$; otherwise write a 0 there. This definition is summarized in the following table.

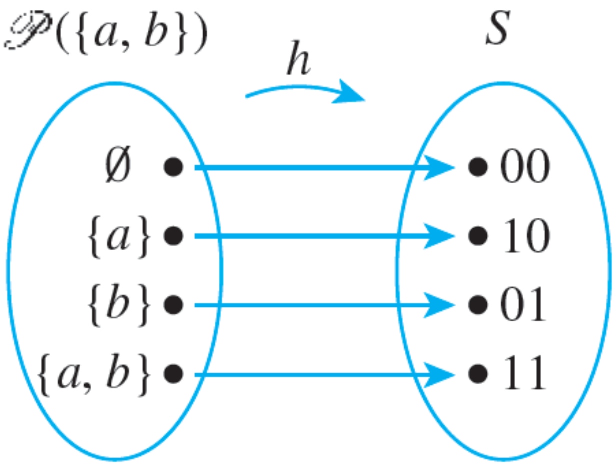
$$\xrightarrow{\quad h \quad}$$

Subset A of $\{a, b\}$	**Status of a in A**	**Status of b in A**	**String $h(A)$ in S**
$\varnothing$	not in	not in	00
$\{a\}$	in	not in	10
$\{b\}$	not in	in	01
$\{a, b\}$	in	in	11

Is h a one-to-one correspondence?

Example 7.2.8 – *Solution*

The arrow diagram shown in Figure 7.2.6 shows clearly that *h* is a one-to-one correspondence.



Figures 7.2.6

Example 7.2.8 – *Solution*

continued

It is onto because each element of S has an arrow pointing to it.

It is one-to-one because each element of S has no more than one arrow pointing to it.

Inverse Functions

Inverse Functions

If F is a one-to-one correspondence from a set X to a set Y, then there is a function from Y to X that "undoes" the action of F; that is, it sends each element of Y back to the element of X that it came from. This function is called the *inverse function* for F.

Theorem 7.2.2

Suppose $F: X \to Y$ is a one-to-one correspondence; in other words, suppose F is one-to-one and onto. Then there is a function $\boldsymbol{F^{-1}: Y \to X}$ that is defined as follows: Given any element y in Y,

$$\boldsymbol{F^{-1}(y)} = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

Or, equivalently,

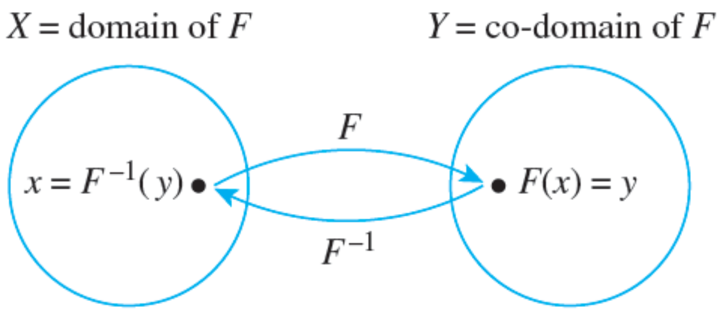
$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

Inverse Functions

Definition

The function F^{-1} is called the **inverse function** for F.

Note that according to this definition, the logarithmic function with base *b* > 0 and *b* ≠ 1 is the inverse of the exponential function with base *b*. The diagram that follows illustrates the fact that an inverse function sends each element back to where it came from.

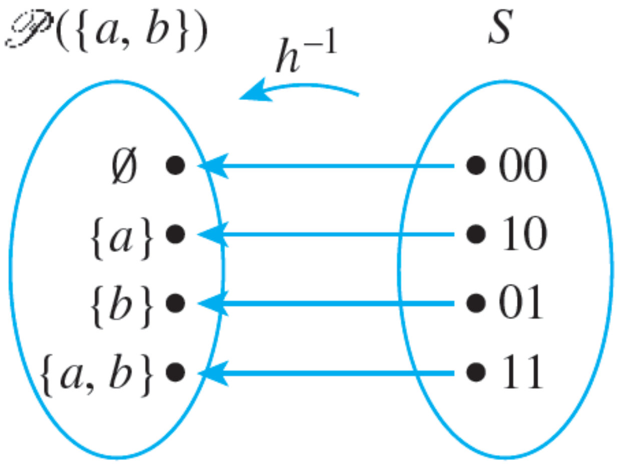


Example 7.2.11 – *Finding an Inverse Function for a Function Given by an Arrow Diagram*

Define the inverse function for the one-to-one correspondence *h* given in Example 7.2.8.

Example 7.2.11 – *Solution*

The arrow diagram for h^{-1} is obtained by tracing the *h*-arrows back from *S* to $\mathscr{P}(\{a, b\})$ as shown below.



$h^{-1}(00) = \emptyset \quad h^{-1}(10) = \{a\}$

$h^{-1}(01) = \{b\} \quad h^{-1}(11) = \{a, b\}$

Example 7.2.13 – *Finding an Inverse Function for a Function Given by a Formula*

The function $f : \mathbf{R} \to \mathbf{R}$ defined by the formula

$$f(x) = 4x - 1 \text{ for each real number } x$$

was shown to be one-to-one in Example 7.2.2 and onto in Example 7.2.5. Find its inverse function.

Example 7.2.13 – *Solution*

For any *[particular but arbitrarily chosen]* y in **R**, by definition of f^{-1},

$$f^{-1}(y) = \text{that unique real number } x \text{ such that } f(x) = y.$$

But $\qquad f(x) = y$

$\Leftrightarrow \quad 4x - 1 = y \qquad$ by definition of f

$\Leftrightarrow \quad x = \dfrac{y+1}{4} \qquad$ by algebra.

Hence $f^{-1}(y) = \dfrac{y+1}{4}$.

Inverse Functions

Theorem 7.2.3

If X and Y are sets and $F: X \to Y$ is one-to-one and onto, then $F^{-1}: Y \to X$ is also one-to-one and onto.