CHAPTER 7

PROPERTIES OF FUNCTIONS

7.1 Functions Defined on General Sets

Functions Defined on General Sets

Definition

A **function f from a set X to a set Y**, denoted $f: X \to Y$, is a relation from X, the **domain** of f, to Y, the **co-domain** of f, that satisfies two properties: (1) every element in X is related to some element in Y, and (2) no element in X is related to more than one element in Y.

Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically:

$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

Given an element y in Y, there may exist elements in X with y as their image. When x is an element such that $f(x) = y$, then x is called **a preimage of y** or **an inverse image of y**. The set of all inverse images of y is called *the inverse image of y*. Symbolically:

$$\textbf{the inverse image of } y = \{x \in X \mid f(x) = y\}.$$

Arrow Diagrams

Arrow Diagrams

If X and Y are finite sets, you can define a function f from X to Y by drawing an arrow diagram. You make a list of elements in X and a list of elements in Y, and draw an arrow from each element in X to the corresponding element in Y, as shown in Figure 7.1.1.

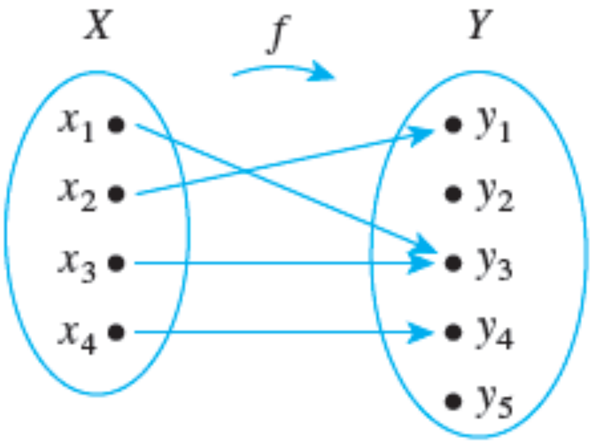


Figure 7.1.1

Example 7.1.1 – *Functions and Nonfunctions*

Which of the arrow diagrams in Figure 7.1.2 define functions from $X = \{a, b, c\}$ to $Y = \{1, 2, 3, 4\}$?

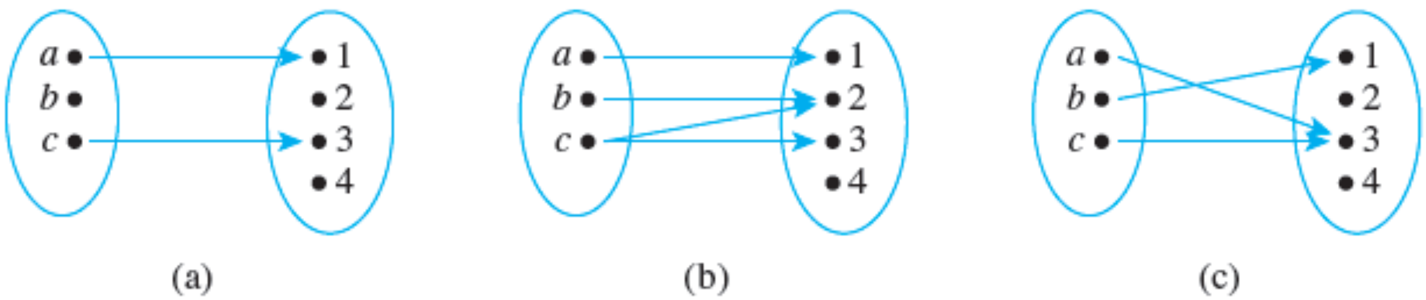


Figure 7.1.2

Example 7.1.1 – *Solution*

Only (c) defines a function.

In (a) the element *b* in *X* is not related to any element of *Y* because there is no arrow that points from *b* to an element in *Y*.

And in (b) the element *c* is not related to a *unique* element of *Y* because from *c* there are two arrows that point to two different elements of *Y*—one toward 2 and the other toward 3.

Arrow Diagrams

Theorem 7.1.1 A Test for Function Equality

If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for every $x \in X$.

Example 7.1.3 – *Equality of Functions*

a. Let $J_3 = \{0, 1, 2\}$, and define functions f and g from J_3 to J_3 as follows:

For every x in J_3,

$$f(x) = (x^2 + x + 1) \; mod \; 3 \quad \text{and} \quad g(x) = (x + 2)^2 \; mod \; 3.$$

Does $f = g$?

Example 7.1.3 – *Equality of Functions* continued

b. Let $F : \mathbf{R} \to \mathbf{R}$ and $G : \mathbf{R} \to \mathbf{R}$ be functions. Define new functions $F + G : \mathbf{R} \to \mathbf{R}$ and $G + F : \mathbf{R} \to \mathbf{R}$ as follows: For every $x \in R$,

$$(F + G)(x) = F(x) + G(x) \quad \text{and} \quad (G + F)(x) = G(x) + F(x).$$

Does $F + G = G + F$?

Example 7.1.3 – *Solution*

a. Yes, the table of values shows that $f(x) = g(x)$ for every x in J_3.

x	x^2+x+1	$f(x) = (x^2+x+1)\ mod\ 3$	$(x+2)^2$	$g(x) = (x+2)^2\ mod\ 3$
0	1	$1\ mod\ 3 = 1$	4	$4\ mod\ 3 = 1$
1	3	$3\ mod\ 3 = 0$	9	$9\ mod\ 3 = 0$
2	7	$7\ mod\ 3 = 1$	16	$16\ mod\ 3 = 1$

Example 7.1.3 – *Solution*

continued

b. Again the answer is yes. For every real number x,

$$(F + G)(x) = F(x) + G(x) \quad \text{by definition of } F + G$$

$$= G(x) + F(x) \quad \text{by the commutative law for addition of real numbers}$$

$$= (G + F)(x) \quad \text{by definition of } G + F.$$

Hence $F + G = G + F$.

Examples of Functions

Examples of Functions

The next examples illustrate some of the wide variety of different types of functions.

Example 7.1.4 – *Solution*

Whatever is input to the identity function comes out unchanged, so $I_X = \left(a_{ij}^k\right) = a_{ij}^k$ and $I_X(\phi(z)) = \phi(z)$.

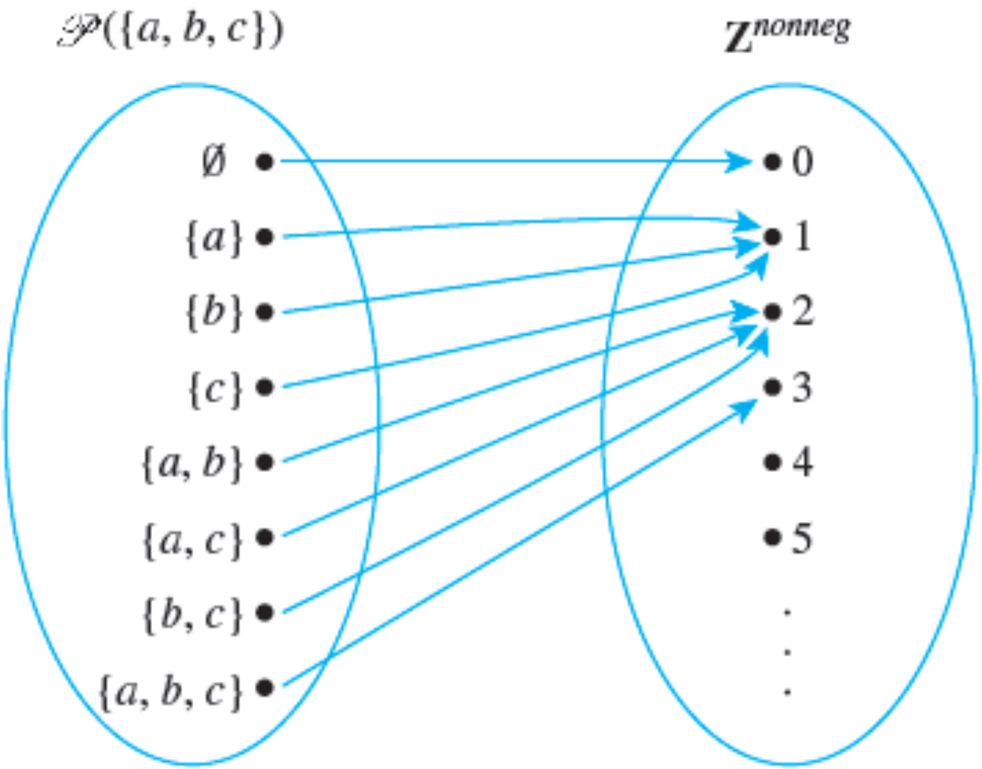
Example 7.1.6 – *A Function Defined on a Power Set*

We know that $\mathscr{P}(A)$ denotes the set of all subsets of the set *A*. Define a function $F : \mathscr{P}(\{a, b, c\}) \to \mathbf{Z}^{nonneg}$ as follows: For each $X \in \mathscr{P}(\{a, b, c\})$,

F(*X*) = the number of elements in *X*.

Draw an arrow diagram for *F*.

Example 7.1.6 – *Solution*



Examples of Functions

Definition Logarithms and Logarithmic Functions

Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x**, written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically:

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

The **logarithmic function with base b** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

Example 7.1.8 – *The Logarithmic Function with Base b*

Find the following:

a. $\log_3 9$

b. $\log_2\left(\frac{1}{2}\right)$

c. $\log_{10}(1)$

d. $\log_2(2^m)$ (*m* is any real number)

e. $2^{\log_2(m)} (m > 0)$

Example 7.1.8 – *Solution*

a. $\log_3 9 = 2$ because $3^2 = 9$.

b. $\log_2\left(\frac{1}{2}\right) = -1$ because $2^{-1} = \frac{1}{2}$.

c. $\log_{10}(1) = 0$ because $10^0 = 1$.

d. $\log_2(2^m) = m$ because the exponent to which 2 must be raised to obtain 2^m is m.

e. $2^{\log_2(m)} = m$ because $\log_2(m)$ is the exponent to which 2 must be raised to obtain m.

Example 7.1.10 – *The Hamming Distance Function*

The Hamming distance function, named after the computer scientist Richard W. Hamming, is very important in coding theory. It gives a measure of the "difference" between two strings of 0's and 1's that have the same length. Let S_n be the set of all strings of 0's and 1's of length n.

Define a function $H: S_n \times S_n \to \mathbf{Z}^{nonneg}$ as follows: For each pair of strings $(s, t) \in S_n \times S_n$,

$H(s, t)$ = the number of positions in which s and t have different values.

Example 7.1.10 – *The Hamming Distance Function* continued

Thus, letting $n = 5$,

$$H(11111, 00000) = 5$$

because 11111 and 00000 differ in all five positions, whereas

$$H(11000, 00000) = 2$$

because 11000 and 00000 differ only in the first two positions.

Example 7.1.10 – *The Hamming Distance Function* continued

a. Find $H(00101, 01110)$.

b. Find $H(10001, 01111)$.

Example 7.1.10 – *Solution*

a. 3

b. 4

Boolean Functions

Boolean Functions

Any input/output table defines a function in the following way: The elements in the input column can be regarded as ordered tuples of 0's and 1's; the set of all such ordered tuples is the domain of the function.

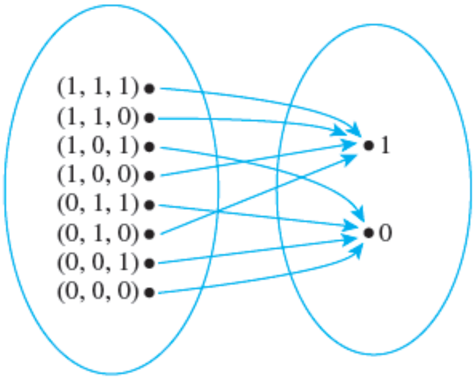
The elements in the output column are all either 0 or 1; thus {0, 1} is taken to be the co-domain of the function. The relation sends each input element to the output element in the same row.

Boolean Functions

Thus, for instance, the input/output table of Figure 7.1.4(a) defines the function with the arrow diagram shown in Figure 7.1.4(b).

Input			Output
P	Q	R	S
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

(a)



(b)

Two Representations of a Boolean Function

Figure 7.1.4

Boolean Functions

More generally, the input/output table corresponding to a circuit with *n* input wires has *n* input columns. Such a table defines a function from the set of all *n*-tuples of 0's and 1's to the set {0, 1}.

Definition

An (*n*-**place) Boolean function** f is a function whose domain is the set of all ordered n-tuples of 0's and 1's and whose co-domain is the set $\{0, 1\}$. More formally, the domain of a Boolean function can be described as the Cartesian product of n copies of the set $\{0, 1\}$, which is denoted $\{0, 1\}^n$. Thus $f: \{0, 1\}^n \rightarrow \{0, 1\}$.

Example 7.1.11 – *A Boolean Function*

Consider the three-place Boolean function defined from the set of all 3-tuples of 0's and 1's to {0, 1} as follows: For each triple (x_1, x_2, x_3) of 0's and 1's,

$$f\,(x_1, x_2, x_3) = (x_1 + x_2 + x_3)\ \mathit{mod}\ 2.$$

Describe f using an input/output table.

Example 7.1.11 – *Solution*

$$f(1, 1, 1) = (1 + 1 + 1) \text{ } mod \text{ } 2 = 3 \text{ } mod \text{ } 2 = 1$$

$$f(1, 1, 0) = (1 + 1 + 0) \text{ } mod \text{ } 2 = 2 \text{ } mod \text{ } 2 = 0$$

The rest of the values of *f* can be calculated similarly to obtain the following table.

Input			Output
x_1	x_2	x_3	$(x_1 + x_2 + x_3)$ *mod* 2
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

Checking Whether a Function Is Well Defined

Checking Whether a Function Is Well Defined

It can sometimes happen that what appears to be a function defined by a rule is not really a function at all. To give an example, suppose we wrote, "Define a function $f : \mathbf{R} \to \mathbf{R}$ by specifying that for each real number x,

$$f(x) \text{ is the real number } y \text{ such that } x^2 + y^2 = 1."$$

There are two distinct reasons why this description does not define a function. For almost all values of x, either (1) there is no y that satisfies the given equation or (2) there are two different values of y that satisfy the equation.

Checking Whether a Function Is Well Defined

For instance, when $x = 2$, there is no real number y such that $2^2 + y^2 = 1$, and when $x = 0$, both $y = -1$ and $y = 1$ satisfy the equation $0^2 + y^2 = 1$.

In general, we say that a "function" is **not well defined** if it fails to satisfy at least one of the requirements for being a function.

Example 7.1.12 – *A Function That Is Not Well Defined*

We know that **Q** represents the set of all rational numbers. Suppose you read that a function $f : \mathbf{Q} \to \mathbf{Z}$ is to be defined by the formula

$$f\left(\frac{m}{n}\right) = m \quad \text{for all integers } m \text{ and } n \text{ with } n \neq 0.$$

That is, the integer associated by f to the number $\frac{m}{n}$ is m.
Is f well defined? Why?

Example 7.1.12 – *Solution*

The function *f* is not well defined. The reason is that fractions have more than one representation as quotients of integers.

For instance, $\frac{1}{2} = \frac{3}{6}$.

Now if *f* were a function, then the definition of a function would imply that $f\left(\frac{1}{2}\right) = \left(\frac{3}{6}\right)$ since $\frac{1}{2} = \frac{3}{6}$.

Example 7.1.12 – *Solution*

continued

But applying the formula for *f*, you find that

$$f\left(\frac{1}{2}\right) = 1 \quad \text{and} \quad f\left(\frac{3}{6}\right) = 3,$$

and so

$$f\left(\frac{1}{2}\right) \neq \left(\frac{3}{6}\right).$$

This contradiction shows that *f* is not well defined and, therefore, is not a function.

Functions Acting on Sets

Functions Acting on Sets

Given a function from a set X to a set Y, you can consider the set of images in Y of all the elements in a subset of X and the set of inverse images in X of all the elements in a subset of Y.

Definition

If $f: X \to Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

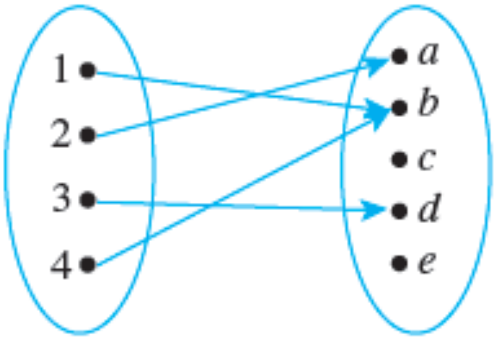
$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

and
$$f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of A**, and $f^{-1}(C)$ is called the **inverse image of C**.

Example 7.1.13 – *The Action of a Function on Subsets of a Set*

Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d, e\}$, and define $F : X \to Y$ by the following arrow diagram:



Let $A = \{1, 4\}$, $C = \{a, b\}$, and $D = \{c, e\}$. Find $F(A)$, $F(X)$, $F^{-1}(C)$, and $F^{-1}(D)$.

Example 7.1.13 – *Solution*

Let $A = \{1, 4\}$, $C = \{a, b\}$, and $D = \{c, e\}$.
Find $F(A)$, $F(X)$,

$F(A) = \{b\}$

$F(X) = \{a, b, d\}$

$F^{-1}(C) = \{1, 2, 4\}$

$F^{-1}(D) = \varnothing$