CHAPTER 5

SEQUENCES, MATHEMATICAL INDUCTION, AND RECURSION

5.1 Sequences

Sequences

Definition

A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

We typically represent a sequence as a set of elements written in a row. In the sequence denoted

$$a_m, a_{m+1}, a_{m+2}, \ldots, a_n,$$

each individual element a_k (read "*a* sub *k*") is called a **term**.

Sequences

The notation

$$a_m, a_{m+1}, a_{m+2}, \ldots$$

denotes an **infinite sequence**. An **explicit formula** or **general formula** for a sequence is a rule that shows how the values of a_k depend on k.

Example 5.1.1 – *Finding Terms of Sequences Given by Explicit Formulas*

Define sequences $a_1, a_2, a_3, \ldots$ and $b_2, b_3, b_4, \ldots$ by the following explicit formulas:

$$a_k = \frac{k}{k+1} \quad \text{for every integer } k \geq 1,$$

$$b_i = \frac{i-1}{i} \quad \text{for every integer } i \geq 2.$$

Compute the first five terms of both sequences.

Example 5.1.1 – *Solution*

$$a_1 = \frac{1}{1+1} = \frac{1}{2} \qquad b_2 = \frac{2-1}{2} = \frac{1}{2}$$

$$a_2 = \frac{2}{2+1} = \frac{2}{3} \qquad b_3 = \frac{3-1}{3} = \frac{2}{3}$$

$$a_3 = \frac{3}{3+1} = \frac{3}{4} \qquad b_4 = \frac{4-1}{4} = \frac{3}{4}$$

$$a_4 = \frac{4}{4+1} = \frac{4}{5} \qquad b_5 = \frac{5-1}{5} = \frac{4}{5}$$

$$a_5 = \frac{5}{5+1} = \frac{5}{6} \qquad b_6 = \frac{6-1}{6} = \frac{5}{6}$$

Example 5.1.1 – *Solution*

continued

As you can see, the first terms of both sequences are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$; in fact, it can be shown that all terms of both sequences are identical.

Example 5.1.2 – *An Alternating Sequence*

Compute the first six terms of the sequence c_0, c_1, c_2, … defined as follows:

$$c_j = (-1)^j \quad \text{for every integer } j \geq 0.$$

Example 5.1.2 – *Solution*

$$\begin{aligned} c_0 &= (-1)^0 = 1 \\ c_1 &= (-1)^1 = -1 \\ c_2 &= (-1)^2 = 1 \\ c_3 &= (-1)^3 = -1 \\ c_4 &= (-1)^4 = 1 \\ c_5 &= (-1)^5 = -1 \end{aligned}$$

Thus the first six terms are 1, −1, 1, −1, 1, −1. Even powers of −1 equal 1 and odd powers of −1 equal −1. It follows that the sequence oscillates endlessly between 1 and −1.

Example 5.1.3 – *Finding an Explicit Formula to Fit Given Initial Terms*

Find an explicit formula for a sequence with the following initial terms:

$$1, \quad -\frac{1}{4}, \quad \frac{1}{9}, \quad -\frac{1}{16}, \quad \frac{1}{25}, \quad -\frac{1}{36}, \ldots$$

Example 5.1.3 – *Solution*

$$\frac{1}{1^2}, \quad \frac{(-1)}{2^2}, \quad \frac{1}{3^2}, \quad \frac{(-1)}{4^2}, \quad \frac{1}{5^2}, \quad \frac{(-1)}{6^2}.$$
$$\updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow \qquad \updownarrow$$
$$a_1 \qquad a_2 \qquad a_3 \qquad a_4 \qquad a_5 \qquad a_6$$

Now note that the denominator of each term equals the square of the subscript of that term, and that the numerator equals ±1.

Example 5.1.3 – *Solution*

continued

Hence

$$a_k = \frac{\pm 1}{k^2}.$$

Also, the numerator oscillates back and forth between +1 and −1; it is +1 when *k* is odd and −1 when *k* is even.

To achieve this oscillation, insert a factor of $(-1)^{k+1}$ (or $(-1)^{k-1}$) into the formula for a_k.

Example 5.1.3 – *Solution*

continued

Consequently, an explicit formula that gives the correct first six terms is

$$a_k = \frac{(-1)^{k+1}}{k^2} \quad \text{for every integer } k \geq 1.$$

Note that making the first term a_0 would have led to the alternative formula

$$a_k = \frac{(-1)^{k}}{(k+1)^2} \quad \text{for every integer } k \geq 0.$$

Sequences

Definition

If m and n are integers and $m \leq n$, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from k equals m to n of a-sub-k**, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We say that $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is the **expanded form** of the sum, and we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

We call k the **index** of the summation, m the **lower limit** of the summation, and n the **upper limit** of the summation.

Example 5.1.4 – *Computing Summations*

Let $a_1 = -2$, $a_2 = -1$, $a_3 = 0$, $a_4 = 1$, and $a_5 = 2$. Compute the following:

a. $\displaystyle\sum_{k=1}^{5} a_k$ b. $\displaystyle\sum_{k=2}^{2} a_k$ c. $\displaystyle\sum_{k=1}^{2} a_{2k}$

Example 5.1.4 – *Solution*

a. $\displaystyle\sum_{k=1}^{5} a_k = a_1 + a_2 + a_3 + a_4 + a_5 = (-2) + (-1) + 0 + 1 + 2 = 0$

b. $\displaystyle\sum_{k=2}^{2} a_k = a_2 = -1$

c. $\displaystyle\sum_{k=1}^{2} a_{2k} = a_{2\cdot 1} + a_{2\cdot 2} = a_2 + a_4 = -1 + 1 = 0$

Sequences

The terms of a summation are expressed using an explicit formula. For instance, it is common to see summations such as

$$\sum_{k=1}^{5} k^2 \quad \text{or} \quad \sum_{i=0}^{8} \frac{(-1)^i}{i+1}.$$

Example 5.1.5 – *When the Terms of a Summation Are Given by a Formula*

Compute

$$\sum_{k=1}^{5} k^2.$$

Example 5.1.5 – *Solution*

$$\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55.$$

Sequences

When the upper limit of a summation is a variable, an ellipsis is used to write the summation in expanded form.

Example 5.1.6 – *Changing from Summation Notation to Expanded Form*

Write $\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$ in expanded form:

Example 5.1.6 – *Solution*

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

Example 5.1.7 – *Changing from Expanded Form to Summation Notation*

Express the following using summation notation:

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n}.$$

Example 5.1.7 – *Solution*

The general term of this summation can be expressed as

$\dfrac{i+1}{n+i}$ for each integer *i* from 0 to *n*.

Hence

$$\frac{1}{n} + \frac{2}{n+1} + \frac{3}{n+2} + \cdots + \frac{n+1}{2n} = \sum_{i=0}^{n} \frac{i+1}{n+i}.$$

Example 5.1.9 – *Using a Single Summation Sign and Separating Off a Final Term*

a. Write $\sum_{k=0}^{n} 2^k + 2^{n+1}$ as a single summation.

b. Rewrite $\sum_{i=1}^{n+1} \frac{1}{i^2}$ by separating off the final term.

Example 5.1.9 – *Solution*

a. $$\sum_{k=0}^{n} 2^k + 2^{k+1} = (2^0 + 2^1 + 2^2 + \cdots + 2^n) + 2^{n+1} = \sum_{k=0}^{n+1} 2^k$$

b. $$\sum_{i=1}^{n+1} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \frac{1}{(n+1)^2} = \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{(n+1)^2}$$

Example 5.1.10 – *A Telescoping Sum*

Some sums can be transformed so that successive cancellation of terms collapses the final result like a telescope. For instance, observe that for every integer $k \geq 1$,

$$\frac{1}{k} - \frac{1}{k+1} = \frac{(k+1)-k}{k(k+1)} = \frac{1}{k(k+1)}.$$

Use this identity to find a simple expression for $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Example 5.1.10 – *Solution*

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}} \right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}} \right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1} \right)$$

$$= 1 - \frac{1}{n+1}$$

Product Notation

Product Notation

Definition

If m and n are integers and $m \leq n$, the symbol $\prod_{k=m}^{n} a_k$, read the **product from *k* equals *m* to *n* of *a*-sub-*k***, is the product of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$.

We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

Product Notation

A recursive definition for the product notation is the following: If *m* is any integer, then

$$\prod_{k=m}^{m} a_k = a_m \quad \text{and} \quad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for every integer } n > m.$$

Example 5.1.11 – *Computing Products*

Compute the following products:

a. $\prod_{k=1}^{5} k$

b. $\prod_{k=1}^{1} \frac{k}{k+1}$

Example 5.1.11 – *Solution*

a. $\displaystyle\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

b. $\displaystyle\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} = \frac{1}{2}$

Properties of Summations and Products

Properties of Summations and Products

The following theorem states general properties of summations and products.

Theorem 5.1.1

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and c is any real number, then the following equations hold for any integer $n \geq m$:

1. $\displaystyle\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (a_k + b_k)$

2. $\displaystyle c \cdot \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \cdot a_k$ generalized distributive law

3. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k).$

Example 5.1.12 – *Using Properties of Summation and Product*

Let $a_k = k + 1$ and $b_k = k - 1$ for every integer k. Write each of the following expressions as a single summation or product:

a. $\displaystyle\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k$

b. $\displaystyle\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right)$

Example 5.1.12 – *Solution*

a. $$\sum_{k=m}^{n} a_k + 2 \cdot \sum_{k=m}^{n} b_k = \sum_{k=m}^{n} (k+1) + 2 \cdot \sum_{k=m}^{n} (k-1)$$ by substitution

$$= \sum_{k=m}^{n} (k+1) + \sum_{k=m}^{n} 2 \cdot (k-1)$$ by Theorem 5.1.1 (2)

$$= \sum_{k=m}^{n} ((k+1) + 2 \cdot (k-1))$$ by Theorem 5.1.1 (1)

$$= \sum_{k=m}^{n} (3k-1)$$ by algebraic simplification

Example 5.1.12 – *Solution*

continued

b. $$\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \left(\prod_{k=m}^{n} (k+1)\right) \cdot \left(\prod_{k=m}^{n} (k-1)\right)$$ by substitution

$$= \prod_{k=m}^{n} ((k+1) \cdot (k-1))$$ by Theorem 5.1.1 (3)

$$= \prod_{k=m}^{n} (k^2 - 1)$$ by algebraic simplification

Change of Variable

Change of Variable

Observe that

$$\sum_{k=1}^{3} k^2 = 1^2 + 2^2 + 3^2$$

and

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2.$$

Hence

$$\sum_{k=1}^{3} k^2 = \sum_{i=1}^{3} i^2.$$

Change of Variable

The appearance of a summation can be altered by more complicated changes of variable as well. For example, observe that

$$\sum_{j=2}^{4}(j-1)^2 = (2-1)^2 + (3-1)^2 + (4-1)^2$$

$$= 1^2 + 2^2 + 3^2$$

$$= \sum_{k=1}^{3} k^2.$$

Example 5.1.13 – *Transforming a Sum by a Change of Variable*

Transform the following summation by making the specified change of variable:

$$\textit{summation: } \sum_{k=0}^{6} \frac{1}{k+1} \qquad \textit{change of variable: } j = k+1$$

Example 5.1.13 – *Solution*

First calculate the lower and upper limits of the new summation:

When $k = 0$, $j = k + 1 = 0 + 1 = 1$.

When $k = 6$, $j = k + 1 = 6 + 1 = 7$.

Thus, the new sum goes from $j = 1$ to $j = 7$.

Example 5.1.13 – *Solution*

continued

Next calculate the general term of the new summation. You will need to replace each occurrence of *k* by an expression in *j*:

Since $j = k + 1$, then $k = j - 1$.

$$\text{Hence } \frac{1}{k+1} = \frac{1}{(j-1)+1} = \frac{1}{j}.$$

Finally, put the steps together to obtain

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j}. \qquad 5.1.1$$

Example 5.1.14 – *When the Upper Limit Appears in the Expression to Be Summed*

Rewrite the summation $\sum_{k=1}^{n+1}\left(\frac{k}{n+k}\right)$ so that the lower limit becomes 0 and the upper limit becomes *n*, but the index of the summation remains *k*.

a. First, transform the summation by making the change of variable $j = k - 1$.

b. Second, transform the summation obtained in part (a) by changing all *j*'s to *k*'s.

Example 5.1.14 – *Solution*

a. The index variable k is bound by the summation symbol to take each of the values from 1 to $n + 1$ in succession.

When $k = 1$, then $j = 1 - 1 = 0$, and when $k = n + 1$, then $j = (n + 1) - 1 = n$.

So, the new lower limit is 0 and the new upper limit is n.

Example 5.1.14 – *Solution*

continued

In addition, since $j = k - 1$, then $k = j + 1$.

Thus

$$\frac{k}{n+k} = \frac{j+1}{n+(j+1)}$$

and so, the general term of the new summation is

$$\frac{j+1}{n+(j+1)}.$$

Therefore,

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{j=0}^{n} \frac{j+1}{n+(j+1)}.$$

5.1.3

Example 5.1.14 – *Solution*

continued

b. Changing all the *j*'s to *k*'s in the right-hand side of equation (5.1.3) gives

$$\sum_{j=0}^{n} \frac{j+1}{n+(j+1)} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}. \quad 5.1.4$$

Combining equations (5.1.3) and (5.1.4) results in

$$\sum_{k=1}^{n+1} \frac{k}{n+k} = \sum_{k=0}^{n} \frac{k+1}{n+(k+1)}.$$

Factorial and “*n* Choose *r*” Notation

Factorial and "*n* Choose *r*" Notation

Definition

For each positive integer n, the quantity ***n* factorial** denoted $\boldsymbol{n!}$, is defined to be the product of all the integers from 1 to n:

$$n! = n\cdot(n-1)\cdots 3\cdot 2\cdot 1.$$

Zero factorial, denoted 0!, is defined to be 1:

$$0! = 1.$$

Factorial and "*n* Choose *r*" Notation

A recursive definition for factorial is the following: Given any nonnegative integer *n*,

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1. \end{cases}$$

The next example illustrates the usefulness of the recursive definition for making computations.

Example 5.1.16 – *Computing with Factorials*

Simplify the following expressions:

a. $\dfrac{8!}{7!}$ b. $\dfrac{5!}{2!\cdot 3!}$ c. $\dfrac{1}{2!\cdot 4!}+\dfrac{1}{3!\cdot 3!}$ d. $\dfrac{(n+1)!}{n!}$ e. $\dfrac{n!}{(n-3)!}$

Example 5.1.16 – *Solution*

a. $\dfrac{8!}{7!} = \dfrac{8 \cdot \cancel{7!}}{\cancel{7!}} = 8$

b. $\dfrac{5!}{2! \cdot 3!} = \dfrac{5 \cdot 4 \cdot \cancel{3!}}{2! \cdot \cancel{3!}} = \dfrac{5 \cdot 4}{2 \cdot 1} = 10$

c. $\dfrac{1}{2! \cdot 4!} + \dfrac{1}{3! \cdot 3!} = \dfrac{1}{2! \cdot 4!} \cdot \dfrac{3}{3} + \dfrac{1}{3! \cdot 3!} \cdot \dfrac{4}{4}$

by multiplying each numerator and denominator by just what is necessary to obtain a common denominator

$= \dfrac{3}{3 \cdot 2! \cdot 4!} + \dfrac{4}{3! \cdot 4 \cdot 3!}$

by rearranging factors

Example 5.1.16 – *Solution*

continued

$$= \frac{3}{3!\cdot 4!} + \frac{4}{3!\cdot 4!}$$

because $3\cdot 2! = 3!$ and $4\cdot 3! = 4!$

$$= \frac{7}{144}$$

by the rule for adding fractions with a common denominator

d. $$\frac{(n+1)!}{n!} = \frac{(n+1)\cdot \cancel{n!}}{\cancel{n!}} = n+1$$

e. $$\frac{n!}{(n-3)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdot\cancel{(n-3)!}}{\cancel{(n-3)!}} = n\cdot(n-1)\cdot(n-2)$$

$$= n^3 - 3n^2 + 2n$$

Factorial and "n Choose r" Notation

Definition

Let n and r be integers with $0 \le r \le n$. The symbol

$$\binom{n}{r}$$

is read "***n* choose *r***" and represents the number of subsets of size r that can be chosen from a set with n elements.

Factorial and “*n* Choose *r*” Notation

Formula for Computing $\binom{n}{r}$

For all integers n and r with $0 \le r \le n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Example 5.1.17 – *Computing* $\binom{n}{r}$

Use the formula for computing $\binom{n}{r}$ to evaluate the following expressions:

a. $\binom{8}{5}$ b. $\binom{4}{4}$ c. $\binom{n+1}{n}$

Example 5.1.17 – *Solution*

a. $\binom{8}{5} = \dfrac{8!}{5!(8-5)!}$

$$= \frac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3 \cdot 2 \cdot 1}}{(\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1) \cdot (\cancel{3 \cdot 2 \cdot 1})}$$

always cancel common factors before multiplying

$= 56.$

Example 5.1.17 – *Solution*

continued

b. $\binom{4}{4} = \dfrac{4!}{4!(4-4)!} = \dfrac{4!}{4!0!} = \dfrac{\cancel{4\cdot 3\cdot 2\cdot 1}}{(\cancel{4\cdot 3\cdot 2\cdot 1})(1)} = 1$

The fact that $0! = 1$ makes this formula computable. It gives the correct value because a set of size 4 has exactly one subset of size 4, namely itself.

c. $\binom{n+1}{n} = \dfrac{(n+1)!}{n!((n+1)-n)!} = \dfrac{(n+1)!}{n!1!} = \dfrac{(n+1)\cdot \cancel{n!}}{\cancel{n!}} = n+1$