CHAPTER 5

SEQUENCES, MATHEMATICAL INDUCTION, AND RECURSION

5.2 Mathematical Induction I: Proving Formulas

Mathematical Induction Ⅰ: Proving Formulas

Once proved by mathematical induction, a theorem is known just as certainly as if were proved by any other mathematical method.

Principle of Mathematical Induction

Let $P(n)$ be a property that is defined for integers n, and let a be a fixed integer. Suppose the following two statements are true:

1. $P(a)$ is true.
2. For every integer $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true.

Then the statement

$$\text{for every integer } n \geq a, P(n)$$

is true.

Mathematical Induction I: Proving Formulas

Proving a statement by mathematical induction is a two-step process. The first step is called the *basis step*, and the second step is called the *inductive step*.

Method of Proof by Mathematical Induction

Consider a statement of the form, "For every integer $n \geq a$, a property $P(n)$ is true." To prove such a statement, perform the following two steps:

Step 1 (basis step): Show that $P(a)$ is true.

Step 2 (inductive step): Show that for every integer $k \geq a$, if $P(k)$ is true then $P(k+1)$ is true. To perform this step,

suppose that $P(k)$ is true, where k is any particular but arbitrarily chosen integer with $k \geq a$.

[This supposition is called the **inductive hypothesis***.]*

Then

show that $P(k+1)$ is true.

Example 5.2.1 – *Sum of the First n Integers*

Use mathematical induction to prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for every integer $n \geq 1$.

Example 5.2.1 – *Solution*

To construct a proof by induction, you must first identify the property $P(n)$. In this case, $P(n)$ is the equation

$$\boxed{1 + 2 + \cdots + n = \frac{n(n+1)}{2}.} \quad \leftarrow \text{the property } (P(n))$$

Note The property is just the equation. The proof will show that the equation is true for every integer $n \geq 1$. *[To see that* $P(n)$ *is a sentence, note that its subject is "the sum of the integers from* 1 *to n" and its verb is "equals."]*

Example 5.2.1 – *Solution*

continued

In the basis step of the proof, you must show that the property is true for $n = 1$, or, in other words, that $P(1)$ is true. Now $P(1)$ is obtained by substituting 1 in place of n in $P(n)$.

The left-hand side of $P(1)$ is the sum of all the successive integers starting at 1 and ending at 1. This is just 1.

Note To write $P(1)$, just copy $P(n)$ and replace each n by 1.

Example 5.2.1 – *Solution*

continued

Thus $P(1)$ is

$$\boxed{1 = \frac{1(1+1)}{2}.} \qquad \leftarrow \text{basis } (P(1))$$

Of course, this equation is true because the right-hand side is

$$\frac{1(1+1)}{2} = \frac{1 \cdot 2}{2} = 1,$$

which equals the left-hand side.

Example 5.2.1 – *Solution*

continued

In the inductive step, you assume that $P(k)$ is true, for a particular but arbitrarily chosen integer k with $k \geq 1$. *[This assumption is the inductive hypothesis.]* You must then show that $P(k + 1)$ is true. What are $P(k)$ and $P(k + 1)$? $P(k)$ is obtained by substituting k for every n in $P(n)$.

Thus $P(k)$ is

$$\boxed{1 + 2 + \cdots + k = \frac{k(k+1)}{2}.} \qquad \leftarrow \text{inductive hypothesis } (P(k))$$

Example 5.2.1 – *Solution*

continued

Similarly, $P(k + 1)$ is obtained by substituting the quantity $(k + 1)$ for every n that appears in $P(n)$. Thus $P(k + 1)$ is

$$1 + 2 + \cdots + (k+1) = \frac{(k+1)((k+1)+1)}{2},$$

or, equivalently,

$$\boxed{1 + 2 + \cdots + (k+1) = \frac{(k+1)(k+2)}{2}.} \quad \leftarrow \text{to show } (P(k+1))$$

Example 5.2.1 – *Solution*

continued

Now the inductive hypothesis is the supposition that $P(k)$ is true. How can this supposition be used to show that $P(k + 1)$ is true? $P(k + 1)$ is an equation, and the truth of an equation can be shown in a variety of ways.

One of the most straightforward is to use the inductive hypothesis along with algebra and other known facts to separately transform the left-hand and right-hand sides until you see that they are the same.

Example 5.2.1 – *Solution* continued

In this case, the left-hand side of $P(k + 1)$ is

$$1 + 2 + \ldots + (k + 1),$$

which equals

$$(1 + 2 + \ldots + k) + (k + 1)$$

The next-to-last term is k because the terms are successive integers and the last term is $k + 1$.

By substitution from the inductive hypothesis,

$$(1 + 2 + \ldots + k) + (k + 1)$$

Example 5.2.1 – *Solution* continued

$$= \frac{k(k+1)}{2} + (k+1)$$

since the inductive hypothesis says that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

by multiplying the numerator and denominator of the second term by 2 to obtain a common denominator

$$= \frac{k^2+k}{2} + \frac{2k+2}{2}$$

by multiplying out the two numerators

$$= \frac{k^2+3k+2}{2}$$

by adding fractions with the same denominator and combining like terms.

Example 5.2.1 – *Solution*

continued

So, the left-hand side of $P(k + 1)$ is $\frac{k^2 + 3k + 2}{2}$.

Now the right-hand side of $P(k + 1)$ is $\frac{(k+1)(k+2)}{2} = \frac{k^2+3k+2}{2}$ by multiplying out the numerator.

Thus the two sides of $P(k + 1)$ are equal to each other, and so the equation $P(k + 1)$ is true.

Mathematical Induction I: Proving Formulas

Theorem 5.2.1 Sum of the First n Integers

For every integer $n \geq 1$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Definition

If a sum with a variable number of terms is shown to equal an expression that does not contain either an ellipsis or a summation symbol, we say that the sum is written **in closed form.**

Example 5.2.2 – *Applying the Formula for the Sum of the First n Integers*

a. Evaluate $2 + 4 + 6 + ... + 500$.

b. Evaluate $5 + 6 + 7 + 8 + ... + 50$.

c. For an integer $h \geq 2$, write $1 + 2 + 3 + ... + (h - 1)$ in closed form.

Example 5.2.2 – *Solution*

a. $2 + 4 + 6 + \ldots + 500 = 2 \cdot (1 + 2 + 3 + \ldots + 250)$

$$= 2 \cdot \left(\frac{250 \cdot 251}{2} \right)$$

by applying the formula for the sum of the first n integers with $n = 250$.

$$= 62{,}750$$

Example 5.2.2 – *Solution*

continued

b. $5 + 6 + 7 + 8 + \ldots + 50 = (1 + 2 + 3 + \ldots + 50) - (1 + 2 + 3 + 4)$

$$= \frac{50 \cdot 51}{2} - 10$$

by applying the formula for the sum of the first n integers with $n = 50$.

$$= 1{,}265$$

Example 5.2.2 – *Solution*

continued

c. $1 + 2 + 3 + \ldots + (h - 1)$

$$= \frac{(h-1)\cdot[(h-1)+1]}{2}$$

by applying the formula for the sum of the first n integers with $n = h - 1$

$$= \frac{(h-1)\cdot h}{2}$$

since $(h - 1) + 1 = h$.

Mathematical Induction I: Proving Formulas

Another famous and important formula in mathematics—the formula for the sum of a geometric sequence.

In a **geometric sequence**, each term is obtained from the preceding one by multiplying by a constant factor. If the first term is 1 and the constant factor is *r*, then the sequence is $1, r, r^2, r^3, \ldots, r^n, \ldots$.

Mathematical Induction I: Proving Formulas

Theorem 5.2.2 Sum of a Geometric Sequence

For any real number r except 1, and any integer $n \geq 0$,

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

Proof (by mathematical induction):
Suppose *r* is a particular but arbitrarily chosen real number that is not equal to 1, and let the property *P*(*n*) be the equation

$$\sum_{i=0}^{n} r^i = \frac{r^{i+1} - 1}{r - 1} \qquad \leftarrow P(n)$$

We must show that *P*(*n*) is true for all integers $n \geq 0$. We do this by mathematical induction on *n*.

Example 5.2.3 – *Solution*

cont'd

Show that *P*(0) is true:

To establish *P*(0), we must show that

$$\sum_{i=0}^{0} r^i = \frac{r^{0+1}-1}{r-1} \qquad \leftarrow P(0)$$

The left-hand side of this equation is $r^0 = 1$ and the right-hand side is

$$\frac{r^{0+1}-1}{r-1} = \frac{r-1}{r-1} = 1$$

Also, because $r^1 = r$ and $r \neq 1$. Hence *P*(0) is true.

Example 5.2.3 – *Solution* cont'd

Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k + 1)$ is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 0. That is:]

Let k be any integer with $k \geq 0$, and suppose that

$$\sum_{i=0}^{k} r^i = \frac{r^{k+1} - 1}{r - 1} \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that P(k + 1) is true. That is:] We must show that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{(k+1)+1} - 1}{r - 1}.$$

Example 5.2.3 – *Solution* cont'd

Or, equivalently, that

$$\sum_{i=0}^{k+1} r^i = \frac{r^{k+2} - 1}{r - 1}. \qquad \leftarrow P(k+1)$$

[We will show that the left-hand side of P(*k* + 1) *equals the right-hand side.]* The left-hand side of *P*(*k* + 1) is

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1}$$

by writing the $(k+1)$st term separately from the first k terms

$$= \frac{r^{k+1} - 1}{r - 1} + r^{k+1}$$

by substitution from the inductive hypothesis

Example 5.2.3 – *Solution* cont'd

$$= \frac{r^{k+1} - 1}{r - 1} + \frac{r^{k+1}(r - 1)}{r - 1}$$

by multiplying the numerator and denominator of the second term by $(r - 1)$ to obtain a common denominator

$$= \frac{(r^{k+1} - 1) + r^{k+1}(r - 1)}{r - 1}$$

by adding fractions

$$= \frac{r^{k+1} - 1 + r^{k+2} - r^{k+1}}{r - 1}$$

by multiplying out and using the fact that $r^{k+1} \cdot r = r^{k+1} \cdot r^1 = r^{k+2}$

$$= \frac{r^{k+2} - 1}{r - 1}$$

by canceling the r^{k+1}'s.

which is the right-hand side of $P(k + 1)$ *[as was to be shown.]*

[Since we have proved the basis step and the inductive step, we conclude that the theorem is true.]

Proving an Equality

Deducing Additional Formulas

Deducing Additional Formulas

The formula for the sum of a geometric sequence can be thought of as a family of different formulas in r, one for each real number r except 1.

Example 5.2.4 – *Applying the Formula for the Sum of a Geometric Sequence*

In each of (a) and (b) below, assume that *m* is an integer that is greater than or equal to 3. Write each of the sums in closed form.

a. $1 + 3 + 3^2 + \cdots + 3^{m-2}$

b. $3^2 + 3^3 + 3^4 + \cdots + 3^m$

Example 5.2.4 – *Solution*

a. $1 + 3 + 3^2 + \cdots + 3^{m-2} = \dfrac{3^{(m-2)+1} - 1}{3 - 1}$ by applying the formula for the sum of a geometric sequence with $r = 3$ and $n = m - 2$

$$= \frac{3^{m-1} - 1}{2}$$

b. $3^2 + 3^3 + 3^4 + \cdots + 3^m = 3^2 \cdot (1 + 3 + 3^2 + \cdots + 3^{m-2})$ by factoring out 3^2

$$= 9 \cdot \left(\frac{3^{m-1} - 1}{2} \right)$$ by part (a).