CHAPTER 8

PROPERTIES OF RELATIONS

8.2

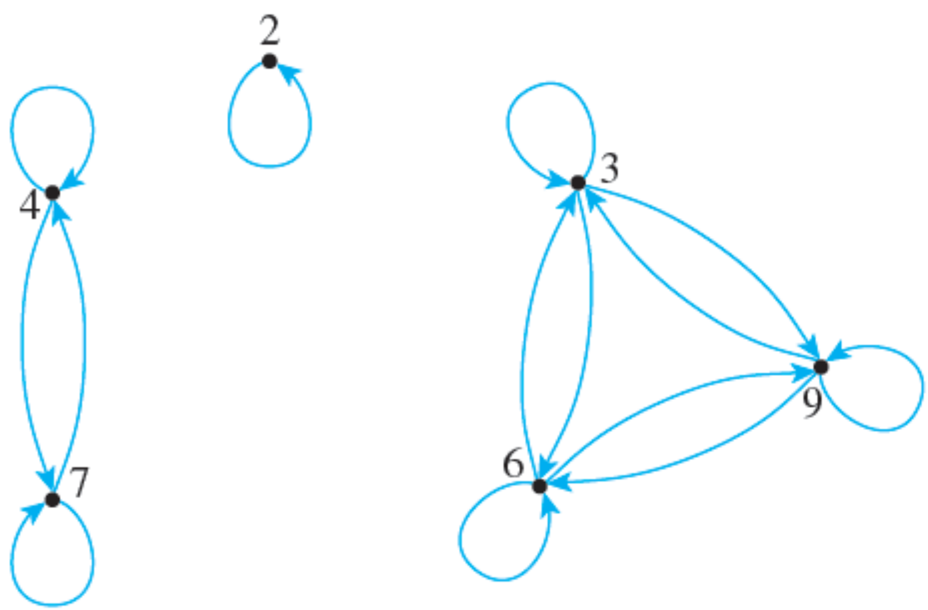
Reflexivity, Symmetry, and Transitivity

Reflexivity, Symmetry, and Transitivity

Definition

Let R be a relation on a set A.

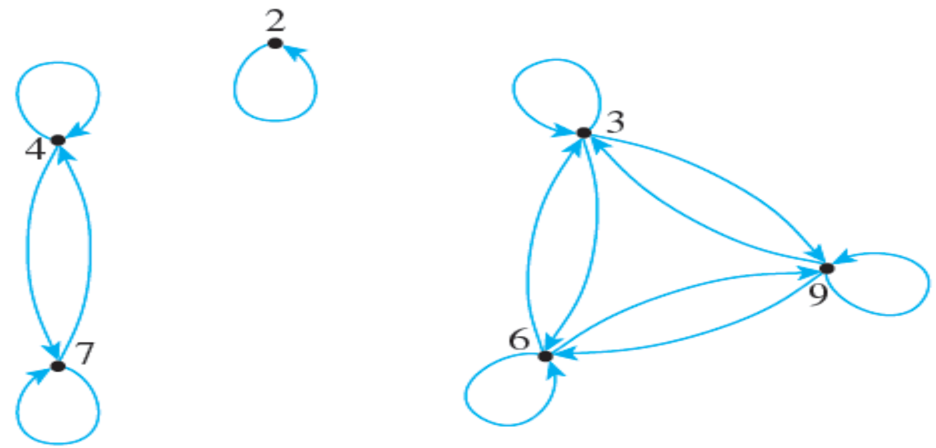
1. R is **reflexive** if, and only if, for every $x \in A$, $x \, R \, x$.
2. R is **symmetric** if, and only if, for every $x, y \in A$, if $x \, R \, y$ then $y \, R \, x$.
3. R is **transitive** if, and only if, for every $x, y, z \in A$, if $x \, R \, y$ and $y \, R \, z$ then $x \, R \, z$.



Reflexivity, Symmetry, and Transitivity

The equivalence of the expressions $x\ R\ y$ and $(x, y) \in R$ for every x and y in A, the reflexive, symmetric, and transitive properties can also be written as follows:

1. R is reflexive $\Leftrightarrow$ for every x in A, $(x, x) \in R$.
2. R is symmetric $\Leftrightarrow$ for every x and y in A, ***if*** $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive $\Leftrightarrow$ for every x, y, and z in A, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.



Reflexivity, Symmetry, and Transitivity

This means that to prove a relation has one of the properties, you use either the method of exhaustion or the method of generalizing from the generic particular.

Recall that the negation of a universal statement is existential. Hence if *R* is a relation on a set *A*, then

1. *R* is **not reflexive** ⇔ there is an element *x* in *A* such that $x \not R x$ *[that is, such that* $(x, x) \notin R$*]*.

Reflexivity, Symmetry, and Transitivity

2. *R* is **not symmetric** $\Leftrightarrow$ there are elements *x* and *y* in *A* such that *x R y* but $y \not R x$ *[that is, such that* (*x*, *y*) ∈ *R but* (*y*, *x*) ∉ *R]*.

3. *R* is **not transitive** $\Leftrightarrow$ there are elements *x*, *y*, and *z* in *A* such that *x R y* and *y R z* but $x \not R z$ *[that is, such that* (*x*, *y*) ∈ *R and* (*y*, *z*) ∈ *R but* (*x*, *z*) ∉ *R]*.

It follows that you can show that a relation does *not* have one of the properties by finding a counterexample.

Example 8.2.1 – *Properties of Relations on Finite Sets*

Let $A = \{0, 1, 2, 3\}$ and define relations R, S, and T on A as follows:

$R = \{(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2), (3, 0), (3, 3)\}$,

$S = \{(0, 0), (0, 2), (0, 3), (2, 3)\}$,

$T = \{(0, 1), (2, 3)\}$.

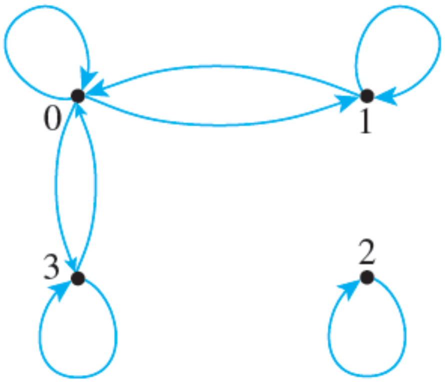
a. Is R reflexive? symmetric? transitive?

b. Is S reflexive? symmetric? transitive?

c. Is T reflexive? symmetric? transitive?

Example 8.2.1 – *Solution*

a. The directed graph of R has the appearance shown below.



R is reflexive: There is a loop at each point of the directed graph. This means that each element of A is related to itself, so R is reflexive.

Example 8.2.1 – *Solution*

continued

R is symmetric: In each case where there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first.

This means that whenever one element of A is related by R to a second, then the second is related to the first. Hence R is symmetric.

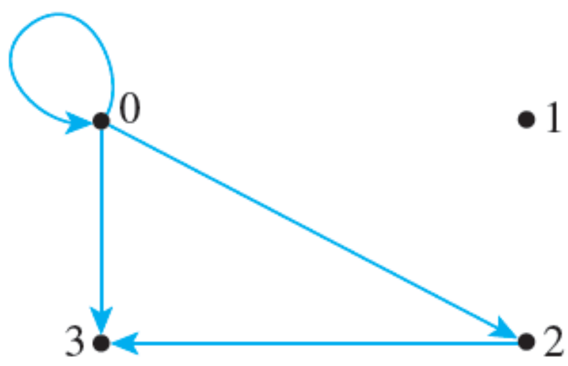
R is not transitive: There is an arrow going from 1 to 0 and an arrow going from 0 to 3, but there is no arrow going from 1 to 3.

Example 8.2.1 – *Solution*

continued

This means that there are elements of A—0, 1, and 3—such that $1\ R\ 0$ and $0\ R\ 3$ but $1 \not R\ 3$. Hence R is not transitive.

b. The directed graph of S has the appearance shown below.



Example 8.2.1 – *Solution*

continued

S is not reflexive: There is no loop at 1, for example. Thus $(1, 1) \notin S$, and so S is not reflexive.

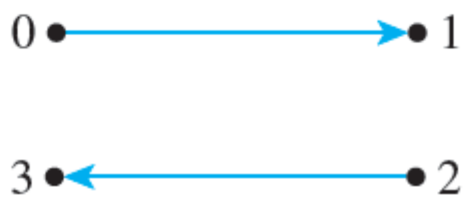
S is not symmetric: There is an arrow from 0 to 2 but not from 2 to 0. Hence $(0, 2) \in S$ but $(2, 0) \notin S$, and so S is not symmetric.

S is transitive: There are three cases for which there is an arrow going from one point of the graph to a second and from the second point to a third.

Example 8.2.1 – *Solution*

continued

c. The directed graph of *T* has the appearance shown below.



T is not reflexive: There is no loop at 0, for example. Thus $(0, 0) \notin T$, so *T* is not reflexive.

T is not symmetric: There is an arrow from 0 to 1 but not from 1 to 0. Thus $(0, 1) \in T$ but $(1, 0) \notin T$, and so *T* is not symmetric.

Example 8.2.1 – *Solution*

continued

T is transitive: The transitivity condition is vacuously true for T. To see this, observe that the transitivity condition says that

For every x, y, $z \in A$, if $(x, y) \in T$ and $(y, z) \in T$ then $(x, z) \in T$.

The only way for this to be false would be for there to exist elements of A that make the hypothesis true and the conclusion false.

Properties of Relations on Infinite Sets

Example 8.2.3 – *Properties of "Less Than"*

Define a relation R on **R** as follows: For all real numbers x and y,

$$\boxed{x\,R\,y \quad \Leftrightarrow \quad x < y.}$$

a. Is R reflexive? b. Is R symmetric? c. Is R transitive?

Example 8.2.3 – *Solution*

a. ***R is not reflexive*: ** R is reflexive if, and only if, $\forall x \in \mathbf{R}$, $x\ R\ x$. By definition of R, this means that $\forall x \in \mathbf{R}$, $x < x$. But this is false: $\exists x \in \mathbf{R}$ such that $x \nless x$.

   As a counterexample, let $x = 0$ and note that $0 \nless 0$. Hence R is not reflexive.

b. ***R is not symmetric*: ** R is symmetric if, and only if, $\forall x, y \in \mathbf{R}$, if $x\ R\ y$ then $y\ R\ x$. By definition of R, this means that $\forall x, y \in \mathbf{R}$, if $x < y$ then $y < x$.

Example 8.2.3 – *Solution*

continued

But this is false: $\exists x, y \in \mathbf{R}$ such that $x < y$ and $y \not< x$. As a counterexample, let $x = 0$ and $y = 1$ and note that $0 < 1$ but $1 \not< 0$. Hence R is not symmetric.

c. ***R is transitive***: R is transitive if, and only if, $\forall x, y, z \in \mathbf{R}$, if $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$. By definition of R, this means that $\forall x, y, z \in \mathbf{R}$, if $x < y$ and $y < z$, then $x < z$. But this statement is true by the transitive law of order for real numbers. Hence R is transitive.

Example 8.2.4 – *Properties of Congruence Modulo 3*

Define a relation *T* on **Z** (the set of all integers) as follows:
For all integers *m* and *n*,

$$\boxed{m\,T\,n \quad \Leftrightarrow \quad 3 \mid (m-n).}$$

This relation is called **congruence modulo 3**.

a. Is *T* reflexive? b. Is *T* symmetric? c. Is *T* transitive?

Example 8.2.4 – *Solution*

a. ***T is reflexive*:** To show that *T* is reflexive, it is necessary to show that For every $m \in \mathbf{Z}$, $m \; T \; m$.

By definition of *T*, this means that

$$\text{For every } m \in \mathbf{Z}, \quad 3 \mid (m - m),$$

which is true because $m - m = 0$ and $3 \mid 0$ (since $0 = 3 \cdot 0$). Hence *T* is reflexive.

b. ***T is symmetric*:** To show that *T* is symmetric, it is necessary to show that

For every $m, n \in \mathbf{Z}$, **if** $m \; T \; n$ then $n \; T \; m$.

Example 8.2.4 – *Solution* continued

By definition of *T* this means that

$$\text{For every } m, n \in \mathbf{Z}, \quad \textbf{if } 3 \mid (m - n) \text{ then } 3 \mid (n - m).$$

Is this true? Suppose *m* and *n* are particular but arbitrarily chosen integers such that $3 \mid (m - n)$.

Must it follow that $3 \mid (n - m)$? *[In other words, can we find an integer so that n − m = 3 ∙ (that integer)?]* By definition of "divides," since

$$3 \mid (m - n),$$

then $m - n = 3k$ for some integer *k*.

Example 8.2.4 – *Solution*

continued

The crucial observation is that $n - m = -(m - n)$. Hence, you can multiply both sides of this equation by -1 to obtain

$$-(m - n) = -3k,$$

which is equivalent to

$$n - m = 3(-k).$$

[Thus, we have found an integer, $-k$, so that $n - m = 3 \cdot$ (that integer).]

Example 8.2.4 – *Solution*

continued

Since $-k$ is an integer, this equation shows that

$$3 \mid (n - m).$$

It follows that T is symmetric.

c. ***T is transitive*:** To show that T is transitive, it is necessary to show that

For every m, n, $p \in \mathbf{Z}$, **if** $m\ T\ n$ and $n\ T\ p$ then $m\ T\ p$.

By definition of T this means that

For every $m, n \in \mathbf{Z}$, **if** $3 \mid (m - n)$ and $3 \mid (n - p)$ then $3 \mid (m - p)$.

Example 8.2.4 – *Solution* continued

Is this true? Suppose *m*, *n*, and *p* are particular but arbitrarily chosen integers such that $3 \mid (m-n)$ and $3 \mid (n-p)$.

Must it follow that $3 \mid (m-p)$? *[In other words, can we find an integer so that m − p = 3 · (that integer)?]* By definition of "divides," since

$$3 \mid (m-n) \quad \text{and} \quad 3 \mid (n-p),$$

then

$$m - n = 3r \qquad \text{for some integer } r,$$

and

$$n - p = 3s \qquad \text{for some integer } s.$$

Example 8.2.4 – *Solution* continued

The crucial observation is that $(m - n) + (n - p) = m - p$. Add these two equations together to obtain

$$(m - n) + (n - p) = 3r + 3s,$$

which is equivalent to

$$m - p = 3(r + s).$$

[Thus, we have found an integer so that $m - p = 3 \cdot$ *(that integer).]* Since r and s are integers, $r + s$ is an integer. So, this equation shows that

$$3 \mid (m - p).$$

It follows that T is transitive.

The Transitive Closure of a Relation

The Transitive Closure of a Relation

The relation obtained by adding the least number of ordered pairs to ensure transitivity is called the *transitive closure* of the relation. More precisely, the transitive closure of a relation is the smallest transitive relation that contains the relation.

Definition

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.

Example 8.2.5 – *Transitive Closure of a Relation*

Let $A = \{0, 1, 2, 3\}$ and consider the relation R defined on A as follows:

$$R = \{(0, 1), (1, 2), (2, 3)\}.$$

Find the transitive closure of R.

Example 8.2.5 – *Solution*

Every ordered pair in *R* is in R^t, so

$$\{(0, 1), (1, 2), (2, 3)\} \subseteq R^t.$$

Thus, the directed graph of *R* contains the arrows shown below.

Since there are arrows going from 0 to 1 and from 1 to 2, R^t must have an arrow going from 0 to 2.

Example 8.2.5 – *Solution* continued

Hence $(0, 2) \in R^t$. Then $(0, 2) \in R^t$ and $(2, 3) \in R^t$, so since R^t is transitive, $(0, 3) \in R^t$. Also, since $(1, 2) \in R^t$ and $(2, 3) \in R^t$, then $(1, 3) \in R^t$. Thus R^t contains at least the following ordered pairs:

$$\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$$

But this relation *is* transitive; hence it equals R^t. The directed graph of R^t is shown below.