CHAPTER 9

COUNTING AND PROBABILITY

9.1 Introduction to Probability

Introduction to Probability

Definition

A **sample space** is the set of all possible outcomes of a random process or experiment. An **event** is a subset of a sample space.

Equally Likely Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the **probability of E**, denoted $\boldsymbol{P(E)}$, is

$$P(E) = \frac{\text{the number of outcomes in } E}{\text{the total number of outcomes in } S}.$$

Introduction to Probability

Notation

For any finite set A, $N(A)$ denotes the number of elements in A.

With this notation, the equally likely probability formula becomes

$$\boxed{P(E) = \frac{N(E)}{N(S)}.}$$

Example 9.1.1 – *Probabilities for a Deck of Cards*

An ordinary deck of cards contains 52 cards divided into four *suits.* The *red suits* are diamonds (♦) and hearts (♥), and the *black suits* are clubs (♣) and spades (♠).

Each suit contains 13 cards of the following *denominations:* 2, 3, 4, 5, 6, 7, 8, 9, 10, J (jack), Q (queen), K (king), and A (ace). The cards J, Q, and K are called *face cards.*

Example 9.1.1 – *Probabilities for a Deck of Cards* continued

Consider picking a card from a well shuffled deck.

a. What is the sample space of outcomes?

b. What is the event that the chosen card is a black face card?

c. What is the probability that the chosen card is a black face card?

Example 9.1.1 – *Solution*

a. The outcomes in the sample space S are the 52 cards in the deck.

b. Let E be the event that a black face card is chosen. The outcomes in E are the jack, queen, and king of clubs and the jack, queen, and king of spades. Symbolically:

$$E = \{J\clubsuit, Q\clubsuit, K\clubsuit, J\spadesuit, Q\spadesuit, K\spadesuit\}.$$

Example 9.1.1 – *Solution*

continued

c. By part (b), $N(E) = 6$, and according to the description of the situation, all 52 outcomes in the sample space are equally likely. Therefore, by the equally likely probability formula, the probability that the chosen card is a black face card is

$$P(E) = \frac{N(E)}{N(S)} = \frac{6}{52} \cong 11.5\%$$

Counting the Elements of a List

Counting the Elements of a List

Some counting problems are as simple as counting the elements of a list. For instance, how many integers are there from 5 through 12? To answer this question, imagine going along the list of integers from 5 to 12, counting each in turn.

list:	5	6	7	8	9	10	11	12
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
count:	1	2	3	4	5	6	7	8

So, the answer is 8.

Counting the Elements of a List

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Example 9.1.4 – *Counting the Elements of a Sublist*

a. How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?

b. What is the probability that a randomly chosen three-digit integer is divisible by 5?

Example 9.1.4 – *Solution*

a. Imagine writing the three-digit integers in a row, noting those that are multiples of 5 and drawing arrows between each such integer and its corresponding multiple of 5.

100	101	102	103	104	105	106	107	108	109	110	...	994	995	996	997	998	999
$\updownarrow$					$\updownarrow$					$\updownarrow$			$\updownarrow$				
$5 \cdot 20$					$5 \cdot 21$					$5 \cdot 22$			$5 \cdot 199$				

From the sketch it is clear that there are as many three-digit integers that are multiples of 5 as there are integers from 20 to 199 inclusive.

Example 9.1.4 – *Solution* continued

By Theorem 9.1.1, there are 199-20+1, or 180, such integers. Hence there are 180 three-digit integers that are divisible by 5.

b. By Theorem 9.1.1 the total number of integers from 100 through 999 is $999 - 100 + 1 = 900$. By part (a), 180 of these are divisible by 5. Hence the probability that a randomly chosen three-digit integer is divisible by 5 is $180/900 = 1/5$.

Example 9.1.5 – *Application: Counting Elements of a One-Dimensional Array*

Analysis of many computer algorithms requires skill at counting the elements of a one-dimensional array. Let $A[1]$, $A[2]$, …, $A[n]$ be a one-dimensional array, where n is a positive integer.

a. Suppose the array is cut at a middle value $A[m]$ so that two subarrays are formed:

   (1) $A[1]$, $A[2]$, …, $A[m]$ and (2) $A[m + 1]$, $A[m + 2]$, …, $A[n]$.

   How many elements does each subarray have?

Example 9.1.5 – *Application: Counting Elements of a One-Dimensional Array* continued

b. What is the probability that a randomly chosen element of the array has an even subscript

(i) if n is even? (ii) if n is odd?

Example 9.1.5 – *Solution*

a. Array (1) has the same number of elements as the list of integers from 1 through m. So, by Theorem 9.1.1, it has m, or $m - 1 + 1$, elements. Array (2) has the same number of elements as the list of integers from $m + 1$ through n. So, by Theorem 9.1.1, it has $n - m$, or $n - (m + 1) + 1$, elements.

Theorem 9.1.1 The Number of Elements in a List

If m and n are integers and $m \leq n$, then there are $n - m + 1$ integers from m to n inclusive.

Example 9.1.5 – *Solution*

continued

b. (i) If *n* is even, each even subscript starting with 2 and ending with *n* can be matched up with an integer from 1 to *n/*2.

$$\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots & n \\
 & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\
 & 2\cdot 1 & & 2\cdot 2 & & 2\cdot 3 & & 2\cdot 4 & & 2\cdot 5 & & 2\cdot(n/2)
\end{array}$$

So, there are *n/*2 array elements with even subscripts. Since the entire array has *n* elements, the probability that a randomly chosen element has an even subscript is

$$\frac{n/2}{n} = \frac{1}{2}.$$

Example 9.1.5 – *Solution*

continued

(ii) If n is odd, then the greatest even subscript of the array is $n - 1$. So, there are as many even subscripts between 1 and n as there are from 2 through $n - 1$.

Then the reasoning of (i) can be used to conclude that there are $(n - 1)/2$ array elements with even subscripts.

$$\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & \ldots & n-1 & & n \\
 & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
 & 2\cdot 1 & & 2\cdot 2 & & 2\cdot 3 & \ldots & 2\cdot[(n-1)/2)] & &
\end{array}$$

Example 9.1.5 – *Solution*

continued

Since the entire array has *n* elements, the probability that a randomly chosen element has an even subscript is

$$\frac{(n-1)/2}{n} = \frac{n-1}{2n}.$$

Observe that as *n* gets larger and larger, this probability gets closer and closer to 1⁄2.

Example 9.1.5 – *Solution*

continued

Note that the answers to (i) and (ii) can be combined using the floor notation.

Theorem 4.6.2 The Floor of $n/2$

For any integer n,

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

By Theorem 4.6.2, the number of array elements with even subscripts is $\lfloor n/2 \rfloor$, so the probability that a randomly chosen element has an even subscript is $\dfrac{\lfloor n/2 \rfloor}{n}$.