CHAPTER 4

ELEMENTARY NUMBER THEORY AND METHODS OF PROOF

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4.7 Indirect Argument: Contradiction and Contraposition

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Indirect Argument: Contradiction and Contraposition

Method of Proof by Contradiction

1. Suppose the statement to be proved is false. That is, suppose that the negation of the statement is true.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement to be proved is true.

Example 4.7.3 – *The Sum of a Rational Number and an Irrational Number*

Use proof by contradiction to show that the sum of any rational number and any irrational number is irrational.

Example 4.7.3 – *Solution*

In this example, the statement to be proved can be written formally as

$\forall$ real numbers *r* and *s*, if *r* is rational and
s is irrational, then *r* + *s* is irrational.

Example 4.7.3 – *Solution*

continued

From this you can see that the negation is

$\exists$ a rational number *r* and an irrational number *s* such that *r* + *s* is rational.

Starting Point: Suppose not. That is, suppose there is a rational number *r* and an irrational number *s* such that *r* + *s* is rational.

Example 4.7.3 – *Solution*

continued

To Show: This supposition leads to a contradiction.

To derive a contradiction, you need to understand what you are supposing: that there are numbers *r* and *s* such that *r* is rational, *s* is irrational, and *r* + *s* is rational. By definition of rational and irrational, this means there are convenient expressions that can be substituted for *r* and *r* + *s*, but all you can say about *s* is that it cannot be written as a quotient of integers

$$r = \frac{a}{b} \quad \text{for some integers } a \text{ and } b \text{ with } b \neq 0, \text{ and} \tag{4.7.1}$$

$$r + s = \frac{c}{d} \quad \text{for some integers } c \text{ and } d \text{ with } d \neq 0. \tag{4.7.2}$$

Example 4.7.3 – *Solution*

continued

If you substitute (4.7.1) into (4.7.2), you obtain

$$\frac{a}{b} + s = \frac{c}{d}.$$

Subtracting *a/b* from both sides gives

$$s = \frac{c}{d} - \frac{a}{b}$$

$$= \frac{bc}{bd} - \frac{ad}{bd} \quad \text{by rewriting } c/d \text{ and } a/b \text{ as equivalent fractions}$$

$$= \frac{bc - ad}{bd} \quad \text{by the rule for subtracting fractions with the same denominator.}$$

Example 4.7.3 – *Solution*

continued

Now both $bc - ad$ and bd are integers because products and differences of integers are integers, and $bd \neq 0$ by the zero product, property.

Hence s can be expressed as a quotient of two integers with a nonzero denominator, and so s is rational, which contradicts the supposition that it is irrational.

Theorem 4.7.3

The sum of any rational number and any irrational number is irrational.

Argument by Contraposition

Argument by Contraposition

A second form of indirect argument, *argument by contraposition*, is based on the logical equivalence between a statement and its contrapositive.

To prove a statement by contraposition, you take the contrapositive of the statement, prove the contrapositive by a direct proof, and conclude that the original statement is true.

Argument by Contraposition

Method of Proof by Contraposition

1. Express the statement to be proved in the form

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x).$$

   (This step may be done mentally.)

2. Rewrite this statement in the contrapositive form

$$\forall x \text{ in } D, \text{ if } Q(x) \text{ is false then } P(x) \text{ is false.}$$

   (This step may also be done mentally.)

3. Prove the contrapositive by a direct proof.

   a. Suppose x is a (particular but arbitrarily chosen) element of D such that $Q(x)$ is false.

   b. Show that $P(x)$ is false.

Example 4.7.4 – *If the Square of an Integer Is Even, Then the Integer Is Even*

Prove that for every integer *n*, if n^2 is even then *n* is even.

Example 4.7.4 – *Solution*

First form the contrapositive of the statement to be proved.

Contrapositive: For every integer *n*, if *n* is not even then n^2 is not even.

By the quotient-remainder theorem with divisor equal to 2, any integer is even or odd, and, by Theorem 4.7.2, no integer is both even and odd.

Theorem 4.7.2

There is no integer that is both even and odd.

Example 4.7.4 – *Solution*

continued

So, if an integer is not even, then it is odd. Thus, the contrapositive can be restated as follows:

Contrapositive: For every integer *n*, if *n* is odd then n^2 is odd.

> **Proposition 4.7.4**
>
> For every integer n, if n^2 is even then n is even.

Proof (by contraposition):
Suppose n is any odd integer. *[We must show that n^2 is odd.]* By definition of odd, $n = 2k + 1$ for some integer k. By substitution and algebra,

$$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

But $2k^2 + 2k$ is an integer because products and sums of integers are integers.

So $n^2 = 2 \cdot (\text{an integer}) + 1$, and thus, by definition of odd, n^2 is odd *[as was to be shown].*