CHAPTER 5

SEQUENCES, MATHEMATICAL INDUCTION, AND RECURSION

5.7 Solving Recurrence Relations by Iteration

Solving Recurrence Relations by Iteration

Suppose you have a sequence that satisfies a certain recurrence relation and initial conditions. It is often helpful to know an explicit formula for the sequence, especially if you need to compute terms with very large subscripts or if you need to examine general properties of the sequence. Such an explicit formula is called a **solution** to the recurrence relation. In this section, we discuss methods for solving recurrence relations.

The Method of Iteration

The Method of Iteration (1/4)

The most basic method for finding an explicit formula for a recursively defined sequence is **iteration**. Iteration works as follows: Given a sequence $a_0, a_1, a_2, \cdots$ defined by a recurrence relation and initial conditions, you start from the initial conditions and calculate successive terms of the sequence until you see a pattern developing. At that point you guess an explicit formula.

Example 5.7.1 – *Finding an Explicit Formula*

Let $a_0, a_1, a_2, \cdots$ be the sequence defined recursively as follows: For each integer $k \geq 1$,

(1) $a_k = a_{k-1} + 2$ recurrence relation

(2) $a_0 = 1$ initial condition.

Use iteration to guess an explicit formula for the sequence.

Example 5.7.1 – *Solution (1/5)*

Recall that to say

$$a_k = a_{k-1} + 2 \qquad \text{for each integer } k \geq 1$$

means

$$a_{\square} = a_{\square - 1} + 2 \qquad \text{no matter what positive integer is placed into the box } \square.$$

In particular,

$$a_1 = a_0 + 2,$$

$$a_2 = a_1 + 2,$$

$$a_3 = a_2 + 2,$$

and so forth.

Example 5.7.1 – *Solution (2/5)*

continued

Now use the initial condition to begin a process of successive substitutions into these equations, not just of numbers but of *numerical expressions*.

Example 5.7.1 – *Solution (3/5)* continued

Here's how the process works for the given sequence:

$a_0 = 1$	the initial condition
$a_1 = a_0 + 2 = 1 + 2$	by substitution
$a_2 = a_1 + 2 = (1 + 2) + 2 \quad = 1 + 2 + 2$	eliminate parentheses
$a_3 = a_2 + 2 = (1 + 2 + 2) + 2 \quad = 1 + 2 + 2 + 2$	eliminate parentheses again; write $3 \cdot 2$ instead of $2 + 2 + 2$?
$a_4 = a_3 + 2 = (1 + 2 + 2 + 2) + 2 = 1 + 2 + 2 + 2 + 2$	eliminate parentheses again; definitely write $4 \cdot 2$ instead of $2 + 2 + 2 + 2$—the length of the string of 2's is getting out of hand.

Example 5.7.1 – *Solution (4/5)* continued

Since it appears helpful to use the shorthand $k \cdot 2$ in place of $2 + 2 + \cdot \cdot \cdot + 2$ (k times), we do so, starting again from a_0.

Example 5.7.1 – *Solution (5/5)*

continued

$$a_0 = 1 \qquad\qquad = 1 + 0 \cdot 2$$

the initial condition

$$a_1 = a_0 + 2 = 1 + 2 \qquad = 1 + 1 \cdot 2$$

by substitution

$$a_2 = a_1 + 2 = (1 + 2) + 2 \quad = 1 + 2 \cdot 2$$

$$a_3 = a_2 + 2 = (1 + 2 \cdot 2) + 2 = 1 + 3 \cdot 2$$

$$a_4 = a_3 + 2 = (1 + 3 \cdot 2) + 2 = 1 + 4 \cdot 2$$

At this point it certainly seems likely that the general pattern is $1 + n \cdot 2$; check whether the next calculation supports this.

$$a_5 = a_4 + 2 = (1 + 4 \cdot 2) + 2 = 1 + 5 \cdot 2$$

It does! So go ahead and write an answer. It's only a guess, after all.

$$\vdots$$

Guess: $a_n = 1 + n \cdot 2 = 1 + 2n$ for every integer n.

The Method of Iteration (2/4)

A sequence like the one in which each term equals the previous term plus a fixed constant, is called an *arithmetic sequence*.

Definition

A sequence $a_0, a_1, a_2, \ldots$ is called an **arithmetic sequence** if, and only if, there is a constant d such that

$$a_k = a_{k-1} + d \quad \text{for each integer } k \geq 1.$$

It follows that

$$a_n = a_0 + dn \qquad \text{for every integer } n \geq 0.$$

Example 5.7.2 – *An Arithmetic Sequence*

Under the force of gravity, an object falling in a vacuum falls about 9.8 meters per second (m/sec) faster each second than it fell the second before. Thus, neglecting air resistance, a skydiver's speed upon leaving an airplane is approximately 9.8 m/sec one second after departure, $9.8 + 9.8 = 19.6$ m/sec two seconds after departure, and so forth. If air resistance is neglected, how fast would the skydiver be falling 60 seconds after leaving the airplane?

Example 5.7.2 – *Solution (1/2)*

Let s_n be the skydiver's speed in m/sec n seconds after exiting the airplane, assuming there is no air resistance. Then s_0 is the initial speed, and since the diver would travel 9.8 m/sec faster each second than the second before,

$$s_k = s_{k-1} + 9.8 \text{ m/sec} \quad \text{for every integer } k \geq 1.$$

It follows that $s_0, s_1, s_2, \cdots$ is an arithmetic sequence with a fixed constant of 9.8, and thus

$$s_n = s_0 + (9.8)n \quad \text{for each integer } n \geq 0.$$

Example 5.7.2 – *Solution (2/2)* continued

Hence sixty seconds after exiting and neglecting air resistance, the skydiver would travel at a speed of

$$s_{60} = 0 + (9.8)(60) = 588 \text{ m/sec}$$

Now 588 m/sec is over half a kilometer per second or over a third of a mile per second, which is very fast for a human being to travel. Fortunately for the skydiver, taking air resistance into account reduces the speed considerably.

Example 5.7.3 – *The Explicit Formula for a Geometric Sequence*

Let r be a fixed nonzero constant, and suppose a sequence $a_0, a_1, a_2, \cdots$ is defined recursively as follows:

$$a_k = ra_{k-1} \text{ for each integer } k \geq 1,$$
$$a_0 = a.$$

Use iteration to guess an explicit formula for this sequence.

Example 5.7.3 – *Solution*

$$
\begin{aligned}
a_0 &= a \\
a_1 &= ra_0 = ra \\
a_2 &= ra_1 = r(ra) = r^2a \\
a_3 &= ra_2 = r(r^2a) = r^3a \\
a_4 &= ra_3 = r(r^3a) = r^4a \\
&\ \vdots
\end{aligned}
$$

Guess: $a_n = r^n a = ar^n$ for any arbitrary integer $n \geq 0$

The Method of Iteration (3/4)

Definition

A sequence $a_0, a_1, a_2, \ldots$ is called a **geometric sequence** if, and only if, there is a constant r such that

$$a_k = ra_{k-1} \quad \text{for each integer } k \geq 1.$$

It follows that

$$a_n = a_0 r^n \quad \text{for each integer } n \geq 0.$$

Example 5.7.4 – *A Geometric Sequence*

As shown in Example 5.6.7, if a bank pays interest at a rate of 4% per year compounded annually and A_n denotes the amount in the account at the end of year n, then

$$A_k = (1.04)\, A_{k-1}, \text{ for each integer } k \geq 1,$$

assuming no deposits or withdrawals during the year. Suppose the initial amount deposited is \$100,000, and assume that no additional deposits or withdrawals are made.

a. How much will the account be worth at the end of 21 years?

b. In how many years will the account be worth \$1,000,000?

Example 5.7.4(a) – *Solution*

A_0, A_1, A_2, · · · is a geometric sequence with initial value 100,000 and constant multiplier 1.04. Hence,

$$A_n = \$100{,}000 \cdot (1.04)^n \quad \text{for every integer } n \geq 0.$$

After 21 years, the amount in the account will be

$$A_{21} = \$100{,}000 \cdot (1.04)^{21} \cong \$227{,}876.81.$$

This is the same answer as that obtained in Example 5.6.7 but is computed much more easily (at least if a calculator with a powering key, such $\boxed{\wedge}$ or $\boxed{x^y}$, is used.

Example 5.7.4(b) – *Solution (1/2)*

Let t be the number of years needed for the account to grow to \$1,000,000. Then

$$\$1{,}000{,}000 = \$100{,}000 \cdot (1.04)^t.$$

Dividing both sides by 100,000 gives

$$10 = (1.04)^t,$$

and taking natural logarithms of both sides results in

$$\ln(10) = \ln(1.04)^t.$$

Example 5.7.4(b) – *Solution (2/2)*

Then

$$\ln(10) \cong t \ln(1.04) \qquad \text{because } \log_b(x^a) = a \log_b(x)$$

and so

$$t = \frac{\ln(10)}{\ln(1.04)} \cong 58.7.$$

Hence the account will grow to $1,000,000 in approximately 58.7 years.

The Method of Iteration (4/4)

The following box indicates some quantities that are approximately equal to certain powers of 10.

$10^7 \cong$	number of seconds in a year
$10^9 \cong$	number of bytes of memory in a personal computer
$10^{11} \cong$	number of neurons in a human brain
$10^{17} \cong$	age of the universe in seconds (according to one theory)
$10^{31} \cong$	number of seconds to process all possible positions of a checkers game if moves are processed at a rate of 1 per billionth of a second
$10^{81} \cong$	number of atoms in the universe
$10^{111} \cong$	number of seconds to process all possible positions of a chess game if moves are processed at a rate of 1 per billionth of a second

Using Formulas to Simplify Solutions Obtained by Iteration

Example 5.7.5 – *An Explicit Formula for the Tower of Hanoi Sequence*

Recall that the Tower of Hanoi sequence $m_1, m_2, m_3, \cdots$ satisfies the recurrence relation

$$m_k = 2m_{k-1} \text{ for each integer } k \geq 2$$

and has the initial condition

$$m_1 = 1.$$

Use iteration to guess an explicit formula for this sequence, and make use of a formula for the Sum of a Geometric Sequence to simplify the answer.

Example 5.7.5 – *Solution (1/3)*

By iteration,

$$
\begin{aligned}
m_1 &= 1 \\
m_2 &= 2m_1 + 1 = 2 \cdot 1 + 1 &&= 2^1 + 1, \\
m_3 &= 2m_2 + 1 = 2(2 + 1) + 1 &&= 2^2 + 2 + 1, \\
m_4 &= 2m_3 + 1 = 2(2^2 + 2 + 1) + 1 &&= 2^3 + 2^2 + 2 + 1, \\
m_5 &= 2m_4 + 1 = 2(2^3 + 2^2 + 2 + 1) + 1 &&= 2^4 + 2^3 + 2^2 + 2 + 1.
\end{aligned}
$$

Example 5.7.5 – *Solution (2/3)*

continued

These calculations show that each term up to m_5 is a sum of successive powers of 2, starting with $2^0 = 1$ and going up to 2^k, where k is 1 less than the subscript of the term. The pattern would seem to continue to higher terms because each term is obtained from the preceding one by multiplying by 2 and adding 1; multiplying by 2 raises the exponent of each component of the sum by 1, and adding 1 adds back the 1 that was lost when the previous 1 was multiplied by 2. For instance, for $n = 6$,

$$\begin{aligned} m_6 = 2m_5 + 1 &= 2(2^4 + 2^3 + 2^2 + 2 + 1) + 1 \\ &= 2^5 + 2^4 + 2^3 + 2^2 + 2 + 1. \end{aligned}$$

Example 5.7.5 – *Solution (3/3)* continued

Thus it seems that, in general,

$$m_n = 2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1.$$

By the formula for the sum of a geometric sequence (Theorem 5.2.2),

$$2^{n-1} + 2^{n-2} + \cdots + 2^2 + 2 + 1 = \frac{2^n - 1}{2-1} = 2^n - 1.$$

Hence the explicit formula seems to be

$$m_n = 2^n - 1 \quad \text{for every integer } n \geq 1.$$

Checking the Correctness of a Formula by Mathematical Induction

Checking the Correctness of a Formula by Mathematical Induction (1/1)

As you can see from some of the previous examples, the process of solving a recurrence relation by iteration can involve complicated calculations. It is all too easy to make a mistake and come up with the wrong formula. That is why it is important to confirm your calculations by checking the correctness of your formula. The most common way to do this is to use mathematical induction.

Example 5.7.7 – *Using Mathematical Induction to Verify the Correctness of a Solution to a Recurrence Relation*

We have obtained a formula for the Tower of Hanoi sequence. Use mathematical induction to show that this formula is correct.

Example 5.7.7 – *Solution (1/6)*

What does it mean to show the correctness of a formula for a recursively defined sequence? Given a sequence of numbers that satisfies a certain recurrence relation and initial condition, the job is to show that each term of the sequence satisfies the proposed explicit formula. In this case, you need to prove the following statement:

> If $m_1, m_2, m_3, \ldots$ is the sequence defined by
>
> $m_k = 2m_{k-1} + 1$ for each integer $k \geq 2$, and
>
> $m_1 = 1$,
>
> then $m_n = 2^n - 1$ for every integer $n \geq 1$.

Example 5.7.7 – *Solution (2/6)*

continued

Proof of Correctness: Let $m_1, m_2, m_3, \cdots$ be the sequence defined by specifying that $m_1 = 1$ and $m_k = 2m_{k-1} + 1$ for each integer $k \geq 2$, and let the property $P(n)$ be the equation

$$m_n = 2^n - 1. \qquad \leftarrow P(n)$$

We will use mathematical induction to prove that for every integer $n \geq 1$, $P(n)$ is true.

Show that P**(1)** ***is true:***

To establish $P(1)$, we must show that

$$m_1 = 2^1 - 1. \qquad \leftarrow P(1)$$

Example 5.7.7 – *Solution* (3/6)

continued

Now the left-hand side of $P(1)$ is

$$m_1 = 1 \qquad \text{by definition of } m_1, m_2, m_3, \cdots,$$

and the right-hand side of $P(1)$ is

$$2^1 - 1 = 2 - 1 = 1.$$

Thus the two sides of $P(1)$ equal the same quantity, and hence $P(1)$ is true.

Example 5.7.7 – *Solution (4/6)* continued

Show that for every integer k* ≥ 1*, if P*(*k*) *is true then P*(*k* + 1) *is also true:

[Suppose that P(k) is true for a particular but arbitrarily chosen integer k ≥ 1*. That is:]*

Suppose that k is any integer with $k \geq 1$ such that

$$m_k = 2^k - 1. \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

[We must show that P(*k* + 1) *is true. That is:]* We must show that

$$m_{k+1} = 2^{k+1} - 1. \qquad \leftarrow P(k+1)$$

Example 5.7.7 – *Solution* (5/6) continued

But the left-hand side of $P(k + 1)$ is

$$m_{k+1} = 2m_{(k+1)-1} + 1$$ by definition of $m_1, m_2, m_3, \cdots,$

$$= 2m_k + 1$$

$$= 2(2^k - 1) + 1$$ by substitution from the inductive hypothesis

$$= 2^{k+1} - 2 + 1$$ by the distributive law and the fact that $2 \cdot 2^k = 2^{k+1}$

$$= 2^{k+1} - 1$$ by basic algebra

Example 5.7.7 – *Solution (6/6)* continued

which equals the right-hand side of $P(k + 1)$. *[Since the basis and inductive steps have been proved, it follows by mathematical induction that the given formula holds for every integer* $n \geq 1$*.]*

Discovering That an Explicit Formula Is Incorrect

Example 5.7.8 – *Using Verification by Mathematical Induction to Find a Mistake*

Let $c_0, c_1, c_2, \cdots$ be the sequence defined as follows:

$$c_k = 2c_{k-1} + k \quad \text{for each integer } k \geq 1,$$

$$c_0 = 1.$$

Suppose your calculations suggest that $c_0, c_1, c_2, \cdots$ satisfies the following explicit formula:

$$c_n = 2^n + n \qquad \text{for every integer } n \geq 0.$$

Is this formula correct?

Example 5.7.8 – *Solution (1/4)*

Start to prove the statement by mathematical induction and see what develops. The proposed formula satisfies the basis step of the inductive proof since on the one hand, $c_0 = 1$ by definition and on the other hand, $2^0 + 0 = 1 + 0 = 1$.

In the inductive step, you suppose that k is any integer with $k \geq 0$ such that

$$c_k = 2^k + k,$$ This is the inductive hypothesis.

and then you must show that

$$c_{k+1} = 2^{k+1} + (k + 1).$$

Example 5.7.8 – *Solution* (2/4)

continued

To do this, you start with c_{k+1}, substitute from the recurrence relation, and use the inductive hypothesis:

$$c_{k+1} = 2c_k + (k+1) \quad \text{by the recurrence relation}$$

$$= 2(2^k + k) + (k+1) \quad \text{by substitution from the inductive hypothesis}$$

$$= 2^{(k+1)} + 3k + 1 \quad \text{by basic algebra.}$$

To finish the verification, therefore, you need to show that

$$2^{k+1} + 3k + 1 = 2^{k+1} + (k+1).$$

Example 5.7.8 – *Solution (3/4)* continued

Now this equation is equivalent to

$2k = 0$ by subtracting $2^{k+1} + k + 1$ from both sides

which is equivalent to

$k = 0$ by dividing both sides by 2.

But this is false since k may be *any* nonnegative integer. For instance, when $k = 1$, then $k + 1 = 2$, and

$$c_2 = 2 \cdot 3 + 2 = 8 \quad \text{whereas} \quad 2^2 + 2 = 4 + 2 = 6.$$

Example 5.7.8 – *Solution (4/4)*

continued

So the formula does not give the correct value for c_2. Hence the sequence $c_0, c_1, c_2, \cdots$ does not satisfy the proposed formula.