CHAPTER 4

ELEMENTARY NUMBER THEORY AND METHODS OF PROOF

4.3 Direct Proof and Counterexample III: Rational Numbers

Direct Proof and Counterexample Ⅲ: Rational Numbers

Definition

A real number r is **rational** if, and only if, it can be expressed as a quotient of two integers with a nonzero denominator. A real number that is not rational is **irrational**. More formally, if r is a real number, then

$$r \text{ is rational} \Leftrightarrow \exists \text{ integers } a \text{ and } b \text{ such that } r = \frac{a}{b} \text{ and } b \neq 0.$$

Example 4.3.1 – *Determining Whether Numbers Are Rational or Irrational*

a. Is 10/3 a rational number?

b. Is $-\frac{5}{39}$ a rational number?

c. Is 0.281 a rational number?

d. Is 7 a rational number?

e. Is 0 a rational number?

f. Is 2/0 a rational number?

g. Is 2/0 an irrational number?

Example 4.3.1 – *Determining Whether Numbers Are Rational or Irrational*
continued

h. Is 0.12121212… a rational number (where the digits 12 are assumed to repeat forever)?

i. If m and n are integers and neither m nor n is zero, is $(m + n)/mn$ a rational number?

Example 4.3.1 – *Solution*

a. Yes, 10/3 is a quotient of the integers 10 and 3 and hence is rational.

b. Yes, $-\frac{5}{39} = \frac{-5}{39}$, which is a quotient of the integers −5 and 39 and hence is rational.

c. Yes, 0.281 = 281/1000. Note that the numbers shown on a typical calculator display are all finite decimals. An explanation similar to the one in this example shows that any such number is rational.

It follows that a calculator with such a display can accurately represent only rational numbers.

Example 4.3.1 – *Solution*

continued

d. Yes, 7 = 7/1.

e. Yes, 0 = 0/1.

f. No, 2/0 is not a real number (division by 0 is not allowed).

g. No, because every irrational number is a real number, and 2/0 is not a real number.

Example 4.3.1 – *Solution* continued

h. Yes. Let $x = 0.12121212\ldots$ Then $100x = 12.12121212\ldots$

Thus

$$100x - x = 12.12121212\ldots - 0.12121212\ldots = 12.$$

But also, $100x - x = 99x$ by basic algebra.

Hence $99x = 12,$

and so $x = \dfrac{12}{99}.$

Therefore, $0.12121212\ldots = 12/99$, which is a ratio of two nonzero integers and thus is a rational number.

Example 4.3.1 – *Solution*

continued

Note that you can use an argument similar to this one to show that any repeating decimal is a rational number.

i. Yes, since m and n are integers, so are $m + n$ and mn (because sums and products of integers are integers). Also, $mn \neq 0$ by the *zero-product property*.

One version of this property says the following:

Zero Product Property

If neither of two real numbers is zero, then their product is also not zero.

More on Generalizing from the Generic Particular

More on Generalizing from the Generic Particular

Most problems are stated in informal language, but solving them often requires translating them into more formal terms.

> **Theorem 4.3.1**
>
> Every integer is a rational number.

Proving Properties of Rational Numbers

Example 4.3.2 – *Any Sum of Rational Numbers Is Rational*

Prove that the sum of any two rational numbers is rational.

Example 4.3.2 – *Solution*

Begin by mentally or explicitly rewriting the statement to be proved in the form "∀ ________, if ________ then ________."

Formal Restatement: ∀ real numbers *r* and *s*, if *r* and *s* are rational then *r* + *s* is rational.

Next ask yourself, "Where am I starting from?" or "What am I supposing?" The answer gives you the starting point, or first sentence, of the proof.

Example 4.3.2 – *Solution*

continued

Starting Point: Suppose *r* and *s* are any particular but arbitrarily chosen real numbers such that *r* and *s* are rational; *or, more simply,*

Suppose *r* and *s* are any rational numbers.

Then ask yourself, "What must I show to complete the proof?"

To Show: *r* + *s* is rational.

Finally ask, "How do I get from the starting point to the conclusion?" or "Why must *r* + *s* be rational if both *r* and *s* are rational?" The answer depends in an essential way on the definition of rational.

Example 4.3.2 – *Solution*

continued

Rational numbers are quotients of integers, so to say that *r* and *s* are rational means that

$$r = \frac{a}{b} \quad \text{and} \quad s = \frac{c}{d}$$

for some integers *a*, *b*, *c*, and *d* where $b \neq 0$ and $d \neq 0$.

It follows by substitution that

$$r + s = \frac{a}{b} + \frac{c}{d}. \qquad (4.3.1)$$

Example 4.3.2 – *Solution*

continued

You need to show that *r* + *s* is rational, which means that *r* + *s* can be written as a single fraction or ratio of two integers with a nonzero denominator.

But the right-hand side of equation (4.3.1) is

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd}$$

by rewriting the fraction with a common denominator

$$= \frac{ab + bc}{bd}$$

by adding fractions with a common denominator.

Example 4.3.2 – *Solution*

continued

Is this fraction a ratio of integers? Yes. Because products and sums of integers are integers, $ad + bc$ and bd are both integers.

Is the denominator $bd \neq 0$? Yes, by the zero product property (since $b \neq 0$ and $d \neq 0$). Thus $r + s$ is a rational number.

Theorem 4.3.2

The sum of any two rational numbers is rational.

Corollary 4.2.3

The double of a rational number is rational.

Deriving New Mathematics from Old

Example 4.3.3 – *Deriving Additional Results about Even and Odd Integers*

Suppose that you have already proved the following properties of even and odd integers:

1. The sum, product, and difference of any two even integers are even.
2. The sum and difference of any two odd integers are even.
3. The product of any two odd integers is odd.
4. The product of any even integer and any odd integer is even.

5. The sum of any odd integer and any even integer is odd.

6. The difference of any odd integer minus any even integer is odd.

7. The difference of any even integer minus any odd integer is odd.

Use the properties listed above to prove that if *a* is any even integer and *b* is any odd integer, then $\frac{a^2 + b^2 + 1}{2}$ is an integer.

Example 4.3.3 – *Solution*

Suppose *a* is any even integer and *b* is any odd integer. By property 3, b^2 is odd, and by property 1, a^2 is even. Then by property 5, $a^2 + b^2$ is odd, and because 1 is also odd, the sum $(a^2 + b^2) + 1 = a^2 + b^2 + 1$ is even by property 2.

Hence, by definition of even, there exists an integer *k* such that $a^2 + b^2 + 1 = 2k.$

Dividing both sides by 2 gives $\frac{a^2 + b^2 + 1}{2} = k,$ which is an integer. Thus $\frac{a^2 + b^2 + 1}{2}$ is an integer.