CHAPTER 9

COUNTING AND PROBABILITY

9.5

Counting Subsets of a Set: Combinations

Counting Subsets of a Set: Combinations

The number of subsets of size *r* that can be chosen from *S* equals the number of subsets of size *r* that *S* has. Each individual subset of size *r* is called an *r-combination* of the set.

Definition *r*-combination

Let n and r be nonnegative integers with $r \leq n$. An ***r*-combination** of a set of n elements is a subset of r of the n elements.

Notation $\binom{n}{r}$

The symbol $\binom{n}{r}$, read "n choose r," denotes the number of subsets of size r (or r-combinations) that can be formed from a set of n elements.

Permutations

Theorem 9.2.3

If n and r are integers and $1 \leq r \leq n$, then the number of r-permutations of a set of n elements is given by the formula

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1) \quad \text{first version}$$

or, equivalently,

$$P(n, r) = \frac{n!}{(n-r)!} \quad \text{second version.}$$

Counting Subsets of a Set: Combinations

Theorem 9.5.1 Computational Formula for $\binom{n}{r}$

The number of subsets of size r (or r-combinations) that can be chosen from a set of n elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n, r)}{r!} \qquad \text{first version}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \qquad \text{second version}$$

where n and r are nonnegative integers with $r \leq n$.

Example 9.5.4 – *Calculating the Number of Teams*

Consider again the problem of choosing five members from a group of twelve to work as a team on a special project. How many distinct five-person teams can be chosen?

Example 9.5.4 – *Solution*

By Theorem 9.5.1,

$$\binom{12}{5} = \frac{12!}{5!(12-5)!} = \frac{\cancel{12}\cdot 11\cdot \cancel{10}\cdot 9\cdot 8\cdot \cancel{7!}}{(\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1)\cdot \cancel{7!}} = 11\cdot 9\cdot 8 = 792.$$

Thus, there are 792 distinct five-person teams.

Example 7 – *Teams with Members of Two Types*

Suppose a group of twelve consists of five men and seven women.

a. How many five-person teams can be chosen that consist of three men and two women?

b. How many five-person teams contain at least one man?

c. How many five-person teams contain at most one man?

Solution:

a. To answer this question, think of forming a team as a two-step process:

Step 1: Choose the men.

Example 7 – *Solution*

cont'd

Step 2: Choose the women.

There are $\binom{5}{3}$ ways to choose the three men out of the five and $\binom{7}{2}$ ways to choose the two women out of the seven.

Hence, by the product rule,

$$\begin{bmatrix}\text{number of teams of five that} \\ \text{contain three men and two women}\end{bmatrix} = \binom{5}{3}\binom{7}{2} = \frac{5!}{3!2!}\cdot\frac{7!}{2!5!}$$

$$= \frac{7\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}{\cancel{3}\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1\cdot \cancel{2}\cdot 1}$$

$$= 210.$$

Example 7 – *Solution*

cont'd

b. This question can also be answered either by the addition rule or by the difference rule. The solution by the difference rule is shorter and is shown first.

Observe that the set of five-person teams containing at least one man equals the set difference between the set of all five-person teams and the set of five-person teams that do not contain any men.

Example 7 – *Solution*

cont'd

See Figure 9.5.5.

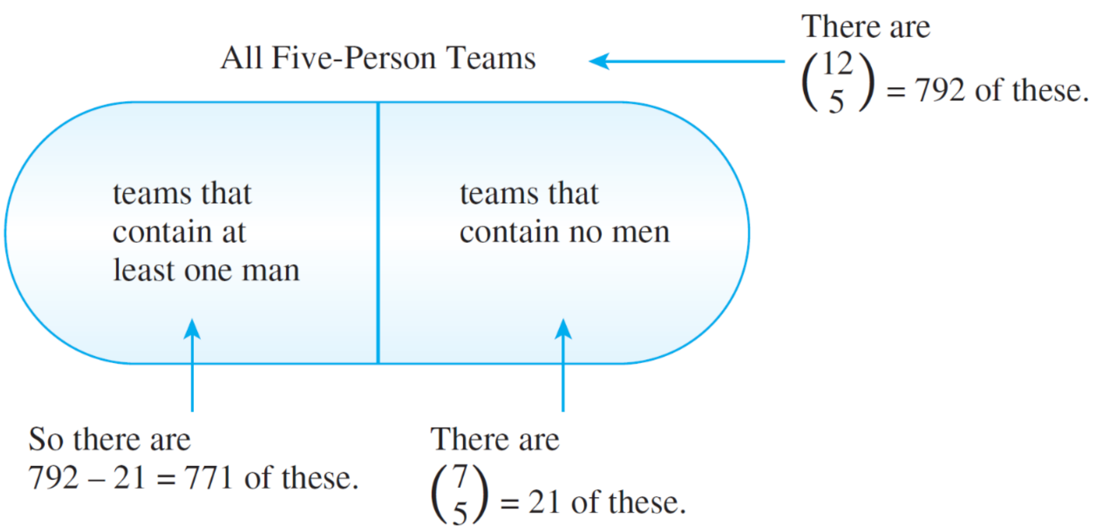


Figure 9.5.5

Example 7 – *Solution*

cont'd

Now a team with no men consists entirely of five women chosen from the seven women in the group, so there are $\binom{7}{5}$ such teams. Also, by Example 4, the total number of five-person teams is $\binom{12}{5}$ = 792.

Hence, by the difference rule,

$$\begin{bmatrix}\text{number of teams}\\ \text{with at least}\\ \text{one man}\end{bmatrix} = \begin{bmatrix}\text{total number}\\ \text{of teams}\\ \text{of five}\end{bmatrix} - \begin{bmatrix}\text{number of teams}\\ \text{of five that do not}\\ \text{contain any men}\end{bmatrix}$$

$$= \binom{12}{5} - \binom{7}{5} = 792 - \frac{7!}{5! \cdot 2!}$$

Example 7 – *Solution*

cont'd

$$= 792 - \frac{7 \cdot \overset{3}{\cancel{6}} \cdot \cancel{5!}}{\cancel{5!} \cdot \cancel{2} \cdot 1} = 792 - 21 = 771.$$

This reasoning is summarized in Figure 9.5.5.

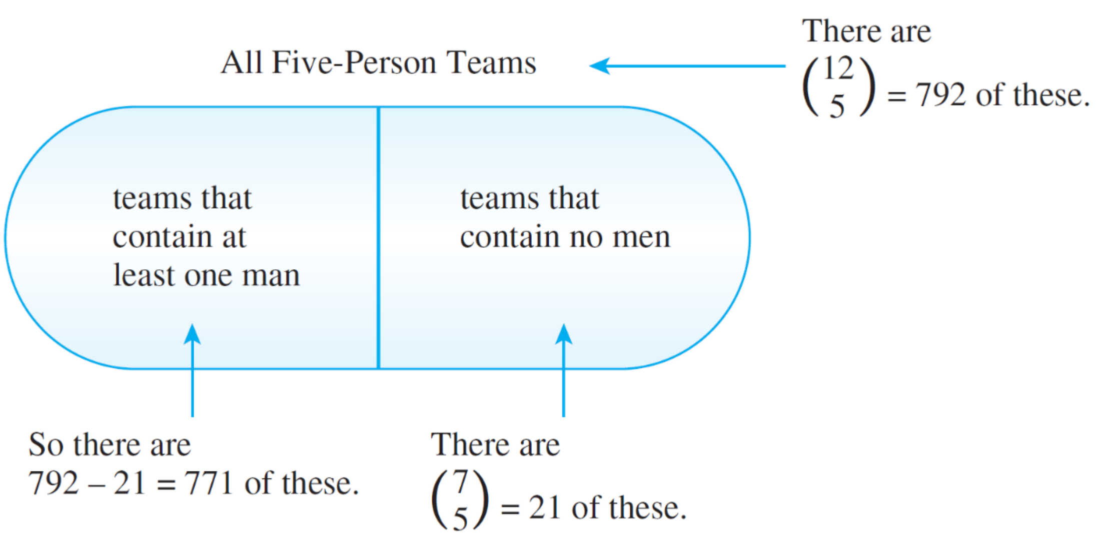
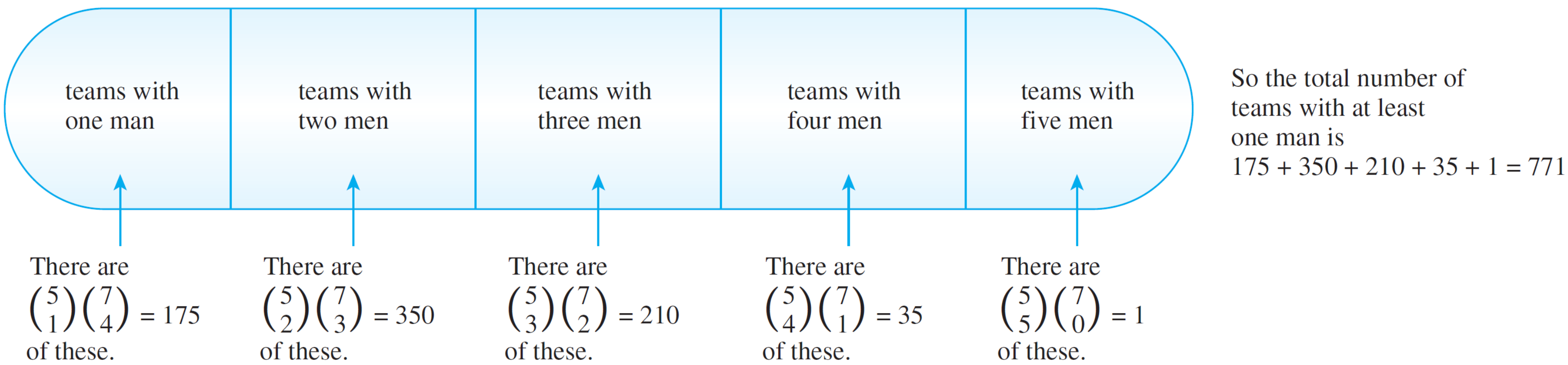


Figure 9.5.5

Example 7 – *Solution*

cont'd

Alternatively, to use the addition rule, observe that the set of teams containing at least one man can be partitioned as shown in Figure 9.5.6.



Teams with At Least One Man

Figure 9.5.6

Example 7 – *Solution*

cont'd

The number of teams in each subset of the partition is calculated using the method illustrated in part (**a**). There are

$$\binom{5}{1}\binom{7}{4} \text{ teams with one man and four women}$$

$$\binom{5}{2}\binom{7}{3} \text{ teams with two men and three women}$$

$$\binom{5}{3}\binom{7}{2} \text{ teams with three men and two women}$$

$$\binom{5}{4}\binom{7}{1} \text{ teams with four men and one woman}$$

Example 7 – *Solution*

cont'd

$$\binom{5}{5}\binom{7}{0} \text{ teams with five men and no women.}$$

Hence, by the addition rule,

$$\begin{bmatrix}\text{number of teams with} \\ \text{at least one man}\end{bmatrix}$$

$$= \binom{5}{1}\binom{7}{4} + \binom{5}{2}\binom{7}{3} + \binom{5}{3}\binom{7}{2} + \binom{5}{4}\binom{7}{1} + \binom{5}{5}\binom{7}{0}$$

$$= \frac{5!}{1!4!} \cdot \frac{7!}{4!3!} + \frac{5!}{2!3!} \cdot \frac{7!}{3!4!} + \frac{5!}{3!2!} \cdot \frac{7!}{2!5!} + \frac{5!}{4!1!} \cdot \frac{7!}{1!6!} + \frac{5!}{5!0!} \cdot \frac{7!}{0!7!}$$

Example 7 – *Solution*

cont'd

$$= \frac{5\cdot\cancel{4!}\cdot 7\cdot\cancel{6}\cdot 5\cdot\cancel{4!}}{\cancel{4!}\cdot\cancel{3\cdot 2}\cdot\cancel{4!}} + \frac{5\cdot\overset{2}{\cancel{4}}\cdot\cancel{3!}\cdot 7\cdot\cancel{6}\cdot 5\cdot\cancel{4!}}{\cancel{3!}\cdot\cancel{2}\cdot\cancel{4!}\cdot\cancel{3\cdot 2}} + \frac{5\cdot\overset{2}{\cancel{4}}\cdot\cancel{3!}\cdot 7\cdot\overset{3}{\cancel{6}}\cdot\cancel{5!}}{\cancel{2}\cdot\cancel{3!}\cdot\cancel{5!}\cdot\cancel{2}}$$

$$+ \frac{5\cdot\cancel{4!}\cdot 7\cdot\cancel{6!}}{\cancel{4!}\cdot\cancel{6!}} + \frac{\cancel{5!}\cdot\cancel{7!}}{\cancel{5!}\cdot\cancel{7!}}$$

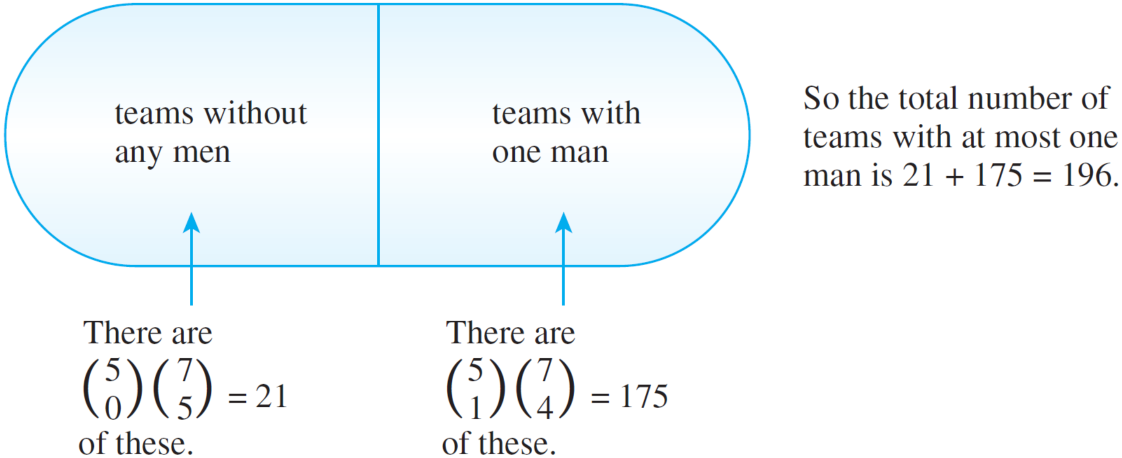
$$= 175 + 350 + 210 + 35 + 1$$
$$= 771.$$

teams with one man	teams with two men	teams with three men	teams with four men	teams with five men
There are $\binom{5}{1}\binom{7}{4} = 175$ of these.	There are $\binom{5}{2}\binom{7}{3} = 350$ of these.	There are $\binom{5}{3}\binom{7}{2} = 210$ of these.	There are $\binom{5}{4}\binom{7}{1} = 35$ of these.	There are $\binom{5}{5}\binom{7}{0} = 1$ of these.

So the total number of teams with at least one man is $175 + 350 + 210 + 35 + 1 = 771$.

c. As shown in Figure 9.5.7, the set of teams containing at most one man can be partitioned into the set that does not contain any men and the set that contains exactly one man.

Hence, by the addition rule,



Teams with At Most One Man

Figure 9.5.7

Example 7 – *Solution*

cont'd

$$\begin{bmatrix}\text{number of teams}\\ \text{with at}\\ \text{most one man}\end{bmatrix} = \begin{bmatrix}\text{number of}\\ \text{teams without}\\ \text{any men}\end{bmatrix} + \begin{bmatrix}\text{number of}\\ \text{teams with}\\ \text{one man}\end{bmatrix}$$

$$= \binom{5}{0}\binom{7}{5} + \binom{5}{1}\binom{7}{4}$$

$$= 21 + 175$$

$$= 196.$$

This reasoning is summarized in Figure 9.5.7.

Example 9.5.11 – *Permutations of a Set with Repeated Elements*

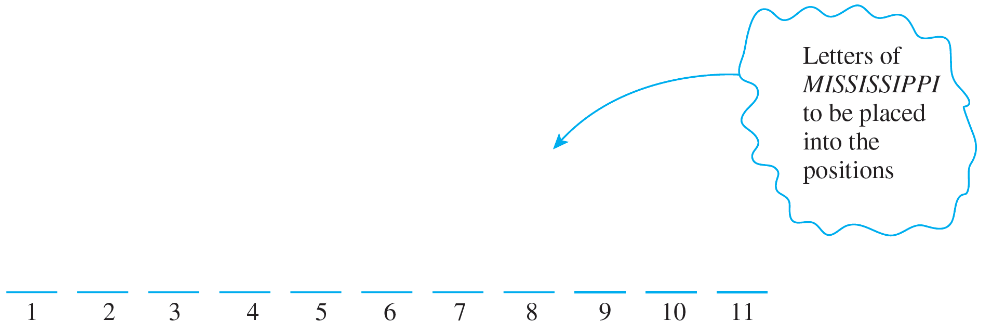
Consider various ways of ordering the letters in the word *MISSISSIPPI*:

IIMSSPISSIP, *ISSSPMIIPIS*, *PIMISSSSIIP*, and so on.

How many distinguishable orderings are there?

Example 9.5.11 – *Solution*

This example generalizes Example 9.5.10. Imagine placing the 11 letters of *MISSISSIPPI* one after another into 11 positions.



Because copies of the same letter cannot be distinguished from one another, once the positions for a certain letter are known, then all copies of the letter can go into the positions in any order.

Example 9.5.11 – *Solution* continued

It follows that constructing an ordering for the letters can be thought of as a four-step process:

Step 1: Choose a subset of four positions for the *S*'s.

Step 2: Choose a subset of four positions for the *I*'s.

Step 3: Choose a subset of two positions for the *P*'s.

Step 4: Choose a subset of one position for the *M*.

Since there are 11 positions in all, there are $\binom{11}{4}$ subsets of four positions for the *S*'s.

Example 9.5.11 – *Solution*

continued

Once the four *S*'s are in place, there are seven positions that remain empty, so there are $\binom{7}{4}$ subsets of four positions for the *I*'s. After the *I*'s are in place, there are three positions left empty, so there are $\binom{3}{2}$ subsets of two positions for the *P*'s. That leaves just one position for the *M*. But $1 = \binom{1}{1}$.

Hence by the multiplication rule,

$$\begin{bmatrix}\text{number of ways to} \\ \text{position all the letters}\end{bmatrix} = \binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1}$$

Example 9.5.11 – *Solution*

continued

$$= \frac{11!}{4!\cdot \cancel{7!}}\cdot\frac{\cancel{7!}}{4!\cdot \cancel{3!}}\cdot\frac{\cancel{3!}}{2!\cdot \cancel{1!}}\cdot\frac{\cancel{1!}}{1!\cdot \cancel{0!}}$$

$$= \frac{11!}{4!\cdot 4!\cdot 2!\cdot 1!}$$

$$= 34{,}650.$$

Counting Subsets of a Set: Combinations

Theorem 9.5.2 Permutations with Sets of Indistinguishable Objects

Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other

n_2 are of type 2 and are indistinguishable from each other

$\vdots$

n_k are of type k and are indistinguishable from each other,

and suppose that $n_1 + n_2 + \cdots + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!n_2!n_3!\cdots n_k!}.$$