CHAPTER 8

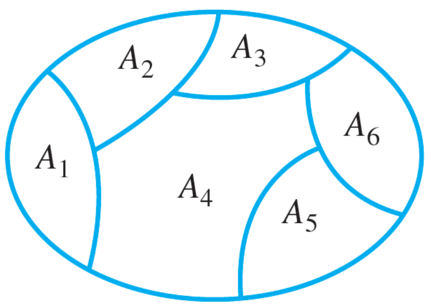
PROPERTIES OF RELATIONS

8.3 Equivalence Relations

The Relation Induced by a Partition

The Relation Induced by a Partition

A **partition** of a set A is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is A. The diagram of Figure 8.3.1 illustrates a partition of a set A by subsets $A_1, A_2, \ldots , A_6$.



$A_i \cap A_j = \varnothing$, whenever $i \neq j$
$A_1 \cup A_2 \cup \cdots \cup A_6 = A$

A Partition of a Set

Figure 8.3.1

The Relation Induced by a Partition

Definition

Given a partition of a set A, the **relation induced by the partition**, R, is defined on A as follows: For every $x, y \in A$,

$$x\, R\, y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$

Example 8.3.1 – *Relation Induced by a Partition*

Let $A = \{0, 1, 2, 3, 4\}$ and consider the following partition of A:

$$\{0, 3, 4\}, \{1\}, \{2\}.$$

Find the relation R induced by this partition.

Example 8.3.1 – *Solution*

Since {0, 3, 4} is a subset of the partition,

0 *R* 3 because both 0 and 3 are in {0, 3, 4}

3 *R* 0 because both 3 and 0 are in {0, 3, 4}

0 *R* 4 because both 0 and 4 are in {0, 3, 4}

4 *R* 0 because both 4 and 0 are in {0, 3, 4}

3 *R* 4 because both 3 and 4 are in {0, 3, 4}

and

4 *R* 3 because both 4 and 3 are in {0, 3, 4}.

Example 8.3.1 – *Solution*

continued

Also,

$0\ R\ 0$ because both 0 and 0 are in $\{0, 3, 4\}$

$3\ R\ 3$ because both 3 and 3 are in $\{0, 3, 4\}$

and

$4\ R\ 4$ because both 4 and 4 are in $\{0, 3, 4\}$.

Example 8.3.1 – *Solution*

continued

Since {1} is a subset of the partition,

$1\ R\ 1$ because both 1 and 1 are in {1},

and since {2} is a subset of the partition,

$2\ R\ 2$ because both 2 and 2 are in {2}.

Hence

$R = \{(0, 0), (0, 3), (0, 4), (1, 1), (2, 2), (3, 0), (3, 3), (3, 4), (4, 0), (4, 3), (4, 4)\}.$

The Relation Induced by a Partition

Theorem 8.3.1

Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Definition of an Equivalence Relation

Definition of an Equivalence Relation

A relation on a set that satisfies the three properties of reflexivity, symmetry, and transitivity is called an *equivalence relation*.

Definition

Let A be a set and R a relation on A. R is an **equivalence relation** if, and only if, R is reflexive, symmetric, and transitive.

Example 8.3.2 – *An Equivalence Relation on a Set of Subsets*

Let X be the set of all nonempty subsets of {1, 2, 3}. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Define a relation **R** on X as follows: For every A and B in X,

A **R** B $\Leftrightarrow$ the least element of A equals the least element of B.

Prove that **R** is an equivalence relation on X.

Example 8.3.2 – *Solution*

R *is reflexive*: Suppose *A* is a nonempty subset of {1, 2, 3}. *[We must show that A* **R** *A.]* It is true to say that the least element of *A* equals the least element of *A*. Thus, by definition of **R**, *A* **R** *A*.

R *is symmetric*: Suppose *A* and *B* are nonempty subsets of {1, 2, 3} and *A* **R** *B*. *[We must show that B* **R** *A.]* Since *A* **R** *B*, the least element of *A* equals the least element of *B*. But this implies that the least element of *B* equals the least element of *A*, and so, by definition of **R**, *B* **R** *A*.

Example 8.3.2 – *Solution*

continued

R *is transitive*: Suppose *A*, *B*, and *C* are nonempty subsets of {1, 2, 3}, *A* **R** *B*, and *B R C*. *[We must show that A* **R** *C.]* Since *A* **R** *B*, the least element of *A* equals the least element of *B* and since *B* **R** *C*, the least element of *B* equals the least element of *C*.

Thus, the least element of *A* equals the least element of *C*, and so, by definition of **R**, *A* **R** *C*.

Equivalence Classes of an Equivalence Relation

Equivalence Classes of an Equivalence Relation

Suppose there is an equivalence relation on a certain set. If *a* is any particular element of the set, then one can ask, "What is the subset of all elements that are related to *a*?" This subset is called the *equivalence class* of *a*.

Definition

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of *a***, denoted **[*a*]** and called the **class of *a*** for short, is the set of all elements x in A such that x is related to a by R.

In symbols:

$$[a] = \{x \in A \mid x \, R \, a\}$$

Equivalence Classes of an Equivalence Relation

The procedural version of this definition is

$$\text{for every } x \in A, \quad x \in [a] \quad \Leftrightarrow \quad x\, R\, a.$$

The notation $[a]_R$ may be used to denote the equivalence class of *a* for the relation *R*.

Example 8.3.5 – *Equivalence Classes of a Relation Given as a Set of Ordered Pairs*

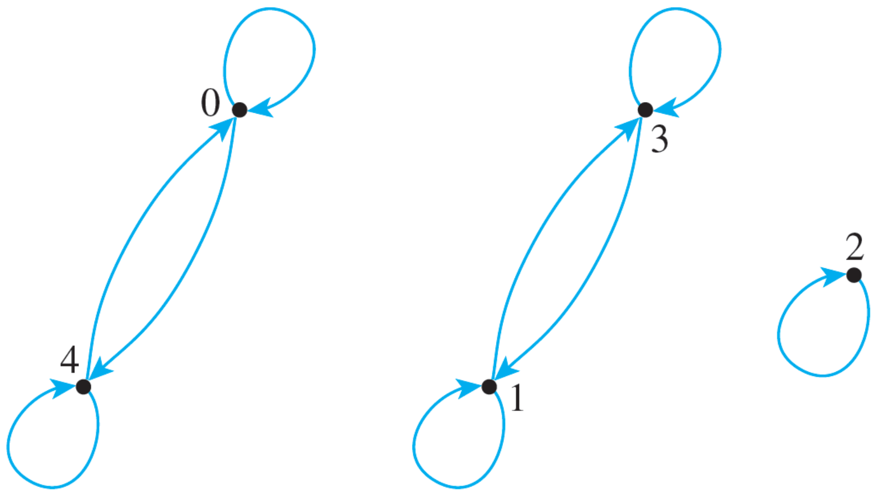
Let $A = \{0, 1, 2, 3, 4\}$ and define a relation R on A as follows:

$$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}.$$

Example 8.3.5 – *Equivalence Classes of a Relation Given as a Set of Ordered Pairs*

continued

The directed graph for *R* is as shown below. As can be seen by inspection, *R* is an equivalence relation on *A*. Find the distinct equivalence classes of *R*.



Example 8.3.5 – *Solution*

First find the equivalence class of every element of *A*.

$$[0] = \{x \in A \mid x\,R\,0\} = \{0, 4\}$$

$$[1] = \{x \in A \mid x\,R\,1\} = \{1, 3\}$$

$$[2] = \{x \in A \mid x\,R\,2\} = \{2\}$$

$$[3] = \{x \in A \mid x\,R\,3\} = \{1, 3\}$$

$$[4] = \{x \in A \mid x\,R\,4\} = \{0, 4\}$$

Note that [0] = [4] and [1] = [3]. Thus the *distinct* equivalence classes of the relation are {0, 4}, {1, 3}, and {2}.

Equivalence Classes of an Equivalence Relation

The first lemma says that if two elements of A are related by an equivalence relation R, then their equivalence classes are the same.

Lemma 8.3.2

Suppose A is a set, R is an equivalence relation on A, and a and b are elements of A. If $a \, R \, b$, then $[a] = [b]$.

Equivalence Classes of an Equivalence Relation

The second lemma says that any two equivalence classes of an equivalence relation are either mutually disjoint or identical.

Lemma 8.3.3

If A is a set, R is an equivalence relation on A, and a and b are elements of A, then

$$\text{either} \quad [a] \cap [b] = \emptyset \quad \text{or} \quad [a] = [b].$$

Equivalence Classes of an Equivalence Relation

Theorem 8.3.4 The Partition Induced by an Equivalence Relation

If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

Congruence Modulo n

Example 8.3.10 – *Equivalence Classes of Congruence Modulo 3*

Let R be the relation of congruence modulo 3 on the set **Z** of all integers. That is, for all integers m and n,

$$\boxed{m\,R\,n \quad \Leftrightarrow \quad 3 \mid (m-n).}$$

Describe the distinct equivalence classes of R.

Example 8.3.10 – *Solution*

For each integer *a*,

$$\begin{aligned}[a] &= \{x \in \mathbf{Z} \mid x\, R\, a\} \\ &= \{x \in \mathbf{Z} \mid 3 \mid (x - a)\} \\ &= \{x \in \mathbf{Z} \mid x - a = 3k, \text{ for some integer } k\}.\end{aligned}$$

Therefore,

$$[a] = \{x \in Z \mid x = 3k + a, \text{ for some integer } k\}.$$

Example 8.3.10 – *Solution*

continued

In particular,

$$
\begin{aligned}
[0] &= \{x \in \mathbf{Z} \mid x = 3k + 0, \text{ for some integer } k\} \\
&= \{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\} \\
&= \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\}, \\
[1] &= \{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\} \\
&= \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\}, \\
[2] &= \{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\} \\
&= \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}.
\end{aligned}
$$

Example 8.3.10 – *Solution* continued

Now since 3 R 0, then by Lemma 8.3.2,

$$[3] = [0].$$

More generally, by the same reasoning,

$$[0] = [3] = [-3] = [6] = [-6] = \dots \text{, and so on.}$$

Similarly,

$$[1] = [4] = [-2] = [7] = [-5] = \dots \text{, and so on.}$$

And

$$[2] = [5] = [-1] = [8] = [-4] = \dots \text{, and so on.}$$

Example 8.3.10 – *Solution*

continued

Notice that every integer is in class [0], [1], or [2]. Hence the distinct equivalence classes are

$$\{x \in \mathbf{Z} \mid x = 3k, \text{ for some integer } k\},$$

$$\{x \in \mathbf{Z} \mid x = 3k + 1, \text{ for some integer } k\}, \quad \text{and}$$

$$\{x \in \mathbf{Z} \mid x = 3k + 2, \text{ for some integer } k\}.$$

In words, the three classes of congruence modulo 3 are (1) the set of all integers that are divisible by 3, (2) the set of all integers that leave a remainder of 1 when divided by 3, and (3) the set of all integers that leave a remainder of 2 when divided by 3.

Congruence Modulo n

Definition

Suppose R is an equivalence relation on a set A and S is an equivalence class of R. A **representative** of the class S is any element a such that $[a] = S$.

Congruence Modulo n

Definition

Let m and n be integers and let d be a positive integer. We say that ***m* is congruent to *n* modulo *d*** and write

$$\boldsymbol{m \equiv n \ (\mathrm{mod}\ d)}$$

if, and only if, $d \mid (m - n)$.

Symbolically: $m \equiv n (\mathrm{mod}\ d) \quad \Leftrightarrow \quad d \mid (m - n)$.

Example 8.3.11 – *Evaluating Congruences*

Determine which of the following congruences are true and which are false.

a. $12 \equiv 7 \pmod{5}$ b. $6 \equiv -8 \pmod{4}$ c. $3 \equiv 3 \pmod{7}$

Example 8.3.11 – *Solution*

a. True. $12 - 7 = 5 = 5 \cdot 1$. Hence $5 \mid (12 - 7)$, and so $12 \equiv 7 \pmod 5$.

b. False. $6 - (-8) = 14$, and $4 \nmid 14$ because $14 \neq 4 \cdot k$ for any integer k. Consequently, $6 \not\equiv -8 \pmod 4$.

c. True. $3 - 3 = 0 = 7 \cdot 0$. Hence $7 \mid (3 - 3)$, and so, $3 \equiv 3 \pmod 7$.

A Definition for Rational Numbers

A Definition for Rational Numbers

For a moment, forget what you know about fractional arithmetic and look at the numbers

$$\frac{1}{3} \quad \text{and} \quad \frac{2}{6}$$

as *symbols*. Considered as symbolic expressions, these *appear* quite different. In fact, if they were written as ordered pairs (1, 3) and (2, 6) they would *be* different.

A Definition for Rational Numbers

The fact that we regard them as “the same” is a specific instance of our general agreement to regard any two numbers

$$\frac{a}{b} \quad \text{and} \quad \frac{c}{d}$$

as equal provided the *cross products* are equal; in other words, if, and only if, $ad = bc$.

Example 8.3.12 – *Rational Numbers Are Really Equivalence Classes*

Let A be the set of all ordered pairs of integers for which the second element of the pair is nonzero. Symbolically:

$$A = \mathbf{Z} \times (\mathbf{Z} - \{0\}).$$

Define a relation R on A as follows: For all pairs (a, b) and (c, d) in A,

$$\boxed{(a, b)\, R\, (c, d) \quad \Leftrightarrow \quad ad = bc.}$$

Example 8.3.12 – *Rational Numbers Are Really Equivalence Classes*
continued

The fact is that *R* is an equivalence relation.

a. Prove that *R* is transitive.

b. Describe the distinct equivalence classes of *R*.

Example 8.3.12 – *Solution*

a. By definition of R, (a, b)=(c, d) and (c, d) = (e, f). We want to show that (a, b) = (e, f)

$$(1)\ ad = bc \quad \text{and} \quad (2)\ cf = de.$$

Since the second elements of all ordered pairs in A are nonzero, $b \neq 0$, $d \neq 0$, and $f \neq 0$. Multiply both sides of equation (1) by f and both sides of equation (2) by b to obtain

$$(1')\ adf = bcf \quad \text{and} \quad (2')\ bcf = bde.$$

Example 8.3.12 – *Solution*

continued

Because both equal bcf,

$$adf = bde,$$

and, since $d \neq 0$, it follows from the cancellation law for multiplication that

$$af = be.$$

Hence, by definition of R, $(a, b)\ R\ (e, f)$.

Example 8.3.12 – *Solution*

continued

b. There is one equivalence class for each distinct rational number. Each equivalence class consists of all ordered pairs (a, b) that, if written as fractions a/b, would equal each other.

The reason is that the condition for two rational numbers to be equal is the same as the condition for two ordered pairs to be related.

Example 8.3.12 – *Solution*

continued

For instance, the class of (1, 2) is

$[(1, 2)] = \{(1, 2), (-1, -2), (2, 4), (-2, -4), (3, 6),$
$(-3, -6), \ldots\}$

since $\frac{1}{2} = \frac{-1}{-2} = \frac{2}{4} = \frac{-2}{-4} = \frac{3}{6} = \frac{-3}{-6}$ and so forth.