CHAPTER 8

PROPERTIES OF RELATIONS

8.1 Relations on Sets

Example 8.1.1 – *The Less-than Relation for Real Numbers*

Define a relation L from **R** to **R** as follows: For all real numbers x and y,

$$x \; L \; y \Leftrightarrow x < y.$$

a. Is $57 \; L \; 53$?

b. Is $(-17) \; L \; (-14)$?

c. Is $143 \; L \; 143$?

d. Is $(-35) \; L \; 1$?

e. Draw the graph of L as a subset of the Cartesian plane $\mathbf{R} \times \mathbf{R}$.

Example 8.1.1 – *Solution*

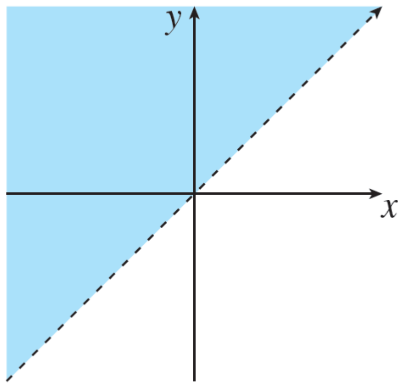
a. No, 57 > 53.

b. Yes, −17 < −14.

c. No, 143 = 143.

d. Yes, −35 < 1.

e. For each value of x, all the points (x, y) with $y > x$ are on the graph. So, the graph consists of all the points above the line $x = y$.



Example 8.1.2 – *The Congruence Modulo 2 Relation*

Define a relation E from $\mathbf{Z}$ to $\mathbf{Z}$ as follows: For every $(m, n) \in \mathbf{Z} \times \mathbf{Z}$,

$$m\ E\ n \Leftrightarrow m - n \text{ is even.}$$

a. Is $4\ E\ 0$? Is $2\ E\ 6$? Is $3\ E\ (-3)$? Is $5\ E\ 2$?

b. List five integers that are related by E to 1.

c. Prove that if n is any odd integer, then $n\ E\ 1$.

Example 8.1.2 – *Solution*

a. Yes, $4\ E\ 0$ because $4 - 0 = 4$ and 4 is even.

Yes, $2\ E\ 6$ because $2 - 6 = -4$ and -4 is even.

Yes, $3\ E\ (-3)$ because $3 - (-3) = 6$ and 6 is even.

No, $5 \not E\ 2$ because $5 - 2 = 3$ and 3 is not even.

Example 8.1.2 – *Solution*

continued

b. There are many such lists. One is

$$1 \text{ because } 1 - 1 = 0 \text{ is even.}$$

$$3 \text{ because } 3 - 1 = 2 \text{ is even.}$$

$$5 \text{ because } 5 - 1 = 4 \text{ is even.}$$

$$-1 \text{ because } -1 - 1 = -2 \text{ is even.}$$

$$-3 \text{ because } -3 - 1 = -4 \text{ is even.}$$

Example 8.1.2 – *Solution*

continued

c. **Proof:** Suppose n is any odd integer. Then $n = 2k + 1$ for some integer k. Now by definition of E, $n\ E\ 1$ if, and only if, $n - 1$ is even.

But by substitution,

$$n - 1 = (2k + 1) - 1 = 2k,$$

and since k is an integer, $2k$ is even. Hence $n\ E\ 1$ *[as was to be shown]*.

Example 8.1.2 – *Solution*

continued

It can be shown that integers *m* and *n* are related by *E* if, and only if, *m mod* 2 = *n mod* 2 (that is, both are even or both are odd).

When this occurs *m* and *n* are said to be **congruent modulo 2**.

Example 8.1.3 – *A Relation on a Power Set*

Let $X = \{a, b, c\}$.

Then $\mathscr{P}(X) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Define a relation **S** from $\mathscr{P}(X)$ to $\mathscr{P}(X)$ as follows: For all sets A and B in $\mathscr{P}(X)$ (that is, for all subsets A and B of X),

$$A \mathbf{\ S\ } B \Leftrightarrow A \text{ has at least as many elements as } B.$$

a. Is $\{a, b\}$ **S** $\{b, c\}$?

b. Is $\{a\}$ **S** $\emptyset$?

c. Is $\{b, c\}$ **S** $\{a, b, c\}$?

d. Is $\{c\}$ **S** $\{a\}$?

Example 8.1.3 – *Solution*

a. Yes, both sets have two elements.

b. Yes, $\{a\}$ has one element and $\varnothing$ has zero elements, and $1 \geq 0$.

c. No, $\{b, c\}$ has two elements and $\{a, b, c\}$ has three elements and $2 < 3$.

d. Yes, both sets have one element.

The Inverse of a Relation

The Inverse of a Relation

If R is a relation from A to B, then a relation R^{-1} from B to A can be defined by interchanging the elements of all the ordered pairs of R.

Definition

Let R be a relation from A to B. Define the inverse relation R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

This definition can be written operationally as follows:

$$\text{For all } x \in A \text{ and } y \in B, \quad (y, x) \in R^{-1} \quad \Leftrightarrow \quad (x, y) \in R.$$

Example 8.1.4 – *The Inverse of a Finite Relation*

Let A = {2, 3, 4} and B = {2, 6, 8}, and let R be the "divides" relation from A to B: For every ordered pair $(x, y) \in A \times B$,

$$x\,R\,y \quad \Leftrightarrow \quad x \mid y \qquad x \text{ divides } y.$$

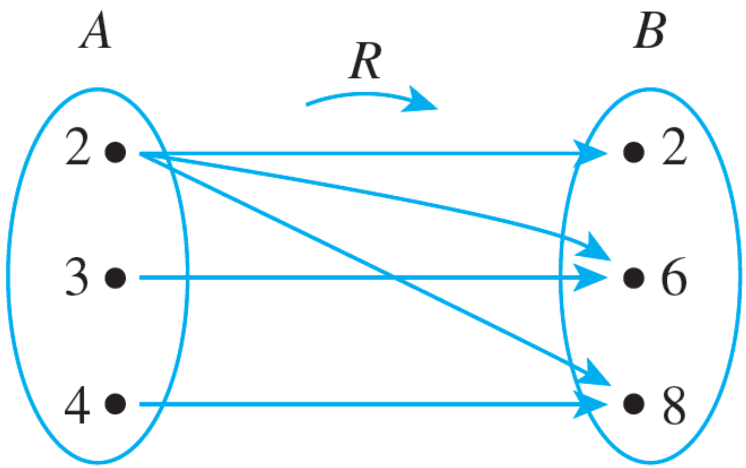
a. State explicitly which ordered pairs are in R and R^{-1}, and draw arrow diagrams for R and R^{-1}.

b. Describe R^{-1} in words.

Example 8.1.4 – *Solution*

a. $R = \{(2, 2), (2, 6), (2, 8), (3, 6), (4, 8)\}$

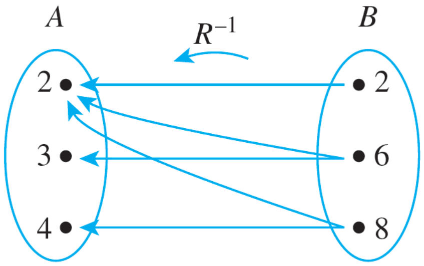
$$R^{-1} = \{(2, 2), (6, 2), (8, 2), (6, 3), (8, 4)\}$$



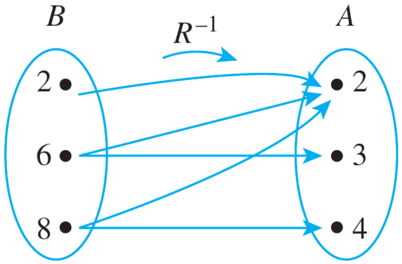
Example 8.1.4 – *Solution*

continued

To draw the arrow diagram for R^{-1}, you can copy the arrow diagram for *R* but reverse the directions of the arrows.



Or you can redraw the diagram so that *B* is on the left.



Example 8.1.4 – *Solution*

continued

b. R^{-1} can be described in words as follows: For every ordered pair $(y, x) \in B \times A$,

$$y \, R^{-1} x \Leftrightarrow y \text{ is a multiple of } x.$$

Directed Graph of a Relation

Directed Graph of a Relation

Definition

A **relation on a set** A is a relation from A to A.

When a relation R is defined *on* a set A, the arrow diagram of the relation can be modified so that it becomes a **directed graph**.

Instead of representing A as two separate sets of points, represent A only once, and draw an arrow from each point of A to each related point.

Directed Graph of a Relation

As with an ordinary arrow diagram,

For all points x and y in A,

$$\text{there is an arrow from } x \text{ to } y \quad \Leftrightarrow \quad x\, R\, y \quad \Leftrightarrow \quad (x, y) \in R.$$

Example 8.1.6 – *Directed Graph of a Relation*

Let $A = \{3, 4, 5, 6, 7, 8\}$ and define a relation R on A as follows: For every $x, y \in A$,

$$x \, R \, y \quad \Leftrightarrow \quad 2 \mid (x - y).$$

Draw the directed graph of R.

Example 8.1.6 – *Solution*

Note that 3 R 3 because 3 − 3 = 0 and $2 | 0$ since 0 = 2 · 0. Thus, there is a loop from 3 to itself. Similarly, there is a loop from 4 to itself, from 5 to itself, and so forth, since the difference of each integer with itself is 0, and $2 | 0$.

Note also that 3 R 5 because 3 − 5 = −2 = 2 · (−1). And 5 R 3 because 5 − 3 = 2 = 2 · 1. Hence there is an arrow from 3 to 5 and also an arrow from 5 to 3.

Example 8.1.6 – *Solution* continued

The other arrows in the directed graph, as shown below, are obtained by similar reasoning.