CHAPTER 5

SEQUENCES, MATHEMATICAL INDUCTION, AND RECURSION

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5.6 Defining Sequences Recursively

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Defining Sequences Recursively (1/2)

A sequence can be defined in a variety of different ways. One informal way is to write the first few terms with the expectation that the general pattern will be obvious.

We might say, for instance, “consider the sequence 3, 5, 7, … .” Unfortunately, misunderstandings can occur when this approach is used.

The next term of the sequence could be 9 if we mean a sequence of odd integers, or it could be 11 if we mean the sequence of odd prime numbers.

Defining Sequences Recursively (2/2)

The second way to define a sequence is to give an explicit formula for its *n*th term. For example, a sequence $a_0, a_1, a_2 \ldots$ can be specified by writing

$$a_n = \frac{(-1)^n}{n+1} \quad \text{for every integer } n \geq 0.$$

The advantage of defining a sequence by such an explicit formula is that each term of the sequence is uniquely determined and can be computed in a fixed, finite number of steps by substitution.

The third way to define a sequence is to use recursion.

Defining Sequences Recursively

This requires giving both an equation, called a *recurrence relation*, that defines each later term in the sequence by reference to earlier terms and also one or more initial values for the sequence. Sometimes it is very difficult or impossible to find an explicit formula for a sequence, but it *is* possible to define the sequence using recursion.

Definition

A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots$ is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$, where i is an integer with $k - i \geq 0$. If i is a fixed integer, the **initial conditions** for such a recurrence relation specify the values of $a_0, a_1, a_2, \ldots, a_{i-1}$. If i depends on k, the initial conditions specify the values of $a_0, a_1, \ldots, a_m$, where m is an integer with $m \geq 0$.

Example 5.6.1 – *Computing Terms of a Recursively Defined Sequence*

Define a sequence $c_0, c_1, c_2, \ldots$ recursively as follows: For every integer $k \geq 2$,

(1) $c_k = c_{k-1} + kc_{k-2} + 1$ recurrence relation

(2) $c_0 = 1$ and $c_1 = 2$ initial conditions

Find c_2, c_3, and c_4.

Example 5.6.1 – *Solution (1/2)*

$c_2 = c_1 + 2c_0 + 1$ by substituting $k = 2$ into (1)

$= 2 + 2 \cdot 1 + 1$ since $c_1 = 2$ and $c_0 = 1$ by (2)

(3) $\therefore c_2 = 5$

$c_3 = c_2 + 3c_1 + 1$ by substituting $k = 3$ into (1)

$= 5 + 3 \cdot 2 + 1$ since $c_2 = 5$ by (3) and $c_1 = 2$ by (2)

Example 5.6.1 – *Solution (2/2)* continued

(4) $\therefore c_3 = 12$

$c_4 = c_3 + 4c_2 + 1$ by substituting $k = 4$ into (1)

$= 12 + 4 \cdot 5 + 1$ since $c_3 = 12$ by (4) and $c_2 = 5$ by (3)

(5) $\therefore c_4 = 33$

Example 5.6.2 – *Writing a Recurrence Relation in More Than One Way*

Let $s_0, s_1, s_2,\ldots$ be a sequence that satisfies the following recurrence relation:

For every integer $k \geq 1$, $\quad s_k = 3s_{k-1} - 1.$

Explain why the following statement is true:

For every integer $k \geq 0$, $\quad s_{k+1} = 3s_k - 1.$

Example 5.6.2 – *Solution*

In informal language, the recurrence relation says that any term of the sequence equals 3 times the previous term minus 1.

Now for any integer $k \geq 0$, the term previous to s_{k+1} is s_k.

Thus for any integer $k \geq 0$, $s_{k+1} = 3s_k - 1$.

Example 5.6.3 – *Sequences That Satisfy the Same Recurrence Relation*

Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ satisfy the recurrence relation that the kth term equals 3 times the $(k - 1)$st term for every integer $k \geq 2$:

(1) $a_k = 3a_{k-1}$ and $b_k = 3b_{k-1}$.

But suppose that the initial conditions for the sequences are different:

(2) $a_1 = 2$ and $b_1 = 1$.

Find (a) a_2, a_3, a_4 and (b) b_2, b_3, b_4.

Example 5.6.3 – *Solution*

a. $a_2 = 3a_1 = 3 \cdot 2 = 6$

$a_3 = 3a_2 = 3 \cdot 6 = 18$

$a_4 = 3a_3 = 3 \cdot 18 = 54$

b. $b_2 = 3b_1 = 3 \cdot 1 = 3$

$b_3 = 3b_2 = 3 \cdot 3 = 9$

$b_4 = 3b_3 = 3 \cdot 9 = 27$

Thus

$a_1, a_2, a_3, \ldots$ begins 2, 6, 18, 54,… and

$b_1, b_2, b_3, \ldots$ begins 1, 3, 9, 27,….

Example 5.6.4 – *Showing That a Sequence Given by an Explicit Formula Satisfies a Certain Recurrence Relation*

The sequence of **Catalan numbers**, named after the Belgian mathematician Eugène Catalan (1814–1894), arises in a remarkable variety of different contexts in discrete mathematics. It can be defined as follows: For each integer $n \geq 1$,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

a. Find C_1, C_2, and C_3.

b. Show that this sequence satisfies the recurrence relation

$C_k = \frac{4k-2}{k+1}C_{k-1}$ for every integer $k \geq 2$.

Example 5.6.4 – *Solution (1/4)*

a. $$C_1 = \frac{1}{2}\binom{2}{1} = \frac{1}{2}\cdot 2 = 1,$$

$$C_2 = \frac{1}{3}\binom{4}{2} = \frac{1}{3}\cdot 6 = 2,$$

$$C_3 = \frac{1}{4}\binom{6}{3} = \frac{1}{4}\cdot 20 = 5$$

Example 5.6.4 – *Solution (2/4)* continued

b. To obtain the kth and $(k - 1)$st terms of the sequence, just substitute k and $k - 1$ in place of n in the explicit formula for $C_1, C_2, C_3, \ldots$.

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

$$C_{k-1} = \frac{1}{(k-1)+1}\binom{2(k-1)}{k-1}$$

$$= \frac{1}{k}\binom{2k-2}{k-1}.$$

Example 5.6.4 – *Solution (3/4)*

continued

Then start with the right-hand side of the recurrence relation and transform it into the left-hand side: For each integer $k \geq 2$,

$$\frac{4k-2}{k+1}C_{k-1} = \frac{4k-2}{k+1}\left[\frac{1}{k}\binom{2k-2}{k-1}\right]$$

by substituting

$$= \frac{2(2k-1)}{k+1} \cdot \frac{1}{k} \cdot \frac{(2k-2)!}{(k-1)!(2k-2-(k-1))!}$$

by the formula for *n* choose *r*

$$= \frac{1}{k+1} \cdot (2(2k-1)) \cdot \frac{(2k-2)!}{(k(k-1)!)(k-1)!}$$

by rearranging the factors

Example 5.6.4 – *Solution (4/4)*

continued

$$= \frac{1}{k+1} \cdot (2(2k-1)) \cdot \frac{1}{k!(k-1)!} \cdot (2k-2)! \cdot \frac{1}{2} \cdot \frac{1}{k} \cdot 2k$$ because $\frac{1}{2} \cdot \frac{1}{k} \cdot 2k = 1$

$$= \frac{1}{k+1} \cdot \frac{2}{2} \cdot \frac{1}{k!} \cdot \frac{1}{(k-1)!} \cdot \frac{1}{k} \cdot (2k) \cdot (2k-1) \cdot (2k-2)!$$ by rearranging the factors

$$= \frac{1}{k+1} \cdot \frac{(2k)!}{k!k!}$$ because $k(k-1)! = k!$, $\frac{2}{2} = 1$, and $2k \cdot (2k-1) \cdot (2k-2)! = (2k)!$

$$= \frac{1}{k+1} \binom{2k}{k}$$ by the formula for *n* choose *r*

$$= C_k$$ by definition of C_1, C_2, C_3, …

Examples of Recursively Defined Sequences

Examples of Recursively Defined Sequences (1/1)

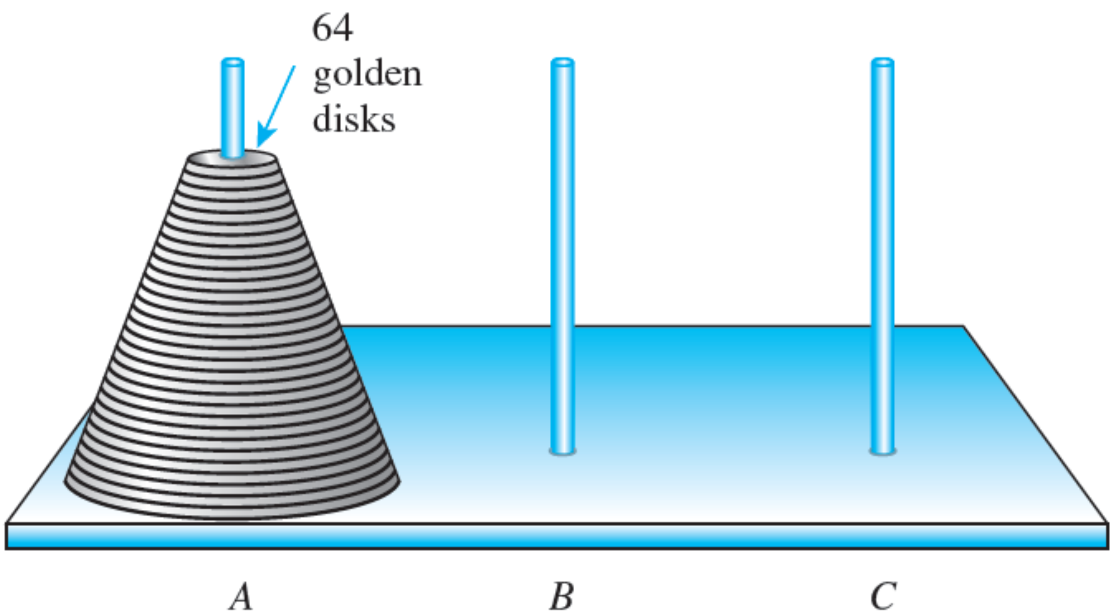


Figure 5.6.1

Example 5.6.6 – *The Fibonacci Numbers (1/2)*

One of the earliest examples of a recursively defined sequence occurs in the writings of Leonardo of Pisa, commonly known as Fibonacci, who was the greatest European mathematician of the Middle Ages and promoted the use of Hindu-Arabic numerals for calculation. In 1202 Fibonacci posed the following problem:

A single pair of rabbits (male and female) is born at the beginning of a year.

Example 5.6.6 – *The Fibonacci Numbers (2/2)*

Assume the following conditions:

1. Rabbit pairs are not fertile during their first month of life but thereafter give birth to one new male/female pair at the end of every month.

2. No rabbits die.

How many rabbits will there be at the end of the year?

Example 5.6.6 – *Solution (1/6)*

One way to solve this problem is to plunge right into the middle of it using recursion. Suppose you know how many rabbit pairs there were at the ends of previous months. How many will there be at the end of the current month?

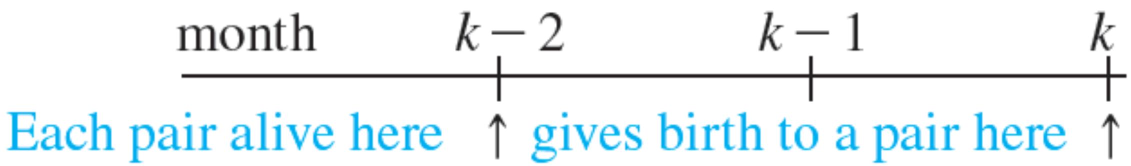
The crucial observation is that the number of rabbit pairs born at the end of month k is the same as the number of pairs alive at the end of month $k - 2$. Why?

Because it is exactly the rabbit pairs that were alive at the end of month $k - 2$ that were fertile during month k.

Example 5.6.6 – *Solution* (2/6)

continued

The rabbits born at the end of month $k - 1$ were not.



Now the number of rabbit pairs alive at the end of month k equals the ones alive at the end of month $k - 1$ plus the pairs newly born at the end of the month. Thus

$$\begin{bmatrix}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k\end{bmatrix} = \begin{bmatrix}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k-1\end{bmatrix} + \begin{bmatrix}\text{the number}\\ \text{of rabbit}\\ \text{pairs born}\\ \text{at the end}\\ \text{of month } k\end{bmatrix}$$

Example 5.6.6 – *Solution (3/6)* continued

$$= \begin{bmatrix}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k-1\end{bmatrix} + \begin{bmatrix}\text{the number}\\ \text{of rabbit}\\ \text{pairs alive}\\ \text{at the end}\\ \text{of month } k-2\end{bmatrix}. \qquad 5.6.2$$

For each integer $n \geq 1$, let

$$F_n = \begin{bmatrix}\text{the number of rabbit pairs}\\ \text{alive at the end of month } n\end{bmatrix}$$

and let

F_0 = the initial number of rabbit pairs
= 1.

Example 5.6.6 – *Solution* (4/6)

continued

Then by substitution into equation (5.6.2), for every integer $k \geq 2$,

$$F_k = F_{k-1} + F_{k-2}.$$

Now $F_0 = 1$, as already noted, and $F_1 = 1$ also, because the first pair of rabbits is not fertile until the second month.

Hence the complete specification of the Fibonacci sequence is as follows: For every integer $k \geq 2$,

(1) $F_k = F_{k-1} + F_{k-2}$ recurrence relation

(2) $F_0 = 1, \quad F_1 = 1$ initial conditions.

Example 5.6.6 – *Solution* (5/6)

continued

To answer Fibonacci's question, compute F_2, F_3, and so forth through F_{12}:

(3) $F_2 = F_1 + F_0 = 1 + 1 \quad = 2$ by (1) and (2)

(4) $F_3 = F_2 + F_1 = 2 + 1 \quad = 3$ by (1), (2), and (3)

(5) $F_4 = F_3 + F_2 = 3 + 2 \quad = 5$ by (1), (3), and (4)

(6) $F_5 = F_4 + F_3 = 5 + 3 \quad = 8$ by (1), (4), and (5)

(7) $F_6 = F_5 + F_4 = 8 + 5 \quad = 13$ by (1), (5), and (6)

(8) $F_7 = F_6 + F_5 = 13 + 8 \quad = 21$ by (1), (6), and (7)

Example 5.6.6 – *Solution (6/6)*

continued

(9) $F_8 = F_7 + F_6 = 21 + 13 \quad = 34$ by (1), (7), and (8)

(10) $F_9 = F_8 + F_7 = 34 + 21 \quad = 55$ by (1), (8), and (9)

(11) $F_{10} = F_9 + F_8 = 55 + 34 \quad = 89$ by (1), (9), and (10)

(12) $F_{11} = F_{10} + F_9 = 89 + 55 \quad = 144$ by (1), (10), and (11)

(13) $F_{12} = F_{11} + F_{10} = 144 + 89 \quad = 233$ by (1), (11), and (12)

At the end of the twelfth month there are 233 rabbit pairs, or 466 rabbits in all.

Example 5.6.8 – *Compound Interest with Compounding Several Times a Year (1/3)*

When an annual interest rate of i is compounded m times per year, the interest rate paid per period is i/m.

(For instance, if 3% = 0.03 annual interest is compounded quarterly, then the interest rate paid per quarter is 0.03/4 = 0.0075.)

For each integer $k \geq 1$, let P_k = the amount on deposit at the end of the kth period, assuming no additional deposits or withdrawals.

Example 5.6.8 – *Compound Interest with Compounding Several Times a Year (2/3)*

Then the interest earned during the *k*th period equals the amount on deposit at the end of the $(k-1)$st period times the interest rate for the period:

$$\text{interest earned during } k\text{th period} = P_{k-1}\left(\frac{i}{m}\right).$$

The amount on deposit at the end of the *k*th period, P_k, equals the amount at the end of the $(k-1)$st period, P_{k-1}, plus the interest earned during the *k*th period:

$$P_k = P_{k-1} + P_{k-1}\left(\frac{i}{m}\right) = P_{k-1}\left(1 + \frac{i}{m}\right). \qquad 5.6.4$$

Example 5.6.8 – *Compound Interest with Compounding Several Times a Year (3/3)*

Suppose $10,000 is left on deposit at 3% compounded quarterly.

a. How much will the account be worth at the end of one year, assuming no additional deposits or withdrawals?

b. The **annual percentage yield (APY)** is the percentage increase in the value of the account over a one-year period. What is the APY for this account?

Example 5.6.8 – *Solution* (1/3)

a. For each integer $n \geq 1$, let P_n = the amount on deposit after n consecutive quarters, assuming no additional deposits or withdrawals, and let P_0 be the initial \$10,000.

Then by equation (5.6.4) with $i = 0.03$ and $m = 4$, a recurrence relation for the sequence $P_0, P_1, P_2, \ldots$ is

(1) $P_k = P_{k-1}(1 + 0.0075) = (1.0075) \cdot P_{k-1}$

for every integer $k \geq 1$.

Example 5.6.8 – *Solution (2/3)*

continued

The amount on deposit at the end of one year (four quarters), P_4, can be found by successive substitution:

(2) $P_0 = \$10{,}000$

(3) $P_1 = 1.0075 \cdot P_0 = (1.0075) \cdot \$10{,}000.00 = \$10{,}075.00$

by (1) and (2)

(4) $P_2 = 1.0075 \cdot P_1 = (1.0075) \cdot \$10{,}075.00 \cong \$10{,}150.56$

by (1) and (3)

(5) $P_3 = 1.0075 \cdot P_2 \cong (1.0075) \cdot \$10{,}150.56 \cong \$10{,}226.69$

by (1) and (4)

Example 5.6.8 – *Solution (3/3)*

continued

(6) $P_4 = 1.0075 \cdot P_3 \cong (1.0075) \cdot \$10{,}226.69 \cong \$10{,}303.39$

by (1) and (5)

Hence after one year there is \$10,303.39 (to the nearest cent) in the account.

b. The percentage increase in the value of the account, or APY, is

$$\frac{10{,}303.39 - 10{,}000}{10{,}000} = 0.03034 = 3.034\%.$$

Recursive Definitions of Sum and Product

Recursive Definitions of Sum and Product (1/1)

Addition and multiplication are called *binary* operations because only two numbers can be added or multiplied at a time. Careful definitions of sums and products of more than two numbers use recursion.

Definition

Given numbers $a_1, a_2, \ldots, a_n$, where n is a positive integer, the **summation from $i = 1$ to n of the a_i**, denoted $\sum_{i=1}^{n} a_i$, is defined as follows:

$$\sum_{i=1}^{1} a_i = a_1 \quad \text{and} \quad \sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n-1} a_i\right) + a_n, \quad \text{if } n > 1.$$

The **product from $i = 1$ to n of the a_i**, denoted $\prod_{i=1}^{n} a_i$, is defined by

$$\prod_{i=1}^{1} a_i = a_1 \quad \text{and} \quad \prod_{i=1}^{n} a_i = \left(\prod_{i=1}^{n-1} a_i\right) \cdot a_n, \quad \text{if } n > 1.$$

Example 5.6.9 – *A Sum of Sums*

Prove that for any positive integer n, if a_1, a_2,… , a_n and b_1, b_2,… , b_n are real numbers, then

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i.$$

Example 5.6.9 – *Solution (1/5)*

The proof is by mathematical induction. Let the property $P(n)$ be the equation

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i. \qquad \leftarrow P(n)$$

We must show that $P(n)$ is true for every integer $n \geq 1$. We do this by mathematical induction on n.

Show that P*(1) *is true: To establish $P(1)$, we must show that

$$\sum_{i=1}^{1}(a_i + b_i) = \sum_{i=1}^{1} a_i + \sum_{i=1}^{1} b_i. \qquad \leftarrow P(1)$$

Example 5.6.9 – *Solution* (2/5)

continued

Now

$$\sum_{i=1}^{1}(a_i + b_i) = a_1 + b_1 \qquad \text{by definition of } \Sigma$$

$$= \sum_{i=1}^{1} a_i + \sum_{i=1}^{1} b_i \qquad \text{also by definition of } \Sigma.$$

Hence *P*(1) is true.

Example 5.6.9 – *Solution* (3/5) continued

Show that for every integer k **≥ 1,** ***if P*****(*****k*****)** ***is true then P*****(*****k*** **+ 1)** ***is also true:***

Suppose that k is any integer with $k \geq 1$ and that $a_1, a_2, \ldots, a_k, a_{k+1}$ and $b_1, b_2, \ldots, b_k, b_{k+1}$ are real numbers such that

$$\sum_{i=1}^{k}(a_i + b_i) = \sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i. \qquad \leftarrow P(k) \text{ inductive hypothesis}$$

We must show that

$$\sum_{i=1}^{k+1}(a_i + b_i) = \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i. \qquad \leftarrow P(k+1)$$

Example 5.6.9 – *Solution (4/5)* continued

[We will show that the left-hand side of this equation equals the right-hand side.]

Now the left-hand side of the equation is

$$\sum_{i=1}^{k+1}(a_i + b_i) = \sum_{i=1}^{k}(a_i + b_i) + (a_{k+1} + b_{k+1}) \qquad \text{by definition of } \Sigma$$

$$= \left(\sum_{i=1}^{k} a_i + \sum_{i=1}^{k} b_i\right) + (a_{k+1} + b_{k+1}) \qquad \text{by inductive hypothesis}$$

Example 5.6.9 – *Solution (5/5)*

continued

$$= \left(\sum_{i=1}^{k} a_i + a_{k+1}\right) + \left(\sum_{i=1}^{k} b_i + b_{k+1}\right)$$

by the associative and commutative laws of algebra

$$= \sum_{i=1}^{k+1} a_i + \sum_{i=1}^{k+1} b_i$$

by definition of Σ

which equals the right-hand side of the equation. *[This is what was to be shown.]*