February 6th

Tuesday, February 6, 2024 1:45 PM

- Set up Discord Server for communication

Gaussian Elimination

We are trying to solve

$$Ax = b$$

is through elementary operations.

Set an upper triangular matrix U such that $Ux = y$ is an equivalent linear system.

A_k — k^{th} leading principal submatrix of A obtained by intersecting the first k rows and columns.

To proceed we must assume that A_k is nonsingular $k = 1, 2, \dots n$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

<u>WANT</u>

Note

$$a_{21} - M_{21} a_{11} = 0$$

$$a_{21} = M_{21} a_{11}$$

$$\frac{a_{21}}{a_{11}} = M_{21}$$

In general, For the ith row

$$a_{i1} - M_{i1} a_{11} = 0$$

$$M_{i1} = \frac{a_{i1}}{a_{11}}$$

These are the multiples we need to get the zeros in the first column

$$Ax = b$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ M_{21} & a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ M_{31} & \vdots & & & \\ \vdots & & & & \\ M_{n1} & a_{n2}^{(1)} & a_{n3}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}$$

$(n-1) \times (n-1)$

$$\begin{bmatrix} a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & & \\ a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{bmatrix}$$

$$M_{i2} = \frac{a_{i2}^{(1)}}{a_{22}^{(1)}}$$

$$\begin{bmatrix} a_{22}^{(1)} & a_{23}^{(1)} & \cdots & a_{2n}^{(1)} \\ M_{32} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ M_{42} & \vdots & & \\ \vdots & & & \\ M_{n2} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} \in$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ M_{21} & a_{22}^{(1)} & a_{23}^{(1)} & & a_{2n}^{(1)} \\ M_{31} & M_{32} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & M_{42} & \vdots & & \end{bmatrix} b_3$$

After $n-1$ - iterations we get an upper triangular matrix

$$A = \begin{bmatrix} m_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ m_{31} & m_{32} & a_{33}^{(2)} & \cdots & a_{3n}^{(2)} \\ \vdots & m_{42} & \vdots & & \\ & \vdots & & & \\ m_{n1} & m_{n2} & a_{n3}^{(2)} & \cdots & a_{nn}^{(2)} \end{bmatrix} b_3$$

we get an upper triangular matrix

What happens to b?

$$b_i = b_i + \Sigma$$

$b_1 = b_1$

$b_2^{(1)} = b_2 - m_{21} b_1$

$b_3^{(2)} = b_3 - m_{32} b_2^{(1)} - m_{31} b_1$

$b_3^{(2)} = b_3 - m_{32}(b_2 - m_{21} b_1) - m_{31} b_1$

$b_3^{(2)} = b_3 - m_{32} b_2 + m_{32} m_{21} b_1 - b_{31} b_1$

What is the flop count to get the upper triangluar form?

For each row (eliminating 1st column)

$2n$ flops and $n-1$ rows

$$2n(n-1) + 2(n-1)(n-2) + \cdots = 2\sum_{k=1}^{n} k(k-1)$$

first iteration

The sum is approximately $\frac{2}{3}n^3$ flops

If you have an augmented matrix

$$2n^2 + 2(n-1)^2 + \cdots = 2\sum_{k=1}^{n} k^2 \approx \frac{2}{3}n^3$$

with back substitution there are n^2 flops.

The total cost of solving $Ax = b$ using this method is

$\frac{2}{3}n^3$ flops

Example

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 9 \\ 9 \\ 16 \end{bmatrix}$$

We want solve $Ax = b$

$$[A \mid b] = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 9 \\ 2 & 2 & -1 & 9 \\ 4 & -1 & 6 & 16 \end{array}\right]$$

$m_{i1} = \frac{a_{i1}}{a_{11}}$

$m_{21} = \frac{2}{2} = 1$

$m_{31} = \frac{4}{2} = 2$

$$\left[\begin{array}{c|cc|c} 2 & 1 & 1 & 9 \\ 1 & 1 & -2 & 0 \\ 2 & -3 & 4 & -2 \end{array}\right]$$

row3 - 2(row1)

$$\text{row3} - 2(\text{row1}) \rightarrow \left[\begin{array}{c|ccc} 2 & -3 & 4 & -2 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{c|cc|c} 2 & 1 & 1 & 9 \\ 1 & 1 & -2 & 0 \\ 2 & -3 & -2 & -2 \end{array}\right]$$

$$m_{32} = \frac{a_{32}^{(1)}}{a_{22}^{(1)}} = \frac{-3}{1} = -3$$

The upper triangular matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix} x = \begin{bmatrix} 9 \\ 0 \\ -2 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$-2x_3 = -2$$
$$x_3 = 1$$

$$x_2 - 2x_3 = 0$$
$$x_2 - 2(1) = 0$$
$$x_2 = 2$$

$$2x_1 + x_2 + x_3 = 9$$
$$2x_1 + 2 + 1 = 9$$
$$2x_1 = 6$$
$$x_1 = 3$$

$$\boxed{x = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}}$$

Interpretation of the Multipliers

The operations of b

$$b_i^{(1)} = b_i - m_{i1} b_1 \qquad i = 2, 3, \ldots n$$
$$b_i^{(2)} = b_i^{(1)} - m_{i2} b_2^{(1)} \qquad i = 3, 4, \ldots n$$
$$\vdots$$
$$b_i^{(n-1)} = b_i^{(n-2)} - m_{i,n-1} b_{n-1}^{(n-2)} \qquad i = n$$

At the end we have

$$\begin{bmatrix} b_1 \\ b_2^{(1)} \\ b_3^{(2)} \\ \vdots \\ b_n^{(n-1)} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = y$$

If we have $Ax = b$ could be solved using an equivalent upper-triangular system $Ux = y$. ... in the original

If we have $Ax = b$ ... system $Ux = y$.
an equivalent upper-triangular system
where U is the upper triangular matrix obtained in the original reduction of A.

Example Suppose we want to solve
$A x = \hat{b}$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix} \qquad \hat{b} = \begin{bmatrix} 7 \\ 3 \\ 20 \end{bmatrix}$$

We already know $U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix}$

$$b_2^{(1)} = 3 - (1)(7) = -4$$

$$b_3^{(1)} = 20 - 2(7) = 6 \Rightarrow b_3^{(2)} = 6 - (-3)(b_2^{(1)})$$
$$= 6 + 3(-4)$$
$$= -6$$

$$Ux = \begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix} x = \begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$-2x_3 = -6$
$x_3 = 3$

$x_2 - 2(3) = -4$
$x_2 = 2$

$2x_1 + 2 + 3 = 7$
$2x_1 = 2$
$x_1 = 1$

$$\boxed{x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

We start out with $Ax = b$
and end up with $Ux = y$

we know
$$b_i^{(1)} = b_i - m_{i1} y_1 \qquad i = 2, 3, \ldots n$$
$$b_i^{(2)} = b_i^{(1)} - m_{i2} y_2 \qquad i = 3, 4, \ldots n$$
$$\vdots$$
$$b_i^{(n-1)} = b_i^{(n)} - m_{i,n-1} y_{n-1} \qquad i = n$$

This leads us to

$$y_i = b_i - \sum_{j=1}^{i-1} m_{ij}\, y_j \qquad i = 1, 2, \ldots n$$

Let's set $L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \cdots & 0 \\ m_{31} & m_{32} & 1 \cdots & 0 \\ \vdots & & & \vdots \\ m_{n1} & m_{n2} & m_{n3} \cdots & 1 \end{bmatrix}$ unit Lower triangular matrix

$$Ly = b$$

In summary, we start with $Ax = b$

we get $Ux = y$

where U is upper triangular and y is the solution of a unit lower triangular system

$$Ly = b$$

$$Ux = y$$

$$LUx = Ly$$

$$LUx = b$$

$$Ax = b$$

Thus $A = LU$

<u>Example</u> Solve the system $Ax = b$ where

$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 0 \\ 11 \end{bmatrix}$

$U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix}$ $\qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ \cdots \end{bmatrix} \begin{bmatrix} y_1 \\ \cdots \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ \cdots \end{bmatrix}$

$Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 11 \end{bmatrix}$$

Forward substitution

$y_1 = 3$

$y_1 + y_2 = 0$

$y_2 = -3$

$2y_1 - 3y_2 + y_3 = 11$

$2(3) - 3(-3) + y_3 = 11$

$15 + y_3 = 11$

$y_3 = -4$

$Ux = y$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -4 \end{bmatrix}$$

$x_3 = 2$

$x_2 - 2(2) = -3$

$x_2 = 1$

$2x_1 + x_2 + x_3 = 3$

$2x_1 + 1 + 2 = 3$

$x_1 = 0$

$$\boxed{x = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}$$

Theorem (LU decomposition Theorem)

Let A be an $n \times n$ matrix whose leading principal submatrices are all nonsingular. Then A can be decomposed in exactly one way into a product $A = LU$ such that L is a unit lower triangular and U is upper triangular.

Proof start of class on Thursday.

Cholesky Decomposition

If we have a positive definite matrix

That A is $n \times n$ real symmetric satisfies the property

$x^T A x > 0$

that A is ...

property $x^T A x > 0$.

<u>Theorem</u> (Cholesky Decomposition Theorem) Let A be positive definite. Then A can be decomposed in exactly one way into a product

$$A = R^T R$$

such that R is upper triangular and has all main diagonal entries r_{ii} positive. R is called the Cholesky factor of A.

<u>Example</u> Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

and $R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ show $A = R^T R$

$$R^T = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$