Feb22

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$$A = \begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix}$$

We know this is a symmetric positive definite matrix. Using the product formulation we find

$$R = \begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Outer-Product Form

$$\begin{bmatrix} a_{11} & b^T \\ b & \hat{A} \end{bmatrix} = \begin{bmatrix} r_{11} & 0 \\ s & \hat{R}^T \end{bmatrix} \cdot \begin{bmatrix} r_{11} & s^T \\ 0 & \hat{R} \end{bmatrix}$$

$$A = R^T R$$

- $r_{11} = \sqrt{a_{11}}$
- $s^T = r_{11}^{-1} b^T$
- $\tilde{A} = \hat{A} - ss^T$

-> Goal becomes find $\hat{R}$ such that $\tilde{A} = \hat{R}^T \hat{R}$

$$\begin{pmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{pmatrix}$$

Then $a_{11} = 4$ and $b = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$ $\hat{A} = \begin{bmatrix} 10 & -2 & -7 \\ -2 & 8 & 4 \\ -7 & 4 & 7 \end{bmatrix}$

Then $r_{11} = \sqrt{4} = 2$

$s^T = r_{11}^{-1} b^T = \frac{1}{2}(-2 \;\; 4 \;\; 2)$
$= (-1 \;\; 2 \;\; 1)$

$\tilde{A} = \hat{A} - ss^T$

$$= \begin{bmatrix} 10 & -2 & -7 \\ -2 & 8 & 4 \\ -7 & 4 & 7 \end{bmatrix} - \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}(-1 \;\; 2 \;\; 1)$$

$$= \begin{bmatrix} 10 & -2 & -7 \\ -2 & 8 & 4 \\ -7 & 4 & 7 \end{bmatrix} - \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 0 & -6 \\ 0 & 4 & 2 \\ -6 & 2 & 6 \end{bmatrix}$$

$$R = \begin{bmatrix} r_{11} & s^T \\ 0 & \hat{R} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & & & \\ 0 & & \hat{R} & \\ 0 & & & \end{bmatrix}$$

$$\tilde{A} = \begin{bmatrix} 9 & 0 & -6 \\ 0 & 4 & 2 \\ -6 & 2 & 6 \end{bmatrix}$$

$a_{11} = 9$ $b = \begin{pmatrix} 0 \\ -6 \end{pmatrix}$

$$\hat{A} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

$r_{11} = \sqrt{a_{11}} = \sqrt{9} = 3$

$s^T = r_{11}^{-1} b^T = \frac{1}{3}(0 \;\; -6)$
$= (0 \;\; -2)$

$\hat{\hat{A}} = \hat{A} - ss^T$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} - \begin{pmatrix} 0 \\ -2 \end{pmatrix}(0 \;\; -2)$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$$\hat{R} = \begin{bmatrix} r_{11} & s^T \\ 0 & \hat{\hat{R}} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & -2 \\ 0 & & \\ 0 & \hat{\hat{R}} & \end{bmatrix}$$

where $\hat{\hat{A}} = \hat{\hat{R}}^T \hat{\hat{R}}$

$$\hat{R} = \begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{\hat{A}} = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$$

$a_{11} = 4$ $b = (2)$

$\hat{A} = 2$ $r_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$

$s^T = r_{11}^{-1} b^T = \frac{1}{2}(2) = 1$

$\bar{A} = \hat{A} - s^T s = 2 - (1)(1) = 1$

$$R = \begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

That is.

$\hat{\hat{A}}$ $\begin{bmatrix} 2 & 1 \end{bmatrix}$

$\bar{A} = \hat{A} - s^T s = 2 - (1)(1) = 1$

$\hat{R} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

That is,

$$R = \begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Bordered Form

$A = R^T R$

$$\begin{bmatrix} A_{j-1} & c \\ c^T & a_{jj} \end{bmatrix} = \begin{bmatrix} R_{j-1}^T & 0 \\ h^T & r_{jj} \end{bmatrix} \begin{bmatrix} R_{j-1} & h \\ 0 & r_{jj} \end{bmatrix}$$

$\rightarrow \cdot\ c = R_{j-1}^T h$ (using forward substitution)

$\Rightarrow \cdot\ r_{jj} = \sqrt{a_{jj} - h^T h}$

$$A = \begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix}$$

$R^T \ R$

$4 = [2][2]$

$R_1 = 2$

We know $a_{11} = 4$ $\quad r_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$

$R_{j-1}^T = 2$

Considerd A_2

$\begin{bmatrix} 4 & -2 \\ -2 & 10 \end{bmatrix}$

$-2 = 2h$

$-1 = h$

$c = -2 \quad a_{22} = 10$

$r_{22} = \sqrt{a_{22} - (-1)(-1)}$

$= \sqrt{10 - 1} = \sqrt{9} = 3$

$R_2 = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$

Now consider ~~A_2~~ A_3

$$A_3 = \begin{bmatrix} 4 & -2 & 4 \\ -2 & 10 & -2 \\ 4 & -2 & 8 \end{bmatrix}$$

c, c^T, a_{33}

$c = \begin{pmatrix} 4 \\ -2 \end{pmatrix} \quad a_{33} = 8$

$c = R_2^T h$

$\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$

$2h_1 = 4$

$h_1 = 2$

$-h_1 + 3h_2 = -2$

$-2 + 3h_2 = -2$

$3h_2 = 0$

$h_2 = 0$

$h = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$r_{33} = \sqrt{a_{33} - h^T h}$

$= \sqrt{8 - [2\ 0]\begin{bmatrix} 2 \\ 0 \end{bmatrix}} = \sqrt{8 - 4} = \sqrt{4} = 2$

$$R_3 = \begin{bmatrix} R_2 & h \\ 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Now consider A_4

$$A_4 = A = \begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix}$$

A_{j-1}, c, a_{44}

$R \quad -1 \quad 2 \,]$

$$A_4 = A = \begin{bmatrix} 4 & -2 & 4 & 2 \\ -2 & 10 & -2 & -7 \\ 4 & -2 & 8 & 4 \\ 2 & -7 & 4 & 7 \end{bmatrix} \quad a_{44}$$

$c = R_{j-1}^T h$

$$\begin{bmatrix} 2 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

$$R_{j-1} = R_3 = \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$2h_1 = 2$, $h_1 = 1$

$-1 + 3h_2 = -7$, $3h_2 = -6$, $h_2 = -2$

$2 + 0 + 2h_3 = 4$, $2h_3 = 2$, $h_3 = 1$

$$h = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$r_{44} = \sqrt{a_{44} - h^T h} = \sqrt{7 - \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}} = \sqrt{7 - (1+4+1)} = \sqrt{7-6} = \sqrt{1} = 1$$

$$R = \begin{bmatrix} R_{j-1} & h \\ 0 & r_{jj} \end{bmatrix} = \begin{bmatrix} 2 & -1 & 2 & 1 \\ 0 & 3 & 0 & -2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Gaussian Elimination with pivoting

Partial Pivot – means only the rows are interchanged

Complete Pivot – both rows and columns can be interchanged.

Note: Solving the system $Ax = b$ is the same as solving the system $\hat{A}x = \hat{b}$ obtained by interchanging equations.

Example We will solve the system

$$\overset{A}{\begin{bmatrix} \boxed{0} & 4 & 1 \\ 1 & 1 & 3 \\ 2 & -2 & 1 \end{bmatrix}} \overset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 9 \\ 6 \\ -1 \end{bmatrix}}$$

by Gaussian elimination with partial pivoting.

exchange rows 1 and 3

$$\begin{bmatrix} 2 & -2 & 1 \\ \boxed{1} & 1 & 3 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 5/2 \\ 0 & 4 & 1 \end{bmatrix}$$

$R_2 - \frac{1}{2}R_1$

$1 - \frac{1}{2}(-2)$

or

$$\begin{bmatrix} 2 & -2 & 1 \\ \frac{1}{2} & 2 & 5/2 \\ 0 & 4 & 1 \end{bmatrix}$$

swap Rows 2 and 3 →

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ \frac{1}{2} & 2 & 5/2 \end{bmatrix} \xrightarrow{R_3 - \frac{1}{2}R_2} \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ \frac{1}{2} & \frac{1}{2} & 2 \end{bmatrix}$$

Then
$$\hat{A} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{array}{l} \leftarrow \text{Row 3 of } A \\ \leftarrow \text{Row 1 } A \\ \leftarrow \text{Row 2 of } A \end{array}$$

We found the LU-decomposition of $\hat{A}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \ldots \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ \ldots \end{bmatrix}$$

[illegible]

we found the LU-decomposition of A

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & \frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

we also need to transform b to $\hat{b}$

$b = \begin{pmatrix} 9 \\ 6 \\ -1 \end{pmatrix}$ then $\hat{b} = \begin{pmatrix} -1 \\ 9 \\ 6 \end{pmatrix}$

$LUx = b$

$Ux = y$

$Ly = b$

$Ly = \hat{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 6 \end{bmatrix}$$

$y_1 = -1$

$y_2 = 9$

$\frac{1}{2}(-1) + \frac{1}{2}(9) + y_3 = 6$

$-\frac{1}{2} + \frac{9}{2} + y_3 = 6$

$4 + y_3 = 6$

$y_3 = 2$

$$y = \begin{pmatrix} -1 \\ 9 \\ 2 \end{pmatrix}$$

$Ux = y$

$$\begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 2 \end{bmatrix}$$

$2x_3 = 2$

$x_3 = 1$

$4x_2 + x_3 = 9$

$x_2 = 2$

$2x_1 - 2x_2 + x_3 = -1$

$2x_1 - 2(2) + 1 = -1$

$2x_1 = 2$

$x_1 = 1$

$$x = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

<u>Exercise</u> Let $A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 10 \\ -2 \\ 5 \end{bmatrix}$

Use Gaussian elimination (by hand) to find matrices L and U such that U is upper triangular L is unit lower triangular with $\boxed{|l_{ij}| \le 1}$ for all $i > j$ and $LU = \hat{A}$, where $\hat{A}$ is obtained by making row interchanges. Use your decomposition to solve $Ax = b$.