February 20th

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Practice Exam 1 should be available this week.

- Written Assignment 2 is due Thursday. It is graded on effort.

Cholesky Decomposition

Inner-product formulation.

$$r_{ii} = +\sqrt{a_{ii} - \sum_{k=1}^{i-1} r_{ki}^2}$$

$$r_{ij} = \left(a_{ij} - \sum_{k=1}^{i-1} r_{ki} r_{kj}\right) / r_{ii} \quad j = i+1, \dots, n$$

Flop Count (Inner-product formulation.

Cholesky's Algorithm (inner product form)

for $i = 1, \dots n$
- for $k = 1, \dots i-1$ (not executed when $i=1$)
  - $a_{ii} \leftarrow a_{ii} - a_{ki}^2$
- if $a_{ii} \le 0$ (A is not positive definite) end program.
- else
  - $a_{ii} \leftarrow \sqrt{a_{ii}}$ (This is r_{ii})
- for $j = i+1, \dots, n$ (do not executed if $i=n$)
  - for $k = 1, \dots, i-1$ (do not execute if $i=1$)
    - $a_{ij} \leftarrow a_{ij} - a_{ki} a_{kj}$
  - $a_{ij} \leftarrow a_{ij} / a_{ii}$ (This is r_{ij})

Proposition Cholesky's algorithm (above) applied to an $n \times n$ matrix performs about $n^3/3$ flops.

Since the flop count is $O(n^3)$ when we double the size of the matrix, the Cholesky factor

Since the flop count is $O(n^3)$ when we double the size of the matrix the Cholesky factor will be multiplied by 8.

Outer - Product Form

where A is a symmetric positive definite matrix $A = R^T R$ in the form

$$\begin{bmatrix} a_{11} & b^T \\ b & \hat{A} \end{bmatrix} = \begin{bmatrix} r_{11} & 0 \\ s & \hat{R}^T \end{bmatrix} \begin{bmatrix} r_{11} & s^T \\ 0 & \hat{R} \end{bmatrix}$$

Using "Block" Multiplication of RHS

$$\begin{bmatrix} r_{11}^2 & r_{11} s^T \\ s r_{11} & s s^T + \hat{R}^T \hat{R} \end{bmatrix}$$

Then $a_{11} = r_{11}^2$

- $r_{11} = \sqrt{a_{11}}$
- $b^T = r_{11} s^T$
- $\hat{A} = s s^T + \hat{R}^T \hat{R}$

we know what s we know $\hat{A}$

$r_{11}^{-1} b^T = s^T$

(1) $r_{11} = \sqrt{a_{11}}$

(2) $s^T = r_{11}^{-1} b^T$ ϵ

(3) $\tilde{A}$ has a decomposition of the form $\hat{R}^T \hat{R}$

Let $\tilde{A} = \hat{A} - s s^T$

This is the outer product formulation

Bordered Form of Cholesky's Method

$A_j \equiv$ it is a $j \times j$ submatrix of A. consisting of the intersection of the first j rows and j columns.

A_j is called the j^{th} leading principal submatrix of A.

It can be show that if A is positive definite then A_j is also positive definite

Let R be Cholesky factor of A, Then R has leading principal submatrices R_j, $j = 1, \ldots n$ that are upper triangular and have positive

has leading principal submatrices R_j, $j=1, \ldots n$
that are upper triangular and have positive
entries on the main diagonal.

We know $R_1 = [r_{11}]$ since $a_{11} = r_{11}^2$ we should
be able to figure out R_j
and eventually arrive at $R_n = R$.

$$A_j = R_j^T R_j$$

Solving for A_j principal submatrix

$$\begin{bmatrix} A_{j-1} & c \\ c^T & a_{jj} \end{bmatrix} = \begin{bmatrix} R_{j-1}^T & 0 \\ h^T & r_{jj} \end{bmatrix} \begin{bmatrix} R_{j-1} & h \\ 0 & r_{jj} \end{bmatrix}.$$

Multiplication leads us to.

$$A_{j-1} = R_{j-1}^T R_{j-1} \qquad c = R_{j-1}^T h \qquad a_{jj} = h^T h + r_{jj}^2$$

We assume we have already found R_{j-1}.

R_{j-1}^T is a lower diagonal matrix

we can sove the system $C = R_{j-1}^T h$

Using forward substitution. to get h.

$$r_{jj}^2 = a_{jj} - h^T h$$

$$r_{jj} = \sqrt{a_{jj} - h^T h}$$

An algorithm built using the method is known
as the bordered form of Cholesky' Method

LU decomposition variants

Theorem (LDV decomposition) Let A be an $n \times n$ matrix.
whose leading principal submatrices are all non-singular.
Then A can be decomposed in exactly one way
as a product

whose leading principal submatrices are all nonsingular. Then A can be decomposed in exactly one way as a product

$$A = L D V$$

such that L is unit lower triangular, D is diagonal, V unit upper triangular.

Theorem Let A be a symmetric matrix whose leading principal submatrices are nonsingular. Then A can be expressed exactly one way as a product

$A = L D L^T$ such that L is unit lower triangular and D is diagonal.

Proof We know from the previous Theorem that $A = L D V$

$(A A^T)^T = A^T A$

If A is symmetric $A^T A = A^2$

we need only show that $V = L^T$.

Since A is symmetric

$$A = A^T = (L D V)^T$$

$$= V^T D^T L^T \qquad D^T = D$$

$$= V^T D L^T$$

$L D V = V^T D L^T$ or $V = L^T$

because our decomposition is unique.

Theorem Let A be positive definite. Then A can be expressed in exactly one way as product

$A = L D L^T$, such that

L is unit lower triangular and D is a diagonal matrix whose main-diagonal entries are positive.

Theorem Let A be positive definite. Then A can be expressed in exactly one way as a product

$A = M D^{-1} M^T$, such that M is lower triangular

expressed in exactly one way as a product
$A = M D^{-1} M^T$, such that M is lower triangular
D is a diagonal matrix whose main diagonal entries are positive and the main diagonal entries of M are the same as those of D

Proof: we know $A = (L D) L^T$

Let $M = L D$ then $A = M D^{-1} M^T$

Can you show this decomposition is unique?

$$M^T = D^T L^T$$

$$M^T = D L^T$$

$$D^{-1} M^T = L^T$$