$$A = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ -1 & -2 & 1 \end{pmatrix} \qquad x = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\cdot -1 + (-2)\cdot 2 + -3\cdot 0 \\ -1\cdot 0 + 2\cdot 1 + 2\cdot 0 \\ -1\cdot -1 + -2\cdot 2 + 1\cdot 0 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ -3 \end{pmatrix}$$

$$x^T x = (-1 \ 2 \ 0) \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = (-1\cdot -1 + 2\cdot 2 + 0\cdot 0) = (5)$$

$$x x^T = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} (-1 \ 2 \ 0) = \begin{pmatrix} -1\cdot -1 & -1\cdot 2 & -1\cdot 0 \\ 2\cdot -1 & 2\cdot 2 & 2\cdot 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x^T A x$:

$$(-1 \ 2 \ 0) \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$(-1\cdot 1 + 0 + 0, \ -2\cdot -1 + 2\cdot 1 + 0, \ -1 - 3, \ 2\cdot 2 + 0) = (-1, 4, 7)$$

$$(-1, 4, 7) \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = (-1 - 1 + 4\cdot 2 + 7\cdot 0) = (9)$$

$$A = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ -1 & -2 & 1 \end{pmatrix} \quad B = \begin{pmatrix} -1 & 0 & 4 \\ -2 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 4 \\ -2 & 1 & 2 \\ 0 & -2 & 1 \end{pmatrix} =$$

2

$$\begin{pmatrix} 1\cdot -1 + -2\cdot -2 + -3\cdot 0 & 1\cdot 0 + -2\cdot 1 + (-3)(-2) & 1\cdot 4 + -2\cdot 2 + (-3)\cdot 1 \\ 0\cdot -1 + 1\cdot -2 + 2\cdot 0 & 0\cdot 0 + 1\cdot 1 + 2(-2) & 4\cdot 1 + 2\cdot -2 + 1\cdot 1 \\ -1\cdot -1 + (-2\cdot -2) + 1\cdot 0 & -1\cdot 0 + -2\cdot 1 + -2\cdot 1 & -1\cdot 4 + (-2)(2) + 1\cdot 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 4 & -3 \\ -2 & -3 & -5 \\ 6 & -4 & -7 \end{pmatrix}$$

b, c, d → similar process, all double

3) MULTIPLY $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}$ Effeciently

$$= \begin{pmatrix} A \cdot C + 0 \cdot 0 & A0 + 0D \\ 0 \cdot C + B \cdot 0 & 0 \cdot 0 + B \cdot D \end{pmatrix} = \begin{pmatrix} A \cdot C & 0 \\ 0 & B \cdot D \end{pmatrix}$$

Partition the matrix, multiply $A \cdot C$, which will be the first 3x3 quadrant, similar with $B \cdot D$ in lower quadrant

6) $\begin{pmatrix} A^T & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} C^T & 0 \\ 0 & D \end{pmatrix}$ → Take answer above, transposing the partition at the first row first col

c, d → not sure of this question
→ generally more involved

4) Flop count → take a number $\boxed{\text{multiply}}$ by another number $\boxed{\text{divide}}$ by 3? Flop count = 2

5) $O(n^3)$ → increase from (100 x 100) to a (300 x 300)

$$\left(\frac{300}{100}\right)^3 = 3^3 = 27$$ → time will increase by approximately 27x

7) Given $x = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 4 \end{pmatrix}$

$||x||_1 = ||1|| + ||-1|| + ||0|| + ||4|| = 6$

$||x||_2 = \sqrt{1^2 + (-1)^2 + 0^2 + 4^2} = \sqrt{18}$

$||x||_\infty = \max(|1|, |-1|, |0|, |4|) = 4$

8) Given $\cancel{x}A = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$

$||A||_1 = |-1| + |2| = 3$

$|1| + |3| = 4$ $\quad \max(3, 4) = 4$ (max col sum)

$||A||_2 = \sqrt{-1^2 + 1^2 + 2^2 + 3^2} = \sqrt{1 + 1 + 4 + 9} = \sqrt{15}$

$||A||_\infty = |-1| + |1| = 2$

$|2| + |3| = 5$ $\quad \max(2, 5) = 5$ (max row sum)

~~9) $\kappa_p(A)$ is condition number using p-norm, $A^{-1} = \Big( \frac{-3}{5}$~~

~~$\kappa_1(A) =$~~ new page

9) $A = \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$, $A^{-1} \begin{pmatrix} -\frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} \end{pmatrix}$

$\kappa_1(A) =$

$\|A^{-1}\|_1 = \left|\frac{-3}{5}\right| + \left|\frac{2}{5}\right| = \frac{5}{5}$ $\quad \max(1, \frac{2}{5}) = 1$

$\left|\frac{1}{5}\right| + \left|\frac{1}{5}\right| = \frac{2}{5}$

$\|A\|_1 \cdot \|A^{-1}\|_1 = 4 \cdot 1 = 4$

$\kappa_2(A)$ :

$\|A^{-1}\|_2 = \sqrt{\frac{-3}{5}^2 + \frac{1}{5}^2 + \frac{2}{5}^2 + \frac{1}{5}^2} = \sqrt{\frac{9}{25} + \frac{1}{25} + \frac{4}{25} + \frac{1}{25}}$

$= \sqrt{\frac{15}{25}} = \sqrt{\frac{3}{5}}$

$\|A\|_2 \cdot \|A^{-1}\|_2 = \sqrt{\frac{3}{5}} \cdot \sqrt{15} = 3$

$\kappa_\infty(A)$:

$\|A^{-1}\|_\infty = \left|\frac{-3}{5}\right| + \left|\frac{1}{5}\right| = \frac{4}{5}$ $\quad \max(\frac{3}{5}, \frac{4}{5}) = \frac{4}{5}$

$\left|\frac{2}{5}\right| + \left|\frac{1}{5}\right| = \frac{3}{5}$

$\|A\|_\infty \cdot \|A^{-1}\|_\infty = 5 \cdot \frac{4}{5} = 4$

10) $Ax = b \rightarrow$ Gaussian elimination

$A = \begin{pmatrix} 2 & 1 \\ 4 & 0 \end{pmatrix}$ $\quad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 0 & -1 \end{array}\right]$ $\quad x = \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{2} \end{pmatrix}$

$R_2 = \frac{1}{4} R_2$ $\quad R_1 = R_1 - 2R_2$

$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 0 & -\frac{1}{4} \end{array}\right]$ $\quad \left[\begin{array}{cc|c} 0 & 1 & \frac{3}{2} \\ 1 & 0 & -\frac{1}{4} \end{array}\right]$

Similar process for Part b + c

11a) What are the conditions that make LU-decomposition unique?

- principle leading minors of the matrix are all non zero
  - ↳ so, no pivoting will be required
- non singular
  - ↳ det $\neq 0$ (but if it were, point above would not be true)
- normalization
  - ↳ either L or U must be normalized
    - ↳ otherwise infinite options

b) ~~[illegible]~~ $Ly = b$ ← solved using backwards substitution

$Ux = y$ ← solved using forward substitution

c) What must be true about A to guarentee this kind of composition?

↳ must be a positive definite matrix, first two points from part A.

Solve $Ax = 6$ using $LU$-decomposition

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \quad 6 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 0 \\ \boxed{-\frac{1}{2}} & 1 \end{pmatrix}$$

$$R_2 = R_2 - \left(\boxed{\tfrac{-1}{2}}\right) R_1$$

$$= \begin{pmatrix} 2 & 1 \\ 0 & \frac{9}{2} \end{pmatrix}$$

$$2y = B$$

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$y_1 = 1$ $\quad y = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$

$\frac{-1}{2}(1) + y_2 = -1$

$y_2 = \frac{-1}{2}$

$$Ux = y$$

$$\begin{pmatrix} 2 & 1 \\ 0 & \frac{9}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$$

$\frac{9}{2} x_2 = \frac{-1}{2}$ $\qquad 2x_1 + \left(\frac{-1}{9}\right)(1) = 1$

$x_2 = \frac{-1}{9}$ $\qquad x_1 = \frac{5}{9}$

$$\boxed{x = \begin{pmatrix} \frac{-1}{9} \\ \frac{5}{9} \end{pmatrix}}$$

Solve $Ax = b$ using LU-decomposition 126

$$A = \begin{pmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$R_2 = R_2 - \left(\boxed{\tfrac{1}{3}}\right) R_1$$

$$R_3 = R_3 - \left(\boxed{\tfrac{2}{3}}\right) R_1$$

$$= \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 3 & 1 \end{pmatrix}$$

$$R_3 = R_3 - \left(\boxed{\tfrac{1}{2}}\right) R_2$$

$$= \begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{\tfrac{1}{3}} & 1 & 0 \\ \boxed{\tfrac{2}{3}} & \boxed{\tfrac{1}{2}} & 1 \end{pmatrix}$$

$$Ly = B$$

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$y_1 = 1$$

$$\frac{1}{3} + y_2 = -1 \rightarrow y_2 = -\frac{4}{3}$$

$$\frac{2}{3}(1) + \left(\frac{1}{2}\right)\left(\frac{-4}{3}\right) + y_3 = 2$$

$$y = \begin{pmatrix} 1 \\ -\frac{4}{3} \\ 2 \end{pmatrix}$$

$$Ux = y$$

$$\begin{pmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{4}{3} \\ 2 \end{pmatrix}$$

$$x_3 = 2$$

$$6x_2 = -\frac{4}{3} \rightarrow x_2 = \frac{-2}{9}$$

$$6x_1 + 18\left(\frac{-2}{9}\right) + 3(2) = 1$$

$$x_1 = \frac{-1}{6}$$

$$\boxed{x = \begin{pmatrix} -\frac{1}{6} \\ -\frac{2}{9} \\ 2 \end{pmatrix}}$$

Solve: $Ax = b$ using LU-decomposition

$$A = \begin{pmatrix} -1 & 1 & 2 \\ -2 & 4 & 7 \\ -3 & 11 & 23 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$R_2 = R_2 \boxed{-(+2)} R_1$

$R_3 = R_3 \boxed{-(+3)} R_1$

$$= \begin{pmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 8 & 17 \end{pmatrix}$$

$R_3 = R_3 - \boxed{(4)} R_2$

$$= \begin{pmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ \boxed{3} & \boxed{4} & 1 \end{pmatrix}$$

$Ly = B$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

$\boxed{y_1 = 2}$

$2y_1 + y_2 = -1 \rightarrow 2(2) + y_2 = -1 \rightarrow \boxed{y_2 = -5}$

$3(2) + 4(-5) + y_3 = 2$

$\boxed{y_3 = 16}$

$Ux = y$

$$\begin{pmatrix} -1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 16 \end{pmatrix}$$

$5x_3 = 16$

$x_3 = \frac{16}{5}$

$2x_2 + 3\left(\frac{16}{5}\right) = -5$

$x_2 = -\frac{73}{10}$

$x_1 = -\frac{29}{10}$

$$\boxed{x = \begin{pmatrix} -\frac{29}{10} \\ -\frac{73}{10} \\ \frac{16}{5} \end{pmatrix}}$$

# Inner Product



Along diagonal:

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

Below diagonal:

$$l_{ik} = \frac{1}{l_{kk}} \left(a_{ik} - \sum_{j=1}^{k-1} l_{ij} l_{kj}\right.$$

---

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 6 \\ 2 & 6 & 45 \end{pmatrix} \qquad U = \begin{pmatrix} \boxed{1} & \boxed{-1} & \boxed{2} \\ 0 & \boxed{2} & \boxed{4} \\ 0 & 0 & \boxed{5} \end{pmatrix}$$

---

$$R_{00} = \sqrt{1} = \boxed{1}$$

$$R_{01} = \frac{a_{01}}{r_{00}} = \frac{-1}{1} = \boxed{-1}$$

$$R_{02} = \frac{a_{02}}{r_{00}} = \frac{2}{1} = \boxed{2}$$

$$R_{11} = \sqrt{a_{11} - a_{10}^2} = \sqrt{5 - (-1)^2} = \sqrt{4} = \boxed{2}$$

$$R_{12} = \frac{1}{r_{11}} \left( A_{12} - R_{01} \cdot R_{02} \right) = \frac{1}{2} \left( 6 - (-1 \cdot 2) \right) = \frac{1}{2} 8 = \boxed{4}$$

$$R_{22} = \sqrt{a_{22} - R_{02}^2 - R_{12}^2} = \sqrt{45 - 2^2 - 4^2} = \sqrt{25} = \boxed{5}$$

# Outer Product:

$$A = R^T R$$

$$\begin{bmatrix} a_{11} & b^T \\ b & \hat{A} \end{bmatrix} = \begin{bmatrix} r_{11} & 0 \\ s & \hat{R}^T \end{bmatrix} \begin{bmatrix} r_{11} & s^T \\ 0 & \hat{R} \end{bmatrix}$$

---

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 6 \\ 2 & 6 & 45 \end{pmatrix} \qquad v = \begin{pmatrix} \boxed{1} & \boxed{-1 \quad 2} \\ 0 & \boxed{2} \quad \boxed{4} \\ 0 & 0 \quad \boxed{5} \end{pmatrix}$$

---

$a_{11} = 1$, $\sqrt{1} = 1$ $r_{11} = \boxed{1}$

$$b = \begin{pmatrix} \boxed{\begin{matrix} -1 \\ 2 \end{matrix}} \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

$$\tilde{A} = \hat{A} - ss^T = \begin{bmatrix} 5 & 6 \\ 6 & 45 \end{bmatrix} - \boxed{\begin{pmatrix} -1 \\ 2 \end{pmatrix}} (-1 \quad 2)$$

$$= \begin{bmatrix} 5 & 6 \\ 6 & 45 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \boxed{4} & 8 \\ 8 & 41 \end{bmatrix}$$

$\cancel{a_{22}}$ new $A_{11} = 4$, $r_{11} = \sqrt{4} = \boxed{2}$

$b = (8)$

$$S_r = r_{11}^{-1} b^T = \frac{1}{2} \cdot 8 = \boxed{4} = s_T$$

$$\hat{A} - s^T s = 41 - 4 \cdot 4 = 25$$

$\cancel{a_{33}}$ new $A_{11} = 25$, $r_{11} = \sqrt{25} = \boxed{5}$

Boardered Form:

$A = R^T R$

$$\begin{bmatrix} A_{j-1} & c \\ c_T & a_{jj} \end{bmatrix} = \begin{bmatrix} R_{j-1}^T & 0 \\ h^T & r_{jj} \end{bmatrix} \begin{bmatrix} R_{j-1} & h \\ 0 & r_{jj} \end{bmatrix}$$

- $c = r_{j-1}^T h$

$r_{jj} = \sqrt{a_{jj} - h^T h}$

---

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 6 \\ 2 & 6 & 45 \end{pmatrix} \qquad r = \begin{pmatrix} \boxed{1} & \boxed{-1} & \boxed{2} \\ 0 & \boxed{2} & \boxed{4} \\ 0 & 0 & \boxed{5} \end{pmatrix}$$

---

$A_1$: $a_{11} = 1$, $r_{11} = \sqrt{1} = \boxed{1}$

$A_2$ $\begin{bmatrix} 1 & \boxed{-1} \\ -1 & 5 \end{bmatrix}$ (c)

$r_{j-1}^T = 1$ $\quad \boxed{-1} = 1h$

$-1 = h$

$r_{22} = \sqrt{a_{22} - h^T h} = \sqrt{5 - (-1)\cdot(-1)} = \sqrt{4} = 2$

$$R_2 = \begin{bmatrix} 1 & \boxed{-1} \\ 0 & \boxed{2} \end{bmatrix}$$

$A_2$ $A_3$: $\begin{bmatrix} 1 & -1 & 2 \\ -1 & 5 & 6 \\ 2 & 6 & 45 \end{bmatrix}$ c, $c_T$, $a_{33}$

$C = R_2^T h$:

$$\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

$1h_1 = 2$

$h_1 = 2$

$-1(2) + 2(h_2) = 6$

$h_2 = 4$

$$h = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$r_3 = \sqrt{45 - (2\ 4)\begin{pmatrix} \cancel{42} \\ 24 \end{pmatrix}} = \sqrt{45 - (4+16)}$

$= \sqrt{45-20} = \sqrt{25} = \boxed{5}$