Feb 27th

Tuesday, February 27, 2024 2:01 PM

There is no quiz this week.

Exercise Let $A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix}$ and $b = \begin{bmatrix} 10 \\ -2 \\ -5 \end{bmatrix}$

Use Gaussian elimination with partial pivoting find L and U, where U is upper triangular and L is unit lower triangular with $|l_{ij}| \le 1$ for all $i > j$ and $LU = \hat{A}$, where $\hat{A}$ can be obtained by making row interchanges, Use the LU decomposition to solve the system $Ax = b$.

$L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix}$

max

$\begin{bmatrix} \boxed{2} & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix}$

U $\qquad L$

Row2 $- \frac{1}{2}$ row1

$= \begin{bmatrix} 2 & 2 & -4 \\ 0 & 0 & 3 \\ 1 & 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Row3 $- \frac{1}{2}$ Row1

$= \begin{bmatrix} 2 & 2 & -4 \\ 0 & \boxed{0} & 3 \\ 0 & 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$

Row 2 $\leftrightarrow$ Row 3

$= \begin{bmatrix} 2 & 2 & -4 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$

$\hat{A} = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 3 & 6 \\ 1 & 1 & 5 \end{bmatrix}$ (matrix A with rows 2 and 3 interchanged)

$\hat{b} = \begin{bmatrix} 10 \\ -5 \\ -2 \end{bmatrix}$

$\hat{A}x = \hat{b}$

$\hat{A} = LU = \begin{bmatrix} 2 & 2 & -4 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$

Let $Ux = y$

then $Ly = \hat{b}$

$Ux = y$

$\begin{bmatrix} 2 & 2 & -4 \\ 0 & 2 & 4 \\ \ldots \end{bmatrix} x = \begin{bmatrix} 10 \\ -10 \\ -7 \end{bmatrix}$

$$\begin{bmatrix} 2 & 2 & -4 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix} x = \begin{bmatrix} 10 \\ -10 \\ -7 \end{bmatrix}$$

$3x_3 = -7$

$x_3 = -\frac{7}{3}$

$2x_2 + 4x_3 = -10$

$2x_2 = -10 + \frac{28}{3}$

$2x_2 = -\frac{2}{3}$

$x_2 = -\frac{1}{3}$

$2x_1 + 2x_2 - 4x_3 = 10$

$2x_1 + 2(-\frac{1}{3}) - 4(-\frac{7}{3}) = 10$

$2x_1 - \frac{2}{3} + \frac{28}{3} = 10$

$2x_1 = \frac{30}{3} - \frac{26}{3}$

$2x_1 = \frac{4}{3}$

$x_1 = \frac{2}{3}$

then $Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix} y = \begin{bmatrix} 10 \\ -5 \\ -2 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$y_1 = 10$

$\frac{1}{2}y_1 + y_2 = -5$

$5 + y_2 = -5$

$y_2 = -10$

$\frac{1}{2}y_1 + y_3 = -2$

$5 + y_3 = -2$

$y_3 = -7$

$$y = \begin{bmatrix} 10 \\ -10 \\ -7 \end{bmatrix}$$

$$x = \begin{bmatrix} 2/3 \\ -1/3 \\ -7/3 \end{bmatrix}$$

Cholesky Decomposition - works if A is symmetric and positive definite

Gaussian Elimination (without pivots) - works to find solution under certain conditions

Gaussian Elimination (with pivots) — works well to find solutions to $Ax=b$ If we do not have an ill-conditioned system.

The most extreme case of an ill-conditioned system is where A is singular,

- $\det(A) = 0$

$Ax=b$ does not have one unique solution,

in A is a singular matrix in the process of unique solution,

If A is a singular matrix in the process of Gaussian elimination there will be a step where all possible pivots are zero.

But with each step of the algorithm the entries of the matrix are subject to round off. And we may arrive at a "solution". But it is not in fact a solution.

The additional cost of row interchanges is not great we make $n^2/2$ comparisons is relatively small compared $\Theta(n^3)$ of the algorithm.

Roughly the cost of Gaussian elimination with partial pivoting is $\frac{2}{3}n^3$ flops.

Back to example from last Thursday

$$A = \begin{bmatrix} 0 & 4 & 1 \\ 1 & 1 & 3 \\ 2 & -2 & 1 \end{bmatrix} \quad \text{and we found} \quad \hat{A} = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

<u>Goal</u> Find P such that $\hat{A} = PA$

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 & 1 \\ 1 & 1 & 3 \\ \boxed{2} & -2 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{2} & \boxed{-2} & \boxed{1} \\ 0 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

<u>Definition</u> A <u>permutation matrix</u> is a matrix that has exactly one 1 in each row and in each column, all other entries are zero.

<u>Exercise</u> Show that if P is a permutation matrix then $P^T P = P P^T = I$, Thus P is nonsingular

Exercise

then $P^T P = P P^T = I$, Thus P is nonsingular
and $P^{-1} = P^T$.

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad P^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

32

$$PP^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let P be a permutation matrix that is $n \times n$
Let $Q = P^T$

$PQ_{ij} = (\text{row } i \text{ of } P) \times (\text{column } j \text{ of } Q)$

This is nonzero if and only if
Since 1 one in each row P and 1 one in each column of Q.
Since $Q = P^T$ it would be nonzero if $i = j$.

$$PQ_{ij} = \sum_{k=1}^{n} P_{ik} Q_{kj}$$

Let's assume $i \neq j$ and $PQ_{ij} = 1$
then there exists a k $1 \leq k \leq n$
Such that $P_{ik} = Q_{kj} = 1$

Since $Q_{kj} = 1$ then $P_{jk} = 1$ since $Q = P^T$

But we said $P_{ik} = 1$ and $i \neq j$.
So P has two 1's in column k. This contradicts P being a permutation matrix.
And $i = j$ must be the for PQ_{ij} to be nonzero.

Theorem Gaussian elimination with partial pivoting ... matrix A produces

Theorem Gaussian elimination with partial pivoting applied to an $n \times n$ matrix A produces a unit lower triangular matrix L such that $|l_{ij}| \leq 1$, an upper triangular matrix U, and a permutation matrix P such that

$$A = P^T L U$$

Exercise $A = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 1 & 5 \\ 1 & 3 & 6 \end{bmatrix}$

Find P, L, and U such that $A = P^T L U$

we already know $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$

$LU = \hat{A} = \begin{bmatrix} 2 & 2 & -4 \\ 1 & 3 & 6 \\ 1 & 1 & 5 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 2 & -4 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ 2,3 3,2

Thus $P^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$A = P^T L U$ where $P^T = P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}$

and $U = \begin{bmatrix} 2 & 2 & -4 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

Computing A^{-1}

We know $A^{-1} A = I$

$$Ix = x$$

$$(A^{-1}A)x = x$$

$$AX = I$$ where X is the columns of some matrix X

$$A[x_1\ x_2\ x_3\ x_4 \ldots x_n] = [e_1 \ldots e_n]$$ and e gives the columns of I.

$$Ax_i = e_i \quad i = 1, \ldots n$$

If we solve those n-systems we obtain A^{-1}.

Complete Pivoting

Here both row and column interchanges are allowed,

We start by finding the largest $|a_{ij}|$ and move to a_{11}.

after zeroing out the first column we know focus on an $(n-1)\times(n-1)$ matrix and repeat the process

- Provides extra protection from roundoff error
- The disadvantage is seeking out the largest entry.

<u>step 1</u> You must compare n^2 entries

<u>step 2</u> You must compare $(n-1)^2$ entries.

$$\sum_{k=0}^{n-2}(n-k)^2 \approx \frac{1}{3}n^3. \text{ Comparisons,}$$

Since partial pivoting works fairly this is the more often used algorithm,

Error Discussion in Gaussian Elimination

Error Discussion in Gaussian Elimination

- Propagation of roundoff error in Gaussian Elimination.

<u>Example</u> Consider the ill-conditioned matrix

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}$$

How would we get L and U?

$$\text{Row } 2 - \frac{999}{1000}\text{Row } 1$$

$$\begin{bmatrix} 1000 & 999 \\ 0 & -0.001 \end{bmatrix} \qquad 998 - \frac{999}{100}(999)$$

$$998 - 998.001$$
$$= -0.001$$

<u>Example</u> The ill-conditioned Hilbert matrices, defined by $h_{ij} = \frac{1}{(i+j-1)}$

For example

$$H_4 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}$$

See if you can find L and U.

$$U = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{12} & \frac{1}{12} & \frac{3}{40} \\ 0 & 0 & & \\ & & & \end{bmatrix}$$