February 8th

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Theorem (LU decomposition Theorem)
Let A be an $n \times n$ matrix whose leading principal submatrices are all nonsingular. Then A can be decomposed in exactly ~~one~~ way into a product $A = LU$ such that L is a unit lower triangular and U is upper triangular.

1. LU decomposition without pivoting
2. Cholesky decomposition
3. Write a code using cholesky decomposition
4. LU decomposition with pivoting
5. Write a code LU decomposition
6. Error propagation of these decomposition.

proof We have already it is possible to rewrite A as LU
we have not shown this is unique.

Let
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ l_{21} & 1 & 0 & \cdots & 0 \\ l_{31} & l_{32} & 1 & 0 & \cdots 0 \\ \vdots & & & & \\ l_{n1} & l_{n2} & l_{n3} & \cdots & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & & & & \\ \vdots & & & & \\ 0 & \cdots & & 0 & u_{nn} \end{bmatrix}$$

Let's say we want to calculate a_{1j} (in matrix A first row) by considering LU

$$\rightarrow a_{1j} = u_{1j} + 0u_{2j} + \cdots + 0u_{nj} \quad j = 1 \ldots n$$

Thus $\boxed{u_{1j} = a_{1j}}$ so the first row of u is uniquely determined by A

What about a_{i1}?

$$\boxed{a_{i1} = l_{i1} u_{11}} \quad i = 2, \ldots n$$

ith row of L
1st column of U

FACT (see if you can prove) If A is non singular

FACT (see if you can prove) If A is non singular then U is non singular.

Thus $u_{11} \neq 0$

$$\boxed{l_{i1} = \frac{a_{i1}}{u_{11}}}$$ is uniquely determined $i = 2, \ldots n$

Exercise Do the same for u_{2j} where $(j \geq 2)$ and second row of U (write in terms of a_{2j} and column of L)

Proof by Induction Suppose that the first $k-1$ rows of U and $k-1$ columns of L are uniquely determined.

Goal Show k^{th} row of U and k^{th} column of L are uniquely determined.

k^{th} row of L is $[l_{k1}\ l_{k2}\ l_{k3} \cdots l_{k,k-1}\ \boxed{1}\ 0 \cdots 0]$

Since the first $k-1$ columns of L are uniquely determined we already know what this row is

Multiply the k^{th} row of L by the j^{th} column of U $\boxed{(j \geq k)}$

$$a_{kj} = \underbrace{\sum_{m=1}^{k-1} l_{km} u_{mj}} + u_{kj} \qquad j = k, k+1 \ldots n$$

We already know these values are uniquely determined

Since u_{mj} comes from a row less than k

Then $$\boxed{u_{kj} = a_{kj} - \sum_{m=1}^{k-1} l_{km} u_{mj}}$$ which must be uniquely determined be uniquely determined

Let's multiply the ith row of L by the k^{th} column of U

$$a_{ik} = \sum_{m=1}^{k-1} l_{im} u_{mk} + l_{ik} u_{kk}$$

Columns are less than k

rows of U less than k

$$l_{ik} u_{kk} = a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mk}$$

$$l_{ik} = \frac{a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mk}}{u_{kk}} = u_{kk}^{-1}\left(a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mk}\right)$$

We have shown the k^{th} column of L and k^{th} row of U are uniquely for any $k = 1 \ldots n$

The algorithm established in this proof is the inner product formulation of Gaussian elimination. And is sometimes referred to as the Doolittle reduction.

<u>Example</u> Let $A = \begin{bmatrix} 2 & 4 & 2 & 3 \\ -2 & -5 & -3 & -2 \\ 4 & 7 & 6 & 8 \\ 6 & 10 & 1 & 12 \end{bmatrix}$

and $b = \begin{bmatrix} -3 \\ 3 \\ -1 \\ -16 \end{bmatrix}$ We will calculate L and U such that $A = LU$ by two different methods.

Gaussian Elimination by row operations (outer product formulation)

$m_{i1} = \frac{a_{i1}}{a_{11}}$

step 1 $\begin{bmatrix} 2 & 4 & 2 & 3 \end{bmatrix}$

step 1

$$\left[\begin{array}{c|cccc} 2 & 4 & 2 & 3 \\ \hline -1 & -1 & \boxed{-1} & 1 \\ 2 & -1 & \boxed{2} & 2 \\ 3 & -2 & \boxed{-5} & 3 \end{array}\right]$$

$$m_{i1} = \frac{a_{i1}}{a_{11}}$$

$$-\frac{2}{2} = -1$$

$$\text{Row } i - m_{i1}(\text{Row } 1)$$

$$4 - (-1)(-5)$$

step 2

$$\begin{bmatrix} 2 & 4 & 2 & 3 \\ -1 & -1 & -1 & 1 \\ 2 & 1 & 3 & 1 \\ 3 & 2 & -3 & 1 \end{bmatrix}$$

$$m_{i2} = \frac{a_{i2}}{a_{22}}$$

$$\text{Row } i - m_{i2}(\text{Row } 2)$$

$$2 - (1)(-1)$$

$$-5 - 2(-1)$$

step 3

$$\begin{bmatrix} 2 & 4 & 2 & 3 \\ -1 & -1 & -1 & 1 \\ 2 & 1 & 3 & 1 \\ 3 & 2 & -1 & 2 \end{bmatrix}$$

$$m_{i3} = \frac{a_{i3}}{a_{33}} = \frac{-3}{3}$$

$$\text{Row } i - m_{i3}(\text{Row } 3)$$

$$1 - (-1)(1)$$

we found $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & -1 & 1 \end{bmatrix}$

$$U = \begin{bmatrix} 2 & 4 & 2 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Inner Product Formulation

$$A = \begin{bmatrix} 2 & 4 & 2 & 3 \\ -2 & -5 & -3 & -2 \\ 4 & 7 & 6 & 8 \\ 6 & 10 & 1 & 12 \end{bmatrix}$$

- $U_{1j} = a_{1j}$
- $l_{i1} = \dfrac{a_{i1}}{U_{11}}$
- $U_{kj} = a_{kj} - \sum_{m=1}^{k-1} l_{km} U_{mj}$

Step 1

$$\left[\begin{array}{c|ccc} 2 & 4 & 2 & 3 \\ \hline -1 & -5 & -3 & -2 \\ 2 & 7 & 6 & 8 \\ 3 & 10 & 1 & 12 \end{array}\right]$$

Step 2

$$\left[\begin{array}{c|ccc} 2 & 4 & 2 & 3 \\ \hline -1 & -1 & -1 & 1 \\ 2 & & 6 & 8 \\ 3 & & 1 & 12 \end{array}\right]$$

- $u_{kj} = a_{kj} - \sum_{m=1}^{k-1} l_{km} u_{mj}$
- $l_{ik} = u_{kk}^{-1}\left(a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mk}\right)$

$$u_{22} = a_{22} - \sum_{m=1}^{1} l_{2m} u_{m2}$$
$$= a_{22} - l_{21} u_{12}$$
$$= -5 - (-1)(4)$$
$$= -5 + 4 = -1$$

$$u_{23} = a_{23} - l_{21} u_{13}$$
$$= -3 - (-1)(2)$$

$$u_{24} = a_{24} - l_{21} u_{14}$$
$$= (-2) - (-1)(3)$$