Vandermonde Matrix

Let $\{x_1, \ldots x_m\}$ be a sequence of numbers. If p and q are polynomials of degree $< n$ and α is a scalar

then $p+q$ is of degree $< n$.
αp is of degree $< n$

The values of the polynomials at x_i satisfy the linearity properties

$$(p+q)(x_i) = p(x_i) + q(x_i)$$
$$(\alpha p)(x_i) = \alpha(p(x_i))$$

Thus the map of vector coefficient p to $(p(x_1), p(x_2), \ldots p(x_m))$ is linear

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix}$$

$$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots + c_{n-1} x^{n-1}$$

$$x = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

$$Ax = \begin{bmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_m) \end{bmatrix}$$

Multiplying a matrix by a matrix

Let A be m by n matrix
X be n by p matrix

Let's say $B = AX$
Then B is an m by p matrix

$$\boxed{b_{ij} = \sum_{k=1}^{n} a_{ik} x_{kj}}$$

Thus b_{ij} is the dot product / inner product of the ith row of A and jth column of X,

psuedo-code for calculating B

$B \leftarrow 0$
for $i = 1, \ldots m$
for $j = 1, \ldots p$
for. $k = 1, \ldots n$

for $j = 1, \ldots$

for $k = 1, \ldots n$

$$\boxed{b_{ij} \leftarrow b_{ij} + a_{ik} x_{kj}}$$

If A is $m \times n$ X is $n \times p$ how flops are there when we calculate AX

$$2mnp$$

If A and X are n by n matrices

AX has how many flops?

$$2n^3$$

We have an $O(n^3)$

Block matrices

Consider $AX = B$

A is an m by n matrix

X is an n by p matrix

$$A = \begin{array}{cc} & \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \end{array}$$

$m = m_1 + m_2$

$n = n_1 + n_2$

We label the matrices in such a way that

A_{ij} has dimension m_i by n_j

$$X = \begin{array}{cc} & \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \end{array} & \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \end{array}$$

$n = n_1 + n_2$

$p = p_1 + p_2$

Then $$B = \begin{array}{cc} & \begin{array}{cc} p_1 & p_2 \end{array} \\ \begin{array}{c} m_1 \\ m_2 \end{array} & \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \end{array}$$

$$\begin{array}{c} \\ m_2 \end{array}\begin{bmatrix} & \\ B_{21} & B_{22} \end{bmatrix}$$

That is,

$$A\,X = B$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

We suspect $B_{11} = A_{11}X_{11} + A_{12}X_{21}$

Theorem Let A, X, B be as partitioned

Then $AX = B$ if and only if

$$A_{i1}X_{1j} + A_{i2}X_{2j} = B_{ij} \quad i,j = 1,2$$

See homework as an exercise to convince yourself this Theorem is true.

$$A = \begin{array}{c} m_1 \\ \vdots \\ m_r \end{array}\overset{\begin{array}{ccc} n_1 & \cdots & n_s \end{array}}{\begin{bmatrix} A_{11} & \cdots & A_{1s} \\ \vdots & & \vdots \\ A_{r1} & & A_{rs} \end{bmatrix}}$$

$m = m_1 + m_2 + \cdots + m_r$

$n = n_1 + n_2 + \cdots + n_s$

$$X = \begin{array}{c} n_1 \\ \vdots \\ n_s \end{array}\overset{\begin{array}{ccc} p_1 & \cdots & p_t \end{array}}{\begin{bmatrix} X_{11} & \cdots & X_{1t} \\ \vdots & & \vdots \\ X_{s1} & \cdots & X_{st} \end{bmatrix}}$$

$n = n_1 + n_2 + \cdots + n_s$

$p = p_1 + p_2 + \cdots p_t$

$$B = \begin{array}{c} m_1 \\ \vdots \\ m_r \end{array}\overset{\begin{array}{ccc} p_1 & \cdots & p_t \end{array}}{\begin{bmatrix} B_{11} & \cdots & B_{1t} \\ \vdots & \ddots & \vdots \\ B_{r1} & \cdots & B_{rt} \end{bmatrix}}$$

$m = m_1 + \cdots m_r$

$p = p_1 + \cdots p_t$

Theorem Let A, X, and B be as partitioned above

$B = AX$ if and only if

$$B_{ij} = \sum^{s} A_{i\cdot} X_{\cdot j} \quad i = 1, \cdots r$$

$B = AX$ if and only if

$$B_{ij} = \sum_{k=1}^{s} A_{ik} X_{kj} \qquad \begin{matrix} i = 1, \dots r \\ j = 1, \dots t \end{matrix}$$

Psuedo code

```
B ← 0
for i = 1 ... r
    for j = 1 ... t
        for k = 1 ... s
            B_ij ← B_ij + A_ik * X_kj
```

What does this do to our flop count?

It does not change!

It is still $2mnp$.

Varying the block size will not affect the total flop count

Let's A, X, and B are n by n matrices

the flop count would remain as n^3.

What changes is how objects are stored in the cache.

Fast Matrix Multiplication

"Standard" matrix multiplication requires $2n^3$ flops

In 1969 V. Strassen developed another method that has $\Theta(n^s)$ where $s = \log_2(7) \approx 2.81$

The current record holder $\Theta(n^{2.376})$

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however there is a catch $Cn^{2.376}$ flops

C is very large.

Perturbation and Condition number

Perturbation Theory $\equiv$ study of how much the solution of a problem is changed (perturbed) if the input data is slightly perturbed.

Example Let f be a real valued differentiable function

Goal: To calculate $f(x)$, but we do not know (precisely) what x is.

we instead have $x+\Delta x$ (δx)

(Δx is bounded)

We can compute $f(x+\Delta x)$

The absolute error $|f(x+\Delta x) - f(x)|$

$$f(t) \approx L(t) = f(x) + f'(x)(t-x) \quad \text{at } x$$

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x$$

$$\underbrace{|f(x+\Delta x) - f(x)|}_{\text{Absolute error}} \approx |f(x) + f'(x)\Delta x - f(x)|$$

$$= |f'(x)\Delta x| = |\Delta x|\,|f'(x)|$$

$|f'(x)|$ is the absolute condition number

If $|f'(x)|$ is large enough that the error may be large even for small Δx

the error may be large even for small Δx
then f is ill-conditioned at x.

Relative Error

$$\frac{|f(x+\Delta x) - f(x)|}{|f(x)|}$$

error in input $\frac{|\Delta x|}{|x|}$

We are ~~looking~~ for C where

$$\frac{|f(x+\Delta x) - f(x)|}{|f(x)|} \approx \frac{|\Delta x|}{|x|} \underline{\underline{C}}$$

← Relative condition number

Finding C

We know: $|f(x+\Delta x) - f(x)| \approx |\Delta x| \, |f'(x)|$

Divide by $|f(x)|$

$$\frac{|f(x+\Delta x) - f(x)|}{|f(x)|} \approx \frac{|\Delta x| \, |f'(x)|}{|f(x)|} \left(\frac{|x|}{|x|}\right)$$

$$\frac{|f(x+\Delta x) - f(x)|}{|f(x)|} \approx \frac{|\Delta x|}{|x|} \, \frac{|f'(x)| \, |x|}{|f(x)|}$$

The relative condition number is $\boxed{\dfrac{|f'(x)| \, |x|}{|f(x)|}}$

Vector and matrix norms

Def

A norm on $\mathbb{R}^n$ is a function that assigns to each $x \in \mathbb{R}^n$ a non-negative real number $||x||$ such the following 3 properties are satisfied for $x, y \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$

positive definite property → ① $||x|| > 0$ if $x \neq 0$ and $||0|| = 0$

← absolute

positive definite property → ① $\|x\| > 0$ if $x \neq 0$ and $\|0\| = 0$

② $\|\alpha x\| = |\alpha| \, \|x\|$ ← absolute homogeneity

③ $\|x+y\| \leq \|x\| + \|y\|$ ← triangle inequality

Example The Euclidean norm is defined by

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$$

$\|x-y\|_2$ ← Euclidean distance between x and y.

Exercise: Verify this is a norm.