Thursday, March 21, 2024 7:38 AM

QR - decompositions

Rotators and Reflectors

In each case we will go into detail about 2×2 case and then extrapolate to an $n \times n$ case.

To keep things simple we will $\mathbb{R}^{n \times n}$ any differences that occur in $\mathbb{C}^{n \times n}$ we will discuss later.

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inner product

let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Then the inner product denoted by $\langle x, y \rangle$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

Inner product has the following properties:

$$\langle x, y \rangle = \langle y, x \rangle$$

$$\langle \alpha_1 x_1 + \alpha_2 x_2 , y \rangle = \alpha_1 \langle x_1, y \rangle + \alpha_2 \langle x_2 , y \rangle$$

$$\langle x, \alpha_1 y_1 + \alpha_2 y_2 \rangle = \alpha_1 \langle x, y_1 \rangle + \alpha_2 \langle x, y_2 \rangle$$

$\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$.

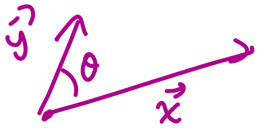
Relation with Euclidean norm.

$$\|x\|_2 = \sqrt{\langle x, x \rangle}$$

When $n = 2$ (or 3) inner product coincides with the dot product

When $n=2$ (or 3) inner product coincides with the dot product

$$\cos(\theta) = \frac{\langle x, y\rangle}{\|x\|_2 \|y\|_2}$$



$$\theta = \arccos\left(\frac{\langle x, y\rangle}{\|x\|_2 \|y\|_2}\right)$$

If $x=0$ or $y=0$ we define $\theta = \frac{\pi}{2}$ radians

Two vectors x and y are orthogonal if the angle between them is $\frac{\pi}{2}$

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x and y are orthogonal if and only if $\langle x, y\rangle = 0$

$\langle x, y\rangle = x^T y$

Orthogonal Matrices

(see last class as well)

Def $Q \in \mathbb{R}^{n\times n}$ is said to be orthogonal if $QQ^T = I$

$$\boxed{\begin{array}{l}\textbf{Thm} \text{ If } Q \in \mathbb{R}^{n\times n} \text{ is orthogonal then for all } x, y \in \mathbb{R}^n \\ \text{(a) } \langle Qx, Qy\rangle = \langle x, y\rangle \quad \text{(b) } \|Qx\|_2 = \|x\|_2\end{array}}$$

proof (a) $\langle Qx, Qy\rangle = (Qy)^T (Qx)$

$$= y^T Q^T Q x = y^T I x$$

$$= y^T x = \langle x, y\rangle$$

(b) $\|Qx\|_2 = \sqrt{\langle Qx, Qx\rangle} \underset{\substack{\uparrow \\ \text{from part} \\ \text{(a)}}}{=} \sqrt{\langle x, x\rangle} = \|x\|_2$

Part (b) tells us Qx and x have the same length.

Part (b) tells us ... the same length.

Orthogonal transformations preserve lengths

$$\arccos \frac{\langle Qx, Qy\rangle}{\|Qx\|_2 \, \|Qy\|_2} = \arccos \frac{\langle x, y\rangle}{\|x\|_2 \, \|y\|_2}$$

Orthogonal transformations preserve angles

<u>Recall</u> The least squares problem

<u>Goal</u> Find an x that minimizes

$$\|b - Ax\|_2$$

By the theorem above $\|b - Ax\|_2 = \|Q(b - Ax)\|_2$

$$= \|Qb - QAx\|_2$$

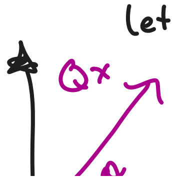
The solution is unchanged if we perform an "orthogonal transformation".

We will show we can find a QA that makes solving (minimizing) $\|Qb - QAx\|_2$ is relatively simple.

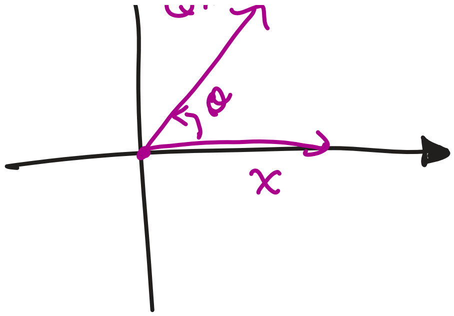
We will show this using rotators and reflectors

<u>Rotators</u>

let's consider vector in $\mathbb{R}^2$



We want to find an operator rotates each vector through a fixed



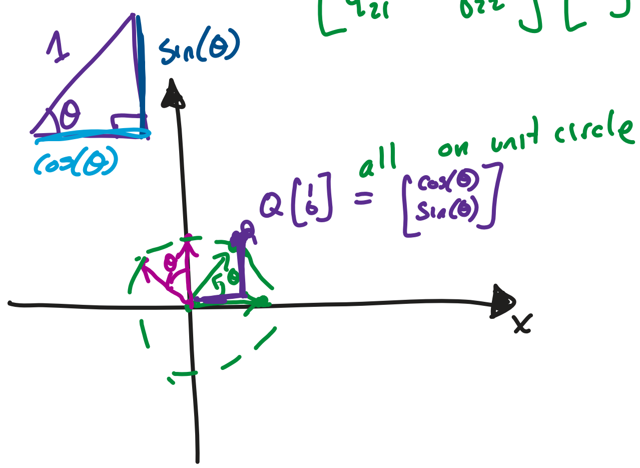
rotates each vector through a fixed angle θ.

This is a linear transformation

Let $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$

Q is completely determined by its actions on the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ above

$$\begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} \qquad \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} q_{12} \\ q_{22} \end{bmatrix}$$



- $Q \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$
- $Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$

$$\begin{bmatrix} q_{11} \\ q_{21} \end{bmatrix} = Q \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$q_{11} = \cos(\theta)$
$q_{21} = \sin(\theta)$

$$\begin{bmatrix} q_{12} \\ q_{22} \end{bmatrix} = Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

$q_{12} = -\sin(\theta)$
$q_{22} = \cos(\theta)$

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

A matrix of this form is called a rotator or rotation.

Exercise Verify Q is orthogonal with determinant of 1.
What is the inverse of Q?

Exercise ...

What is the inverse of Q?

Rotators can be used to zero in vectors or matrices.

Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_2 \neq 0$

$$Q = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Find Q (that is θ) such that

$Q^T x = \begin{bmatrix} y \\ 0 \end{bmatrix}$ for some y.

Since $Q^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$

$$Q^T x = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \cos\theta + x_2 \sin\theta \\ -x_1 \sin\theta + x_2 \cos\theta \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix}$$

Then $0 = -x_1 \sin\theta + x_2 \cos(\theta)$

$$\boxed{x_1 \boxed{\sin\theta} = x_2 \boxed{\cos\theta}}$$

$$\frac{x_1}{x_2} = \frac{\cos(\theta)}{\sin\theta}$$

$$\frac{x_1}{x_2} = \cot(\theta)$$

$$\theta = \operatorname{arc\,cot}\left(\frac{x_1}{x_2}\right)$$

We can find Q with actually knowing what θ is.

$\cos(\theta) = x_1$ and $\sin(\theta) = x_2$

But x_1 and x_2 may not be in $[-1, 1]$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{x_2^2 + x_1^2}{x_2^2 + x_1^2} = 1$$

$\sin^2\theta = \frac{x_2^2}{x_1^2 + x_2^2}$ or $\sin\theta = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}$

and $\cos(\theta) = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$

and $\cos(\theta)$ $\sqrt{x_1^2+x_2^2}$

Thus,
$$Q^T = \begin{bmatrix} \frac{x_1}{\sqrt{x_1^2+x_2^2}} & \frac{x_2}{\sqrt{x_1^2+x_2^2}} \\ -\frac{x_2}{\sqrt{x_1^2+x_2^2}} & \frac{x_1}{\sqrt{x_1^2+x_2^2}} \end{bmatrix} = \frac{1}{\sqrt{x_1^2+x_2^2}} \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}$$

Then
$$Q^T x = \frac{1}{\sqrt{x_1^2+x_2^2}} \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{x_1^2+x_2^2}{\sqrt{x_1^2+x_2^2}} \\ 0 \end{bmatrix}$$

Using a rotator to simplify a matrix.

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Goal: Find R such that $R = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

$$Q^T \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} \frac{a_{11}^2 + a_{21}^2}{\sqrt{a_{11}^2+a_{21}^2}} \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{a_{11}^2+a_{21}^2} \\ 0 \end{bmatrix}$$

$$Q^T \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} r_{12} \\ r_{22} \end{bmatrix}$$

Let $r_{11} = \sqrt{a_{11}^2+a_{21}^2}$

$$R = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$$

where
$$Q^T A = R$$

Linear system
$$Ax = b$$
$$Q^T A x = Q^T b$$
$$Rx = Q^T b = c$$
$$Rx = c$$

since R is upper triangular we can solve for x using back substitution.

We can solve for x using back substitution.

$$Q^T A = R$$

$$Q Q^T A = QR$$

$$IA = QR$$

$$A = QR \quad \swarrow \text{ We have a } QR\text{-decomposition}$$

<u>Example</u> Use a QR decomposition to solve the system $\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

We need a Q such that

$$Q^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix} \qquad \cos\theta \qquad \sin\theta$$

$$Q = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$Q^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow x_1 \sin\theta = x_2 \cos\theta$

here $x_1 = x_2 = 1$

$\Rightarrow \sin\theta = \cos\theta = \frac{\sqrt{2}}{2}$

Thus

$$R = Q^T A = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \frac{\sqrt{2}}{2} \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$$

$$Q^T \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Q^T \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Q^T \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$Q^T A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = Q^T \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\frac{\sqrt{2}}{2} \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\frac{1}{2}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} z_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad z_2 = 1$$

$$2z_1 + 5z_2 = 3 \qquad z_1 = -1$$

$$2z_1 = -2$$

$$z_1 = -1$$

Solution is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Exercise Use a QR decomposition to solve linear system

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 29 \end{bmatrix}$$

Beyond the 2×2 case.

A plane rotator is a matrix of the form

$$Q = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{ccccccccc} & & & \text{Column } i & & & \text{Column } j & & \\ \end{array}$$

$$Q = \begin{array}{cc} \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & c & & & -s & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & s & & & c & & \\ & & & & & & & 1 & \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \text{Row } i \\ \\ \\ \\ \text{Row } j \\ \\ \\ \\ \end{array} \end{array}$$

$c = \cos(\theta) \qquad s = \sin(\theta)$

All other entries in the matrix are zeros.

These matrices are sometimes called Given rotators or Jacobi rotators. For simplicity we will just refer to these matrices as rotators.

We will see we can transform any vector x to one whose j^{th} entry is zero by applying Q^T where x_i $s = \frac{x_j}{\ldots}$

to one whose j^{th} entry ...

where $c = \frac{x_i}{\sqrt{x_i^2 + x_j^2}}$ $s = \frac{x_j}{\sqrt{x_i^2 + x_j^2}}$

If $x_i = x_j = 0$ take $c = 1$ and $s = 0$

Geometric Interpretation

All vectors lying in $x_i x_j$ - plane are rotated through an angle θ.

All vectors orthogonal to $x_i x_j$ - plane are left fixed.

A typical vector is neither on the $x_i x_j$-plane nor perpendicular.

but it can expressed uniquely as a sum

$x = p + p^{\perp}$ where p is in the $x_i x_j$-plane

$p^{\perp}$ - orthogonal to p.

<u>Theorem</u> Let $A \in \mathbb{R}^{n \times n}$. Then there exists an orthogonal matrix Q and an upper triangular matrix R such that

$A = QR$.

<u>proof</u>