January 23rd

Tuesday, January 23, 2024 7:43 AM

General Techniques

1. Matrix Factorization and/or algorithm
2. Perbuturbation theory and condition numbers.
3. Effect of round off error (ie. flops).
4. Analysis of speed of algorithm
5. Utilize software.

Matrix Multiplication

By convention, a matrix will be indicated by a capital letter $(A, B, C, \ldots)$
a vector will be indicated using lower case letters.

Matrix $$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

this matrix has m columns and n rows

The entries of the matrix may be real or complex

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$ is an m-tuple of real numbers

we can multiply A by x. Let's say $b = Ax$

b is an n-tuple $$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$b_i = a_{i1} x_1 + a_{i2} x_2 + \cdots a_{im} x_m$$
$$= \sum^{n} a_{ij} x_j$$

$b_i = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{im}x_m$

$= \sum_{j=1}^{n} \boxed{a_{ij} x_j}$

In other words, the ith component of b the inner product (dot product) of the ith row of A with x.

<u>Example</u> is

Here $n = 2$ and $m = 3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}$$

Psuedo-code for this type of multiplication

row-oriented multiplication →

```
b ← 0
for i = 1, 2, .. n
    for j = 1, 2, ... , m
        b_i ← b_i + a_ij x_j
```

Another way to view matrix-vector multiplication

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$b = Ax$

$$b = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1m} \\ \vdots \\ a_{nm} \end{bmatrix} x_m$$

That b is a linear combination of the columns of A.

Back to the example above

of A.

Back to the example above

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 50 \\ 122 \end{bmatrix}$$

$$\begin{bmatrix} 50 \\ 122 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} 7 + \begin{bmatrix} 2 \\ 5 \end{bmatrix} 8 + \begin{bmatrix} 3 \\ 6 \end{bmatrix} 9$$

Proposition If $b = Ax$, then b is a linear combination of the columns A.

If we let A_j describe the j^{th} column of A we have

$$b = \sum_{j=1}^{m} A_j x_j$$

Psuedo code

$b \leftarrow 0$

for $j = 1, \ldots m$

$\quad b \leftarrow b + A_j x_j$

---

column-oriented multiplication

$b \leftarrow 0$

for $j = 1, \ldots m$

$\quad$ for $i = 1, \ldots n$

$\quad\quad b_i \leftarrow b_i + a_{ij} x_j$

Flop Counts

Real numbers are stored in floating point format.

FLOPs $\equiv$ Floating-point arithmetic operators.

$b_i \leftarrow b_i + a_{ij} x_j$ this involves 2 flops because there are 2 operations,

To determine/estimate "how long" a computer may take to run we may count the number of flops

Let's say A is an n by m matrix

$b \leftarrow 0$
for $i = 1, \dots n$
for $j = 1, \dots m$
$b_i \leftarrow b_i + A_{ij} x_j \leftarrow 2$ flops

since we repeat the line $b_i \leftarrow b_i + A_{ij} x_j$ nm times there are $2nm$ flops

Let B be a 300 by 400 matrix
C be a 600 by 400 matrix
column vector x is the correct length

$Bx \quad 2nm = 2(300)(400)$

Big O $\quad Cx \quad 2(600)(400)$ twice as much as Bx

A lot of calculations/algorithms involve square matrices. That is if B is an n by n matrix

$O(n^2)$ the process is order n^2

The notation emphasizes the dependence on n.

The notation emphasize ...

To get set up

1. Install Python and /or Anaconda
2. Install language intepreter
   example: VS code
   Jupyter
3. Install NumPy
   If using VS code
   In the command window type:
   Python3 -m pip install numpy
   Py

Vandermonde Matrix

Let $\{x_1, \dots x_m\}$ be a sequence of numbers. If p and q are polynomials of degree $< n$ and α is a scalar

then $p+q$ is of degree $< n$

αp is of degree $< n$.

The values of the polynomials at x_i satisfy the linearity properties

$$(p+q)(x_i) = p(x_i) + q(x_i)$$

$$(\alpha p)(x_i) = \alpha (p(x_i))$$

Thus the map of vector coefficient p to $(p(x_1), p(x_2), \dots p(x_m))$ is linear

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & & & & \\ & & & & {}^{n-1} \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix}$$