February 1st

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Linear system $Ax = b$

A small error on b

$b + \Delta b$ (idelly Δb is close to zero)

Instead of getting solution x we get

$$A\hat{x} = b + \Delta b$$

Then $\hat{x} = x + \Delta x$

But how does Δx compare to Δb?

Let's take any vector norm $\|\cdot\|$

The size of Δb relative to b is

$$\frac{\|\Delta b\|}{\|b\|}$$

we would like to say if $\frac{\|\Delta b\|}{\|b\|}$ is small then

$\frac{\|\Delta x\|}{\|x\|}$ is also small.

$Ax = b$ and $A\hat{x} = b + \Delta b$

$A(x + \Delta x) = b + \Delta b$

$A(x + \Delta x) = Ax + \Delta b$

$A\Delta x = \Delta b$

or $\Delta x = A^{-1}\Delta b$

$\|A^{-1}\| \|\Delta b\|$

$\|\Delta x\| = \|A^{-1}\Delta b\| \leq \|A^{-1}\| \|\Delta b\|$

$\|\Delta x\| \leq \|A^{-1}\| \|\Delta b\|$

$Ax = b$

$\|b\| \leq \|A\| \|x\|$

$\frac{1}{\|b\|} \geq \frac{1}{\|A\| \|x\|}$

$$\frac{1}{\|b\|} \geq \frac{1}{\|A\| \, \|x\|}$$

$$\frac{1}{\|b\|} \|A\| \geq \frac{1}{\|x\|}$$

$$\frac{1}{\|x\|} \leq \frac{1}{\|b\|} \|A\|$$

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\|\Delta b\|}{\|b\|} \|A\| \, \|A^{-1}\|$$

$$\frac{\|\Delta x\|}{\|x\|} \leq \|A\| \, \|A^{-1}\| \frac{\|\Delta b\|}{\|b\|}$$

The factor $\|A\| \, \|A^{-1}\|$ is the condition number of A denoted by $K(A)$.

$$\boxed{K(A) = \|A\| \, \|A^{-1}\|}$$

Theorem Let A be nonsingular and let x and $\hat{x} = x + \Delta x$ be the solutions of $Ax = b$ and $A\hat{x} = b + \Delta b$.

Then

$$\frac{\|\Delta x\|}{\|x\|} \leq K(A) \frac{\|\Delta b\|}{\|b\|}$$

this inequality is sharp. That is, there exists b and Δb such that the equality holds.

Proposition For any induced matrix norm

(a) $\|I\| = 1$

(b) $K(A) \geq 1$

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Best possible condition number is 1.

The condition number is dependent on the choice of norm.

We will many use 1-, 2-, ∞- norms

$$K_p(A) = \|A\|_p \|A^{-1}\|_p \quad \text{for } 1 \leq p \leq \infty$$

Example Let $A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}$

then $A^{-1} = \begin{bmatrix} -998 & 999 \\ 999 & -1000 \end{bmatrix}$

$\|\cdot\|_\infty$ – row sum norm

$\|\cdot\|_1$ – column sum norm

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

$\|A\|_\infty = 1999 \qquad \|A^{-1}\|_\infty = 1999$

$\|A\|_1 = 1999 \qquad \|A^{-1}\|_1 = 1999$

$$\boxed{K_\infty(A) = K_1(A) = (1999)^2}$$

$$A = \begin{bmatrix} 1000 & 999 \\ 999 & 998 \end{bmatrix}$$

$$Ax = b \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$1000x_1 + 999x_2 = b_1$$

$$999x_1 + 998x_2 = b_2$$

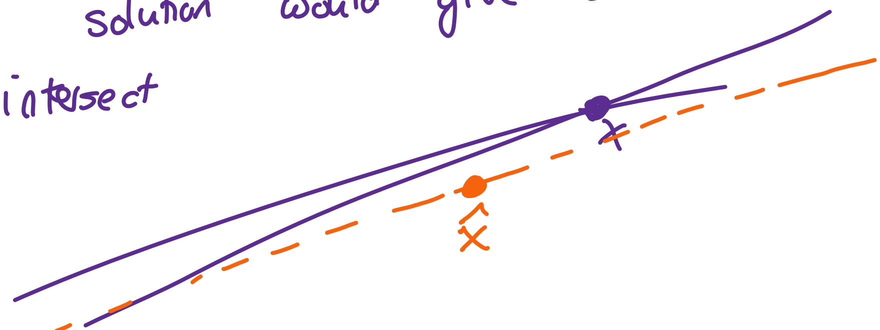
→ slope of line is $m_1 = -\frac{1000}{999} \approx -1.001001$

→ Slope of line is $m_2 = -\frac{999}{998} \approx -1.001002$

Slope of line is $m_2 = \frac{-999}{998} \approx -1.001002$

The two lines are almost parallel

The solution would give where the lines intersect



There is no number where a system goes from well-conditioned to ill-conditioned, But it is generally agreed that the number is somewhere between 10^2 to 10^4.

Let's say we are using system that only stores out to 7 decimal places,

and we have $\frac{\|\Delta b\|}{\|b\|} \approx 10^{-7}$

If $K(A) \approx 10^7$ we can not be sure our answer is reasonable,

<u>Example</u> <u>Hilbert Matrices</u>

A <u>Hilbert Matrix</u> defined by $h_{ij} = \frac{1}{i+j-1}$

and H_n denote the $n \times n$ Hilbert matrix

$$H_4 = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 \\ 1/2 & 1/3 & 1/4 & 1/5 \\ 1/3 & 1/4 & 1/5 & 1/6 \\ 1/4 & 1/5 & 1/6 & 1/7 \end{bmatrix}$$

The matrix is symmetric and positive

The matrix is symmetric and positive definite, but is ill conditioned as n increases

$K_2(H_4) \approx 1.6 \times 10^4$ $\quad K_2(H_8) = 1.5 \times 10^{10}$

Gaussian Elimination

(without pivoting)

We can always transform a linear system of equations using 3 elementary operations

1. Add a multiple of one equation to another
2. Interchange two equations
3. Multiply an equation by a nonzero constant.

Proposition If $\hat{A} x = \hat{b}$ is obtained from $Ax = b$ by an elementary operation of type 1, 2, or 3 then the system $Ax = b$ and $\hat{A} x = \hat{b}$ are equivalent.

Augmented Matrix

$Ax = b$ it is sometimes convenient to write $[A \mid b]$

Proposition Suppose $\hat{A}$ is obtained from A by an elementary row operation. Then $\hat{A}$ is nonsingular if and only if A is nonsingular.

Apply Gaussian Elimination to find

Example Apply Gaussian Elimination to find x where

$$\begin{bmatrix} 9 & -6 & 6 \\ -6 & 5 & -1 \\ 6 & -1 & 15 \end{bmatrix} x = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$$

Augmented Matrix

$$\left[\begin{array}{ccc|c} 9 & -6 & 6 & 3 \\ -6 & 5 & -1 & -3 \\ 6 & -1 & 15 & 0 \end{array}\right] \xrightarrow{\text{Row①} \times \frac{1}{9}} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -6 & 5 & -1 & -3 \\ 6 & -1 & 15 & 0 \end{array}\right]$$

$$\xrightarrow{6\times\text{Row①} + \text{Row②}} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 3 & -1 \\ 6 & -1 & 15 & 0 \end{array}\right] \xrightarrow{-6\times\text{Row①}+\text{Row③}} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 3 & -1 \\ 0 & 3 & 11 & -2 \end{array}\right]$$

$$\xrightarrow{-3\times\text{Row②}+\text{Row③}} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 2 & 1 \end{array}\right] \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Back substitution

$2x_3 = 1$

$x_3 = \frac{1}{2}$

$x_2 + 3\left(\frac{1}{2}\right) = -1$

$x_2 = -1 - \frac{3}{2}$

$x_2 = -\frac{5}{2}$

$x_1 - \frac{2}{3}\left(-\frac{5}{2}\right) + \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3}$

$x_1 + \frac{5}{3} + \frac{1}{3} = \frac{1}{3} \rightarrow x_1 = \frac{1}{3} - \frac{5}{3} - \frac{1}{3}$

$x_1 = -\frac{5}{3}$

$$\boxed{x = \begin{bmatrix} -\frac{5}{3} \\ -\frac{5}{2} \\ \frac{1}{2} \end{bmatrix}}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$m_{i1} = a_{i1}/a_{11}$

i^{th} row $\leftarrow$ i^{th} row $- m_{i1}(1^{st}$ row$)$

$\downarrow$

$[a_{11} \quad \cdots \quad a_{1n}]$

we may find it helpful to keep track of the multiplier

$$\left[\begin{array}{c|cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & a_{22}^{*} & \cdots & a_{2n}^{*} \\ 0 & \vdots & \ddots & \vdots \\ \vdots & & & a_{nn}^{*} \\ 0 & a_{n2}^{*} & \cdots & \end{array}\right]$$

we may ... it helpful to ... of the multiplier

$$\left[\begin{array}{c|cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline m_{21} & a_{22}^{*} & \cdots & a_{2n}^{*} \\ m_{31} & \vdots & & \\ \vdots & a_{n2}^{*} & \cdots & a_{nn}^{*} \\ m_{n1} & a_{n2} & \cdots & a_{nn} \end{array}\right]$$