A plane rotator is a matrix

$$\begin{array}{cc} & \begin{array}{ccccc} & \text{Column } i & & \text{Column } j & \end{array} \\ Q = & \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & c & & -s & & \\ & & & 1 \ \ddots & & & \\ & & s & & c & & \\ & & & & & 1 \ \ddots & \\ & & & & & & 1 \end{bmatrix} \begin{array}{l} \\ \\ \text{Row } i \\ \\ \text{Row } j \\ \\ \\ \end{array} \end{array}$$

$$QQ^T = I$$

Theorem Let $A \in \mathbb{R}^{n \times n}$. Then there exists an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

proof

Q is taken to be product of rotators.

Let Q_{21} be a rotator acting on the x_1, x_2 plane., such that Q_{21}^T make the transformation

$$\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{n1} \end{bmatrix} \rightarrow \begin{bmatrix} * \\ 0 \\ a_{31} \\ a_{41} \\ \vdots \\ a_{n1} \end{bmatrix}$$

Then $Q_{21}^T A$ has a zero in the $(2,1)$ position

Similarly we can find a plane rotator Q_{31} acting in the $x_1 x_3$ - plane such that $Q_{31}^T (Q_{21}^T A)$ has a zero in the $(3,1)$ position. Does not disturb the $(2,1)$ - position Q_{31}^T leaves the second row of $Q_{21}^T A$ unchanged.

$Q_{41}, Q_{51}, \dots Q_{n1}$

$Q_{n1}^T \dots Q_{21}^T A$ has zeros in the entire first column except the $(1,1)$ - position.

Q_{32} be a plane rotator acting in $x_2 x_3$ - plane such that $(3,2)$ of $Q_{32}^T (Q_{n1}^T \cdots Q_{21}^T A)$ is zero.

This rotator does not disturb any zeros in the first column.

This rotator does not disturb any zeros in the first column.

$$R = Q_{n,n-1}^T Q_{n,n-2}^T \cdots Q_{21}^T A$$

is upper triangular.

Q is the product of orthogonal matrices

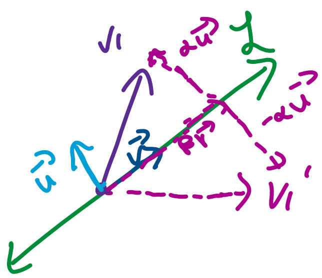
thus is itself orthogonal.

and $R = Q^T A$, that is $QR = A$

Reflectors

We will start with $n = 2$

in $\mathbb{R}^2$



Goal: To get an operator that reflects each vector in $\mathbb{R}^2$ over $\mathcal{L}$.

This operation is a linear transformation (can be represented by a matrix)

$\vec{v}$ a nonzero vector on $\mathcal{L}$

$\vec{u}$ a nonzero vector orthogonal to $\mathcal{L}$.

and $\|\vec{u}\|_2 = 1$

Vectors $\vec{u}$ and $\vec{v}$ form a basis in $\mathbb{R}^2$

Then $\vec{v}_1 = \alpha\vec{u} + \boxed{\beta\vec{v}}$

the reflection of $\vec{v}_1$ is

$\vec{v}_1{}' = -\alpha\vec{u} + \beta\vec{v}$

So we have a matrix Q such that

$$Q(\alpha\vec{u} + \beta\vec{v}) = -\alpha\vec{u} + \beta\vec{v}$$

for all α and β.

Then it is necessary (and sufficient)

① $Q\vec{u} = -\vec{u}$

② $Q\vec{v} = \vec{v}$

2×1

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\langle \vec{w}, \vec{w} \rangle = \vec{w}^T\vec{w}$

$\langle w, w \rangle = w \cdot w$

Let's try $P = \vec{u}\vec{u}^T \in$ a 2×2 matrix.

$$P\vec{v} = \vec{u}\vec{u}^T\vec{v} = \vec{u}(\vec{u}^T v) = 0$$

because $\vec{u}$ and $\vec{v}$ are orthogonal

$$P\vec{u} = (\vec{u}\vec{u}^T)\vec{u}) = \vec{u}(\vec{u}^T\vec{u}) = \vec{u}\,\|\vec{u}\|_2^2$$
$$= \vec{u}$$

$$I\vec{u} = \vec{u}$$
$$I\vec{v} = \vec{v}$$

Let $Q = I - 2P$ then

$$Q\vec{u} = (I - 2P)\vec{u} = I\vec{u} - 2P\vec{u}$$
$$= \vec{u} - 2\vec{u} = -\vec{u}$$

$$Q\vec{v} = (I - 2P)\vec{v} = I\vec{v} - 2P\vec{v}$$
$$= \vec{v} - 2(0) = \vec{v}$$

In the 2×2 case where $\vec{u}$ is orthogonal to $\mathcal{L}$ we want to reflect any vector through $\mathcal{L}$

$$Q = I - 2\vec{u}\vec{u}^T$$

Moving on to the $n\times n$ case:

Theorem Let $\vec{u} \in \mathbb{R}^n$ with $\|\vec{u}\|_2 = 1$ and defined $P \in \mathbb{R}^{n\times n}$ by $P = \vec{u}\vec{u}^T$

Then

(a) $P\vec{u} = \vec{u}$

(b) $P\vec{v} = 0$ if $\langle \vec{u}, \vec{v} \rangle = 0$

(c) $P^2 = P$

(d) $P^T = P$

$P^2 = P$ is called a projector

Note A matrix $P^2 = P$ is called a projector or idempotent.

A projector that is also symmetric $(P^T = P)$ is called an orthoprojector

The matrix $P = \vec{u}\,\vec{u}^T$ has rank 1 (its range consists of multiples of $\vec{u}$)

We can summarize the properties as

P is a rank-1 orthoprojector.

Theorem Let $\vec{u} \in \mathbb{R}^n$ with $\|\vec{u}\|_2 = 1$ and define $Q \in \mathbb{R}^{n \times n}$ by $Q = I - 2\vec{u}\,\vec{u}^T$

Then

(a) $Q\vec{u} = -\vec{u}$

(b) $Q\vec{v} = \vec{v}$ if $\langle \vec{u}, \vec{v} \rangle = 0$

(c) $Q = Q^T$ (Q is symmetric)

(d) $Q^T = Q^{-1}$ (Q is orthogonal)

(e) $Q^{-1} = Q$ (Q is an involution)

$Q = I - 2\vec{u}\,\vec{u}$ $(\|\vec{u}\|_2 = 1)$ are called reflectors or Householder transformations.

Set $\mathcal{H} = \{\vec{v} \in \mathbb{R}^n \mid \langle \vec{u}, \vec{v} \rangle = 0\}$ is an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ known as a hyperplane.

The matrix Q maps each vector $\vec{v}_1$ to its reflection through the hyperplane $\mathcal{H}$.

In the case $n = 3$ $\mathcal{H}$ is an ordinary plane through the origin.

In the case ... plane through the origin.

Proposition Let $\vec{u}$ be a nonzero vector in $\mathbb{R}^n$ and define $\gamma = 2/\|\vec{u}\|_2^2$ and $Q = I - \gamma \vec{u}\vec{u}^T$

Then Q is a reflector satisfying

(a) $Q\vec{u} = -\vec{u}$

(b) $Q\vec{v} = \vec{v}$ if $\langle \vec{u}, \vec{v} \rangle = 0$

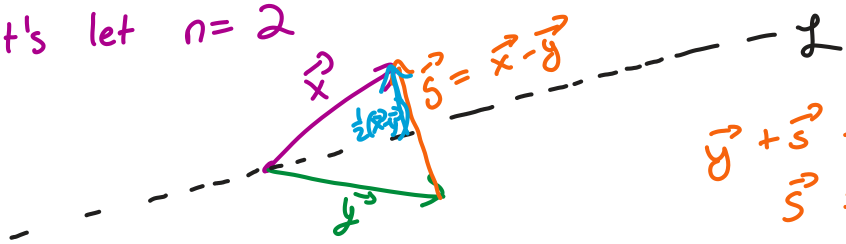
Theorem Let $x, y \in \mathbb{R}^n$ with $\vec{x} \neq \vec{y}$ but $\|\vec{x}\|_2 = \|\vec{y}\|_2$. Then there is a unique reflector Q such $Q\vec{x} = \vec{y}$

proof we will skip uniqueness and instead show Q exists.

We need to show/find a $\vec{u}$ such that

$$Q\vec{x} = (I - \gamma \vec{u}\vec{u}^T)\vec{x} = \vec{y} \quad \text{where } \gamma = \frac{2}{\|\vec{u}\|_2^2}$$

Let's let $n = 2$



$\vec{y} + \vec{s} = \vec{x}$

$\vec{s} = \vec{x} - \vec{y}$

let $\vec{u} = \frac{1}{2}(\vec{x} - \vec{y})$ or any multiple

let's let $\vec{u} = \vec{x} - \vec{y}$ and $\gamma = \frac{2}{\|\vec{u}\|_2^2}$

$Q = I - \gamma \vec{u}\vec{u}^T$

Note: $\vec{x} = \frac{1}{2}(\vec{x} - \vec{y}) + \frac{1}{2}(\vec{x} + \vec{y})$

Note: $\vec{x} = \frac{1}{2}(\vec{x}-\vec{y}) + \frac{1}{2}(\vec{x}+\vec{y})$

<u>Question</u> Is $\vec{x}+\vec{y}$ orthogonal to $\vec{x}-\vec{y}$?

$$\langle \vec{x}+\vec{y}, \vec{x}-\vec{y} \rangle = \langle \vec{x}, \vec{x}-\vec{y} \rangle + \langle \vec{y}, \vec{x}-\vec{y} \rangle$$

$$= \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, -\vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, -\vec{y} \rangle$$

$$= \langle \vec{x}, \vec{x} \rangle - \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{y} \rangle - \langle \vec{y}, \vec{y} \rangle$$

$$= \|\vec{x}\|_2^2 - \|\vec{y}\|_2^2 \qquad \text{since } \|\vec{x}\|_2 = \|\vec{y}\|_2$$

$$= 0$$

$\vec{u} = \vec{x} - \vec{y}$

$$Q(\vec{x}+\vec{y}) = \vec{x}+\vec{y}$$

$$Q\vec{x} = Q\left(\tfrac{1}{2}(\vec{x}+\vec{y}) + \tfrac{1}{2}(\vec{x}-\vec{y})\right) \qquad \vec{u} = \vec{x}-\vec{y}$$

$$= \tfrac{1}{2}Q(\vec{x}+\vec{y}) + \tfrac{1}{2}Q(\vec{x}-\vec{y})$$

$$= \tfrac{1}{2}(\vec{x}+\vec{y}) - \tfrac{1}{2}(\vec{x}-\vec{y})$$

$$= \tfrac{1}{2}\vec{x} + \tfrac{1}{2}\vec{y} - \tfrac{1}{2}\vec{x} + \tfrac{1}{2}\vec{y}$$

$$= \vec{y}$$

<u>Corollary</u> Let $x \in \mathbb{R}^n$ be any nonzero vector. Then there exists a reflector Q such that

$$Q \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} * \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

proof Let $u = \begin{bmatrix} -\tau \\ 0 \end{bmatrix}$ where $\tau = \pm \|\vec{x}\|_2$

proof Let $y = \begin{bmatrix} -\tau \\ 0 \\ \vdots \end{bmatrix}$ where $\boxed{\tau = \pm \|\vec{x}\|_2}$

we allow for $\pm$ so that $\vec{x} \neq \vec{y}$.

$\|\vec{x}\|_2 = \|\vec{y}\|_2$ Thus we can define Q

Such that $Q\vec{x} = \vec{y}$

Reflector $Q = I - \gamma \vec{u}\vec{u}^T$

$$\vec{u} = \vec{x} - \vec{y} = \begin{bmatrix} \tau + x_1 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and } \gamma = \frac{2}{\|\vec{u}\|_2^2}$$

$\vec{u}$ can be any multiple of $\vec{x} - \vec{y}$

Let's normalize $\vec{u}$ so its first entry is 1.

$$\vec{u} = (\vec{x} - \vec{y}) / \tau + x_1 = \begin{bmatrix} 1 \\ x_2/\tau + x_1 \\ \vdots \\ x_n/\tau + x_1 \end{bmatrix}$$

Recall $\tau = \pm \|\vec{x}\|_2$

the sign is permitted to vary

(we do not want $\vec{x} = \vec{y}$)

In practice we could take τ to have the same sign as x_1. (No risk of dividing by zero when dividing by $\tau + x_1$)

Calculation of γ

$$\|\vec{u}\|_2^2 = \frac{(\tau + x_1)^2 + x_2^2 + \cdots + x_n^2}{(\tau + x_1)^2}$$

$$= \frac{\tau^2 + 2\tau x_1 + \boxed{x_1^2 + x_2^2 + \cdots x_n^2}}{(\tau + x_1)^2}$$

$\|\vec{x}\|_2^2$

τ^2

$$= \frac{\tau^2 + 2\tau x_1 + \underbrace{x_1^2 + x_2^2 + \cdots + x_n^2}_{\tau^2}}{(\tau + x_1)^2}$$

$$= \frac{\tau^2 + 2\tau x_1 + \|\vec{x}\|_2^2}{(\tau + x_1)^2}$$

But $\tau^2 = \|\vec{x}\|_2^2$

$$\Rightarrow = \frac{2\tau^2 + 2\tau x_1}{(\tau + x_1)^2}$$

$$= \frac{2\tau(\tau + x_1)}{(\tau + x_1)^2}$$

$$= \frac{2\tau}{\tau + x_1}$$

and $\boxed{\gamma = \frac{(\tau + x_1)}{\tau}}$

Given $\vec{x} \in \mathbb{R}^n$ this algorithm will calculate τ, γ, and $\vec{u}$ such that $Q = I - \gamma \vec{u}\vec{u}^T$ is a reflector of for which $Q\vec{x} = \begin{bmatrix} -\tau \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

If $\vec{x} = 0$ set $\gamma = 0$ and thus $Q = I$

Otherwise γ and $\vec{u}$ are produced

$\vec{u}$ is stored over $\vec{x}$.

$\beta \leftarrow \max_{1 \le i \le n} |x_i|$

If $\beta = 0$

then $\gamma \leftarrow 0$

else

$x \leftarrow x / \beta$ (vector divided by β)

$\tau \leftarrow \sqrt{x_1^2 + x_2^2 + \cdots x_n^2}$

if $(x_1 < 0)$ $\tau \leftarrow -\tau$

$x_1 \leftarrow \tau + x_1$

$$x_1 \leftarrow \tau + x_1$$

$$\gamma \leftarrow x_1 / \tau$$

$$x(1 \text{ to } n-1) \leftarrow \frac{x(1 \text{ to } n-1)}{x_1}$$

$$x_1 \leftarrow 1$$

$$\tau \leftarrow \tau \beta$$