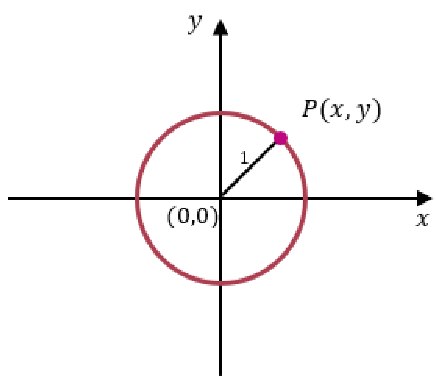
5.2 Circular Trigonometry

THE SINE FUNCTION ON THE UNIT CIRCLE

In computer games, objects typically move up-and-down and left-to-right. These movements can be produced using the sine and cosine functions.

Draw a circle with radius 1 unit and on its circumference, place a point, let's call it P.

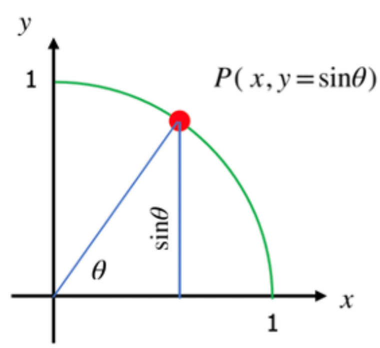
The circle centered at the origin with radius 1 is called the unit-circle.



From our presentation of the sine and cosine function using right triangles, we can see that

$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{1} = y$. That is, $y = \sin\theta$.

This tells us that the sine of the angle θ determines the vertical distance of the point P from the horizontal axis.



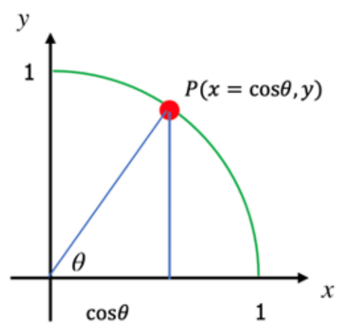
THE COSINE FUNCTION ON THE UNIT CIRCLE

To define cosine function, place a point $P(x, y)$ on the circumference of unit-circle.

Once again, from our presentation of the cosine functions using right triangles, we can see that

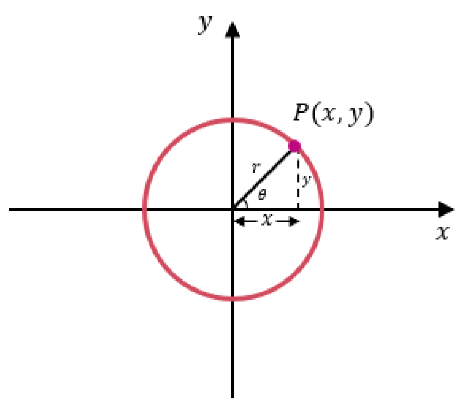
$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{1} = x. \text{ That is, } x = \cos\theta.$$

This tells us that the cosine of the angle θ determines the horizontal distance of the point P from the vertical axis.



THE SINE AND COSINE FUNCTIONS ON ANY CIRCLE

We can extend this idea by making the radius of the circle r units rather than just 1 unit.



Using the same reasoning we just used with the unit circle, we see that

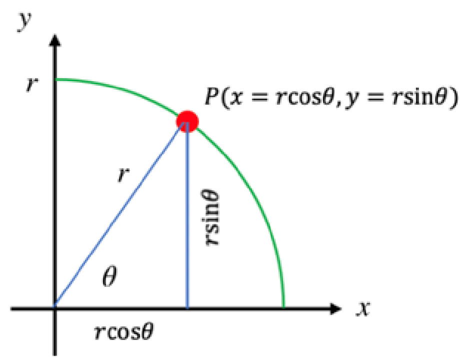
$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r} \rightarrow r \cdot \sin\theta = y \rightarrow y = r \cdot \sin\theta$$

and

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r} \rightarrow r \cdot \cos\theta \rightarrow x = r \cdot \cos\theta$$

which, again, tells us that the sine of the angle θ determines the vertical distance of the point P from the horizontal axis and that the cosine of the angle θ determines the horizontal distance of the point P from the vertical axis.

If P represents an object, that object's height y off the ground (the horizontal axis) is given by $r \cdot \sin\theta$, and that object's horizontal distance x from some reference point is given by $r \cdot \cos\theta$. The height of the object is controlled by some number r times $\sin\theta$, and its horizontal distance is controlled by some number r times $\cos\theta$.



Example (1)

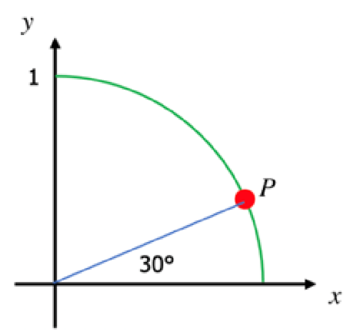
An object lies on the circumference of a unit circle. Find its coordinates if the line segment from the origin to the object makes angle of 30° with the horizontal.

Because the object is on the circumference of unit circle, we can use

$x = r\cos\theta$ and $y = r\sin\theta$, with $r = 1$, $\theta = 30°$.

$$x = 1\cos 30° \text{ and } y = 1\sin 30°$$
$$x = \cos 30° \text{ and } y = \sin 30°$$
$$x = 0.8660 \text{ and } y = 0.5$$

The coordinates of the object are (0.8660, 0.5).



Example (2)

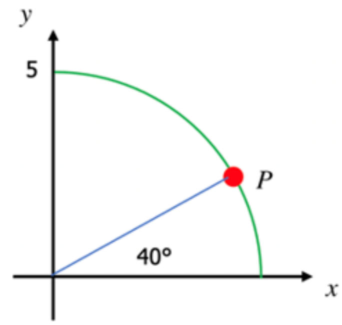
An object lies on the circumference of a circle of radius 5 cm. Find its coordinates if the line segment from the origin to the object makes angle of 40° with the horizontal.

Because the object is on the circumference of circle of radius 5 cm, we can use

$x = r\cos\theta$ and $y = r\sin\theta$, with $r = 5$, $\theta = 40°$.

$$\begin{aligned} x &= 5\cos 40° && \text{and} \quad y = 5\sin 40° \\ x &= 5(0.7660) && \text{and} \quad y = 5(0.6428) \\ x &= 3.8302 && \text{and} \quad y = 3.2139 \end{aligned}$$

The coordinates of the object are (3.8302, 3.2139).



Example (3)

The coordinates of an object are (2.1, 3.6373). Find its distance from the origin.

We can use the Pythagorean Theorem, $a^2 + b^2 = c^2$, where c is the hypotenuse, the radius of the circle in our case.

$$2.1^2 + 3.6373^2 = r^2$$
$$4.41 + 13.2300 = r^2$$
$$17.64 = r^2$$
$$\sqrt{17.64} = \sqrt{r^2}$$
$$4.2 = r$$

We conclude that the object is about 4.2 cm from the origin.

TRY THESE

1. An object lies on the circumference of a unit circle. Find its coordinates if the line segment from the origin to the object makes angle of 45° with the horizontal.

ANS: (0.7071, 0.7071)

2. An object lies on the circumference of a unit circle. Find its coordinates if the line segment from the origin to the object makes angle of 5° with the horizontal.

ANS: (0.9962, 0.0872)

3. An object lies on the circumference of a circle of radius 25 cm. Find its coordinates if the line segment from the origin to the object makes angle of 75° with the horizontal.

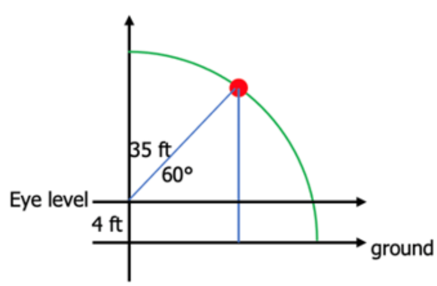
ANS: (6.4705, 4.8396)

4. An object lies on the circumference of a circle of radius 10 feet. Find its coordinates if the line segment from the origin to the object makes angle of 135° with the horizontal.

ANS: (-7.0711, 7.0711)

5. How high above the ground is an object that makes an angle of 60° with a 4-foot-tall observer's eyes and is 35 feet away from that observer's eyes? Round to two decimals place.

ANS: 34.31 ft



6. The coordinates of an object are (5.682, 2.0521). Find its distance from the origin if it makes an angle of 60° with the horizontal.

ANS: 6 units

NOTE TO THE INSTRUCTOR

We are using degrees rather than radians as most game players think in terms of degrees, not radians.

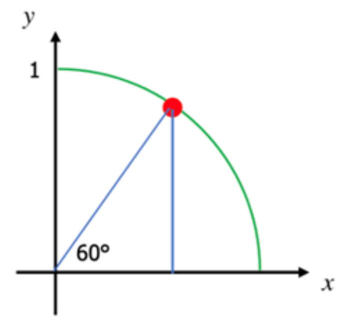
Consider demonstrating these examples:

1. An object lies on the circumference of a unit circle. Find its coordinates if the line segment from the origin to the object makes angle of 60° with the horizontal.

Because the object is on the circumference of unit circle, we can use
$x = r\cos\theta$ and $y = r\sin\theta$, with $r = 1$, $\theta = 60°$.

$$x = 1\cos 60° \text{ and } y = 1\sin 60°$$
$$x = \cos 60° \text{ and } y = \sin 60°$$
$$x = 0.5 \text{ and } y = 0.8660$$

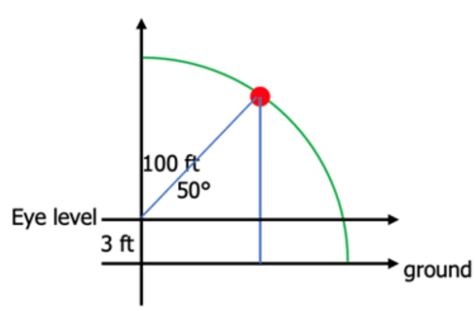
The coordinates of the object are (0.5, 0.8660).



2. An object lies on the circumference of a circle of radius 25 m. Find its coordinates if the line segment from the origin to the object makes angle of 120° with the horizontal.

ANS: (-12.5, 21.6506)

3. How high above the ground is an object that makes an angle of 50° with a 3-foot-tall observer's eyes and is 100 feet away from that observer's eyes? Round to two decimals places.



ANS: 79.60 ft

5. The coordinates of an object are (5, 12). Find its distance from the origin.

We can use the Pythagorean Theorem, $a^2 + b^2 = c^2$, where c is the hypotenuse, the radius of the circle in our case.

$$5^2 + 12^2 = r^2$$
$$25 + 144 = r^2$$
$$169 = r^2$$
$$\sqrt{169} = \sqrt{r^2}$$
$$13 = r$$

We conclude that the object is about 13 units from the origin.