3.5 The Dot Product, Length of a Vector, and the Angle between Two Vectors in Three Dimensions

THE DOT PRODUCT OF TWO VECTORS

The dot product of two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ in two dimensions is nicely extended to three dimensions.

The dot product of vectors $\vec{u} = \langle u_x, u_y, u_z \rangle$ and $\vec{v} = \langle v_x, v_y, v_z \rangle$ is a scalar (real number) and is defined to be

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$$

Since $u_x, u_y, u_z, v_x,$ $v_y$and v_z are real numbers, you can see that the dot product is itself a real number and not a vector.

Example (1)

To compute the dot product of the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$, we compute

$$\vec{u} \cdot \vec{v} = 5 \cdot 3 + 2 \cdot 4 + 4 \cdot (-7) = 15 + 8 - 28 = -5$$

Since the dot product is a scalar, it follows the properties of real numbers.

PROPERTIES OF THE DOT PRODUCT

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, the dot product is commutative
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, the dot product distributes over vector addition
3. $\vec{u} \cdot \vec{0} = 0$, the dot product with the zero vector, $\vec{0}$, is the scalar 0.
4. $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$

Example (2)

Compute the dot product $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, where

$\vec{u} = \langle 5, -2, -3 \rangle$, $\vec{v} = \langle 6, 4, 1 \rangle$, and $\vec{w} = \langle -3, 7, -2 \rangle$,

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \langle 5, -2, -3 \rangle \cdot \langle 6, 4, 1 \rangle + \langle 5, -2, -3 \rangle \cdot \langle -3, 7, -2 \rangle$$

$$= (5 \cdot 6 + (-2) \cdot 4 + (-3) \cdot 1) + (5 \cdot (-3) + (-2) \cdot 7) + (-3) \cdot (-2))$$
$$= 30 - 8 - 3 - 15 - 14 + 6$$
$$= -4$$

THE LENGTH OF A VECTOR IN THREE DIMENSIONS

The length (magnitude) of a vector in two dimensions is nicely extended to three dimensions.

The dot product of a vector $\vec{v} = \langle v_x, v_y \rangle$ with itself gives the length of the vector.

$$||\vec{v}\,|| = \sqrt{{v_x}^2 + {v_y}^2}$$

You can see that the length of the vector is the square root of the sum of the squares of each of the vector's components. The same is true for the length of a vector in three dimensions.

The dot product of a vector $\vec{v} = \langle v_x, v_y, v_z \rangle$ with itself gives the length of the vector.

$$||\vec{v}\,|| = \sqrt{{v_x}^2 + {v_y}^2 + \ {v_z}^2}$$

Example (3)

Use the dot product to find the length of the vector $\vec{v} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$.

In this case, $v_x = 4$, $v_y = 2$, and $v_z = 6$

Using $||\vec{v}\,|| = \sqrt{{v_x}^2 + {v_y}^2 + \ {v_z}^2}$, we get

$$||\vec{v}\,|| = \sqrt{4^2 + 2^2 + \ 6^2}$$
$$||\vec{v}\,|| = \sqrt{56}$$
$$||\vec{v}\,|| = \sqrt{4 \cdot 14}$$
$$||\vec{v}\,|| = \sqrt{4} \cdot \sqrt{14}$$
$$||\vec{v}\,|| = 2\sqrt{14}$$

The length of the vector $\vec{v} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ is $2\sqrt{14}$ units.

THE ANGLE BETWEEN TWO VECTORS

The formula for the angle between two vectors in two dimensions is nicely extended to three dimensions.

If θ is the smallest nonnegative angle between two non-zero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} \text{ or } \theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$$

where $0 \leq \theta \leq 2\pi$ and $\|\vec{u}\| = \sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2}$ and $\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$

Example (4)

Find the angle between the vectors $\vec{u} = \langle 5, -3, -1\rangle$ and $\vec{v} = \langle 2, 4, -5\rangle$.

Using $\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$, we get

$$\theta = \cos^{-1}\frac{\langle 5, -3, -1\rangle \cdot \langle 2, 4, -5\rangle}{\sqrt{5^2 + (-3)^2 + (-1)^2}\cdot\sqrt{2^2 + 4^2 + (-5)^2}}$$

$$\theta = \cos^{-1}\frac{5\cdot 2 + (-3)\cdot 4 + (-1)\cdot(-5)}{\sqrt{25 + 9 + 1}\cdot\sqrt{4 + 16 + 25}}$$

$$\theta = \cos^{-1}\frac{3}{\sqrt{35}\cdot\sqrt{45}}$$

$$\theta = 85.66$$

We conclude that the angle between these two vectors is close to 85.7° rounded to one decimal place.

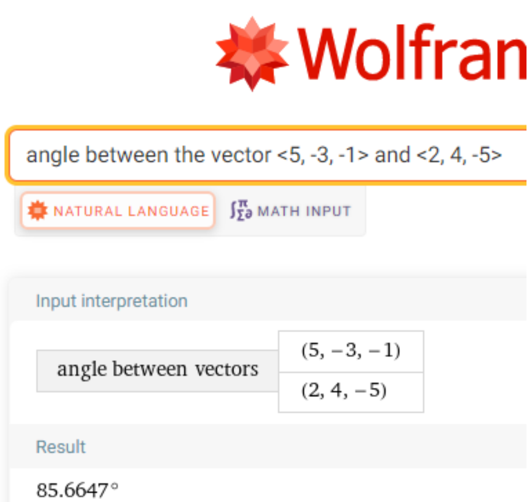
USING TECHNOLOGY

We can use technology to find the magnitude of the vector and the angle θ between two vectors.

Go to www.wolframalpha.com.

To find the magnitude (length) of the vector $\vec{v} = \langle 4, 2, 6 \rangle$, enter magnitude of <4, 2, 6> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $||\vec{v}|| = 2\sqrt{14}$.

To find the angle between the vectors $\vec{u} = \langle 5, -3, -1 \rangle$ and $\vec{v} = \langle 2, 4, -5 \rangle$, enter angle between the vector <5, -3, -1> and <2, 4, -5> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\theta = 85.7°$, rounded to one decimal place.



TRY THESE

1. Find the dot product of the vectors $\vec{u} = \langle -2, 3, -9 \rangle$ and $\vec{v} = \langle 5, -1, 2 \rangle$.

ANS: $\vec{u} \cdot \vec{v} = -31$

2. Find the dot product of the vectors $\vec{u} = \langle 6, 2, -1 \rangle$ and $\vec{v} = \langle 2, -7, -2 \rangle$.

ANS: $\vec{u} \cdot \vec{v} = 0$

3. Find the length of the vector $\vec{u} = \langle 4, -7, -6 \rangle$.

ANS: $\sqrt{101}$

4. Find the length of the vector $\vec{v} = \langle 0, 5, 0 \rangle$.

ANS: 5

5. Find the angle between the vectors $\vec{u} = \langle 3, 4, 5 \rangle$ and $\vec{v} = \langle -3, -1, 8 \rangle$.

ANS: $63.6°$

6. Find the angle between the vectors $\vec{u} = \langle 1, -2, 1 \rangle$ and $\vec{v} = \langle 3, 5, 7 \rangle$.

ANS: $90°$

NOTE TO INSTRUCTOR

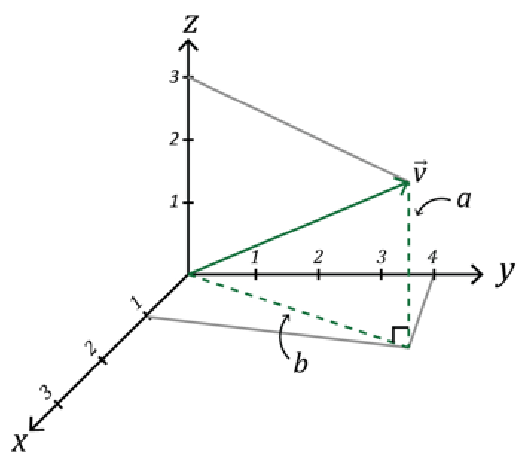
1. Define the dot product of two vectors. Note that it is just a definition, and not derived. Then follow with an example.

Find the dot product of the vectors $\vec{u} = \langle -6, 2, 4\rangle$ and $\vec{v} = \langle 3, 4, -2\rangle$.

$$\vec{u} \cdot \vec{v} = -6 \cdot 3 + \ 2 \cdot 4 + 4 \cdot (-2) = -18 + 8 - 8 = -18$$

2. Consider using an example to demonstrate why $\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$.
Show that the length of the vector $\vec{v} = \langle 1, 4, 3\rangle = \sqrt{1^2 + 4^2 + 3^2}$.

Let's start by drawing this vector.



Notice that for the vector $\vec{v}$, there is a right triangle with base b and height h.
We can use the Pythagorean Theorem to find the length of $\vec{v}$.

$$\|\vec{v}\| = \sqrt{b^2 + h^2}$$

Since the height of the triangle from the xy-plane to the tip of v is 3 units, so, $h = 3$, we have

$$\|\vec{v}\| = \sqrt{b^2 + 3^2}$$

Now b is itself the hypotenuse to its own triangle in the xy-plane. So,

$$b^2 = 1^2 + 4^2$$

Now, we have the length of $\vec{v}$.

$$\|\vec{v}\| = \sqrt{1^2 + 4^2 + 3^2}$$

In general, you can see that the length of $\vec{v}$ is the square root of the sum of the squares of $\vec{v}$'s components.

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

Consider as examples:

1. Find the dot product of the vectors $\vec{u} = \langle 2, 3, -4 \rangle$ and $\vec{v} = \langle 5, 5, 3 \rangle$.

ANS: $\vec{u} \cdot \vec{v} = 13$

2. Find the length of the vector $\vec{u} = \langle -2, -8, 5 \rangle$.

ANS: $\sqrt{93}$

3. Find the length of the vector $\vec{v} = 5\langle 4, 1, -3 \rangle - 3\langle -1, -2, 4 \rangle$.

ANS: $\sqrt{83}$

4. Find the angle between the vectors $\vec{u} = \langle 2, -6, 3 \rangle$ and $\vec{v} = \langle -3, 10, 2 \rangle$.

ANS: $143.7°$

5. Find the angle between the vectors $\vec{u} = \langle -3, -2, 5 \rangle$ and $\vec{v} = \langle 6, 4, 10 \rangle$.

ANS: $71.6°$

6. Find a vector $\vec{v}$ that is perpendicular to $\vec{u} = \langle 4, 6, 8 \rangle$.

Two vectors are perpendicular to each other if the angle between them is 90° = $\pi/2$. The cosine of the angle between two vectors is given by $\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$. We know that $\theta = \pi/2$.

The $\cos(\pi/2) = 0$ gives us $\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} = 0$

A fraction is 0 only when the numerator is 0, so we are looking for a vector $\vec{v}$ such that $\vec{u} \cdot \vec{v} = 0$. (Recall this from a previous section.)

Let the unknown vector $\vec{v} = \langle a, b, c \rangle$. Then

$$\vec{u} \cdot \vec{v} = \langle 4, 6, 8 \rangle \cdot \langle a, b, c \rangle = 0$$

$$4a + 6b + 8c = 0$$

Choose any numbers at all for a and b. Suppose $a = 5, b = 6$, then

$$4 \cdot 5 + 6 \cdot 5 + 8c = 0$$

$$20 + 30 + 8c = 0$$

$$50 + 8c = 0$$

$$8c = -50$$

$$c = -\frac{50}{8} = -\frac{25}{4}$$

So, a vector $\vec{v}$ that is perpendicular to $\vec{u} = \langle 4, 6, 8 \rangle$ is the vector $\vec{v} = \langle 5, 6, -\frac{25}{4} \rangle$.