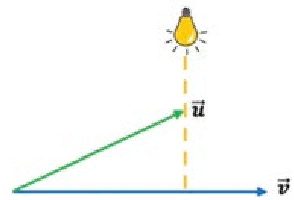
2.6 The Vector Projection of One Vector onto Another

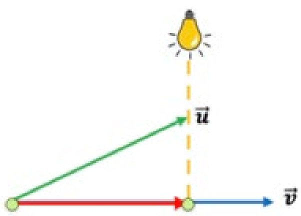
PROJECTION

Let's project vector $\vec{u} = \langle u_x, u_y \rangle$ onto the vector $\vec{v} = \langle v_x, v_y \rangle$.

To do so, imagine a light bulb above $\vec{u}$ shining perpendicular onto $\vec{v}$.



The light from the bulb will cast a shadow of $\vec{u}$ onto $\vec{v}$, and it is this shadow that we are looking for. The shadow is the projection of $\vec{u}$ onto $\vec{v}$.



The red vector is the projection of $\vec{u}$ onto $\vec{v}$. The notation commonly used to represent the projection of $\vec{u}$ onto $\vec{v}$ is $\text{proj}_{\vec{v}}\vec{u}$.

Vector parallel to $\vec{v}$ with magnitude $\frac{\vec{u}\cdot\vec{v}}{||\vec{v}||}$ in the direction of $\vec{v}$ is called projection of $\vec{u}$ onto $\vec{v}$.

The formula for $\text{proj}_{\vec{v}}\vec{u}$ is

$$\boxed{\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{||\vec{v}||^2}\vec{v}}$$

Example (1)

To find the projection of $\vec{u} = \langle 4, 3 \rangle$ onto $\vec{v} = \langle 2, 8 \rangle$, we need to compute both the dot product of $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{v}$, then apply the formula.

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}$$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{\langle 4, 3 \rangle \cdot \langle 2, 8 \rangle}{\|\langle 2, 8 \rangle\|^2}\langle 2, 8 \rangle$$

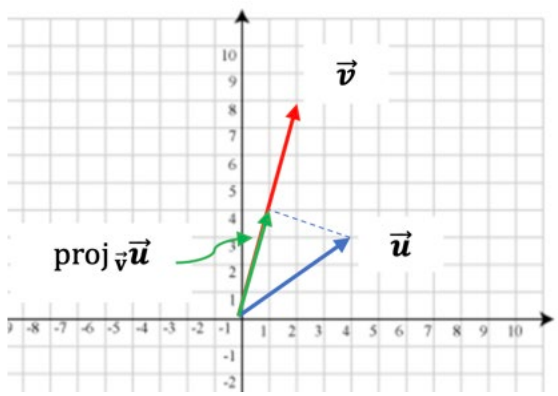
$$\text{proj}_{\vec{v}}\vec{u} = \frac{4 \cdot 2 + 3 \cdot 8}{\left(\sqrt{2^2 + 8^2}\right)^2}\langle 2, 8 \rangle$$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{32}{\left(\sqrt{4 + 64}\right)^2}\langle 2, 8 \rangle$$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{32}{68}\langle 2, 8 \rangle$$

$$\text{proj}_{\vec{v}}\vec{u} = \frac{8}{17}\langle 2, 8 \rangle$$

$$\text{proj}_{\vec{v}}\vec{u} = \left\langle \frac{16}{17}, \frac{64}{17} \right\rangle$$

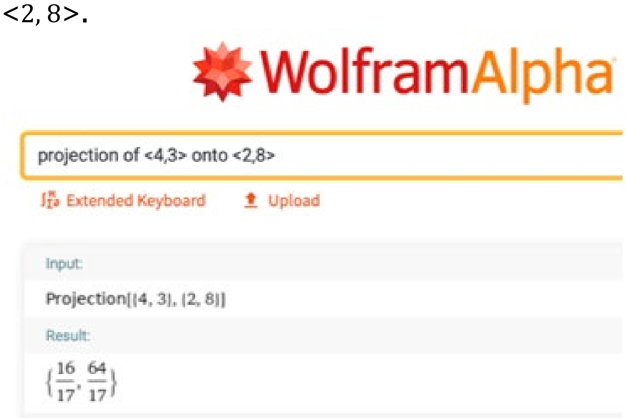


USING TECHNOLOGY

We can use technology to determine the projection of one vector onto another.

Go to www.wolframalpha.com.

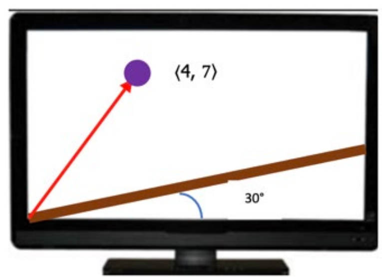
To find the projection of $\vec{u} = \langle 4, 3 \rangle$ onto $\vec{v} = \langle 2, 8 \rangle$, use the "projection" command. In the entry field enter projection of $<4, 3>$ onto $<2, 8>$.



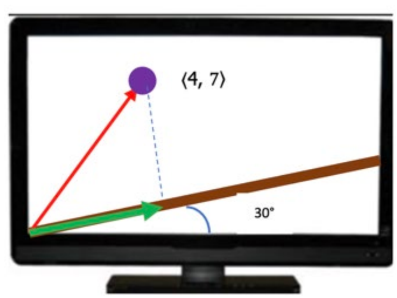
Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\left\langle \frac{16}{17}, \frac{64}{17} \right\rangle$.

Example (2)

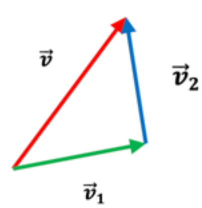
As an applied example, suppose a video game has a ball moving near a wall.



We take the origin at the bottom-left-most corner of the screen. The wall is at a 30° angle to the horizontal, and at a point in time, the ball is at position $\vec{v} = \langle 4, 7 \rangle$. To find the perpendicular distance from the ball to the wall, we use the projection formula to project the vector $\vec{v} = \langle 4, 7 \rangle$ onto the wall.

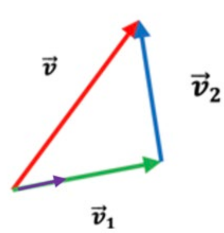


We begin by decomposing $\vec{v}$ into two vectors $\vec{v}_1$ and $\vec{v}_2$ so that $\vec{v} = \vec{v}_1 + \vec{v}_2$ and $\vec{v}_1$ lies along the wall**.**

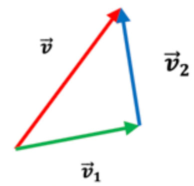


The length (magnitude) of the vector $\vec{v}$ is then the distance from the ball to the wall.

The vector $\vec{v}_1$ is the projection of $\vec{v}$ onto the wall. We can get $\vec{v}_1$ by scaling (multiplying) a unit vector $\vec{w}$ that lies along the wall and, thus, along with $\vec{v}_1$.



Since $\vec{w}$ lies at a 30° angle to the horizontal, $\vec{w} = \langle \cos 30°,\ \sin 30° \rangle = \langle 0.866, 0.5 \rangle$, using the projection formula, we get the projection of $\vec{v}$ that lies along the wall.



$$\vec{v}_1 = \mathrm{proj}_{\vec{w}}\vec{v} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w}$$

$$\vec{v}_1 \frac{\langle 4, 7 \rangle \cdot \langle 0.866, 0.5 \rangle}{\left\| \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle \right\|^2} \langle 0.866, 0.5 \rangle$$

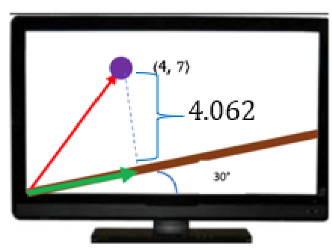
$$= \frac{4 \cdot (0.866) + 7 \cdot (0.5)}{\left(\sqrt{(0.866)^2 + (.5)^2}\right)^2} \langle 0.866, 0.5 \rangle$$

$$\vec{v}_1 = \frac{6.964}{\left(\sqrt{1}\right)^2} \langle 0.866, 0.5 \rangle$$

$$\vec{v}_1 = (6.964)\langle 0.866, 0.5 \rangle$$

$$\vec{v}_1 = \langle 6.031, 3.482 \rangle$$

Since that $\boldsymbol{\vec{v}} = \boldsymbol{\vec{v}_1} + \boldsymbol{\vec{v}_2}$, subtraction get us



$$\boldsymbol{\vec{v}_2} = \boldsymbol{\vec{v}} - \boldsymbol{\vec{v}_1}$$

$$\boldsymbol{\vec{v}_2} = \langle 4, 7 \rangle - \langle 6.031, 3.482 \rangle$$

$$\boldsymbol{\vec{v}_2} = \langle 4 - 6.031, 7 - 3.482 \rangle$$

$$\boldsymbol{\vec{v}_2} = \langle -2.031, 3.518 \rangle$$

To get the magnitude of $\boldsymbol{\vec{v}_2}$, we use

$$\|\boldsymbol{\vec{v}_2}\| = \sqrt{{v_x}^2 + {v_y}^2}$$

$$\|\boldsymbol{\vec{v}_2}\| = \sqrt{(-2.031)^2 + 3.518^2}$$

$$\|\boldsymbol{\vec{v}_2}\| = \sqrt{4.125 + 12.376}$$

$$\left\|\overrightarrow{\boldsymbol{v}_2}\right\| = 4.062$$

TRY THESE

1. Find the projection of the vector $\vec{v} = \langle 3, 5 \rangle$ onto the vector $\vec{u} = \langle 6, 2 \rangle$.

ANS: $\left\langle \frac{21}{5}, \frac{7}{5} \right\rangle$

2. Find $\mathrm{proj}_{\vec{v}}\vec{u}$, where of $\vec{u} = \langle -2, 5 \rangle$ onto $\vec{v} = \langle 6, -5 \rangle$.

ANS: $\left\langle \frac{-222}{61}, \frac{185}{61} \right\rangle$

NOTE TO INSTRUCTOR

When presenting the projection formula, consider pointing out that the numerator is a dot product and a scalar (a real number). The denominator is a length, so it, too, is a scalar. A scalar divided by a scalar is also a scalar, so the formula shows a vector is multiplied by a scalar. That is, it shows a vector scaled longer or shorter. That scaled vector is the projection.

Consider working through these problems as examples.

1. Find the projection of the vector $\vec{v} = \langle 1, 4 \rangle$ onto the vector $\vec{u} = \langle 2, 5 \rangle$.

ANS: $\left\langle \frac{44}{29}, \frac{110}{29} \right\rangle$

2. Find $\mathrm{proj}_{\vec{v}}\vec{u}$, where of $\vec{u} = \langle 2, -4 \rangle$ onto $\vec{v} = \langle 5, 5 \rangle$.

ANS: $\langle -1, -1 \rangle$

3. Find $\mathrm{proj}_{\vec{v}}\vec{u}$, where of $\vec{u} = \langle 5, 10 \rangle$ onto $\vec{v} = \langle 6, -3 \rangle$.

ANS: $\langle 0, 0 \rangle$ These two vectors are orthogonal (perpendicular to each other).