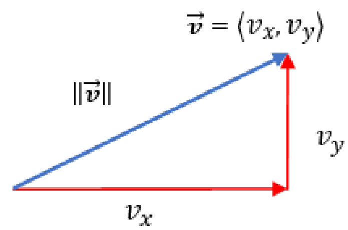
2.3 Magnitude, Direction, and Components of a Vector

THE MAGNITUDE OF A VECTOR

It is productive to represent the horizontal and vertical components of a vector $\vec{v}$ as v_x and v_y, respectively.

The magnitude (the length) of a vector $\vec{v} = \langle v_x, v_y \rangle$ is

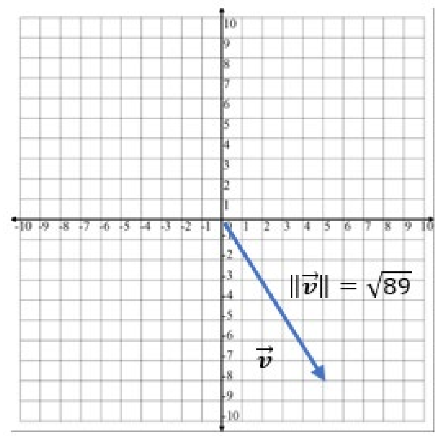
$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$



The vector $\vec{v} = \langle 5, -8 \rangle$ has magnitude:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$
$$= \sqrt{5^2 + (-8)^2} = \sqrt{25 + 64} = \sqrt{89}$$

Interpret this as the length of the vector $\vec{v} = \langle 5, -8 \rangle$ is $\sqrt{89}$ units.



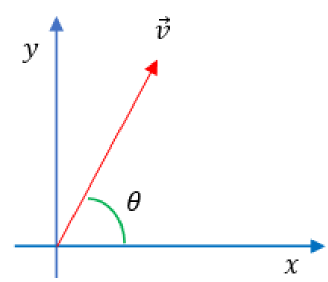
THE DIRECTION OF A VECTOR

The direction of a vector $\vec{v}$ is the angle the vector makes with the positive x-axis.

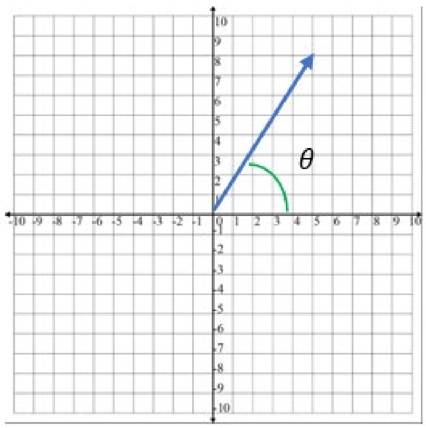
It is typically represented with the uppercase Greek letter theta θ. We use some trigonometry to determine the angle θ.

$\tan\theta = \frac{y}{x}$ or $\theta = \tan^{-1}\frac{y}{x}$

The angle θ is always between 0° and 360°.



To approximate the direction of the vector $\vec{v} = \langle 5, 8\rangle$, use $\theta = \tan^{-1}\frac{y}{x}$, with $x = 5$ and $y = 8$.



$$\vec{v} = \langle 5, 8\rangle$$

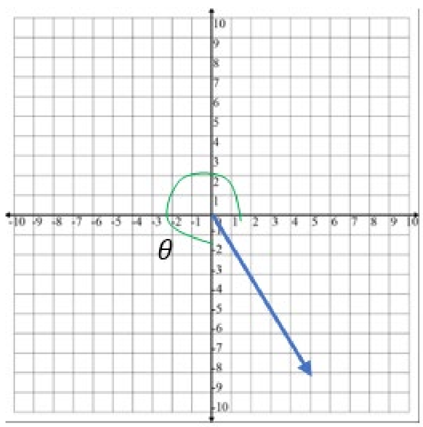
$$\theta = \tan^{-1}\frac{y}{x}$$

$$\theta = \tan^{-1}\frac{8}{5}$$

Using a calculator, we get

$\theta = 57.99°$
$\theta = 58°$

To approximate the direction of the vector $\vec{v} = \langle 5, -8\rangle$, use $\theta = \tan^{-1}\frac{y}{x}$, with $x = 5$ and $y = -8$.



$$\vec{v} = \langle 5, -8\rangle$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\theta = \tan^{-1}\frac{-8}{5}$$

Using a calculator, we get

$\theta = -57.99°$

Vertical component is in Quadrant IV and θ must be in the interval $[0, 360)$, therefore we calculate θ by

$$\theta = 360° - 57.99° = 302.005°$$

$$\theta = 302°.$$

THE COMPONENTS OF A VECTOR

The lengths of the x- and y- components of a vector $\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$ in two dimensions can be found using trigonometric ratios.

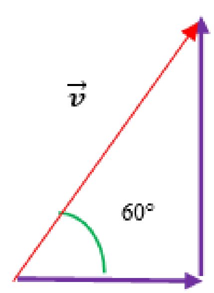
$$\vec{v}_x = \|\vec{v}\|\cos\theta \quad \text{and} \quad \vec{v}_y = \|\vec{v}\|\sin\theta$$

$\vec{v}_x$ is the horizontal component of $\vec{v}$ and $\vec{v}_y$ is the vertical component.

The angle θ is always between 0° and 360°.

Suppose the magnitude of a vector $\vec{v} = \langle v_x, v_y \rangle$ is 20 units, and that $\vec{v}$ makes a 60° angle with the horizontal. Then, the components of $\vec{v}$ are

$$\begin{aligned} \vec{v}_x &= \|\vec{v}\|\cos\theta \\ &= 20\cos 60° \\ &= 20 \cdot \frac{1}{2} \\ &= 10 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \vec{v}_y &= \|\vec{v}\|\sin\theta \\ &= 20\sin 60° \\ &= 20 \cdot \frac{\sqrt{3}}{2} \\ &= 10\sqrt{3} \end{aligned}$$



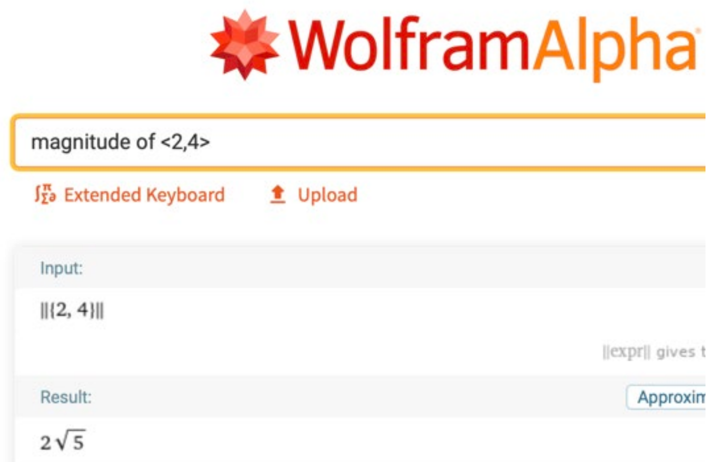
So, we could write $\vec{v} = \langle v_x, v_y \rangle$ as $\vec{v} = \langle 10, 10\sqrt{3} \rangle$

USING TECHNOLOGY

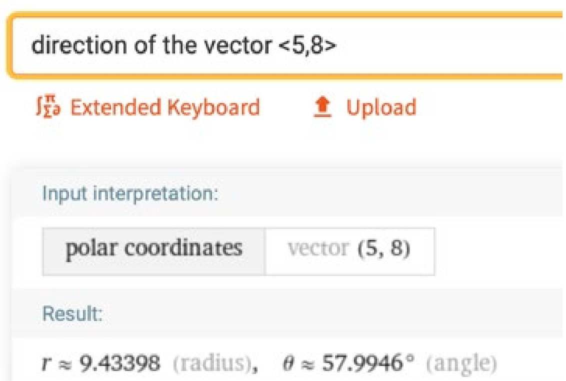
We can use technology to determine the magnitude of a vector.

Go to www.wolframalpha.com.

To find the magnitude of the vector $\vec{v} = \langle 2, 4 \rangle$, enter magnitude of <2,4> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\|\vec{v}\| = 2\sqrt{5}$.



To find the direction of the vector $\vec{v} = \langle 5, 8 \rangle$, enter direction of the vector <5,8> in the entry field. Wolframalpha answers $57.9946° \approx 58°$.



TRY THESE

1. Find the magnitude of the vector $\vec{v} = \langle 3, -4 \rangle$.

ANS: $||\vec{v}|| = 5$

2. Find the magnitude of the vector $\vec{v} = \langle -3, -3 \rangle$.

ANS: $||\vec{v}|| = 3\sqrt{2}$

3. Find the components of the vector $\vec{v}$ if the magnitude of $\vec{v}$ is 6 and it makes a 30° angle with the horizontal.

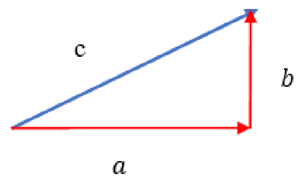
ANS: $\vec{v}_x = 3\sqrt{3}$ and $\vec{v}_y = 3$

4. Approximate the direction of the vector $\vec{v} = \langle 3, 10 \rangle$.

ANS: $\theta \approx 73.3008°$

NOTE TO INSTRUCTOR

1. Remind students of the Pythagorean Theorem.



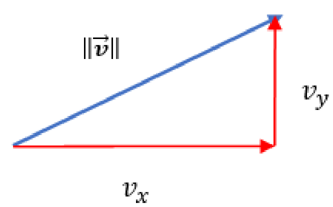
$$a^2 + b^2 = c^2$$

2. Consider deriving the magnitude of a vector $\vec{v} = \langle v_x, v_y \rangle$ using the Pythagorean Theorem. Note that the v_x and the v_y in $\langle v_x, v_y \rangle$ represent the lengths of the horizontal and vertical components, respectively, of $\vec{v}$.

$$||\vec{v}||^2 = {v_x}^2 + {v_y}^2$$

$$\sqrt{||\vec{v}||^2} = \sqrt{{v_x}^2 + {v_y}^2}$$

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$



Use as an example of the vector $\vec{v} = \langle 6, 3 \rangle$. The magnitude of $\vec{v} = \langle 6, 3 \rangle$ is

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$
$$||\vec{v}|| = \sqrt{6^2 + 3^2}$$
$$||\vec{v}|| = \sqrt{36 + 9}$$
$$||\vec{v}|| = \sqrt{45}$$
$$||\vec{v}|| = \sqrt{9 \cdot 5}$$
$$||\vec{v}|| = \sqrt{9} \cdot \sqrt{5}$$
$$||\vec{v}|| = 3\sqrt{5}$$

3. Demonstrate how to find the magnitude of $\vec{v} = \langle -5, 4 \rangle$.

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$
$$||\vec{v}|| = \sqrt{(-5)^2 + 4^2}$$
$$||\vec{v}|| = \sqrt{25 + 16}$$
$$||\vec{v}|| = \sqrt{41}$$

4. Find the components of the vector $\vec{v}$ if the magnitude of $\vec{v}$ is 7 and it makes a 30° angle with the horizontal.

$$\begin{aligned}\vec{v}_x &= \|\vec{v}\|\cos\theta \\ &= 7\cos 30° \\ &= 7\cdot\frac{\sqrt{3}}{2} \\ &= \frac{7\sqrt{3}}{2}\end{aligned} \qquad \begin{aligned}\vec{v}_y &= \|\vec{v}\|\sin\theta \\ &= 7\sin 30° \\ &= 7\cdot\frac{1}{2} \\ &= \frac{7}{2}\end{aligned}$$

So, $\vec{v}_x = \frac{7\sqrt{3}}{2}$ and $\vec{v}_y = \frac{7}{2}$

5. Approximate the components of the vector $\vec{v}$ if the magnitude of $\vec{v}$ is 16 and it makes a 128° angle with the horizontal.

$$\begin{aligned}\vec{v}_x &= \|\vec{v}\|\cos\theta \\ &= 16\cos 128° \\ &\approx 16\cdot(-0.616) \\ &\approx -9.86\end{aligned} \qquad \begin{aligned}\vec{v}_y &= \|\vec{v}\|\sin\theta \\ &= 16\sin 128° \\ &\approx 16\cdot(0.788) \\ &\approx 12.61\end{aligned}$$

So, $\vec{v}_x = -9.86$ and $\vec{v}_y = 12.61$

6. Approximate the direction of the vector $\vec{v} = \langle 2, 7\rangle$.

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\theta = \tan^{-1}\frac{7}{2}$$

Using a calculator, we get

$$\theta = 74.0546041°$$
$$\theta = 74.05°$$