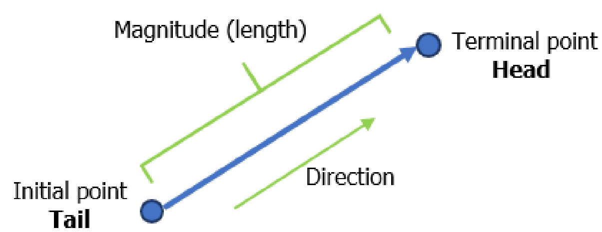
UNIT 2 VECTORS IN TWO DIMENSIONS

2.1 Vectors

Vectors are fundamental objects in applied mathematics; they efficiently convey information about a mathematical or physical object. Let's get a sense of what they are.

A **VECTOR** is a representation of an object that has both direction and magnitude. By direction, we mean the place toward which something faces, and by magnitude, we mean the size of something.

A vector can be depicted visually by an arrow, with an initial point called the tail and a terminal point called the head. The length of the arrow represents the vector's magnitude.

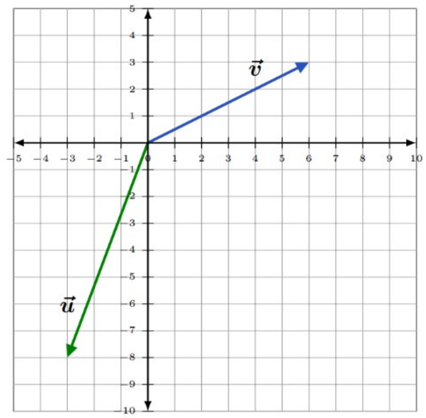


Vectors are often named using a bold-typed letter with an arrow on top of it. For example, the vector in the picture could be named $\vec{V}$ or $\vec{v}$.

An example of a vector is a car’s velocity. Velocity is a vector since it has both magnitude (speed) and direction. A car might be moving west at 60 mph. Other examples of vectors are displacement, acceleration, and force.

The temperature of some medium is not a vector since it has only magnitude. But if the medium is being heated, its temperature is increasing and has a direction; it is going upward. The increase or decrease in temperature is a vector.

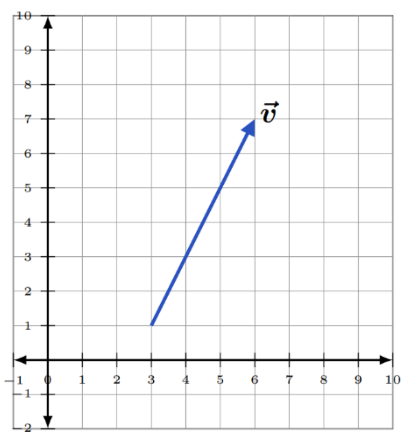
VECTORS IN STANDARD POSITION



A vector with its initial point at the origin in a Cartesian coordinate system is said to be in STANDARD POSITION. The vector $\vec{v}$ in the diagram has its initial point at the origin $(0,0)$, and its terminal point at $(6,3)$.

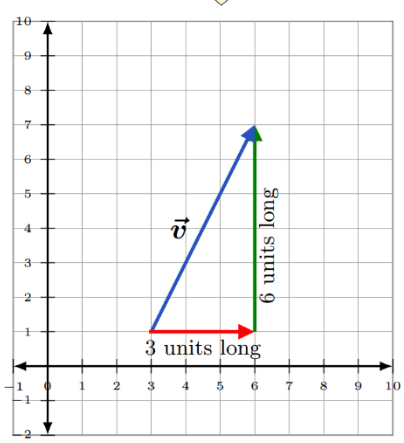
COMPONENTS OF A VECTOR

Vectors in the xy-plane can be broken into their **horizontal** and **vertical** components.



For example, the vector $\vec{v}$ in the diagram can be broken into two components,

1. its horizontal, or x-component, and
2. its vertical, or y-component.



The vector $\vec{v}$ in component form is expressed using angle brackets as $\vec{v} = \langle 3, 6 \rangle$, where

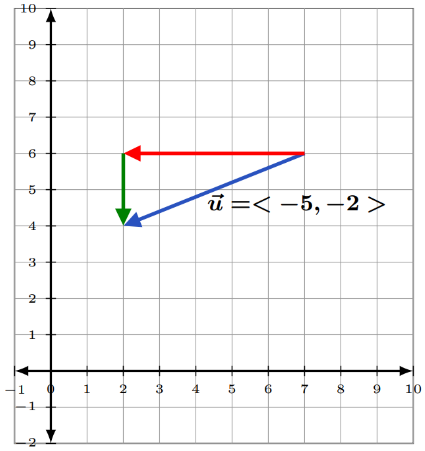
1. the first component, 3 is the length and direction of its x-component, and
2. the second component, 6 is the length and direction of its y-component.

The vector $\vec{u}$ in the picture below has

FIRST COMPONENT = (terminal x-value) – (initial x-value) $= 2 - 7 = -5$, and

SECOND COMPONENT = (terminal y-value) – (initial y-value) $= 4 - 6 = -2$,

so that $\vec{u} = \langle -5, -2 \rangle$.



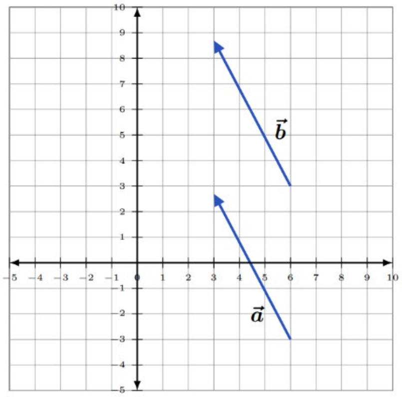
ROW AND COLUMN FORMS OF A VECTOR

Vectors are represented by a single column matrix or a single row matrix. The vectors $\vec{v} = \langle 3, 6 \rangle$, and $\vec{u} = \langle -5, -2 \rangle$ above, can be represented by the 2x1 row matrix and the 1x2 column matrix, respectively as

$$\vec{v} = \begin{bmatrix} 3 & 6 \end{bmatrix} \text{ and } \vec{u} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

EQUAL VECTORS

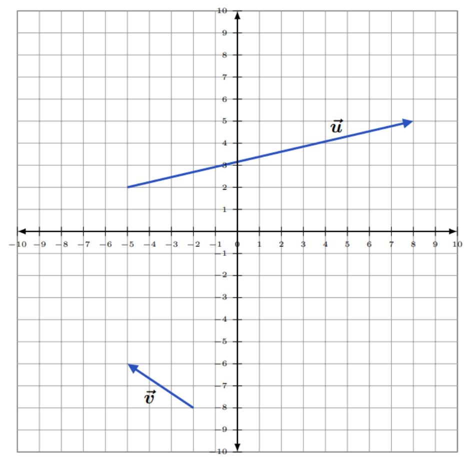
Two vectors are EQUAL if they have the same direction and magnitude. They may start and end at different positions, but their representing arrows will be parallel.



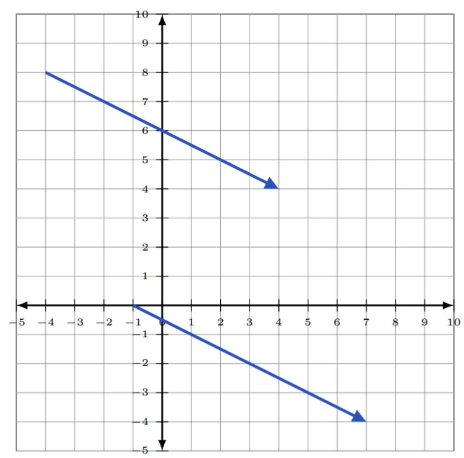
In the diagram vectors $\vec{a}$ and $\vec{b}$ are equal but appear in different locations in the xy-plane.

2.1 TRY THESE

1. Express the vectors $\vec{v}$ and $\vec{u}$ in component form.



2. Explain why the two vectors are equal.



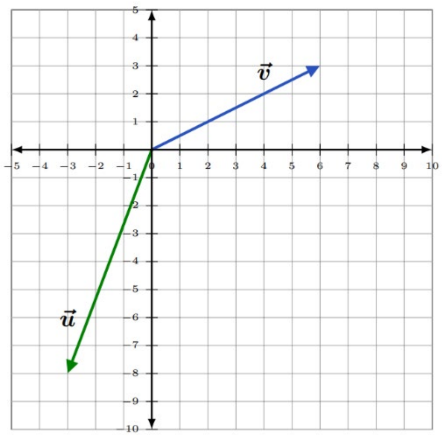
2.2 Addition, Subtraction, and Scalar Multiplication of Vectors

ADDITION & SUBTRACTION OF VECTORS

To add or subtract two vectors, add, or subtract their corresponding components.

Example (1)

To **ADD** the vectors $\vec{u}$ and $\vec{v}$, begin by writing each in component form.



$\vec{u} = \langle -3, -8 \rangle$ and $\vec{v} = \langle 6, 3 \rangle$

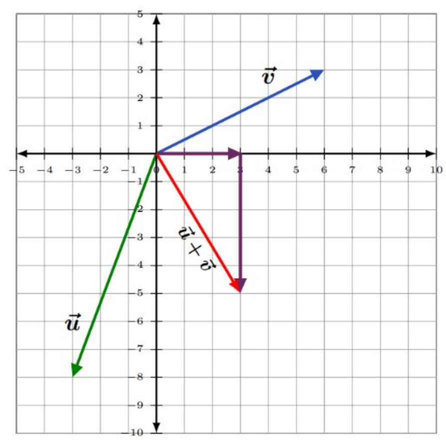
ADD their corresponding components.

$$\vec{u} + \vec{v} = \langle -3 + 6, -8 + 3 \rangle = \langle 3, -5 \rangle$$

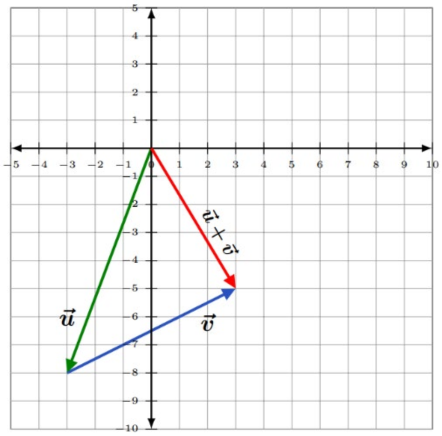
So, $\vec{u} + \vec{v} = \langle 3, -5 \rangle$

Now, graph this sum.

- Start at the origin.
- Since the <u>horizontal component</u> is 3, move 3 units to the *right*.
- Since the <u>vertical component</u> is -5, move 5 units *downward*.

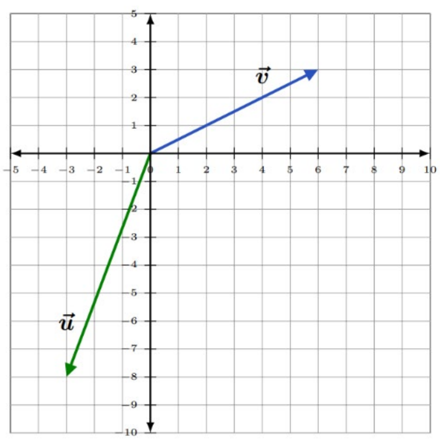


The addition of two vectors $\vec{u}$ and $\vec{v}$ can be demonstrated by placing the tail of one vector at the head of the other. Then connect the tail of $\vec{u}$ to the head of $\vec{v}$.



Example (2)

To **SUBTRACT** the vector $\vec{u}$ from the vector $\vec{v}$, begin by writing each in component form.



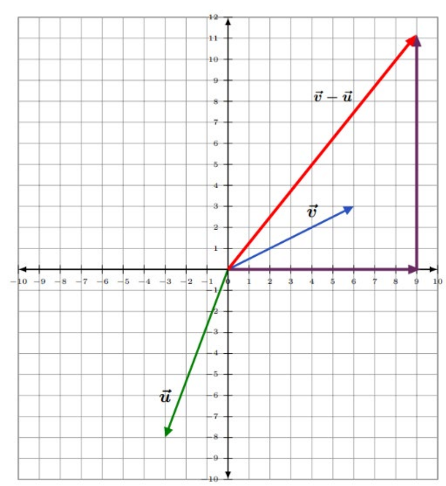
$\vec{u} = \langle -3, -8 \rangle$ and $\vec{v} = \langle 6, 3 \rangle$

SUBTRACT the components of $\vec{u}$ from the corresponding components of $\vec{v}$.

$$\vec{v} - \vec{u} = \langle 6-(-3), 3-(-8) \rangle = \langle 6+3, 3+8 \rangle = \langle 9, 11 \rangle$$

So, $\vec{v} - \vec{u} = \langle 9, 11 \rangle$

Now, graph this sum.
- Start at the origin.
- Since the horizontal component is 9, move 9 units to the *right*.
- Since the vertical component is 11, move 11 units *upward*.



SCALARS

In contrast to a vector, and having both direction and magnitude, a SCALAR is a physical quantity defined by only its magnitude.

Examples are speed, time, distance, density, and temperature. They are represented by real numbers (both positive and negative), and they can be operated on using the regular laws of algebra.

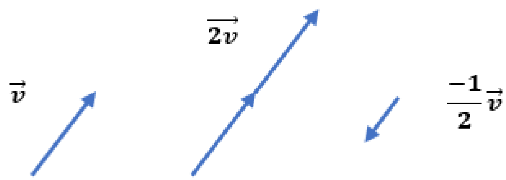
The term *scalar* derives from this usage: *a scalar is that which scales, resizes a vector*.

Scalar multiplication is the multiplication of a vector by a real number (a scalar).

Suppose we let the letter k represent a real number and $\vec{v}$ be the vector $\langle x, y\rangle$. Then, the scalar multiple of the vector $\vec{v}$ is

$$k\vec{v} = \langle kx, ky\rangle$$

To multiply a vector by a scalar (a constant), multiply each of its components by the constant.



1. Suppose $\vec{u} = \langle -3, -8\rangle$ and $k = 3$.

   Then $k\vec{u} = 3\vec{u} = 3\langle -3, -8\rangle = \langle 3(-3), 3(-8)\rangle = \langle -9, -24\rangle$

2. Suppose $\vec{v} = \langle 6, 3\rangle$ and $k = \frac{-1}{3}$.

   Then $k\vec{u} = \frac{-1}{3}\vec{u} = \frac{-1}{3}\langle 6, 3\rangle = \left\langle \frac{-1}{3}(6), \frac{-1}{3}(3)\right\rangle = \langle -2, -1\rangle$

3. Suppose $\vec{u} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

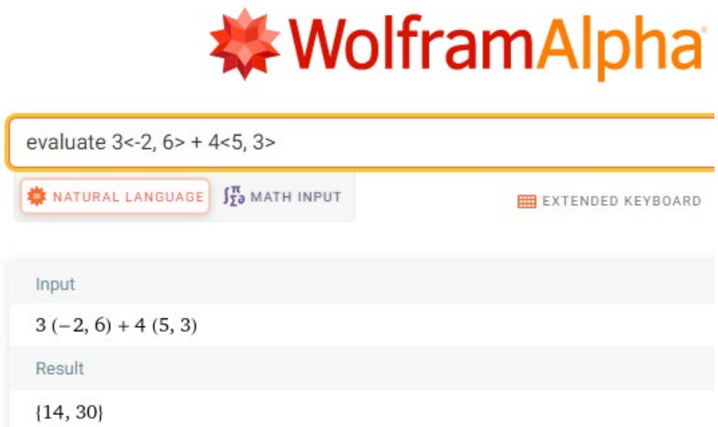
   Then $3\vec{u} + 4\vec{v} = 3\begin{bmatrix} -2 \\ 6 \end{bmatrix} + 4\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 18 \end{bmatrix} + \begin{bmatrix} 20 \\ 12 \end{bmatrix} = \begin{bmatrix} 14 \\ 30 \end{bmatrix}$

USING TECHNOLOGY

We can use technology to add and subtract vectors and to multiply a vector by a scalar.

Go to www.wolframalpha.com.

For the vectors $\vec{u} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, use WolframAlpha to find $3\vec{u} + 4\vec{v}$. Enter evaluate 3<-2, 6> + 4<5, 3> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $< 14, 30 >$.



2.2 TRY THESE

1. Find the sum of the two vectors $\vec{u} = \langle -5, 2 \rangle$ and $\vec{v} = \langle 10, -1 \rangle$.

2. Subtract the vector $\vec{u} = \langle -5, 2 \rangle$ from the vector $\vec{v} = \langle 10, -1 \rangle$.

3. Suppose $\vec{u} = \langle -5, 2 \rangle$, $\vec{v} = \langle 1, 6 \rangle$, and $\vec{w} = \langle 4, -3 \rangle$. Perform the operation $2\vec{u} - 4\vec{v} + 3\vec{w}$.

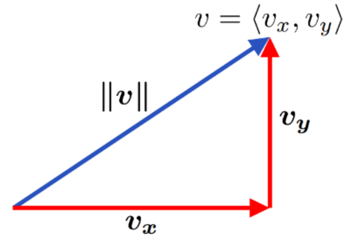
2.3 Magnitude, Direction, and Components of a Vector

THE MAGNITUDE OF A VECTOR

It is productive to represent the horizontal and vertical components of a vector $\vec{v}$ as v_x and v_y, respectively.

The magnitude (the length) of a vector $\vec{v} = \langle v_x, v_y \rangle$ is

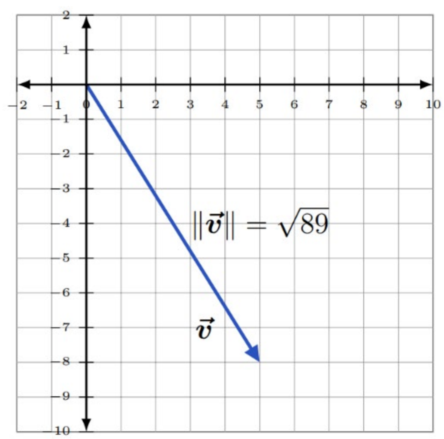
$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$



The vector $\vec{v} = \langle 5, -8 \rangle$ has magnitude:

$$\begin{aligned} \|\vec{v}\| &= \sqrt{{v_x}^2 + {v_y}^2} \\ &= \sqrt{5^2 + (-8)^2} = \sqrt{25 + 64} = \sqrt{89} \end{aligned}$$

Interpret this as the length of the vector $\vec{v} = \langle 5, -8 \rangle$ is $\sqrt{89}$ units.



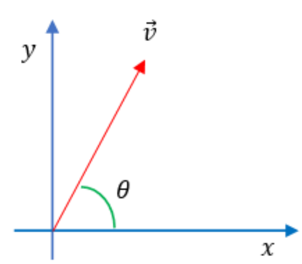
THE DIRECTION OF A VECTOR

The direction of a vector $\vec{v}$ is the angle the vector makes with the positive x-axis.

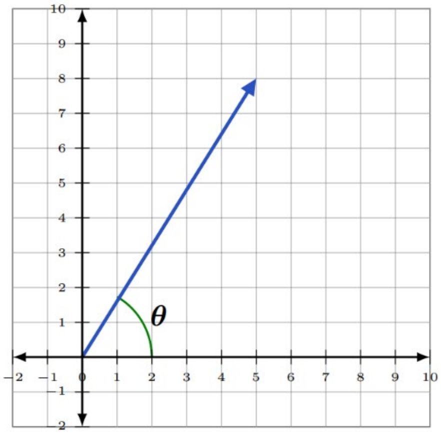
It is typically represented with the uppercase Greek letter theta θ. We use some trigonometry to determine the angle θ.

$$\tan\theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1}\frac{y}{x}$$

The angle θ is always between 0° and 360°.



To approximate the direction of the vector $\vec{v} = \langle 5, 8 \rangle$, use $\theta = \tan^{-1}\frac{y}{x}$, with $x = 5$ and $y = 8$.



$$\vec{v} = \langle 5, 8 \rangle$$

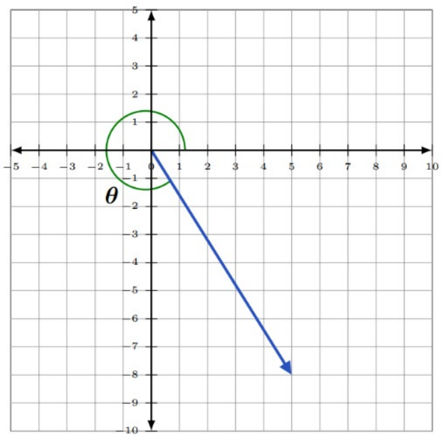
$$\theta = \tan^{-1}\frac{y}{x}$$

$$\theta = \tan^{-1}\frac{8}{5}$$

Using a calculator, we get

$\theta = 57.99°$
$\theta = 58°$

To approximate the direction of the vector $\vec{v} = \langle 5, -8 \rangle$, use $\theta = \tan^{-1}\frac{y}{x}$, with $x = 5$ and $y = -8$.



$$\vec{v} = \langle 5, -8 \rangle$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\theta = \tan^{-1}\frac{-8}{5}$$

Using a calculator, we get

$\theta = -57.99°$

Vertical component is in Quadrant IV and θ must be in the interval $[0, 360)$, therefore we calculate θ by

$$\theta = 360° - 57.99° = 302.005°$$

$$\theta = 302°.$$

THE COMPONENTS OF A VECTOR

The lengths of the x- and y- components of a vector $\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$ in two dimensions can be found using trigonometric ratios.

$\vec{v}_x = \|\vec{v}\|\cos\theta$ and $\vec{v}_y = \|\vec{v}\|\sin\theta$

$\vec{v}_x$ is the horizontal component of $\vec{v}$ and $\vec{v}_y$ is the vertical component.

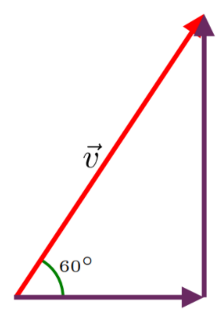
The angle θ is always between 0° and 360°.

Suppose the magnitude of a vector $\vec{v} = \langle v_x, v_y \rangle$ is 20 units, and that $\vec{v}$ makes a 60° angle with the horizontal. Then, the components of $\vec{v}$ are

$$\vec{v}_x = \|\vec{v}\|\cos\theta$$
$$= 20\cos 60°$$
$$= 20 \cdot \frac{1}{2}$$
$$= 10$$

and

$$\vec{v}_y = \|\vec{v}\|\sin\theta$$
$$= 20\sin 60°$$
$$= 20 \cdot \frac{\sqrt{3}}{2}$$
$$= 10\sqrt{3}$$



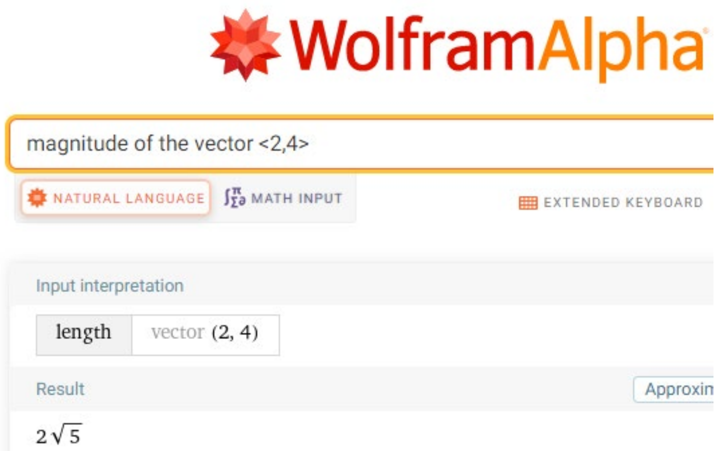
So, we could write $\vec{v} = \langle v_x, v_y \rangle$ as $\vec{v} = \langle 10, 10\sqrt{3} \rangle$

USING TECHNOLOGY

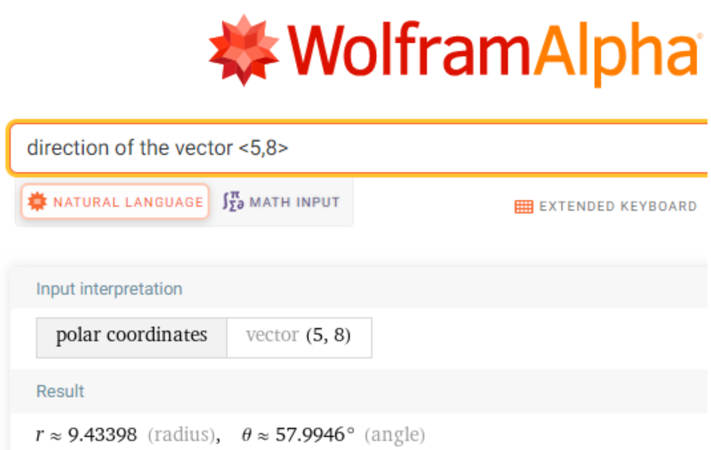
We can use technology to determine the magnitude of a vector.

Go to www.wolframalpha.com.

To find the magnitude of the vector $\vec{v} = \langle 2, 4 \rangle$, enter magnitude of the vector <2,4> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\|\vec{v}\| = 2\sqrt{5}$.



To find the direction of the vector $\vec{v} = \langle 5, 8 \rangle$, enter direction of the vector <5,8> in the entry field. Wolframalpha answers $57.9946° \approx 58°$.



2.3 TRY THESE

1. Find the magnitude of the vector $\vec{v} = \langle 3, -4 \rangle$.

2. Find the magnitude of the vector $\vec{v} = \langle -3, -3 \rangle$.

3. Find the components of the vector $\vec{v}$ if the magnitude of $\vec{v}$ is 6 and it makes a 30° angle with the horizontal.

4. Approximate the direction of the vector $\vec{v} = \langle 3, 10 \rangle$.

2.4 The Dot Product of Two Vectors, the Length of a Vector, and the Angle Between Two Vectors

THE DOT PRODUCT OF TWO VECTORS

The length of a vector or the angle between two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ can be found using the dot product.

The dot product of vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ is a scalar (real number) and is defined to be

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y$$

Since u_x, u_y, v_x and v_y are real numbers, you can see that the dot product is itself a real number and not a vector.

Example (1)

To compute the dot product of the vectors $\vec{u} = \langle 5, 2 \rangle$ and $\vec{v} = \langle 3, 4 \rangle$, we compute

$$\vec{u} \cdot \vec{v} = 5 \cdot 3 + 2 \cdot 4 = 15 + 8 = 23$$

Since the dot product is a scalar, it follows the properties of real numbers.

PROPERTIES OF THE DOT PRODUCT

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, the dot product is commutative
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, the dot product distributes over vector addition
3. $\vec{u} \cdot \vec{0} = 0$, the dot product with the zero vector $\vec{0}$, is the scalar 0.
4. $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$

Example (2)

Compute the dot product $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$,
where $\vec{u} = \langle 5, -2 \rangle$, $\vec{v} = \langle 6, 4 \rangle$, and $\vec{w} = \langle -3, 7 \rangle$.

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$\begin{aligned} \vec{u} \cdot (\vec{v} + \vec{w}) &= \langle 5, -2 \rangle \cdot \langle 6, 4 \rangle + \langle 5, -2 \rangle \cdot \langle -3, 7 \rangle \\ &= (5 \cdot 6 + (-2) \cdot 4) + (5 \cdot (-3) + (-2) \cdot 7) \\ &= 30 - 8 - 15 - 14 \\ &= -7 \end{aligned}$$

THE LENGTH OF A VECTOR

The length (magnitude) of a vector you know is given by $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$. The length can also be found using the dot product. If we dot a vector $\vec{v} = \langle v_x, v_y \rangle$ with itself, we get

$$\vec{v} \cdot \vec{v} = \langle v_x, v_y \rangle \cdot \langle v_x, v_y \rangle$$
$$\vec{v} \cdot \vec{v} = v_x \cdot v_x + v_y \cdot v_y$$
$$\vec{v} \cdot \vec{v} = {v_x}^2 + {v_y}^2$$

By Vector Property 4, $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$. This gives $||\vec{v}||^2 = {v_x}^2 + {v_y}^2$.

Taking the square root of each side produces

$$\sqrt{||\vec{v}||^2} = \sqrt{{v_x}^2 + {v_y}^2}$$

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$

Which is the length of the vector $\vec{v}$.

The dot product of a vector $\vec{v} = \langle v_x, v_y \rangle$ with itself gives the length of the vector.

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$

Example (3)

Use the dot product to find the length of the vector $\vec{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.
In this case, $v_x = 2$ and $v_y = 6$.

Using $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$, we get

$$||\vec{v}|| = \sqrt{2^2 + 6^2}$$

$$||\vec{v}|| = \sqrt{40}$$

$$||\vec{v}|| = \sqrt{4 \cdot 10}$$

$$||\vec{v}|| = \sqrt{4} \cdot \sqrt{10}$$

$$||\vec{v}|| = 2\sqrt{10}$$

The length of the vector $\vec{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is $2\sqrt{10}$ units.

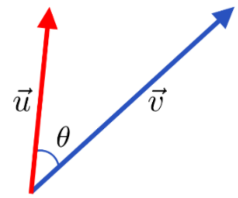
THE ANGLE BETWEEN TWO VECTORS

The dot product and elementary trigonometry can be used to find the angle θ between two vectors.

If θ is the smallest nonnegative angle between two non-zero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||} \text{ or } \theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||}$$

where $0 \le \theta \le 2\pi$ and $||\vec{u}|| = \sqrt{{u_x}^2 + {u_y}^2}$ and $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$



Example (4)

Find the angle between the vectors $\vec{u} = \langle 5, -3\rangle$ and $\vec{v} = \langle 2, 4\rangle$.

Using $\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||}$, we get

$$\theta = \cos^{-1}\frac{\langle 5, -3\rangle \cdot \langle 2, 4\rangle}{\sqrt{5^2 + (-3)^2}\cdot\sqrt{2^2 + 4^2}}$$

$$\theta = \cos^{-1}\frac{5\cdot 2 + (-3)\cdot 4}{\sqrt{25+9}\cdot\sqrt{4+16}}$$

$$\theta = \cos^{-1}\frac{-2}{\sqrt{34}\cdot\sqrt{20}}$$

$$\theta = 94.4$$

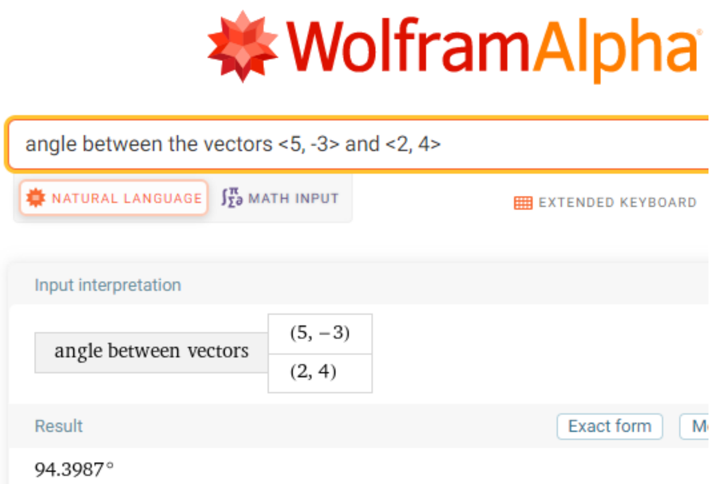
We conclude that the angle between these two vectors is close to 94.4°.

USING TECHNOLOGY

We can use technology to find the angle θ between two vectors.

Go to www.wolframalpha.com.

To find the angle between the vectors $\vec{u} = \langle 5, -3 \rangle$ and $\vec{v} = \langle 2, 4 \rangle$, enter angle between the vectors <5, -3> and <2, 4> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\theta = 94.4$, rounded to one decimal place.



2.4 TRY THESE

1. Find the dot product of the vectors $\vec{u} = \langle -2, 3\rangle$ and $\vec{v} = \langle 5, -1\rangle$.

2. Find the dot product of the vectors $\vec{u} = \langle -4, 6\rangle$ and $\vec{v} = \langle 3, 2\rangle$.

3. Find the length of the vector $\vec{u} = \langle 4, -7\rangle$.

4. Find the length of the vector $\vec{v} = \langle 0,5\rangle$.

5. Find the angle between the vectors $\vec{u} = \langle -2, 3\rangle$ and $\vec{v} = \langle 5, -1\rangle$.

6. Find the angle between the vectors $\vec{u} = \langle -4, 6\rangle$ and $\vec{v} = \langle 3, 2\rangle$.

2.5 Parallel and Perpendicular Vectors, The Unit Vector

PARALLEL AND ORTHOGONAL VECTORS

Two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ are **parallel** if the angle between them is 0° or 180°.

Also, two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ are parallel to each other if the vector $\vec{u}$ is some multiple of the vector $\vec{v}$. That is, they will be parallel if the vector $\vec{u} = c\vec{v}$, for some real number c. That is, $\vec{u}$ is some multiple of $\vec{v}$.

Two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ are **orthogonal** (perpendicular to each other) if the angle between them is 90° or 180°.

Use this shortcut: *Two vectors are perpendicular to each other if their dot product is 0.*

Example (1)

The two vectors $\vec{u} = \langle 2, -3 \rangle$ and $\vec{v} = \langle -8, 12 \rangle$ are parallel to each other since the angle between them is $180°$.

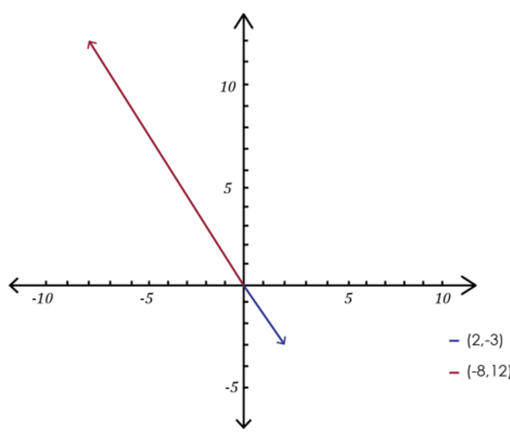
$$\theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \cdot ||\vec{v}||}$$

$$\theta = \cos^{-1} \frac{\langle 2, -3 \rangle \cdot \langle -8, 12 \rangle}{\sqrt{2^2 + (-3)^2} \cdot \sqrt{(-8)^2 + 12^2}}$$

$$\theta = \cos^{-1} \frac{2 \cdot (-8) + (-3) \cdot 12}{\sqrt{4 + 9} \cdot \sqrt{64 + 144}}$$

$$\theta = \cos^{-1} \frac{-52}{\sqrt{13} \cdot \sqrt{208}}$$

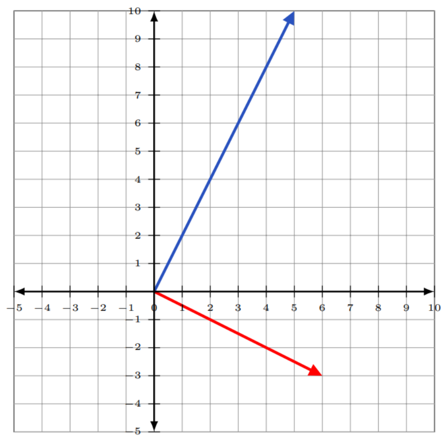
$$\theta = 180°$$



Example (2)

To show that the two vectors $\vec{u} = \langle 5,10 \rangle$ and $\vec{v} = \langle 6,-3 \rangle$ are orthogonal (perpendicular to each other), we just need to show that their dot product is 0.

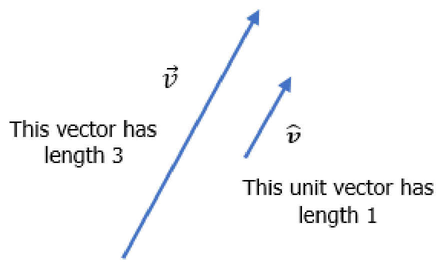
$$\langle 5,10 \rangle \cdot \langle 6,-3 \rangle = 5 \cdot 6 + 10 \cdot (-3) = 30 - 30 = 0$$



THE UNIT VECTOR

A unit vector is a vector of length 1.

A unit vector in the same direction as the vector $\vec{v}$ is often denoted with a "hat" on it as in $\hat{v}$. We call this vector "v hat."



The unit vector $\hat{v}$ corresponding to the vector $\vec{v}$ is defined to be

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}$$

Example (3)

The unit vector corresponding to the vector $\vec{v} = \langle -8, 12 \rangle$ is

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{(-8)^2 + (12)^2}}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{64 + 144}}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{208}}$$

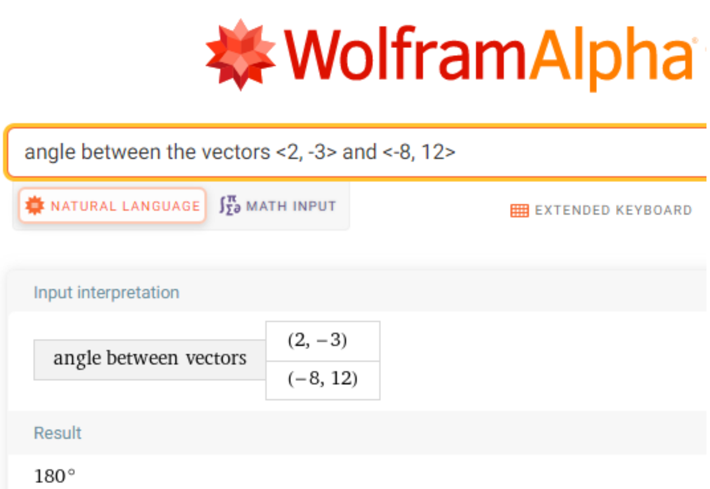
$$\hat{v} = \left\langle \frac{-8}{\sqrt{208}}, \frac{12}{\sqrt{208}} \right\rangle$$

USING TECHNOLOGY

We can use technology to find the angle θ between two vectors.

Go to www.wolframalpha.com.

To show that the vectors $\vec{u} = \langle 2, -3 \rangle$ and $\vec{v} = \langle -8, 12 \rangle$ are parallel, enter angle between the vectors <2, -3> and <-8, 12> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\theta = 180°$, indicating the two vectors are parallel.



2.5 TRY THESE

1. Determine if the vectors $\vec{u} = \langle 2, 1 \rangle$ and $\vec{v} = \langle 3\,, -6 \rangle$ are parallel to each other, perpendicular to each other, or neither parallel nor perpendicular to each other.

2. Determine if the vectors $\vec{u} = \langle 2, 16 \rangle$ and $\vec{v} = \langle \frac{1}{2}, 4 \rangle$ are parallel to each other, perpendicular to each other, or neither parallel nor perpendicular to each other.

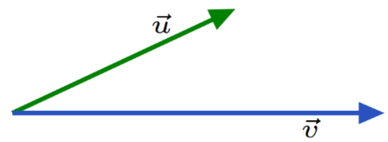
3. Determine if the vectors $\vec{u} = \langle 7, 6 \rangle$ and $\vec{v} = \langle 2, -1 \rangle$ are parallel to each other, perpendicular to each other, or neither parallel nor perpendicular to each other.

4. Find the unit vector corresponding to the vector $\vec{v} = \langle 2, -1 \rangle$.

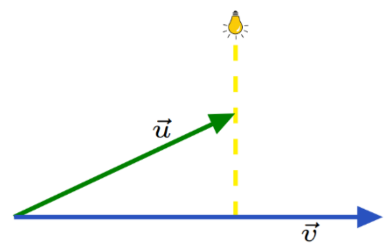
2.6 The Vector Projection of One Vector onto Another

PROJECTION

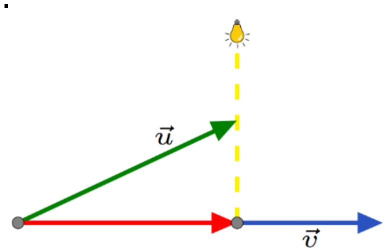
Let's project vector $\vec{u} = \langle u_x, u_y \rangle$ onto the vector $\vec{v} = \langle v_x, v_y \rangle$.



To do so, imagine a light bulb above $\vec{u}$ shining perpendicular onto $\vec{v}$.



The light from the bulb will cast a shadow of $\vec{u}$ onto $\vec{v}$, and it is this shadow that we are looking for. The shadow is the projection of $\vec{u}$ onto $\vec{v}$.



The red vector is the projection of $\vec{u}$ onto $\vec{v}$. The notation commonly used to represent the projection of $\vec{u}$ onto $\vec{v}$ is $\mathrm{proj}_{\vec{v}}\vec{u}$.

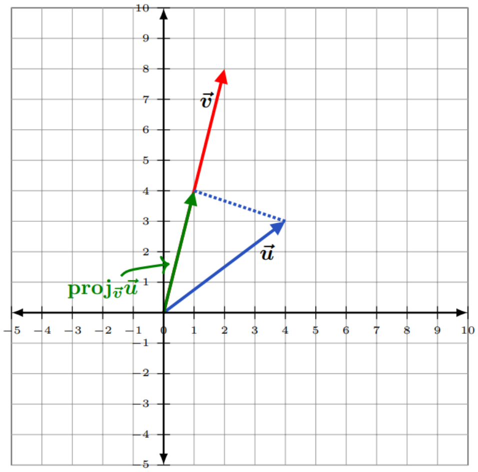
Vector parallel to $\vec{v}$ with magnitude $\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$ in the direction of $\vec{v}$ is called projection of $\vec{u}$ onto $\vec{v}$.

The formula for $\mathrm{proj}_{\vec{v}}\vec{u}$ is

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

Example (1)

To find the projection of $\vec{u} = \langle 4, 3 \rangle$ onto $\vec{v} = \langle 2, 8 \rangle$, we need to compute both the dot product of $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{v}$, then apply the formula.



$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}$$

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\langle 4, 3 \rangle \cdot \langle 2, 8 \rangle}{\|\langle 2, 8 \rangle\|^2}\langle 2, 8 \rangle$$

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{4 \cdot 2 + 3 \cdot 8}{\left(\sqrt{2^2 + 8^2}\right)^2}\langle 2, 8 \rangle$$

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{32}{\left(\sqrt{4 + 64}\right)^2}\langle 2, 8 \rangle$$

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{32}{68}\langle 2, 8 \rangle$$

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{8}{17}\langle 2, 8 \rangle$$

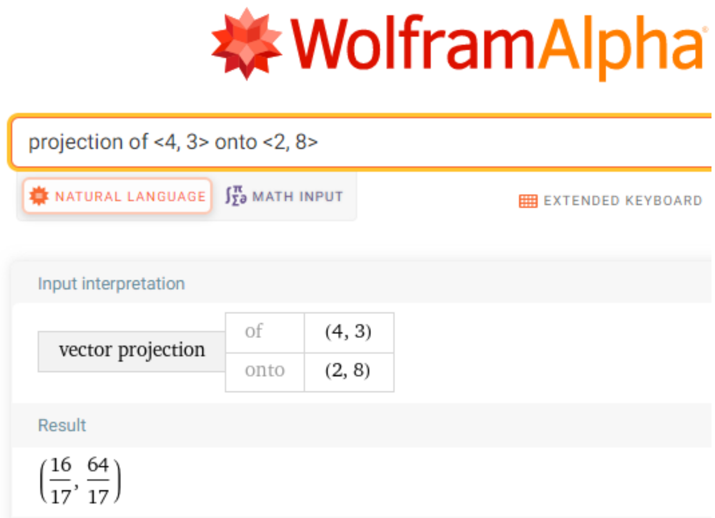
$$\mathrm{proj}_{\vec{v}}\vec{u} = \left\langle \frac{16}{17}, \frac{64}{17} \right\rangle$$

USING TECHNOLOGY

We can use technology to determine the projection of one vector onto another.

Go to www.wolframalpha.com.

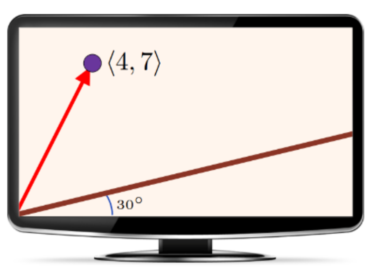
To find the projection of $\vec{u} = \langle 4, 3 \rangle$ onto $\vec{v} = \langle 2, 8 \rangle$, use the "projection" command. In the entry field enter projection of $<4, 3>$ onto $<2, 8>$.



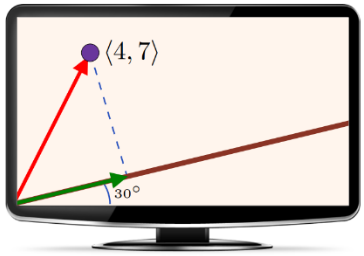
Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\left\langle \frac{16}{17}, \frac{64}{17} \right\rangle$.

Example (2)

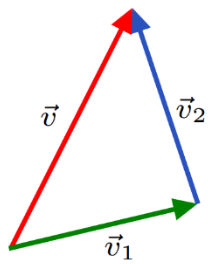
As an applied example, suppose a video game has a ball moving near a wall.



We take the origin at the bottom-left-most corner of the screen. The wall is at a 30° angle to the horizontal, and at a point in time, the ball is at position $\vec{v} = \langle 4, 7 \rangle$. To find the perpendicular distance from the ball to the wall, we use the projection formula to project the vector $\vec{v} = \langle 4, 7 \rangle$ onto the wall.

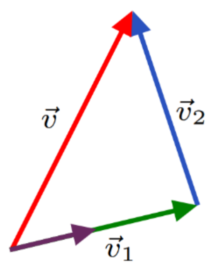


We begin by decomposing $\vec{v}$ into two vectors $\vec{v}_1$ and $\vec{v}_2$ so that $\vec{v} = \vec{v}_1 + \vec{v}_2$ and $\vec{v}_1$ lies along the wall.



The length (magnitude) of the vector $\vec{v}$ is then the distance from the ball to the wall.

The vector $\vec{v}_1$ is the projection of $\vec{v}$ onto the wall. We can get $\vec{v}_1$ by scaling (multiplying) a unit vector $\vec{w}$ that lies along the wall and, thus, along with $\vec{v}_1$.



Since $\vec{w}$ lies at a 30° angle to the horizontal, $\vec{w} = \langle \cos 30°,\ \sin 30°\rangle = \langle 0.866, 0.5\rangle$, using the projection formula, we get the projection of $\vec{v}$ that lies along the wall.

$$\vec{v}_1 = \mathrm{proj}_{\vec{w}}\vec{v} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w}$$

$$\vec{v}_1 \frac{\langle 4, 7\rangle \cdot \langle 0.866, 0.5\rangle}{\left\|\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\right\|^2}\langle 0.866, 0.5\rangle$$

$$= \frac{4 \cdot (0.866) + 7 \cdot (0.5)}{\left(\sqrt{(0.866)^2 + (.5)^2}\right)^2}\langle 0.866, 0.5\rangle$$

$$\vec{v}_1 = \frac{6.964}{\left(\sqrt{1}\right)^2}\langle 0.866, 0.5\rangle$$

$$\vec{v}_1 = (6.964)\langle 0.866, 0.5\rangle$$

$$\vec{v}_1 = \langle 6.031, 3.482\rangle$$

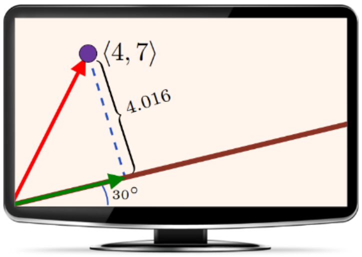
Since that $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2$, subtraction get us

$$\vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_1$$

$$\vec{\boldsymbol{v}}_2 = \langle 4, 7\rangle - \langle 6.031, 3.482\rangle$$

$$\vec{\boldsymbol{v}}_2 = \langle 4 - 6.031, 7 - 3.482\rangle$$

$$\vec{\boldsymbol{v}}_2 = \langle -2.031, 3.518\rangle$$



To get the magnitude of $\vec{\boldsymbol{v}}_2$, we use

$$\|\vec{\boldsymbol{v}}_2\| = \sqrt{{v_x}^2 + {v_y}^2}$$

$$\|\vec{\boldsymbol{v}}_2\| = \sqrt{(-2.031)^2 + 3.518^2}$$

$$\|\vec{\boldsymbol{v}}_2\| = \sqrt{4.125 + 12.376}$$

$$\left\|\overrightarrow{\boldsymbol{v}_2}\right\| = 4.062$$

2.6 TRY THESE

1. Find the projection of the vector $\vec{v} = \langle 3, 5\rangle$ onto the vector $\vec{u} = \langle 6, 2\rangle$.

2. Find $\mathrm{proj}_{\vec{v}}\vec{u}$, with $\vec{u} = \langle -2, 5\rangle$ and $\vec{v} = \langle 6, -5\rangle$.