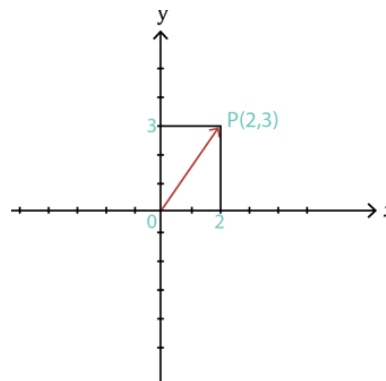# UNIT 3   VECTORS IN THREE DIMENSIONS

3.1 Three Dimensional Vectors

3-DIMENSIONAL SPACE

To this point, we have been working with vectors in 2-dimenisional space. Now, we will expand our discussion to 3-dimensional space.

The **2-dimensional coordinate system** is built around a set of two axes that intersect at right angles and one particular point called the origin. Points in the plane are described by ordered pairs (x, y) and vectors in standard position by $\langle x, y \rangle$.

The **3-dimensional coordinate system** is built around a set of three axes that intersect at right angles and one particular point again called the origin. Points in the plane are described by ordered triples (x, y, z) and vectors in standard position by $\langle x, y, z \rangle$.
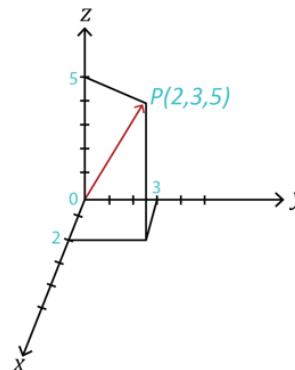
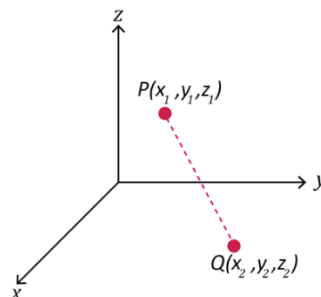### THE DISTANCE BETWEEN TWO POINTS IN 2 & 3-DIMENSIONAL SPACE

In **two-dimensional space**, the distance d between two points say $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the distance formula
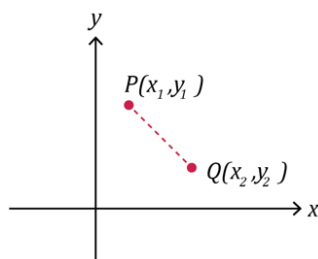
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In **three-dimensional space**, the distance d between two points say $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example (1)

The distance between the two points $P(2, 2, 5)$ and $Q(5, 6, 2)$ is
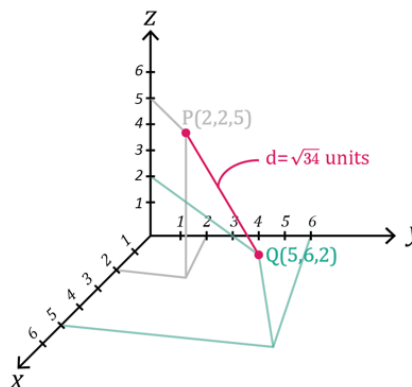
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d = \sqrt{(5 - 2)^2 + (6 - 2)^2 + (2 - 5)^2}$$

$$d = \sqrt{(3)^2 + (4)^2 + (-3)^2}$$

$$d = \sqrt{9 + 16 + 9}$$

$$d = \sqrt{34} \approx 5.8 \text{ units}$$

The distance between the two points $P(2, 2, 5)$ and $Q(5, 6, 2)$ is $\sqrt{34} \approx 5.8$ units.

USING TECHNOLOGY

We can use technology to find the distance between points.

Go to www.wolframalpha.com.

To find the distance between the two points $(-3, 5)$ and $(-7, 4)$ enter distance $(-3, 5)$ and $(-7, 4)$ in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\sqrt{17} \approx 4.12311$.

WolframAlpha

distance (-3,5) and (-7,4)

NATURAL LANGUAGE   MATH INPUT   EXTENDED KEYBOARD

Input interpretation

distance   point   coordinates $(-3, 5)$
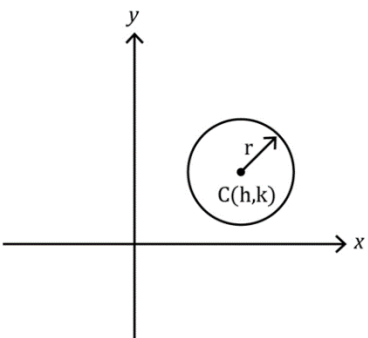   point   coordinates $(-7, 4)$

Result

$\sqrt{17} \approx 4.12311$

THE EQUATION OF A CIRCLE AND A SPHERE

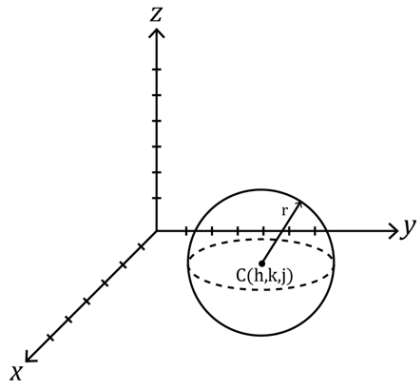We can use the distance formulas to get equations of circles and spheres.

The center-radius form of a circle with center at the point $C(h, k)$ and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

The center-radius form of a sphere with center at the point $C(h, k, j)$ and radius r is

$$(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2$$

Example (2)
To write the equation of a circle that has the point $C(4,7)$ as its center and radius 8, we use the center-radius form $(x - h)^2 + (y - k)^2 = r^2$ with $h = 4$, $k = 7$, and $r = 8$.

$$(x - h)^2 + (y - k)^2 = r^2$$

$$(x - 4)^2 + (y - 7)^2 = 8^2$$

$$(x - 4)^2 + (y - 7)^2 = 64$$

Example (3)
To write the equation of a sphere that has the point $C(4,7,1)$ as its center and radius 8, we use the center-radius form $(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2$ with $h = 4$, $k = 7$, $j = 1$, and $r = 8$.

$$(x - h)^2 + (y - k)^2 + (z - j)^2 = r^2$$

$$(x - 4)^2 + (y - 7)^2 + (z - 1)^2 = 8^2$$

$$(x - 4)^2 + (y - 7)^2 + (z - 1)^2 = 64$$

3.1 TRY THESE

1. Find the distance between the two points $(2, 4)$ and $(-3, 6)$. Round to one decimal place.

2. Find distance between the two points $(-3, 5, -6)$ and $(7, -4, 2)$. Round to one decimal place.

3. Write the equation of a circle that has the point $C(2, 9)$ as its center and radius 1.

4. Write the equation of a sphere that has the point $C(-2, 5, -7)$ as its center and radius 4.

3.2 Magnitude and Direction Cosines of a Vector

THE MAGNITUDE OF A VECTOR

You likely recall that the magnitude (the length) of a vector $\vec{v} = \langle v_x, v_y \rangle$ in **2-dimensions** is

$$\|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2}$$
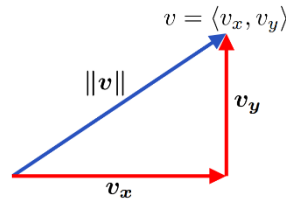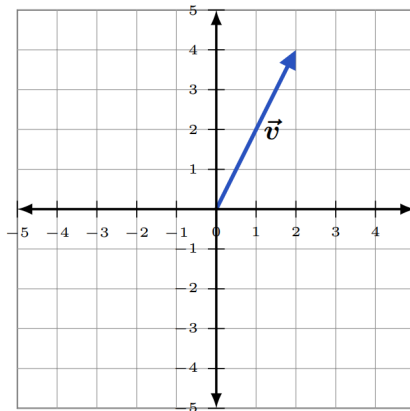

Example (1)

The vector $\vec{v} = \langle 2, 4 \rangle$ has magnitude

$$\|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2}$$

$$\sqrt{2^2 + 4^2} = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$$

$$\|\vec{v}\| = 2\sqrt{5}$$

Interpret this as the length of the vector $\vec{v} = \langle 2, 4 \rangle$ is $2\sqrt{5}$ units.

The formula for the length of the vector $\vec{v} = \langle v_x, v_y, v_z \rangle$ in **3-dimensions** is

$$\|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2 + v_z{}^2}$$

Example (2)

The vector $\vec{v} = \langle 2, 4, -6 \rangle$ has magnitude

$$\|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2 + v_z{}^2}$$

$$\|\vec{v}\| = \sqrt{2^2 + 4^2 + (-6)^2}$$

$$\|\vec{v}\| = \sqrt{4 + 16 + 36}$$

$$\|\vec{v}\| = \sqrt{56}$$

$$\|\vec{v}\| = 2\sqrt{14}$$

Interpret this as the length of the vector $\vec{v} = \langle 2, 4, -6 \rangle$ is $2\sqrt{14}$ units.

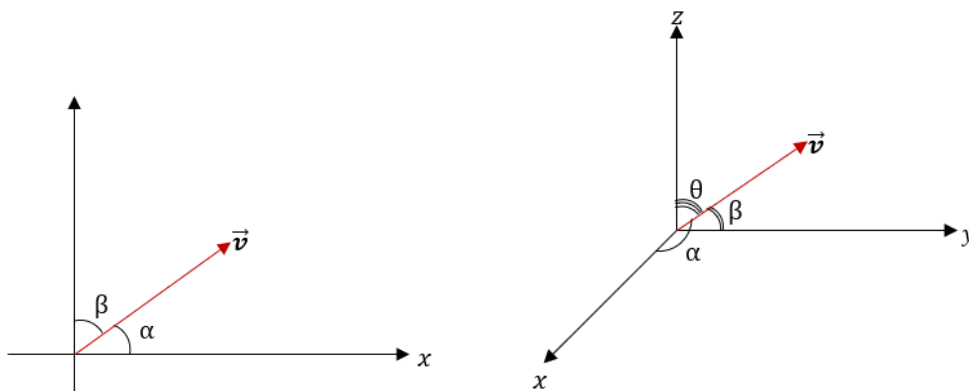## THE DIRECTION COSINES OF VECTORS IN 2- AND 3-DIMENSIONS

The direction cosines of a vector $\vec{v} = \langle v_x, v_y \rangle$ or $\vec{v} = \langle v_x, v_y, v_z \rangle$ are the cosines of the angles the vector forms with the coordinate axes.

The direction cosines are important as they uniquely determine the direction of the vector.

Direction cosines are found by dividing each component of the vector by the magnitude (length) of the vector.

$$\cos\alpha = \frac{v_x}{\|\vec{v}\|}, \quad \cos\beta = \frac{v_y}{\|\vec{v}\|}$$

$$\cos\alpha = \frac{v_x}{\|\vec{v}\|}, \quad \cos\beta = \frac{v_y}{\|\vec{v}\|}, \quad \cos\theta = \frac{v_z}{\|\vec{v}\|}$$

Example (3)

Find the direction cosines of the vector $\vec{v} = \langle 4, 5, 2 \rangle$.

First, find the magnitude of the vector $\vec{v} = \langle 4, 5, 2 \rangle$

$$\|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2 + v_z{}^2} = \sqrt{4^2 + 5^2 + 2^2} = \sqrt{16 + 25 + 4} = \sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}$$

Get the direction cosines by dividing each component, 4, 5, and 2, by this magnitude.

$$\cos\alpha = \frac{v_x}{\|\vec{v}\|} = \frac{4}{3\sqrt{5}} \approx 0.596$$

$$\cos\beta = \frac{v_y}{\|\vec{v}\|} = \frac{5}{3\sqrt{5}} \approx 0.745$$

$$\cos\theta = \frac{v_z}{\|\vec{v}\|} = \frac{2}{3\sqrt{5}} \approx 0.298$$

Example (4)

Find the vector $\vec{v}$ that has magnitude 32 and direction cosines $\cos\alpha = 5/8$ and $\cos\beta = -3/8$.

Since $\cos\alpha = \frac{v_x}{\|\vec{v}\|}$ and $\cos\beta = \frac{v_y}{\|\vec{v}\|}$,

$$v_x = \|\vec{v}\| \cdot \cos\alpha = 32 \cdot \frac{5}{8} = 20, \text{ and}$$

$$v_y = \|\vec{v}\| \cdot \cos\beta = 32 \cdot \frac{-3}{8} = -12.$$

So, $\vec{v} = \langle 20, -12 \rangle$.

USING TECHNOLOGY

We can use technology to determine the magnitude of a vector.

> Go to www.wolframalpha.com.

To find the magnitude of the vector $\vec{v} = \langle 2, 4, -6 \rangle$, enter magnitude of <2, 4, -6> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\|\vec{v}\| = 2\sqrt{14}$.

To find the direction cosines of the vector $\vec{v} = \langle 4, 5, 2 \rangle$, enter evaluate 4/(magnitude of <4,5,2>), 5/(magnitude of <4,5,2>), 2/(magnitude of <4,5,2>) in the entry field. WolframAlpha answers $\{0.596285, 0.745356, 0.298142\}$.

We can use WolframAlpha to approximate a vector give its magnitude and direction cosines.

3.2 TRY THESE

1. Find the magnitude of the vector $\vec{v} = \langle -3, 4, -2 \rangle$.

2. Find the magnitude of the vector $\vec{v} = \langle 1, -1 \rangle$.

3. Find the cosines of the vector $\vec{v} = \langle 3, -1, 2 \rangle$. Round to three decimal places.

4. Approximate the vector $\vec{v}$ that has magnitude 24 and direction cosines $\cos\alpha = -3/4, \cos\beta = -1/4, \cos\theta = 7/8$.

3.3 Arithmetic on Vectors in 3-Dimensional Space

ADDITION & SUBTRACTION OF VECTORS

To add or subtract two vectors, add or subtract their corresponding components.

Example (1)   To **add** the vectors $\vec{u} = \langle 2, 5, 4 \rangle$ and $\vec{v} = \langle 4, 2, 1 \rangle$, add their corresponding components.

$$\vec{u} + \vec{v} = \langle 2 + 4, 5 + 2, 4 + 1 \rangle = \langle 6, 7, 5 \rangle$$

$$\text{So, } \vec{u} + \vec{v} = \langle 6, 7, 5 \rangle$$
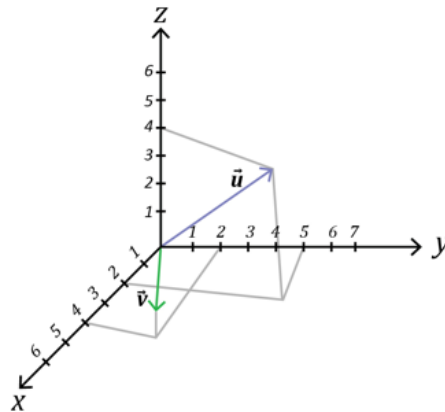
Now, graph this sum. Start at the origin.

Since the x −component is 6, move 6 units in the x −direction.
Since the y −component is 7, move 7 units in the y −direction.
Since the z −component is 5, move 5 units upward.

Example (2)

To **subtract** the vectors $\vec{u} = \langle 2, 5, 4 \rangle$ and $\vec{v} = \langle 4, 2, 1 \rangle$ subtract their corresponding components.

$$\vec{u} - \vec{v} = \langle 2 - 4, 5 - 2, 4 - 1 \rangle = \langle -2, 3, 3 \rangle$$

$$\text{So, } \vec{u} - \vec{v} = \langle -2, 3, 3 \rangle$$

SCALAR MULTIPLICATION

Scalar multiplication is the multiplication of a vector by a real number (a scalar).

Suppose we let the letter k represent a real number and $\vec{v}$ be the vector $\langle x, y, z \rangle$. Then, the scalar multiple of the vector $\vec{v}$ is
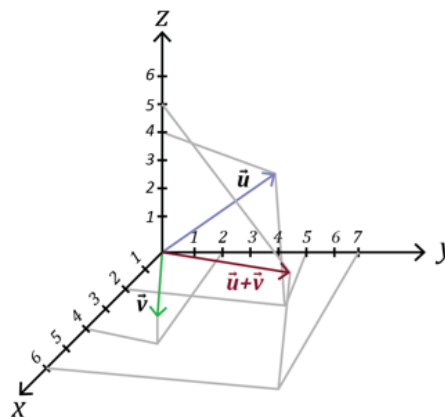
$$k\vec{v} = \langle kx, ky, kz \rangle$$

Example (3)

Suppose $\vec{u} = \langle -3, -8, 5 \rangle$ and $k = 3$.

Then $k\vec{u} = 3\vec{u} = 3\langle -3, -8, 5 \rangle = \langle 3(-3), 3(-8), 3(5) \rangle = \langle -9, -24, 15 \rangle$

Example (4)

Suppose $\vec{v} = \langle 6, 3, -12 \rangle$ and $k = \frac{-1}{3}$.

Then $k\vec{u} = \frac{-1}{3}\vec{u} = \frac{-1}{3}\langle 6, 3, -12 \rangle = \left\langle \frac{-1}{3}(6), \frac{-1}{3}(3), \frac{-1}{3}(-12) \right\rangle = \langle -2, -1, 4 \rangle$

Example (5)

Suppose $\vec{u} = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$. Find $3\vec{u} + 4\vec{v} - 2\vec{w}$.

Then $3\vec{u} + 4\vec{v} - 2\vec{w} = 3\begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix} - 2\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 18 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \\ -32 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 28 \\ -36 \end{bmatrix}$

USING TECHNOLOGY

We can use technology to determine the value of adding or subtracting vectors.

Go to www.wolframalpha.com.

Suppose $\vec{u} = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$. Use WolframAlpha to find $3\vec{u} + 4\vec{v} - 2\vec{w}$. In the entry field enter evaluate 3*[-2,6,0] + 4*[1,2,-8] − 2*[-3,-1,2].

WolframAlpha

evaluate 3*[-2,6,0] + 4*[1,2,-8] − 2*[-3,-1,2]

NATURAL LANGUAGE   MATH INPUT   EXTENDED KEYBOARD

Input

$3\,(-2,\,6,\,0) + 4\,(1,\,2,\,-8) - 2\,(-3,\,-1,\,2)$

Result

$\{4,\,28,\,-36\}$

WolframAlpha answers $(4, 28, -36)$ which is WolframAlpha's notation for $\begin{bmatrix} 4 \\ 28 \\ -36 \end{bmatrix}$.

3.3 TRY THESE

1. Add the vectors $\vec{u} = \langle -3, 4, 6 \rangle$ and $\vec{v} = \langle 8, 7, -5 \rangle$.

2. Subtract the vector $\vec{v} = \langle 8, 7, -5 \rangle$ from the vector $\vec{u} = \langle -3, 4, 6 \rangle$.

3. Given the three vectors, $\vec{u} = \langle 2, 4, -5 \rangle$, $\vec{v} = \langle -3, 4, -8 \rangle$, and $\vec{w} = \langle 0, 1, 2 \rangle$, find $2\vec{u} + 3\vec{v} - 4\vec{w}$.

4. Suppose $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$, find $4\vec{u} - 4\vec{v} - \vec{w}$.

3.4 The Unit Vector in 3-Dimensions and Vectors in Standard Position

THE UNIT VECTOR IN 3-DIMENSIONS

The unit vector, as you will likely remember, in 2-dimensions is a vector of length 1. A unit vector in the same direction as the vector $\vec{v}$ is often denoted with a "hat" on it as in $\hat{v}$. We call this vector "v hat."

> The unit vector $\hat{v}$ corresponding to the vector $\vec{v}$ is defined to be
> $$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

Example (1)

The unit vector corresponding to the vector $\vec{v} = \langle -8, 12 \rangle$ is

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{(-8)^2 + (12)^2}}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{64 + 144}}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{208}}$$

$$\hat{v} = \left\langle \frac{-8}{\sqrt{208}}, \frac{12}{\sqrt{208}} \right\rangle$$

A unit vector in 3-dimensions and in the same direction as the vector $\vec{v}$ is defined in the same way as the unit vector in 2-dimensions.

The unit vector $\hat{v}$ corresponding to the vector $\vec{v}$ is defined to be $\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$, where $\vec{v} = \langle x, y, z \rangle$.

For example, the unit vector corresponding to the vector $\vec{v} = \langle 5, -3, 4 \rangle$ is

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} = \frac{\langle 5, -3, 4 \rangle}{\sqrt{5^2 + (-3)^2 + 4^2}}$$

$$\hat{v} = \frac{\langle 5, -3, 4 \rangle}{\sqrt{25 + 9 + 16}}$$

$$\hat{v} = \frac{\langle 5, -3, 4 \rangle}{\sqrt{50}}$$

$$\hat{v} = \frac{\langle 5, -3, 4 \rangle}{5\sqrt{2}}$$

$$\hat{v} = \left\langle \frac{5}{5\sqrt{2}}, \frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}} \right\rangle$$

VECTORS IN STANDARD POSITION

A vector with its initial point at the origin in a Cartesian coordinate system is said to be in *standard position*. A common notation for a unit vector in standard position uses the lowercase letters i, j, and k is to represent the unit vector in

the x-direction with the vector $\hat{\imath}$, where $\hat{\imath} = \langle 1, 0, 0 \rangle$, and
the y-direction with the vector $\hat{\jmath}$, where $\hat{\jmath} = \langle 0, 1, 0 \rangle$, and
the z-direction with the vector $\hat{k}$, where $\hat{k} = \langle 0, 0, 1 \rangle$.
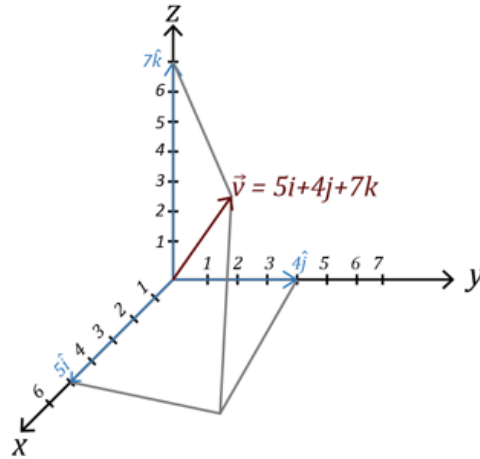
The figure shows these three unit vectors.
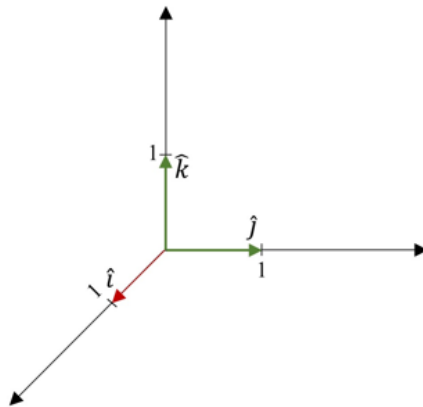
Any vector can be expressed as a combination of these three unit vectors.

Example (2)

The vector $\vec{v} = \langle 5, 4, 7 \rangle$ can be expressed as $\vec{v} = 5\hat{\imath} + 4\hat{\jmath} + 7\hat{k}$.

Now, the unit-vector in the direction of $\vec{v} = 5\hat{\imath} + 4\hat{\jmath} + 7\hat{k}$ is

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} = \frac{\langle 5, 4, 7 \rangle}{\sqrt{5^2 + 4^2 + 7^2}}$$

$$\hat{v} = \frac{\langle 5, 4, 7 \rangle}{\sqrt{25 + 16 + 49}}$$

$$\hat{v} = \frac{\langle 5, -3, 4 \rangle}{\sqrt{90}}$$

$$\hat{v} = \frac{\langle 5, 4, 7 \rangle}{\sqrt{9 \cdot 10}}$$

$$\hat{v} = \frac{\langle 5, 4, 7 \rangle}{3\sqrt{10}}$$

$$\hat{v} = \left\langle \frac{5}{3\sqrt{10}}, \frac{4}{3\sqrt{10}}, \frac{7}{3\sqrt{10}} \right\rangle$$

$$\vec{v} = \frac{5}{3\sqrt{10}}\hat{\imath} + \frac{4}{3\sqrt{10}}\hat{\jmath} + \frac{7}{3\sqrt{10}}\hat{k}$$
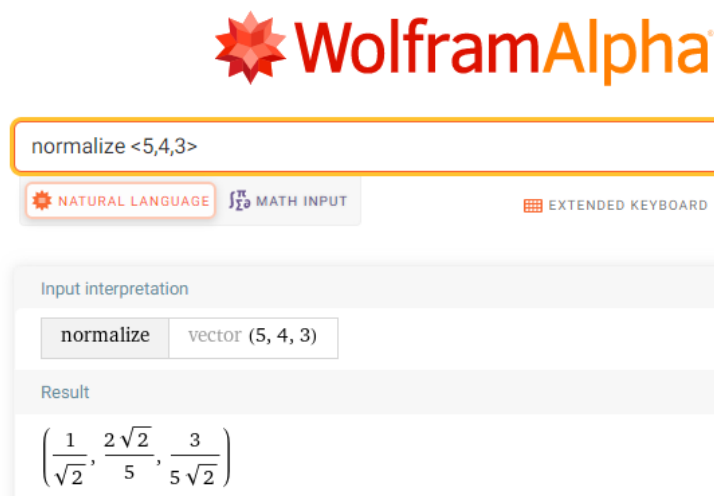
NORMALIZING A VECTOR

Normalizing a vector is a common practice in mathematics and it also has practical applications in computer graphics. Normalizing a vector $\vec{v}$ is the process of identifying the unit vector of a vector $\vec{v}$.

USING TECHNOLOGY

We can use technology to find the unit vector in the direction of the given vector.

Go to www.wolframalpha.com.

Use WolframAlpha to find the unit vector in the direction of $\vec{u} = \langle 5, 4, 3 \rangle$. Enter normalize <5,4,3> in the entry field and WolframAlpha gives you an answer.

Translate WolframAlpha's answer to $\frac{1}{\sqrt{2}}\hat{\imath} + \frac{2\sqrt{2}}{5}\hat{\jmath} + \frac{3}{5\sqrt{2}}\hat{k}$.

3.4 TRY THESE

1. Write the unit vector that corresponds to $\vec{v} = \langle 2, -3, 4 \rangle$.

2. Write the unit vector that corresponds to $\vec{v} = \langle 1, -1, 1 \rangle$.

3. Write the unit vector that corresponds to $\vec{v} - \vec{u} = \langle 6, 7, 2 \rangle - \langle 2, 7, 6 \rangle$.

4. Normalize the vector $\vec{v} = \langle 4, 3, 2 \rangle$.

3.5 The Dot Product, Length of a Vector, and the Angle between Two Vectors in Three Dimensions

THE DOT PRODUCT OF TWO VECTORS

The dot product of two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ in two dimensions is nicely extended to three dimensions.

> The dot product of vectors $\vec{u} = \langle u_x, u_y, u_z \rangle$ and $\vec{v} = \langle v_x, v_y, v_z \rangle$
> is a scalar (real number) and is defined to be
>
> $$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$$

Since $u_x, u_y, u_z, v_x, v_y,$ and v_z are real numbers, you can see that the dot product is itself a real number and not a vector.

Example (1)

To compute the dot product of the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$, we compute

$$\vec{u} \cdot \vec{v} = 5 \cdot 3 + 2 \cdot 4 + 4 \cdot (-7) = 15 + 8 - 28 = -5$$

> Since the dot product is a scalar, it follows the properties of real numbers.

> **PROPERTIES OF THE DOT PRODUCT**
> 1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, the dot product is commutative
> 2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, the dot product distributes over vector addition
> 3. $\vec{u} \cdot \vec{0} = 0$, the dot product with the zero vector $\vec{0}$, is the scalar 0.
> 4. $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

Example (2)

Compute the dot product $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, where

$$\vec{u} = \langle 5, -2, -3 \rangle, \vec{v} = \langle 6, 4, 1 \rangle, \text{ and } \vec{w} = \langle -3, 7, -2 \rangle,$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \langle 5, -2, -3 \rangle \cdot \langle 6, 4, 1 \rangle + \langle 5, -2, -3 \rangle \cdot \langle -3, 7, -2 \rangle$$
$$= (5 \cdot 6 + (-2) \cdot 4 + (-3) \cdot 1) + (5 \cdot (-3) + (-2) \cdot 7) + (-3) \cdot (-2))$$

$$= 30 - 8 - 3 - 15 - 14 + 6$$
$$= -4$$

THE LENGTH OF A VECTOR IN THREE DIMENSIONS

The length (magnitude) of a vector in two dimensions is nicely extended to three dimensions.

The dot product of a vector $\vec{v} = \langle v_x, v_y \rangle$ with itself gives the length of the vector.

$$\|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2}$$

You can see that the length of the vector is the square root of the sum of the squares of each of the vector's components. The same is true for the length of a vector in three dimensions.

The dot product of a vector $\vec{v} = \langle v_x, v_y, v_z \rangle$ with itself gives the length of the vector.

$$\|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2 + v_z{}^2}$$

Example (3)

Use the dot product to find the length of the vector $\vec{v} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$.

In this case, $v_x = 4$, $v_y = 2$, and $v_z = 6$

Using $\|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2 + v_z{}^2}$, we get

$$\|\vec{v}\| = \sqrt{4^2 + 2^2 + 6^2}$$
$$\|\vec{v}\| = \sqrt{56}$$
$$\|\vec{v}\| = \sqrt{4 \cdot 14}$$
$$\|\vec{v}\| = \sqrt{4} \cdot \sqrt{14}$$
$$\|\vec{v}\| = 2\sqrt{14}$$

The length of the vector $\vec{v} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ is $2\sqrt{14}$ units.

THE ANGLE BETWEEN TWO VECTORS

The formula for the angle between two vectors in two dimensions is nicely extended to three dimensions.

If θ is the smallest nonnegative angle between two non-zero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} \text{ or } \theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$$

where $0 \le \theta \le 2\pi$ and $\|\vec{u}\| = \sqrt{u_x{}^2 + u_y{}^2 + u_z{}^2}$ and $\|\vec{v}\| = \sqrt{v_x{}^2 + v_y{}^2 + v_z{}^2}$

Example (4)

Find the angle between the vectors $\vec{u} = \langle 5, -3, -1\rangle$ and $\vec{v} = \langle 2, 4, -5\rangle$.

Using $\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$, we get

$$\theta = \cos^{-1}\frac{\langle 5, -3, -1\rangle \cdot \langle 2, 4, -5\rangle}{\sqrt{5^2 + (-3)^2 + (-1)^2}\cdot\sqrt{2^2 + 4^2 + (-5)^2}}$$

$$\theta = \cos^{-1}\frac{5\cdot 2 + (-3)\cdot 4 + (-1)\cdot(-5)}{\sqrt{25 + 9 + 1}\cdot\sqrt{4 + 16 + 25}}$$

$$\theta = \cos^{-1}\frac{3}{\sqrt{35}\cdot\sqrt{45}}$$

$$\theta = 85.66$$

We conclude that the angle between these two vectors is close to 85.7° rounded to one decimal place.

USING TECHNOLOGY

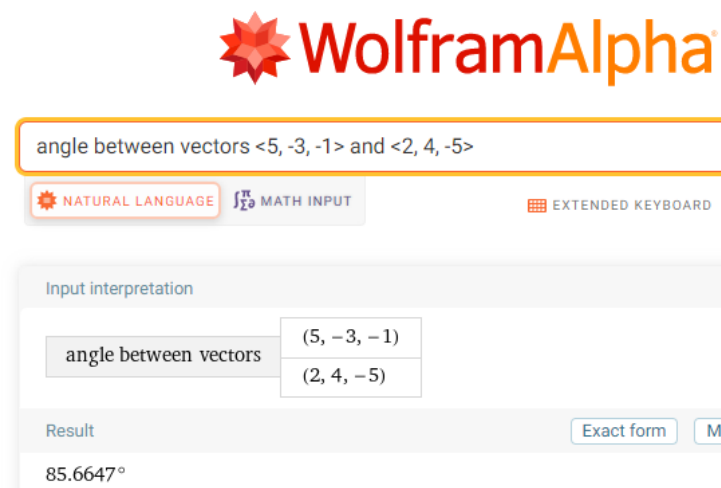We can use technology to find the magnitude of the vector and the angle θ between two vectors.

> Go to www.wolframalpha.com.

To find the magnitude (length) of the vector $\vec{v} = \langle 4, 2, 4 \rangle$, enter magnitude of <4, 2, 4> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\|\vec{v}\| = 6$.

To find the angle between the vectors $\vec{u} = \langle 5, -3, -1 \rangle$ and $\vec{v} = \langle 2, 4, -5 \rangle$, enter angle between vectors <5, -3, -1> and <2, 4, -5> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\theta = 85.7°$, rounded to one decimal place.

3.5 TRY THESE

1. Find the dot product of the vectors $\vec{u} = \langle -2, 3, -9 \rangle$ and $\vec{v} = \langle 5, -1, 2 \rangle$.

2. Find the dot product of the vectors $\vec{u} = \langle 6, 2, -1 \rangle$ and $\vec{v} = \langle 2, -7, -2 \rangle$.

3. Find the length of the vector $\vec{u} = \langle 4, -7, -6 \rangle$.

4. Find the length of the vector $\vec{v} = \langle 0, 5, 0 \rangle$.

5. Find the angle between the vectors $\vec{u} = \langle 3, 4, 5 \rangle$ and $\vec{v} = \langle -3, -1, 8 \rangle$.

6. Find the angle between the vectors $\vec{u} = \langle 1, -2, 1 \rangle$ and $\vec{v} = \langle 3, 5, 7 \rangle$.

3.6 The Cross Product: Algebra

THE CROSS PRODUCT OF TWO VECTORS

A vector that is perpendicular to both vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$, can be found using the cross product. The cross product requires that both vectors be in three-dimensional space.

> The cross product of vectors $\vec{u} = \langle u_x, u_y, u_z \rangle$ and $\vec{v} = \langle v_x, v_y, v_z \rangle$
> is a vector and is defined to be
>
> $$\vec{u} \times \vec{v} = \langle u_y v_z - u_z v_y, \ u_z v_x - u_x v_z, \ u_x v_y - u_y v_z \rangle$$

This formula is challenging to remember. A nice device to help you remember both this formula and the dot product formula is to visualize them in a 3x3 square of components. The square shows how vectors can interact with one another.

	x	y	z
x	Dot	Cross	Cross
y	Cross	Dot	Cross
z	Cross	Cross	Dot

For the cross product,
The x-component has a product that involves no x-components: $u_y v_z - u_z v_y$
The y-component has a product that involves no y-components: $u_z v_x - u_x v_z$
The z-component has a product that involves no z-components: $u_x v_y - u_y v_z$

Each component is a difference of two diagonal products.

	x	y	z
x	Dot	x*y	x*z
y	y*x	Dot	y*z
z	z*x	z*y	Dot

To produce the x-component,
(top right) - (bottom left) = y*z − z*y

To produce the y-component,
(bottom left) - (top right) = z*x − x*z

To produce the z-component,
(top right) − (bottom left) = x*y − y*x

The **DOT** product is the interaction between two vectors having **similar** components:
$$x \cdot x, \quad y \cdot y, \quad z \cdot z$$

The dot product measures similarity since it combines only interactions of matching components.

The **CROSS** product is the interaction between two vectors having **different** components:
$$x \cdot y, \; x \cdot z, \; y \cdot x, \; y \cdot z, \; z \cdot x, \; z \cdot y$$

The cross product measures cross interactions since it combines interactions of different components.

Example (1)

Find the cross product of the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$.

	3	4	-7
5	Dot	5*4	5*-7
2	2*3	Dot	2*-7
4	4*3	4*4	Dot

	3	4	-7
5	Dot	20	-35
2	6	Dot	-14
4	12	16	Dot

To produce the x-component, (top right) - (bottom left) = 2*(-7) − 4*4 = -30

To produce the y-component, (bottom left) - (top right) = 4*3 − 5*(-7) = 47

To produce the z-component, (top right) − (bottom left) = 5*4 − 2*3 = 14

$$\vec{u} \times \vec{v} = \langle -30, 47, 14 \rangle$$

*Be careful with the computation. It goes (bottom left) − (top right) while the first and last go (top right) − (bottom left).

USING TECHNOLOGY

We can use technology to find the cross product between two vectors.

Go to www.wolframalpha.com.

To find the cross product of the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$, use either the "cross" or the x command. Wolframalpha tells you what it thinks you entered, then it tells you its answer. In this case, $\vec{u} \times \vec{v} = \langle -30, 47, 14 \rangle$.

WolframAlpha

<5, 2, 4> cross <3, 4, -7>

NATURAL LANGUAGE   MATH INPUT   EXTENDED KEYBOARD

Input interpretation

$(5, 2, 4) \times (3, 4, -7)$

$a \times b$

Result

$(-30, 47, 14)$

THE RIGHT-HAND RULE

You can see that the cross product of the two vectors $\vec{u}$ and $\vec{v}$, is itself a vector. But where is this vector $\vec{u} \times \vec{v}$? The cross product of two vectors is a vector that is perpendicular to the plane formed by the two vectors. What about the two perpendicular directions? Does this perpendicular vector lie above or below the plane formed by the two vectors? We use the **right-hand rule**.

Hold your hand as shown in the picture, your index and middle fingers extended. Your thumb points in the direction of the cross product.
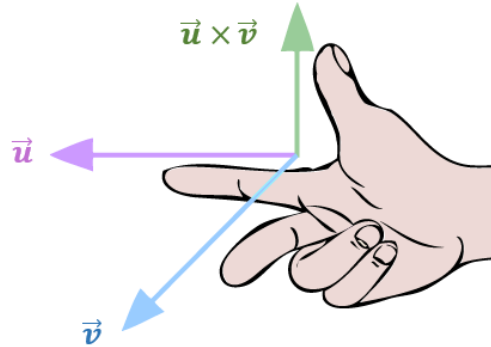
Since the dot product is a scalar, it follows the properties of real numbers.

PROPERTIES OF THE CROSS PRODUCT

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$, the cross product is **anti-commutative**

2. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$, the cross product distributes over vector addition

3. $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{u})$

4. $\vec{u} \times \vec{0} = \vec{0}$, the cross product with the zero vector $\vec{0}$, is the zero vector $\vec{0}$

USING TECHNOLOGY

For example, use WolframAlpha to compute both the cross product $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$, with $\vec{u} = \langle 5, -2, -3 \rangle$ and $\vec{v} = \langle 6, 4, 1 \rangle$, to show that one is the opposite of the other.

WolframAlpha

<5, -2, -3> cross <6, 4, 1>

NATURAL LANGUAGE   MATH INPUT   EXTENDED KEYBOARD

Input interpretation

$(5, -2, -3) \times (6, 4, 1)$

$a \times b$

Result

$(10, -23, 32)$

WolframAlpha

<6, 4, 1> cross <5, -2, -3>

NATURAL LANGUAGE   MATH INPUT   EXTENDED KEYBOARD

Input interpretation

$(6, 4, 1) \times (5, -2, -3)$

$a \times b$

Result

$(-10, 23, -32)$

Notice that $\langle 10, -23, -32 \rangle = -\langle -10, 23, -32 \rangle$, verifying property 1.
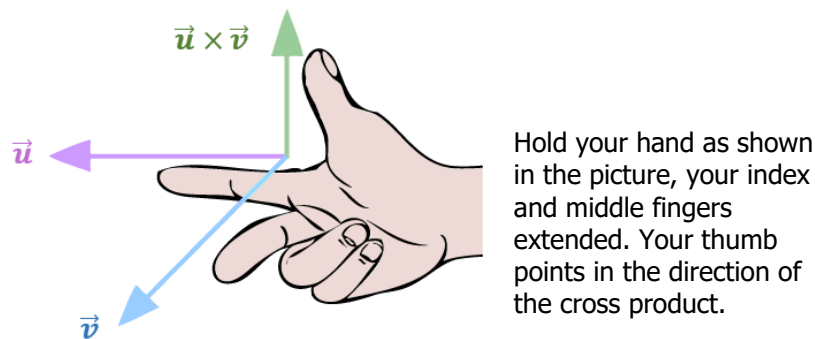
3.6 TRY THESE

1. Find the cross product of the vectors $\vec{u} = \langle 4, -2, 1 \rangle$ and $\vec{v} = \langle 5, -1, 3 \rangle$.

2. Find the cross product of the vectors $\vec{u} = \langle -2, 3, -9 \rangle$ and $\vec{v} = \langle -8, 12, -36 \rangle$.

3. Find $\vec{u} \times \vec{v} \cdot \vec{w}$, where $\vec{u} = \langle -2, 5, 3 \rangle$, $\vec{v} = \langle 4, 4, -2 \rangle$, and $\vec{w} = \langle 2, 6, -5 \rangle$.

* Note that the cross product must be computed first since if it is not, we would be crossing a vector with a scalar.

3.7 The Cross Product: Geometry

THE CROSS PRODUCT OF TWO VECTORS AND THE RIGHT-HAND RULE

The cross product of the two vectors $\vec{u}$ and $\vec{v}$, is itself a vector. Where is this vector $\vec{u} \times \vec{v}$? The cross product of two vectors is a vector perpendicular to the plane formed by the two vectors. What if there are two perpendicular directions? Does this perpendicular vector lie above or below the plane formed by the two vectors? Let's use the **right-hand rule**.
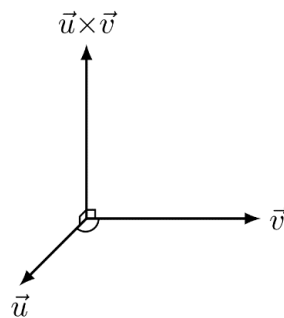

Hold your hand as shown in the picture, your index and middle fingers extended. Your thumb points in the direction of the cross product.

THE GEOMETRY OF THE CROSS PRODUCT

If θ is the angle between the two vectors $\vec{u} = \langle u_x, u_y, u_z \rangle$ and $\vec{v} = \langle v_x, v_y, v_z \rangle$, then the length (magnitude) of the cross product $\vec{u} \times \vec{v}$ is

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$$

$$\|\vec{u}\| = \sqrt{u_x^2 + u_y^2 + u_z^2} \text{ and } \|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$$

Example (1)

The length of the vector $\vec{u} \times \vec{v}$, where $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$ is

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$$

$$\sqrt{u_x^2 + u_y^2 + u_z^2} \cdot \sqrt{v_x^2 + v_y^2 + v_z^2} \cdot \sin\theta$$

We now need to get $\sin\theta$. We'll use the formula for the angle between two vectors.

$$\theta = \cos^{-1}\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

$$\theta = \cos^{-1}\frac{\langle 5, 2, 4 \rangle \cdot \langle 3, 4, -7 \rangle}{\sqrt{5^2 + 2^2 + 4^2} \cdot \sqrt{3^2 + 4^2 + (-7)^2}}$$

$$\theta = \cos^{-1}\frac{5 \cdot 3 + 2 \cdot 4 + 4 \cdot (-7)}{\sqrt{45} \cdot \sqrt{74}}$$

$$\theta = \cos^{-1}\frac{-5}{\sqrt{45} \cdot \sqrt{74}}$$

$$\theta = 94.97°$$

Now we compute $\|\vec{u} \times \vec{v}\| = \sqrt{45} \cdot \sqrt{74}\sin(94.97°) = 57.49$ units.

USING TECHNOLOGY

We can use technology to find the magnitude of the cross product of two vectors.

Go to www.wolframalpha.com.

To find the length of the cross product of the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$ enter magnitude of <5, 2, 4> cross <3, 4, -7> in the entry field. Wolframalpha tells you what it thinks you entered and its answer. In this case it shows you result of $\sqrt{3305}$. Click on the approximate form button to get the result in decimal form as 57.49.

Result   Approximate form

$\sqrt{3305}$

WolframAlpha

magnitude of <5,2,4> cross <3,4,-7>

NATURAL LANGUAGE   MATH INPUT   EXTENDED KEYBOARD

Input interpretation

$\|(5, 2, 4) \times (3, 4, -7)\|$

$a \times b$

$\|expr\|$ gives t

Result   Ex

57.4891

AREA OF A PARALLELOGRAM

Geometrically, $\|\vec{u} \times \vec{v}\|$ produces the area of a parallelogram determined by $\vec{u}$ and $\vec{v}$.

$$\text{Area} = \|\vec{u} \times \vec{v}\| = \|\vec{u}\|\|\vec{v}\|\sin\theta$$
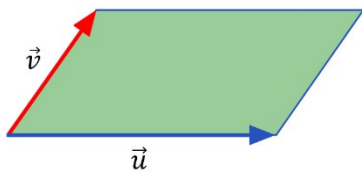
The area of the parallelogram determined by the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$ from the above example is 57.49 square units.

THE CROSS PRODUCT OF PERPENDICULAR AND PARALLEL VECTORS

> If vectors $\vec{u}$ and $\vec{v}$ are **perpendicular** to each other, then the angle between them is 90° and $\sin(90°) = 1$, so that
> $$\vec{u} \times \vec{v} = \|\vec{u}\| \|\vec{v}\|$$

> If vectors $\vec{u}$ and $\vec{v}$ are **parallel** to each other, then the angle between them is 0° and $\sin(0°) = 0$.

It makes sense then to define the cross product of parallel vectors to be the zero vector, $\vec{0}$. Also, if at least one of the vectors $\vec{u}$ and $\vec{v}$ is the zero vector $\vec{0}$, then the cross product $\vec{u} \times \vec{v}$ is defined to be the zero vector. We can say that if the cross product of two vectors is zero, then the vectors are parallel to each other. Also, if two vectors are parallel to each other, then their cross product is zero. We combine these statements together in an *if-and-only-if* statement.

> Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel to each other if and only if $\vec{u} \times \vec{v} = 0$.

> **PROPERTIES OF THE CROSS PRODUCT**
>
> 5. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$, the cross product is **anti-commutative**
>
> 6. $k(\vec{u} \times \vec{v}) = k\vec{u} \times \vec{v} = \vec{u} \times k\vec{v}$, multiplication by a scalar
>
> 7. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$, the cross product distributes over vector addition
>
> 8. $\vec{u} \times \vec{0} = \vec{0}$, the cross product with the zero vector $\vec{0}$, is the zero vector $\vec{0}$.

Example (2)

Use WolframAlpha to verify that

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

where $\vec{u} = \langle 5, -2, -3 \rangle$, $\vec{v} = \langle 6, 4, 1 \rangle$, and $\vec{w} = \langle -3, 7, 2 \rangle$.

Use W|A to first compute $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u}$ and then $\vec{u} \times \vec{v} + \vec{u} \times \vec{w}$. Determine if the results do or do not match. We can do this in one step by entering

<5,-2,-3> x (<6,4,1> + <-3,7,2>) = <5,-2,-3> x <6,4,1> + <5,-2,-3> x <-3,7,2>

If the statement on the left of the = equals the statement on the right, W|A responds with True. If the statement on the left of the ≠ equals the statement on the right, W|A responds with False.

In this case, we get a True response and have verified the truth of the statement.

WolframAlpha

<5,-2,-3> x (<6,4,1> + <-3,7,2>) = <5,-2,-3> x <6,4,1> + <5,-2,-3> x <-3,7,2>

NATURAL LANGUAGE   MATH INPUT   EXTENDED KEYBOARD

Input

$\{5, -2, -3\} \times (\{6, 4, 1\} + \{-3, 7, 2\}) =$
   $\{5, -2, -3\} \times \{6, 4, 1\} + \{5, -2, -3\} \times \{-3, 7, 2\}$

Result

True

3.7 TRY THESE

1. Find the cross product of the vectors $\vec{u} = \langle -4, 3, 5 \rangle$ and $\vec{v} = \langle 5, -1, 2 \rangle$.

2. Find the cross product of the vectors $\vec{u} = \langle -2, 3, -9 \rangle$ and $\vec{v} = \langle 6, -9, 27 \rangle$.

3. Find the length of the vector formed by the cross product of the vectors $\vec{u} = \langle 3, -5, 4 \rangle$ and $\vec{v} = \langle 2, -4, 1 \rangle$.

4. Find the angle between the vectors $\vec{u} = \langle 4, -7, -6 \rangle$ and $\vec{v} = \langle 5, -1, 2 \rangle$.

5. Determine if the vectors $\vec{u} = \langle 3, -2, 1 \rangle$ and $\vec{v} = \langle 0, 2, 4 \rangle$ are perpendicular or parallel to each other.

6. Find the area of the parallelogram and the triangle formed by the vectors $\vec{u} = \langle 1, -2, -4 \rangle$ and $\vec{v} = \langle 4, 3, -5 \rangle$.