2.4 The Dot Product of Two Vectors, the Length of a Vector, and the Angle Between Two Vectors

THE DOT PRODUCT OF TWO VECTORS

The length of a vector or the angle between two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ can be found using the dot product.

The dot product of vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ is a scalar (real number) and is defined to be

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y$$

Since u_x, u_y, v_x and v_y are real numbers, you can see that the dot product is itself a real number and not a vector.

Example (1)

To compute the dot product of the vectors $\vec{u} = \langle 5, 2 \rangle$ and $\vec{v} = \langle 3, 4 \rangle$, we compute

$$\vec{u} \cdot \vec{v} = 5 \cdot 3 + 2 \cdot 4 = 15 + 8 = 23$$

Since the dot product is a scalar, it follows the properties of real numbers.

PROPERTIES OF THE DOT PRODUCT

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, the dot product is commutative
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, the dot product distributes over vector addition
3. $\vec{u} \cdot \vec{0} = 0$, the dot product with the zero vector, $\vec{0}$, is the scalar 0.
4. $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

Example (2)

Compute the dot product $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, where $\vec{u} = \langle 5, -2 \rangle$, $\vec{v} = \langle 6, 4 \rangle$, and $\vec{w} = \langle -3, 7 \rangle$.

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$\begin{aligned}\vec{u} \cdot (\vec{v} + \vec{w}) &= \langle 5, -2 \rangle \cdot \langle 6, 4 \rangle + \langle 5, -2 \rangle \cdot \langle -3, 7 \rangle \\ &= (5 \cdot 6 + (-2) \cdot 4) + (5 \cdot (-3) + (-2) \cdot 7) \\ &= 30 - 8 - 15 - 14 \\ &= -7\end{aligned}$$

THE LENGTH OF A VECTOR

The length (magnitude) of a vector you know is given by $\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$. The length can also be found using the dot product. If we dot a vector $\vec{v} = \langle v_x, v_y \rangle$ with itself, we get

$$\vec{v} \cdot \vec{v} = \langle v_x, v_y \rangle \cdot \langle v_x, v_y \rangle$$
$$\vec{v} \cdot \vec{v} = v_x \cdot v_x + v_y \cdot v_y$$
$$\vec{v} \cdot \vec{v} = {v_x}^2 + {v_y}^2$$

By Vector Property 4, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$. This gives $\|\vec{v}\|^2 = {v_x}^2 + {v_y}^2$.

Taking the square root of each side produces

$$\sqrt{\|\vec{v}\,\|^2} = \sqrt{{v_x}^2 + {v_y}^2}$$

$$\|\vec{v}\,\| = \sqrt{{v_x}^2 + {v_y}^2}$$

Which is the length of the vector $\vec{v}$.

> The dot product of a vector $\vec{v} = \langle v_x, v_y \rangle$ with itself gives the length of the vector.
>
> $$\|\vec{v}\,\| = \sqrt{{v_x}^2 + {v_y}^2}$$

Example (3)

Use the dot product to find the length of the vector $\vec{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

In this case, $v_x = 2$ and $v_y = 6$.

Using $\|\vec{v}\,\| = \sqrt{{v_x}^2 + {v_y}^2}$, we get

$$\|\vec{v}\,\| = \sqrt{2^2 + 6^2}$$

$$\|\vec{v}\,\| = \sqrt{40}$$

$$\|\vec{v}\,\| = \sqrt{4 \cdot 10}$$

$$\|\vec{v}\,\| = \sqrt{4} \cdot \sqrt{10}$$

$$\|\vec{v}\,\| = 2\sqrt{10}$$

The length of the vector $\vec{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is $2\sqrt{10}$ units.

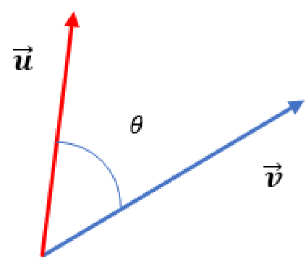
THE ANGLE BETWEEN TWO VECTORS

The dot product and elementary trigonometry can be used to find the angle θ between two vectors.

If θ is the smallest nonnegative angle between two non-zero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} \text{ or } \theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$$

where $0 \leq \theta \leq 2\pi$ and $\|\vec{u}\| = \sqrt{{u_x}^2 + {u_y}^2}$ and $\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$



Example (4)

Find the angle between the vectors $\vec{u} = \langle 5, -3\rangle$ and $\vec{v} = \langle 2, 4\rangle$.

Using $\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$, we get

$$\theta = \cos^{-1}\frac{\langle 5, -3\rangle \cdot \langle 2, 4\rangle}{\sqrt{5^2 + (-3)^2}\cdot\sqrt{2^2 + 4^2}}$$

$$\theta = \cos^{-1}\frac{5\cdot 2 + (-3)\cdot 4}{\sqrt{25 + 9}\cdot\sqrt{4 + 16}}$$

$$\theta = \cos^{-1}\frac{-2}{\sqrt{34}\cdot\sqrt{20}}$$

$$\theta = 94.4$$

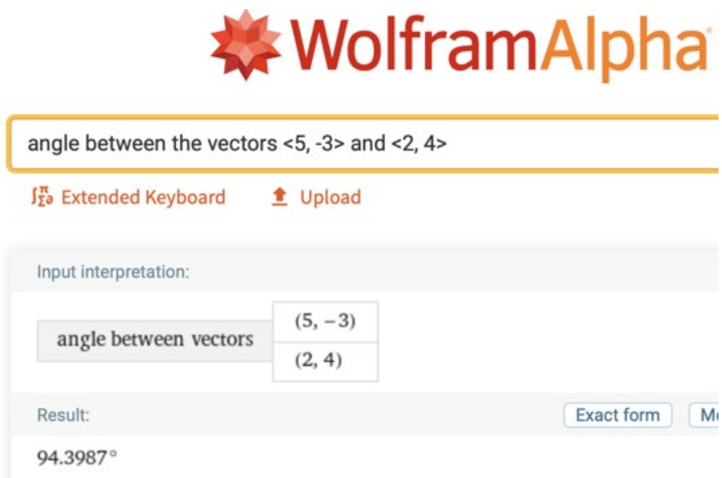
We conclude that the angle between these two vectors is close to 94.4°.

USING TECHNOLOGY

We can use technology to find the angle θ between two vectors.

Go to www.wolframalpha.com.

To find the angle between the vectors $\vec{u} = \langle 5, -3 \rangle$ and $\vec{v} = \langle 2, 4 \rangle$, enter angle between the vectors <5, -3> and <2, 4> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\theta = 94.4$, rounded to one decimal place.



TRY THESE

1. Find the dot product of the vectors $\vec{u} = \langle -2, 3 \rangle$ and $\vec{v} = \langle 5, -1 \rangle$.

ANS: $\vec{u} \cdot \vec{v} = -13$

2. Find the dot product of the vectors $\vec{u} = \langle -4, 6 \rangle$ and $\vec{v} = \langle 3, 2 \rangle$.

ANS: $\vec{u} \cdot \vec{v} = 0$

3. Find the length of the vector $\vec{u} = \langle 4, -7 \rangle$.

ANS: $\sqrt{65}$

4. Find the length of the vector $\vec{v} = \langle 0,5 \rangle$.

ANS: 5

5. Find the angle between the vectors $\vec{u} = \langle -2, 3 \rangle$ and $\vec{v} = \langle 5, -1 \rangle$.

ANS: $135°$

6. Find angle between the vectors $\vec{u} = \langle -4, 6 \rangle$ and $\vec{v} = \langle 3, 2 \rangle$.

ANS: $90°$

NOTE TO INSTRUCTOR

1. Perhaps begin by discussing the zero vector $\vec{0} = < 0{,}0 >$. This vector is represented by a single point. It has a length of measure 0.

2. Define the dot product of two vectors. Note that it is just a definition, and not derived. Then follow with an example.

Find the dot product of the vectors $\vec{u} = \langle -6, 2\rangle$ and $\vec{v} = \langle 3, 4\rangle$.

$$\vec{u} \cdot \vec{v} = -6 \cdot 3 + \ 3 \cdot 4 = -18 + 12 = -6$$

3. Although it is developed at the beginning of the chapter, consider proving that $||\vec{v}\,|| = \sqrt{{v_x}^2 + {v_y}^2}$. It is instructive for students to see proofs as it helps to develop their logic.

Proof: We want to show for a vector $\vec{v} = \langle v_x, v_y\rangle$, that $||\vec{v}\,|| = \sqrt{{v_x}^2 + {v_y}^2}$.

For a vector $||\vec{v}\,|| = \sqrt{{v_x}^2 + {v_y}^2}$,

$$\vec{v} \cdot \vec{v} = \langle v_x, v_y\rangle \cdot \langle v_x, v_y\rangle$$

$$\vec{v} \cdot \vec{v} = v_x \cdot v_x + v_y \cdot v_y$$

$$\vec{v} \cdot \vec{v} = {v_x}^2 + {v_y}^2$$

By Vector Property 4, $\vec{v} \cdot \vec{v} = \ ||\vec{v}||^2$. This gives $||\vec{v}||^2 = {v_x}^2 + {v_y}^2$.

Taking the square root of each side produces

$$\sqrt{||\vec{v}\,||^2} = \sqrt{{v_x}^2 + {v_y}^2}$$

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$

4. Find the length of the vector $\vec{v} = < -4, -3 >$.

Using $||\vec{v}|| = \sqrt{v_x{}^2 + v_y{}^2}$ with $v_x = -4$ and $v_y = -3$,

$$||\vec{v}|| = \sqrt{(-4)^2 + (-3)^2}$$

$$||\vec{v}|| = \sqrt{16 + 9}$$

$$||\vec{v}|| = \sqrt{25}$$

$$||\vec{v}|| = 5$$

Make a conclusion. The length of the vector $\vec{v} = < -4, -3 >$ is 5 units.

5. Discuss the angle between two vectors and show an example of how to use the inverse cosine on the calculator.

Example

Find the angle between the vectors $\vec{u} = \langle -7, 2\rangle$ and $\vec{v} = \langle 6, 3\rangle$.

Use $\theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{||\vec{u}\,|| \cdot ||\vec{v}\,||}$ with $\vec{u} = \langle -7, 2\rangle$, and $\vec{v} = \langle 6, 3\rangle$.

$$\vec{u} \cdot \vec{v} = -7 \cdot 6 + \ 2 \cdot 3 = -42 + 6 = -36$$

$$||\vec{u}|| = \sqrt{(-7)^2 + 2^2} = \sqrt{53},\ ||\vec{v}|| = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$$

$$\theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{||\vec{u}\,|| \cdot ||\vec{v}\,||} = \frac{-36}{\sqrt{53} \cdot 3\sqrt{5}} = 137.48$$

On the TI-84, input 2nd $\cos$(–36/(2nd x^2 $53 * 3 * 2\text{nd}x^2\ 5)$