Teacher Notes

Our approach is to present common problems found in a standard algebra course and "Try These" that closely match the problems given in the example set. Teacher notes include problems similar to the examples in the textbook; the numbers are different. We present ideas on possible ways to discuss a particular topic or concept in some cases.

You might:

1. Present example problems

2. Have students work through the "Try These" problems

3. Emphasize concepts

1.1 CONSTANTS, VARIABLES, AND EXPRESSIONS

Consider showing this on the board. You can use it to discuss the meanings of the terms constants, variables, and expressions.

> To produce a textbook, suppose the publisher spent $140,000 for typesetting and $5.50 per book for printing and binding.

a) Which of the two quantities is the variable quantity?
b) Which of the two quantities is the constant?
c) Write the expression that produces the cost of producing x number of books.
d) What is the cost of producing 1000 books?
e) What is the cost of producing the 1001st book?

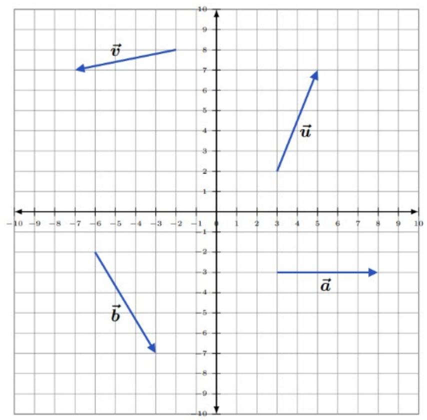
ANSWERS:

a) The variable is the number of textbooks to be printed and bound.
b) The constant is $140,000 for typesetting.
c) The expression that produces the cost of producing x number of books is
   5.50x$ + 140,000.
d) The cost of producing 1000 books is $5.50*(1000) + 140,000 = $145,500.
e) The cost of producing the 1001st books is
   (The cost of producing 1001 books) – (The cost of producing 1000 books)
   = ($5.50*(1001) + 140,000) – $145,500
   = $145,505.50 – $145,500
   = $5.5

2.1 VECTORS

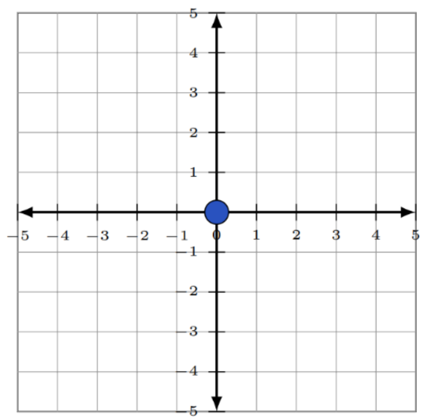
Consider showing these examples on the board.

1. Express the vectors $\vec{v}$, $\vec{u}$, $\vec{a}$, and $\vec{b}$ in component form.

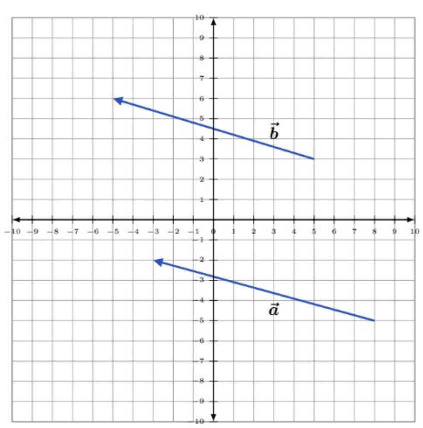


ANSWERS:
$\vec{v} = \langle -5, -1 \rangle$,
$\vec{u} = \langle 2, 5 \rangle$,
$\vec{a} = \langle 3, -5 \rangle$,
$\vec{b} = \langle 5, 0 \rangle$

2. Illustrate the zero vector, $\vec{0} = \langle 0, 0 \rangle$. This vector has zero magnitude and no direction.



3. Illustrate why the two vectors $\vec{a}$ and $\vec{b}$ are equal.



ANSWER:
They are equal because they have the same magnitude and direction.

2.2 ADDITION, SUBTRACTION, AND SCALAR MULTIPLICATION OF VECTORS

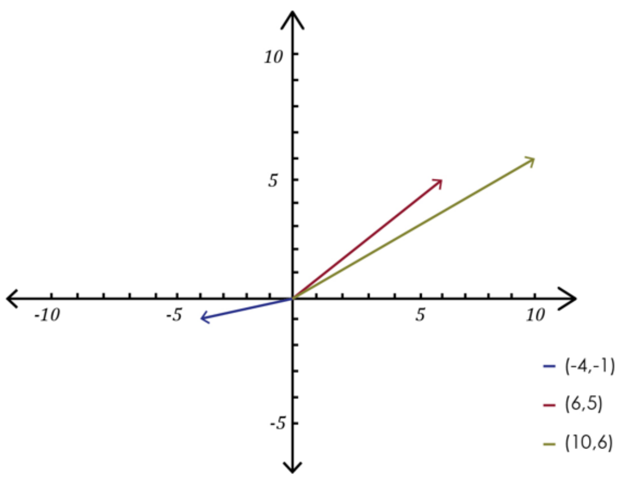
Consider showing these examples on the board.

1. Using the vectors $\vec{u} = \langle -4, -1 \rangle$ and $\vec{v} = \langle 6, 5 \rangle$, show addition using both the arrows originating at the origin and then by placing the tail of $\vec{u}$ onto the head of $\vec{v}$.

ANS: $\vec{u} + \vec{v} = \langle -4 + 6, -1 + 5 \rangle = \langle 2, 4 \rangle$

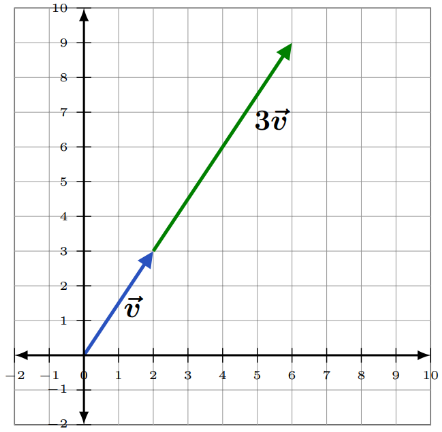
2. Subtract the vector $\vec{u} = \langle -4, -1 \rangle$ from the vector $\vec{v} = \langle 6, 5 \rangle$.

$$\vec{v} - \vec{u} = \langle 6 - (-4), 5 - (-1) \rangle = \langle 6 + 4, 5 + 1 \rangle = \langle 10, 6 \rangle$$



ANS: $\vec{u} - \vec{v} = \langle -10, -6 \rangle$

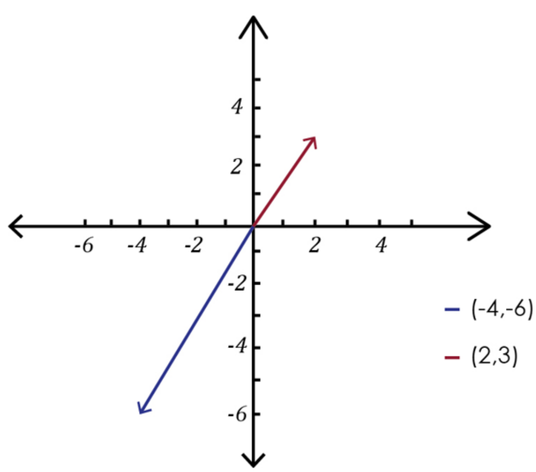
3. Multiply the vector $\vec{v} = \langle 2, 3 \rangle$ by the scalar 3.



ANS: $3\vec{v} = 3\langle 2, 3 \rangle = \langle 6, 9 \rangle$

Draw $\vec{v}$ in one color and $3\vec{v}$ in another color. Point out how the length of vector $\vec{v}$ tripled. That is, $3\vec{v}$ should look 3 times as long as $\vec{v}$. It can be a bit hard to show because the vectors will appear to be on top of the other.

4. Multiply the vector $\vec{v} = \langle 2, 3 \rangle$ by the scalar -2.

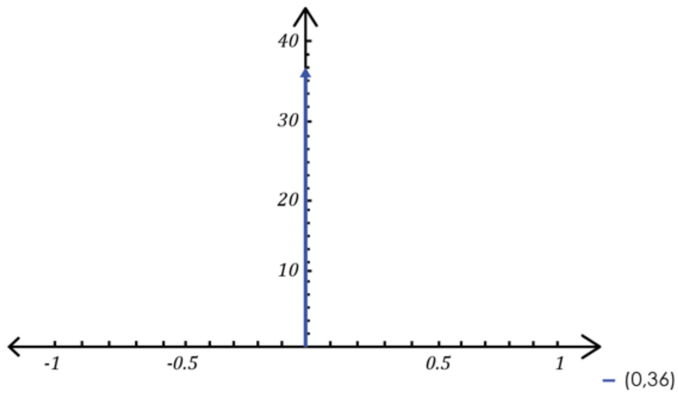


ANS:$-2\vec{v} = -2\langle 2, 3 \rangle = \langle -4, -6 \rangle$

Draw $\vec{v}$ in one color and $-2\vec{v}$ in another color. Point out how the length of vector $\vec{v}$ doubled and points in the opposite direction of $\vec{v}$. That is, -2$\vec{v}$ should look twice as long as $\vec{v}$ but pointing in the opposite direction.

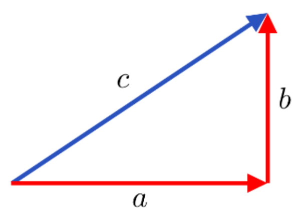
5. Suppose $\vec{u} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$. Find $5\vec{u} + 2\vec{v}$.

$$5\vec{u} + 2\vec{v} = 5\begin{bmatrix} -2 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -10 \\ 30 \end{bmatrix} + \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 36 \end{bmatrix}$$



2.3 MAGNITUDE, DIRECTION, AND COMPONENTS OF A VECTOR

1. Remind students of the Pythagorean Theorem.



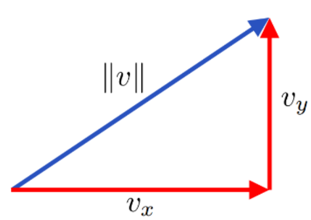
$$a^2 + b^2 = c^2$$

2. Consider deriving the magnitude of a vector $\vec{v} = \langle v_x, v_y \rangle$ using the Pythagorean Theorem. Note that the v_x and the v_y in $\langle v_x, v_y \rangle$ represent the lengths of the horizontal and vertical components, respectively, of $\vec{v}$.

$$||\vec{v}||^2 = {v_x}^2 + {v_y}^2$$

$$\sqrt{||\vec{v}||^2} = \sqrt{{v_x}^2 + {v_y}^2}$$

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$



Use as an example of the vector $\vec{v} = \langle 6, 3 \rangle$. The magnitude of $\vec{v} = \langle 6, 3 \rangle$ is

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$
$$||\vec{v}|| = \sqrt{6^2 + 3^2}$$
$$||\vec{v}|| = \sqrt{36 + 9}$$
$$||\vec{v}|| = \sqrt{45}$$
$$||\vec{v}|| = \sqrt{9 \cdot 5}$$
$$||\vec{v}|| = \sqrt{9} \cdot \sqrt{5}$$
$$||\vec{v}|| = 3\sqrt{5}$$

3. Demonstrate how to find the magnitude of $\vec{v} = \langle -5, 4 \rangle$.

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$
$$||\vec{v}|| = \sqrt{(-5)^2 + 4^2}$$
$$||\vec{v}|| = \sqrt{25 + 16}$$
$$||\vec{v}|| = \sqrt{41}$$

4. Find the components of the vector $\vec{v}$ if the magnitude of $\vec{v}$ is 7 and it makes a 30° angle with the horizontal.

$$\vec{v}_x = \|\vec{v}\|\cos\theta = 7\cos30° = 7\cdot\frac{\sqrt{3}}{2} = \frac{7\sqrt{3}}{2}$$

$$\vec{v}_y = \|\vec{v}\|\sin\theta = 7\sin30° = 7\cdot\frac{1}{2} = \frac{7}{2}$$

So, $\vec{v}_x = \frac{7\sqrt{3}}{2}$ and $\vec{v}_y = \frac{7}{2}$

5. Approximate the components of the vector $\vec{v}$ if the magnitude of $\vec{v}$ is 16 and it makes a 128° angle with the horizontal.

$$\vec{v}_x = \|\vec{v}\|\cos\theta = 16\cos128° \approx 16\cdot(-0.616) \approx -9.86$$

$$\vec{v}_y = \|\vec{v}\|\sin\theta = 16\sin128° \approx 16\cdot(0.788) \approx 12.61$$

So, $\vec{v}_x = -9.86$ and $\vec{v}_y = 12.61$

6. Approximate the direction of the vector $\vec{v} = \langle 2, 7\rangle$.

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\theta = \tan^{-1}\frac{7}{2}$$

Using a calculator, we get

$$\theta = 74.0546041°$$
$$\theta = 74.05°$$

2.4 THE DOT PRODUCT OF TWO VECTORS, THE LENGTH OF A VECTOR, AND THE ANGLE BETWEEN TWO VECTORS

1. Perhaps begin by discussing the zero vector $\vec{0} = < 0,0 >$. This vector is represented by a single point. It has a length of measure 0.

2. Define the dot product of two vectors. Note that it is just a definition, and not derived. Then follow with an example.

Find the dot product of the vectors $\vec{u} = \langle -6, 2 \rangle$ and $\vec{v} = \langle 3, 4 \rangle$.

$$\vec{u} \cdot \vec{v} = -6 \cdot 3 + \ 3 \cdot 4 = -18 + 12 = -6$$

3. Although it is developed at the beginning of the chapter, consider proving that $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$. It is instructive for students to see proofs as it helps to develop their logic.

Proof: We want to show for a vector $\vec{v} = \langle v_x, v_y \rangle$, that $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$.

For a vector $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$,

$$\vec{v} \cdot \vec{v} = \langle v_x, v_y \rangle \cdot \langle v_x, v_y \rangle$$

$$\vec{v} \cdot \vec{v} = v_x \cdot v_x + v_y \cdot v_y$$

$$\vec{v} \cdot \vec{v} = {v_x}^2 + {v_y}^2$$

By Vector Property 4, $\vec{v} \cdot \vec{v} = \ ||\vec{v}||^2$. This gives $||\vec{v}||^2 = {v_x}^2 + {v_y}^2$.

Taking the square root of each side produces

$$\sqrt{||\vec{v}||^2} = \sqrt{{v_x}^2 + {v_y}^2}$$

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$

4. Find the length of the vector $\vec{v} = < -4, -3 >$.

Using $\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$ with $v_x = -4$ and $v_y = -3$,

$$\|\vec{v}\| = \sqrt{(-4)^2 + (-3)^2}$$

$$\|\vec{v}\| = \sqrt{16 + 9}$$

$$\|\vec{v}\| = \sqrt{25}$$

$$\|\vec{v}\| = 5$$

Make a conclusion. The length of the vector $\vec{v} = < -4, -3 >$ is 5 units.

5. Discuss the angle between two vectors and show an example of how to use the inverse cosine on the calculator.

Find the angle between the vectors $\vec{u} = \langle -7, 2 \rangle$ and $\vec{v} = \langle 6, 3 \rangle$.

Use $\theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$ with $\vec{u} = \langle -7, 2 \rangle$, and $\vec{v} = \langle 6, 3 \rangle$.

$$\vec{u} \cdot \vec{v} = -7 \cdot 6 + 2 \cdot 3 = -42 + 6 = -36$$

$$\|\vec{u}\| = \sqrt{(-7)^2 + 2^2} = \sqrt{53}, \ \|\vec{v}\| = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$$

$$\theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|} = \frac{-36}{\sqrt{53} \cdot 3\sqrt{5}} = 137.48$$

On the TI-84, input $2^{\text{nd}}\cos(-36/(2^{\text{nd}}x^2\ 53 * 3 * 2\text{nd}x^2\ 5)$

2.5 PARALLEL AND PERPENDICULAR VECTORS, THE UNIT VECTOR

1. Show that $\vec{u} = \langle 1, 4\rangle$ and $\vec{v} = \langle 4, 16\rangle$ are parallel to each other.

Method 1

$$\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||}$$

$$\theta = \cos^{-1}\frac{\langle 1,4\rangle\cdot\langle 4,16\rangle}{\sqrt{1^2+4^2}\cdot\sqrt{4^2+16^2}}$$

$$\theta = \cos^{-1}\frac{1\cdot 4+4\cdot 16}{\sqrt{1+16}\cdot\sqrt{16+256}}$$

$$\theta = \cos^{-1}\frac{68}{\sqrt{17}\cdot\sqrt{272}}$$

$$\theta = 0°$$

Method 2

Make sure your calculator is in degree mode, not radian mode.

Show that $\vec{u} = c\vec{v}$. Notice that $\vec{v} = 4\vec{u}$.

$$\langle 4, 16\rangle = 4\langle 1,4\rangle$$

2. Show that the vectors $\vec{u} = \langle 2,\ 1\rangle$ and $\vec{v} = \langle 3, -6\rangle$ are perpendicular to each other.

Method 1

$$\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||}$$

$\theta = \cos^{-1}((\langle 2,1\rangle\cdot\langle 3,-6\rangle)/(2\text{^}2+1\text{^}2)\cdot\sqrt{}(3\text{^}2+$
$6\text{^}2\ ))$

$$\theta = \cos^{-1}\frac{2\cdot 3+1\cdot(-6)}{\sqrt{4+1}\cdot\sqrt{9+36}}$$

$$\theta = \cos^{-1}\frac{0}{\sqrt{5}\cdot\sqrt{45}} = 0$$

$$\theta = 90°$$

Method 2

The dot product of these two vectors is 0.

3. Find the unit vector corresponding to the vector $\vec{v} = \langle 4, 3 \rangle$.

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} = \frac{\langle 4, 3 \rangle}{\sqrt{4^2 + 3^2}}$$

$$\hat{v} = \frac{\langle 4, 3 \rangle}{\sqrt{16 + 9}}$$

$$\hat{v} = \frac{\langle 4, 3 \rangle}{\sqrt{25}}$$

$$\hat{v} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$$

2.6 THE VECTOR PROJECTION OF ONE VECTOR ONTO ANOTHER

When presenting the projection formula, consider pointing out that the numerator is a dot product and a scalar (a real number). The denominator is a length, so it, too, is a scalar. A scalar divided by a scalar is also a scalar, so the formula shows a vector is multiplied by a scalar. That is, it shows a vector scaled longer or shorter. That scaled vector is the projection.

Consider working through these problems as examples.

1. Find the projection of the vector $\vec{v} = \langle 1, 4 \rangle$ onto the vector $\vec{u} = \langle 2, 5 \rangle$.

ANS: $\left\langle \frac{44}{29}, \frac{110}{29} \right\rangle$

2. Find $\text{proj}_{\vec{v}}\vec{u}$, where of $\vec{u} = \langle 2, -4 \rangle$ onto $\vec{v} = \langle 5, 5 \rangle$.

ANS: $\langle -1, -1 \rangle$

3. Find $\text{proj}_{\vec{v}}\vec{u}$, where of $\vec{u} = \langle 5, 10 \rangle$ onto $\vec{v} = \langle 6, -3 \rangle$.

ANS: $\langle 0, 0 \rangle$ These two vectors are orthogonal (perpendicular to each other).

3.1 THREE DIMENSIONAL VECTORS

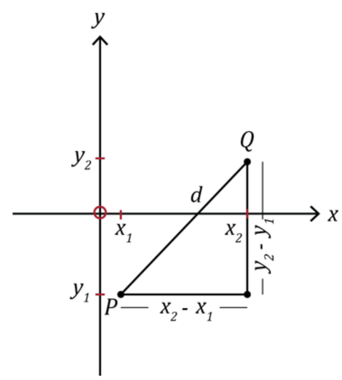
Consider deriving the formula for the distance between two points. Let the two points be $P(x_1, y_1)$ and $Q(x_2, y_2)$. Draw the two points and use the Pythagorean Theorem.

By the Pythagorean Theorem,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Take square roots to get

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



The equation of a circle comes from the distance formula by using one of the points, say $P(x_1, y_1)$, as the center $C(h, k)$ and the other point, say $Q(x_2, y_2)$, as a general point (x, y) on the circle. A circle is defined as a closed plane curve consisting of all points (x, y) at a given distance r from a point within the curve. Use the distance formula replacing d with r and $(x_2 - x_1)^2$ with $(x - h)^2$ and $(y_2 - y_1)^2$ with $(y - k)^2$.

Consider working through these problems as examples.

1. Find the distance between the two points $(3, -5)$ and $(2, -3)$. Round to one decimal place.

ANS: $\sqrt{5} \approx 2.2$ units

2. Find the distance between the two points $(-1, -2, -3)$ and $(4, -6, 1)$. Round to one decimal place.

ANS: $\sqrt{57} \approx 7.5$ units

3. Write the equation of a circle that has the point $C(3, 6)$ as its center and radius 2.

ANS: $(x - 3)^2 + (y - 6)^2 = 4$

4. Write the equation of a sphere that has the point $C(-4, -3, 7)$ as its center and radius 3.

ANS: $\big(x - (-4)\big)^2 + \big(y - (-3)\big)^2 + (y - 7)^2 = 3^2$
$(x + 4)^2 + (y + 3)^2 + (y - 7)^2 = 9$

3.2 MAGNITUDE AND DIRECTION COSINES OF A VECTOR

Consider presenting the formulas then working through these example problems.

1. Find the magnitude of the vector $\vec{v} = \langle 4, 1, -3 \rangle$.

ANS: $||\vec{v}|| = \sqrt{26}$

2. Find the magnitude of the vector $\vec{v} = \langle 8, -8 \rangle$.

ANS: $||\vec{v}|| = 8\sqrt{2}$

3. Find the direction cosines of the vector $\vec{v} = \langle 4, 1, -3 \rangle$. Round to three decimal places.

ANS: {0.784, 0.196, -0.588}

4. Approximate the vector $\vec{v}$ that has magnitude 30 and direction cosines $\cos\alpha = -3/5$, $cos\beta = -1/2$, $\cos\theta = 7/10$.

ANS: $< -18, -15, 21 >$

3.3 ARITHMETIC ON VECTORS IN 3-DIMENSIONAL SPACE

Consider working through these examples.

1. Add the vectors $\vec{u} = \langle 3, -4, 5\rangle$ and $\vec{v} = \langle -1, 4, 2\rangle$.

ANS: $\vec{u} + \vec{v} = \langle 2, 0, 7\rangle$

2. Subtract the vector $\vec{v} = \langle -5, 2, 1\rangle$ from the vector $\vec{u} = \langle -9, 4, -3\rangle$.

ANS: $\vec{u} - \vec{v} = \langle -4, 2, -4\rangle$

3. Given the three vectors, $\vec{u} = \langle 1, -2, -3\rangle$, $\vec{v} = \langle 4, 3, 2\rangle$, and $\vec{w} = \langle 1, -1, 1\rangle$,
   a. Find $2\vec{u} + 3\vec{v} - 4\vec{w}$.
   b. Find the length of the vector $2\vec{u} + 3\vec{v} - 4\vec{w}$.

ANS: a. $2\vec{u} + 3\vec{v} - 4\vec{w} = \langle 10, 9, -4\rangle$
b. $\sqrt{197}$

4. Suppose $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -5 \\ 2 \\ 6 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 6 \\ 0 \\ -6 \end{bmatrix}$, find $3\vec{u} - 4\vec{v} - 2\vec{w}$.

ANS: $3\vec{u} - 4\vec{v} - 2\vec{w} = \langle 11, -5, -21\rangle$

3.4 THE UNIT VECTOR IN 3-DIMENSIONS AND VECTORS IN STANDARD POSITION

Consider working through these examples.

1. Write the unit vector that corresponds to $\vec{v} = \langle 9, -2, 6 \rangle$.

ANS: $\frac{9}{11}\hat{\imath} - \frac{2}{11}\hat{\jmath} + \frac{6}{11}\hat{k}$

2. Write the unit vector that corresponds to $\vec{v} - \vec{u} = \langle 7, -2, 8 \rangle - \langle 3, 3, 6 \rangle$.

ANS: $\frac{4}{3\sqrt{5}}\hat{\imath} - \frac{\sqrt{5}}{3}\hat{\jmath} + \frac{2}{3\sqrt{5}}\hat{k}$

3. Normalize the vector $\vec{v} = \langle 4, -4, 5 \rangle$.

ANS: $\frac{4}{\sqrt{57}}\hat{\imath} - \frac{4}{\sqrt{57}}\hat{\jmath} + \frac{5}{\sqrt{57}}\hat{k}$

3.5 THE DOT PRODUCT, LENGTH OF A VECTOR, AND THE ANGLE BETWEEN TWO VECTORS IN THREE DIMENSIONS

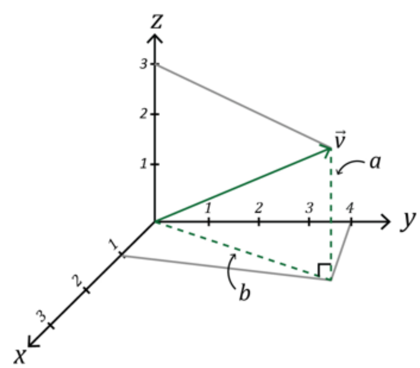
1. Define the dot product of two vectors. Note that it is just a definition, and not derived. Then follow with an example.

Find the dot product of the vectors $\vec{u} = \langle -6, 2, 4\rangle$ and $\vec{v} = \langle 3, 4, -2\rangle$.

$$\vec{u} \cdot \vec{v} = -6 \cdot 3 + \ 2 \cdot 4 + 4 \cdot (-2) = -18 + 8 - 8 = -18$$

2. Consider using an example to demonstrate why $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$. Show that the length of the vector $\vec{v} = \langle 1, 4, 3\rangle = \sqrt{1^2 + 4^2 + 3^2}$.

Let's start by drawing this vector.



Notice that for the vector $\vec{v}$, there is a right triangle with base b and height a.
We can use the Pythagorean Theorem to find the length of $\vec{v}$.

$$||\vec{v}|| = \sqrt{b^2 + a^2}$$

Since the height of the triangle from the xy-plane to the tip of v is 3 units, so, $a = 3$, we have

$$||\vec{v}|| = \sqrt{b^2 + 3^2}$$

Now b is itself the hypotenuse to its own triangle in the xy-plane. So,

$$b^2 = 1^2 + 4^2$$

Now, we have the length of $\vec{v}$.

$$||\vec{v}|| = \sqrt{1^2 + 4^2 + \ 3^2}$$

In general, you can see that the length of $\vec{v}$ is the square root of the sum of the squares of $\vec{v}$'s components.

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

Consider as examples:

1. Find the dot product of the vectors $\vec{u} = \langle 2, 3, -4\rangle$ and $\vec{v} = \langle 5, 5, 3\rangle$.

ANS: $\vec{u} \cdot \vec{v} = 13$

2. Find the length of the vector $\vec{u} = \langle -2, -8, 5\rangle$.

ANS: $\sqrt{93}$

3. Find the length of the vector $\vec{v} = 5\langle 4, 1, -3\rangle - 3\langle -1, -2, 4\rangle$.

ANS: $\sqrt{83}$

4. Find the angle between the vectors $\vec{u} = \langle 2, -6, 3\rangle$ and $\vec{v} = \langle -3, 10, 2\rangle$.

ANS: $143.7°$

5. Find the angle between the vectors $\vec{u} = \langle -3, -2, 5\rangle$ and $\vec{v} = \langle 6, 4, 10\rangle$.

ANS: $71.6°$

6. Find a vector $\vec{v}$ that is perpendicular to $\vec{u} = \langle 4, 6, 8\rangle$.

Two vectors are perpendicular to each other if the angle between them is 90° = $\pi/2$. The cosine of the angle between two vectors is given by $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$. We know that $\theta = \pi/2$.

The $\cos(\pi/2) = 0$ gives us $\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} = 0$

A fraction is 0 only when the numerator is 0, so we are looking for a vector $\vec{v}$ such that $\vec{u} \cdot \vec{v} = 0$. (Recall this from a previous section.)
Let the unknown vector be $\vec{v} = \langle a, b, c\rangle$. Then

$$\vec{u} \cdot \vec{v} = \langle 4, 6, 8 \rangle \cdot \langle a, b, c \rangle = 0$$

$$4a + 6b + 8c = 0$$

Choose any numbers at all for a and b. Suppose $a = 5, b = 6$, then

$$4 \cdot 5 + 6 \cdot 5 + 8c = 0$$

$$20 + 30 + 8c = 0$$

$$50 + 8c = 0$$

$$8c = -50$$

$$c = -\frac{50}{8} = -\frac{25}{4}$$

So, a vector $\vec{v}$ that is perpendicular to $\vec{u} = \langle 4, 6, 8 \rangle$ is the vector $\vec{v} = \langle 5, 6, -\frac{25}{4} \rangle$.

3.6 THE CROSS PRODUCT: ALGEBRA

Consider demonstrating these examples.

1. Find the cross product of the vectors $\vec{u} = \langle 5, -2, 6 \rangle$ and $\vec{v} = \langle 2, -1, 3 \rangle$.

ANS: $\vec{u} \times \vec{v} = \langle 0, -3, -1 \rangle$

2. Find the cross product of the vectors $\vec{u} = \langle -5, 6, -2 \rangle$ and $\vec{v} = \langle 15, -18, 6 \rangle$.

ANS: $\vec{u} \times \vec{v} = \vec{0}$

3.7 THE CROSS PRODUCT: GEOMETRY

Consider as examples:

1. Find the cross product of the vectors $\vec{u} = \langle 2, 3, -4 \rangle$ and $\vec{v} = \langle 5, 5, 3 \rangle$.

ANS: $\vec{u} \times \vec{v} = \langle 29, -26, -5 \rangle$

2. Find the cross product of the vectors $\vec{u} = \langle -4, -2, 4 \rangle$ and $\vec{v} = \langle 8, 4, -8 \rangle$.

ANS: $\vec{u} \times \vec{v} = \langle 0, 0, 0 \rangle = \vec{0}$

3. Find the length of the vector $\vec{v} = 3\langle 5, -1, 3 \rangle - 2\langle 2, -3, 1 \rangle$.

ANS: $\sqrt{17}$

4. Find the angle between the vectors $\vec{u} = \langle 2, -6, 3 \rangle$ and $\vec{v} = \langle -3, 10, 2 \rangle$.

ANS: $\theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$

$\theta = \cos^{-1} \frac{\langle 2, -6, 3 \rangle \cdot \langle -3, 10, 2 \rangle}{\sqrt{2^2 + (-6)^2 + 3^2} \cdot \sqrt{(-3)^2 + (10)^2 + 2^2}}$

$\theta = \cos^{-1} \frac{2 \cdot (-3) + (-6) \cdot (10) + 3 \cdot 2}{\sqrt{49} \cdot \sqrt{113}}$

$\theta = \cos^{-1} \frac{-60}{\sqrt{49} \cdot \sqrt{113}}$

$\theta = 143.74°$

5. Find the angle between the vectors $\vec{u} = \langle -3, -2, 5 \rangle$ and $\vec{v} = \langle 6, 4, 10 \rangle$.

ANS: $71.59°$

4.2 ADDITION, SUBTRACTION, SCALAR MULTIPLICATION, AND PRODUCTS OF ROW AND COLUMN MATRICES

1. Consider using the matrices

$$A = \begin{bmatrix} 1 & 4 & -5 \\ 2 & 0 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 & -2 \\ 3 & 1 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$$

a. Illustrate matrix addition, $A + B$
b. Illustrate matrix subtraction, $A - B$
c. Illustrate an addition that is not defined, $A + C$
d. Illustrate the commutativity of matrix addition, $A + B = B + A$

2. Consider using the matrices

$$A = [4 \quad 2 \quad -2 \quad 3], \quad B = \begin{bmatrix} 2 \\ 6 \\ 1 \\ 5 \end{bmatrix}, \quad C = [2 \quad 3 \quad 4]$$

a. Illustrate row and column matrix multiplication, $A \cdot B$
b. Illustrate the commutativity of row and column matrix
c. Illustrate multiplication, $A \cdot B = B \cdot A$
d. Illustrate how dimension matters by showing that $B \cdot C$ is not defined

3. Perhaps use the following example as a motivation for row and column matrix multiplication.

Suppose your business sells three sizes of artist's paint brushes, small-sized brushes, medium-sized brushes, and large-sized brushes. Small brushes sell for \$15 each, medium brushes for \$20 each, and large brushes for \$25 each. What would your total revenue be if you sold 50 small-sized artist's brushes, 40 medium-sized brushes, and 30 large-sized brushes?

Using $R = np$, your revenue from the sale of the
small brushes is $R = 50 \cdot \$15 = \$750$
medium brushes is $R = 40 \cdot \$20 = \$800$
large brushes is $R = 30 \cdot \$25 = \$750$

The total revenue is just the sum of these three products,
$50 \cdot \$15 + 40 \cdot \$20 + 30 \cdot \$25 = \$750 + \$800 + \$750 = \$2300.$

We can compute the total revenue using two matrices and matrix multiplication.
Let the first matrix be the row matrix of the number of brushes sold,

$$N = [50 \quad 40 \quad 30]$$

and the second matrix be the column matrix of the number of boxes sold.

$$P = \begin{bmatrix} \$15 \\ \$20 \\ \$25 \end{bmatrix}$$

The total revenue is the matrix product $R = NP$.

$$R = \begin{bmatrix} 50 & 40 & 30 \end{bmatrix} \cdot \begin{bmatrix} \$15 \\ \$20 \\ \$25 \end{bmatrix}$$

$$= [50 \cdot \$15 + 40 \cdot \$20 + \ 30 \cdot \$25]$$

$$= [\$2300]$$

4.3 MATRIX MULTIPLICATION

1. Consider using the matrices $A = \begin{bmatrix} 4 & 2 & -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 6 \\ 1 \\ 5 \end{bmatrix}$ to remind us of the process of row and column matrix multiplication, $A \cdot B$.

2. Consider using the matrices

$$A = \begin{bmatrix} 1 & 4 & -5 \\ 2 & 0 & 3 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 0 & 5 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}, \ D = \begin{bmatrix} 4 & 2 \\ 7 & 3 \end{bmatrix}$$

   a. To illustrate matrix multiplication, $A \cdot B$
   b. And another multiplication, C^2
   c. That a matrix multiplication may not be defined, $B \cdot A$
   d. That a matrix multiplication is not necessarily commutative, $C \cdot D \neq D \cdot C$

4.4 ROTATION MATRICES IN 2-DIMENSIONS

Note that we plan to rotate some vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ through some angle θ to the new position given by the vector $v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and to do so, we will use the rotation matrix, a matrix that rotates points in the xy-plane counterclockwise through an angle θ relative to the x-axis.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Consider demonstrating these rotations:

1. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$ is rotated 90° counterclockwise.

Using the rotation formula $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ with $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$ and $\theta = 90°$, we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \cdot 5 + (-1) \cdot (-5) \\ 1 \cdot 5 + 0 \cdot (-5) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

When rotated counterclockwise 90°, the vector $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$ becomes $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

- If your class knows some trig, you can show the conversion of

  $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix}$ to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

  Since $\cos 90° = 0$ and $\sin 90° = 1$

- If trig is a challenge, use WolframAlpha to perform the matrix multiplication.

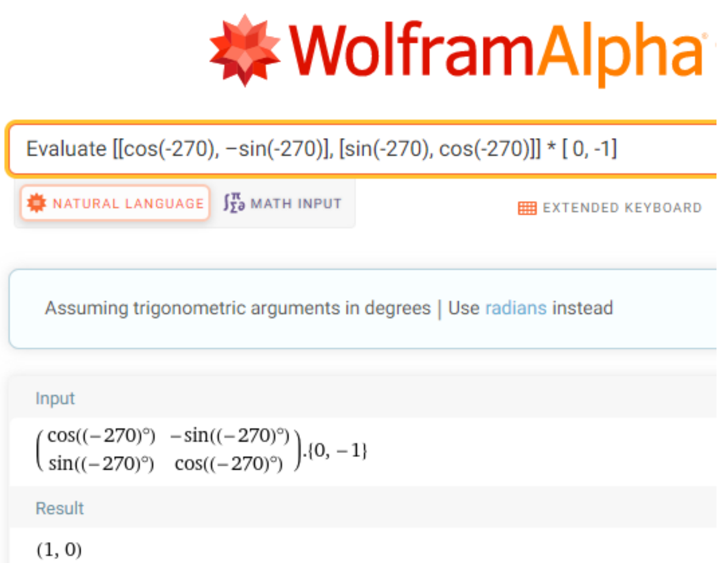
Go to www.wolframalpha.com.

To find rotation of the vector, enter Evaluate [[cos(90), –sin(90)], [sin(90), cos(90)]] * [5, -5] into the entry field. WolframAlpha tells you what it thinks you entered, then tells you its answer.

Be sure to write a conclusion so your students know to do so.

When rotated counterclockwise 90°, the vector $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$ becomes $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

2. The rotation formula works for clockwise rotations. We just need to make the angle of rotation negative.
   Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is rotated –270°.



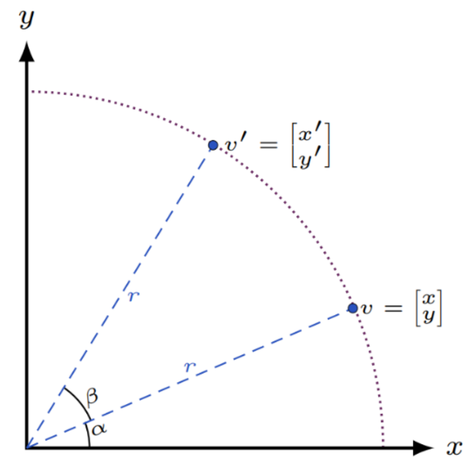
When rotated clockwise 90°, the vector $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ becomes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

DERIVING THE ROTATION FORMULA

If your class knows some trig, you may wish to derive the rotation formula.

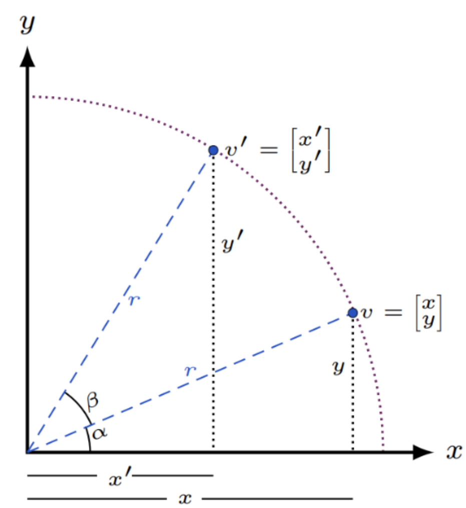
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

We wish to derive a formula that rotates a vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ counterclockwise through some angle θ to the new position given by the vector $v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$.



We wish to rotate the vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ through an angle β around the origin.
We know that in general,

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \& \quad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$



The figure shows that for angle α,

$$\begin{cases} \cos\alpha = \dfrac{x}{r} \\ \sin\alpha = \dfrac{y}{r} \end{cases} \rightarrow \begin{cases} x = r\cdot\cos\alpha \\ y = r\cdot\sin\alpha \end{cases}$$

Also

$$\begin{cases} \cos(\alpha+\beta) = \dfrac{x'}{r} \\ \sin(\alpha+\beta) = \dfrac{y'}{r} \end{cases} \rightarrow \begin{cases} x' = r\cdot\cos(\alpha+\beta) \\ y' = r\cdot\sin(\alpha+\beta) \end{cases}$$

By the trigonometry addition identity,

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta}$$

$$\begin{aligned} x' &= r\cdot\cos(\alpha+\beta) \\ &= r\cdot(\cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta) \\ &= r\cdot\cos\alpha\cos\beta - r\cdot\sin\alpha\sin\beta \end{aligned}$$

Then, since $x = r\cdot\cos\alpha$ and $y = r\cdot\sin\alpha$

replace with x and with y

$x' = x\cdot\cos\beta - y\cdot\sin\beta$

Similarly, by the trigonometry addition identity,

$$\boxed{\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta}$$

$$\begin{aligned} y' &= r\cdot\sin(\alpha+\beta) \\ &= r\cdot(\sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta) \\ &= r\cdot\sin\alpha\cos\beta + r\cdot\cos\alpha\sin\beta \end{aligned}$$

Then, since $y = r\cdot\sin\alpha$ and $x = r\cdot\cos\alpha$

replace with y and with x

$y' = y\cdot\cos\beta + x\cdot\sin\beta$

Rewrite this as $y' = x\cdot\sin\beta + y\cdot\cos\beta$

Now we have $\begin{cases} x' = x\cdot\cos\beta - y\cdot\sin\beta \\ y' = x\cdot\sin\beta + y\cdot\cos\beta \end{cases}$

Putting these two results into matrix form, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$.

Replacing β with θ to match our notation, we get $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

And we have produced the rotation formula.

4.5 FINDING THE ANGLE OF ROTATION BETWEEN TWO ROTATED VECTORS IN 2-DIMENSIONS

Note that to find the angle θ between the two vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$, we use the rotation formula in reverse.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

produces the system of equations

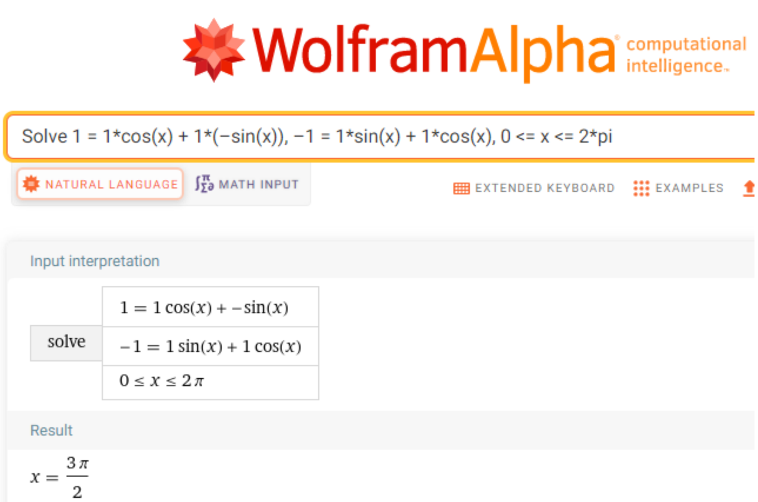
$$\begin{cases} x' = x \cdot \cos\theta + y \cdot (-\sin\theta) \\ y' = x \cdot \sin\theta + y \cdot \cos\theta \end{cases}$$

Consider demonstrating this example.

Find the angle θ through which the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is rotated to become $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\begin{cases} x' = x \cdot \cos\theta + y \cdot (-\sin\theta) \\ y' = x \cdot \sin\theta + y \cdot \cos\theta \end{cases} \Rightarrow \begin{cases} 1 = 1 \cdot \cos\theta + 1 \cdot (-\sin\theta) \\ -1 = 1 \cdot \sin\theta + 1 \cdot \cos\theta \end{cases}$$

Use W|A to solve this system. Go to www.wolframalpha.com and enter
Solve 1 = 1*cos(x) + 1*(–sin(x)), –1 = 1*sin(x) + 1*cos(x), 0 <= x <= 2*pi into the entry field.
Both entries and rows are separated by commas as W|A does not see spaces.
Wolframalpha tells you what it thinks you entered, then tells you its answer



We conclude that the angle of rotation is $\frac{3\pi}{2} = 270° = -90°$.

USING THE DOT PRODUCT TO FIND THE ANGLE BETWEEN TWO VECTORS

You may wish to point out that the angle of rotation can also be found using the dot product formula from Chapter 2.4.

If θ is the smallest nonnegative angle between two non-zero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} \quad \text{or} \quad \theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$$

where $0 \le \theta \le 2\pi$ and $\|\vec{u}\| = \sqrt{{u_x}^2 + {u_y}^2}$ and $\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$.

Find the angle between the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$.

Using $\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$, we get

$$\theta = \cos^{-1}\frac{\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\end{bmatrix}}{\sqrt{1^2+1^2}\cdot\sqrt{1^2+(-1)^2}}$$

$$\theta = \cos^{-1}\frac{1\cdot 1 + 1\cdot(-1)}{\sqrt{1+1}\cdot\sqrt{1+1}}$$

$$\theta = \cos^{-1}\frac{0}{\sqrt{1}\cdot\sqrt{1}}$$

$$\theta = \cos^{-1}0$$

$$\theta = \frac{\pi}{2} = 90°$$

4.6 ROTATION MATRICES IN 3-DIMENSIONS

Consider noting that if we are familiar with the 2-D rotation matrices of Chapter 4.4, then the 3-D rotation matrix for rotating a vector around the x–axis may not be a surprise.

- For 2-D, the rotation matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- For 3-D, the rotation matrix around the x–axis is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

Look closely at the last two rows and two columns.

The 2-D rotation matrix shows up in the 3-D rotation matrix. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

The fist column in the 3-D rotation matrix is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

These are the numbers that when the multiplication is performed, keep the x-component the same.

$$\begin{bmatrix} 1 & \square & \square \\ 0 & \square & \square \\ 0 & \square & \square \end{bmatrix}$$

What happens in the y and z components is just a rotation in the plane. The y-axis rotates counterclockwise toward the z-axis, so in the lower right part of the 3x3 matrix, we will have the standard rotation matrix that we saw for the plane in Chapter 4.4.

Consider demonstrating these examples.

Find the vector $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ that results when the given vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is rotated the given angle θ counterclockwise around the given axis.

1. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ through 90° around the x–axis.

Using the rotation formula $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ and $\theta = 90°$, we get

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° \\ 0 & \sin 90° & \cos 90° \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$$

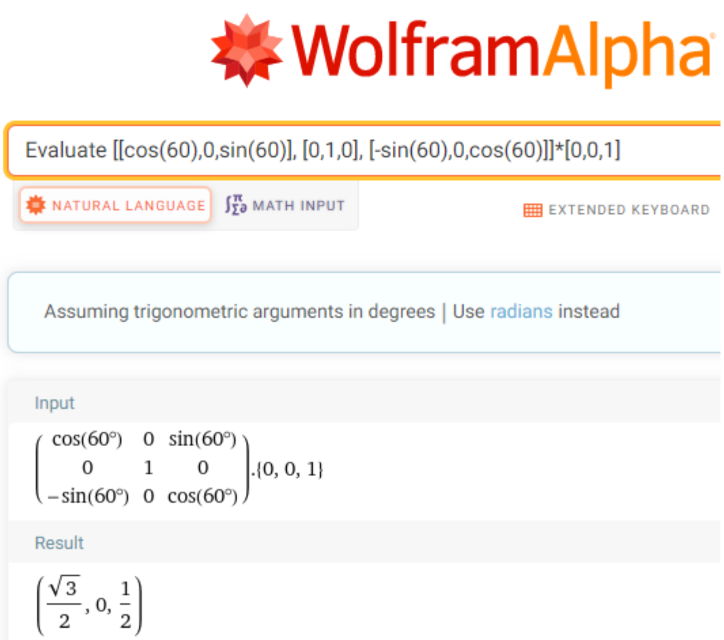
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 0 \cdot 4 + 0 \cdot 3 \\ 0 \cdot 5 + 0 \cdot 4 + (-1) \cdot 3 \\ 0 \cdot 5 + 1 \cdot 4 + 0 \cdot 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix}$$

When rotated counterclockwise 90° around the x–axis, the vector $\begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ becomes $\begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix}$.

2. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ through 60° around the y–axis.

Using the rotation formula $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\theta = 60°$, and WolframAlpha we get



When rotated counterclockwise 60° around the y–axis, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ becomes $\begin{bmatrix} \sqrt{3}/2 \\ 0 \\ 1/2 \end{bmatrix}$.

5.1 THE BASIC TRIGONOMETRIC FUNCTIONS

This section is a brief introduction to right angle trigonometry. We present the sine and cosine functions because they are used to control an object's vertical and horizontal motion. These two functions best serve the needs of gaming programmers at an introductory level. We begin sine and cosine using right triangles to give a visual understanding of both functions.

The following section presents circular trigonometry. In computer games, objects' vertical and horizontal motion takes place over time. The graphs of sine and cosine, as time increases, help us to see how these two functions control vertical and horizontal movement.

The last section presents how the amplitude and period of the sine and cosine functions determine the height of an object and the speed at which an object changes height.

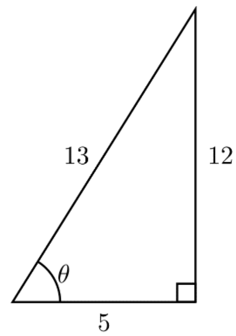
Consider demonstrating these examples:

Example (1)

Find both $\sin\theta$ and $\cos\theta$ for the 5-12-13 triangle.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{12}{13} = 0.9231$$

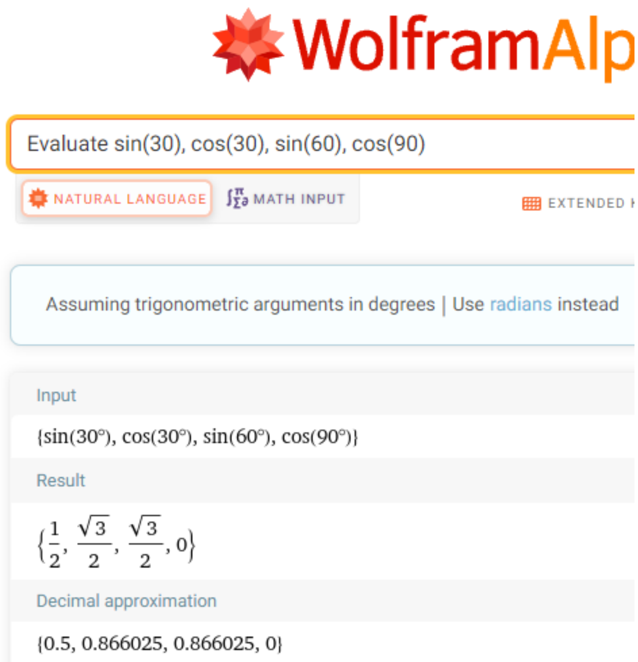
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{5}{13} = 0.3846$$



Example (2)

Find $\sin 30°$, $\cos 30°$, $\sin 60°$, $\cos 90°$.

To compute these ratios, enter Evaluate sin(30), cos(30), sin(60), cos(90) into the entry field.
Separate the entries with commas. W|A does not see spaces.
WolframAlpha tells you what it thinks you entered, then tells you its answers.



We conclude that $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, $\sin 60° = \frac{\sqrt{3}}{2}$, and $\cos 60° = \frac{1}{2}$

W|A also provides us with decimal approximations to these ratios.

$\sin 30° = 0.5$, $\cos 30° = 0.8660$, $\sin 60° = 0.8660$, and $\cos 60° = 0.5$.

5.2 CIRCULAR TRIGONOMETRY

We use degrees rather than radians as most game players think in terms of degrees, not radians.

Consider demonstrating these examples:

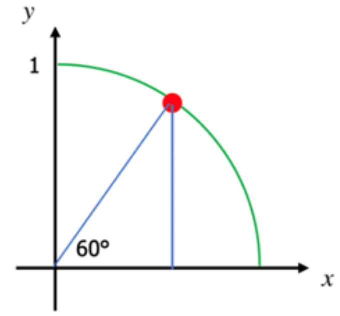
1. An object lies on the circumference of a unit circle. Find its coordinates if the line segment from the origin to the object makes angle of 60° with the horizontal.

Because the object is on the circumference of unit circle, we can use

$x = r\cos\theta$ and $y = r\sin\theta$, with $r = 1$, $\theta = 60^\circ$.

$$x = 1\cos 60^\circ \text{ and } y = 1\sin 60^\circ$$
$$x = \cos 60^\circ \text{ and } y = \sin 60^\circ$$
$$x = 0.5 \text{ and } y = 0.8660$$

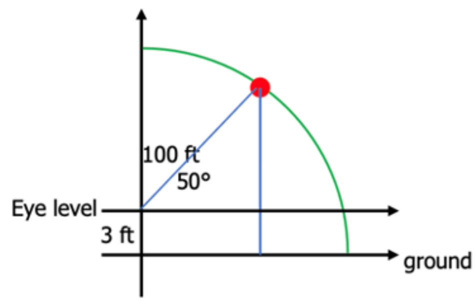
The coordinates of the object are (0.5, 0.8660).



2. An object lies on the circumference of a circle of radius 25 m. Find its coordinates if the line segment from the origin to the object makes angle of 120° with the horizontal.

ANS: (-12.5, 21.6506)

3. How high above the ground is an object that makes an angle of 50° with a 3-foot-tall observer's eyes and is 100 feet away from that observer's eyes? Round to two decimals places.



ANS: 79.60 ft

4. The coordinates of an object are (5, 12). Find its distance from the origin.

We can use the Pythagorean Theorem, $a^2 + b^2 = c^2$, where c is the hypotenuse, the radius of the circle in our case.

$$5^2 + 12^2 = r^2$$
$$25 + 144 = r^2$$
$$169 = r^2$$
$$\sqrt{169} = \sqrt{r^2}$$
$$13 = r$$

We conclude that the object is about 13 units from the origin.

5.3 GRAPHS OF THE SINE FUNCTION

The intent of this section is for students to see how the sine function controls an object's vertical distance from the horizontal.

Be careful to explain that the graph shows how the height of the object increases and decreases as the angle of view increases. Point out that the curve is NOT the path of the object. A common error students make is to think that the curve is the path of the object.

5.4 GRAPHS OF THE COSINE FUNCTION

The intent of this section is for students to see how the cosine function controls an object's horizontal distance from the observer. Point out that we should think of the cosine values in terms of absolute values since they represent distances to the left or right of the observer.

Be careful to explain that the graph shows how the horizontal distance of the object increases and decreases as the angle of view increases. Point out that the curve is NOT the path of the object. A common error students make is to think that the curve is the path of the object.

5.5 AMPLITUDE AND PERIOD OF THE SINE AND COSINE FUNCTIONS

First, consider presenting the **amplitude** of the sine and cosine function.

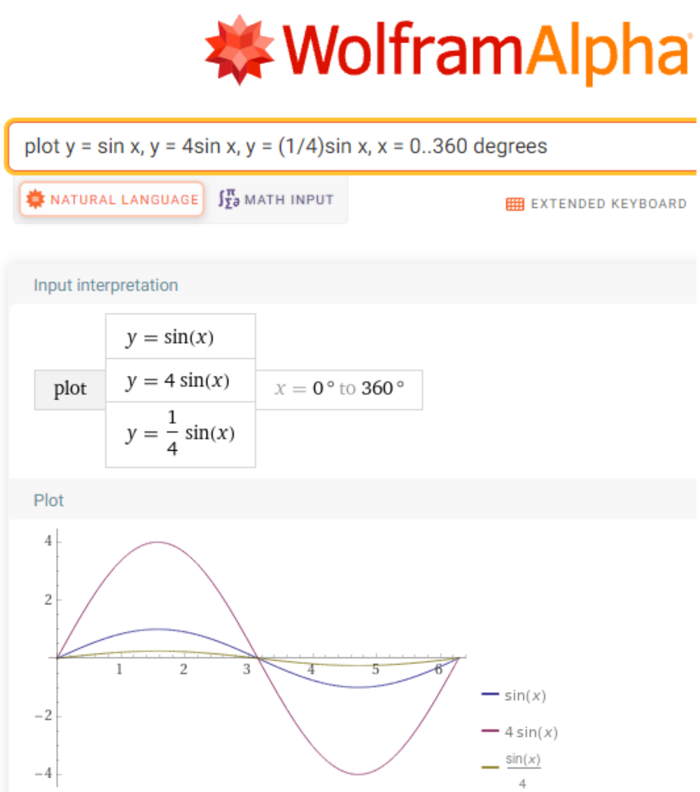
Ask what would happen if we multiplied $y = \sin\theta$ by 4.

If we were to multiply the sine function $y = \sin\theta$ by 4, getting $y = 4\sin\theta$, each of the sine values would be multiplied by 4 making each value 4 times what it was. Each height would be quadrupled. The amplitude of $y = 4\sin\theta$ is 4.

Now discuss what would happen if we multiplied $y = \sin\theta$ by 1/4.

If we were to multiply the sine function $y = \sin\theta$ by $\frac{1}{4}$, getting $y = \frac{1}{4}\sin\theta$, each of the sine values would be multiplied by $\frac{1}{4}$, making each value $\frac{1}{4}$ of what it was. Each height of $y = \sin\theta$ would be $\frac{1}{4}$ of what it was in $y = \sin\theta$. The amplitude of $y = \frac{1}{4}\sin\theta$ is $\frac{1}{4}$.

To compare the graphs, use WolframAlpha or Desmos to construct the graphs of $y = \sin\theta$, $y = 4\sin\theta$, and $y = \frac{1}{4}\sin\theta$ all on the same coordinate system.



Now present the **period** of the sine and cosine function.

Suppose B represents a positive number. Then the period of both $y = \sin(B\theta)$ and $y = \cos(\mathrm{B}\theta)$ is $\frac{360°}{B}$. As B gets bigger, $\frac{360°}{B}$ gets smaller and the period increases.

Ask what would happen if we were to multiply the angle θ by 4.

If we were to multiply the angle in the sine function $y = \sin\theta$ by 4, getting $y = \sin 4\theta$, each of the angle's values would be multiplied by 4 making each value 4 times what it was. Each angle would be quadrupled and there would be 4 cycles in the interval 0° to 360°. The period of $y = \sin 4\theta$ is $\frac{360°}{4} = 90°$. The period of $y = \sin 4\theta$ is smaller than that of $y = \sin\theta$.

Ask what would happen if we were to multiply the angle θ is multiplied by $1/4$.

If we were to multiply the angle in the sine function $y = \sin\theta$ by $1/4$, getting $y = \sin\left(\frac{1}{4}\theta\right)$. Each of the angle's values would be multiplied by 1/4 making each value 1/4 what it was and there would be only 1/4 of a cycle in the interval 0° to 360°. The period of $y = \sin\left(\frac{1}{4}\theta\right)$ is $\frac{360°}{1/4} = 360° \times 4 = 1440°$. The period of $y = \sin\left(\frac{1}{4}\theta\right)$ is greater than that of $y = \sin\theta$.

For comparison, you could use WolframAlpha or Desmos to construct the graph of each of these functions on the same coordinate system.