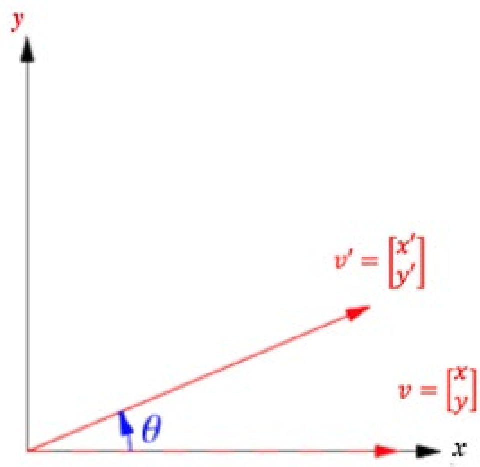
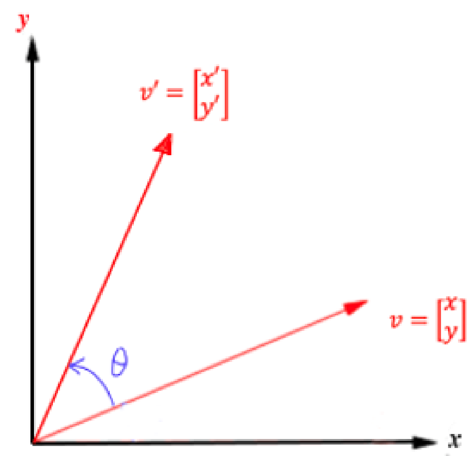
4.4 Rotation Matrices in 2-Dimensions

THE ROTATION MATRIX

To this point, we worked with vectors and with matrices. Now, we will put them together to see how to use a matrix multiplication to rotate a vector in the counterclockwise direction through some angle θ in 2-dimensions.





Our plan is to rotate the vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ counterclockwise through some angle θ to the new position given by the vector $v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$. To do so, we use the rotation matrix, a matrix that rotates points in the xy-plane counterclockwise through an angle θ relative to the x-axis.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

THE ROTATION PROCESS

To get the coordinates of the new vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$, perform the matrix multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example (1)

Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is rotated 90° counterclockwise.

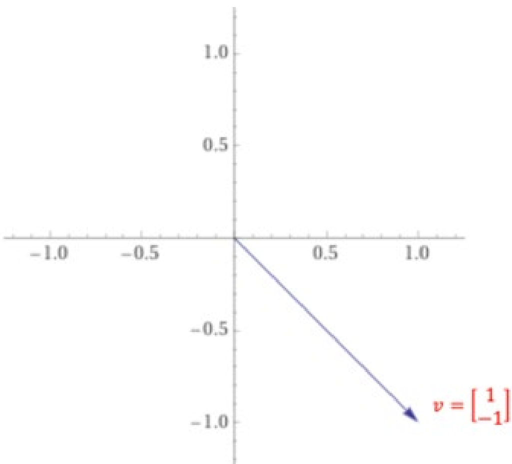
Using the rotation formula $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ with $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\theta = 90°$, we get

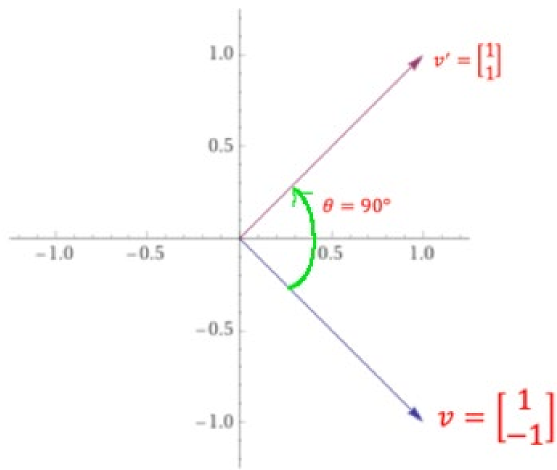
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + (-1) \cdot (-1) \\ 1 \cdot 1 + 0 \cdot (-1) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When rotated counterclockwise 90°, the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ becomes $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.





Example (2)

Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is rotated 60° counterclockwise.

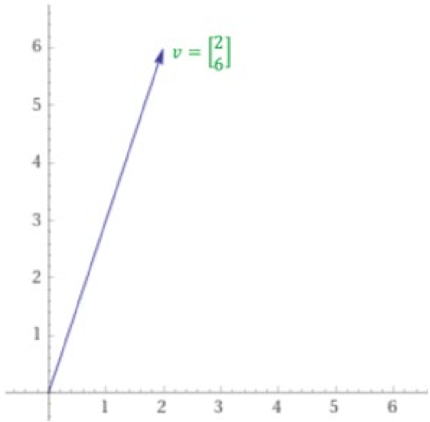
Using the rotation formula $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ with $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\theta = 60°$, we get

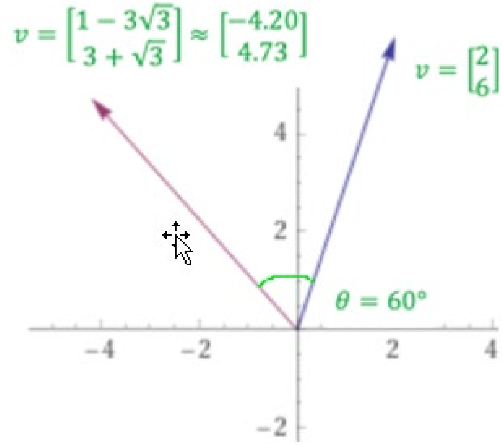
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1/2 \cdot 2 + \left(-\sqrt{3}/2\right) \cdot 6 \\ \sqrt{3}/2 \cdot 2 + 1/2 \cdot 6 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - 3\sqrt{3} \\ 3 + \sqrt{3} \end{bmatrix}$$

When rotated counterclockwise 60°, the vector $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ becomes $\begin{bmatrix} 1 - 3\sqrt{3} \\ 3 + \sqrt{3} \end{bmatrix}$.



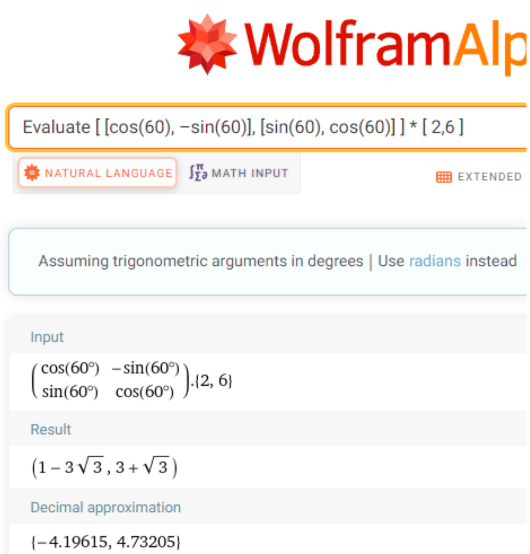


USING TECHNOLOGY

We can use technology to help us find the rotation. WolframAlpha evaluates the trig functions for us.

Go to www.wolframalpha.com.

We can check the above problem from Example 2 by using WolframAlpha. Find the vector $\begin{bmatrix}x'\\y'\end{bmatrix}$ that results when the vector $\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2\\6\end{bmatrix}$ is rotated 60° counterclockwise.
To find rotation of the vector enter Evaluate [[cos(60), –sin(60)], [sin(60), cos(60)]] * [2,6] into the entry field. Both entries and rows are separated by commas and W|A does not see spaces. WolframAlpha tells you what it thinks you entered, then it shows you the answer.



When rotated counterclockwise 60°, the vector $\begin{bmatrix}2\\6\end{bmatrix}$ becomes $\begin{bmatrix}1-3\sqrt{3}\\3+\sqrt{3}\end{bmatrix} \approx \begin{bmatrix}-4.20\\4.73\end{bmatrix}$

TRY THESE

1. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ through 90°. ANS: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

2. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ through 180°. ANS: $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ through 270°. ANS: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

4. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ through 90°. ANS: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

5. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ through 45°. ANS: $\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$

6. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ through 45°. ANS: $\begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}$

7. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2.20205 \\ 4.48898 \end{bmatrix}$ through $-63°$. ANS: $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

8. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$ through $-90°$. ANS: $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$

9. Approximate, to five decimal places, the coordinates of the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ when it is rotated counterclockwise 30°.

   ANS: $\begin{bmatrix} -1.36603 \\ 0.36603 \end{bmatrix}$

NOTE TO INSTRUCTOR

Note that we plan to rotate some vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ through some angle θ to the new position given by the vector $v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, and to do so, we will use the rotation matrix, a matrix that rotates points in the xy-plane counterclockwise through an angle θ relative to the x-axis.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Consider demonstrating these rotations:

1. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$ is rotated 90° counterclockwise.

Using the rotation formula $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ with $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$ and $\theta = 90°$, we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix}\begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \cdot 5 + (-1) \cdot (-5) \\ 1 \cdot 5 + 0 \cdot (-5) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

When rotated counterclockwise 90°, the vector $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$ becomes $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

- If your class knows some trig, you can show the conversion of

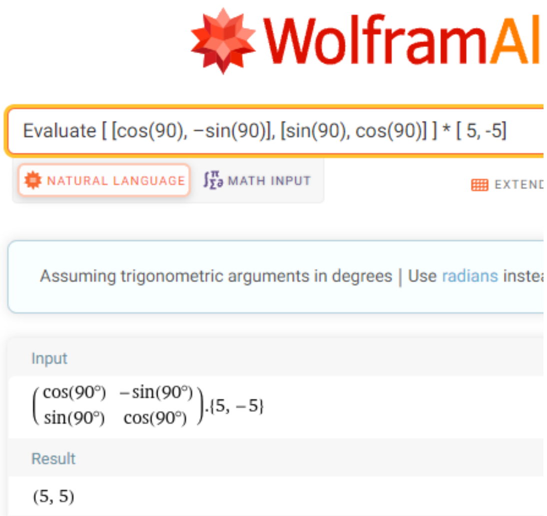
  $\begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix}$ to $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

  Since $\cos 90° = 0$ and $\sin 90° = 1$

- If trig is a challenge, use WolframAlpha to perform the matrix multiplication.

Go to www.wolframalpha.com.

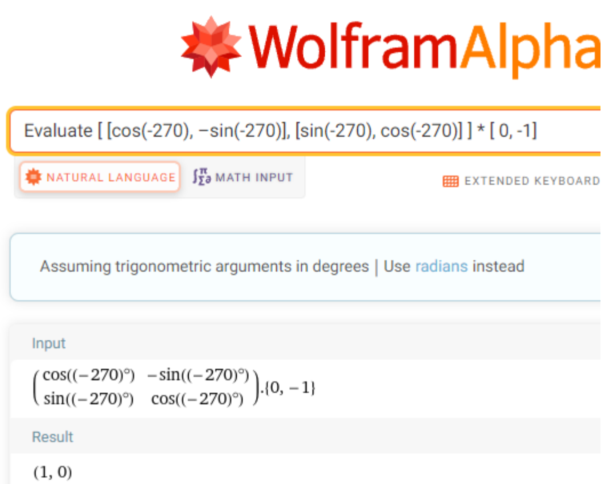
To find rotation of the vector, enter Evaluate [[cos(90), –sin(90)], [sin(90), cos(90)]] * [5, -5] into the entry field. WolframAlpha tells you what it thinks you entered, then tells you its answer.



Be sure to write a conclusion so your students know to do so.

When rotated counterclockwise 90°, the vector $\begin{bmatrix} 5 \\ -5 \end{bmatrix}$ becomes $\begin{bmatrix} 5 \\ 5 \end{bmatrix}$.

2. The rotation formula works for clockwise rotations. We just need to make the angle of rotation negative.
   Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ is rotated –270°.



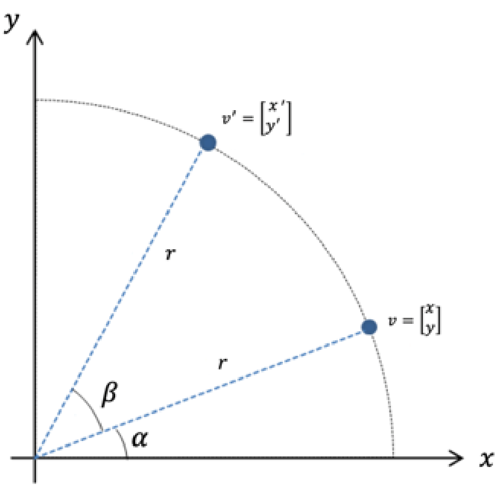
When rotated clockwise 90°, the vector $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ becomes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

DERIVING THE ROTATION FORMULA

If your class knows some trig, you may wish to derive the rotation formula.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

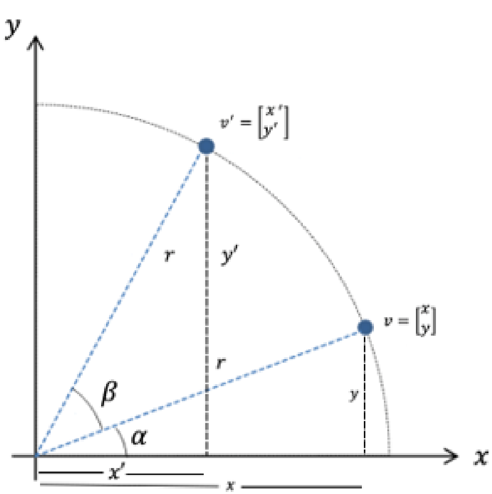
We wish to derive a formula that rotates a vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ counterclockwise through some angle θ to the new position given by the vector $v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$.



We wish to rotate the vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ through an angle β around the origin.

We know that in general,

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \& \quad \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$



The figure shows that for angle α,

$$\begin{cases} \cos\alpha = \dfrac{x}{r} \\ \sin\alpha = \dfrac{y}{r} \end{cases} \rightarrow \begin{cases} x = r \cdot \cos\alpha \\ y = r \cdot \sin\alpha \end{cases}$$

Also

$$\begin{cases} \cos(\alpha+\beta) = \dfrac{x'}{r} \\ \sin(\alpha+\beta) = \dfrac{y'}{r} \end{cases} \rightarrow \begin{cases} x' = r \cdot \cos(\alpha+\beta) \\ y' = r \cdot \sin(\alpha+\beta) \end{cases}$$

By the trigonometry addition identity,

$$\boxed{\cos(\alpha+\beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta}$$

$$\begin{aligned} x' &= r \cdot \cos(\alpha+\beta) \\ &= r \cdot (\cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta) \\ &= \underbrace{r \cdot \cos\alpha}\cos\beta - \underbrace{r \cdot \sin\alpha}\sin\beta \end{aligned}$$

Then, since $x = r \cdot \cos\alpha$ and $y = r \cdot \sin\alpha$

Replace with x and with y

$x' = x \cdot \cos\beta - y \cdot \sin\beta$

Similarly, by the trigonometry addition identity,

$$\boxed{\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta}$$

$$\begin{aligned} y' &= r \cdot \sin(\alpha+\beta) \\ &= r \cdot (\sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta) \\ &= \underbrace{r \cdot \sin\alpha}\cos\beta + \underbrace{r \cdot \cos\alpha}\sin\beta \end{aligned}$$

Then, since $y = r \cdot \sin\alpha$ and $x = r \cdot \cos\alpha$

Replace with y and with x

$y' = y \cdot \cos\beta + x \cdot \sin\beta$

Rewrite this as $y' = x \cdot \sin\beta + y \cdot \cos\beta$

Now we have $\begin{cases} x' = x \cdot \cos\beta - y \cdot \sin\beta \\ y' = x \cdot \sin\beta + y \cdot \cos\beta \end{cases}$

Putting these two results into matrix form, $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

Replacing β with θ to match our notation, we get $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

And we have produced the rotation formula.