MATHEMATICS FOR GAME DEVELOPERS

Mathematics for Game Developers

DOWNEY UNIFIED SCHOOL DISTRICT
PUBLISHER

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DOWNEY UNIFIED SCHOOL DISTRICT (DUSD) TEAM

DUSD Vision
All students graduate with a 21st Century education that ensures they are college and career ready, globally competitive and citizens of strong character.

DUSD Mission
Downey Unified School District is committed to developing all students to be self-motivated learners and productive, responsible, and compassionate members of an ever-changing global society. Our highly qualified staff foster meaningful relationships with students, parents, and the community while providing a relevant and rigorous curriculum in facilities that advance teaching and learning.

DUSD Administration
Superintendent John A. Garcia, Jr., Ph.D.
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Additional Support
California Community College K12 Srong Workforce Program Round 3 award: Open-Source Downey taps into DUSD’s existing ecosystem and builds a culture of open education resources (OER) and industry collaborative work-based learning to be shared with high schools across the state.

OPEN EDUCATION RESOURCES (OER) TEAM

Denny Burzynski, Author
Kelly Cooper, Project Manager
Veronika Focht, Book Designer, Technical Editor
June Bayha, Thought Partner
Muhammad Awais, Technical Reviewer

ABOUT THE AUTHOR

Denny Burzynski has a Master's Degree in Mathematics from the California State University, Long Beach, and has taught mathematics at California and Nevada community colleges for 45 years. His programming experience started when Fortran IV was popular and used line numbers and punch cards. With some of his friends, Denny authored six college mathematics textbooks from basic arithmetic, elementary, and intermediate algebra, college algebra, to applied calculus. Denny presented at many mathematics conferences all over the country, spent a semester interning as a program director at the National Science Foundation in Washington, D.C., and served as president of both the California Mathematics Council Community Colleges and the Nevada Mathematical Association of Community Colleges.

INTRODUCTION

Are you reading this note because you want to develop computer games or write computer programs that perform particular tasks or create whimsical illusions? How is your background in mathematics? Good? Okay? Not sure? To design and program fun and adventurous games, you may not need deep knowledge of mathematics to understand what programming commands do and how to use them to your advantage. In this book, we focus on a few of the mathematical instruments used in programming languages such as C# and C++. Programming languages have libraries of commands that you can use to write programs, and their commands come from mathematical instruments. In Mathematics for Game Developers, we explore the basic ideas of how mathematical instruments control action and motion in the games you develop. And we introduce technology that performs all the mathematical calculations you might need. How awesome is that!

We clearly define four mathematical instruments: vectors, matrices, and the sine and cosine functions and describe what they do. We intend to add practical insight to your programming experience. Programmers use commands to direct computers to perform specific tasks. Commands commonly used in C# and C++ include mathematical instruments such as the Quaternion, the Euler angle, and the rotation matrix that have been around a long time. Quaternions were developed in 1843 by the Irish Mathematician William Rowan Hamilton and matrices in 1850 by the English mathematician James Sylvester. Hipparchus of Nicea (ancient Greece), who lived from 180-125 BCE, likely compiled the first trigonometric table.

Directing your computer to rotate an object or move it vertically or horizontally uses a command born from one or more of these instruments. Can you imagine what Hamilton, Euler, and Hipparchus would think if they knew what you are planning to do with their mathematics!

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UNIT 1 SOME BASIC ALGEBRA

1.1 Constants, Variables, and Expressions

Constants and variables, at least one of these objects appear in every mathematical expression one can imagine. Let's get a sense of just what they are.

A **VARIABLE** is a quantity that has a capacity for change in a particular context.

A **CONSTANT** is a quantity that has no capacity for change in a particular context.

Let's put both in the context of hiring a programmer to write a program that performs some particular task. Suppose there are three programmers, *A*, *B*, and *C*, we are considering.

Programmer A charges a flat fee of $25,000 for writing the program.
Programmer *A*'s fee is constant. In this context, the fee has no capacity for change. The fee is $25,000, no more, no less. Outside this context, maybe writing a less complicated program, the flat fee may be less than $25,000.

Programmer B charges $100/hour for writing the program.
Programmer *B's* fee is variable. In this context, the fee has the capacity for change. The total fee varies with the amount of time taken to write the program.

Programmer C charges a flat fee of $15,000 and $100/whole hour (1, 2, 3, ..., 50) for writing the program.
Programmer *C*'s fee is variable since the total fee varies since it has the capacity for change. The fee varies with the amount of time taken to write the program. Programmer *C*'s fee structure comprises both a constant, the flat fee of $15,000, and a variable, the $100/ whole hour. But because it contains a variable, the entire quantity is variable.

A BIT MORE DETAIL

CONSTANTS are represented with numerals (1, 2, 3, …) and, in special cases, letters or symbols. The constant pi is represented with the Greek letter π, where π is the non-repeating and non-terminating decimal number 3.14159….

VARIABLES are typically represented with symbols or a group of symbols. You've seen these. In fact, you usually see a variable represented with the letter x. Why x, you ask?

Check out this short TED talk to hear as good a theory as I have ever heard. It is worth your time. https://tinyurl.com/thevariablex

It is convenient to think of a variable as a container that can hold different objects at different times. In the programmer example, we might let the letter x represent the number of hours Programmer C takes to write the program. The number of hours can vary from, say, 1 to 50. Think of x as a container into which could be placed the numbers 1, 2, 3, and so on up to and including 50.

a. If C takes only 1 hour to write the program, think of the number 1 being placed into the container named letter x. Then C's total fee would be

$$\$15{,}000 + 1\times\$100 = \$15{,}100$$

b. If C takes 2 hours to write the program, think of the number 2 being placed into the container named letter x. Then C's total fee would be

$$\$15{,}000 + 2\times\$100 = \$15{,}200$$

c. If C takes 50 hours to write the program, think of the number 50 being placed into the container named letter x. Then C's total fee would be

$$\$15{,}000 + 50\times\$100 = \$20{,}000$$

MATHEMATICAL EXPRESSIONS

A MATHEMATICAL EXPRESSION is a constant, a variable, or a finite combination of constants and variables constructed together with mathematical operations (like +, −, x ÷).

Think of a mathematical expression (or just expression) as a set of computing instructions that converts an INPUT value to an OUTPUT value.

For example, using the letter x to represent the number of hours C takes to write the program, we could express Programmer Cs total fee with the expression

$$15{,}000 + 100x$$

The variable x (the container named x) can hold, one at a time, any of the fifty numbers 1, 2, 3, …50. See this expression as a set of computing instructions that take an input value, one of the numbers 1, 2, 3, …50, and converts it to a single output value.

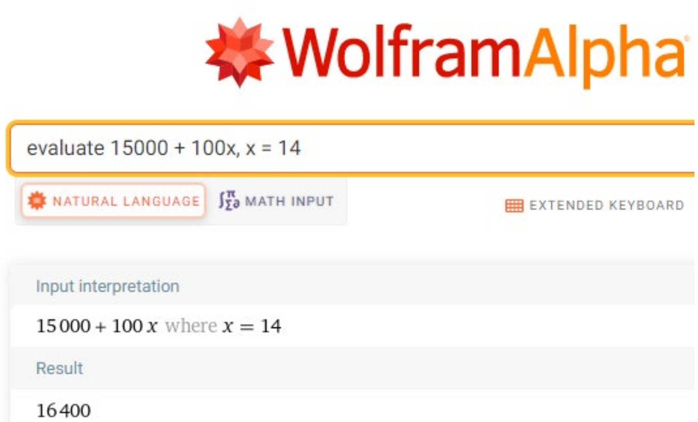
Try to read the instructions in English:
$15{,}000 + 100x \rightarrow$ to get Programmer Cs total fee, multiply the number of hours taken to write the program by 100, then add 15,000. Or, maybe better, Programmer Cs total fee is $15,000 more than $100 times the number of hours worked.

USING TECHNOLOGY

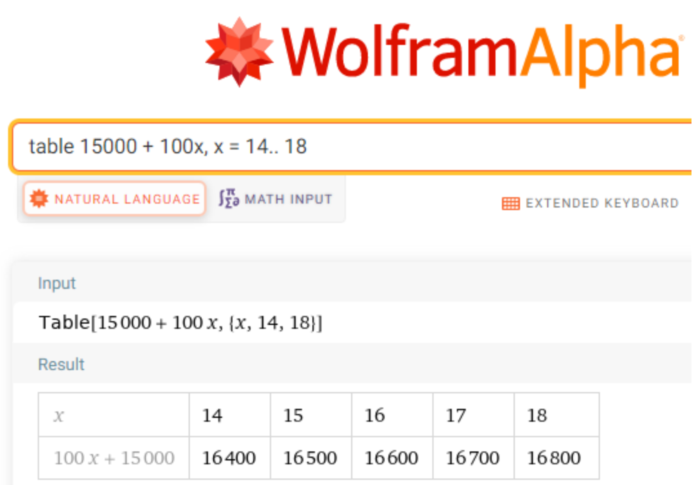
We can use technology to evaluate expressions.

Go to www.wolframalpha.com.

To evaluate 15,000+100x at x = 14, use the "evaluate" command. Enter evaluate 15000 + 100x, x = 14 in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $15000 + 100\text{x},\ \text{x} = 14$.



To evaluate $15{,}000 + 100x$ at x = 14 through 18, use the "table" command. Enter table 15000 + 100x, x = 14.. 18 in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case it shows you a table with answers for $15000 + 100\text{x}, \text{x} = 14..18$.



1.1 TRY THESE

1. Suppose a subscription to a photograph service costs \$50/year and that each downloaded photograph costs \$2.

a) Which of the two quantities is the variable quantity?

b) Which of the two quantities is the constant?

c) Write the expression that produces the annual cost of subscribing and downloading x number of photographs.

d) What is the annual cost of subscribing and downloading 20 photographs?

2. What is the minimum number of cookies a person must eat to be happy? What is the minimum number of cookies beyond that number must eat to feel sick? These numbers are likely different for all of us. Let the variable x represent the minimum number of cookies someone must eat to be happy, and the variable y be the minimum number that makes that person sick.

a) How many variable quantities are in this problem?

b) Are there any constants in this problem?

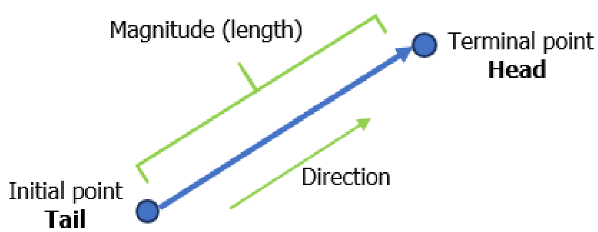
UNIT 2 VECTORS IN TWO DIMENSIONS

2.1 Vectors

Vectors are fundamental objects in applied mathematics; they efficiently convey information about a mathematical or physical object. Let's get a sense of what they are.

> A **VECTOR** is a representation of an object that has both direction and magnitude. By direction, we mean the place toward which something faces, and by magnitude, we mean the size of something.

A vector can be depicted visually by an arrow, with an initial point called the tail and a terminal point called the head. The length of the arrow represents the vector's magnitude.

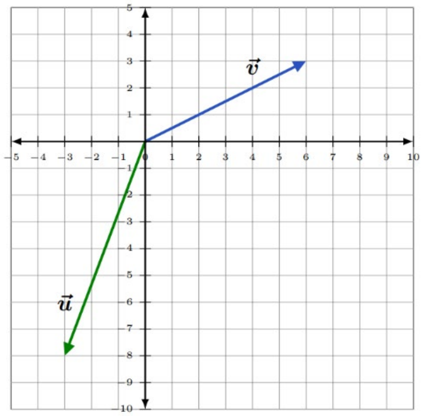


> Vectors are often named using a bold-typed letter with an arrow on top of it. For example, the vector in the picture could be named $\vec{V}$ or $\vec{v}$.

An example of a vector is a car’s velocity. Velocity is a vector since it has both magnitude (speed) and direction. A car might be moving west at 60 mph. Other examples of vectors are displacement, acceleration, and force.

The temperature of some medium is not a vector since it has only magnitude. But if the medium is being heated, its temperature is increasing and has a direction; it is going upward. The increase or decrease in temperature is a vector.

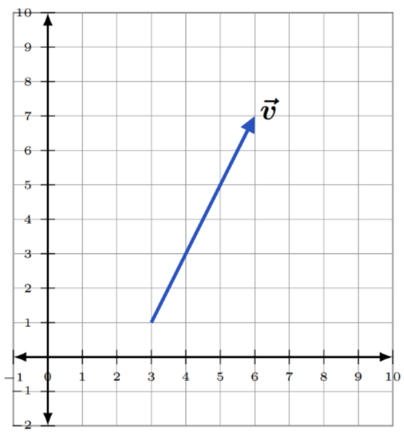
VECTORS IN STANDARD POSITION



A vector with its initial point at the origin in a Cartesian coordinate system is said to be in STANDARD POSITION. The vector $\vec{v}$ in the diagram has its initial point at the origin $(0,0)$, and its terminal point at $(6,3)$.

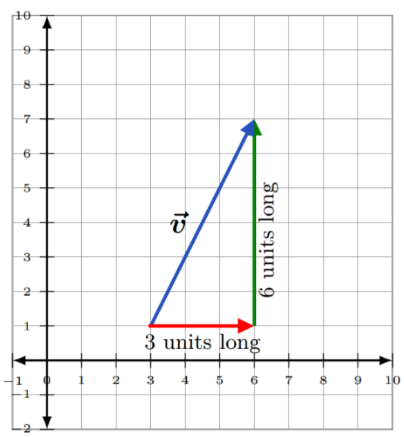
COMPONENTS OF A VECTOR

Vectors in the xy-plane can be broken into their **horizontal** and **vertical** components.



For example, the vector $\vec{v}$ in the diagram can be broken into two components,

1. its horizontal, or x-component, and
2. its vertical, or y-component.



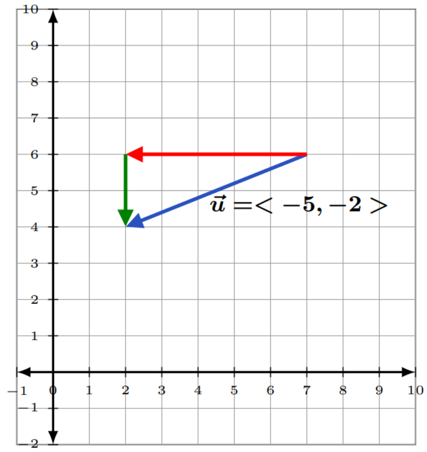
The vector $\vec{v}$ in component form is expressed using angle brackets as $\vec{v} = \langle 3, 6 \rangle$, where

1. the first component, 3 is the length and direction of its x-component, and
2. the second component, 6 is the length and direction of its y-component.

The vector $\vec{u}$ in the picture below has

FIRST COMPONENT = (terminal x-value) – (initial x-value) $= 2 - 7 = -5$, and
SECOND COMPONENT = (terminal y-value) – (initial y-value) $= 4 - 6 = -2$,

so that $\vec{u} = \langle -5, -2 \rangle$.



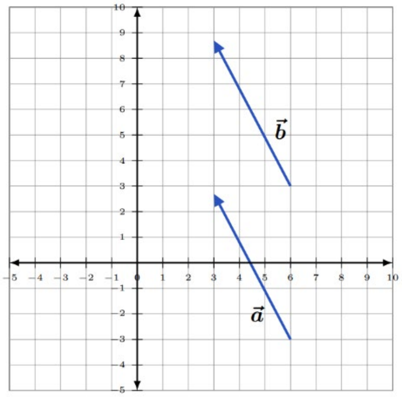
ROW AND COLUMN FORMS OF A VECTOR

Vectors are represented by a single column matrix or a single row matrix. The vectors $\vec{v} = \langle 3, 6 \rangle$, and $\vec{u} = \langle -5, -2 \rangle$ above, can be represented by the 2x1 row matrix and the 1x2 column matrix, respectively as

$$\vec{v} = [3 \quad 6] \text{ and } \vec{u} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

EQUAL VECTORS

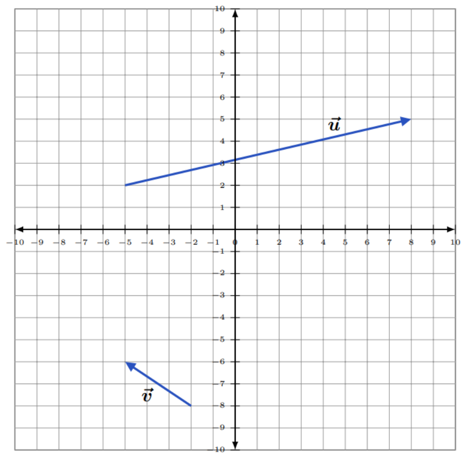
Two vectors are EQUAL if they have the same direction and magnitude. They may start and end at different positions, but their representing arrows will be parallel.



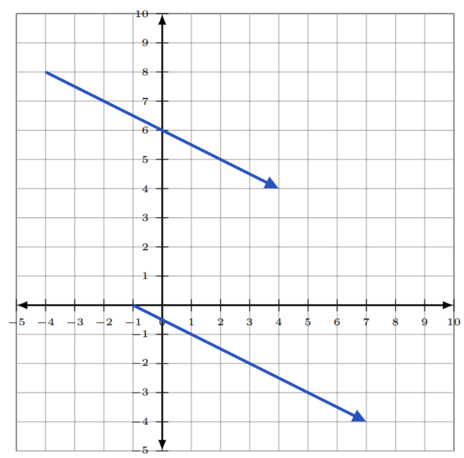
In the diagram vectors $\vec{a}$ and $\vec{b}$ are equal but appear in different locations in the xy-plane.

2.1 TRY THESE

1. Express the vectors $\vec{v}$ and $\vec{u}$ in component form.



2. Explain why the two vectors are equal.



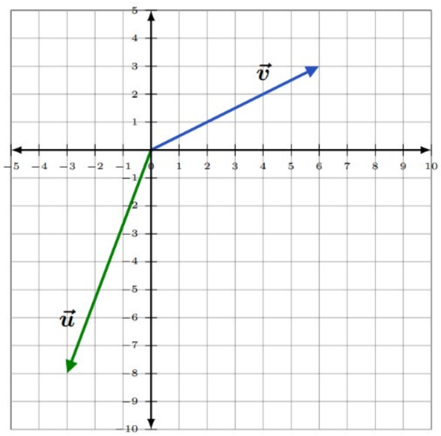
2.2 Addition, Subtraction, and Scalar Multiplication of Vectors

ADDITION & SUBTRACTION OF VECTORS

To add or subtract two vectors, add, or subtract their corresponding components.

Example (1)

To **ADD** the vectors $\vec{u}$ and $\vec{v}$, begin by writing each in component form.



$\vec{u} = \langle -3, -8 \rangle$ and $\vec{v} = \langle 6, 3 \rangle$

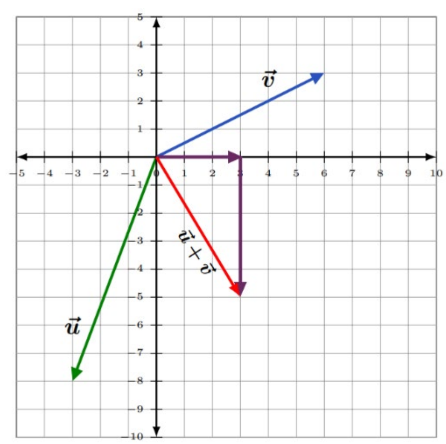
ADD their corresponding components.

$$\vec{u} + \vec{v} = \langle -3 + 6, -8 + 3 \rangle = \langle 3, -5 \rangle$$

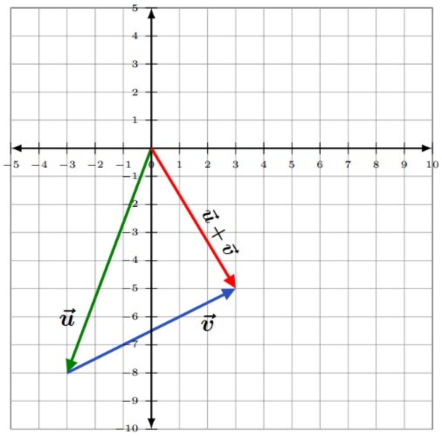
So, $\vec{u} + \vec{v} = \langle 3, -5 \rangle$

Now, graph this sum.

- Start at the origin.
- Since the <u>horizontal component</u> is 3, move 3 units to the *right*.
- Since the <u>vertical component</u> is -5, move 5 units *downward*.

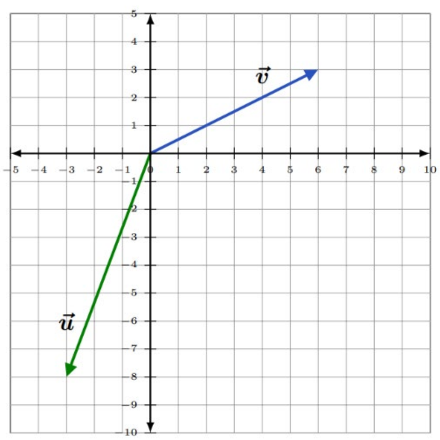


The addition of two vectors $\vec{u}$ and $\vec{v}$ can be demonstrated by placing the tail of one vector at the head of the other. Then connect the tail of $\vec{u}$ to the head of $\vec{v}$.



Example (2)

To **SUBTRACT** the vector $\vec{u}$ from the vector $\vec{v}$, begin by writing each in component form.



$\vec{u} = \langle -3, -8 \rangle$ and $\vec{v} = \langle 6, 3 \rangle$

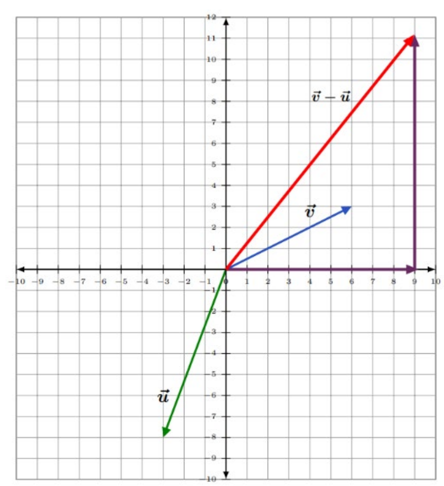
SUBTRACT the components of $\vec{u}$ from the corresponding components of $\vec{v}$.

$$\vec{v} - \vec{u} = \langle 6-(-3), 3-(-8) \rangle = \langle 6+3, 3+8 \rangle = \langle 9, 11 \rangle$$

So, $\vec{v} - \vec{u} = \langle 9, 11 \rangle$

Now, graph this sum.

- Start at the origin.
- Since the <u>horizontal component</u> is 9, move 9 units to the *right*.
- Since the <u>vertical component</u> is 11, move 11 units *upward*.



SCALARS

In contrast to a vector, and having both direction and magnitude, a SCALAR is a physical quantity defined by only its magnitude.

Examples are speed, time, distance, density, and temperature. They are represented by real numbers (both positive and negative), and they can be operated on using the regular laws of algebra.

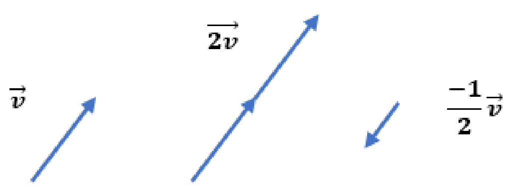
The term *scalar* derives from this usage: *a scalar is that which scales, resizes a vector*.

Scalar multiplication is the multiplication of a vector by a real number (a scalar).

Suppose we let the letter k represent a real number and $\vec{v}$ be the vector $\langle x, y\rangle$. Then, the scalar multiple of the vector $\vec{v}$ is

$$k\vec{v} = \langle kx, ky\rangle$$

To multiply a vector by a scalar (a constant), multiply each of its components by the constant.



1. Suppose $\vec{u} = \langle -3, -8\rangle$ and $k = 3$.

   Then $k\vec{u} = 3\vec{u} = 3\langle -3, -8\rangle = \langle 3(-3), 3(-8)\rangle = \langle -9, -24\rangle$

2. Suppose $\vec{v} = \langle 6, 3\rangle$ and $k = \frac{-1}{3}$.

   Then $k\vec{u} = \frac{-1}{3}\vec{u} = \frac{-1}{3}\langle 6, 3\rangle = \left\langle \frac{-1}{3}(6), \frac{-1}{3}(3)\right\rangle = \langle -2, -1\rangle$

3. Suppose $\vec{u} = \begin{bmatrix}-2\\6\end{bmatrix}$ and $\vec{v} = \begin{bmatrix}5\\3\end{bmatrix}$.

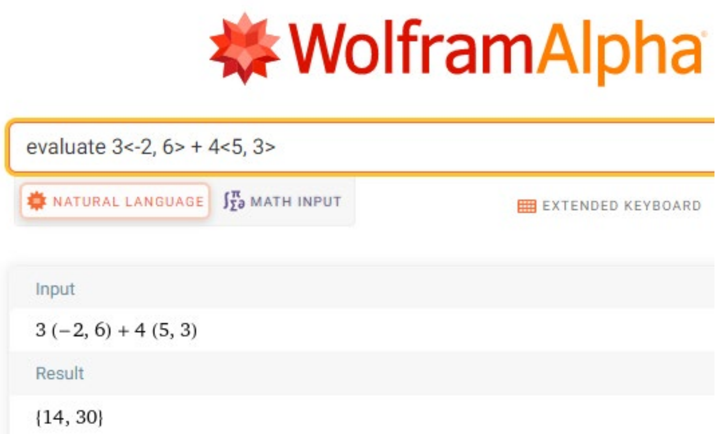
   Then $3\vec{u} + 4\vec{v} = 3\begin{bmatrix}-2\\6\end{bmatrix} + 4\begin{bmatrix}5\\3\end{bmatrix} = \begin{bmatrix}-6\\18\end{bmatrix} + \begin{bmatrix}20\\12\end{bmatrix} = \begin{bmatrix}14\\30\end{bmatrix}$

USING TECHNOLOGY

We can use technology to add and subtract vectors and to multiply a vector by a scalar.

Go to www.wolframalpha.com.

For the vectors $\vec{u} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, use WolframAlpha to find $3\vec{u} + 4\vec{v}$. Enter evaluate 3<-2, 6> + 4<5, 3> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $< 14, 30 >$.



2.2 TRY THESE

1. Find the sum of the two vectors $\vec{u} = \langle -5, 2 \rangle$ and $\vec{v} = \langle 10, -1 \rangle$.

2. Subtract the vector $\vec{u} = \langle -5, 2 \rangle$ from the vector $\vec{v} = \langle 10, -1 \rangle$.

3. Suppose $\vec{u} = \langle -5, 2 \rangle$, $\vec{v} = \langle 1, 6 \rangle$, and $\vec{w} = \langle 4, -3 \rangle$. Perform the operation $2\vec{u} - 4\vec{v} + 3\vec{w}$.

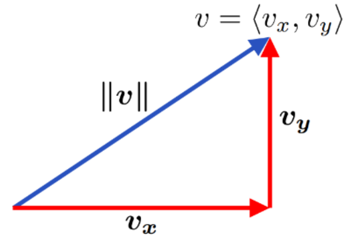
2.3 Magnitude, Direction, and Components of a Vector

THE MAGNITUDE OF A VECTOR

It is productive to represent the horizontal and vertical components of a vector $\vec{v}$ as v_x and v_y, respectively.

The magnitude (the length) of a vector $\vec{v} = \langle v_x, v_y \rangle$ is

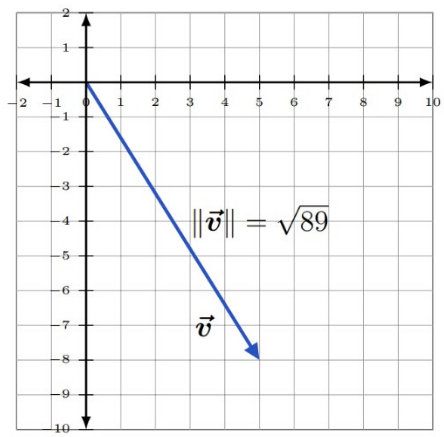
$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$



The vector $\vec{v} = \langle 5, -8 \rangle$ has magnitude:

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$
$$= \sqrt{5^2 + (-8)^2} = \sqrt{25 + 64} = \sqrt{89}$$

Interpret this as the length of the vector $\vec{v} = \langle 5, -8 \rangle$ is $\sqrt{89}$ units.



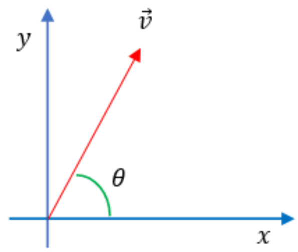
THE DIRECTION OF A VECTOR

The direction of a vector $\vec{v}$ is the angle the vector makes with the positive x-axis.

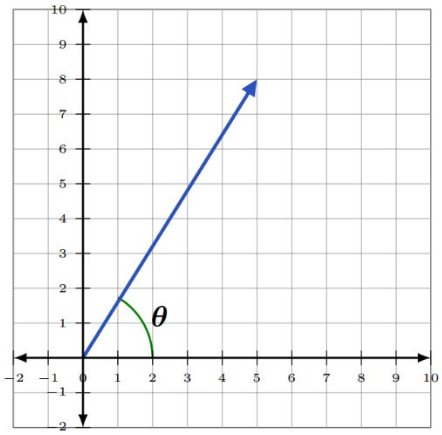
It is typically represented with the uppercase Greek letter theta θ. We use some trigonometry to determine the angle θ.

$$\tan\theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1}\frac{y}{x}$$

The angle θ is always between 0° and 360°.



To approximate the direction of the vector $\vec{v} = \langle 5, 8 \rangle$, use $\theta = \tan^{-1}\frac{y}{x}$, with $x = 5$ and $y = 8$.



$$\vec{v} = \langle 5, 8 \rangle$$

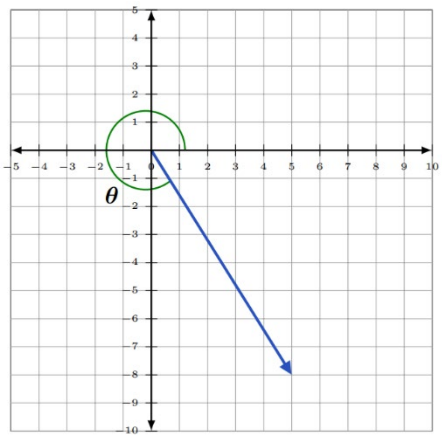
$$\theta = \tan^{-1}\frac{y}{x}$$

$$\theta = \tan^{-1}\frac{8}{5}$$

Using a calculator, we get

$\theta = 57.99°$
$\theta = 58°$

To approximate the direction of the vector $\vec{v} = \langle 5, -8 \rangle$, use $\theta = \tan^{-1}\frac{y}{x}$, with $x = 5$ and $y = -8$.



$$\vec{v} = \langle 5, -8 \rangle$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\theta = \tan^{-1}\frac{-8}{5}$$

Using a calculator, we get

$$\theta = -57.99°$$

Vertical component is in Quadrant IV and θ must be in the interval $[0, 360)$, therefore we calculate θ by

$$\theta = 360° - 57.99° = 302.005°$$

$$\theta = 302°.$$

THE COMPONENTS OF A VECTOR

The lengths of the x- and y- components of a vector $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$ in two dimensions can be found using trigonometric ratios.

$\vec{v}_x = ||\vec{v}||\cos\theta$ and $\vec{v}_y = ||\vec{v}||\sin\theta$

$\vec{v}_x$ is the horizontal component of $\vec{v}$ and $\vec{v}_y$ is the vertical component.

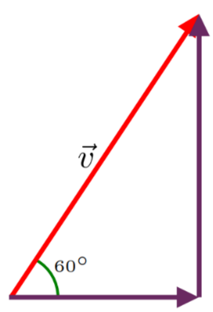
The angle θ is always between 0° and 360°.

Suppose the magnitude of a vector $\vec{v} = \langle v_x, v_y \rangle$ is 20 units, and that $\vec{v}$ makes a 60° angle with the horizontal. Then, the components of $\vec{v}$ are

$$\vec{v}_x = ||\vec{v}||\cos\theta$$
$$= 20\cos 60°$$
$$= 20 \cdot \frac{1}{2}$$
$$= 10$$

and

$$\vec{v}_y = ||\vec{v}||\sin\theta$$
$$= 20\sin 60°$$
$$= 20 \cdot \frac{\sqrt{3}}{2}$$
$$= 10\sqrt{3}$$



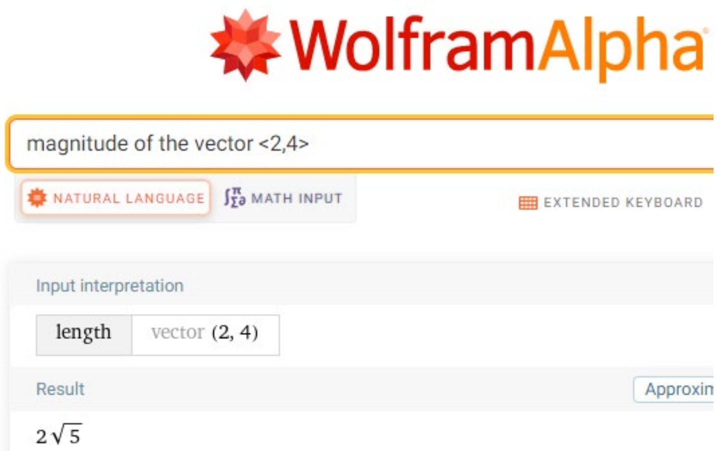
So, we could write $\vec{v} = \langle v_x, v_y \rangle$ as $\vec{v} = \langle 10, 10\sqrt{3} \rangle$

USING TECHNOLOGY

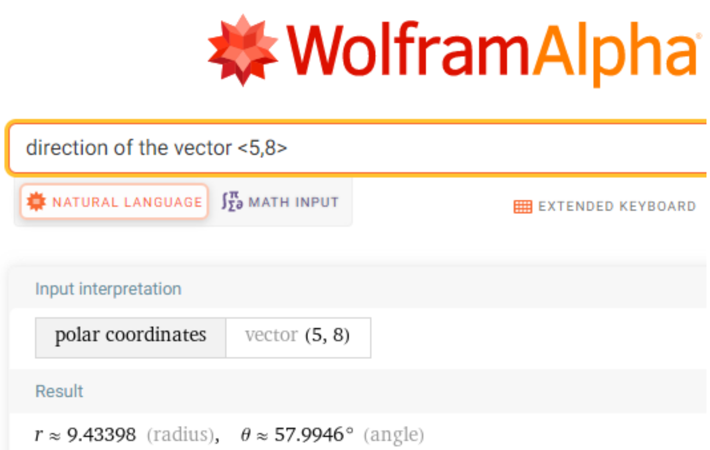
We can use technology to determine the magnitude of a vector.

Go to www.wolframalpha.com.

To find the magnitude of the vector $\vec{v} = \langle 2, 4 \rangle$, enter magnitude of the vector <2,4> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\|\vec{v}\| = 2\sqrt{5}$.



To find the direction of the vector $\vec{v} = \langle 5, 8 \rangle$, enter direction of the vector <5,8> in the entry field. Wolframalpha answers $57.9946° \approx 58°$.



2.3 TRY THESE

1. Find the magnitude of the vector $\vec{v} = \langle 3, -4 \rangle$.

2. Find the magnitude of the vector $\vec{v} = \langle -3, -3 \rangle$.

3. Find the components of the vector $\vec{v}$ if the magnitude of $\vec{v}$ is 6 and it makes a 30° angle with the horizontal.

4. Approximate the direction of the vector $\vec{v} = \langle 3, 10 \rangle$.

2.4 The Dot Product of Two Vectors, the Length of a Vector, and the Angle Between Two Vectors

THE DOT PRODUCT OF TWO VECTORS

The length of a vector or the angle between two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ can be found using the dot product.

The dot product of vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ is a scalar (real number) and is defined to be

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y$$

Since u_x, u_y, v_x and v_y are real numbers, you can see that the dot product is itself a real number and not a vector.

Example (1)

To compute the dot product of the vectors $\vec{u} = \langle 5, 2 \rangle$ and $\vec{v} = \langle 3, 4 \rangle$, we compute

$$\vec{u} \cdot \vec{v} = 5 \cdot 3 + 2 \cdot 4 = 15 + 8 = 23$$

Since the dot product is a scalar, it follows the properties of real numbers.

PROPERTIES OF THE DOT PRODUCT

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, the dot product is commutative
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, the dot product distributes over vector addition
3. $\vec{u} \cdot \vec{0} = 0$, the dot product with the zero vector $\vec{0}$, is the scalar 0.
4. $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$

Example (2)

Compute the dot product $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, where $\vec{u} = \langle 5, -2 \rangle$, $\vec{v} = \langle 6, 4 \rangle$, and $\vec{w} = \langle -3, 7 \rangle$.

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$\begin{aligned} \vec{u} \cdot (\vec{v} + \vec{w}) &= \langle 5, -2 \rangle \cdot \langle 6, 4 \rangle + \langle 5, -2 \rangle \cdot \langle -3, 7 \rangle \\ &= (5 \cdot 6 + (-2) \cdot 4) + (5 \cdot (-3) + (-2) \cdot 7) \\ &= 30 - 8 - 15 - 14 \\ &= -7 \end{aligned}$$

THE LENGTH OF A VECTOR

The length (magnitude) of a vector you know is given by $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$. The length can also be found using the dot product. If we dot a vector $\vec{v} = \langle v_x, v_y \rangle$ with itself, we get

$$\vec{v} \cdot \vec{v} = \langle v_x, v_y \rangle \cdot \langle v_x, v_y \rangle$$
$$\vec{v} \cdot \vec{v} = v_x \cdot v_x + v_y \cdot v_y$$
$$\vec{v} \cdot \vec{v} = {v_x}^2 + {v_y}^2$$

By Vector Property 4, $\vec{v} \cdot \vec{v} = ||\vec{v}||^2$. This gives $||\vec{v}||^2 = {v_x}^2 + {v_y}^2$.

Taking the square root of each side produces

$$\sqrt{||\vec{v}||^2} = \sqrt{{v_x}^2 + {v_y}^2}$$

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$

Which is the length of the vector $\vec{v}$.

The dot product of a vector $\vec{v} = \langle v_x, v_y \rangle$ with itself gives the length of the vector.

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$

Example (3)

Use the dot product to find the length of the vector $\vec{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$.

In this case, $v_x = 2$ and $v_y = 6$.

Using $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$, we get

$$||\vec{v}|| = \sqrt{2^2 + 6^2}$$

$$||\vec{v}|| = \sqrt{40}$$

$$||\vec{v}|| = \sqrt{4 \cdot 10}$$

$$||\vec{v}|| = \sqrt{4} \cdot \sqrt{10}$$

$$||\vec{v}|| = 2\sqrt{10}$$

The length of the vector $\vec{v} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is $2\sqrt{10}$ units.

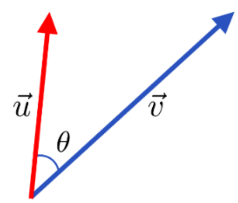
THE ANGLE BETWEEN TWO VECTORS

The dot product and elementary trigonometry can be used to find the angle θ between two vectors.

If θ is the smallest nonnegative angle between two non-zero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} \text{ or } \theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$$

where $0 \le \theta \le 2\pi$ and $\|\vec{u}\| = \sqrt{{u_x}^2 + {u_y}^2}$ and $\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$



Example (4)

Find the angle between the vectors $\vec{u} = \langle 5, -3\rangle$ and $\vec{v} = \langle 2, 4\rangle$.

Using $\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$, we get

$$\theta = \cos^{-1}\frac{\langle 5,-3\rangle\cdot\langle 2,4\rangle}{\sqrt{5^2+(-3)^2}\cdot\sqrt{2^2+4^2}}$$

$$\theta = \cos^{-1}\frac{5\cdot 2+(-3)\cdot 4}{\sqrt{25+9}\cdot\sqrt{4+16}}$$

$$\theta = \cos^{-1}\frac{-2}{\sqrt{34}\cdot\sqrt{20}}$$

$$\theta = 94.4$$

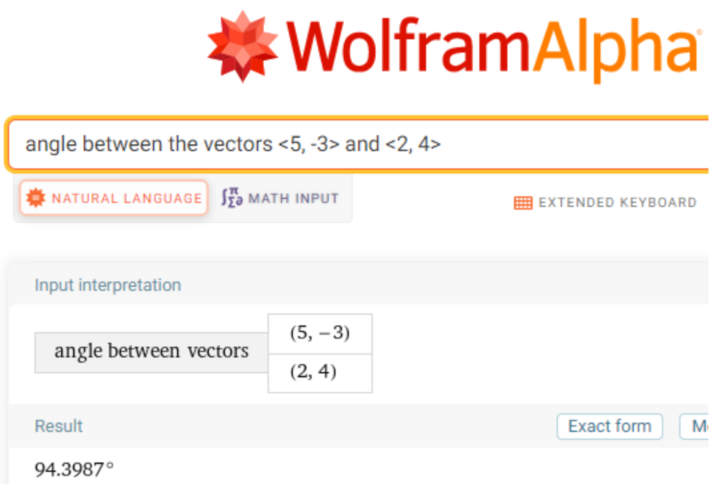
We conclude that the angle between these two vectors is close to 94.4°.

USING TECHNOLOGY

We can use technology to find the angle θ between two vectors.

Go to www.wolframalpha.com.

To find the angle between the vectors $\vec{u} = \langle 5, -3 \rangle$ and $\vec{v} = \langle 2, 4 \rangle$, enter angle between the vectors <5, -3> and <2, 4> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\theta = 94.4$, rounded to one decimal place.



2.4 TRY THESE

1. Find the dot product of the vectors $\vec{u} = \langle -2, 3 \rangle$ and $\vec{v} = \langle 5, -1 \rangle$.

2. Find the dot product of the vectors $\vec{u} = \langle -4, 6 \rangle$ and $\vec{v} = \langle 3, 2 \rangle$.

3. Find the length of the vector $\vec{u} = \langle 4, -7 \rangle$.

4. Find the length of the vector $\vec{v} = \langle 0,5 \rangle$.

5. Find the angle between the vectors $\vec{u} = \langle -2, 3 \rangle$ and $\vec{v} = \langle 5, -1 \rangle$.

6. Find the angle between the vectors $\vec{u} = \langle -4, 6 \rangle$ and $\vec{v} = \langle 3, 2 \rangle$.

2.5 Parallel and Perpendicular Vectors, The Unit Vector

PARALLEL AND ORTHOGONAL VECTORS

Two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ are **parallel** if the angle between them is 0° or 180°.

Also, two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ are parallel to each other if the vector $\vec{u}$ is some multiple of the vector $\vec{v}$. That is, they will be parallel if the vector $\vec{u} = c\vec{v}$, for some real number c. That is, $\vec{u}$ is some multiple of $\vec{v}$.

Two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ are **orthogonal** (perpendicular to each other) if the angle between them is 90° or 270°.

Use this shortcut: *Two vectors are perpendicular to each other if their dot product is 0.*

Example (1)

The two vectors $\vec{u} = \langle 2, -3 \rangle$ and $\vec{v} = \langle -8, 12 \rangle$ are parallel to each other since the angle between them is $180°$.

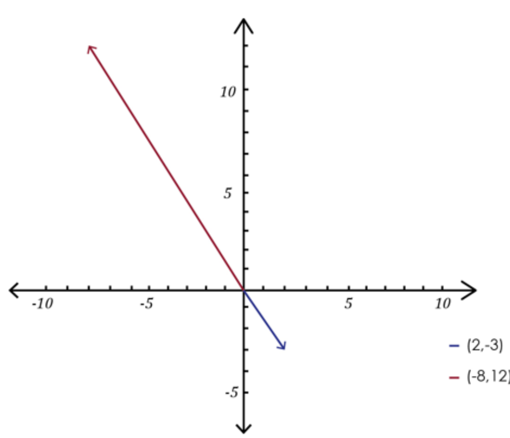
$$\theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \cdot ||\vec{v}||}$$

$$\theta = \cos^{-1} \frac{\langle 2, -3 \rangle \cdot \langle -8, 12 \rangle}{\sqrt{2^2 + (-3)^2} \cdot \sqrt{(-8)^2 + 12^2}}$$

$$\theta = \cos^{-1} \frac{2 \cdot (-8) + (-3) \cdot 12}{\sqrt{4 + 9} \cdot \sqrt{64 + 144}}$$

$$\theta = \cos^{-1} \frac{-52}{\sqrt{13} \cdot \sqrt{208}}$$

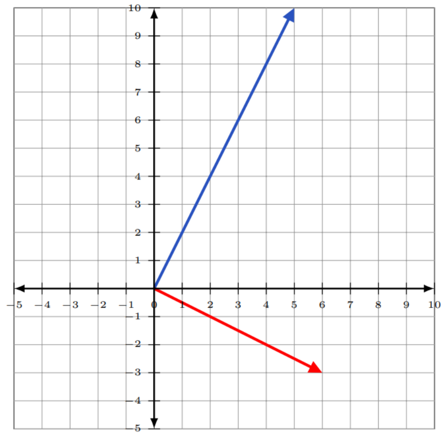
$$\theta = 180°$$



Example (2)

To show that the two vectors $\vec{u} = \langle 5,10\rangle$ and $\vec{v} = \langle 6,-3\rangle$ are orthogonal (perpendicular to each other), we just need to show that their dot product is 0.

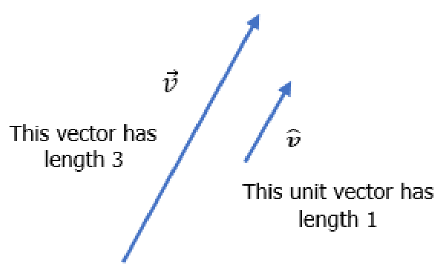
$$\langle 5,10\rangle \cdot \langle 6,-3\rangle = 5\cdot 6 + 10\cdot(-3) = 30 - 30 = 0$$



THE UNIT VECTOR

A unit vector is a vector of length 1.

A unit vector in the same direction as the vector $\vec{v}$ is often denoted with a "hat" on it as in $\hat{v}$. We call this vector "v hat."



The unit vector $\hat{v}$ corresponding to the vector $\vec{v}$ is defined to be

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}$$

Example (3)

The unit vector corresponding to the vector $\vec{v} = \langle -8, 12 \rangle$ is

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{(-8)^2 + (12)^2}}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{64 + 144}}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{208}}$$

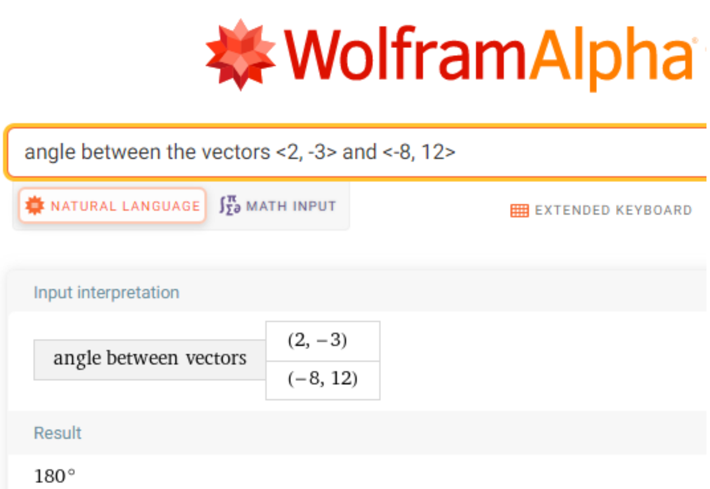
$$\hat{v} = \left\langle \frac{-8}{\sqrt{208}}, \frac{12}{\sqrt{208}} \right\rangle$$

USING TECHNOLOGY

We can use technology to find the angle θ between two vectors.

Go to www.wolframalpha.com.

To show that the vectors $\vec{u} = \langle 2, -3 \rangle$ and $\vec{v} = \langle -8, 12 \rangle$ are parallel, enter angle between the vectors <2, -3> and <-8, 12> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\theta = 180°$, indicating the two vectors are parallel.



2.5 TRY THESE

1. Determine if the vectors $\vec{u} = \langle 2, 1 \rangle$ and $\vec{v} = \langle 3\ , -6 \rangle$ are parallel to each other, perpendicular to each other, or neither parallel nor perpendicular to each other.

2. Determine if the vectors $\vec{u} = \langle 2, 16 \rangle$ and $\vec{v} = \langle \frac{1}{2}, 4 \rangle$ are parallel to each other, perpendicular to each other, or neither parallel nor perpendicular to each other.

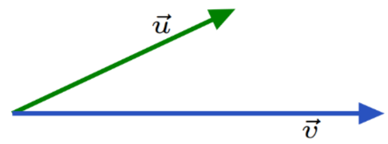
3. Determine if the vectors $\vec{u} = \langle 7, 6 \rangle$ and $\vec{v} = \langle 2, -1 \rangle$ are parallel to each other, perpendicular to each other, or neither parallel nor perpendicular to each other.

4. Find the unit vector corresponding to the vector $\vec{v} = \langle 2, -1 \rangle$.

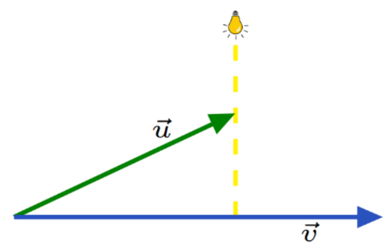
2.6 The Vector Projection of One Vector onto Another

PROJECTION

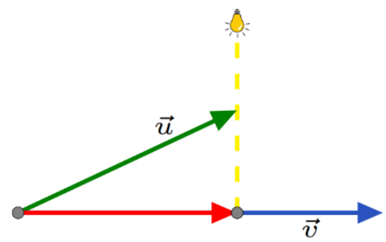
Let's project vector $\vec{u} = \langle u_x, u_y \rangle$ onto the vector $\vec{v} = \langle v_x, v_y \rangle$.



To do so, imagine a light bulb above $\vec{u}$ shining perpendicular onto $\vec{v}$.



The light from the bulb will cast a shadow of $\vec{u}$ onto $\vec{v}$, and it is this shadow that we are looking for. The shadow is the projection of $\vec{u}$ onto $\vec{v}$.



The red vector is the projection of $\vec{u}$ onto $\vec{v}$. The notation commonly used to represent the projection of $\vec{u}$ onto $\vec{v}$ is $\mathrm{proj}_{\vec{v}}\vec{u}$.

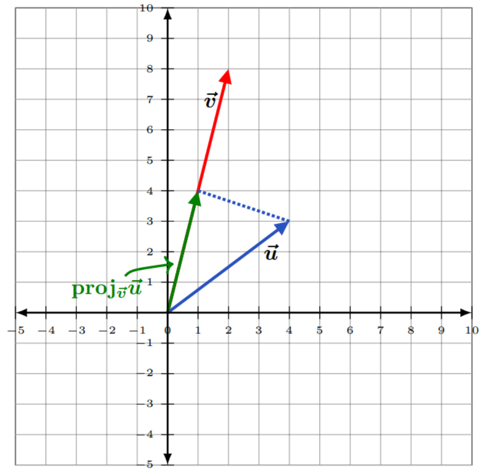
Vector parallel to $\vec{v}$ with magnitude $\frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$ in the direction of $\vec{v}$ is called projection of $\vec{u}$ onto $\vec{v}$.

The formula for $\mathrm{proj}_{\vec{v}}\vec{u}$ is

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\vec{v}$$

Example (1)

To find the projection of $\vec{u} = \langle 4, 3\rangle$ onto $\vec{v} = \langle 2, 8\rangle$, we need to compute both the dot product of $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{v}$, then apply the formula.



$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v}$$

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{\langle 4, 3\rangle \cdot \langle 2, 8\rangle}{\|\langle 2, 8\rangle\|^2}\langle 2, 8\rangle$$

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{4 \cdot 2 + 3 \cdot 8}{\left(\sqrt{2^2 + 8^2}\right)^2}\langle 2, 8\rangle$$

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{32}{\left(\sqrt{4 + 64}\right)^2}\langle 2, 8\rangle$$

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{32}{68}\langle 2, 8\rangle$$

$$\mathrm{proj}_{\vec{v}}\vec{u} = \frac{8}{17}\langle 2, 8\rangle$$

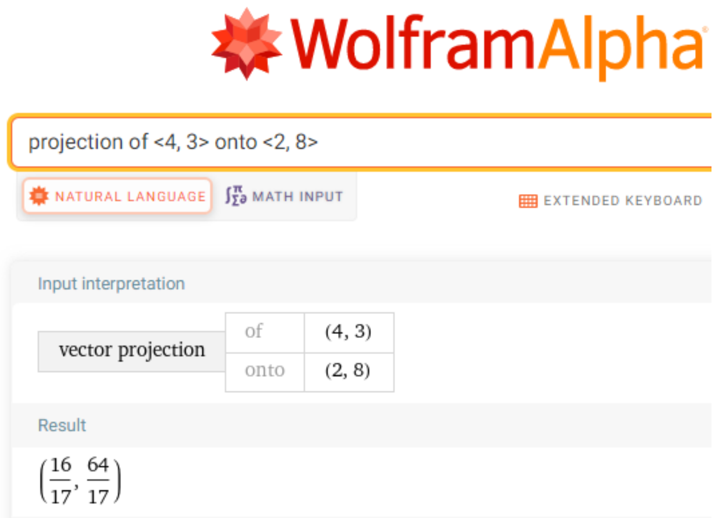
$$\mathrm{proj}_{\vec{v}}\vec{u} = \left\langle \frac{16}{17}, \frac{64}{17}\right\rangle$$

USING TECHNOLOGY

We can use technology to determine the projection of one vector onto another.

Go to www.wolframalpha.com.

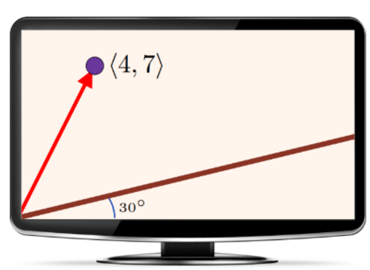
To find the projection of $\vec{u} = \langle 4, 3\rangle$ onto $\vec{v} = \langle 2, 8\rangle$, use the "projection" command. In the entry field enter projection of <$4, 3$> onto <$2, 8$>.



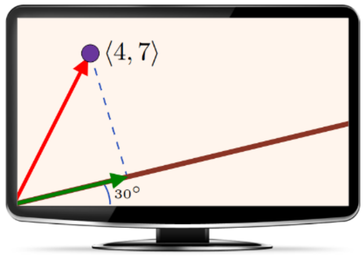
Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\left\langle \frac{16}{17}, \frac{64}{17} \right\rangle$.

Example (2)

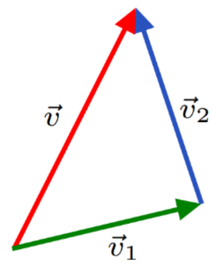
As an applied example, suppose a video game has a ball moving near a wall.



We take the origin at the bottom-left-most corner of the screen. The wall is at a 30° angle to the horizontal, and at a point in time, the ball is at position $\vec{v} = \langle 4, 7\rangle$. To find the perpendicular distance from the ball to the wall, we use the projection formula to project the vector $\vec{v} = \langle 4, 7\rangle$ onto the wall.

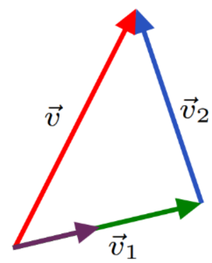


We begin by decomposing $\vec{v}$ into two vectors $\vec{v}_1$ and $\vec{v}_2$ so that $\vec{v} = \vec{v}_1 + \vec{v}_2$ and $\vec{v}_1$ lies along the wall.



The length (magnitude) of the vector $\vec{v}$ is then the distance from the ball to the wall.

The vector $\vec{v}_1$ is the projection of $\vec{v}$ onto the wall. We can get $\vec{v}_1$ by scaling (multiplying) a unit vector $\vec{w}$ that lies along the wall and, thus, along with $\vec{v}_1$.



Since $\vec{w}$ lies at a 30° angle to the horizontal, $\vec{w} = \langle \cos 30°,\ \sin 30°\rangle = \langle 0.866, 0.5\rangle$, using the projection formula, we get the projection of $\vec{v}$ that lies along the wall.

$$\vec{v}_1 = \mathrm{proj}_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\vec{w}$$

$$\vec{v}_1 \frac{\langle 4,7\rangle \cdot \langle 0.866, 0.5\rangle}{\left\|\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\right\|^2}\langle 0.866, 0.5\rangle$$

$$= \frac{4\cdot(0.866) + 7\cdot(0.5)}{\left(\sqrt{(0.866)^2 + (.5)^2}\right)^2}\langle 0.866, 0.5\rangle$$

$$\vec{v}_1 = \frac{6.964}{\left(\sqrt{1}\right)^2}\langle 0.866, 0.5\rangle$$

$$\vec{v}_1 = (6.964)\langle 0.866, 0.5\rangle$$

$$\vec{v}_1 = \langle 6.031, 3.482\rangle$$

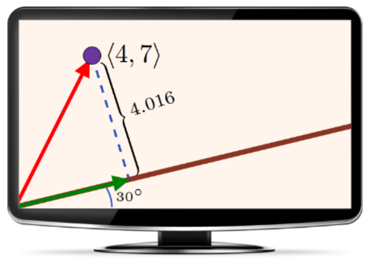
Since that $\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}_1 + \vec{\boldsymbol{v}}_2$, subtraction get us

$$\vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{v}} - \vec{\boldsymbol{v}}_1$$

$$\vec{\boldsymbol{v}}_2 = \langle 4, 7\rangle - \langle 6.031, 3.482\rangle$$

$$\vec{\boldsymbol{v}}_2 = \langle 4 - 6.031, 7 - 3.482\rangle$$

$$\vec{\boldsymbol{v}}_2 = \langle -2.031, 3.518\rangle$$



To get the magnitude of $\vec{\boldsymbol{v}}_2$, we use

$$\|\vec{\boldsymbol{v}}_2\| = \sqrt{{v_x}^2 + {v_y}^2}$$

$$\|\vec{\boldsymbol{v}}_2\| = \sqrt{(-2.031)^2 + 3.518^2}$$

$$\|\vec{\boldsymbol{v}}_2\| = \sqrt{4.125 + 12.376}$$

$$\left\|\overrightarrow{\vec{\boldsymbol{v}}_2}\right\| = 4.062$$

2.6 TRY THESE

1. Find the projection of the vector $\vec{v} = \langle 3, 5\rangle$ onto the vector $\vec{u} = \langle 6, 2\rangle$.

2. Find $\mathrm{proj}_{\vec{v}}\vec{u}$, with $\vec{u} = \langle -2, 5\rangle$ and $\vec{v} = \langle 6, -5\rangle$.

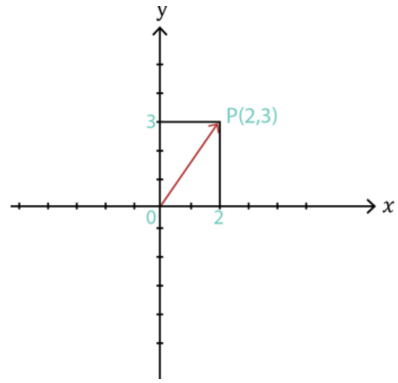
UNIT 3 VECTORS IN THREE DIMENSIONS

3.1 Three Dimensional Vectors

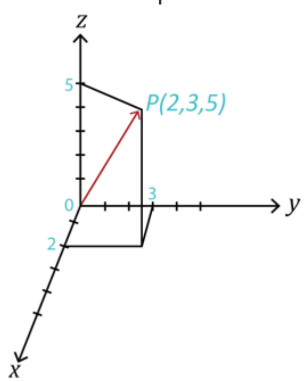
3-DIMENSIONAL SPACE

To this point, we have been working with vectors in 2-dimenisional space. Now, we will expand our discussion to 3-dimensional space.

The **2-dimensional coordinate system** is built around a set of two axes that intersect at right angles and one particular point called the origin. Points in the plane are described by ordered pairs (x, y) and vectors in standard position by $\langle x, y\rangle$.

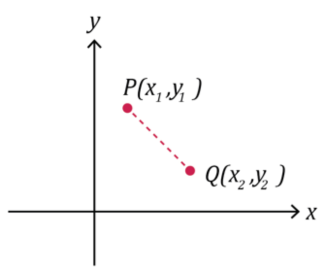


The **3-dimensional coordinate system** is built around a set of three axes that intersect at right angles and one particular point again called the origin. Points in the plane are described by ordered triples (x, y, z) and vectors in standard position by $\langle x, y, z\rangle$.



THE DISTANCE BETWEEN TWO POINTS IN 2 & 3-DIMENSIONAL SPACE

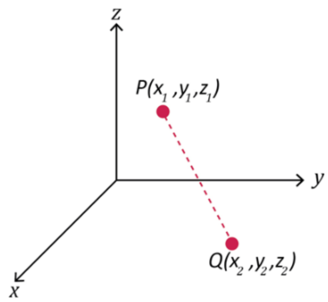
In **two-dimensional space**, the distance d between two points say $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the distance formula



$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In **three-dimensional space**, the distance d between two points say $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Example (1)

The distance between the two points $P(2, 2, 5)$ and $Q(5, 6, 2)$ is

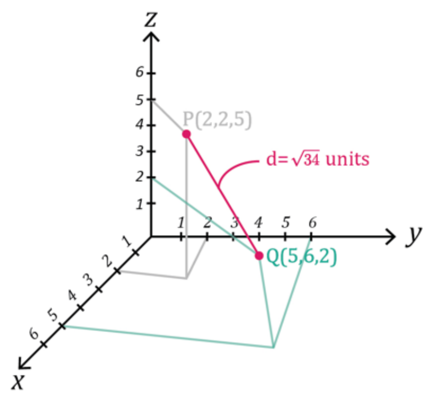
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$d = \sqrt{(5-2)^2 + (6-2)^2 + (2-5)^2}$$

$$d = \sqrt{(3)^2 + (4)^2 + (-3)^2}$$

$$d = \sqrt{9 + 16 + 9}$$

$$d = \sqrt{34} \approx 5.8 \text{ units}$$



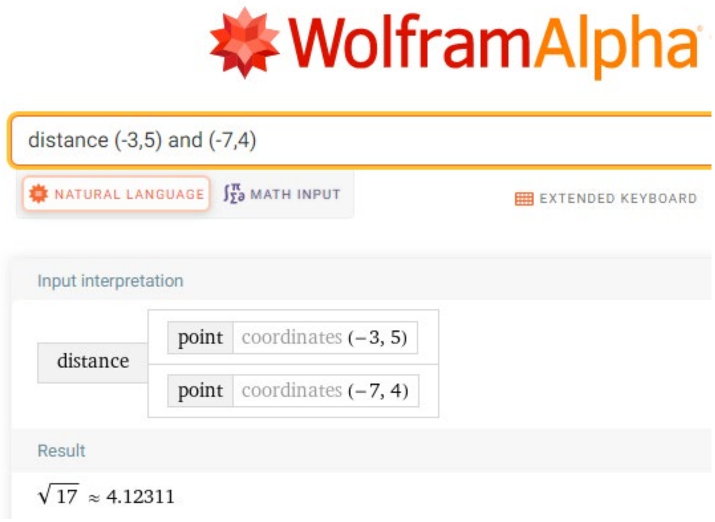
The distance between the two points $P(2, 2, 5)$ and $Q(5, 6, 2)$ is $\sqrt{34} \approx 5.8$ units.

USING TECHNOLOGY

We can use technology to find the distance between points.

Go to www.wolframalpha.com.

To find the distance between the two points $(-3, 5)$ and $(-7, 4)$ enter distance $(-3, 5)$ and $(-7, 4)$ in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\sqrt{17} \approx 4.12311$.

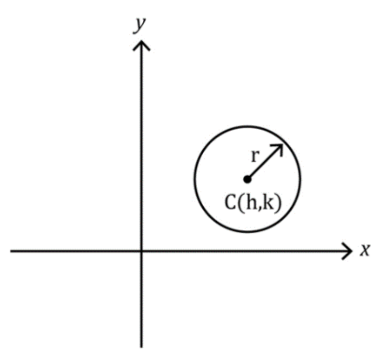


THE EQUATION OF A CIRCLE AND A SPHERE

We can use the distance formulas to get equations of circles and spheres.

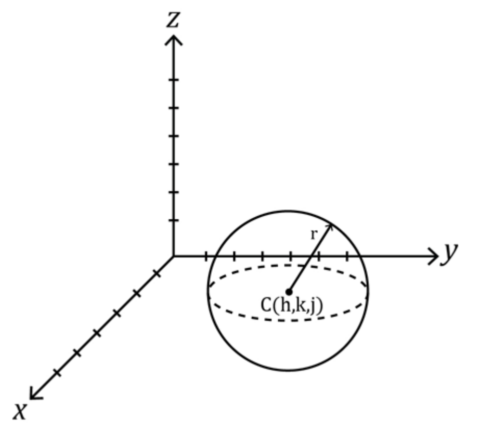
The center-radius form of a circle with center at the point $C(h,k)$ and radius r is

$$(x-h)^2+(y-k)^2=r^2$$



The center-radius form of a sphere with center at the point $C(h,k,j)$ and radius r is

$$(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2$$



Example (2)

To write the equation of a circle that has the point $C(4, 7)$ as its center and radius 8, we use the center-radius form $(x-h)^2 + (y-k)^2 = r^2$ with $h = 4$, $k = 7$, and $r = 8$.

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-4)^2 + (y-7)^2 = 8^2$$

$$(x-4)^2 + (y-7)^2 = 64$$

Example (3)

To write the equation of a sphere that has the point $C(4, 7, 1)$ as its center and radius 8, we use the center-radius form $(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2$ with $h = 4$, $k = 7$, $j = 1$, and $r = 8$.

$$(x-h)^2 + (y-k)^2 + (z-j)^2 = r^2$$

$$(x-4)^2 + (y-7)^2 + (z-1)^2 = 8^2$$

$$(x-4)^2 + (y-7)^2 + (z-1)^2 = 64$$

3.1 TRY THESE

1. Find the distance between the two points $(2, 4)$ and $(-3, 6)$. Round to one decimal place.

2. Find distance between the two points $(-3, 5, -6)$ and $(7, -4, 2)$. Round to one decimal place.

3. Write the equation of a circle that has the point $C(2, 9)$ as its center and radius 1.

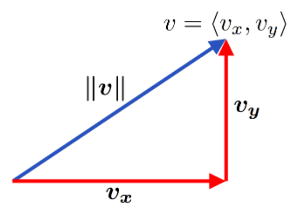
4. Write the equation of a sphere that has the point $C(-2, 5, -7)$ as its center and radius 4.

3.2 Magnitude and Direction Cosines of a Vector

THE MAGNITUDE OF A VECTOR

You likely recall that the magnitude (the length) of a vector $\vec{v} = \langle v_x, v_y \rangle$ in **2-dimensions** is

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$



Example (1)

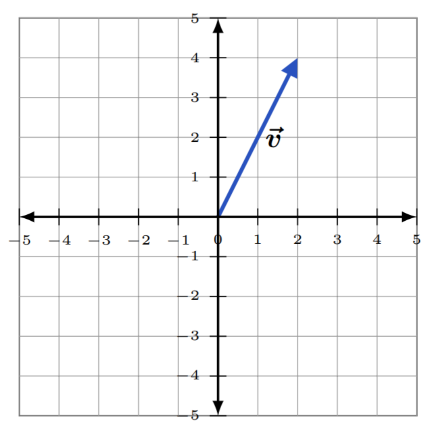
The vector $\vec{v} = \langle 2, 4 \rangle$ has magnitude

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$$

$$\sqrt{2^2 + 4^2} = \sqrt{4 + 16} = \sqrt{20} = 2\sqrt{5}$$

$$\|\vec{v}\| = 2\sqrt{5}$$

Interpret this as the length of the vector $\vec{v} = \langle 2, 4 \rangle$ is $2\sqrt{5}$ units.



The formula for the length of the vector $\vec{v} = \langle v_x, v_y, v_z \rangle$ in **3-dimensions** is

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

Example (2)

The vector $\vec{v} = \langle 2, 4, -6 \rangle$ has magnitude

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$
$$\|\vec{v}\| = \sqrt{2^2 + 4^2 + (-6)^2}$$
$$\|\vec{v}\| = \sqrt{4 + 16 + 36}$$
$$\|\vec{v}\| = \sqrt{56}$$
$$\|\vec{v}\| = 2\sqrt{14}$$

Interpret this as the length of the vector $\vec{v} = \langle 2, 4, -6 \rangle$ is $2\sqrt{14}$ units.

THE DIRECTION COSINES OF VECTORS IN 2- AND 3-DIMENSIONS

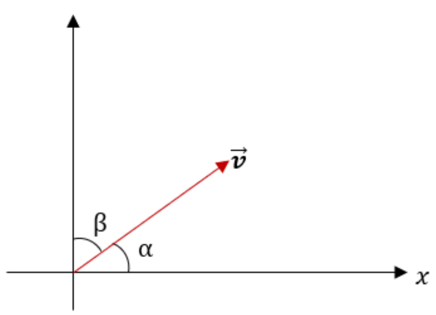
The direction cosines of a vector $\vec{v} = \langle v_x, v_y \rangle$ or $\vec{v} = \langle v_x, v_y, v_z \rangle$ are the cosines of the angles the vector forms with the coordinate axes.

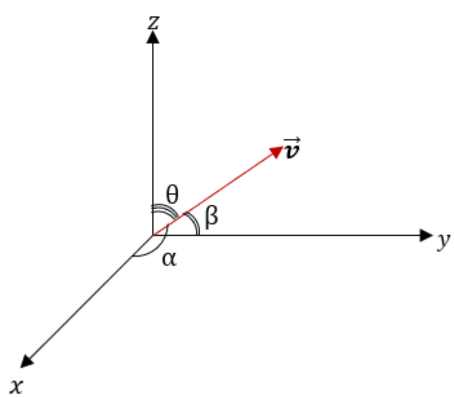
The direction cosines are important as they uniquely determine the direction of the vector.

Direction cosines are found by dividing each component of the vector by the magnitude (length) of the vector.

$$\cos\alpha = \frac{v_x}{\|\vec{v}\|}, \quad \cos\beta = \frac{v_y}{\|\vec{v}\|}$$

$$\cos\alpha = \frac{v_x}{\|\vec{v}\|}, \quad \cos\beta = \frac{v_y}{\|\vec{v}\|}, \quad \cos\theta = \frac{v_z}{\|\vec{v}\|}$$





Example (3)

Find the direction cosines of the vector $\vec{v} = \langle 4, 5, 2 \rangle$.

First, find the magnitude of the vector $\vec{v} = \langle 4, 5, 2 \rangle$

$$\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2} = \sqrt{4^2 + 5^2 + 2^2} = \sqrt{16 + 25 + 4} = \sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}$$

Get the direction cosines by dividing each component, 4, 5, and 2, by this magnitude.

$$\cos\alpha = \frac{v_x}{\|\vec{v}\|} = \frac{4}{3\sqrt{5}} \approx 0.596$$

$$\cos\beta = \frac{v_y}{\|\vec{v}\|} = \frac{5}{3\sqrt{5}} \approx 0.745$$

$$\cos\theta = \frac{v_z}{\|\vec{v}\|} = \frac{2}{3\sqrt{5}} \approx 0.298$$

Example (4)

Find the vector $\vec{v}$ that has magnitude 32 and direction cosines $\cos\alpha = 5/8$ and $\cos\beta = -3/8$.

Since $\cos\alpha = \frac{v_x}{\|\vec{v}\|}$ and $\cos\beta = \frac{v_y}{\|\vec{v}\|}$,

$v_x = \|\vec{v}\| \cdot \cos\alpha = 32 \cdot \frac{5}{8} = 20$, and

$v_y = \|\vec{v}\| \cdot \cos\beta = 32 \cdot \frac{-3}{8} = -12$.

So, $\vec{v} = \langle 20, -12 \rangle$.

USING TECHNOLOGY

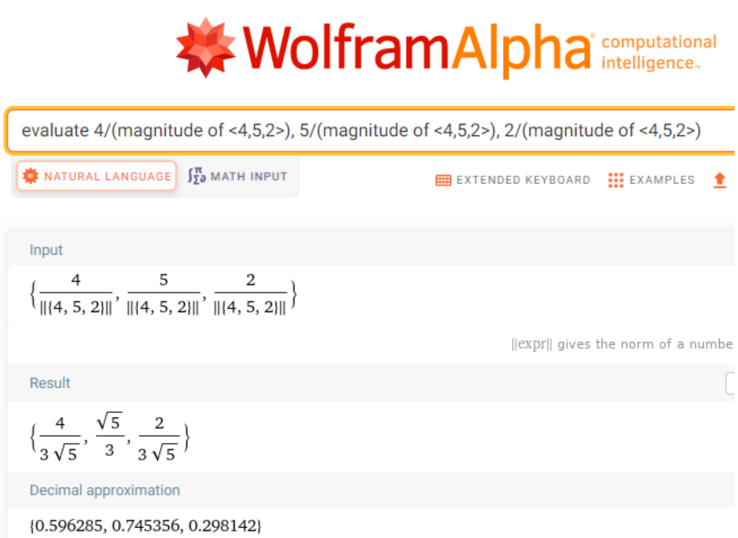
We can use technology to determine the magnitude of a vector.

Go to www.wolframalpha.com.

To find the magnitude of the vector $\vec{v} = \langle 2, 4, -6 \rangle$, enter magnitude of <2, 4, -6> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\|\vec{v}\| = 2\sqrt{14}$.

To find the direction cosines of the vector $\vec{v} = \langle 4, 5, 2 \rangle$, enter evaluate 4/(magnitude of <4,5,2>), 5/(magnitude of <4,5,2>), 2/(magnitude of <4,5,2>) in the entry field. WolframAlpha answers $\{0.596285, 0.745356, 0.298142\}$.

We can use WolframAlpha to approximate a vector give its magnitude and direction cosines.



3.2 TRY THESE

1. Find the magnitude of the vector $\vec{v} = \langle -3, 4, -2 \rangle$.

2. Find the magnitude of the vector $\vec{v} = \langle 1, -1 \rangle$.

3. Find the cosines of the vector $\vec{v} = \langle 3, -1, 2 \rangle$. Round to three decimal places.

4. Approximate the vector $\vec{v}$ that has magnitude 24 and direction cosines $\cos\alpha = -3/4, \cos\beta = -1/4, \cos\theta = 7/8$.

3.3 Arithmetic on Vectors in 3-Dimensional Space

ADDITION & SUBTRACTION OF VECTORS

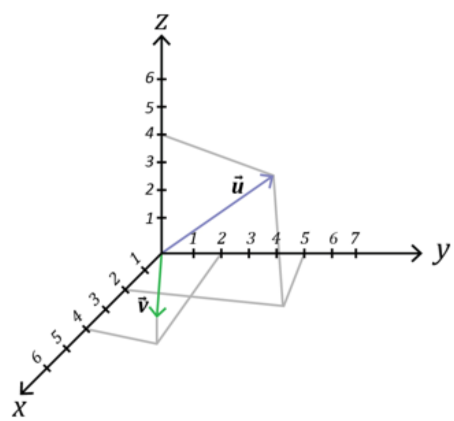
To add or subtract two vectors, add or subtract their corresponding components.

Example (1)

To **add** the vectors $\vec{u} = \langle 2, 5, 4\rangle$ and $\vec{v} = \langle 4, 2, 1\rangle$, add their corresponding components.

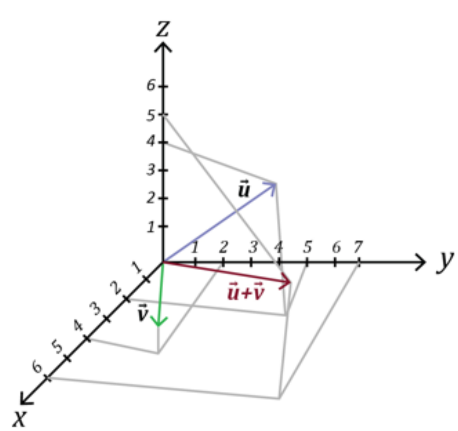
$$\vec{u} + \vec{v} = \langle 2+4, 5+2, 4+1\rangle = \langle 6, 7, 5\rangle$$

So, $\vec{u} + \vec{v} = \langle 6, 7, 5\rangle$



Now, graph this sum. Start at the origin.

Since the x –component is 6, move 6 units in the x –direction.
Since the y –component is 7, move 7 units in the y –direction.
Since the z –component is 5, move 5 units upward.



Example (2)

To **subtract** the vectors $\vec{u} = \langle 2, 5, 4\rangle$ and $\vec{v} = \langle 4, 2, 1\rangle$ subtract their corresponding components.

$$\vec{u} - \vec{v} = \langle 2-4, 5-2, 4-1\rangle = \langle -2, 3, 3\rangle$$

So, $\vec{u} - \vec{v} = \langle -2, 3, 3\rangle$

SCALAR MULTIPLICATION

Scalar multiplication is the multiplication of a vector by a real number (a scalar).

Suppose we let the letter k represent a real number and $\vec{v}$ be the vector $\langle x, y, z\rangle$. Then, the scalar multiple of the vector $\vec{v}$ is

$$k\vec{v} = \langle kx, ky, kz\rangle$$

Example (3)

Suppose $\vec{u} = \langle -3, -8, 5\rangle$ and $k = 3$.

Then $k\vec{u} = 3\vec{u} = 3\langle -3, -8, 5\rangle = \langle 3(-3), 3(-8), 3(5)\rangle = \langle -9, -24, 15\rangle$

Example (4)

Suppose $\vec{v} = \langle 6, 3, -12\rangle$ and $k = \frac{-1}{3}$.

Then $k\vec{u} = \frac{-1}{3}\vec{u} = \frac{-1}{3}\langle 6, 3, -12\rangle = \left\langle \frac{-1}{3}(6), \frac{-1}{3}(3), \frac{-1}{3}(-12)\right\rangle = \langle -2, -1, 4\rangle$

Example (5)

Suppose $\vec{u} = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$. Find $3\vec{u} + 4\vec{v} - 2\vec{w}$.

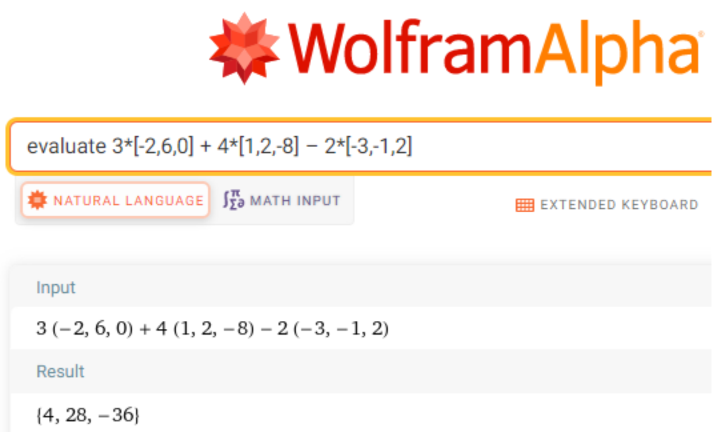
Then $3\vec{u} + 4\vec{v} - 2\vec{w} = 3\begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix} - 2\begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 18 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \\ -32 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ 28 \\ -36 \end{bmatrix}$

USING TECHNOLOGY

We can use technology to determine the value of adding or subtracting vectors.

Go to www.wolframalpha.com.

Suppose $\vec{u} = \begin{bmatrix} -2 \\ 6 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$. Use WolframAlpha to find $3\vec{u} + 4\vec{v} - 2\vec{w}$. In the entry field enter evaluate 3*[-2,6,0] + 4*[1,2,-8] – 2*[-3,-1,2].



WolframAlpha answers $(4, 28, -36)$ which is WolframAlpha's notation for $\begin{bmatrix} 4 \\ 28 \\ -36 \end{bmatrix}$.

3.3 TRY THESE

1. Add the vectors $\vec{u} = \langle -3, 4, 6 \rangle$ and $\vec{v} = \langle 8, 7, -5 \rangle$.

2. Subtract the vector $\vec{v} = \langle 8, 7, -5 \rangle$ from the vector $\vec{u} = \langle -3, 4, 6 \rangle$.

3. Given the three vectors, $\vec{u} = \langle 2, 4, -5 \rangle$, $\vec{v} = \langle -3, 4, -8 \rangle$, and $\vec{w} = \langle 0, 1, 2 \rangle$, find $2\vec{u} + 3\vec{v} - 4\vec{w}$.

4. Suppose $\vec{u} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 6 \\ 4 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$, find $4\vec{u} - 4\vec{v} - \vec{w}$.

3.4 The Unit Vector in 3-Dimensions and Vectors in Standard Position

THE UNIT VECTOR IN 3-DIMENSIONS

The unit vector, as you will likely remember, in 2-dimensions is a vector of length 1. A unit vector in the same direction as the vector $\vec{v}$ is often denoted with a "hat" on it as in $\hat{v}$. We call this vector "v hat."

The unit vector $\hat{v}$ corresponding to the vector $\vec{v}$ is defined to be

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}$$

Example (1)

The unit vector corresponding to the vector $\vec{v} = \langle -8, 12 \rangle$ is

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{(-8)^2 + (12)^2}}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{64 + 144}}$$

$$\hat{v} = \frac{\langle -8, 12 \rangle}{\sqrt{208}}$$

$$\hat{v} = \left\langle \frac{-8}{\sqrt{208}}, \frac{12}{\sqrt{208}} \right\rangle$$

A unit vector in 3-dimensions and in the same direction as the vector $\vec{v}$ is defined in the same way as the unit vector in 2-dimensions.

The unit vector $\hat{v}$ corresponding to the vector $\vec{v}$ is defined to be $\hat{v} = \frac{\vec{v}}{||\vec{v}||}$, where $\vec{v} = \langle x, y, z \rangle$.

For example, the unit vector corresponding to the vector $\vec{v} = \langle 5, -3, 4 \rangle$ is

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}$$

$$\hat{v} = \frac{\langle 5, -3, 4 \rangle}{\sqrt{5^2 + (-3)^2 + 4^2}}$$

$$\hat{v} = \frac{\langle 5, -3, 4 \rangle}{\sqrt{25 + 9 + 16}}$$

$$\hat{v} = \frac{\langle 5, -3, 4 \rangle}{\sqrt{50}}$$

$$\hat{v} = \frac{\langle 5, -3, 4 \rangle}{5\sqrt{2}}$$

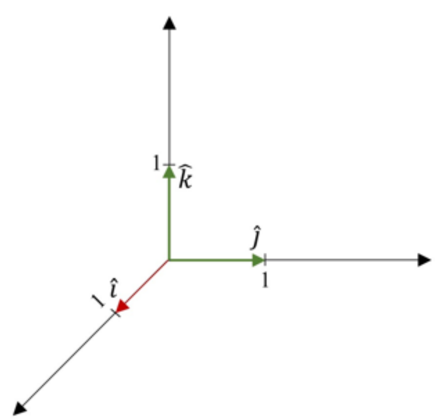
$$\hat{v} = \left\langle \frac{5}{5\sqrt{2}}, \frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}} \right\rangle$$

VECTORS IN STANDARD POSITION

A vector with its initial point at the origin in a Cartesian coordinate system is said to be in *standard position*. A common notation for a unit vector in standard position uses the lowercase letters i, j, and k is to represent the unit vector in

the x-direction with the vector $\hat{\imath}$, where $\hat{\imath} = \langle 1, 0, 0 \rangle$, and
the y-direction with the vector $\hat{\jmath}$, where $\hat{\jmath} = \langle 0, 1, 0 \rangle$, and
the z-direction with the vector $\hat{k}$, where $\hat{k} = \langle 0, 0, 1 \rangle$.

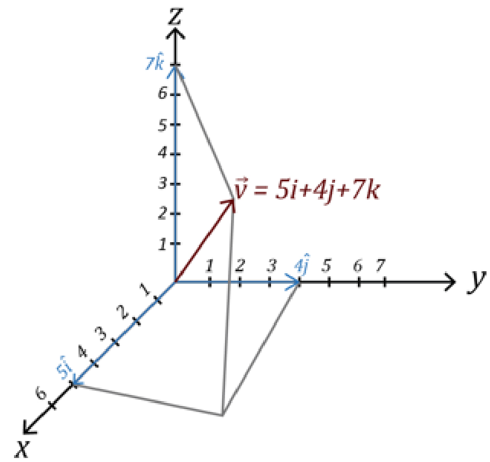
The figure shows these three unit vectors.



Any vector can be expressed as a combination of these three unit vectors.

Example (2)

The vector $\vec{v} = \langle 5, 4, 7 \rangle$ can be expressed as $\vec{v} = 5\hat{\imath} + 4\hat{\jmath} + 7\hat{k}$.



Now, the unit-vector in the direction of $\vec{v} = 5\hat{\imath} + 4\hat{\jmath} + 7\hat{k}$ is

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\hat{v} = \frac{\langle 5, 4, 7 \rangle}{\sqrt{5^2 + 4^2 + 7^2}}$$

$$\hat{v} = \frac{\langle 5, 4, 7 \rangle}{\sqrt{25 + 16 + 49}}$$

$$\hat{v} = \frac{\langle 5, -3, 4 \rangle}{\sqrt{90}}$$

$$\hat{v} = \frac{\langle 5, 4, 7 \rangle}{\sqrt{9 \cdot 10}}$$

$$\hat{v} = \frac{\langle 5, 4, 7 \rangle}{3\sqrt{10}}$$

$$\hat{v} = \left\langle \frac{5}{3\sqrt{10}}, \frac{4}{3\sqrt{10}}, \frac{7}{3\sqrt{10}} \right\rangle$$

$$\vec{v} = \frac{5}{3\sqrt{10}}\hat{\imath} + \frac{4}{3\sqrt{10}}\hat{\jmath} + \frac{7}{3\sqrt{10}}\hat{k}$$

NORMALIZING A VECTOR

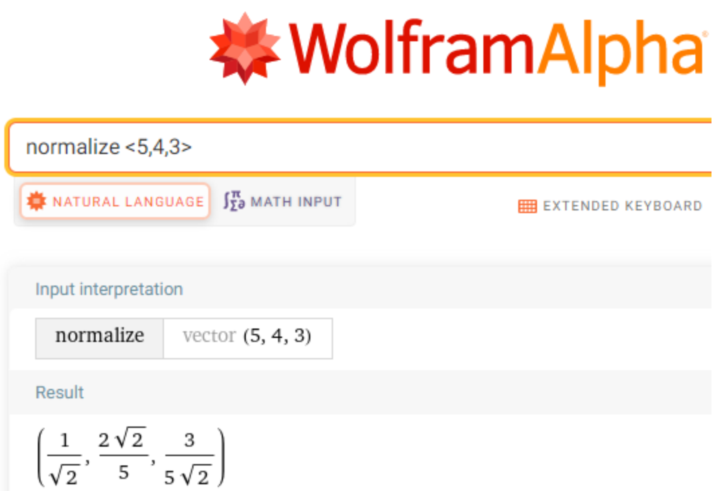
Normalizing a vector is a common practice in mathematics and it also has practical applications in computer graphics. Normalizing a vector $\vec{v}$ is the process of identifying the unit vector of a vector $\vec{v}$.

USING TECHNOLOGY

We can use technology to find the unit vector in the direction of the given vector.

Go to www.wolframalpha.com.

Use WolframAlpha to find the unit vector in the direction of $\vec{u} = \langle 5, 4, 3 \rangle$. Enter normalize <5,4,3> in the entry field and WolframAlpha gives you an answer.



Translate WolframAlpha's answer to $\frac{1}{\sqrt{2}}\hat{\imath} + \frac{2\sqrt{2}}{5}\hat{\jmath} + \frac{3}{5\sqrt{2}}\hat{k}$.

3.4 TRY THESE

1. Write the unit vector that corresponds to $\vec{v} = \langle 2, -3, 4 \rangle$.

2. Write the unit vector that corresponds to $\vec{v} = \langle 1, -1, 1 \rangle$.

3. Write the unit vector that corresponds to $\vec{v} - \vec{u} = \langle 6, 7, 2 \rangle - \langle 2, 7, 6 \rangle$.

4. Normalize the vector $\vec{v} = \langle 4, 3, 2 \rangle$.

3.5 The Dot Product, Length of a Vector, and the Angle between Two Vectors in Three Dimensions

THE DOT PRODUCT OF TWO VECTORS

The dot product of two vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$ in two dimensions is nicely extended to three dimensions.

The dot product of vectors $\vec{u} = \langle u_x, u_y, u_z \rangle$ and $\vec{v} = \langle v_x, v_y, v_z \rangle$ is a scalar (real number) and is defined to be

$$\vec{u} \cdot \vec{v} = u_x v_x + u_y v_y + u_z v_z$$

Since u_x, u_y, u_z, v_x, v_y, and v_z are real numbers, you can see that the dot product is itself a real number and not a vector.

Example (1)

To compute the dot product of the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$, we compute

$$\vec{u} \cdot \vec{v} = 5 \cdot 3 + 2 \cdot 4 + 4 \cdot (-7) = 15 + 8 - 28 = -5$$

Since the dot product is a scalar, it follows the properties of real numbers.

PROPERTIES OF THE DOT PRODUCT

1. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$, the dot product is commutative
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, the dot product distributes over vector addition
3. $\vec{u} \cdot \vec{0} = 0$, the dot product with the zero vector $\vec{0}$, is the scalar 0.
4. $\vec{u} \cdot \vec{u} = ||\vec{u}||^2$

Example (2)

Compute the dot product $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$, where

$\vec{u} = \langle 5, -2, -3 \rangle$, $\vec{v} = \langle 6, 4, 1 \rangle$, and $\vec{w} = \langle -3, 7, -2 \rangle$,

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$\begin{aligned}\vec{u} \cdot (\vec{v} + \vec{w}) &= \langle 5, -2, -3 \rangle \cdot \langle 6, 4, 1 \rangle + \langle 5, -2, -3 \rangle \cdot \langle -3, 7, -2 \rangle \\ &= (5 \cdot 6 + (-2) \cdot 4 + (-3) \cdot 1) + (5 \cdot (-3) + (-2) \cdot 7) + (-3) \cdot (-2))\end{aligned}$$

$$= 30 - 8 - 3 - 15 - 14 + 6$$
$$= -4$$

THE LENGTH OF A VECTOR IN THREE DIMENSIONS

The length (magnitude) of a vector in two dimensions is nicely extended to three dimensions.

The dot product of a vector $\vec{v} = \langle v_x, v_y \rangle$ with itself gives the length of the vector.

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2}$$

You can see that the length of the vector is the square root of the sum of the squares of each of the vector's components. The same is true for the length of a vector in three dimensions.

The dot product of a vector $\vec{v} = \langle v_x, v_y, v_z \rangle$ with itself gives the length of the vector.

$$||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2 + \ {v_z}^2}$$

Example (3)

Use the dot product to find the length of the vector $\vec{v} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$.

In this case, $v_x = 4$, $v_y = 2$, and $v_z = 6$

Using $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2 + \ {v_z}^2}$, we get

$$||\vec{v}|| = \sqrt{4^2 + 2^2 + \ 6^2}$$
$$||\vec{v}|| = \sqrt{56}$$
$$||\vec{v}|| = \sqrt{4 \cdot 14}$$
$$||\vec{v}|| = \sqrt{4} \cdot \sqrt{14}$$
$$||\vec{v}|| = 2\sqrt{14}$$

The length of the vector $\vec{v} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ is $2\sqrt{14}$ units.

THE ANGLE BETWEEN TWO VECTORS

The formula for the angle between two vectors in two dimensions is nicely extended to three dimensions.

If θ is the smallest nonnegative angle between two non-zero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||} \text{ or } \theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||}$$

where $0 \leq \theta \leq 2\pi$ and $||\vec{u}|| = \sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2}$ and $||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$

Example (4)

Find the angle between the vectors $\vec{u} = \langle 5, -3, -1\rangle$ and $\vec{v} = \langle 2, 4, -5\rangle$.

Using $\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{||\vec{u}||\cdot||\vec{v}||}$, we get

$$\theta = \cos^{-1}\frac{\langle 5, -3, -1\rangle\cdot\langle 2, 4, -5\rangle}{\sqrt{5^2 + (-3)^2 + (-1)^2}\cdot\sqrt{2^2 + 4^2 + (-5)^2}}$$

$$\theta = \cos^{-1}\frac{5\cdot 2 + (-3)\cdot 4 + (-1)\cdot(-5)}{\sqrt{25 + 9 + 1}\cdot\sqrt{4 + 16 + 25}}$$

$$\theta = \cos^{-1}\frac{3}{\sqrt{35}\cdot\sqrt{45}}$$

$$\theta = 85.66$$

We conclude that the angle between these two vectors is close to 85.7° rounded to one decimal place.

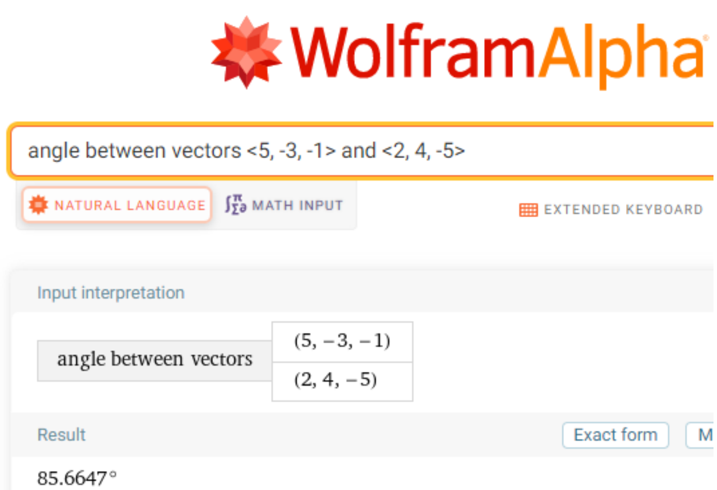
USING TECHNOLOGY

We can use technology to find the magnitude of the vector and the angle θ between two vectors.

Go to www.wolframalpha.com.

To find the magnitude (length) of the vector $\vec{v} = \langle 4, 2, 4 \rangle$, enter magnitude of <4, 2, 4> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\|\vec{v}\| = 6$.

To find the angle between the vectors $\vec{u} = \langle 5, -3, -1 \rangle$ and $\vec{v} = \langle 2, 4, -5 \rangle$, enter angle between vectors <5, -3, -1> and <2, 4, -5> in the entry field. Wolframalpha tells you what it thinks you entered, then tells you its answer. In this case, $\theta = 85.7°$, rounded to one decimal place.



3.5 TRY THESE

1. Find the dot product of the vectors $\vec{u} = \langle -2, 3, -9 \rangle$ and $\vec{v} = \langle 5, -1, 2 \rangle$.

2. Find the dot product of the vectors $\vec{u} = \langle 6, 2, -1 \rangle$ and $\vec{v} = \langle 2, -7, -2 \rangle$.

3. Find the length of the vector $\vec{u} = \langle 4, -7, -6 \rangle$.

4. Find the length of the vector $\vec{v} = \langle 0, 5, 0 \rangle$.

5. Find the angle between the vectors $\vec{u} = \langle 3, 4, 5 \rangle$ and $\vec{v} = \langle -3, -1, 8 \rangle$.

6. Find the angle between the vectors $\vec{u} = \langle 1, -2, 1 \rangle$ and $\vec{v} = \langle 3, 5, 7 \rangle$.

3.6 The Cross Product: Algebra

THE CROSS PRODUCT OF TWO VECTORS

A vector that is perpendicular to both vectors $\vec{u} = \langle u_x, u_y \rangle$ and $\vec{v} = \langle v_x, v_y \rangle$, can be found using the cross product. The cross product requires that both vectors be in three-dimensional space.

The cross product of vectors $\vec{u} = \langle u_x, u_y, u_z \rangle$ and $\vec{v} = \langle v_x, v_y, v_z \rangle$ is a vector and is defined to be

$$\vec{u} \times \vec{v} = \langle u_y v_z - u_z v_y,\; u_z v_x - u_x v_z,\; u_x v_y - u_y v_z \rangle$$

This formula is challenging to remember. A nice device to help you remember both this formula and the dot product formula is to visualize them in a 3x3 square of components. The square shows how vectors can interact with one another.

	x	y	z
x	Dot	Cross	Cross
y	Cross	Dot	Cross
z	Cross	Cross	Dot

For the cross product,
The x-component has a product that involves no x-components: $u_y v_z - u_z v_y$
The y-component has a product that involves no y-components: $u_z v_x - u_x v_z$
The z-component has a product that involves no z-components: $u_x v_y - u_y v_z$

Each component is a difference of two diagonal products.

	x	y	z
x	Dot	x*y	x*z
y	y*x	Dot	y*z
z	z*x	z*y	Dot

To produce the x-component,
(top right) - (bottom left) = y*z – z*y

To produce the y-component,
(bottom left) - (top right) = z*x – x*z

To produce the z-component,
(top right) – (bottom left) = x*y – y*x

The **DOT** product is the interaction between two vectors having **similar** components:

$$x \cdot x, \qquad y \cdot y, \qquad z \cdot z$$

The dot product measures similarity since it combines only interactions of matching components.

The **CROSS** product is the interaction between two vectors having **different** components:

$$x \cdot y,\ x \cdot z,\ y \cdot x,\ y \cdot z,\ z \cdot x,\ z \cdot y$$

The cross product measures cross interactions since it combines interactions of different components.

Example (1)

Find the cross product of the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$.

	3	4	-7
5	Dot	5*4	5*-7
2	2*3	Dot	2*-7
4	4*3	4*4	Dot

	3	4	-7
5	Dot	20	-35
2	6	Dot	-14
4	12	16	Dot

To produce the x-component, (top right) - (bottom left) = 2*(-7) – 4*4 = -30

To produce the y-component, (bottom left) - (top right) = 4*3 – 5*(-7) = 47

To produce the z-component, (top right) – (bottom left) = 5*4 – 2*3 = 14

$$\vec{u} \times \vec{v} = \langle -30, 47, 14 \rangle$$

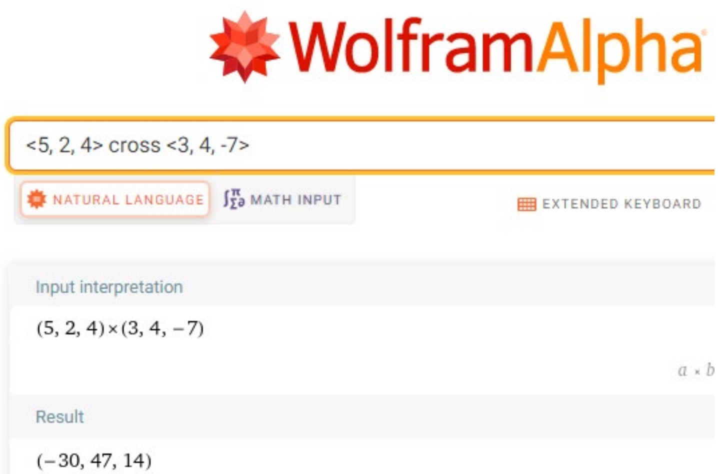
*Be careful with the computation. It goes (bottom left) – (top right) while the first and last go (top right) – (bottom left).

USING TECHNOLOGY

We can use technology to find the cross product between two vectors.

Go to www.wolframalpha.com.

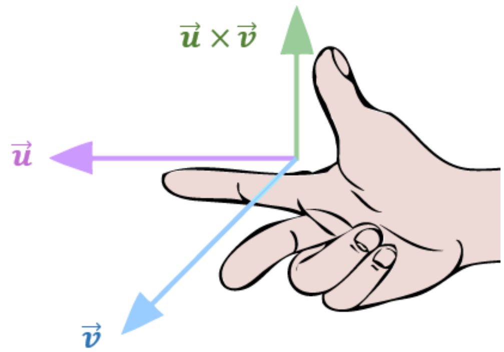
To find the cross product of the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$, use either the "cross" or the x command. Wolframalpha tells you what it thinks you entered, then it tells you its answer. In this case, $\vec{u} \times \vec{v} = \langle -30, 47, 14 \rangle$.



THE RIGHT-HAND RULE

You can see that the cross product of the two vectors $\vec{u}$ and $\vec{v}$, is itself a vector. But where is this vector $\vec{u} \times \vec{v}$? The cross product of two vectors is a vector that is perpendicular to the plane formed by the two vectors. What about the two perpendicular directions? Does this perpendicular vector lie above or below the plane formed by the two vectors? We use the **right-hand rule**.

Hold your hand as shown in the picture, your index and middle fingers extended. Your thumb points in the direction of the cross product.



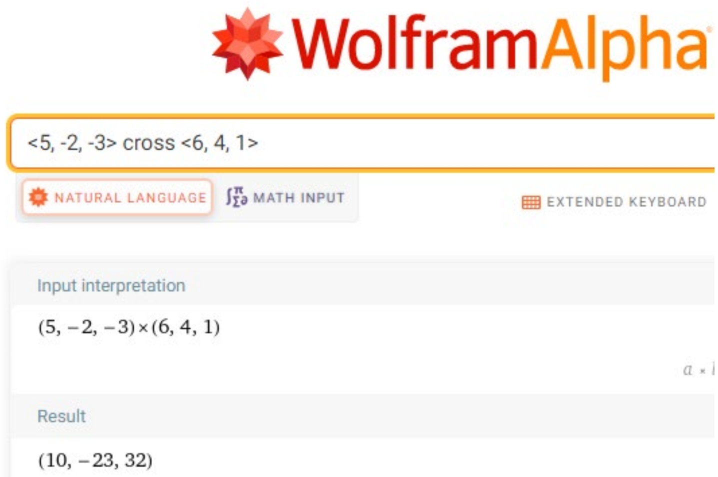
Since the dot product is a scalar, it follows the properties of real numbers.

PROPERTIES OF THE CROSS PRODUCT

1. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$, the cross product is **anti-commutative**
2. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$, the cross product distributes over vector addition
3. $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v} = \vec{u} \times (k\vec{u})$
4. $\vec{u} \times \vec{0} = \vec{0}$, the cross product with the zero vector $\vec{0}$, is the zero vector $\vec{0}$

USING TECHNOLOGY

For example, use WolframAlpha to compute both the cross product $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$, with $\vec{u} = \langle 5, -2, -3 \rangle$ and $\vec{v} = \langle 6, 4, 1 \rangle$, to show that one is the opposite of the other.



Notice that $\langle 10, -23, -32\rangle = -\langle -10, 23, -32\rangle$, verifying property 1.

3.6 TRY THESE

1. Find the cross product of the vectors $\vec{u} = \langle 4, -2, 1\rangle$ and $\vec{v} = \langle 5, -1, 3\rangle$.

2. Find the cross product of the vectors $\vec{u} = \langle -2, 3, -9\rangle$ and $\vec{v} = \langle -8,\ 12, -36\rangle$.

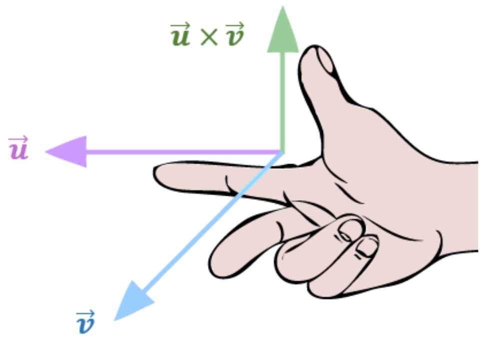
3. Find $\vec{u} \times \vec{v} \cdot \vec{w}$, where $\vec{u} = \langle -2, 5, 3\rangle$, $\vec{v} = \langle 4, 4, -2\rangle$, and $\vec{w} = \langle 2, 6, -5\rangle$.

* Note that the cross product must be computed first since if it is not, we would be crossing a vector with a scalar.

3.7 The Cross Product: Geometry

THE CROSS PRODUCT OF TWO VECTORS AND THE RIGHT-HAND RULE

The cross product of the two vectors $\vec{u}$ and $\vec{v}$, is itself a vector. Where is this vector $\vec{u} \times \vec{v}$? The cross product of two vectors is a vector perpendicular to the plane formed by the two vectors. What if there are two perpendicular directions? Does this perpendicular vector lie above or below the plane formed by the two vectors? Let's use the **right-hand rule**.



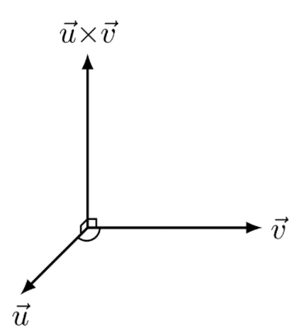
Hold your hand as shown in the picture, your index and middle fingers extended. Your thumb points in the direction of the cross product.

THE GEOMETRY OF THE CROSS PRODUCT

If θ is the angle between the two vectors $\vec{u} = \langle u_x, u_y, u_z \rangle$ and $\vec{v} = \langle v_x, v_y, v_z \rangle$, then the length (magnitude) of the cross product $\vec{u} \times \vec{v}$ is

$$||\vec{u} \times \vec{v}|| = ||\vec{u}||\,||\vec{v}||\sin\theta$$

$$||\vec{u}|| = \sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2} \text{ and } ||\vec{v}|| = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$



Example (1)

The length of the vector $\vec{u} \times \vec{v}$, where $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$ is

$$||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}||\sin\theta$$

$$\sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2} \cdot \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2} \cdot \sin\theta$$

We now need to get $\sin\theta$. We'll use the formula for the angle between two vectors.

$$\theta = \cos^{-1} \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \cdot ||\vec{v}||}$$

$$\theta = \cos^{-1} \frac{\langle 5, 2,4 \rangle \cdot \langle 3, 4, -7 \rangle}{\sqrt{5^2 + 2^2 + 4^2} \cdot \sqrt{3^2 + 4^2 + (-7)^2}}$$

$$\theta = \cos^{-1} \frac{5 \cdot 3 + 2 \cdot 4 + 4 \cdot (-7)}{\sqrt{45} \cdot \sqrt{74}}$$

$$\theta = \cos^{-1} \frac{-5}{\sqrt{45} \cdot \sqrt{74}}$$

$$\theta = 94.97°$$

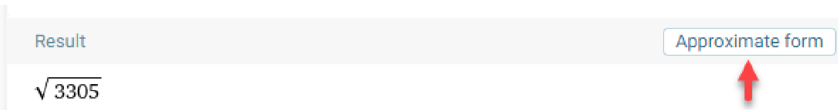
Now we compute $||\vec{u} \times \vec{v}|| = \sqrt{45} \cdot \sqrt{74} \sin(94.97°) = 57.49$ units.

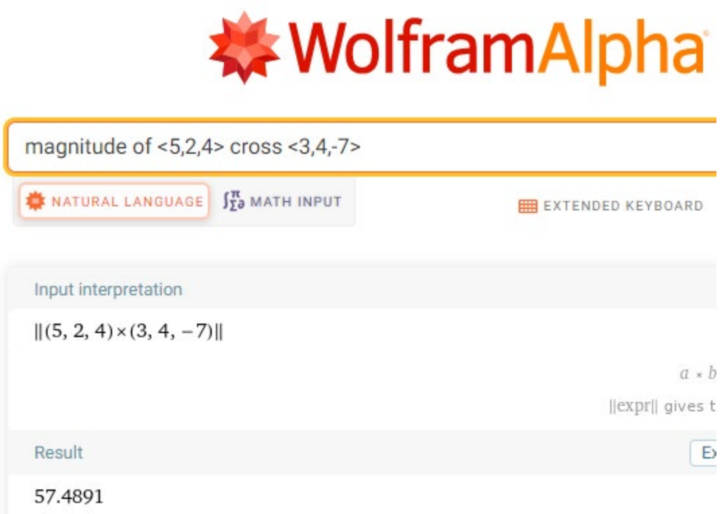
USING TECHNOLOGY

We can use technology to find the magnitude of the cross product of two vectors.

Go to www.wolframalpha.com.

To find the length of the cross product of the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$ enter magnitude of <5, 2, 4> cross <3, 4, -7> in the entry field. Wolframalpha tells you what it thinks you entered and its answer. In this case it shows you result of $\sqrt{3305}$. Click on the approximate form button to get the result in decimal form as 57.49.

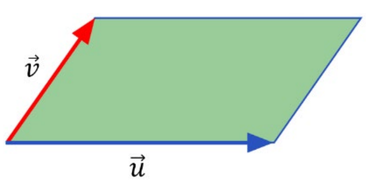




AREA OF A PARALLELOGRAM

Geometrically, $||\vec{u} \times \vec{v}||$ produces the area of a parallelogram determined by $\vec{u}$ and $\vec{v}$.

Area = $||\vec{u} \times \vec{v}|| = ||\vec{u}||||\vec{v}||\sin\theta$



The area of the parallelogram determined by the vectors $\vec{u} = \langle 5, 2, 4 \rangle$ and $\vec{v} = \langle 3, 4, -7 \rangle$ from the above example is 57.49 square units.

THE CROSS PRODUCT OF PERPENDICULAR AND PARALLEL VECTORS

If vectors $\vec{u}$ and $\vec{v}$ are **perpendicular** to each other, then the angle between them is 90° and $\sin(90°) = 1$, so that $\vec{u} \times \vec{v} = \|\vec{u}\|\|\vec{v}\|$

If vectors $\vec{u}$ and $\vec{v}$ are **parallel** to each other, then the angle between them is 0° and $\sin(0°) = 0$.

It makes sense then to define the cross product of parallel vectors to be the zero vector, $\vec{0}$. Also, if at least one of the vectors $\vec{u}$ and $\vec{v}$ is the zero vector $\vec{0}$, then the cross product $\vec{u} \times \vec{v}$ is defined to be the zero vector. We can say that if the cross product of two vectors is zero, then the vectors are parallel to each other. Also, if two vectors are parallel to each other, then their cross product is zero. We combine these statements together in an *if-and-only-if* statement.

Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel to each other if and only if $\vec{u} \times \vec{v} = 0$.

PROPERTIES OF THE CROSS PRODUCT

5. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$, the cross product is **anti-commutative**
6. $k(\vec{u} \times \vec{v}) = k\vec{u} \times \vec{v} = \vec{u} \times k\vec{v}$, multiplication by a scalar
7. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$, the cross product distributes over vector addition
8. $\vec{u} \times \vec{0} = \vec{0}$, the cross product with the zero vector $\vec{0}$, is the zero vector $\vec{0}$.

Example (2)

Use WolframAlpha to verify that

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

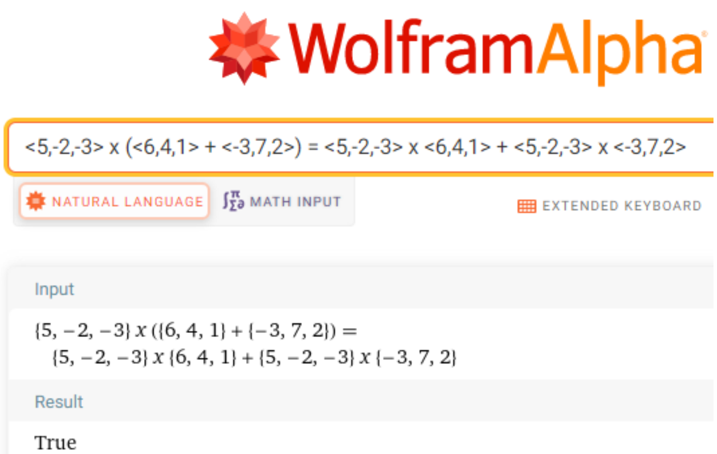
where $\vec{u} = \langle 5, -2, -3 \rangle$, $\vec{v} = \langle 6, 4, 1 \rangle$, and $\vec{w} = \langle -3, 7, 2 \rangle$.

Use W|A to first compute $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u}$ and then $\vec{u} \times \vec{v} + \vec{u} \times \vec{w}$. Determine if the results do or do not match. We can do this in one step by entering

<5,-2,-3> x (<6,4,1> + <-3,7,2>) = <5,-2,-3> x <6,4,1> + <5,-2,-3> x <-3,7,2>

If the statement on the left of the = equals the statement on the right, W|A responds with True.
If the statement on the left of the ≠ equals the statement on the right, W|A responds with False.

In this case, we get a True response and have verified the truth of the statement.



3.7 TRY THESE

1. Find the cross product of the vectors $\vec{u} = \langle -4, 3, 5 \rangle$ and $\vec{v} = \langle 5, -1, 2 \rangle$.

2. Find the cross product of the vectors $\vec{u} = \langle -2, 3, -9 \rangle$ and $\vec{v} = \langle 6, -9, 27 \rangle$.

3. Find the length of the vector formed by the cross product of the vectors $\vec{u} = \langle 3, -5, 4 \rangle$ and $\vec{v} = \langle 2, -4, 1 \rangle$.

4. Find the angle between the vectors $\vec{u} = \langle 4, -7, -6 \rangle$ and $\vec{v} = \langle 5, -1, 2 \rangle$.

5. Determine if the vectors $\vec{u} = \langle 3, -2, 1 \rangle$ and $\vec{v} = \langle 0, 2, 4 \rangle$ are perpendicular or parallel to each other.

6. Find the area of the parallelogram and the triangle formed by the vectors $\vec{u} = \langle 1, -2, -4 \rangle$ and $\vec{v} = \langle 4, 3, -5 \rangle$.

UNIT 4 MATRICES

4.1 Matrices

MATRIX

A matrix is a rectangular array of objects, often numbers.

Example (1)

The rectangular array of numbers $\begin{bmatrix} 5 & -2 \\ 0 & 4 \\ -6 & 3 \end{bmatrix}$ is a matrix having 3 rows and 2 columns.

DIMENSION OF A MATRIX

Matrix having m number of rows and n number of columns has dimension (size) $m \times n$ (pronounced as "m by n") and is called an $m \times n$ matrix.

The matrix in Example (1) is a 3×2 matrix since it is composed of 3 rows and 2 columns. When specifying the dimension of a matrix, the number of rows is stated first and the number of columns second.

ELEMENTS OF A MATRIX

It is common to use an uppercase letter of the alphabet to name a matrix and the corresponding lowercase letter to name an element (entry or member) of the matrix. Subscripts are attached to the lowercase letter to specify its position in the matrix.

The first number in subscript indicates the row in which the element resides and the second number the column.

The subscript numbers appear adjacent to each other and typically without a comma separating them.

We could name the matrix of Example 1 with the uppercase letter A and write $A = \begin{bmatrix} 5 & -2 \\ 0 & 4 \\ -6 & 3 \end{bmatrix}$.

We specify the element -2 in row 1, column 2, with the notation a_{12}. The lowercase a is used to indicate that the element is from matrix A and the subscripts indicate we are observing the entry in row 1, column 2. The subscript is not the number 12, but rather the two individual numbers, 1 and 2.

In general, the notation a_{ij} denotes the entry in row i and column j.

Some other elements of A are
$a_{11} = 5$, the number in row 1, column 1
$a_{31} = -6$, the number in row 3, column 1
$a_{22} = 4$, the number in row 2, column 2

In general, an $m \times n$ matrix has the form $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$. For some number m, the element a_{m2} is the number in row m, column 2.

YOUR TURN:

In the matrix $B = \begin{bmatrix} 0 & -4 & 2 \\ 1 & -1 & 5 \\ -3 & 3 & 8 \end{bmatrix}$,

a) Specify the size of B.
b) Find the value of b_{11}.
c) Find the value of b_{13}.
d) Find the value of b_{32}.

ANS: (a) 3×3, (b) 0, (c) 2, (d) 3

EQUAL MATRICES

Two matrices A and B are said to be equal, written as $A = B$, if they are the same size and all the corresponding entries are equal.

In matrix notation, for all i and j, $A = B$ if $a_{ij} = b_{ij}$. The notation a_{ij} names the element in row i and column j of matrix A. Similarly, the notation b_{ij} names the element in row i and column j of matrix B. The notation $a_{ij} = b_{ij}$ indicates that the element in row i and column j of matrix A is the same as the element in row i and column j of matrix B.

SQUARE MATRICES

A matrix is called square if it has the same number of columns as rows.

Example (2)

The 2×2 matrices A and B are both equal and square.

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$$

MAIN DIAGONAL OF A SQUARE MATRIX

Consider a square matrix, say $A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -5 \\ -1 & 6 & 7 \end{bmatrix}$. Imagine a line passing from the top left element to the bottom right element as in the picture.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -5 \\ -1 & 6 & 7 \end{bmatrix}.$$

This diagonal set of elements from the top left element to the bottom right is called the main diagonal of the matrix.

DIAGONAL AND NON-DIAGONAL ELEMENTS OF A MATRIX

The elements lying on the main diagonal of matrix A are called the diagonal elements of matrix A. The elements 2, 4, and 7 are the diagonal elements of matrix A. The elements lying off the main diagonal of matrix A are called the non-diagonal or off-diagonal elements of matrix A. The elements 1, 0, 3, -5, -1, and 6 are the non-diagonal elements of matrix A.

THE IDENTITY MATRIX

An Identity matrix is a square matrix that has only 1's on its main diagonal and 0's everywhere else.

A matrix in which every diagonal element is 1 and every non-diagonal element is 0 is an identity matrix. Identity matrices are typically named with the uppercase letter I. It is not uncommon to write the size of the matrix as a subscript on the I.

Example (3)

The square matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the 3×3. We could write $I_{3\times3}$ to indicate the 3×3 identity matrix.

THE ZERO MATRIX

The zero matrix is a matrix, in which every element is 0.

Zero matrices are commonly named with a 0.

Example (4)

The matrix $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a zero matrix.

THE TRANSPOSE OF A MATRIX

Consider some $m \times n$ matrix A. For example, suppose A is the 2×3 matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$.
Form a new matrix, call it A-transpose and denote it by A^T, by making

- The first row of A the first column of A^T,
- The second row of A the second column of A^T.

Then $A^T = \begin{bmatrix} 2 & 5 \\ 3 & 6 \\ 4 & 7 \end{bmatrix}$ is the transpose of $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$.

The rows of a matrix are the columns of its transpose. If the matrix A is size $m \times n$, then dimension of A^T is $n \times m$.

ROW MATRICES AND COLUMN MATRICES

A row matrix is a matrix with only one row and any number of columns.

The matrix $R = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$ is a row matrix with 3 columns. It is a 1×3 matrix.

A column matrix is a matrix with only one column and any number of rows.

The matrix $C = \begin{bmatrix} 6 \\ 7 \end{bmatrix}$ is a column matrix with 2 rows. It is a 2×1 matrix.

VECTORS AS MATRICES

When we first described vectors, we expressed them using the bracket notation. For example, we could write a vector as $\langle 2, 4, 6 \rangle$. We can just as easily describe this vector using a row matrix $\begin{bmatrix} 2 & 4 & 6 \end{bmatrix}$ or column matrix $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$.

4.1 TRY THESE

1. Specify the dimension of each matrix.

   a. $S = \begin{bmatrix} 0 & 2 & 5 \\ -6 & -3 & 2 \\ 1 & 9 & 2 \\ 8 & -1 & 4 \end{bmatrix}$

   b. $T = \begin{bmatrix} 5 & 6 & -3 \\ 0 & 0 & -3 \end{bmatrix}$

   c. $Q = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$

2. True or False: The transpose of a square matrix is also a square matrix.

3. In the matrix $S = \begin{bmatrix} 0 & 2 & 5 \\ -6 & -3 & 2 \\ 1 & 9 & 2 \\ 8 & -1 & 4 \end{bmatrix}$

   a. Find the value of s_{13}.
   b. Find the value of s_{23}.
   c. Find the value of s_{31}.
   d. Find the value of s_{43}.

4. Construct and name the transpose of $S = \begin{bmatrix} 0 & 2 & 5 \\ -6 & -3 & 2 \\ 1 & 9 & 2 \\ 8 & -1 & 4 \end{bmatrix}$.

5. Construct $I_{4\times4}$.

6. Construct the transpose of $I_{3\times3}$.

7. Write the column matrix $\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ using vector bracket notation, < >.

8. Construct a 2×2 matrix in which the diagonal elements are 5 and 6 and the non-diagonal elements are 0 and 2.

4.2 Addition, Subtraction, Scalar Multiplication, and Products of Row and Column Matrices

ADDITION AND SUBTRACTION OF MATRICES

Let A and B be $m \times n$ matrices. Then the sum, $A + B$, is the new matrix formed by adding corresponding entries together. The difference, $A - B$, is the new matrix formed by subtracting each entry in matrix B from its corresponding entry in matrix A.

To add or subtract two or more matrices together, they all must be of the same size. That is, they all must have the same number of rows and the same numbers of columns. To add them together, add the corresponding elements together. To subtract one from the other, subtract corresponding elements from each other.

Example (1)

If the addition and subtraction is defined (if it is possible), perform each operation.

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 4 \\ 6 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 \\ 0 & 4 \\ 2 & 7 \end{bmatrix}, C = \begin{bmatrix} 9 & -4 \\ 2 & 6 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 2 & 5 \\ -1 & 4 \\ 6 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 2+3 & 5+1 \\ -1+0 & 4+4 \\ 6+2 & 0+7 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -1 & 8 \\ 8 & 7 \end{bmatrix}$$

$$A - B = = \begin{bmatrix} 2 & 5 \\ -1 & 4 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 0 & 4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 2-3 & 5-1 \\ -1-0 & 4-4 \\ 6-2 & 0-7 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ -1 & 0 \\ 4 & -7 \end{bmatrix}$$

$A + C$ is not defined as they are different sizes. Matrix A is a 3×2 matrix whereas matrix B is a 2×2 matrix.

YOUR TURN: Compute $B - A$.

SCALAR MULTIPLICATION

You might recall that a scalar is a physical quantity that is defined by only its magnitude and that some examples are speed, time, distance, density, and temperature. They are represented by real numbers (both positive and negative), and they can be operated on using the regular laws of algebra.

To multiply a matrix by a scalar, multiply every element of the matrix by the scalar.

Symbolically, $k \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} k \cdot a_{11} & k \cdot a_{12} & \cdots & k \cdot a_{1n} \\ k \cdot a_{21} & k \cdot a_{22} & \cdots & k \cdot a_{2n} \\ \vdots & \vdots & & \vdots \\ k \cdot a_{m1} & k \cdot a_{m2} & \cdots & k \cdot a_{mn} \end{bmatrix}$

Example (2)

$$6 \cdot \begin{bmatrix} 4 & 1 & -3 \\ 0 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 6 \cdot 4 & 6 \cdot 1 & 6 \cdot (-3) \\ 6 \cdot 0 & 6 \cdot 3 & 6 \cdot 5 \end{bmatrix} = \begin{bmatrix} 24 & 6 & -18 \\ 0 & 18 & 30 \end{bmatrix}$$

YOUR TURN: Multiply $7 \cdot \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$.

MULTIPLICATION WITH ROW AND COLUMN MATRICES

Suppose we have two matrices, A and B, where A is a $1 \times n$ matrix and B is an $n \times 1$ matrix. That is, A has one row and n columns and B has n rows and only 1 column.

$$A = [a_1 \quad a_2 \ \cdots \ a_n] \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

The product $A \cdot B$ is the new matrix obtained by multiplying together the corresponding elements of each matrix then adding those sums together.

$$A \cdot B = [a_1 \quad a_2 \ \cdots \ a_n] \cdot \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$A \cdot B = [a_1 \cdot b_1 + a_2 \cdot b_2 + \cdots + a_n \cdot b_n]$$

This product is the sum (addition) of the
first entry in A times the first entry in B
second entry in A times the second entry in B
⋮
last entry in A times the last entry in B

Example (3)

Suppose $A = [2 \quad 4 \quad 5]$ and $B = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$. Then,

$$A \cdot B = [2 \quad 4 \quad 5] \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$$A \cdot B = [2 \cdot 1 + 4 \cdot 4 + 5 \cdot 3]$$

$$A \cdot B = [2 + 16 + 15]$$

$$A \cdot B = [33]$$

Notice the dimensions of the two matrices. The number of rows of B, is 3 which is equal to the number of columns of A, which is also 3. The product is a 1×1 matrix whose dimension is the (number of rows of A) $\times$ (number of columns of B).

YOUR TURN: Suppose $A = \begin{bmatrix} 3 & 1 & -2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ -6 \\ 0 \\ 4 \end{bmatrix}$. Show that $A \cdot B = [17]$.

MOTIVATION FOR THE PROCESS OF MULTIPLICATION WITH ROW AND COLUMN MATRICES

This process of multiplication may not seem intuitive; however, we can motivate it with an example. You probably know, or at least believe, that the revenue R realized by selling n number of units of some product for p dollars per unit is given by $R = np$. Revenue equals (the number of units sold) times the (price of each unit).

$$\boxed{R = n \cdot p}$$

Example (4)

Suppose your business sells three sizes of boxes, small-sized boxes, medium-sized boxes, and large-sized boxes. Small boxes sell for \$3 each, medium boxes for \$5 each, and large boxes for \$7 each. What would your total revenue be if you sold 20 small-sized boxes, 30 medium-sized boxes, and 40 large-sized boxes?

Using $R = n \cdot p$, your revenue from the sale of the
small boxes is $R = 20 \cdot \$3 = \$60$
medium boxes is $R = 30 \cdot \$5 = \$150$
large boxes is $R = 40 \cdot \$7 = \$280$

The total revenue is the sum of these three products,
$20 \cdot \$3 + 30 \cdot \$5 + 40 \cdot \$7 = \$60 + \$150 + \$280 = \$490$.

We can compute the total revenue using two matrices and matrix multiplication.

Let the first matrix be the row matrix of the number of boxes sold N,

$$N = \begin{bmatrix} 20 & 30 & 40 \end{bmatrix}$$

and the second matrix be the column matrix of the price per boxes sold P.

$$P = \begin{bmatrix} \$3 \\ \$5 \\ \$7 \end{bmatrix}$$

The total revenue is the matrix product

$$R = N \cdot P$$

$$R = \begin{bmatrix} 20 & 30 & 40 \end{bmatrix} \cdot \begin{bmatrix} \$3 \\ \$5 \\ \$7 \end{bmatrix}$$

$$= [20 \cdot \$3 + 30 \cdot \$5 + 40 \cdot \$7]$$

$$= [\$490]$$

IMPORTANT OBSERVATION – SEE THIS

Notice that the result of a row and column matrix multiplication is a matrix with exactly one entry. That entry is the sum of a collection of products. In Example 4, the result of the row and column matrix multiplication is a matrix with exactly one entry, \$490. The \$490 is the sum of the products $20 \cdot \$3$, $30 \cdot \$5$, and $40 \cdot \$7$. Don't let the phrase the sum of a collection of products befuddle you. It means it is the addition (the sum) of a collection of multiplications (products). This idea will be helpful in the next section when we discuss multiplication of matrices of larger dimensions.

$$\$490 = \underbrace{20 \cdot \$3}_{\text{Product}} + \underbrace{30 \cdot \$5}_{\text{Product}} + \underbrace{40 \cdot \$7}_{\text{Product}}$$

Sum of Products

DIMENSION MATTERS

Notice the dimensions of the two matrices N and P from Example 4. The number of rows of P is 3, which is equal to the number columns of N, which is also 3. The product is a 1×1 matrix whose dimension is (the number of rows of N) $\times$ (the number of columns of P).

$$(1 \times 3) \cdot (3 \times 1)$$

=

Dimension of the Product

> To multiply a row matrix A and column matrix B together, it must be that the
>
> (number of rows of B) = (number of columns of A)

Symbolically, if A has n number of columns, B must have n number of rows

Example (5)

Suppose $A = \begin{bmatrix} 2 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix}$.

This multiplication will not work, it is not defined. Matrix B has 4 rows, but A has only 3 columns.

$$A \cdot B = \begin{bmatrix} 2 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix} = [2 \cdot 1 + 4 \cdot 4 + 5 \cdot 3 + Now\ what?]$$

4.2 TRY THESE

Using these six matrices, perform each operation if it is defined (if it is possible).

$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ -1 & -2 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 4 & 9 \end{bmatrix}$ $E = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $F = \begin{bmatrix} -2 \end{bmatrix}$

1. $A + B$
2. $B - A$
3. $D + E$
4. $D \cdot E$
5. $E \cdot D$
6. $-3 \cdot \begin{bmatrix} -4 & 0 \\ 2 & -1 \end{bmatrix}$
7. $A + C$
8. $3 \cdot (D \cdot E)$
9. $(D \cdot E) \cdot F$
10. F^2

4.3 Matrix Multiplication

COMPATIBLE MATRICES

We are going to multiply together two matrices, one of size $m \times n$, and one of size $n \times p$. The multiplication will be possible, and the product exists because the sizes make them compatible with each other.

$$(m \times n) \cdot (n \times p)$$

=

Dimension of the Product

Notice the number of columns of the leftmost matrix is equal to the number of rows of the rightmost matrix.

> For the product, $A \cdot B$, of two matrices to exist it must be that
>
> (the number of columns of matrix A) = (the number of rows of matrix B)
>
> Matrices for which this is true are said to be compatible with each other.

MATRICES AS COLLECTIONS OF ROW AND COLUMN MATRICES

It is productive to think of a matrix as a collection of individual row matrices and column matrices. For example, we can think of the matrix $A = \begin{bmatrix} 3 & 1 \\ -4 & 2 \\ 0 & 5 \end{bmatrix}$ as being composed of

- the three row matrices, $[3 \quad 1]$, $[-4 \quad 2]$, and $[0 \quad 5]$, and
- the two column matrices $\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

  (If you need a review of row and column matrices, see Section 4.2)

MULTIPLICATION OF TWO MATRICES

To multiply two compatible matrices A and B together, multiply every row matrix of A through every column matrix of B.

Suppose the size of matrix A is 3×4 and the size of matrix B is 4×5. The matrices are compatible with each other and the size of the product is 3×5

Some of the entries of the product $A \cdot B$ are

a_{11}: The entry in row 1, column 1, is the result of multiplying the 1st row of matrix A through the 1st column of matrix B.

a_{12}: The entry in row 1, column 2, is the result of multiplying the 1st row of matrix A through the 2nd column of matrix B.

a_{24}: The entry in row 2, column 4, is the result of multiplying the 2nd row of matrix A through the 4th column of matrix B.

a_{35}: The entry in row 3, column 5, is the result of multiplying the 3rd row of matrix A through the 5th column of matrix B.

a_{33}: The entry in row 3, column 3, is the result of multiplying the 3rd row of matrix A through the 3rd column of matrix B.

Do you see the general rule for producing any particular entry?

To get the entry in row i and column j, a_{ij}, multiply the ith row of matrix A through the jth column of matrix B.

Example (1)

Compute the product of the matrices $A = \begin{bmatrix} 3 & 1 \\ -4 & 2 \\ 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$.

First note that the two matrices are compatible

$$\begin{array}{ccc} A & \cdot & B \\ (3 \times 2) & \cdot & (2 \times 2) \end{array}$$

=

Dimension of the Product is 3 x 2

$$A \cdot B = \begin{bmatrix} 3 & 1 \\ -4 & 2 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$$

The product is the 3×2 matrix of the form $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

Since we are multiplying 3 rows through 2 columns, there will be 6 entries. The six entries of $A \cdot B$ are

$a_{11} =$ the 1st row of A times the 1st column of B
$= [3 \quad 1] \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [3 \cdot 3 + 1 \cdot 4] = [13]$

$a_{12} =$ the 1st row of A times the 2nd column of B
$= [3 \quad 1] \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = [3 \cdot 2 + 1 \cdot 1] = [7]$

$a_{21} =$ the 2nd row of A times the 1st column of B
$= [-4 \quad 2] \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [-4 \cdot 3 + 2 \cdot 4] = [-4]$

$a_{22} =$ the 2nd row of A times the 2nd column of B
$= [-4 \quad 2] \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = [-4 \cdot 2 + 2 \cdot 1] = [-6]$

$a_{31} =$ the 3rd row of A times the 1st column of B
$= [0 \quad 5] \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [0 \cdot 3 + 5 \cdot 4] = [20]$

$a_{32} =$ the 3rd row of A times the 2nd column of B
$= [0 \quad 5] \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = [0 \cdot 2 + 5 \cdot 1] = [5]$

So, $A \cdot B = \begin{bmatrix} 13 & 7 \\ -4 & -6 \\ 20 & 5 \end{bmatrix}$

YOUR TURN: Show that the product of the matrices $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & 4 \end{bmatrix}$ is $\begin{bmatrix} 7 & 12 & 12 \\ 9 & 14 & 4 \end{bmatrix}$.

USING TECHNOLOGY

You can see that multiplying matrices together involves a lot of arithmetic and can be cumbersome. We can use technology to help us through the process.

Go to www.wolframalpha.com.

To find the product of the two matrices of above Your Turn Example, enter [[2,3], [4,1]] * [[2,3,0], [1,2,4]] in the entry field. WolframAlpha sees a matrix as a collection of row matrices.

These outer square brackets begin and end the actual matrix.

[[2,3], [4,1]]

These inner square brackets begin and end each row of the matrix.

Both entries and rows are separated by commas and W|A does not see spaces.

Wolframalpha tells you what it thinks you entered, then tells you its answer $\begin{bmatrix} 7 & 12 & 12 \\ 9 & 14 & 4 \end{bmatrix}$.

4.3 TRY THESE

Perform each multiplication if it is defined. If it is not defined, write "not defined."

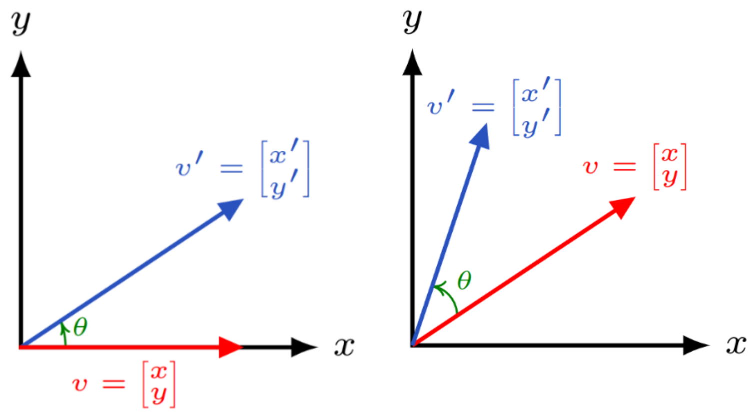
$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ -1 & -2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -2 & 6 \\ 4 & 1 \end{bmatrix} \quad E = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \quad F = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

1. $A \cdot C$
2. $C \cdot A$
3. Compare your answers to question 1 and 2. If you got them right, would you say that matrix multiplication is or is not commutative?
4. $D \cdot C$
5. $C \cdot F$
6. $A \cdot E$
7. D^2
8. $D \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
9. $B \cdot D$
10. $B \cdot D \cdot C$
11. $D \cdot B$

4.4 Rotation Matrices in 2-Dimensions

THE ROTATION MATRIX

To this point, we worked with vectors and with matrices. Now, we will put them together to see how to use a matrix multiplication to rotate a vector in the counterclockwise direction through some angle θ in 2-dimensions.



Our plan is to rotate the vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ counterclockwise through some angle θ to the new position given by the vector $v' = \begin{bmatrix} x' \\ y' \end{bmatrix}$. To do so, we use the rotation matrix, a matrix that rotates points in the xy-plane counterclockwise through an angle θ relative to the x-axis.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

THE ROTATION PROCESS

To get the coordinates of the new vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$, perform the matrix multiplication

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example (1)

Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is rotated 90° counterclockwise.

Using the rotation formula $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ with $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

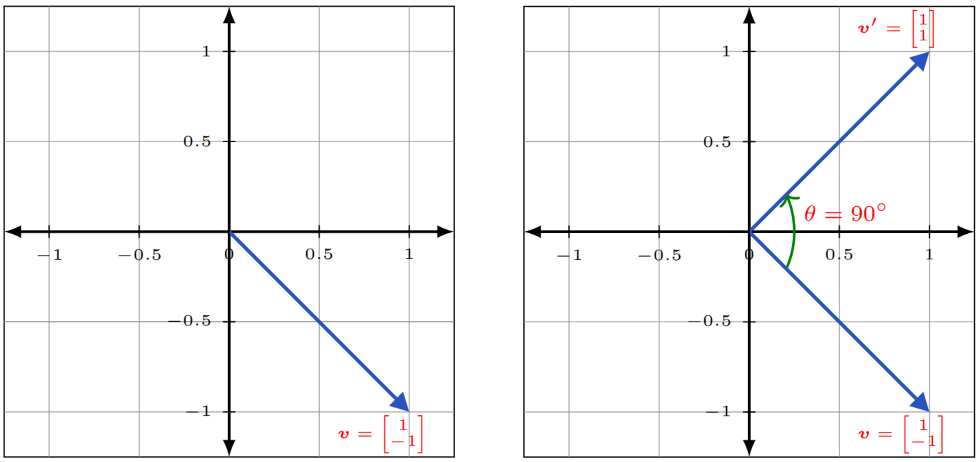
and $\theta = 90°$, we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 90° & -\sin 90° \\ \sin 90° & \cos 90° \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + (-1) \cdot (-1) \\ 1 \cdot 1 + 0 \cdot (-1) \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

When rotated counterclockwise 90°, the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ becomes $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.



Example (2)

Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is rotated 60° counterclockwise.

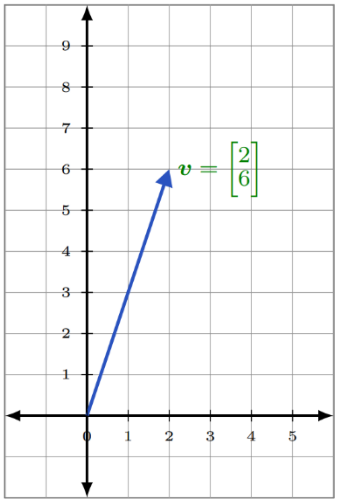
Using the rotation formula $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ with $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\theta = 60°$, we get

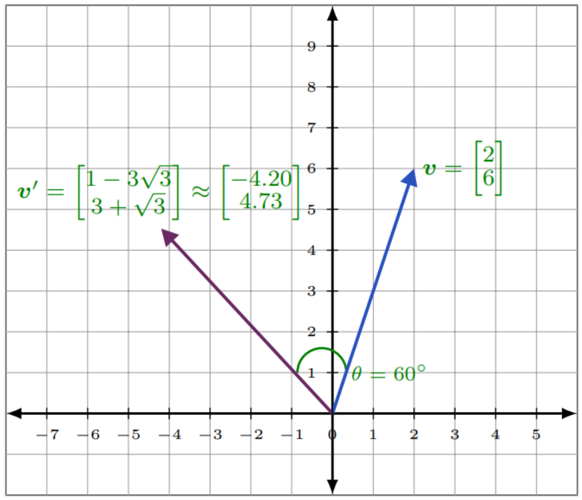
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 1/2 \cdot 2 + (-\sqrt{3}/2) \cdot 6 \\ \sqrt{3}/2 \cdot 2 + 1/2 \cdot 6 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - 3\sqrt{3} \\ 3 + \sqrt{3} \end{bmatrix}$$

When rotated counterclockwise 60°, the vector $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ becomes $\begin{bmatrix} 1 - 3\sqrt{3} \\ 3 + \sqrt{3} \end{bmatrix}$.



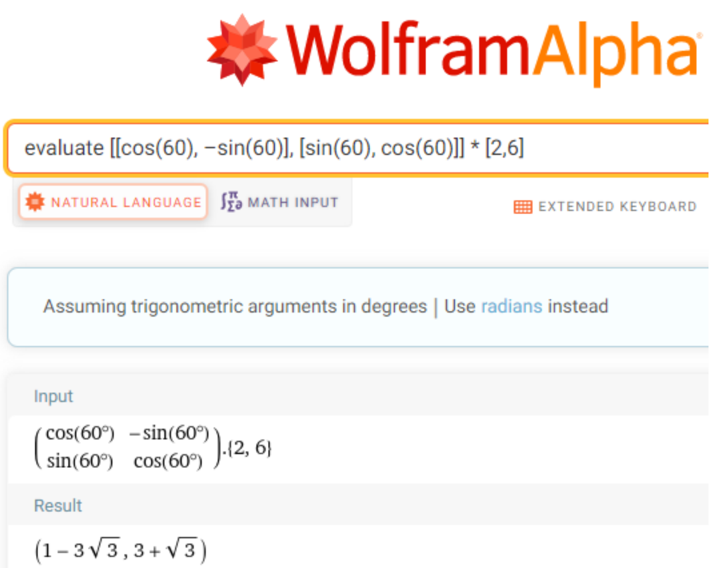


USING TECHNOLOGY

We can use technology to help us find the rotation. WolframAlpha evaluates the trig functions for us.

Go to www.wolframalpha.com.

We can check the above problem from Example 2 by using WolframAlpha. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ is rotated 60° counterclockwise. To find rotation of the vector enter evaluate [[cos(60), –sin(60)], [sin(60), cos(60)]] * [2,6] into the entry field.



When rotated counterclockwise 60°, the vector $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ becomes $\begin{bmatrix} 1-3\sqrt{3} \\ 3+\sqrt{3} \end{bmatrix}$.

4.4 TRY THESE

1. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is rotated 90° counterclockwise.

2. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is rotated 180° counterclockwise.

3. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is rotated 270° counterclockwise.

4. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is rotated 90° counterclockwise.

5. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is rotated 45° counterclockwise.

6. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is rotated 45° counterclockwise.

7. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2.20205 \\ 4.48898 \end{bmatrix}$ is rotated $-63°$ counterclockwise.

8. Find the vector $\begin{bmatrix} x' \\ y' \end{bmatrix}$ that results when $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$ is rotated $-90°$ counterclockwise.

9. Approximate, to five decimal places, the coordinates of the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ when it is rotated counterclockwise 30°.

4.5 Finding the Angle of Rotation Between Two Rotated Vectors in 2-Dimensions

GIVEN THE ROTATED VECTOR, FIND THE ANGLE OF ROTATION

Suppose we did not know the angle θ of rotation. We can get it by working backwards and solving a system of equations. The rotation formula

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

produces the system of equations

$$\begin{cases} x' = x \cdot \cos\theta + y \cdot (-\sin\theta) \\ y' = x \cdot \sin\theta + y \cdot \cos\theta \end{cases}$$

Example (1)

In Example 1 of Chapter 4.4, we found that when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ was rotated counterclockwise by 90°, it became the vector $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We got this rotated vector by applying the rotation formula $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot \cos\theta + (-1) \cdot (-\sin\theta) \\ 1 \cdot \sin\theta + (-1) \cdot \cos\theta \end{bmatrix}$$

Since two vectors are equal only if their corresponding components are equal, we have the system of two equations

$$\begin{cases} 1 = 1 \cdot \cos\theta + (-1) \cdot (-\sin\theta) \\ 1 = 1 \cdot \sin\theta + (-1) \cdot \cos\theta \end{cases}$$

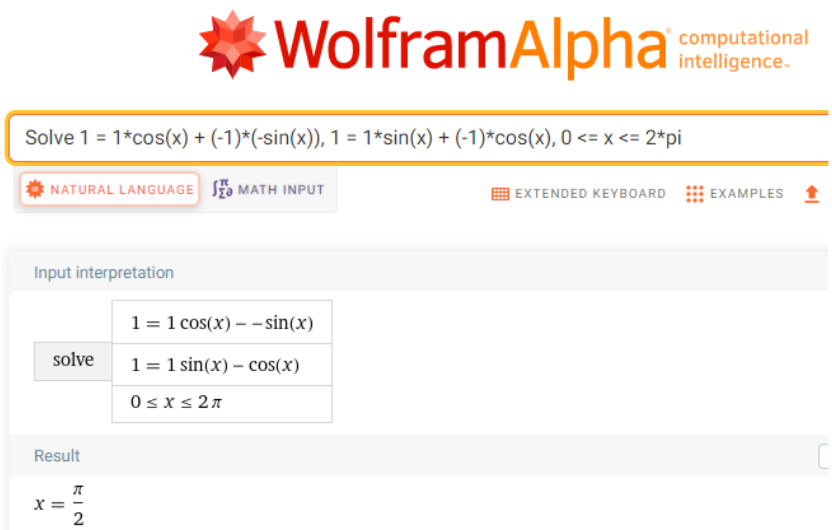
USING TECHNOLOGY

We can use WolframAlpha to help us solve above system for the angle of rotation, θ.

Go to www.wolframalpha.com.

Since we want to rotate only one time around the coordinate system, we want to instruct W|A to give us solutions only where the angle θ is between 0 and 2π.

Using the English letter x in place of the Greek letter θ, enter
Solve 1 = 1*cos(x) + (-1)*(-sin(x)), 1 = 1*sin(x) + (-1)*cos(x), 0 <= x <= 2*pi in the entry field.



W|A shows the angle of rotation is $\theta = \frac{\pi}{2}$, which is 90°. We conclude that the angle of rotation is 90°.

Example (2)

In Example 2 of Chapter 4.4, we found that when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ was rotated counterclockwise by 60°, it became the vector $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1-3\sqrt{3} \\ 3+\sqrt{3} \end{bmatrix}$. We got this rotated vector by applying the rotation formula $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1-3\sqrt{3} \\ 3+\sqrt{3} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

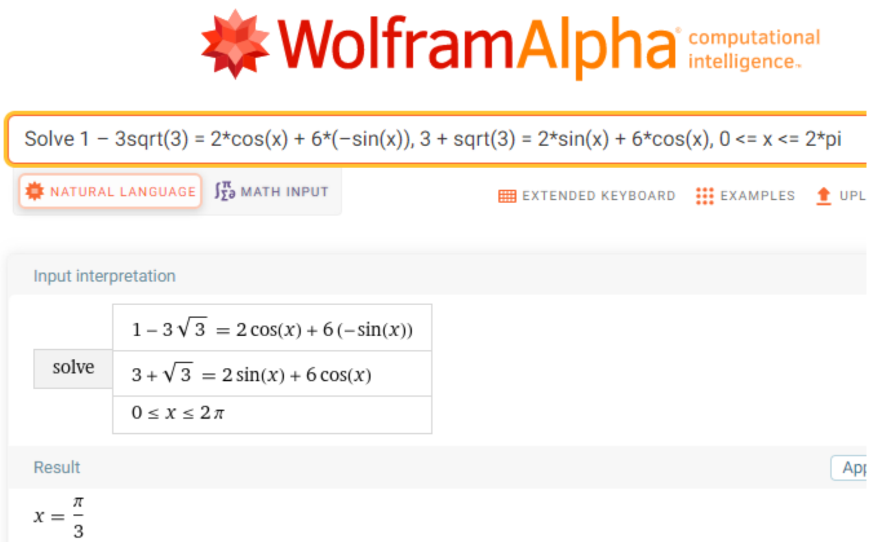
$$\begin{bmatrix} 1-3\sqrt{3} \\ 3+\sqrt{3} \end{bmatrix} = \begin{bmatrix} 2\cdot\cos\theta + 6\cdot(-\sin\theta) \\ 2\cdot\sin\theta + 6\cdot\cos\theta \end{bmatrix}$$

Since two vectors are equal only if their corresponding components are equal, we have the system of two equations

$$\begin{cases} 1 - 3\sqrt{3} = 2 \cdot \cos\theta + 6 \cdot (-\sin\theta) \\ 3 + \sqrt{3} = 2 \cdot \sin\theta + 6 \cdot \cos\theta \end{cases}$$

We will use WolframAlpha to help us solve this system for the angle of rotation, θ.

Using the English letter x in place of the Greek letter θ, enter
Solve 1 – 3sqrt(3) = 2*cos(x) + 6*(–sin(x)), 3 + sqrt(3) = 2*sin(x) + 6*cos(x), 0 <= x <= 2*pi
in the entry field. Separate the two equations with a comma.



W|A shows the angle of rotation is $\theta = \frac{\pi}{3}$, which is 60°. We conclude that the angle of rotation is 60°.

4.5 TRY THESE

1. Find the angle θ through which the vector $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$ is rotated to become $\begin{bmatrix} 0 \\ 3\sqrt{2} \end{bmatrix}$.

2. Find the angle θ through which the vector $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is rotated to become $\begin{bmatrix} 1+\sqrt{3} \\ -1+\sqrt{3} \end{bmatrix}$.

3. Find the angle θ through which the vector $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is rotated to become $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$.

4. Find the angle θ through which the vector $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ is rotated to become $\begin{bmatrix} -1+\sqrt{3} \\ -1-\sqrt{3} \end{bmatrix}$.

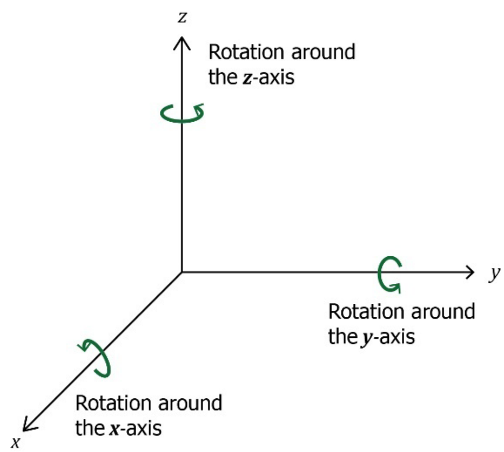
5. Find the angle θ through which the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is rotated to become $\begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$.

4.6 Rotation Matrices in 3-Dimensions

THE THREE BASIC ROTATIONS

A basic rotation of a vector in 3-dimensions is a rotation around one of the coordinate axes. We can rotate a vector counterclockwise through an angle θ around the x–axis, the y–axis, or the z–axis.

To get a counterclockwise view, imagine looking at an axis straight on toward the origin.



Our plan is to rotate the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ counterclockwise around one of the axes through some angle θ to the new position given by the vector $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$. To do so, we will use one of the three rotation matrices.

⇩

THE ROTATION MATRICES

The rotation matrices for x, y, and z axes are, respectively,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

THE ROTATION PROCESS

x–axis

To rotate the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ counterclockwise through an angle θ around the x–axis to a new position $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$, perform the matrix multiplication,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

y–axis

To rotate the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ counterclockwise through an angle θ around the y–axis to a new position $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$, perform the matrix multiplication,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

z–axis

To rotate the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ counterclockwise through an angle θ around the z–axis to a new position $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$, perform the matrix multiplication,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Example (1)

Find the vector $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ that results when the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is rotated 90° counterclockwise around x–axis.

Using the rotation formula $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\theta = 90°$, we get

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90° & -\sin 90° \\ 0 & \sin 90° & \cos 90° \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 2 + 0 \cdot 3 \\ 0 \cdot 1 + 0 \cdot 2 + (-1) \cdot 3 \\ 0 \cdot 1 + 1 \cdot 2 + 0 \cdot 3 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

When rotated counterclockwise 90° around the x–axis, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ becomes $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$.

USING TECHNOLOGY

We can use technology to help us find the rotation. WolframAlpha evaluates the trig functions for us.

Go to www.wolframalpha.com.

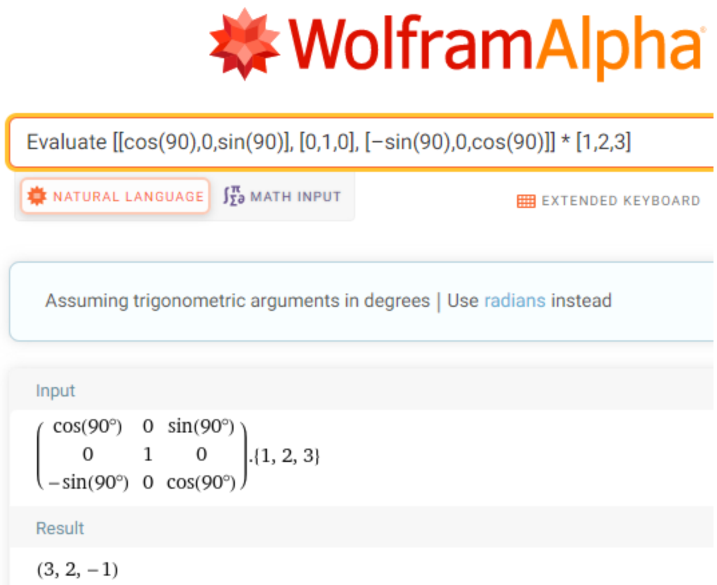
Example (2)

In Example 1, we rotated the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 90° around the x–axis to get $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$.

Now we will use WolframAlpha to rotate vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 90° around the y–axis. We use the y–axis rotation matrix $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$.

To perform the rotation, enter Evaluate [[cos(90),0,sin(90)], [0,1,0], [–sin(90),0,cos(90)]] * [1,2,3] into the entry field.

Both entries and rows are separated by commas as W|A does not see spaces.
Wolframalpha tells you what it thinks you entered, then tells you its answer.

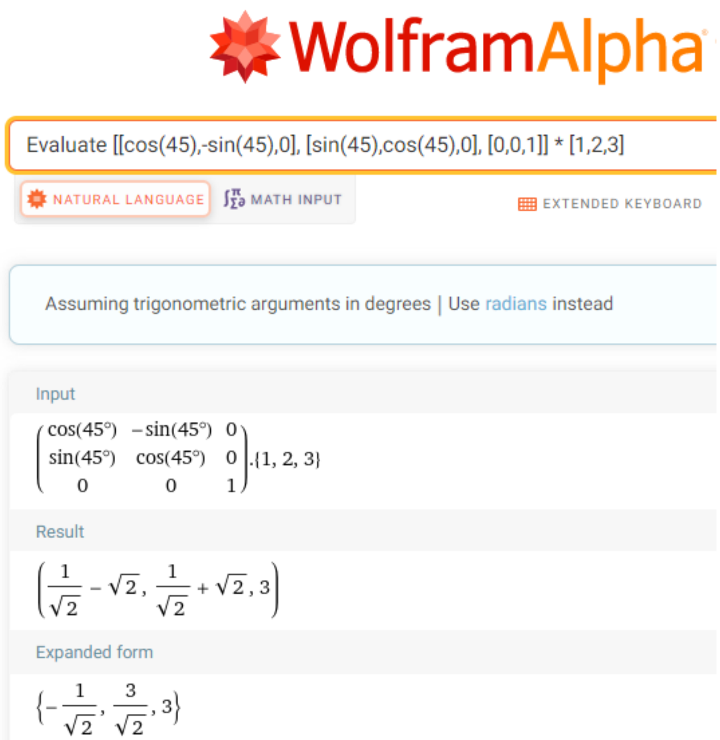


When rotated counterclockwise 90° around the y–axis, the vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ becomes $\begin{bmatrix}3\\2\\-1\end{bmatrix}$.

Example (3) Find the vector $\begin{bmatrix}x'\\y'\\z'\end{bmatrix}$ that results when the vector $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ is rotated 45° counterclockwise around the z–axis.

Since we are rotating the vector around the z–axis, we use the z–axis rotation matrix $\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

Using WolframAlpha with $\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\theta = 45°$, we get



When rotated counterclockwise 45° around the z–axis, the vector $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ becomes $\begin{bmatrix} -1/\sqrt{2} \\ 3/\sqrt{2} \\ 3 \end{bmatrix}$.

4.6 TRY THESE

Find the vector $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ that results when the given vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is rotated the given angle θ counterclockwise around the given axis.

1. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ through 90° around the x–axis.

2. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ through 45° around the z–axis.

3. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$ through 30° around the y–axis.

UNIT 5 SOME BASIC TRIGONOMETRY

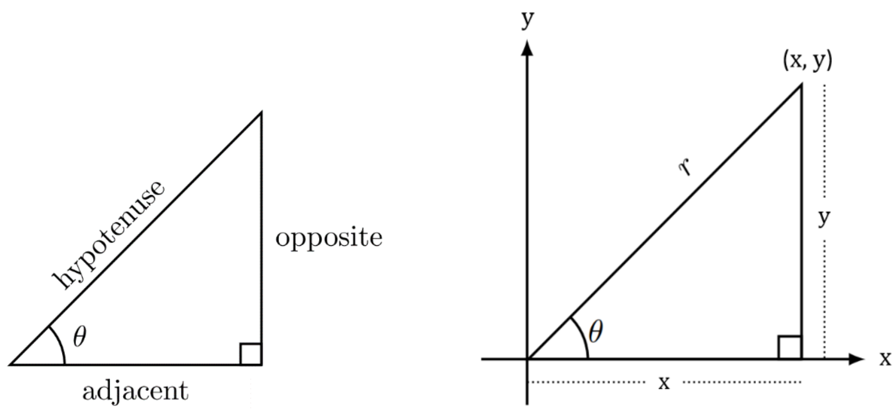
5.1 The Basic Trigonometric Functions

RIGHT TRIANGLE TRIGONOMETRY

There are six trigonometric functions associated with right triangles. Since our focus is on the mathematics of games, we will concentrate on only three of them, the sine function, the cosine function, and the tangent function.

> The sine function is useful for producing the vertical motion of an object and the cosine function for producing the horizontal motion.

The figures just below show right triangles with angle θ, and sides opposite angle θ, adjacent to angle θ, and the hypotenuse of the triangle.



The angle θ has two measures associated with it:

1. Its degree measure, which we can label $\theta°$, and
2. Its trigonometric measure.

> A trigonometric measure of an angle is a ratio (quotient) of two of the sides of the triangle.

We will discuss all three of these ratios, the sine, the cosine, and the tangent of an angle.

THE SINE OF AN ANGLE

In words: In a right triangle, the *sine* of angle θ is the ratio of the length of the side opposite θ to the length of the hypotenuse. We abbreviate the phrase "the *sine* of angle θ" with $\sin\theta$.

Then, $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$. That is $\sin\theta = \frac{y}{r}$.

THE COSINE OF AN ANGLE

In words: In a right triangle, the *cosine* of angle θ is the ratio of the length of the side adjacent to θ to the length of the hypotenuse. We abbreviate the phrase "the *cosine* of angle θ" with $\cos\theta$.

Then, $\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$. That is $\cos\theta = \frac{x}{r}$.

THE TANGENT OF AN ANGLE

In words: In a right triangle, the *tangent* of angle θ is the ratio of the length of the side opposite θ to the length of the side adjacent to θ. We abbreviate the phrase "the *tangent* of angle θ" with $\tan\theta$.

Then, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$. That is $\tan\theta = \frac{y}{x}$.

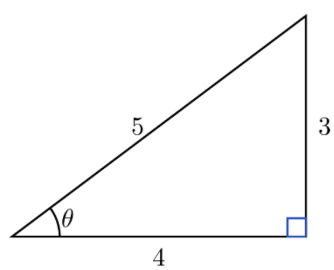
Example (1)

Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the 3-4-5 triangle.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5} = 0.6$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5} = 0.8$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4} = 0.75$$



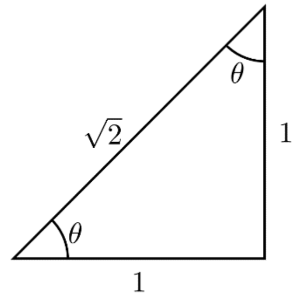
Example (2)

Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for the triangle.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \approx 0.7071$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \approx 0.7071$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1$$



USING TECHNOLOGY

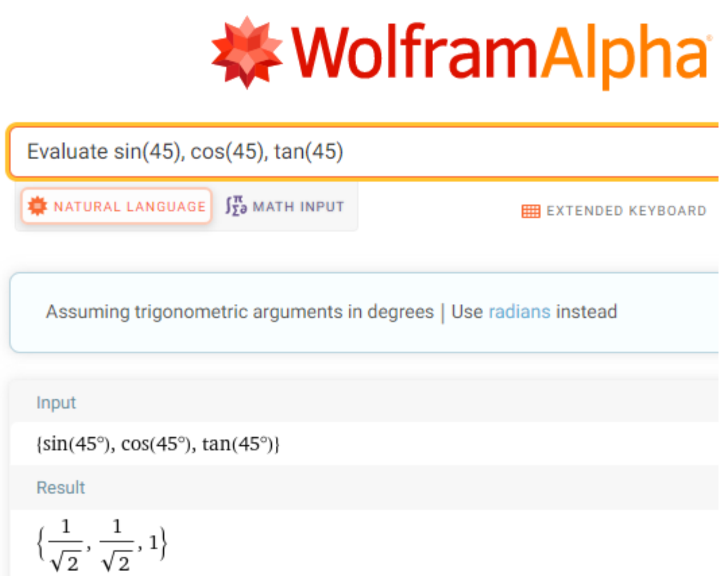
WolframAlpha evaluates the sines, cosines, and tangents of angles for us.

Go to www.wolframalpha.com.

Example (3)

Find $\sin 45°$, $\cos 45°$, and $\tan 45°$.

To compute these ratios, enter Evaluate sin(45), cos(45), tan(45) into the entry field.
Separate the entries with commas. W|A does not see spaces.
WolframAlpha tells you what it thinks you entered, then tells you its answers.



We conclude that $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$, and $\tan 45° = 1$.

W|A also provides us with decimal approximations to these ratios.

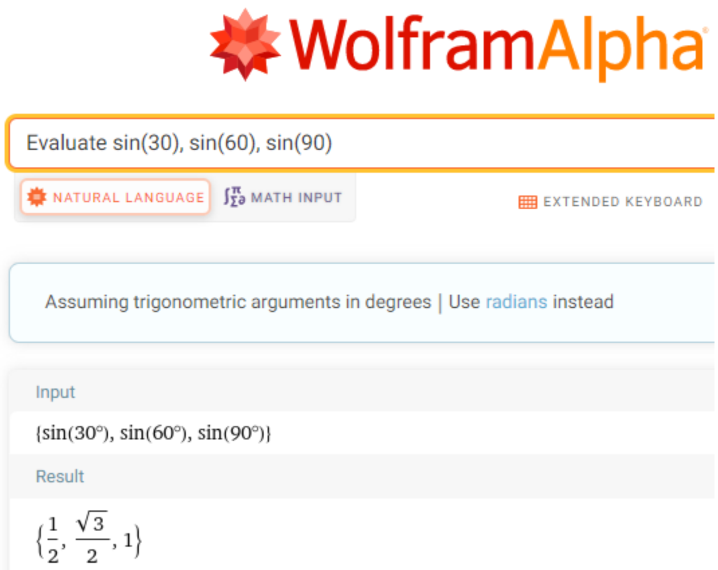
$\sin 45° = 0.7070107,\ \cos 45° = 0.7070107$, and $\tan 45° = 1$

Notice that these are the same values we got in Example 2.

Example (4)

Find $\sin 30°,\ \sin 60°,\ \sin 90°$.

To compute these ratios, enter Evaluate sin(30), sin(60), sin(90) into the entry field.
Separate the entries with commas. W|A does not see spaces.
WolframAlpha tells you what it thinks you entered, then tells you its answers.



We conclude that $\sin 30° = \frac{1}{2},\ \sin 60° = \frac{\sqrt{3}}{2}$, and $\sin 90° = 1$.

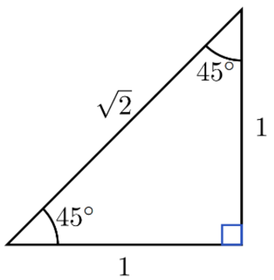
W|A also provides us with decimal approximations to these ratios.

$\sin 30° = 0.5,\ \sin 60° = 0.866025$, and $\sin 90° = 1$.

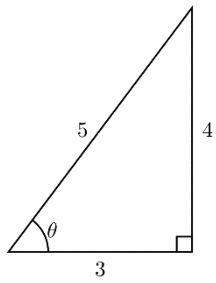
5.1 TRY THESE

1. Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ for each triangle. Write your answers as decimal numbers rounded to 4 places.

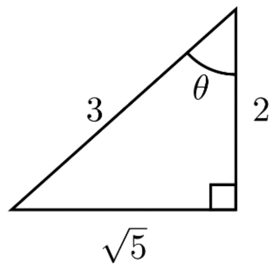
a)



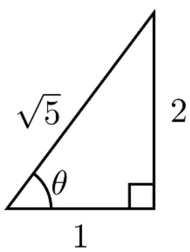
b)



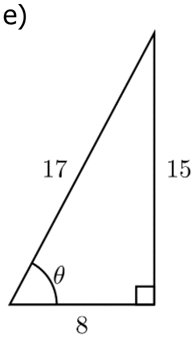
c)



d)



e)



2. Find each value. Write your answers as decimal numbers rounded to 4 places.

a) $\sin 30°$, $\cos 30°$, $\tan 30°$

b) $\sin 90°$, $\cos 90°$

c) $\sin 0°$, $\cos 0°$, $\tan 0°$

d) $\sin 180°$, $\cos 180°$

e) $\sin 120°$, $\cos 120°$

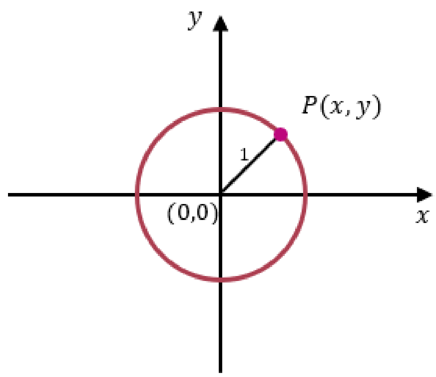
5.2 Circular Trigonometry

THE SINE FUNCTION ON THE UNIT CIRCLE

In computer games, objects typically move up-and-down and left-to-right. These movements can be produced using the sine and cosine functions.

Draw a circle with radius 1 unit and on its circumference, place a point, let's call it P.

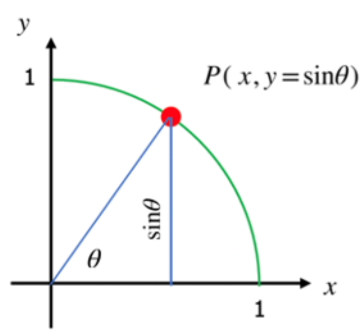
The circle centered at the origin with radius 1 is called the unit-circle.



From our presentation of the sine and cosine function using right triangles, we can see that

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{1} = y. \text{ That is, } y = \sin\theta.$$

This tells us that the sine of the angle θ determines the vertical distance of the point P from the horizontal axis.



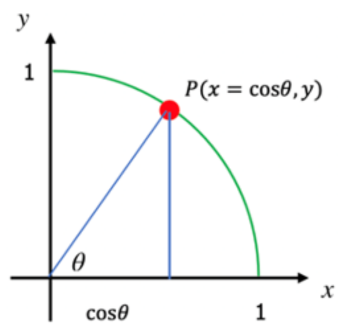
THE COSINE FUNCTION ON THE UNIT CIRCLE

To define cosine function, place a point $P(x, y)$ on the circumference of unit-circle.

Once again, from our presentation of the cosine functions using right triangles, we can see that

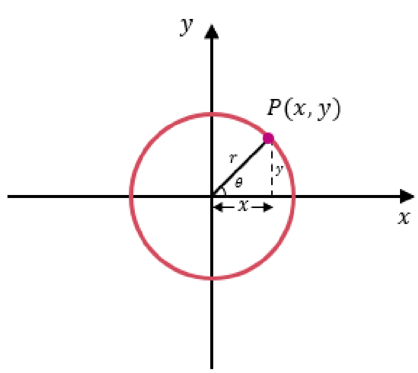
$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{1} = x. \text{ That is, } x = \cos\theta.$$

This tells us that the cosine of the angle θ determines the horizontal distance of the point P from the vertical axis.



THE SINE AND COSINE FUNCTIONS ON ANY CIRCLE

We can extend this idea by making the radius of the circle r units rather than just 1 unit.



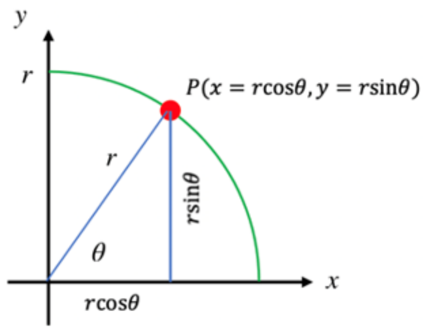
Using the same reasoning we just used with the unit circle, we see that

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{y}{r} \rightarrow r \cdot \sin\theta = y \rightarrow y = r \cdot \sin\theta$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{x}{r} \rightarrow r \cdot \cos\theta \rightarrow x = r \cdot \cos\theta$$

which, again, tells us that the sine of the angle θ determines the vertical distance of the point P from the horizontal axis and that the cosine of the angle θ determines the horizontal distance of the point P from the vertical axis.

If P represents an object, that object's height y off the ground (the horizontal axis) is given by $r \cdot \sin\theta$, and that object's horizontal distance x from some reference point is given by $r \cdot \cos\theta$. The height of the object is controlled by some number r times $\sin\theta$, and its horizontal distance is controlled by some number r times $\cos\theta$.



Example (1)

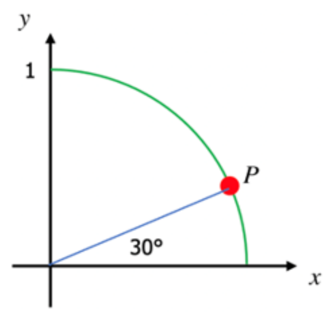
An object lies on the circumference of a unit circle. Find its coordinates if the line segment from the origin to the object makes angle of 30° with the horizontal.

Because the object is on the circumference of unit circle, we can use

$x = r\cos\theta$ and $y = r\sin\theta$, with $r = 1$, $\theta = 30°$.

$$x = 1\cos 30° \text{ and } y = 1\sin 30°$$
$$x = \cos 30° \text{ and } y = \sin 30°$$
$$x = 0.8660 \text{ and } y = 0.5$$

The coordinates of the object are (0.8660, 0.5).



Example (2)

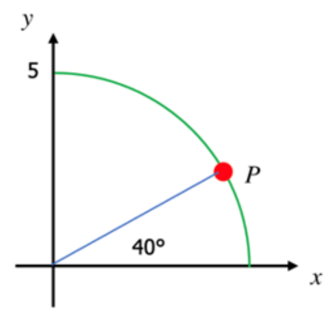
An object lies on the circumference of a circle of radius 5 cm. Find its coordinates if the line segment from the origin to the object makes angle of 40° with the horizontal.

Because the object is on the circumference of circle of radius 5 cm, we can use

$x = r\cos\theta$ and $y = r\sin\theta$, with $r = 5$, $\theta = 40°$.

$$\begin{aligned} x &= 5\cos 40° \quad &\text{and} \quad y &= 5\sin 40° \\ x &= 5(0.7660) \quad &\text{and} \quad y &= 5(0.6428) \\ x &= 3.8302 \quad &\text{and} \quad y &= 3.2139 \end{aligned}$$

The coordinates of the object are (3.8302, 3.2139).



Example (3)

The coordinates of an object are (2.1, 3.6373). Find its distance from the origin.

We can use the Pythagorean Theorem, $a^2 + b^2 = c^2$, where c is the hypotenuse, the radius of the circle in our case.

$$\begin{aligned} 2.1^2 + 3.6373^2 &= r^2 \\ 4.41 + 13.2300 &= r^2 \\ 17.64 &= r^2 \\ \sqrt{17.64} &= \sqrt{r^2} \\ 4.2 &= r \end{aligned}$$

We conclude that the object is about 4.2 cm from the origin.

5.2 TRY THESE

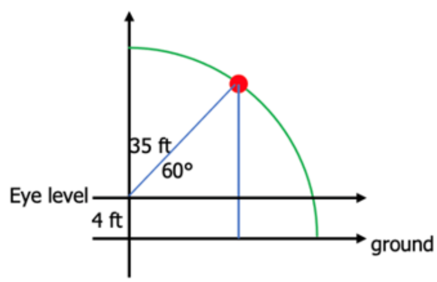
1. An object lies on the circumference of a unit circle. Find its coordinates if the line segment from the origin to the object makes angle of 45° with the horizontal.

2. An object lies on the circumference of a unit circle. Find its coordinates if the line segment from the origin to the object makes angle of 5° with the horizontal.

3. An object lies on the circumference of a circle of radius 25 cm. Find its coordinates if the line segment from the origin to the object makes angle of 75° with the horizontal.

4. An object lies on the circumference of a circle of radius 10 feet. Find its coordinates if the line segment from the origin to the object makes angle of 135° with the horizontal.

5. How high above the ground is an object that makes an angle of 60° with a 4-foot-tall observer's eyes and is 35 feet away from that observer's eyes? Round to two decimals place.



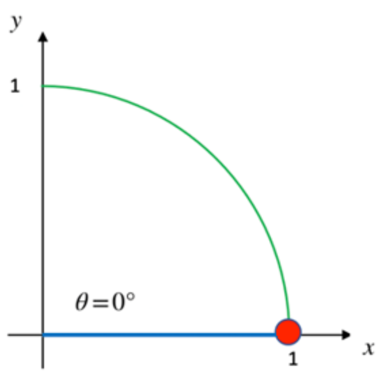
6. The coordinates of an object are (5.682, 2.0521). Find its distance from the origin if it makes an angle of 60° with the horizontal.

5.3 Graphs of the Sine Function

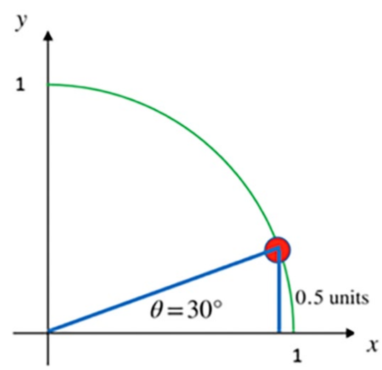
DISCRETE GRAPH OF THE SINE FUNCTION FROM 0° TO 90°

The graph of the sine function gives a visual illustration of how it determines the height of an object from a horizontal axis.

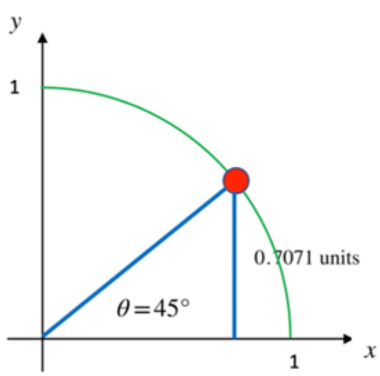
Imagine an object moving counterclockwise along the circumference of the unit circle. Start the object's motion at the point (1,0), then measure its height from the horizontal axis as its angle from origin increases from 0° to 90°.



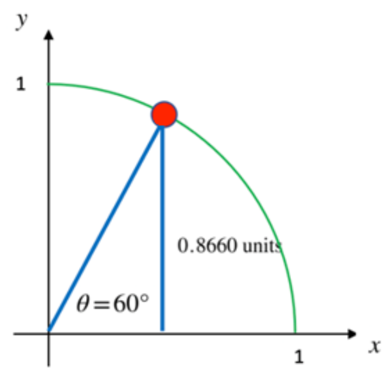
Height of object from horizontal
$= \sin 0° = 0$ units



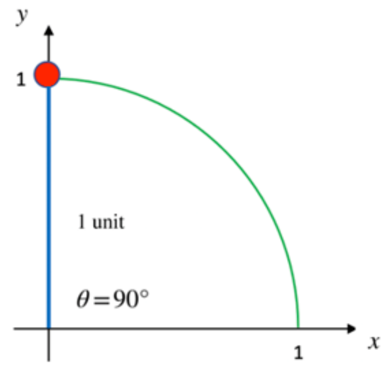
Height of object from horizontal
$= \sin 30° = 0.5$ units



Height of object from horizontal
$= \sin 45° \approx 0.7071$ units



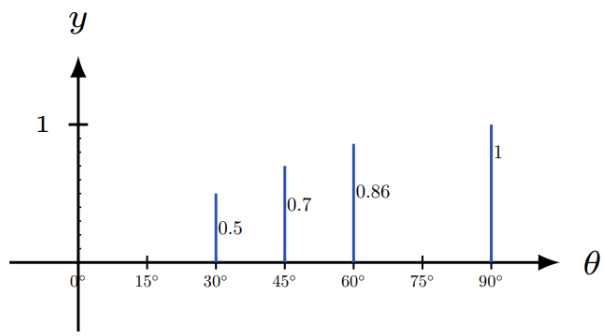
Height of object from horizontal
$= \sin 60° \approx 0.8660$ units



Height of object from horizontal
= $\sin 90° \approx 1$ unit

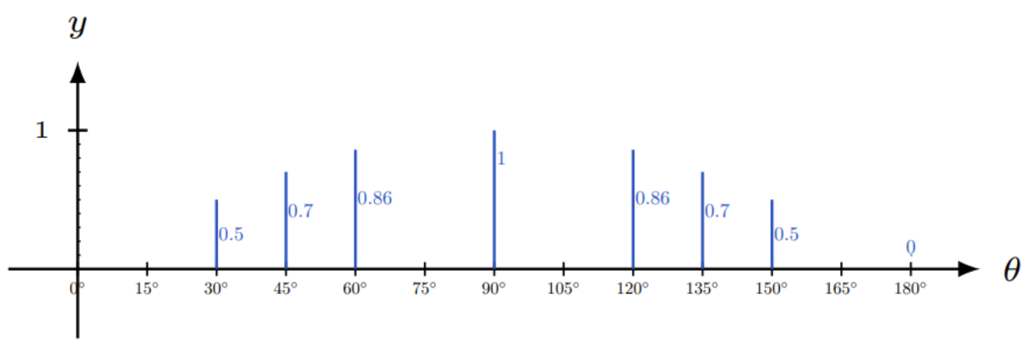
GRAPHS OF THE HEIGHTS

If the angle is between 0° and 90°, the graph of the heights looks like this



We can see that from 0° to 90°, as the angle from the observer to the object increases, the height of the object from the horizontal increases. That is, *the object moves vertically upward.*

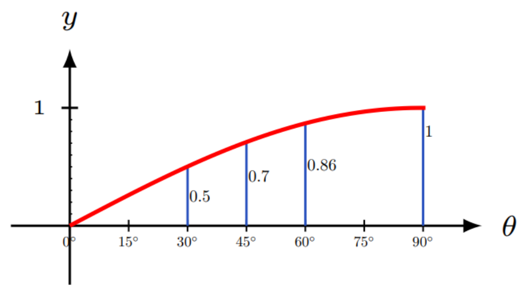
If the angle goes past 90°, say all the way to 180°, the graph of the heights looks like this



The object moves vertically upward, then vertically downward.

THE CONTINUOUS SINE CURVE FROM 0° TO 90°

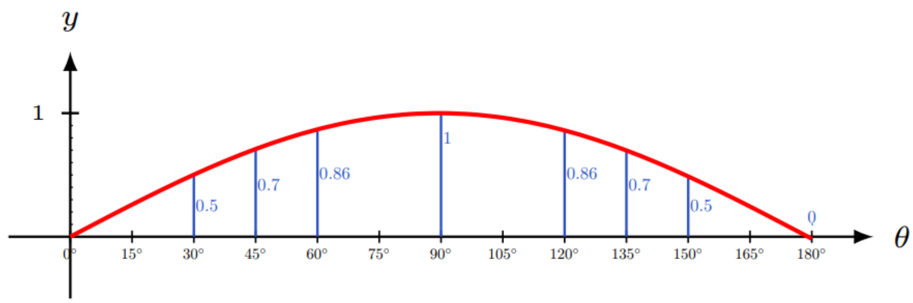
If we plotted all the heights for all the infinitely many angles between 0° and 90°, we would get this continuous graph



$y = \sin\theta$ for $0 \leq \theta \leq 90°$

THE CONTINUOUS SINE CURVE FROM 0° TO 180°

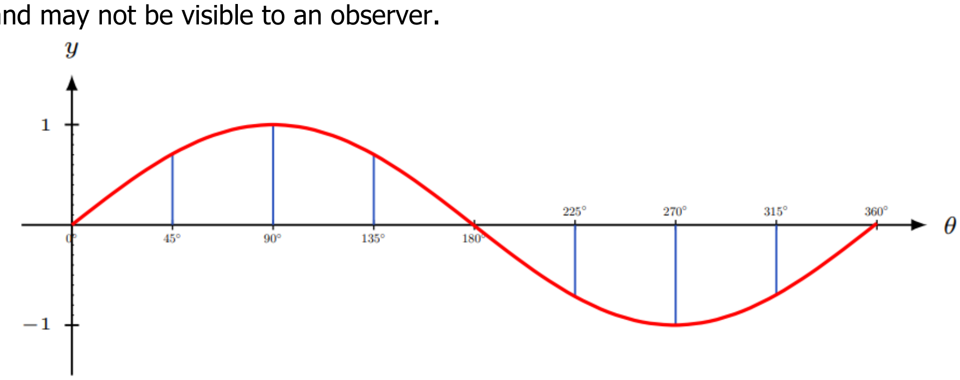
If we plotted all the heights for all the angles between 0° and 180°, we would get the continuous graph below



$y = \sin\theta$ for $0 \leq \theta \leq 180°$

THE CONTINUOUS SINE CURVE FROM 0° TO 360°

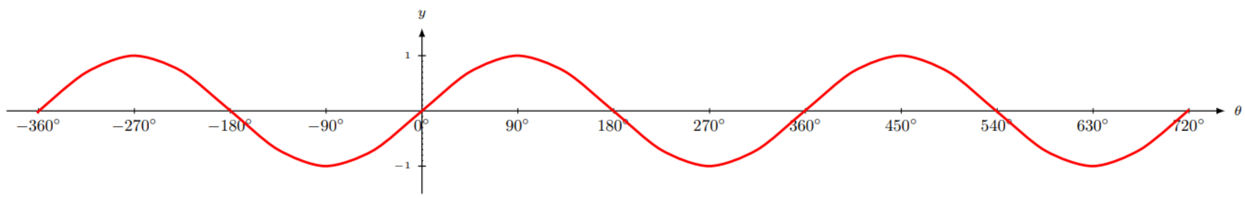
If we were to let the object travel all the way around the circle, we get the graph of the sine curve from 0° to 360°. You can see that when the angle θ is between 180° and 360°, the object is below the horizontal and may not be visible to an observer.



$y = \sin\theta$ for $0 \leq \theta \leq 360°$

THE EXTENDED SINE CURVE

If we were to let the object keep travelling around the circle, we would see that the height of the curve just oscillates between –1 and 1.



$y = \sin\theta$

It may now be visually apparent that

> The sine function controls the vertical distance of an object above or below the horizontal.

WHAT TO SEE

The graph shows how an object's vertical distance from the horizontal changes as the angle of view increases. As the angle of view increases, the vertical distance from the horizontal increases and decreases.

WHAT NOT TO SEE

The graph does not show how an object moves horizontally as the angle of view increases. The object is not moving up and down horizontally along the curve as time goes by. The horizontal axis is the angle of view, not time.

5.3 TRY THESE

1. An object moves along the circumference of a unit circle. Find its height from the horizontal if the angle it makes from the origin is

   a. 225°
   b. 270°
   c. 315°
   d. 360°

2. An object moves along the circumference of a unit circle. Find its height from the horizontal if the angle it makes from the origin is

   a. 390°
   b. 405°
   c. 420°
   d. 450°

3. Determine if each statement is true or false.

   a. Height at 87° > height at 78°
   b. Height at 155° > height at 145°
   c. Height at 30° ≥ height at 150°
   d. Height at 90° ≥ height at 270°

4. Keeping in mind that the sine function determines vertical distance, and the cosine function determines horizontal distance, determine if each statement is true or false. The observer is at the origin.

   a. Vertical height at 87° > horizontal distance at 87°
   b. Vertical height at 155° > horizontal distance at 55°
   c. Vertical height at 20° < horizontal distance at 20°
   d. Vertical height at 135° = horizontal distance at 315°

5.4 Graphs of the Cosine Function

DISCRETE GRAPH OF THE COSINE FUNCTION FROM 0° TO 360°

Just as the sine function determines the vertical distance of an object from an observer, the cosine function determines the horizontal distance of an object from that observer.

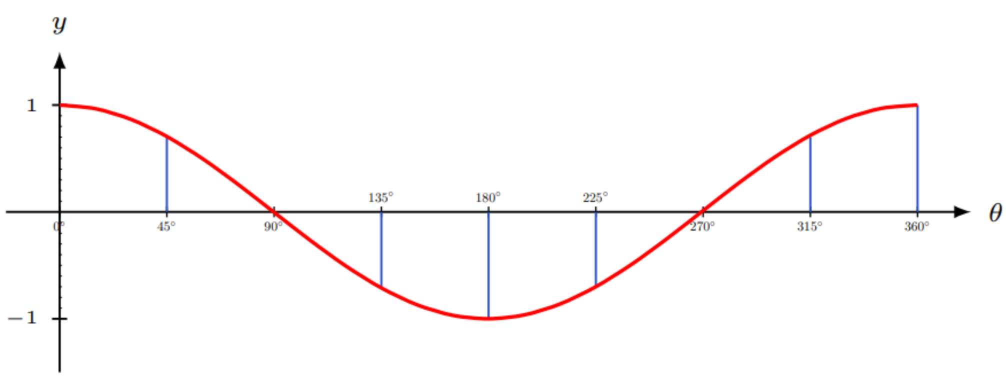
Here is table of values for the cosine function for angles between 0° and 360° followed by a graph of the cosine function for all angles from 0° to 360°.

Angle θ	Cosine θ (Horizontal Distance from Observer)
0°	1
45°	0.7071
90°	0
135°	–0.7071
180°	–1
225°	–0.7071
270°	0
315°	0.7071
360°	1

The positive cosine values indicate that the object is to the front of the observer whereas the negative values indicate that the object is to the back of the observer.

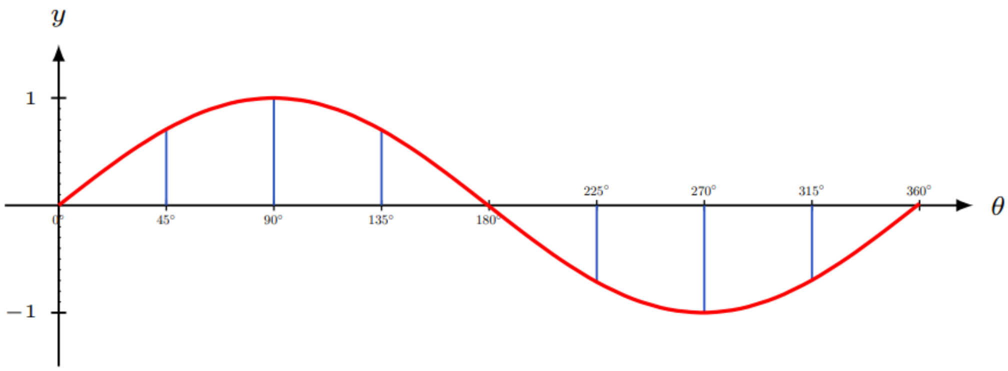
For example, at an angle of 45° from the observer's eye, the object is 0.7071 units in front of the observer.

At an angle of 135° from the observer's eye, the object is 0.7071 units behind the observer.



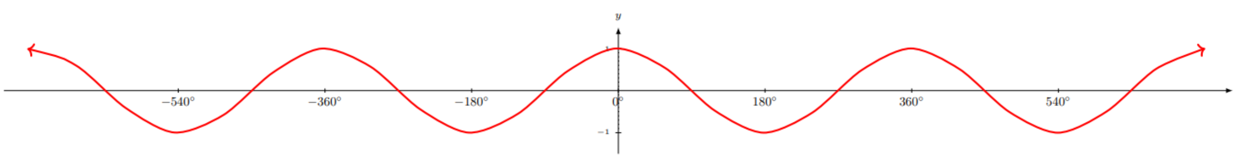
$y = \cos\theta$, for $0° \leq \theta \leq 360°$

If you think that this graph looks like the graph of the sine function, but shifted to the left by 90°, you would be right.



THE EXTENDED COSINE CURVE

Just as the sine curve does, the heights of the cosine curve oscillate between –1 and 1.



$y = \cos\theta$

The graph of the cosine function gives a visual illustration of how it determines the horizontal distance of an object from a vertical axis.

It may now be visually apparent that

> The cosine function determines the horizontal distance of an object to the left or right of an observer.

WHAT TO SEE

The graph shows how an object's horizontal distance from the observer changes as the angle of view increases. As the angle of view increases, the horizontal distance from the vertical increases (moves away from the observer) and decreases (moves toward the observer).

WHAT NOT TO SEE

The graph does not show how an object moves vertically as angle of view increases. The object is not moving along the curve.

5.4 TRY THESE

1. An object moves along the circumference of a unit circle. Find its horizontal distance from an observer if the angle it makes from observer's eye is

a. 225°
b. 270°
c. 315°
d. 360°

2. An object moves along the circumference of a unit circle. Find its horizontal distance from an observer if the angle it makes from observer's eye is

a. 390°
b. 405°
c. 420°
d. 450°

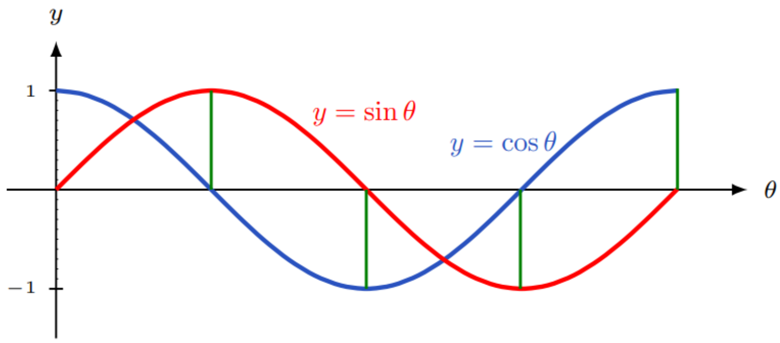
3. Determine if each statement is true or false.

a. Horizontal distance at 87° > Horizontal distance at 78°
b. Horizontal distance at 45° > Horizontal distance at 145°
c. Horizontal distance at 30° ≥ Horizontal distance at 150°
d. Horizontal distance at 90° = Horizontal distance at 270°

5.5 Amplitude and Period of the Sine and Cosine Functions

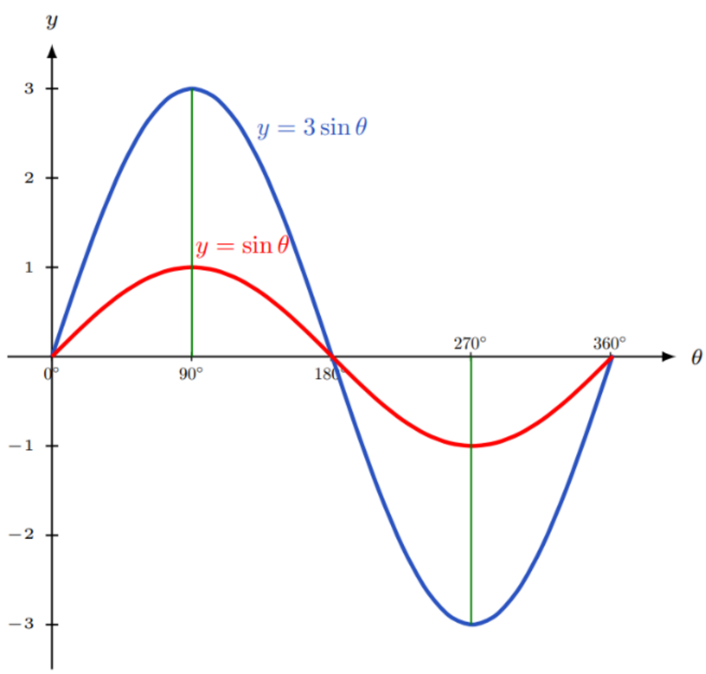
AMPLITUDE

We have seen how the graphs of both the sine function, $y = \sin\theta$ and the cosine function $y = \cos\theta$, oscillate between -1 and $+1$. That is, the heights oscillate between –1 and 1.

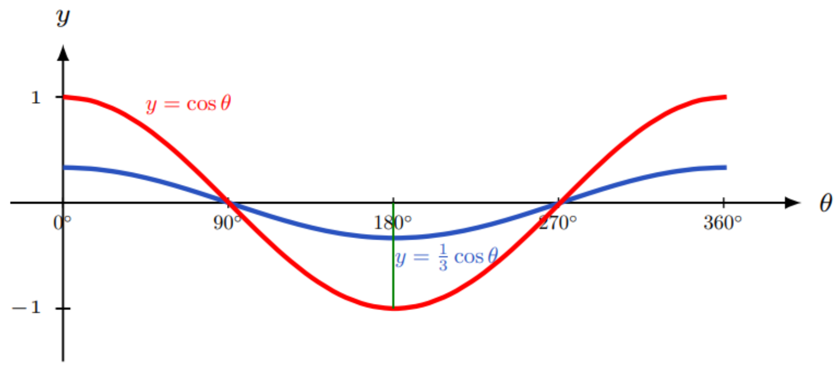


The height from the horizontal axis to the peak (or through) of a sine or cosine function is called the **amplitude** of the function. Each of the curves $y = \sin\theta$ and $y = \cos\theta$ has amplitude 1.

If we were to multiply the sine function $y = \sin\theta$ by 3, getting $y = 3\sin\theta$, each of the sine values would be multiplied by 3, making each value 3 times what it was. Each height would be tripled. The amplitude of $y = 3\sin\theta$ is 3.



If we were to multiply the cosine function $y = \cos\theta$ by $1/3$, getting $y = 1/3\cos\theta$, each of the cosine values would be multiplied by 1/3 making each value 1/3 of what it was. Each height of $y = \cos\ \theta$ would be 1/3 of what it was.
The amplitude of $y = 1/3\cos\theta$ is 1/3.



THE AMPLITUDE OF $y = A\mathrm{SIN}\theta$ AND $y = A\mathrm{COS}\theta$

Suppose A represents a positive number. Then the **amplitude** of both $y = A\sin\theta$ and $y = A\cos\theta$ is A and it represents height from the horizontal axis to the peak of the curve.

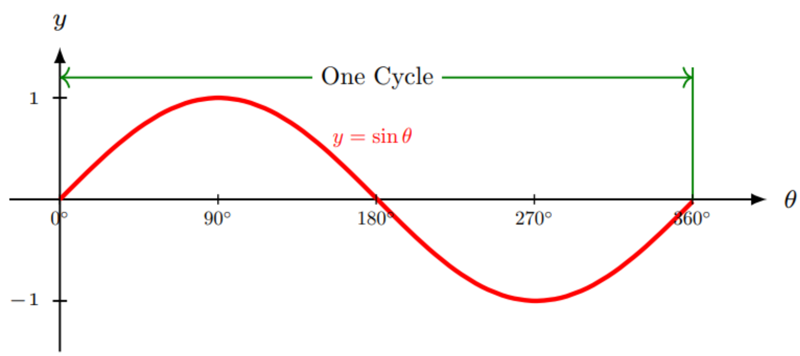
Examples

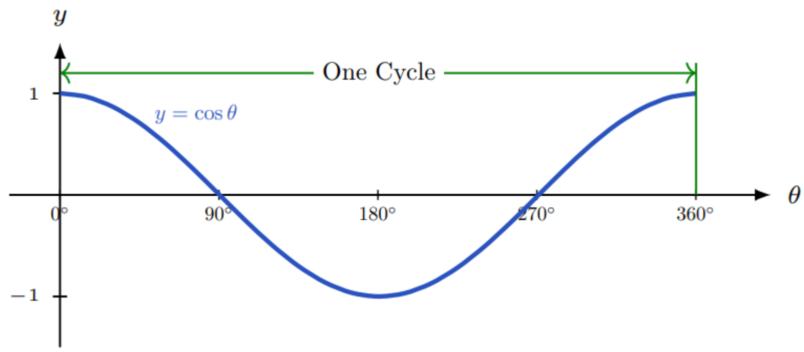
The amplitude of $y = 5/8\sin\theta$ is 5/8. This means that the peak of the curve is 5/8 of a unit above the horizontal axis.

The amplitude of $y = 3\sin\theta$ is 3. This means that the peak of the curve is 3 units above the horizontal axis.

PERIOD

Both the sine function and cosine function, $y = \sin\theta$ and $y = \cos\theta$, go through exactly one cycle from 0° to 360°. The **period** of the sine function and cosine functions, $y = \sin\theta$ and $y = \cos\theta$, is the “time” required for one complete cycle.





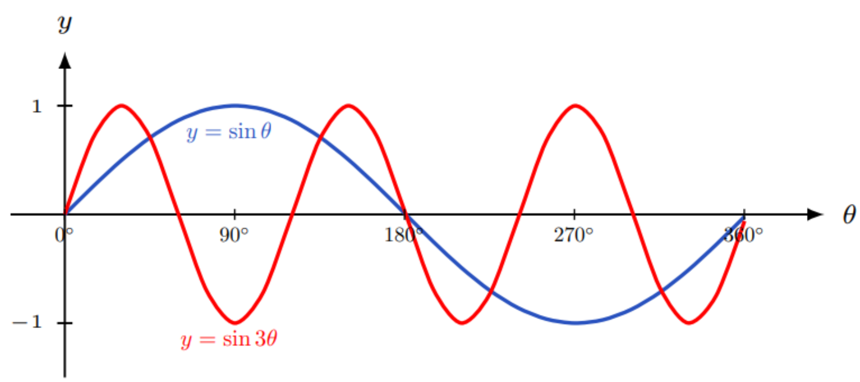
An interesting thing happens to the curves $y = \sin\theta$ and $y = \cos\theta$ when the angle θ is multiplied by some positive number, B. If the number B is greater than 1, the number of cycles on 0° to 360° increases for both $y = \sin\theta$ and $y = \cos\theta$. That is, the peaks of the curve are closer together, meaning their periods decrease. If the number B is strictly between 0 and 1, the peaks of the curve are farther apart, meaning their periods increase.

THE PERIOD OF $y = \mathrm{SIN}(B\theta)$ AND $y = \mathrm{COS}(B\theta)$

Suppose B represents a positive number. Then the **period** of both $y = \sin(B\theta)$ and $y = \cos(\mathrm{B}\theta)$ is $\frac{360°}{B}$. As B gets bigger, $\frac{360°}{B}$ gets smaller and the period increases.

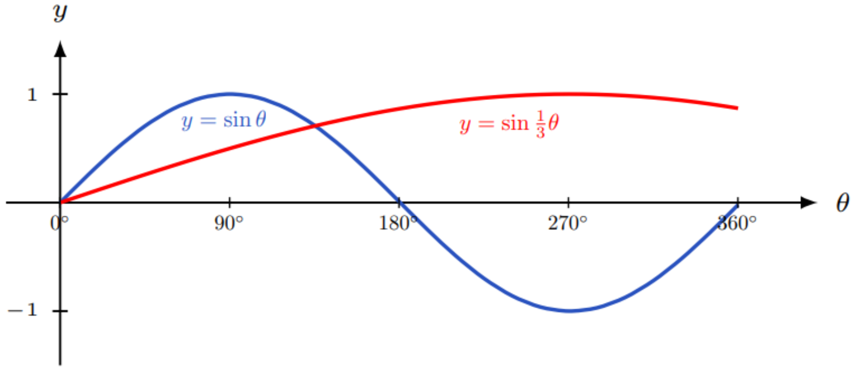
If we were to multiply the angle in the sine function $y = \sin\theta$ by 3, getting $y = \sin 3\theta$, each of the angle's values would be multiplied by 3 making each value 3 times what it was. Each angle would be tripled and there would be 3 cycles in the interval 0° to 360°.

The period of $y = \sin 3\theta$ is $\frac{360°}{3} = 120°$. The period of $y = \sin 3\theta$ is smaller than that of $y = \sin\theta$.



If we were to multiply the angle in the sine function $y = \sin\theta$ by $1/3$, getting $y = \sin\left(\frac{1}{3}\theta\right)$. Each of the angle's values would be multiplied by 1/3 making each value 1/3 what it was and there would be only 1/3 of a cycle in the interval 0° to 360°.

The period of $y = \sin\left(\frac{1}{3}\theta\right)$ is $\frac{360°}{1/3} = 360° \times 3 = 1080°$. The period of $y = \sin\left(\frac{1}{3}\theta\right)$ is greater than that of $y = \sin\theta$.



USING TECHNOLOGY

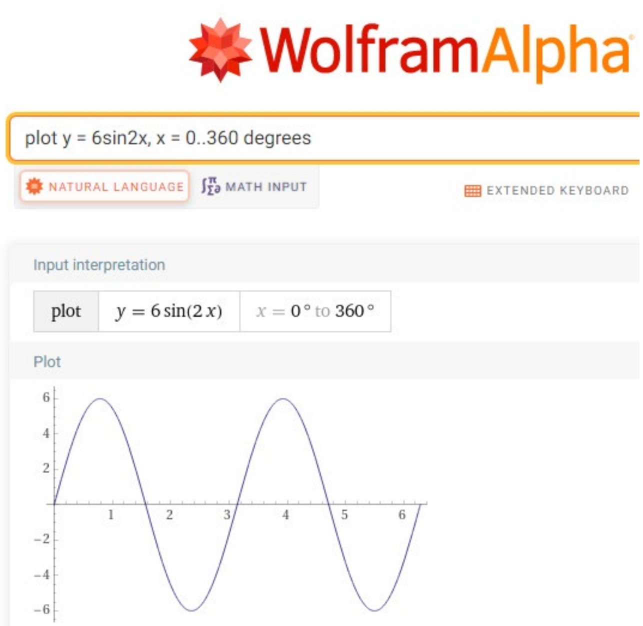
We can use technology to help us construct the graph of a sine or cosine function.

Go to www.wolframalpha.com.

Example (1)

Plot two complete cycles of $y = 6\sin 2\theta$ from 0° to 360°.

Type plot y = 6sin2x, x = 0..360 degrees in the entry field.
WolframAlpha tells you what it thinks you entered, then produces the graph.



You can see that WolframAlpha has plotted two complete cycles from 0° to 360° with amplitude 6.

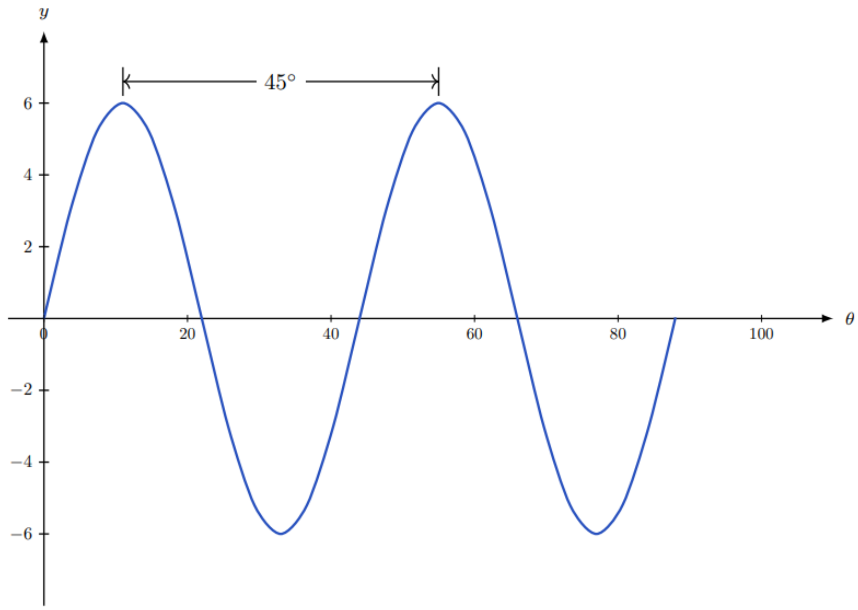
Example (2)

Find the period of $y = 6\sin 8\theta$.

We just need to evaluate $\frac{360°}{B}$ with $B = 8$.

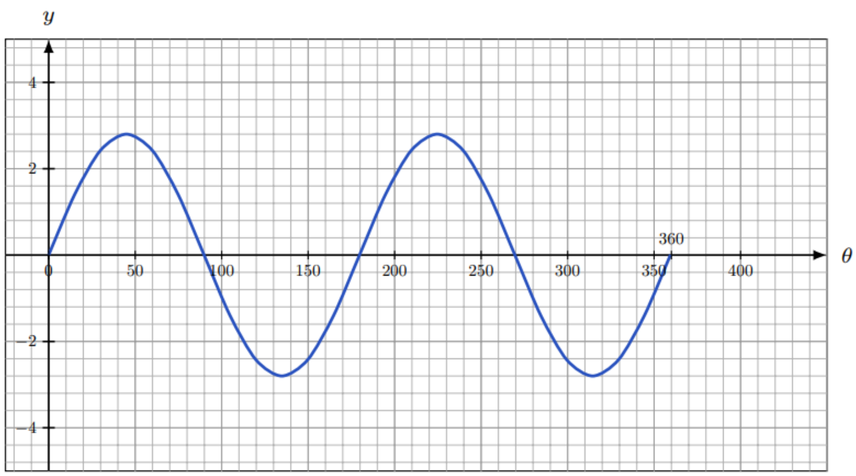
$\frac{360°}{8} = 45°$

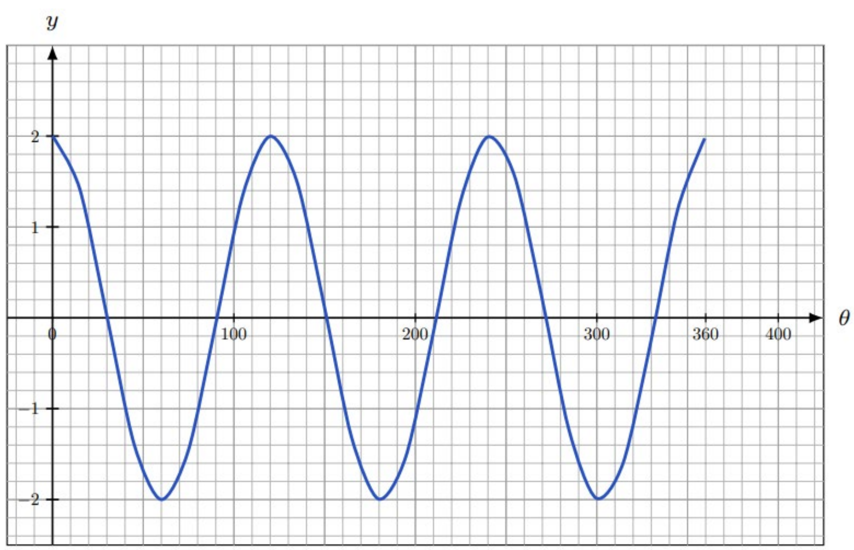
The period of $y = 6\sin 8\theta$ is $45°$
The graph of $y = 6\sin 8\theta$ helps us visualize this 45° period. You can see that the peaks differ by 45°.



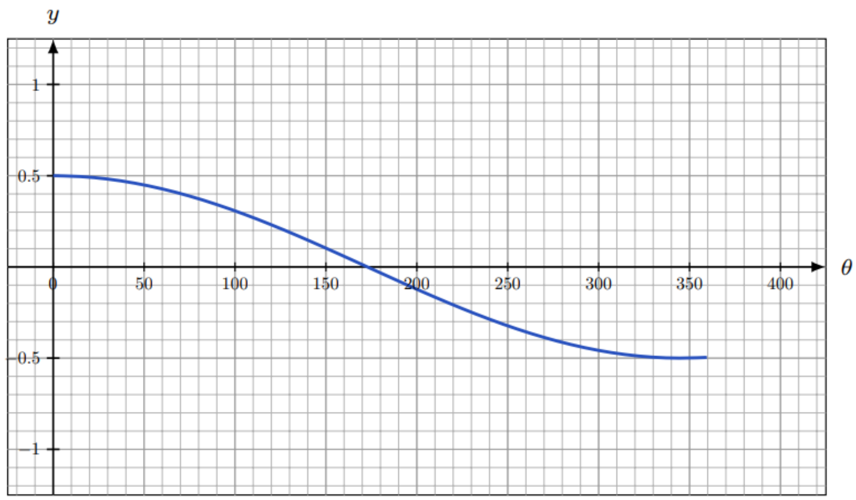
5.5 TRY THESE

1. Write the equation of each graph.

a. 

b. 

c.



2. How many complete cycles are there in the graph of $y = 4\cos(3\theta)$ from 0° to 360°? What is the period and amplitude of this function?

3. How many complete cycles are there in the graph of $y = 5\sin\left(\frac{4}{5}\theta\right)$ from 0° to 360°? What is the period and amplitude of this function?

4. Write the equation of a sine curve that has amplitude 15 and period 50°. You need to specify both A and B in $y = A\sin(B\theta)$. Keep in mind that the period of this function is $\frac{360°}{B}$.

5. Write the equation of a cosine curve that has amplitude 100 and period 12°. You need to specify both A and B in $y = A\cos(B\theta)$. Keep in mind that the period of this function is $\frac{360°}{B}$.

6. Write the equation of a cosine function that has amplitude 3 and makes two complete cycles from 0° to 180°.

7. Write the equation of a sine function that has amplitude 4 and makes three complete cycles from 0° to 90°.

ANSWERS TO TRY THESE

1.1 Constants, Variables, and Expressions
(page 5)

1.

a) The variable quantity is the download cost.

b) The constant is the fixed service cost.

c) Annual cost = 50 + $2x$, where x represents the number of downloaded photographs.

d) Annual cost = 50 + 2·20 = 50 + 40 = 90

The annual cost of downloading 20 photos is $90.

3.

a) There are 2 variable quantities in this problem.

b) There are no constants in this problem.

2.1 Vectors
(page 9)

1. $\vec{v} = \langle -3, 2 \rangle$ and $\vec{u} = \langle 13, 3 \rangle$
2. Two vectors are equal because they have the same direction and magnitude.

2.2 Addition, Subtraction, and Scalar Multiplication of Vectors
(page 13)

1. $\vec{u} + \vec{v} = \langle -5, 1 \rangle$
2. $\vec{v} - \vec{u} = \langle 15, -3 \rangle$

2.3 Magnitude, Direction, and Components of a Vector
(page 18)

1. $\|\vec{v}\| = 5$
2. $\|\vec{v}\| = 3\sqrt{2}$
3. $\vec{v}_x = 3\sqrt{3}$ and $\vec{v}_y = 3$
4. $\theta \approx 73.3008°$

2.4 The Dot Product of Two Vectors, the Length of a Vector, and the Angle Between Two Vectors
(page 23)

1. $\vec{u} \cdot \vec{v} = -13$
2. $\vec{u} \cdot \vec{v} = 0$
3. $\sqrt{65}$
4. 5
5. $135°$
6. $90°$

2.5 Parallel and Perpendicular Vectors, The Unit Vector
(page 27)

1. Parallel
2. Perpendicular
3. Neither parallel nor perpendicular
4. $\hat{v} = \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$

2.6 The Vector Projection of One Vector onto Another
(page 32)

1. $\left\langle \frac{21}{5}, \frac{7}{5} \right\rangle$
2. $\left\langle \frac{-222}{61}, \frac{185}{61} \right\rangle$

3.1 Three Dimensional Vectors
(page 36)

1. $\sqrt{29} \approx 5.4$ units
2. $7\sqrt{5} \approx 15.6$ units
3. $(x-2)^2 + (y-9)^2 = 1$
4. $(x+2)^2 + (y-5)^2 + (z+7)^2 = 16$

3.2 Magnitude and Direction Cosines of a Vector
(page 41)

1. $\|\vec{v}\| = \sqrt{29}$
2. $\|\vec{v}\| = \sqrt{2}$
3. {0.802, -0.267, 0.535}
4. $< -18, -6, 21 >$

3.3 Arithmetic on Vectors in 3-Dimensional Space

(page 45)

1. $\vec{u} + \vec{v} = \langle 5, 11, 1\rangle$
2. $\vec{u} - \vec{v} = \langle -11, -3, 11\rangle$
3. $2\vec{u} + 3\vec{v} - 4\vec{w} = \langle -5, 16, -42\rangle$
4. $4\vec{u} - 4\vec{v} - \vec{w} = \langle 16, -13, -26\rangle$

3.4 The Unit Vector in 3-Dimensions and Vectors in Standard Position

(page 49)

1. $\frac{2}{\sqrt{29}}\hat{\imath} - \frac{3}{\sqrt{29}}\hat{\jmath} + \frac{4}{\sqrt{29}}\hat{k}$
2. $\frac{1}{\sqrt{3}}\hat{\imath} - \frac{1}{\sqrt{3}}\hat{\jmath} + \frac{1}{\sqrt{3}}\hat{k}$
3. $\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{k}$
4. $\frac{4}{\sqrt{29}}\hat{\imath} + \frac{3}{\sqrt{29}}\hat{\jmath} + \frac{2}{\sqrt{29}}\hat{k}$

3.5 The Dot Product, Length of a Vector, and the Angle between Two Vectors in Three Dimensions

(page 54)

1. $\vec{u} \cdot \vec{v} = -31$
2. $\vec{u} \cdot \vec{v} = 0$
3. $\sqrt{101}$
4. 5
5. $63.6°$
6. $90°$

3.6 The Cross Product: Algebra

(page 59)

1. $\vec{u} \times \vec{v} = \langle -5, -7, 6\rangle$
2. $\vec{u} \times \vec{v} = \vec{0}$
3. 144

3.7 The Cross Product: Geometry

(page 65)

1. $\vec{u} \times \vec{v} = \langle 11, 33, -11\rangle$
2. $\vec{u} \times \vec{v} = \langle 0, 0, 0\rangle$ **=** $\vec{0}$
3. $5\sqrt{6}$ = 12.2 units

4.

$$\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$$
$$\theta = \cos^{-1}\frac{\langle 4,-7,6\rangle\cdot\langle 5,-1,2\rangle}{\sqrt{4^2+(-7)^2+\ 6^2}\cdot\sqrt{5^2+(-1)^2+\ 2^2}}$$
$$\theta = \cos^{-1}\frac{4\cdot 5+(-7)\cdot(-1)+6\cdot 2}{\sqrt{91}\cdot\sqrt{30}}$$
$$\theta = \cos^{-1}\frac{25}{\sqrt{101}\cdot\sqrt{30}}$$

$$\theta = 74.19°$$

5. Perpendicular since $\vec{u}\cdot\vec{v} = 0$

6. Parallelogram is 26.94 square units. Triangle is (½ of 26.94) = 13.47 square units.

4.1 Matrices

(page 70)

1.
   a) 4×3
   b) 2×3
   c) 1×3
2. True
3.
   a) 5
   b) 2
   c) 1
   d) 4
4. $S^T = \begin{bmatrix} 0 & -6 & 1 & 8 \\ 2 & -3 & 9 & -1 \\ 5 & 2 & 2 & 4 \end{bmatrix}$
5. $I_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
6. $I_{3\times 3}^T = I_{3\times 3}$
7. <4,3,2>
8. $\begin{bmatrix} 5 & 0 \\ 2 & 6 \end{bmatrix}$ or $\begin{bmatrix} 5 & 2 \\ 0 & 6 \end{bmatrix}$

4.2 Addition, Subtraction, Scalar Multiplication, and Products of Row and Column Matrices
(page 75)

1. $\begin{bmatrix} 4 & 4 \\ 0 & 3 \\ -1 & 2 \end{bmatrix}$
2. $\begin{bmatrix} -2 & 0 \\ 2 & -3 \\ -1 & -6 \end{bmatrix}$
3. Not possible
4. $[38]$
5. $[38]$
6. $\begin{bmatrix} 12 & 0 \\ -6 & 3 \end{bmatrix}$
7. Not defined
8. $[114]$
9. $[-76]$
10. $[4]$

4.3 Matrix Multiplication
(page 79)

1. $\begin{bmatrix} 10 & 5 & 5 \\ 10 & 5 & 0 \\ 16 & 8 & 4 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 19 \\ 2 & 18 \end{bmatrix}$
3. Is not commutative
4. $\begin{bmatrix} 20 & 10 & 0 \\ 12 & 6 & 13 \end{bmatrix}$
5. $\begin{bmatrix} 20 \\ 15 \end{bmatrix}$
6. $\begin{bmatrix} 9 \\ 1 \\ 8 \end{bmatrix}$
7. $\begin{bmatrix} 28 & -6 \\ -4 & 25 \end{bmatrix}$
8. D
9. $\begin{bmatrix} 2 & 20 \\ -2 & 6 \\ -6 & -8 \end{bmatrix}$
10. $\begin{bmatrix} 84 & 42 & 26 \\ 20 & 10 & 0 \\ -44 & -22 & -26 \end{bmatrix}$
11. Not defined

4.4 Rotation Matrices in 2-Dimensions
(page 83)

1. $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$
3. $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
4. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
5. $\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$
6. $\begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}$
7. $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$
8. $\begin{bmatrix} -3 \\ 3 \end{bmatrix}$
9. $\begin{bmatrix} -1.36603 \\ 0.36603 \end{bmatrix}$

4.5 Finding the Angle of Rotation Between Two Rotated Vectors in 2-Dimensions
(page 87)

1. $\theta = \frac{\pi}{4} = 45°$
2. $\theta = \frac{\pi}{3} = 60°$
3. $\theta = \frac{3\pi}{2} = 270°$
4. $\theta = \frac{\pi}{3} = 60°$
5. $\theta = \frac{7\pi}{4} = 315° = -45°$

4.6 Rotation Matrices in 3-Dimensions
(page 92)

1. $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$
2. $\begin{bmatrix} \sqrt{2} \\ 0 \\ 1 \end{bmatrix}$
3. $\begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$

5.1 The Basic Trigonometric Functions
(page 97)

1.
   a) $\sin 45° = \frac{1}{\sqrt{2}} = 0.7071, \cos 45° = \frac{1}{\sqrt{2}} = 0.7071, \tan 45° = 1$

   b) $\sin\theta = \frac{4}{5} = 0.8, \cos\theta = \frac{3}{5} = 0.6, \tan\theta = \frac{4}{3} = 1.3333$

   c) $\sin\theta = \frac{\sqrt{5}}{3} = 0.7454, \cos\theta = \frac{2}{3} = 0.6666, \tan\theta = \frac{\sqrt{5}}{2} = 1.1180$

   d) $\sin\theta = \frac{2}{\sqrt{5}} = 0.8944, \cos\theta = \frac{1}{\sqrt{5}} = 0.4472, \tan\theta = \frac{2}{1} = 2$

   e) $\sin\theta = \frac{15}{17} = 0.8834, \cos\theta = \frac{8}{17} = 0.4705, \tan\theta = \frac{15}{8} = 1.875$

2.
   a) $\sin 30° = 0.5,\ \cos 30° = 0.8661,\ \tan 30° = 0.5774$

   b) $\sin 90° = 1,\ \cos 90° = 0$

   c) $\sin 0° = 0,\ \cos 0° = 1,\ \tan 0° = 0$

   d) $\sin 180° = 0,\ \cos 180° = -1$

   e) $\sin 120° = 0.8660,\ \cos 120° = -0.5$

5.2 Circular Trigonometry
(page 102)

1. (0.7071, 0.7071)
2. (0.9962, 0.0872)
3. (6.4705, 4.8396)
4. (-7.0711, 7.0711)
5. 34.31 ft
6. 6 units

5.3 Graphs of the Sine Function
(page 107)

1.
   a) -0.7071

   b) -1

   c) -0.7071

   d) 0

2.
   a) 0.5

   b) 0.7071

   c) 0.8660

   d) 1

3.

a) True, since 0.9986 > 0.9781

b) False, since 0.4226 < 0.5736

c) True, since 0.5 = 0.5

d) True, since $1 \geq -1$

4.

a) True, since $\sin(87°) = 0.9986 > \cos(87°) = 0.0523$

b) False, since $\sin(155°) = 0.4226 < \cos(55°) = 0.5736$

c) True, since $\sin(20°) = 0.3420 < \cos(20°) = 0.9396$

d) True, since $\sin(135°) = 0.7071 = \cos(315°) = 0.7071$

5.4 Graphs of the Cosine Function

(page 110)

1.

a) -0.7071

b) 0

c) 0.7071

d) 1

2.

a) 0.8660

b) 0.7071

c) 0.5

d) 0

3.

a) False, since 0.0523 < 0.2079

b) False, since 0.7071 < 0.9063 (Be careful here: 0.7071 > –0.9063, but the negative sign tells us the object is the left of the observer. Think absolute value. At 45°, the object is 0.7071 to the right of the observer. At 145°, the object is 0.9063 units to the left of the observer, and, therefore, farther from the observer.)

c) False, since is |0.8660| =|-0.8660|

d) True, since 0 = 0

5.5 Amplitude and Period of the Sine and Cosine Functions
(page 116)

1.

a) $y = 3\sin(2x)$

b) $y = 2\cos(3x)$

c) $y = 7\cos(x)$

2. 3 complete cycles. Period is $\frac{360°}{3} = 120°$. Amplitude is 4.

3. $\frac{4}{5}$ of a complete cycle. Period is $\frac{360°}{4/5} = 360° \times \frac{5}{4} = 450°$. Amplitude is 5.

4. $y = 15\sin(7.2\theta)$, where $\frac{360°}{B} = 50° \rightarrow B = \frac{360°}{50°} = 7.2$

5. $y = 100\cos(30\theta)$, where $\frac{360°}{B} = 12° \rightarrow B = \frac{360°}{12°} = 30$

6. $y = 3\cos(4\theta)$

We need to specify both A and B in $y = A\cos(B\theta)$. Since the amplitude is 3, $A = 3$. Since the curve makes two complete cycles from 0° to 180°, it must make 4 complete cycles from 0° to 360°. So, $B = 4$.

7. $y = 4\sin(12\theta)$

We need to specify both A and B in $y = A\cos(B\theta)$. Since the amplitude is 4, $A = 4$. Since the curve makes three complete cycles from 0° to 90°, it must make 12 complete cycles from 0° to 360°. So, $B = 12$.

Thanks for exploring Mathematics for Game Developers.

Please reach out with ideas and edits.

We hope to connect and collaborate!

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