4.5 Finding the Angle of Rotation Between Two Rotated Vectors in 2-Dimensions

GIVEN THE ROTATED VECTOR, FIND THE ANGLE OF ROTATION

Suppose we did not know the angle θ of rotation. We can get it by working backwards and solving a system of equations. The rotation formula

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

produces the system of equations

$$\begin{cases} x' = x \cdot \cos\theta + y \cdot (-\sin\theta) \\ y' = x \cdot \sin\theta + y \cdot \cos\theta \end{cases}$$

Example (1)

In Example 1 of Chapter 4.4, we found that when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ was rotated counterclockwise by 90°, it became the vector $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We got this rotated vector by applying the rotation formula $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot \cos\theta + (-1) \cdot (-\sin\theta) \\ 1 \cdot \sin\theta + (-1) \cdot \cos\theta \end{bmatrix}$$

Since two vectors are equal only if their corresponding components are equal, we have the system of two equations

$$\begin{cases} 1 = 1 \cdot \cos\theta + (-1) \cdot (-\sin\theta) \\ 1 = 1 \cdot \sin\theta + (-1) \cdot \cos\theta \end{cases}$$

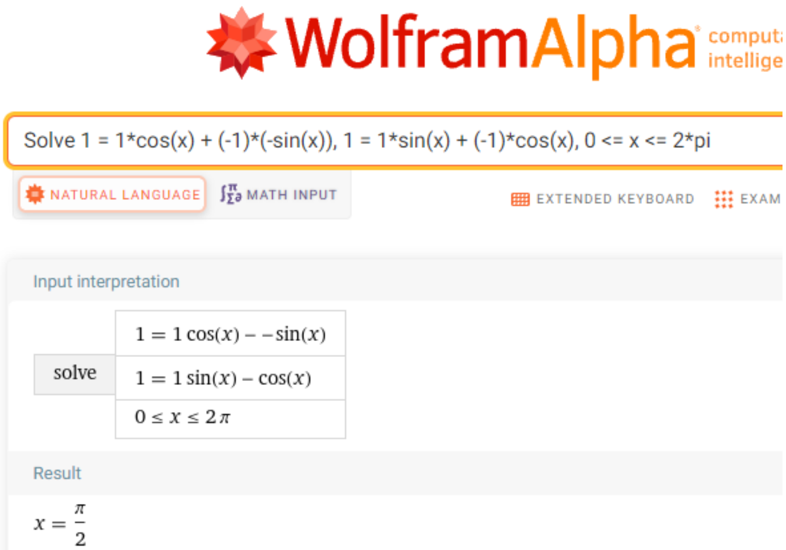
USING TECHNOLOGY

We can use WolframAlpha to help us solve above system for the angle of rotation, θ.

Go to www.wolframalpha.com.

Since we want to rotate only one time around the coordinate system, we want to instruct W|A to give us solutions only where the angle θ is between 0 and 2π.

Using the English letter x in place of the Greek letter θ, enter
Solve 1 = 1*cos(x) + (-1)*(-sin(x)), 1 = 1*sin(x) + (-1)*cos(x), 0 <= x <= 2*pi in the entry field.



W|A shows the angle of rotation is $\theta = \frac{\pi}{2}$, which is 90°. We conclude that the angle of rotation is 90°.

Example (2)

In Example 2 of Chapter 4.4, we found that when the vector $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ was rotated counterclockwise by 60°, it became the vector $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - 3\sqrt{3} \\ 3 + \sqrt{3} \end{bmatrix}$. We got this rotated vector by applying the rotation formula $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} 1 - 3\sqrt{3} \\ 3 + \sqrt{3} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

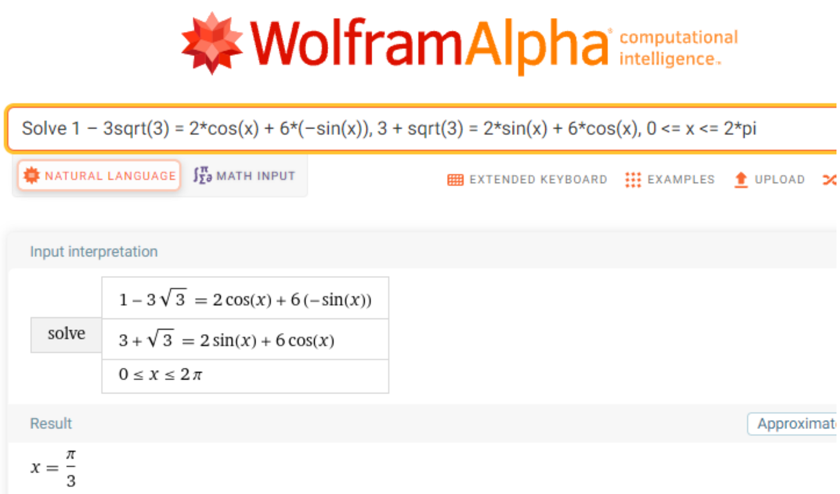
$$\begin{bmatrix} 1 - 3\sqrt{3} \\ 3 + \sqrt{3} \end{bmatrix} = \begin{bmatrix} 2 \cdot \cos\theta + 6 \cdot (-\sin\theta) \\ 2 \cdot \sin\theta + 6 \cdot \cos\theta \end{bmatrix}$$

Since two vectors are equal only if their corresponding components are equal, we have the system of two equations

$$\begin{cases} 1 - 3\sqrt{3} = 2 \cdot \cos\theta + 6 \cdot (-\sin\theta) \\ 3 + \sqrt{3} = 2 \cdot \sin\theta + 6 \cdot \cos\theta \end{cases}$$

We will use WolframAlpha to help us solve this system for the angle of rotation, θ.

Using the English letter x in place of the Greek letter θ, enter
Solve 1 – 3sqrt(3) = 2*cos(x) + 6*(–sin(x)), 3 + sqrt(3) = 2*sin(x) + 6*cos(x), 0 <= x <= 2*pi
in the entry field. Separate the two equations with a comma.



W|A shows the angle of rotation is $\theta = \frac{\pi}{3}$, which is 60°. We conclude that the angle of rotation is 60°.

TRY THESE

1. Find the angle θ through which the vector $\begin{bmatrix}3\\3\end{bmatrix}$ is rotated to become $\begin{bmatrix}0\\3\sqrt{2}\end{bmatrix}$.

ANS: $\theta = \frac{\pi}{4} = 45°$

2. Find the angle θ through which the vector $\begin{bmatrix}2\\-2\end{bmatrix}$ is rotated to become $\begin{bmatrix}1+\sqrt{3}\\-1+\sqrt{3}\end{bmatrix}$.

ANS: $\theta = \frac{\pi}{3} = 60°$

3. Find the angle θ through which the vector $\begin{bmatrix}2\\0\end{bmatrix}$ is rotated to become $\begin{bmatrix}0\\-2\end{bmatrix}$.

ANS: $\theta = \frac{3\pi}{2} = 270°$

4. Find the angle θ through which the vector $\begin{bmatrix}-2\\-2\end{bmatrix}$ is rotated to become $\begin{bmatrix}-1+\sqrt{3}\\-1-\sqrt{3}\end{bmatrix}$.

ANS: $\theta = \frac{\pi}{3} = 60°$

5. Find the angle θ through which the vector $\begin{bmatrix}1\\1\end{bmatrix}$ is rotated to become $\begin{bmatrix}\sqrt{2}\\0\end{bmatrix}$.

ANS: $\theta = \frac{7\pi}{4} = 315°$ = -45°

NOTE TO INSTRUCTOR

Note that to find the angle θ between the two vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} x' \\ y' \end{bmatrix}$, we use the rotation formula in reverse.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

produces the system of equations

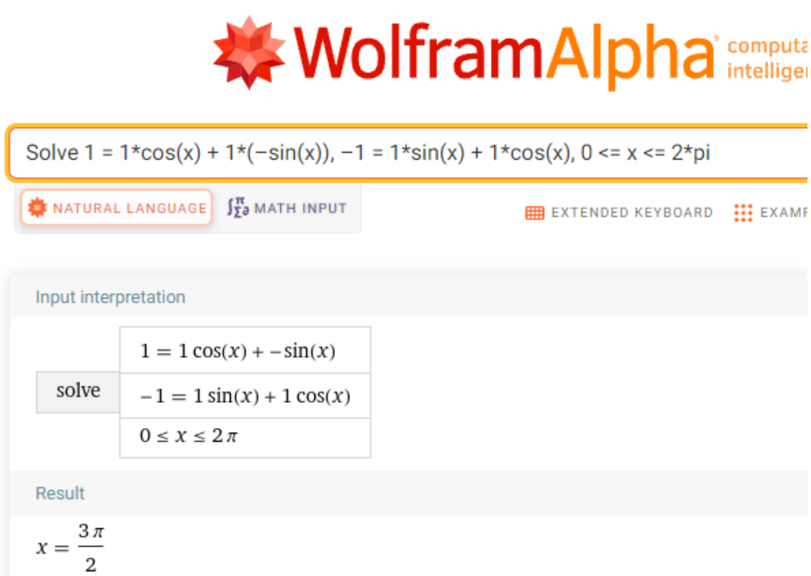
$$\begin{cases} x' = x \cdot \cos\theta + y \cdot (-\sin\theta) \\ y' = x \cdot \sin\theta + y \cdot \cos\theta \end{cases}$$

Consider demonstrating this example.

Find the angle θ through which the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is rotated to become $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$$\begin{cases} x' = x \cdot \cos\theta + y \cdot (-\sin\theta) \\ y' = x \cdot \sin\theta + y \cdot \cos\theta \end{cases} \quad \Rightarrow \quad \begin{cases} 1 = 1 \cdot \cos\theta + 1 \cdot (-\sin\theta) \\ -1 = 1 \cdot \sin\theta + 1 \cdot \cos\theta \end{cases}$$

Use W|A to solve this system. Go to www.wolframalpha.com and enter
Solve 1 = 1*cos(x) + 1*(–sin(x)), –1 = 1*sin(x) + 1*cos(x), 0 <= x <= 2*pi into the entry field.
Both entries and rows are separated by commas as W|A does not see spaces.
Wolframalpha tells you what it thinks you entered, then tells you its answer



We conclude that the angle of rotation is $\frac{3\pi}{2} = 270° = -90°$.

USING THE DOT PRODUCT TO FIND THE ANGLE BETWEEN TWO VECTORS

You may wish to point out that the angle of rotation can also be found using the dot product formula from Chapter 2.4.

If θ is the smallest nonnegative angle between two non-zero vectors $\vec{u}$ and $\vec{v}$, then

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|} \quad \text{or} \quad \theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$$

where $0 \le \theta \le 2\pi$ and $\|\vec{u}\| = \sqrt{{u_x}^2 + {u_y}^2}$ and $\|\vec{v}\| = \sqrt{{v_x}^2 + {v_y}^2}$.

Find the angle between the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$.

Using $\theta = \cos^{-1}\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$, we get

$$\theta = \cos^{-1}\frac{\begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\end{bmatrix}}{\sqrt{1^2+1^2}\cdot\sqrt{1^2+(-1)^2}}$$

$$\theta = \cos^{-1}\frac{1\cdot1+1\cdot(-1)}{\sqrt{1+1}\cdot\sqrt{1+1}}$$

$$\theta = \cos^{-1}\frac{0}{\sqrt{1}\cdot\sqrt{1}}$$

$$\theta = \cos^{-1}0$$

$$\theta = \frac{\pi}{2} = 90°$$