Quiz 02 (60 mins.)

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Department: 資工系 **Name:** 張碩文 **ID:** B09304007

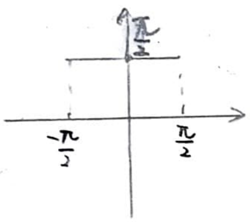
Problem 1: (30 points)

$-\frac{\pi}{2} < x < \frac{\pi}{2}$

Please use Fourier Integral representation to show that $\int_0^{\infty} \frac{\cos(\pi\omega/2)\cos \pi\omega}{1-\omega^2} = \begin{cases} \frac{\pi}{2}, & |x| < \frac{\pi}{2} \\ 0, & |x| > \frac{\pi}{2} \end{cases}$

Answer:

$$A(w) = \int_{-\infty}^{\infty} f(x) \cos wx\, dx = 2\int_0^{\infty} f(x) \cos wx\, dx$$



$$= 2\int_0^{\frac{\pi}{2}} f(x) \cos wx\, dx = 2\int_0^{\frac{\pi}{2}} \frac{\pi}{2} \cos wx\, dx = \pi \int_0^{\frac{\pi}{2}} \cos wx\, dx = \pi \cdot \frac{\sin wx}{w}\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi \sin\frac{\pi w}{2}}{w}$$

$\because f(x) \in$ even

$$B(w) = 0$$

$$\therefore f(x) = \frac{1}{\pi}\int_0^{\infty}\left(\frac{\pi \sin\frac{\pi w}{2}}{w} \cdot \cos wx\right) dw = \int_0^{\infty}\left(\frac{\sin\frac{\pi w}{2}}{w} \cdot \cos wx\right) dw$$

Problem 2: (30 points)

In the first quadrant, there is a vector at a 45-degree angle to the origin and the vector length is 1.

(1) Please define the vector on a complex plane.

(2) If we rotate this vector eight times counterclockwise and each rotation angle is 45 degree as well, please find the final length of this vector.

Answer:

(1) $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$

(2) 1

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Problem 3: (20 points)

The complex number z_1 and z_2 are given by $z_1 = p+2i$ and $z_2 = 1-2i$ where p is an integer.

(1) Find $\frac{z_1}{z_2}$ in the form $a+bi$ where a and b are both real. Give your answer in its simplest form in terms of p.

(2) Given that $\left|\frac{z_1}{z_2}\right| = 13$. Please find the possible value of p.

Answer:

(1) $\frac{p+2i}{1-2i} = \frac{(p+2i)(1-2i)}{5} = \frac{(p+4)+2(1-p)i}{5} = \frac{p+4}{5} + \frac{2(1-p)}{5}i$

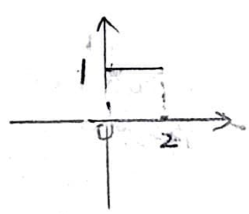
(2) $\frac{\sqrt{4+p^2}}{\sqrt{5}} = 13$ $\quad p = \pm 29$

$4+p^2 = 5 \cdot 169$

$= 845$

Problem 4: (20 points)

A function $f(x)$ is defined as $f(x) = \begin{cases} 1, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$.



(1) Please find the Fourier integral representation

(2) Please use the result in (1) to find $\int_0^\infty \frac{1}{\omega} \sin\omega \, d\omega$

Answer:

(1) $A(w) = \int_{-\infty}^{\infty} f(x) \cos wx\, dx = \int_0^2 \cos wx\, dx = \frac{\sin wx}{w}\Big|_0^2 = \frac{\sin 2w}{w}$

$B(w) = \int_{-\infty}^{\infty} f(x) \sin wx\, dx = \int_0^2 \sin wx\, dx = \frac{-\cos wx}{w}\Big|_0^2 = \frac{-\cos 2w}{w} + \frac{1}{w}$

$\therefore f(x) = \frac{1}{\pi}\int_0^\infty \left(\frac{\sin 2w}{w} \cos wx + \left(\frac{-\cos 2w}{w} + \frac{1}{w}\right) \cdot \sin wx \right) dw$

(2) $f(1) = \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\, dw = 1$

$\therefore \int_0^\infty \frac{\sin w}{w}\, dw = \frac{\pi}{2}$

Quiz 02 (60 mins.)

Department: 海工系 **Name:** 黃育翔 **ID:** B083040007

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Problem 1: (30 points)

A function $f(x) = e^{-a|x|}$, $a > 0$, please find 《hint: $\int_0^{\infty} e^{-ax}\cos\omega x dx = \frac{a}{a^2+\omega^2}$ 》

(1) Fourier integral of *f(x)*

(2) Calculate $\int_0^{\infty} \frac{\cos(2x)}{x^2+4} dx$

$f(x) = \frac{1}{\pi}\int_0^{\infty} (A(w)\cos wx + B(w)\sin wx) dw$

Answer: $f(x) = f(-x) \Rightarrow f(x)$ is an even func. $\Rightarrow B(w) = 0$

(1) $A(w) = \int_{-\infty}^{\infty} f(x)\cdot\cos wx\cdot dx = 2\int_0^{\infty} e^{-ax}\cdot\cos wx\cdot dx = 2\mathcal{L}[\cos wx]_{s=a} = \frac{2a}{a^2+w^2}$

$\therefore f(x) = \frac{1}{\pi}\int_0^{\infty} \frac{2a}{a^2+w^2}\cos wx\cdot dw = \frac{2a}{\pi}\int_0^{\infty}\frac{\cos wx}{a^2+w^2}\cdot dw$ #

(2) $a=2, x=2 \Rightarrow f(2) = \frac{4}{\pi}\int_0^{\infty}\frac{\cos 2w}{w^2+4}\cdot dw = e^{-4}$

$\Rightarrow \int_0^{\infty}\frac{\cos 2w}{w^2+4}\cdot dw = \frac{\pi}{4e^4}$ #

送分 **Problem 2: (30 points)**

Please use Fourier Integral representation to show that $\int_0^{\infty}\frac{\cos(\cancel{\pi\omega}\,\omega x/2)\cos \cancel{\pi\omega}\,\omega x}{1-\omega^2} d\omega = \begin{cases} \frac{\pi}{2} & ,|x| < \frac{\pi}{2} \\ 0 & ,|x| > \frac{\pi}{2} \end{cases}$

Answer: $f(x) = \frac{1}{2}$

$A(w) = \int_{-\infty}^{\infty} f(x)\cdot\cos wx\cdot dx = \int_0^{\frac{\pi}{2}}\cos wx\cdot dx =$

Quiz 02 (60 mins.)

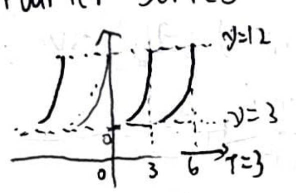
Problem 3: (20 points)

You can expand the function defined by $f(x) = x^2 + 3,\ 0 < x < 3$ in a Fourier series, a cosine series, or a sine series. Please choose the correct answers.

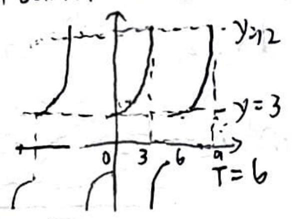
(a) $f(6) = 3$ for Fourier sine series; (b) $f(3) = 12$ for Fourier cosine series; (c) $f(0) = 3$ for Fourier series; (d) $f(-1) = 4$ for Fourier sine series; (e) $f(-3) = 12$ for Fourier cosine series

Answer:

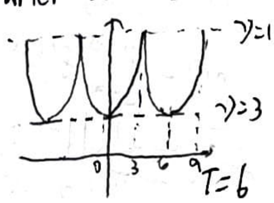
Fourier series



Fourier sine series



Fourier cosine series



(a) X, $f(6)=f(0)=0$

(b) O

(c) X,

(d) X, $f(-1)=-f(1)=-4$

(e) O

A: (b), (e)

Problem 4: (20 points)

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You can expand the function defined by $f(x) = x,\ 0 < x < \pi$ in a Fourier sine series and Fourier cosine series. Give a set of criteria, and there by judge which series is the better. Please explain.

Answer:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Fourier sine series is the better since the error is smaller.

Fourier cosine series: $b_n = 0$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cdot dx = \frac{1}{\pi}\int_0^{\pi} x\cdot dx = \frac{1}{\pi}\cdot\frac{1}{2}x^2\Big]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos nx\cdot dx = \frac{2}{\pi}\int_0^{\pi} x\cdot\cos nx\cdot dx$$

$$= \frac{2}{\pi}\cdot\cancel{\frac{1}{n}x\sin nx}\Big]_0^{\pi} - \frac{2}{\pi}\cdot\frac{1}{n}\int_0^{\pi}\sin nx\cdot dx = \frac{2}{n^2\pi}\cos nx\Big]_0^{\pi}$$

$$= \frac{2}{n^2\pi}((-1)^n - 1) = \begin{cases} -\frac{4}{n^2\pi}, & n \in \text{odd} \\ 0, & n \in \text{even} \end{cases}$$

$$\Rightarrow f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{4}{(2n-1)^2\pi}\cos(2n-1)x$$

Fourier sine series: $a_n = a_0 = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\cdot dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\cdot dx = \frac{2}{\pi}\cdot\frac{-1}{n}\cdot x\cos nx\Big]_0^{\pi} + \cancel{\frac{2}{n\pi}\int_0^{\pi}\cos nx\cdot dx} = -\frac{2}{n\pi}\cdot\pi\cdot\cos n\pi = -\frac{2\cos n\pi}{n} = -\frac{2(-1)^n}{n}$$

$$\Rightarrow f(x) = -\sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\cdot\sin nx$$

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Quiz 02 (60 mins.)

Department: 資工111 **Name:** 陳文揚 **ID:** B073040007

Problem 1: (30 points)

A function is defined as $f(x)=|x|$ when $-2<x<2$. Besides, the *f(x + 4)* is equal to *f(x)*.

(1) Please find the Fourier series of *f(x)*

(2) Applying your result to find the summation value of $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\dots$

+28 **Answer:** $f(x)=1+\sum_{n=1}^{\infty}\frac{-8}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi}{2}x$

(1) $T=4$

$f(x)\in$ even function

$\Rightarrow b_n=0$

$a_0=\frac{1}{4}\int_{-2}^{2}f(x)\,dx$

$=\frac{1}{2}\int_0^2 x\,dx$

$=\frac{1}{2}\left(\frac{1}{2}x^2\Big|_0^2\right)=1$

$a_n=\frac{1}{2}\int_{-2}^{2}f(x)\cos\frac{2n\pi}{4}x\,dx$

$=\int_0^2 x\cos\frac{n\pi}{2}dx$

$=\left[\left(x\cdot\frac{2}{n\pi}\sin\frac{n\pi}{2}x\Big|_0^2\right)-\int_0^2\frac{2}{n\pi}\sin\frac{n\pi}{2}x\,dx\right]$

$=\frac{-2}{n\pi}\int_0^2\sin\frac{n\pi}{2}x\,dx$

$=\frac{+4}{(n\pi)^2}\left(\cos\frac{n\pi}{2}x\Big|_0^2\right)$

$=\frac{4}{(n\pi)^2}\left((-1)^n-1\right)=\begin{cases}\frac{-8}{(n\pi)^2}, & n\in odd\\ 0, & n\in even\end{cases}$

(2) x代4

$f(x)=0=1+\frac{-8}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$

$\Rightarrow\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$

$=-1\times\frac{\pi^2}{-8}$

$=\frac{\pi^2}{8}$

Problem 2: (30 points)

A function is defined as $f(x)=3\cos x+4\text{in}\,x$ with the period of L (0<x<L) is $f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L})$. Please calculate the approximation error when we separately use Fourier Series and Degree-3 Taylor Series at *x=0*.

Answer:

$f'(x)=-3\sin x+4\cos x$

$f''(x)=-3\cos x-4\sin x$

$f(0)=3\cos 0+4\sin 0=3$

$f'''(x)=3\sin x-4\cos x$

$f^{(4)}(x)=3\cos x+4\sin x$

+30 $f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3$

$=3+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3=3$

$\Rightarrow x=0$

$\Rightarrow\frac{f^{(4)}(0)}{4!}x^4$

$=\frac{3}{4!}\times 0=0$

Quiz 02 (60 mins.)

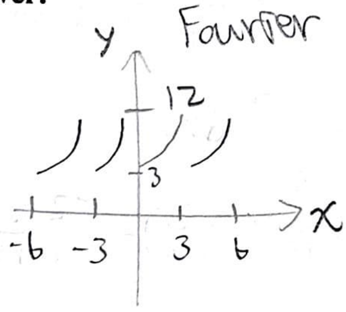
Problem 3: (20 points)

You can expand the function defined by $f(x) = x^2 + 3,\ 0 < x < 3$ in a Fourier series, a cosine series, or a sine series. Please choose the correct answers.

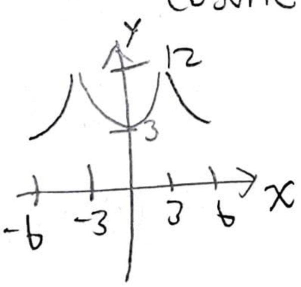
(a) $f(6) = 3$ for Fourier sine series; (b) $f(3) = 12$ for Fourier cosine series; (c) $f(0) = 3$ for Fourier series; (d) $f(-1) = 4$ for Fourier sine series; (e) $f(-3) = 12$ for Fourier cosine series

Answer:

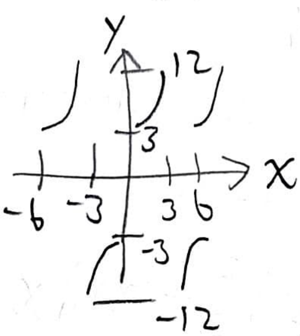
Fourier



cosine



sine



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(a) $f(6) = \frac{3-3}{2} = 0$

✓ (b) $f(3) = 12$

(c) $f(0) = \frac{3+12}{2} = \frac{15}{2}$

(d) $f(-1) < 0 \neq 4$

✓ (e) $f(-3) = 12$

A: (b), (e)

Problem 4: (20 points)

You can expand the function defined by $f(x) = x,\ 0 < x < \pi$ in a Fourier sine series and Fourier cosine series. Give a set of criteria, and there by judge which series is the better. Please explain.

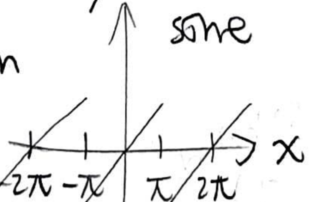
+15

Answer:

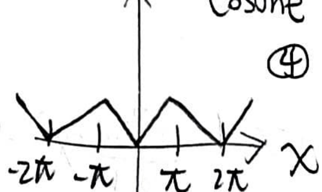
Fourier sine series

$f(x) \in$ odd function

$\Rightarrow a_0 = a_n = 0$



$T = 2\pi$? $\Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{\pi(2n-1)^2} \cos(2n-1)x$



$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$

$= \frac{2}{\pi}\left[\left(x \cdot \frac{1}{n} \cdot -\cos nx\Big|_0^{\pi}\right) - \int_0^{\pi} \frac{1}{n} \cdot -\cos nx\, dx\right]$

$= \frac{2}{\pi}\left(-\frac{\pi}{n} \cdot (-1)^n + \frac{1}{n}\left(\frac{1}{n} \sin nx\Big|_0^{\pi}\right)\right) = \frac{-2}{n}(-1)^n$

$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{-2}{n}(-1)^n \sin nx$ 1

Fourier cosine series

$f(x) \in$ even function

$\Rightarrow b_n = 0$ ①

④ $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$= \frac{1}{\pi}\int_0^{\pi} x\, dx$

$= \frac{1}{2\pi}\left(x^2\Big|_0^{\pi}\right)$

$= \frac{\pi}{2}$ ②

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx$

$= \frac{2}{\pi}\left[\left(x \cdot \frac{1}{n} \sin nx\Big|_0^{\pi}\right) - \int_0^{\pi} \frac{1}{n} \sin nx\, dx\right]$

$= \frac{-2}{\pi n}\left(\frac{1}{n} - \cos nx\Big|_0^{\pi}\right)$

$= \begin{cases} \frac{-4}{\pi n^2}, & n \in \text{odd} \\ 0, & n \in \text{even} \end{cases}$ ③

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Quiz 02 (60 mins.)

Department: 資工 **Name:** [illegible] **ID:** B063040007

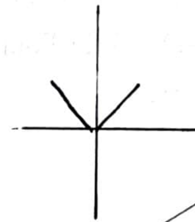
Problem 1: (30 points)

A function is defined as $f(x)=|x|$ when $-2<x<2$. Besides, the $f(x+4)$ is equal to $f(x)$.

(1) Please find the Fourier series of $f(x)$

(2) Applying your result to find the summation value of $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots$

Answer: $T=4$. $F(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{2n\pi}{T}x+b_n\sin\frac{2n\pi}{T}x)$



$f(x)$為 even $\Rightarrow b_n=0$

$a_0=\frac{1}{4}\int_{-2}^{2}|x|\,dx=\frac{1}{2}\int_0^2 x\,dx=\frac{1}{4}x^2\Big]_0^2=1$

$a_n=\frac{2}{4}\int_{-2}^{2}|x|\cos\frac{2n\pi}{4}x\,dx$

$=\int_0^2 x\cos\frac{2n\pi}{4}x\,dx=\frac{2}{n\pi}\sin\frac{n\pi}{2}x\cdot x\Big]_0^2-\frac{2}{n\pi}\int_0^2\sin\frac{n\pi}{2}x\,dx$

$=+\frac{2}{n\pi}\times\frac{2}{n\pi}\times\cos\frac{n\pi}{2}x\Big]_0^2$

$=\frac{4}{(n\pi)^2}((-1)^n-1)$ $\begin{cases}\frac{-8}{(n\pi)^2} & n\in\text{odd}\\ 0 & n\in\text{even}\end{cases}$

$+30$

$F(0)=1+\frac{-8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=0$

$=1+\frac{-8}{\pi^2}(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots)=0$ $\qquad -1=\frac{-8}{\pi^2}(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots)$

① A: $F(x)=1+\sum_{n=1}^{\infty}(\frac{-8}{[(2n-1)\pi]^2}\cos\frac{(2n-1)\pi x}{2})$ ② $\frac{\pi^2}{8}=(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots)$

Problem 2: (30 points)

The expansion of the periodic function $f(x)=x^2$ with the period of L $(0<x<L)$ is $f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L})$. Please determine the value of b_n.

Answer: $T=L$

$x\cos\frac{2n\pi}{L}$

$b_n=\frac{1}{L}\int_0^L(x^2\cdot\sin\frac{2n\pi x}{L})$

$+10$

$=\frac{1}{L}(\frac{-L}{2n\pi}\cos\frac{2n\pi}{L}x\cdot x^2\Big]_0^L-\int_0^L-\frac{L}{2n\pi}\cos\frac{2n\pi}{L}\cdot 2x\,dx$

$=\frac{1}{L}(\frac{-L}{2n\pi}\cos\frac{2n\pi}{L}x\cdot x^2\Big]_0^L+\frac{2L}{2n\pi}(\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\cdot x\Big]_0^L-\int_0^L\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}dx)$

$=\frac{-1}{2n\pi}\cos\frac{2n\pi x}{L}\cdot x^2\Big]_0^L+\frac{1}{n\pi}(\frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\cdot x\Big]_0^L+\frac{L}{2n\pi}\times\frac{L}{2n\pi}\cos\frac{2n\pi x}{L}\Big]_0^L$

$=\frac{-1}{2n\pi}(L^2)$

A: $\frac{-L^2}{2n\pi}$

Quiz 02 (60 mins.)

Problem 3: (20 points)

You can expand the function defined by $f(x) = x^2 + 3,\ 0 < x < 3$ in a Fourier series, a cosine series, or a sine series. Please choose the correct answers.

(a) $f(6) = 3$ for Fourier sine series; (b) $f(3) = 12$ for Fourier cosine series; (c) $f(0) = 3$ for Fourier series; (d) $f(-1) = 4$ for Fourier sine series; (e) $f(-3) = 12$ for Fourier cosine series

Answer:

Fourier sine series ⇒ f(x) 為 odd.

Fourier cosine series ⇒ f(x) 為 even

Fourier serieres ⇒ 無法得知，因不確定為 even or odd.

A: b, e.

×20

Problem 4: (20 points)

You can expand the function defined by $f(x) = x,\ 0 < x < \pi$ in a Fourier sine series and Fourier cosine series. Give a set of criteria, and there by judge which series is the better. Please explain.

Answer:

$f(x) = x$

$T = 2\pi$

×10

sine series $F(x) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{2n\pi x}{T}\right)$

f(x) is odd

$b_n \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx$

$= \frac{2}{\pi}\int_0^{\pi} \sin nx \cdot x\, dx = \frac{2}{\pi}\left(-\frac{\cos nx}{n} \cdot x\Big]_0^{\pi} - \int_0^{\pi} \frac{-\cos nx}{n} dx\right)$

$= -\frac{2}{n\pi}(\pi(-1)^n) = \frac{(-2)^{n+1}}{n}$.

Fourier sine series ⇒ $F(x) = \sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n} \sin nx$

Fourier cosine series $= \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{(2n-1)^2\pi} \cos(2n-1)x\right)$

cosine series f(x) is even

$F(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx)$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\, dx = \frac{1}{\pi}\int_0^{\pi} x dx = \frac{1}{2\pi}(x^2)\Big]_0^{\pi} = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$

A: sine series is better because the error is smaller.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cdot \cos nx\, dx = \frac{2}{\pi}\int_0^{\pi}(x \cos nx\, dx)$

$= \frac{2}{\pi}\left(\frac{\sin nx}{n} x\Big]_0^{\pi} - \int_0^{\pi} \frac{\sin nx}{n} dx\right) = -\frac{2}{\pi} \times \frac{1}{n^2} \times \cos nx\Big]_0^{\pi} = -\frac{2}{n^2\pi}((-1)^n - 1)$

$\begin{cases} \frac{4}{n^2\pi} & n \in \text{odd} \\ 0 & n \in \text{even} \end{cases}$

<u>**Quiz 02 (60 mins.)**</u>

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Department: 資工系 **Name:** [illegible] **ID:** 308304000 7

Problem 1: (30 points)

The complex number z_1 and z_2 are given by $z_1 = p + 2i$ and $z_2 = 1 - 2i$ where p is an integer.

(1) Find $\frac{z_1}{z_2}$ in the form $a + bi$ where a and b are both real. Give your answer in its simplest form in terms of p.

(2) Given that $\left|\frac{z_1}{z_2}\right| = 13$. Please find the possible value of p.

Answer:

(1) $\frac{z_1}{z_2} = \frac{p+2i}{1-2i} = \frac{(p+2i)(1+2i)}{(1-2i)(1+2i)} = \frac{p+2i+2pi-4}{5} = \frac{(p-4)+(2p+2)i}{5} = \frac{p-4}{5} + \frac{2(p+1)}{5}i$ #

(2) $\frac{p^2-8p+16}{25} + \frac{4p^2+8p+4}{25} = \frac{5p^2+20}{25} + \frac{p^2+4}{5} = 169$

$p^2+4 = 845$

$p^2 = 841 \Rightarrow p = \pm 29$ #

Problem 2: (30 points)

In the first quadrant, there is a vector at a 45-degree angle to the origin and the vector length is 1.

(1) Please define the vector on a complex plane.

(2) If we rotate this vector eight times counterclockwise and each rotation angle is 45 degree as well, please find the final length of this vector.

Answer:

(1) $(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ #

(2)

A: 1 #

Quiz 02 (60 mins.)

Problem 3: (20 points)

In the practical engineering way, we can use Taylor series to approximate arbitrary functions. Please consider a function $f(x) = 1 + x + x^2$ and determine the Taylor polynomial with zero approximation error when x is equal to a.

Answer:

$f'(x) = 1+2x \Rightarrow f'(x-a) = 1+2(x-a)$

$f''(x) = 2$

$f'''(x) = 0$

$\therefore f(x) = f(x-a) + f'(x-a)x + f''(x-a) \cdot \frac{x^2}{2!}$

$= 1+x-a+(x-a)^2+x+2x(x-a)+x^2$

$= 1+(2x+1)(x-a)+(x-a)^2+x+x^2$ #

Problem 4: (20 points)

Find the Fourier Transform of $f(x) = e^{-|x+3|} - 2e^{-|x|}$.

《hint: $f(x) = e^{-a|x|}, a > 0$. Then, the fourier transform of $f(x)$ is $\frac{2a}{a^2 + w^2}$.》

Answer: $f(x) = e^{-i\omega(-3)} \cdot \frac{2}{1+\omega^2} - 2 \cdot \frac{2}{1+\omega^2}$

$= \frac{2e^{i\omega 3} - 4}{1+\omega^2}$ #

30

Quiz 02 (60 mins.)

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Problem 1: (30 points)

The complex Fourier series of the function, $f(t) = (\cos(5\pi t))^3$, is $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i5n\pi t}$. Please find the value of c_1.

《hint: $\cos(2\theta) = 2\cos^2\theta - 1$》

$[\cos(10\pi t)]^2 + 1 = \cos(20\pi t) + 2$

$\frac{2n\pi t}{T} = 5n\pi t$

$[\cos(5\pi t)]^4 \to 4[\cos(5\pi t)]^3 \cdot 5\pi \cdot [-\sin(5\pi t)]$

$T = \frac{2}{5}$

Answer:

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i5n\pi t}\, dt = \frac{f^{(n)}(t)}{n!}(t-a)^n$$

$$C_1 = \frac{5}{2}\int_{-1/5}^{1/5} [\cos(5\pi t)]^3 e^{-i5\pi t} dt = \frac{5}{2}\left[\int_{-1/5}^{1/5} [\cos(5\pi t)]^4 dt - \int_{-1/5}^{1/5} [\cos(5\pi t)]^3 \cdot (-i\sin(5\pi t))\right]$$

$$= \frac{5}{2}\left[\int_{-1/5}^{1/5} \cos(20\pi t) dt + \int_{-1/5}^{1/5} 2\, dt + \frac{i}{20\pi}\cdot [\cos(5\pi t)]^4 \Big|_{-1/5}^{1/5}\right]$$

$$= \frac{5}{2}\left[-\sin(4\pi) + \sin(-4\pi) + \frac{2}{5} + \frac{2}{5} + \frac{i}{20\pi}\cdot[(-1)^4 - (-1)^4]\right]$$

$$= 2 \ \#$$

Problem 2: (30 points)

30

The definition of Binomial Series is

$$(1+x)^p = 1 + C_1^p x + C_2^p x^2 + C_3^p x^3 + \ldots + C_p^p x^p, \ \forall p \in R \text{ and } |x| < 1$$

(1) Please apply the Maclaurin Series property to prove the correctness of Binomial Series.

(2) Please find the Maclaurin Series of $(1-x)^{-2}$ by using the result of (1).

Answer:

$f(x) = (1+x)^p$

$f'(x) = p(1+x)^{p-1}$

$f''(x) = p(p-1)(1+x)^{p-2}$

$\vdots$

(1)

$$f(x) = \frac{f^{(n)}(a)}{n!}\cdot (x-a)^n \xRightarrow{a=0} \frac{f^{(n)}(x)}{n!}\cdot x^n$$

$$(1+x)^p = \frac{1^p}{0!} + \frac{p(1+x)^{p-1}}{1!}x^1 + \frac{p(p-1)(1+x)^{p-2}}{2!}x^2 + \cdots$$

$$\because |x| < 1 \ \therefore (1+x)^p = \frac{1}{0!} + \frac{p}{1!}x^1 + \frac{p(p-1)}{2!}x^2 + \cdots$$

$$= 1 + C_1^p x + C_2^p x^2 + C_3^p x^3 + \cdots + C_p^p x^p$$

(2)

$$(1-x)^{-2} = 1 - C_1^{-2} x + C_2^{-2} x^2 - C_3^{-2} x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} C_n^{-2} x^n \cdot (-1)^n \quad \#$$

Problem 3: (20 points) $\int u\,dv = uv - \int v\,du$

You can expand the function defined by $f(x) = x^2 + 3,\ 0 < x < 3$ in a Fourier series, a cosine series, or a sine series. Please choose the correct answers.

(a) $f(6) = 3$ for Fourier sine series; (b) $f(3) = 12$ for Fourier cosine series; (c) $f(0) = 3$ for Fourier series; (d) $f(-1) = 4$ for Fourier sine series; (e) $f(-3) = 12$ for Fourier cosine series

$\frac{1}{6}\left[\frac{2}{648}+1\right] = \frac{1}{6}\times\frac{325}{324} =$

Answer:

$f(x) = x^2+3,\ 0<x<3$

$T=6$

cosine: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3}$

$a_0 = \frac{1}{6}\int_{-1/6}^{1/6}(x^2+3)dx = \frac{1}{6}\left[\frac{x^3}{3}\Big|_{-1/6}^{1/6} + 3x\Big|_{-1/6}^{1/6}\right] =$

$a_n = \frac{1}{3}\int_{-1/6}^{1/6}(x^2+3)\cos\frac{n\pi x}{3}dx = \frac{1}{3}\left[\left(\frac{+3}{n\pi}x^2\sin\frac{n\pi x}{3}\right)\Big|_{-1/6}^{1/6} - \int_{-1/6}^{1/6}\frac{+6}{n\pi}x\cdot\sin\frac{n\pi x}{3}dx + 3\int_{-1/6}^{1/6}\cos\frac{n\pi x}{3}\right.$

$= \frac{1}{3}\left(\frac{+3}{n\pi}\cdot\frac{1}{36}\cdot\sin\frac{n\pi}{18}\times 2 + \frac{18}{(n\pi)^2}\int_{-1/6}^{1/6}\cos\frac{n\pi x}{3}dx +\right.$

0

sine: $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3}$

$b_n = \frac{1}{3}\int_{-1/6}^{1/6}(x^2+3)\sin\frac{n\pi x}{3}dx$

$= \frac{1}{3}\left[\left(\frac{-3}{n\pi}\cdot x^2\cdot\cos\frac{n\pi x}{3}\right)\Big|_{-1/6}^{1/6} - \int_{-1/6}^{1/6}\frac{-6}{n\pi}\cdot x\cdot\cos\frac{n\pi x}{3}dx + 3\int_{-1/6}^{1/6}\sin\frac{n\pi x}{3}dx\right]$

$0 \quad 0$

$= \frac{1}{3}\left[0 - \left(\frac{-18}{(n\pi)^2}x\cdot\sin\frac{n\pi x}{3}\right)\Big|_{-1/6}^{1/6} + \int_{-1/6}^{1/6}\frac{-18}{n\pi}\sin\frac{n\pi x}{3}dx\right]$

$= \frac{6}{(n\pi)^2}\left[\sin\frac{n\pi}{18} + \sin\frac{n\pi}{18}\right]$

Problem 4: (20 points)

You can expand the function defined by $f(x) = x,\ 0 < x < \pi$ in a Fourier sine series and Fourier cosine series. Give a set of criteria, and there by judge which series is the better. Please explain.

Answer: <hint: smaller approximation error>

$\sum_{n=1}^{\infty}\left(\frac{-48}{n\pi}(-1)^n + \frac{36(-1)^n}{(n\pi)^2}\right)\sin\frac{n\pi}{3}x$