Solution

109-2

Homework 01

(due day in two weeks, 4/7)

Problem 1: (20 points)

Please determine whether the following functions belong to periodic function. If yes, please fine the period.

(1) $\sin\frac{n\pi x}{l}$ (2) $\cos\frac{n\pi x}{l}$ (3) $f(x)=a_0\cdot e^{-x}$

(4) $f(x)=a_0+a_1\cos\frac{\pi x}{l}+b_1\sin\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}+b_2\sin\frac{2\pi x}{l}+\ldots+a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}+\ldots$

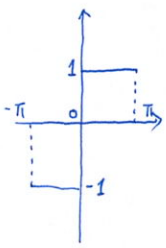
(1)(2) $\frac{2\pi}{n\pi/l}=\frac{2l}{n}$

(3) f(x) is not a periodic function

(4) $T=2l$

Problem 2: (10 points)

If $f(x+2\pi)=f(x)$ and $f(x)=\begin{cases}-1, -\pi\le x<0\\ 1, 0<x<\pi\end{cases}$, please find f(x)'s Fourier Series.



$T=2\pi$

$f(x)\in$ odd

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin nx\,dx=\frac{-2}{n\pi}\cos nx\Big|_0^{\pi}=\frac{2}{n\pi}(1-(-1)^n)=\begin{cases}0, & n\in\text{even}\\ \frac{4}{n\pi}, & n\in\text{odd}\end{cases}$$

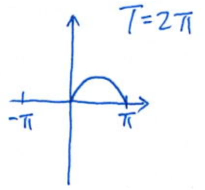
$$\therefore f(x)=\sum_{\substack{n=1\\ n\in\text{odd}}}^{\infty}\frac{4}{n\pi}\sin nx$$ #

or

$$\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin(2n-1)x$$ #

Problem 3: 40 points)

Given the function $f(x)=\begin{cases}0, -\pi<x<0\\ \sin x, 0\le x<\pi\end{cases}$



(1) Find its Fourier series

(2) Show that $\frac{1}{2}+\frac{1}{1\times3}-\frac{1}{3\times5}+\frac{1}{5\times7}-\frac{1}{7\times9}+\ldots=\frac{\pi}{4}$

(1) $f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{2\pi}\int_0^{\pi}\sin x\,dx=\frac{1}{\pi}$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$

$=\frac{1}{2\pi}\int_0^{\pi}[\sin(1+n)x+\sin(1-n)x]\,dx$

$=\frac{1}{2\pi}\left[\frac{-1}{1+n}\cos(1+n)x\Big|_0^{\pi}+\frac{-1}{1-n}\cos(1-n)x\Big|_0^{\pi}\right]$

$=\frac{1}{2\pi}\left[\frac{-1}{1+n}(\underbrace{\cos(1+n)\pi}_{=(-1)^{1+n}}-1)+\frac{-1}{1-n}(\underbrace{\cos(1-n)\pi}_{=(-1)^{1-n}}-1)\right]$

$=\frac{1}{2\pi}\frac{2}{1-n^2}(1-(-1)^{n+1}),\ n\neq1$

$\therefore a_n=\begin{cases}\frac{2}{\pi(1-n^2)}, & n\in\text{even}\\ 0, & n\in\text{odd}\end{cases}$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi}\sin x\sin nx\,dx$

$=\frac{1}{\pi}\left\{\frac{1}{2}\left[\underbrace{\int_0^{\pi}\cos(1-n)x\,dx}_{①}-\underbrace{\int_0^{\pi}\cos(1+n)x\,dx}_{②}\right]\right.$

$n=1$:

第②項 $=\int_0^{\pi}\cos2x\,dx=0$

第①項: $\int_0^{\pi}1\,dx=\pi$

$n\neq1$:

$\frac{1}{1-n}\overset{0}{\sin(1-n)x}\Big|_0^{\pi}-\frac{1}{1+n}\overset{0}{\sin(1+n)x}\Big|_0^{\pi}=0$

$=\begin{cases}\frac{1}{2}, & n=1\\ 0, & n\neq1\end{cases}$

$\therefore f(x)=\frac{1}{\pi}+\frac{1}{2}\sin nx+\sum_{\substack{n=2\\ n\in\text{even}}}^{\infty}\frac{2}{\pi(1-n^2)}\cos nx$ #

(2) 令 $x=\frac{\pi}{2}$

$\therefore f(x)=\frac{1}{\pi}+\frac{1}{2}+\frac{2}{\pi}\left[-\frac{1}{3}(-1)-\frac{1}{3\times5}(1)-\frac{1}{5\times7}(-1)-\cdots\right]=1$

$\Rightarrow \frac{1}{2}-\frac{1}{\pi}=\frac{2}{\pi}\left[\frac{1}{1\times3}-\frac{1}{3\times5}+\frac{1}{5\times7}-\cdots\right]$

$\Rightarrow \frac{\pi}{4}-\frac{1}{2}=\frac{1}{1\times3}-\frac{1}{3\times5}+\frac{1}{5\times7}-\cdots$

$\Rightarrow \frac{1}{2}+\frac{1}{1\times3}-\frac{1}{3\times5}+\frac{1}{5\times7}-\cdots=\frac{\pi}{4}$ #

Problem 4: (30 points)

Please find the Fourier Sine series expansion of *Cos x* when $0 \leq x \leq \pi$

$\because$ Fourier Sine series $\rightarrow$ 令 $f(x)=\cos x$ 為 odd function，則 $T=2\ell=2\pi$

$$\therefore f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{\ell}x$$

$$b_n=\frac{2}{2\pi}\int_{-\pi}^{\pi}\cos x\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\cos x\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\left[\frac{1}{2}\left(\sin(n+1)x+\sin(n-1)x\right)dx\right]$$

$$=\frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x+\sin(n-1)x\right]dx=\frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1}\Big|_0^{\pi}-\frac{\cos(n-1)x}{n-1}\Big|_0^{\pi}\right],\ n\neq 1$$

$$=\frac{1}{\pi}\left\{\left[-\frac{\cos(n+1)\pi}{n+1}-\frac{\cos(n-1)\pi}{n-1}\right]-\left[-\frac{\cos(n+1)0}{n+1}-\frac{\cos(n-1)0}{n-1}\right]\right\}$$

$$=\frac{1}{\pi}\left\{\left[-\frac{(-1)^{n+1}}{n+1}-\frac{(-1)^{n-1}}{n-1}\right]-\left[-\frac{1}{n+1}-\frac{1}{n-1}\right]\right\}$$

$$=\frac{1}{\pi}\left\{(-1)^n\left[-\frac{(-1)^{n+1}}{n+1}-\frac{(-1)^{n-1}}{n-1}\right]+\frac{n\cancel{-1}+n\cancel{+1}}{n^2-1}\right\}=\frac{1}{\pi}\left\{(-1)^n\left[\frac{1}{n+1}+\frac{1}{n-1}\right]+\frac{2n}{n^2-1}\right\}$$

$$=\frac{1}{\pi}\left\{(-1)^n\frac{2n}{n^2-1}+\frac{2n}{n^2-1}\right\}=\frac{1}{\pi}\cdot\frac{2n}{n^2-1}\left[(-1)^n+1\right],\ n\neq 1 \text{ 且 } n\in\text{even}$$

$n=1$:
$$b_1=\frac{1}{\pi}\int_0^{\pi}\left[\sin 2x+\sin 0x\right]dx=\frac{1}{\pi}\left[-\frac{\cos 2x}{2}\Big|_0^{\pi}\right]=0$$

$$\therefore f(x)=\sum_{\substack{n=2\\ n\in\text{even}}}^{\infty}\frac{2n}{\pi(n^2-1)}\left[(-1)^n+1\right]\sin nx$$

or

$$\sum_{n=1}^{\infty}\frac{2\cdot(2n)}{\pi\left[(2n)^2-1\right]}\left[(-1)^{2n}+1\right]\sin 2nx$$