Introduction to Probability Quiz 2012/5/14

請說明推導或運算過程

1. (10%) We consider n independent tosses of a coin, whose probability of heads, Y, is uniformly distributed in $(0, 1)$. What is the variance of the number of heads X?

2. (10%) Suppose $Y = 2X + 3$, where $X \sim$ **exponential**(1). What is $M_Y(s)$, the MGF of Y?

3. (10%) Suppose that the MGF of X is

$$M_X(s) = \frac{1}{2} \cdot \frac{1}{2-s} + \frac{3}{4} \cdot \frac{1}{1-s}.$$

   What is the PDF of X?

4. (20%) Jane visits bookstores, looking for *Great Expectations*. Any given bookstore carries the book with probability p, independent of the others. In a bookstore visited, Jane spends a random amount of time, exponentially distributed with parameter λ, until she finds the book or she makes sure that the bookstore does not carry it. Assume that Jane will keep visiting bookstores until she finds one. What is the mean, variance, and the PDF of total time Jane spends in bookstores?

5. (30%) The zero-mean bivariate normal PDF is of the form

$$f_{X,Y}(x, y) = ce^{-q(x,y)},$$

   where the exponent term $q(x, y)$ is a quadratic function of x and y,

$$q(x, y) = \frac{\frac{x^2}{\sigma_x^2} - 2\rho\frac{xy}{\sigma_x\sigma_y} + \frac{y^2}{\sigma_y^2}}{2(1-\rho^2)},$$

   σ_x and σ_y are positive constants, $-1 < \rho < 1$ is a constant, and c is a normalization constant.

   (a) By completing the square, re-write $q(x, y)$ in the form $(\alpha x - \beta y)^2 + \gamma y^2$, for some constants α, β, and γ.

   (b) Show that X and Y are zero-mean normal random variables with variance σ_x^2 and σ_y^2, respectively.

(c) Find the normalization constant c.

(d) Show that the conditional PDF of X given $Y = y$ is normal, and identify its conditional mean and variance.

(e) Show that the correlation coefficient of X and Y is ρ.

(f) Show that the estimation error $E[X|Y] - X$ is normal with mean zero and variance $(1-\rho^2)\sigma_x^2$, and is independent of Y.

6. (20%) A machine processes parts, one at a time. The processing times of different parts are i.i.d. random variables, uniformly distributed in $[1, 5]$ (time units). What is approximately the probability that the number of parts processed within 320 units, is at least 100?