Introduction to Probability Quiz 2012/4/9

1. (20%) A gambler makes a sequence of independent bets. In each bet, he wins \$1 with probability p and loses \$1 with probability $1 - p$. Initially, the gambler has \$k, and plays until he either accumulates \$n or has no money left. What is the probability that he ends up with \$n?

2. (10%) Let A and B be independent events. Prove that A and B^c are independent.

3. (10%) Let A, B and C be independent events, with $P(C) > 0$. Prove that A and B^c are conditionally independent given C.

4. (10%) An urn contains n balls, out of which m are red. We select k balls at random without replacement. What is the probability that i of the selected balls are red?

5. (10%) Alvin's database of friends contains n entries. Due to a software glitch, the addresses correspond to the names in a totally random fashion. Alvin writes a holiday card to each to his friends and sends it to the address. What is the probability that at least one of his friends receives the correct card?

6. (20%) A coin has probability of heads equal to p. It is tossed successively and independently until back-to-back heads or back-to-back tails appear. Find the expected value of the number of tosses using the total expectation theorem.

7. (10%) A smoker mathematician carries one matchbox in his left pocket and one in his right pocket. Initially, both boxes have n matches. Each time he wants to light a cigarette, he selects a matchbox in his left or right pocket with equal probability. What is the PMF of the number of remaining matches at the moment he reaches for a matchbox and finds it empty?

8. (10%) A die with r faces is rolled n times. In each roll, face i is up with probability p_i. Let X_i be the number of times that face i is up in n rolls. Find

   (a) the expected value and variance of X_i, and

   (b) $E[X_i X_j]$ for $i \neq j$.