Homework 03

(due day in two weeks, 5/19)

Problem 1: (30 points)

(1) If *f(x)* is an even function, find the Fourier transform of *f(x)*, $F(\alpha) = ?$

(2) Based on (1), if $\int_0^\infty f(x)\cos\alpha x dx = \begin{cases} 1-\alpha, 0 \le \alpha \le 1 \\ 0, \alpha > 1 \end{cases}$, find *f(x)* = ?

Answer

(1) $f(x) \in$ even function.

$$F(\alpha) = F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

$$= \int_{-\infty}^{\infty} f(x)(\cos\alpha x - i\sin\alpha x)dx$$

$$= 2\int_0^\infty f(x)\cos\alpha x dx, \quad F(\alpha) \in \text{even}$$
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(2) $F(\alpha) = \begin{cases} 2(1-\alpha), & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$

$$f(x) = \frac{1}{\pi}\int_0^1 2(1-\alpha)\cos\alpha x \, d\alpha$$

$$= \frac{2}{\pi}\cdot\frac{1}{x}(1-\alpha)\sin\alpha x\Big|_0^1 + \frac{2}{\pi x}\int_0^1 \sin\alpha x \, d\alpha$$

$$= \frac{2}{\pi x^2}[1-\cos x] = \frac{2}{\pi x^2}\cdot 2\sin^2\frac{x}{2}$$

$$= \frac{4}{\pi x^2}\sin^2\frac{x}{2}$$
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Problem 2: (30 points)

One function *f(x)*'s Fourier Transform is defined as $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$. Please find the Fourier Transform of $f(t)\sin\omega_0 t$.

$$f(t) \xrightarrow{F} F(\omega)$$

$$f(t)e^{i\omega_0 t} \xrightarrow{F} F(\omega-\omega_0)$$

$$\therefore F[f(t)\sin\omega_0 t] = F[f(t)\frac{1}{2i}(e^{i\omega_0 t} - e^{-i\omega_0 t})] = \frac{1\cdot i}{2i\cdot i}(F(\omega-\omega_0) - F(\omega+\omega_0)]$$

$$= \frac{-i}{2}[F(\omega-\omega_0) - F(\omega+\omega_0)]$$

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Problem 3: (20 points)

Find the Fourier Transform of $f(x) = e^{-|x+3|} - 2e^{-|x|}$.

Answer

$$\mathcal{F}[e^{-|x+3|} - 2e^{-|x|}] = \mathcal{F}[e^{-|x+3|}] - 2\,\mathcal{F}[e^{-|x|}]$$

$$= e^{i3w}\frac{2}{1+w^2} - 2\cdot\frac{2}{1+w^2}$$

$$= e^{i3w}\frac{2}{1+w^2} - \frac{4}{1+w^2}$$

Problem 4: (20 points)

Using the Fourier transform to solve the following differential equation.

(1) $y''(x) + 4y'(x) + 3y(x) = 3\delta(x)$

(2) $y''(x) + 4y'(x) + 3y(x) = 3\delta(x-3)$

Answer:

(1) $\mathcal{F}\{y'' + 4y' + 3y = 3\delta(x)\}$

$$(i\omega)^2 Y(\omega) + 4(i\omega)Y(\omega) + 3Y(\omega) = 3$$

$$\therefore \left((i\omega)^2 + 4(i\omega) + 3\right)Y(\omega) = 3$$

$$Y(\omega) = \frac{3}{(i\omega)^2 + 4(i\omega) + 3} = \frac{3}{(i\omega+1)(i\omega+3)}$$

$$= \frac{3/2}{1+i\omega} + \frac{-3/2}{3+i\omega}$$

$$\therefore y(x) = \frac{3}{2}e^{-x}H(x) - \frac{3}{2}e^{-3x}H(x)$$

(2) $\mathcal{F}\{y'' + 4y' + 3y = 3\delta(x-3)\}$

$$\Rightarrow (i\omega)^2 Y(\omega) + 4(i\omega)Y(\omega) + 3Y(\omega) = 3\cdot 1\cdot e^{-i3\omega}$$

$$\Rightarrow \left[(i\omega)^2 + 4(i\omega) + 3\right]Y(\omega) = 3\cdot e^{-i3\omega}$$

$$Y(\omega) = \frac{3}{(i\omega)^2 + 4(i\omega) + 3}e^{-i3\omega}$$

$$= \left(\frac{3/2}{1+i\omega} + \frac{-3/2}{3+i\omega}\right)e^{-i3\omega}$$

$$\therefore y(x) = \left(\frac{3}{2}e^{-(x-3)}H(x-3) - \frac{3}{2}e^{-3(x-3)}H(x-3)\right)$$