Quiz 2019.05.01

1. (20%) Two balls are drawn without replacement from an urn with 2 white balls and 3 black balls. If the drawn balls are of different colors, a fair coin is flipped once. If they are of the same color, a fair coin is flipped twice.

   - Find the mean of the number of heads in the coin flips.
   - Find the variance of the number of heads in the coin flips.

2. (20%) The number of queries arriving at a call center per minute is Poisson random variable with mean $\frac{3}{2}$.

   - Find the probability of fewer than 2 queries in a minute.
   - Find the expected waiting time for the first call-in query.

3. (20%) The CDF of a random variable is assumed to be

$$F_X(t) = \frac{1}{1+e^{-at}},\ t \in \mathbb{R}$$

   where $a > 0$ controls the functional behavior near $t = 0$.

   - Find the corresponding PDF.
   - Which CDF corresponds to a larger variance, $a = \frac{1}{2}$ or $a = 2$? Explain.

4. (20%) Suppose $X \sim \mathcal{N}(0,1)$ and

$$g(t) = \mathbf{P}(X \le 2t),\ t \in \mathbb{R}$$

   Find $g(1)$ and $g'(1)$.

5. (20%) Find the mean and variance of H with PDF

$$f_H(t) = e^{-2|t+1|},\ t \in \mathbb{R}$$

$$X \sim \textbf{Poisson}(\lambda),\ \ p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!},\ k = 0, 1, \ldots \quad (E[X] = \lambda, \text{var}(X) = \lambda)$$

$$X \sim \textbf{exponential}(\lambda),\ \ f_X(x) = \lambda e^{-\lambda x},\ x \ge 0 \quad \left(E[X] = \frac{1}{\lambda}, \text{var}(X) = \frac{1}{\lambda^2}\right)$$

$$X \sim \mathcal{N}(\mu, \sigma^2),\ \ f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ x \in \mathbb{R} \quad \left(E[X] = \mu, \text{var}(X) = \sigma^2\right)$$

Answers must be derived, computed, or properly explained.