Quiz 04 (60 mins.)

Department:___________ Name:____________ ID:____________

Problem 1: (30 points)

Using the Fourier transform to solve the following differential equation.

(1) $y''(x)+4y'(x)+3y(x)=3\delta(x)$

(2) $y''(x)+4y'(x)+3y(x)=3\delta(x-3)$

Answer

(1) $\mathcal{F}\{y''+4y'+3y=3\delta(x)\}$

$(i\omega)^2Y(\omega)+4(i\omega)Y(\omega)+3Y(\omega)=3$

$\therefore\ ((i\omega)^2+4(i\omega)+3)\,Y(\omega)=3$

$Y(\omega)=\dfrac{3}{(i\omega)^2+4(i\omega)+3}=\dfrac{3}{(i\omega+1)(i\omega+3)}$

$=\dfrac{3/2}{1+i\omega}+\dfrac{-3/2}{3+i\omega}$

$\therefore\ y(x)=\frac{3}{2}e^{-x}H(x)-\frac{3}{2}e^{-3x}H(x)$

(2) $\mathcal{F}\{y''+4y'+3y=3\delta(x-3)\}$

$\Rightarrow (i\omega)^2Y(\omega)+4(i\omega)Y(\omega)+3Y(\omega)=3\cdot 1\cdot e^{-i3\omega}$

$\Rightarrow [(i\omega)^2+4(i\omega)+3]\,Y(\omega)=3\cdot e^{-i3\omega}$

$Y(\omega)=\dfrac{3}{(i\omega)^2+4(i\omega)+3}e^{-i3\omega}$

$=\left(\dfrac{3/2}{1+i\omega}+\dfrac{-3/2}{3+i\omega}\right)e^{-i3\omega}$

$\therefore\ y(x)=\left(\frac{3}{2}e^{-(x-3)}H(x-3)-\frac{3}{2}e^{-3(x-3)}H(x-3)\right)$ #

Problem 2: (30 points)

One function *f(x)*'s Fourier Transform is defined as $F(\omega)=\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt$. Please find the Fourier Transform of $f(t)\sin\omega_0 t$.

Answer:

$f(t)\xrightarrow{\mathcal{F}}F(\omega)$

$f(t)e^{i\omega_0 t}\xrightarrow{\mathcal{F}}F(\omega-\omega_0)$

$\therefore\ \mathcal{F}[f(t)\sin\omega_0 t]=\mathcal{F}[f(t)\frac{1}{2i}(e^{i\omega_0 t}-e^{-i\omega_0 t})]=\frac{1\cdot i}{2i\cdot i}(F(\omega-\omega_0)-F(\omega+\omega_0))$

$=\frac{-i}{2}[F(\omega-\omega_0)-F(\omega+\omega_0)]$

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Problem 3: (20 points)

You decide to transfer one signal *s(x)* (*i.e.*, $s(x)=e^{-2|x|}$)and make the *f(x)* $= \cos 3X$ as the carrier wave. Based on the modulation technique, you can create one new signal *r(x)* containing the *s(x)* and *f(x)*. Please describe

Answer

$S(x)=e^{-2|x|} \xrightarrow{\mathcal{F}} S(\omega)=\frac{4}{4+\omega^2}$

$f(x)=\cos 3x \xrightarrow{\mathcal{F}} F(\omega)=\pi[\delta(\omega-3)+\delta(\omega+3)]$

f(x) = cos 3x
未給，有寫出Modulation算式的即給分

$\mathcal{F}[S(x)f(x)]=\frac{1}{2\pi}S(\omega)*F(\omega)=\frac{1}{2\pi}\left[\frac{4}{4+\omega^2}*(\pi(\delta(\omega-3)+\delta(\omega+3)))\right]$

$=\frac{1}{2}\left[\frac{4}{4+\omega^2}*\delta(\omega-3)+\frac{4}{4+\omega^2}*\delta(\omega+3)\right]$ ..... Note.

$=\frac{2}{4+(\omega-3)^2}+\frac{2}{4+(\omega+3)^2}$ 解

Note: $f(x)*\delta(x)=f(x) \Rightarrow f(x)*\delta(x-a)=f(x-a)$

pf. $\mathcal{F}^{-1}(\mathcal{F}[f(x)*\delta(x)]=\mathcal{F}[f(x)]\cdot\mathcal{F}[\overset{=1}{\delta(x)}]=\mathcal{F}[f(x)]) \Rightarrow f(x)*\delta(x$

$\mathcal{F}^{-1}(\mathcal{F}[f(x)*\delta(x-a)]=\mathcal{F}[f(x)]\mathcal{F}[\delta(x-a)]=\mathcal{F}[f(x)]\cdot e^{i\omega a})$

$\Rightarrow f(x)*\delta(x-a)=f(x-a)$ Q.E.D

Problem 4: (20 points)

Please use the following property $F(\omega)=\Im[f(x)]=\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$ to find the Fourier Transform of

$f(t)=4e^{-3t^2}\sin(2t)$

Answer

$\because \mathcal{F}[4e^{-3t^2}]=4\sqrt{\frac{\pi}{3}}e^{-\frac{\omega^2}{4\cdot3}}=4\sqrt{\frac{\pi}{3}}e^{-\frac{\omega^2}{12}}$, $\mathcal{F}[\sin 2t]=\mathcal{F}\left[\frac{1}{2i}(e^{i2t}-e^{-i2t})\right]=\frac{1}{2i}[2\pi\delta(\omega-2)-2\pi\delta(\omega+2)]$

$=\frac{\pi}{i}[\delta(\omega-2)-\delta(\omega+2)]$

Adopting the Concept of modulation

$\mathcal{F}[h(t)\cdot g(t)]=\frac{1}{2\pi}H(\omega)*G(\omega)$

$\therefore \mathcal{F}[f(x)]=F(\omega)=\mathcal{F}[4e^{-3t^2}\sin 2t]=\frac{1}{2\pi}\left[4\sqrt{\frac{\pi}{3}}e^{-\frac{\omega^2}{12}}*\frac{\pi}{i}(\delta(\omega-2)-\delta(\omega+2))\right]$

$=\frac{2}{i}\sqrt{\frac{\pi}{3}}\left(e^{-\frac{(\omega-2)^2}{12}}-e^{-\frac{(\omega+2)^2}{12}}\right)$ 解

$\because f(x)*\delta(x)=f(x)$

$f(x)*\delta(x-a)=f(x-a)$