機率學期中考 2015.05.20

1. We load a plane 100 packages whose weights are independent random variables that are exponentially distributed with mean 25 pounds. What is the probability that the total weight will exceed 2800 pounds? Express your answer in terms of the CDF of a standard Gaussian. (hint: decide the mean and variance of an exponential random variable, and apply the central limit theorem.)

2. A machine processes parts, one part at a time. The processing times of different parts are independent Poisson random variables, each with parameter $\lambda = 3$. Approximate the probability that the number of parts processed within 320 time units, denoted by N_{320}, is at least 100. (Express your answer in terms of the CDF of a standard Gaussian.)

3. Consider an i.i.d. sequence of random variables $\{X_n\} = X_1, X_2, \ldots$ that are uniformly distributed in the interval $[0, 1]$, and let
$$\{Z_n\} = \{\max(X_1, \ldots, X_n)\}.$$
Show that
$$Z_n \xrightarrow{\mathbf{P}} 1$$
and
$$Z_n \xrightarrow{\text{a.s.}} 1.$$

4. A remote village has 2 gas stations. Each gas station is open on any given day with probability 2/3, independent of the other. The amount of gas available in each station is uniformly distributed between 0 and 500 gallons. What is the transform associated with the total amount of gas available at the gas stations that are open?

5. Jane visits bookstores, looking for *Great Expectations* until she finds it. Any given bookstore carries the book with probability $1/3$, independent of the others. In a bookstore visited, Jane spends a random amount of time, geometrically distributed with parameter $2/3$, until she either finds the book or she determines that the bookstore does not carry it. What are the variance and the PDF of the total time spent in bookstores.

6. We are told that the transform associated with a random variable X is
$$M_X(s) = \frac{0.5e^s}{1 - 0.4e^s}.$$
Show that this is not possible.

7. Suppose

$$M_X(s) = \exp\left\{\frac{3s^2}{2} + 5s\right\}.$$

What is $E[X]$ and $\mathrm{var}(X)$?

8. We consider n independent tosses of a biased coin whose probability of heads Y is uniformly distributed over the interval $[0, 1]$. What is the expectation and variance of X, the number of heads obtained?

9. Consider the hat problem discussed earlier, where 52 people throw their hats in a box and then pick a hat at random. What are the expectation and the variance of the number of people who pick their own hat?

10. The **covariance** of two random variables X and Y is defined by

$$\mathrm{cov}(X, Y) \equiv E[(X - E[X])(Y - E[Y])].$$

Show that

$$\mathrm{cov}(X, Y) = E[XY] - E[X]E[Y].$$

11. Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a random delay (X for Juliet, Y for Romeo) that is uniformly and independently distributed in $[0, 1]$. What is the PDF of the difference $Z = X - Y$ between their times of arrival?