Midterm 2019.04.10

1. (10%) Suppose there are 7 white balls and 3 black balls in an urn. 3 balls are drawn without replacement. Find the probability that the last drawn ball is black by the total probability theorem.

2. (20%) There are 3 coins. If coin 1 is flipped, the probability of head is 0.3. If coin 2 is flipped, the probability of head is 0.5. If coin 3 is flipped, the probability of head is 0.3. A coin is chosen with equal probability and flipped. If it is head, the next coin (in cyclic order) is flipped twice. If it is tail, the other coin is flipped twice.

   - Find the mean of the number of heads in 3 flips.
   - Find the variance of the number of heads in 3 flips.

3. (20%) The number of queries arriving at a call center per minute is Poisson random variable with mean 2.

   - Find the probability of fewer than 2 queries in a minute.
   - Find the expected waiting time for the first call-in query.

4. (20%) A sigmoid function is defined by

$$\sigma(x) = \frac{1}{1+e^{-x}},\ x \in \mathbb{R}$$

   - Why is it a valid CDF?
   - What is the corresponding PDF?

5. (20%) Suppose X is a standard normal random variable and

$$g(x) = \mathbf{P}(X \le x^2),\ x \in \mathbb{R}$$

   - Find $\min g(x)$.
   - Find $g'(1)$.

6. (10%) Find the variance of H with PDF

$$f_H(h) = e^{-2|h|},\ h \in \mathbb{R}$$

$$X \sim \textbf{Poisson}(\lambda),\ \ p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!},\ k = 0, 1, \ldots \quad (E[X] = \lambda, \text{var}(X) = \lambda)$$

$$X \sim \textbf{exponential}(\lambda),\ \ f_X(x) = \lambda e^{-\lambda x},\ x \ge 0 \quad \left(E[X] = \frac{1}{\lambda}, \text{var}(X) = \frac{1}{\lambda^2}\right)$$

$$X \sim \mathcal{N}(\mu, \sigma^2),\ \ f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ x \in \mathbb{R} \quad \left(E[X] = \mu, \text{var}(X) = \sigma^2\right)$$

Answers must be derived, computed, or properly explained.