Solution

109-2

Quiz 02 (60 mins.)

Department:___________ **Name:**____________ **ID:**____________

Problem 1: (30 points)

〈送分〉

Please use Fourier Integral representation to show that $\int_0^\infty \frac{\cos(\pi\omega/2)\cos \cancel{\pi\omega}\ \omega x}{1-\omega^2} = \begin{cases} \cancel{\frac{\pi}{2}}\ \frac{\pi}{2}\cos x & ,|x|<\frac{\pi}{2} \\ 0 & ,|x|>\frac{\pi}{2} \end{cases}$

分析: ∵有 $\cos wx$ ∴為 even function

兩邊同乘 $\frac{2}{\pi}$ 可知該式為 $\cos x$ ∴ $f(x)=\cos x$ 且 $|x|<\frac{\pi}{2}$

Answer:

$$f(x)=\frac{2}{\pi}\int_0^\infty A_1(w)\cos wx\,dw$$

$$A_1(w)=\int_0^\infty f(x)\cos wx\,dx=\int_0^{\frac{\pi}{2}}\cos x\cos wx\,dx$$

$$=\int_0^{\frac{\pi}{2}}\frac{1}{2}[\cos(1+w)x+\cos(1-w)x]dx$$

$$=\frac{1}{2}\left[\left(\frac{1}{1+w}\sin(1+w)x\Big|_0^{\frac{\pi}{2}}\right)+\left(\frac{1}{1-w}\sin(1-w)x\Big|_0^{\frac{\pi}{2}}\right)\right]$$

$$=\frac{1}{2}\left[\frac{1}{1+w}\sin(1+w)\cdot\frac{\pi}{2}+\frac{1}{1-w}\sin(1-w)\cdot\frac{\pi}{2}\right]$$

$$=\frac{\cos(\frac{\pi w}{2})}{1-w^2}$$

$$\therefore f(x)=\frac{2}{\pi}\int_0^\infty \frac{\cos(\frac{\pi w}{2})}{1-w^2}\cos wx\,dw=\begin{cases}\cos x, & |x|<\frac{\pi}{2}\\ 0, & |x|>\frac{\pi}{2}\end{cases}$$

$$\therefore \int_0^\infty \frac{\cos(\frac{\pi w}{2})\cos wx}{1-w^2}dw=\begin{cases}\frac{\pi}{2}\cos x, & |x|<\frac{\pi}{2}\\ 0, & |x|>\frac{\pi}{2}\end{cases}$$

Q.E.D.

Problem 2: (30 points)

In the first quadrant, there is a vector at a 45-degree angle to the origin and the vector length is 1.

(1) Please define the vector on a complex plane.

(2) If we rotate this vector eight times counterclockwise and each rotation angle is 45 degree as well, please find the final length of this vector.

Answer:

(1) $1+i$

(2) Rotate 8 times $\Rightarrow (1+i)^8$

$|1+i|=\sqrt{1^2+1^2}=\sqrt{2}$

$(\sqrt{2})^8=2^4=16$ #

Sol 2.

$$(1+i)^8=(\sqrt{2}\cdot(\cos 45^\circ+i\sin 45^\circ))^8$$

$$=(\sqrt{2})^8\cdot(\cos(\underbrace{45^\circ+45^\circ+\cdots+45^\circ}_{8\text{ times}})+i\sin(\underbrace{45^\circ+45^\circ+\cdots+45^\circ}_{8\text{ times}}))$$

$$=16(\cos 360^\circ+i\sin 360^\circ)$$

$$=16$$ #

<u>**Quiz 02 (60 mins.)**</u>

Problem 3: (20 points)

The complex number z_1 and z_2 are given by $z_1 = p + 2i$ and $z_2 = 1 - 2i$ where p is an integer.

(1) Find $\frac{z_1}{z_2}$ in the form $a + bi$ where a and b are both real. Give your answer in its simplest form in terms of p.

(2) Given that $\left|\frac{z_1}{z_2}\right| = 13$. Please find the possible value of p.

Answer:

(1) $\frac{z_1}{z_2} = \frac{p+2i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{p+2pi+2i-4}{1+4}$

$= \frac{p-4}{5} + \frac{2(p+1)i}{5}$ #

(2) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{p^2+4}}{\sqrt{1+4}} = 13$

$\Rightarrow p^2 = 841$

$\therefore p = \pm 29$ #

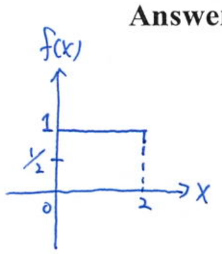
Problem 4: (20 points)

A function $f(x)$ is defined as $f(x) = \begin{cases} 1, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$.

(1) Please find the Fourier integral representation

(2) Please use the result in (1) to find $\int_0^\infty \frac{1}{\omega} \sin \omega \, d\omega$

Answer:



$f(x) = \frac{1}{\pi}\int_0^\infty A(\omega)\cos\omega x\, d\omega + \frac{1}{\pi}\int_0^\infty B(\omega)\sin\omega x\, d\omega$

(1) $A(\omega) = \int_{-\infty}^{\infty} f(x)\cos\omega x\, dx = \int_0^2 1\cdot\cos\omega x\, dx = \frac{1}{\omega}\sin 2\omega$

$B(\omega) = \int_{-\infty}^{\infty} f(x)\sin\omega x\, dx = \int_0^2 1\cdot\sin\omega x\, dx = \frac{1}{\omega}(1-\cos 2\omega)$

$\therefore f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{1}{\omega}\sin 2\omega\cos\omega x + \frac{1}{\omega}(1-\cos 2\omega)\sin\omega x\right] d\omega$

(2) 令 $x=0 \Rightarrow \frac{1}{2} = \frac{1}{\pi}\int_0^\infty \frac{1}{\omega}\sin 2\omega\, d\omega$

$\int_0^\infty \frac{1}{\omega}\sin 2\omega\, d\omega = \frac{\pi}{2}$

$\Rightarrow \int_0^\infty \frac{1}{2\omega}\sin 2\omega\, d2\omega = \frac{\pi}{2} \Rightarrow \int_0^\infty \frac{1}{\omega}\sin\omega\, d\omega = \frac{\pi}{2}$ #