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- **chapter 1**

- Decide appropriate sample spaces for 2 alternative games as follows, both involving 10 successive coin tosses.

  1. Game 1: We receive \$1 each time a head comes up.
  2. Game 2: We receive \$1 for each coin toss, up to and including the first time a head comes up. Then we receive \$2 for each coin toss, up to and including the second time a head comes up. More generally, the dollar amount per toss is doubled each time a head comes up.

- Consider the experiment of rolling a pair of 4-sided dice. We assume the dice are fair, meaning that each of the 16 possible outcomes has the same probability. Calculate the probability of the following events.

  – the sum of the rolls is even.
  – the sum of the rolls is odd.
  – the first roll is equal to the second.
  – the first roll is larger than the second.
  – at least one roll is equal to 4.

- Construct a probability model for an experiment involving a single toss of a fair coin. Then construct a probability model for an experiment involving 3 tosses of a fair coin.

- A **wheel of fortune** is continuously calibrated from 0 to 1. For an experiment involving a single spin, the possible outcomes are the numbers in the interval $\Omega = [0, 1]$. Suppose that the wheel is fair. What is the probability of a single number? What is the probability of an interval $[a, b] \subset [0, 1]$?

- Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays being equally likely. The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived. What is the probability that they will meet?

- We toss a fair coin 3 times. Define events

$$\begin{aligned} \mathcal{A} &= \{\text{more heads then tails come up}\}, \\ \mathcal{B} &= \{\text{1st toss is a head}\}. \end{aligned}$$

  Find the conditional probability $\mathbf{P}(\mathcal{A}|\mathcal{B})$.

- A fair 4-sided die is rolled twice. Let X and Y be the results of the first and the second roll, respectively. Define events

$$\begin{aligned} \mathcal{A}_m &= \{\max(X, Y) = m\}, \\ \mathcal{B} &= \{\min(X, Y) = 2\}. \end{aligned}$$

  What are the conditional probabilities of

$$\mathbf{P}(\mathcal{A}_m | \mathcal{B}), \quad m = 1, 2, 3, 4?$$

- A conservative design team C and an innovative design team N are asked to separately design a new product within a month. From past experience we know that

  - Team C is successful with probability 2/3.
  - Team N is successful with probability 1/2.
  - At least one team is successful with probability 3/4.

  Assuming that exactly one team is successful, what is the probability that it is team N?

- If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present, the radar generates a false alarm with probability 0.10. Suppose that an aircraft is present with probability 0.05. What is the probability of no aircraft presence and a false alarm? What is the probability of aircraft presence and no detection?

- 3 cards are drawn from an ordinary 52-card deck without replacement. We wish to find the probability that none of the 3 cards is a heart.

- A class consisting of 4 graduate and 12 undergraduate students is randomly divided into 4 groups of 4 students. What is the probability that each group includes a graduate student?

- You are told that a prize is equally likely to be found behind any one of 3 closed doors in front of you. You first choose one of the doors. A friend opens for you one of the remaining 2 doors, after making sure that the prize is not behind it. At this point, you can stick with the original choice, or switch to the other unopened door. What is the best strategy?

- In a chess tournament, your probability of winning the first game is 0.3 against half the players, 0.4 against a quarter of the players, and 0.5 against the remaining players. You play the first game against a randomly chosen opponent. What is the probability of winning?

- You roll a fair 4-sided die. If the result is 1 or 2, you roll once more, otherwise you stop. What is the probability that the sum total of your rolls is at least 4?

- Alice is taking a probability class. At the end of each week, she is either up-to-date or fallen-behind. If she is up-to-date in a week, she will be up-to-date the next week with probability 0.8. If she is fallen-behind in a week, she will be fallen-behind the next week with probability 0.6. What is the probability that she is up-to-date after 3 weeks?

- Use the same setting of Example 1.9. Given that an alarm has been generated by the radar, what is the probability that an aircraft is present?

- Use the same setting of Example 1.13. Given that you win the first game, what is the probability that the opponent is of type 1?

- A test for a certain rare disease is correct 95% of the time. A random person has a probability of 0.001 of having the disease. Given that the person just tested positive, what is the probability of actually having the disease?

- Consider 2 independent tosses of a fair coin. Define

$$\begin{aligned} \mathcal{H}_i &= \{\text{toss } i \text{ is a head}\}, \\ \mathcal{D} &= \{\text{the 2 tosses have different results}\}. \end{aligned}$$

  Verify that the events $\mathcal{H}_1, \mathcal{H}_2$ are independent, but not conditionally independent given $\mathcal{D}$.

- There are a blue coin and a red coin. The coins are biased. The probability of head is 0.99 for the blue coin and 0.01 for the red coin. We choose one coin at random with probability 1/2 for each coin, and proceed with 2 tosses of the chosen coin. Define

$$\begin{aligned} \mathcal{H}_i &= \{\text{toss } i \text{ is a head}\}, \\ \mathcal{B} &= \{\text{the blue coin is chosen}\}. \end{aligned}$$

  Verify that the events $\mathcal{H}_1, \mathcal{H}_2$ are not independent, but conditionally independent given $\mathcal{B}$.

- Consider 2 independent tosses of a fair coin. Define

$$\begin{aligned} \mathcal{H}_i &= \{\text{toss } i \text{ is a head}\}, \\ \mathcal{D} &= \{\text{the 2 tosses have different results}\}. \end{aligned}$$

  Show that the events $\mathcal{H}_1, \mathcal{H}_2, \mathcal{D}$ are pair-wise independent, but not independent.

- A **computer network** connects two nodes A and B through nodes C, D, E, F, as shown in Figure 1.15 (a). For every pair of directly connected nodes, say i and j, there is a given probability p_{ij} that the link from i to j is up. It is often assumed that the link failures are independent of each other. What is the probability that there is a path connecting A and B in which all links are up?

- An **internet service provider** has installed c modems to serve n dialup customers. It is estimated that at a given time, each customer will need a connection with probability p, independent of the others. What is the probability that there are more customers needing a connection than there are modems?

- A local telephone number is a 7-digit sequence, but the first digit has to be different from 0 or 1. How many distinct telephone numbers are there?

- Consider an n-element set $\mathcal{S} = \{s_1, \dots, s_n\}$. How many subsets does it have (including itself and the empty set)?

- How many letter sequences are there that consist of 4 distinct letters?

- You have n_1 classical music CDs, n_2 rock music CDs, and n_3 country music CDs. In how many different ways can you arrange them so that the CDs of the same type are contiguous?

- We have a group of n persons. Consider clubs that consist of a special person from the group (the club leader) and a number (possibly 0) of additional club members. How many distinct ways of composition for such a club are there? Consider two ways of counting: choose the members and then choose the leader vs. choose the leader first and then choose the additional members.

- How many different letter sequences can be obtained by rearranging the letters in the word TATTOO?

- A class consisting of 4 graduate and 12 undergraduate students is randomly divided into 4 groups of 4 students. What is the probability that each group includes a graduate student? Let solve this problem by counting.

- **chapter 2**

- Let $Y = |X|$. Find $p_Y(y)$ with

$$p_X(x) = \begin{cases} \frac{1}{9}, & x \text{ is an integer in } [-4, 4], \\ 0, & \text{otherwise.} \end{cases}$$

- Consider 2 independent coin tosses, each with a probability of $\frac{3}{4}$ for a head, and let X be the number of heads obtained. What are the PMF and the mean of X?

- The PMF of random variable X is as follows:

$$p_X(x) = \begin{cases} \frac{1}{9}, & x \text{ is an integer in } [-4, 4], \\ 0, & \text{otherwise}, \end{cases}$$

  Find the variance of X by finding the PMF of $(X - E[X])^2$ first.

- The PMF of random variable X is as follows:

$$p_X(x) = \begin{cases} \frac{1}{9}, & x \text{ is an integer in } [-4, 4], \\ 0, & \text{otherwise}, \end{cases}$$

  Find the variance of X *without* finding the PMF of $(X - E[X])^2$ first.

- If the weather is good (with probability 0.6), Alice walks the 2 miles to school at a speed V of 5 miles per hour. Otherwise, she rides her motorcycle at a speed of 30 miles per hour. What is the mean time T for Alice to get to school?

- Consider the Bernoulli random variable

$$X \sim \textbf{Bernoulli}(p).$$

  Find the mean and variance of X.

- What is the mean and the variance associated with a roll of 6-sided die?

- Find the mean and variance of the Poisson random variable Z with PMF

$$p_Z(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

- Consider a game where a person is given 2 questions and must decide which one to answer first.
  - Question A is answered correctly with probability 0.8, and the prize money is 100.
  - Question B is answered correctly with probability 0.5, and the prize money is 200.

If the first question attempted is answered incorrectly, the quiz terminates. If it is answered correctly, the person is allowed to attempt the second question. What question should be answered first to maximize the expected value of the total prize money received?

- A probability class has 300 students and each student has probability of 1/3 of getting an A, independent of any other student. What is the mean of X, the number of students that get an A?
- Suppose n people throw their hats in a box and then each picks one hat at random. What is the expected value of H, the number of people that get back their own hat?
- Let X be the roll of a fair 6-sided die. Let $\mathcal{A}$ be the event that the roll is an even number. What is the conditional PMF of X given $\mathcal{A}$?
- A student will take a certain test repeatedly, up to a maximum of n times, each time with a probability p of passing, independent of the number of previous attempts. What is the conditional PMF of the number of attempts K, given that the student passes the test?
- Professor May B. Right answers each of her students' questions incorrectly with probability 1/4, independent of other questions. In each lecture, she is asked 0, 1, or 2 questions with equal probability 1/3. Let X and Y be respectively the number of questions she is asked and the number of questions she answers wrong in a given lecture. Find $p_{XY}(x, y)$. What is the probability that she answers at least one question incorrectly?
- Consider a transmitter that is sending messages over a computer network. Let Y be the length of a message, and X be the travel time of the message. Suppose the PMF of Y is

$$\begin{aligned} p_Y(10^2) &= \tfrac{5}{6}, \\ p_Y(10^4) &= \tfrac{1}{6}. \end{aligned}$$

  X depends on Y and the congestion of the network. In particular,

$$\begin{aligned} X = 10^{-4}Y, &\quad \text{with probability } \tfrac{1}{2} \\ X = 10^{-3}Y, &\quad \text{with probability } \tfrac{1}{3} \\ X = 10^{-2}Y, &\quad \text{with probability } \tfrac{1}{6} \end{aligned}$$

  We want to know the PMF of X.

- Message transmitted by a computer in Boston through a data network is destined for New York with probability 0.5, for Chicago with probability 0.3, and for San Francisco with probability 0.2. The transmission time X is random. The mean transmission time is 0.05 seconds for a message destined for New York, 0.1 seconds for a message destined for Chicago, and 0.3 seconds for a message destined for San Francisco. What is $E[X]$?

- You write a program over and over, and each time there is a probability p that it works correctly, independent of previous attempts. What is the mean and variance of X, the number of tries until the program works correctly?

- Consider 2 independent tosses of a fair coin. Let X be the number of heads and $\mathcal{A}$ be the event that the number of heads is even. Show that X and $\mathcal{A}$ are not independent.

- Consider n independent tosses of a coin, whose head comes up in each toss with probability p. Let X_i be the Bernoulli random variable for toss i. Then

  $$X = X_1 + \cdots + X_n$$

  is a binomial random variable. The variance of X is thus

  $$\begin{aligned} \text{var}(X) &= \text{var}(X_1) + \cdots + \text{var}(X_n) \\ &= np(1-p). \end{aligned}$$

- We wish to estimate the **approval rate** of a president. We ask n persons at random from the voter population. Let X_i be defined by

  $$X_i = \begin{cases} 1, & \text{if the } i\text{th person approves the president} \\ 0, & \text{otherwise.} \end{cases}$$

  We average the responses of the n persons by

  $$S_n = \frac{X_1 + \cdots + X_n}{n}.$$

  Suppose $X_1, \ldots, X_n$ are independent Bernoulli random variables with mean p and variance $p(1-p)$. Find the mean and variance of S_n.

- The probability of a well-defined event, say $\mathcal{A}$, can be difficult to compute. Yet, whether $\mathcal{A}$ occurs in a trial is easy to decide. In this case, we can estimate the probability of $\mathcal{A}$ by **simulation**. That is, we run n trials, and record the number of trials with outcomes belonging to $\mathcal{A}$. Consider n independent Bernoulli random variables

$$X_i = \begin{cases} 1, & \text{if outcome } i \text{ is in } \mathcal{A} \\ 0, & \text{otherwise} \end{cases} \quad \Rightarrow \quad \begin{cases} p_{X_i}(1) = \mathbf{P}(\mathcal{A}), \\ p_{X_i}(0) = 1 - \mathbf{P}(\mathcal{A}). \end{cases}$$

Then

$$X = \frac{X_1 + \cdots + X_n}{n}$$

is a good estimate of $\mathbf{P}(\mathcal{A})$, since

$$E[X] = \mathbf{P}(\mathcal{A}), \quad \text{var}(X) = \frac{\mathbf{P}(\mathcal{A})(1 - \mathbf{P}(\mathcal{A}))}{n} \quad \xrightarrow{n\to\infty} \quad 0.$$

- **chapter 3**

- A gambler spins a **wheel of fortune**, which is continuously calibrated between 0 and 1, and observes the resulting number. Assuming that any two subintervals of $[0, 1]$ of the same length have the same probability, this experiment can be modelled in terms of a random variable X with PDF

$$f_X(x) = \begin{cases} c, & 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

The constant c can be decided through normalization

$$1 = \int f_X(x)dx = \int_0^1 c\,dx = c.$$

More generally, for a random variable X uniformly distributed in $[a, b]$,

$$f_X(x) = \frac{1}{b-a}, \quad a \le x \le b.$$

- Alvin's **driving time** to work is between 15 and 20 minutes in a sunny day, and between 20 and 25 minutes in a rainy day, with all times being equally likely in each case. Assume that a day is sunny with probability 2/3, and rainy with probability 1/3. What is the PDF of the driving time, viewed as a random variable X?

- Consider a random variable X with PDF

$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & 0 < x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Although $f_X(x)$ becomes arbitrarily large as x approaches 0,

$$\int f_X(x)dx = \int_0^1 \frac{1}{2\sqrt{x}}dx = 1,$$

so it is nonetheless a valid PDF.

- Consider a continuous random variable X uniformly distributed over the interval $[a, b]$. What is the expectation and variance of X?
- The time until a small **meteorite** first lands anywhere in the Sahara desert is modeled as an exponential random variable with a mean of 10 days. The time is currently midnight. What is the probability that a meteorite first lands sometime between 6 a.m. and 6 p.m. of the first day?
- You are allowed to take a certain test 3 times, and your final score will be the maximum of the test scores. Assume that your score in each test takes one of the values from 1 to 10 with equal probability 1/10, independently of the scores in other tests. What is the PMF of the final score?
- The yearly snowfall at Mountain Rainier is modeled as a normal random variable with a mean of $\mu = 60$ and a standard deviation of $\sigma = 20$. What is the probability that this year's snowfall will be at least 80 inches?
- A binary message is transmitted as a signal S, which is either -1 or $+1$. The communication channel corrupts the transmission with additive normal noise with mean $\mu = 0$ and variance σ^2. The receiver concludes that the signal -1 (or $+1$) was transmitted if the value received is < 0 (or ≥ 0). What is the probability of error?
- Consider the Romeo and Juliet example. What is the joint PDF for the random delays of Romeo and Juliet?
- Suppose that the joint PDF of the random variables X and Y is a constant c on the set S shown in Fig. 3.12 and is zero outside. Determine the value of c and the PDF of X.
- A surface is ruled with parallel lines, which are at distance d from each other. Suppose that we throw a needle of length l on the surface at random. We assume that $l < d$. What is the probability that the needle will intersect one of the lines?
- Let X and Y be described by a uniform PDF on the unit square. It follows that the joint CDF of X and Y is given by

$$F_{XY}(x, y) = xy, \quad 0 \leq x, y \leq 1.$$

- The time T until a new light bulb burns out is an exponential random variable with parameter λ. Alice turns the light on, leaves the room, and when she returns, t time units later, finds that the light bulb is still on (event $\mathcal{A}$). Let X be the additional time until the light bulb burns out. What is the conditional PDF of X?

- The metro train arrives at a station every quarter hour starting at 6 a.m. You walk into the station every morning between 7:10 and 7:30 a.m., and your arrival time is a uniform random variable over this interval. What is the PDF of the time you have to wait for the first train to arrive?

- Ben **throws a dart** at a circular target of radius r. Let the point of impact be (X, Y). We assume that he always hits the target, and that all points of impact are equally likely. What is the conditional PDF $f_{X|Y}(x|y)$?

- The speed of a typical vehicle that drives past a police radar is modeled as an exponential random variable X with mean 50 miles per hour. The police radar's measurement Y of the vehicle's speed has an error which is modeled as a normal random variable with zero mean and standard deviation equal to one tenth of the vehicle's speed. What is the joint PDF of X and Y?

- Let X be a continuous random variable with a piecewise constant PDF

$$f_X(x) = \begin{cases} 1/3, & 0 \le x \le 1, \\ 2/3, & 1 < x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

  Define the events

$$\begin{aligned} \mathcal{A}_1 &= \{X \text{ lies in the interval } [0,1]\}, \\ \mathcal{A}_2 &= \{X \text{ lies in the interval } [1,2]\}. \end{aligned}$$

  Find $E[X]$ and var(X) via the total expectation theorem.

- Let X and Y be independent normal random variables with means μ_x, μ_y and variances σ_x^2, σ_y^2, respectively. What is the joint PDF of X and Y?

- A light bulb is known to have an exponentially distributed lifetime Y. However, the manufacturing company is experiencing quality control problems, so the parameter Λ of the PDF of Y is random, uniformly distributed in the interval $[1, 3/2]$. We test a light bulb and record its lifetime y. What can we say about the parameter Λ?

- A binary signal S is transmitted, and we are given that $\mathbf{P}(S = 1) = p$ and $\mathbf{P}(S = -1) = 1 - p$. The received signal is $Y = S + N$, where N is a normal noise with zero mean and unit variance. What is the probability that $S = 1$, as a function of the observed value y of Y?

- **chapter 4**

- **chapter 5**

- **chapter 6**

- Consider a Bernoulli process.

  - Let U be the number of successes in trials 1 to 5, and V the number of successes in trials 6 to 10. Then, U and V are independent.
  - Let U (respectively, V) be the first odd (respectively, even) time in which we have a success. Then, U and V are independent.

- A computer executes 2 types of jobs, **priority** and **non-priority**, and operates in **slots**. A slot is **busy** if the computer executes a priority job within the slot, and is **idle** otherwise. We call a string of idle (or busy) slots, flanked by busy (or idle) slots, an idle (or busy) **period**.

  A priority job arrives with probability p at the beginning of each slot, independent of other slots, and requires one full slot to execute. A non-priority job is always available and is executed at a given slot if no priority job is available.

  Let us derive the PMF of the following random variables.

  1. T, the time index of the first idle slot
  2. B, the length of the first busy period
  3. I, the length of the first idle period
  4. Z, the number of slots after the first slot of the first busy period up to and including the first subsequent idle slot

- Let N be the first time that we have a success immediately following a previous success. What is the probability $\mathbf{P}(X_{N+1} = X_{N+2} = 0)$ that there are no successes in the two trials that follow?

- It has been observed that after a rainy day, the number of days until it rains again is geometrically distributed with parameter p, independent of the past. Find the probability that it rains on both the 5th and the 8th day of the month.

- In each minute of basketball play, Jeremy Lin commits a foul with probability p and no foul with probability $1 - p$. The number of fouls in different minutes are assumed independent. He will play 30 minutes if he does not foul out. What is the PMF of Lin's playing time Z?

- Gary Kasparov, a world chess champion, plays against 100 amateurs in a large simultaneous exhibition. It has been estimated from past experience that Kasparov wins in such exhibitions 99% of his games on average. What are the probabilities that he will win 100 games, 98 games, 95 games, and 90 games?

- A packet consisting of a string of n symbols is transmitted over a noisy channel. Each symbol has probability $p = 0.0001$ of being transmitted in error, independent of errors in the other symbols. How small should n be in order for the probability of incorrect transmission (occurs whenever at least one symbol is transmitted in error) to be less than 0.001?

- Bill gets e-mails according to a Poisson process at a rate of $\lambda = 0.2$ messages per hour. He checks email every hour. What is the probability of finding 0 or 1 new messages?

- Arrivals of customers at the local supermarket are modeled by a Poisson process with a rate of $\lambda = 10$ customers per minute. Let M be the number of customers arriving between 9:00 and 9:10, and N be the number of customers arriving between 9:30 and 9:35. What is the distribution of $M + N$?

- You and your partner go to a badminton court, and have to wait until the players occupying the court finish playing. Assume that their playing time has an exponential PDF. Then the PDF of your waiting time also have the same exponential PDF, regardless of when they started playing.

- When you enter the bank, you find that all 3 tellers are busy serving other customers, and there are no other customers in queue. No more customers are allowed to enter that day. Assume that the service times for you and for each of the customers being served are i.i.d. exponential random variables. What is the probability that you will be the last to leave?

- You call the IRS hotline and you are told that you are the 56th person in line, excluding the person being served. Callers depart according to a Poisson process with a rate of $\lambda = 2$ per minute. How long will you have to wait on average until your service starts, and what is the probability that you will have to wait for more than 30 minutes?

- A packet that arrives at a node of a data network is either a *local* packet that is destined for that node (this happens with probability p), or else it is a *transit* packet that must be relayed to another node (probability $1-p$). Packets arrive according to a Poisson process with rate λ, and each one is a local or transit packet independent of other packets and of the arrival times. Then the process of *local* packet arrivals is Poisson with rate λp.

- People with letters to mail arrive at the post office according to a Poisson process with rate λ_1, while people with packages to mail arrive at the post office according to an independent Poisson process with rate λ_2. The merged process, which includes arrivals of both types, is Poisson with rate

  $$\lambda_1 + \lambda_2.$$

  Given that an arrival has just occurred, the probability that it is an arrival of a person with a letter to mail is

  $$\frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

- 2 light bulbs have independent exponentially distributed lifetimes T_a and T_b with parameters λ_a and λ_b, respectively. What is the distribution of $Z = \min(T_a, T_b)$, the first time when a bulb burns out?

- 3 light bulbs have i.i.d. exponentially distributed lifetimes with a common parameter λ. What is the expected value of the time until the last bulb burns out?

- **chapter 7**

- **Alice** is taking a probability course.

  - In each week, she can be either **up-to-date** or **fallen-behind**.
  - If she is up-to-date in a given week, the probability that she will be up-to-date in the next week is 0.8.
  - If she is fallen-behind in a given week, the probability that she will be up-to-date in the next week is 0.6.

  Construct a Markov chain.

- **A fly** moves along a straight line in unit increments.

  - At each time period, it moves one unit to the left with probability 0.3, one unit to the right with probability 0.3, and stays in place with probability 0.4.

- Two spiders are lurking at positions 1 and m: if the fly lands there, it is captured by a spider.

We want to construct a Markov chain, assuming that the fly starts in a position between 1 and m.

- **A machine** can be either working or broken down on a given day.
  - If it is working, it will break down the next day with probability b, and will continue working with probability $1 - b$.
  - If it breaks down on a given day, it will be repaired and be working on the next day with probability r, and will continue to be broken down with probability $1 - r$.

  1. Construct a Markov chain.
  2. Suppose whenever the machine remains broken down for l days, it is replaced by a new working machine. What is the new Markov chain?

- Consider **a two-state Markov chain** with transition probabilities

$$p_{11} = 0.8,\ p_{12} = 0.2, \quad p_{21} = 0.6,\ p_{22} = 0.4.$$

  Find the steady-state probabilities.

- An absent-minded professor has 2 umbrellas that he uses when commuting from home to office and back. If it rains and an umbrella is available in his location, he takes it. If it is not raining, he always forgets to take an umbrella. Suppose that it rains with probability p each time he commutes. What is the steady-state probability that he gets wet during a commute?

- A superstitious professor works in a circular building with m doors, where m is odd, and never uses the same door twice in a row. Instead, he uses with probability p (or probability $1 - p$) the door that is adjacent in the clockwise (or counter-clockwise) direction to the door he used last. What is the probability that a given door will be used on some particular day far into the future?

- A person walks along a straight line and, at each time period, takes a step to the right with probability b, and a step to the left with probability $1 - b$. She starts in one of the positions $1, \dots, m$, but if she reaches the position 0 (or position $m + 1$), her step is instantly reflected back to 1 (or m, respectively). We introduce a Markov chain model whose states are the positions $1, \dots, m$. What are the steady-state probabilities?

- Packets arrive at a node of a communication network, where they are stored in a buffer and then transmitted. The storage capacity of the buffer is m: if m packets are already present, any newly arriving packets are discarded. We discretize time in very small periods, and we assume that in each period, at most one event can happen that can change the number of packets stored in the node. We assume that at each period, exactly one of the following occurs:
  - one new packet arrives, with probability $b > 0$;
  - one existing packet completes transmission, with probability $d > 0$ if there is at least one packet in the node, and with probability 0 otherwise;
  - no new packet arrives and no existing packet completes transmission, with probability $1 - b - d$ if there is at least one packet in the node, and with probability $1 - b$ otherwise;
- Consider the Markov chain shown in Fig. 7.17(a). Note that there are 2 recurrent classes, namely $\{1\}$ and $\{4, 5\}$. We would like to calculate the probability that the state eventually enters the recurrent class $\{4, 5\}$ starting from one of the transient states. We can lump these states and treat them as a single absorbing state, call it state 6, as in Fig. 7.17(b).
- A gambler wins \$1 at each round with probability p, and loses \$1 with probability $1-p$. He plays continuously until he either accumulates a target amount of \$m, or loses all his money. What is the probability that he wins?
- Consider the spider-and-fly model of Example 7.2. This corresponds to a Markov chain. Assume $m = 4$. What is the expected number of steps until the fly is captured?
- A machine, once in **production** mode, operates continuously until an alarm signal is generated. The time up to the alarm signal is an exponential random variable with parameter 1. Subsequent to the alarm signal, the machine is in **test** mode for an exponentially distributed amount of time with parameter 5. The test results are positive, with probability 1/2, in which case the machine returns to production mode, or negative, with probability 1/2, in which case the machine is taken for **repair**. The duration of the repair is exponentially distributed with parameter 3. Construct a continuous-time Markov chain. What are the steady-state probabilities?
- Packets arrive at a node of communication network according to a Poisson process with rate λ. The packets are stored at a buffer with room for

up to m packets, and are then transmitted one at a time. However, if a packet finds a full buffer upon arrival, it is discarded. The time required to transmit a packet is exponentially distributed with parameter μ. Construct a continuous-time Markov chain. What are the steady-state probabilities?