Linear Algebra Midterm 2011.11.30

1. (10%) Find a vector x orthogonal to the row space of A, a vector y orthogonal to the column space, and a vector z orthogonal to the nullspace, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 4 \end{bmatrix}.$$

2. (10%) Suppose S is spanned by the vectors $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$. Find two vectors that span $S^{\perp}$.

3. (10%) Prove that the trace of

$$P = \frac{aa^T}{a^Ta},$$

where a is a non-zero vector, is always equal to 1.

4. (10%) Solve $Ax = b$ by least squares, and find $p = A\hat{x}$ for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Verify that $b - p$ is perpendicular to the columns of A.

5. (10%) If $P_C = A(A^TA)^{-1}A^T$ is the projection matrix onto the column space of A, what is the projection matrix P_R onto the row space?

6. (10%) Find cofactor matrices and then multiply C_A^T by A and C_B^T by B, where

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 0 & 0 \end{bmatrix}.$$

7. (10%) Let F_n be the determinant of a tri-diagonal matrix of size $n \times n$,

$$F_n = \det \begin{bmatrix} 1 & -1 & & & \\ 1 & 1 & -1 & & \\ & 1 & 1 & -1 & \\ & & \cdot & \cdot & \cdot \\ & & & 1 & 1 \end{bmatrix}.$$

Show by cofactor expansion along row 1 that $F_n = F_{n-1} + F_{n-2}$.

8. (10%) Suppose the permutation P takes $(1, 2, 3, 4, 5)$ to $(5, 4, 1, 2, 3)$.

   (a) What does P^2 do to $(1, 2, 3, 4, 5)$?

   (b) What does P^{-1} do to $(1, 2, 3, 4, 5)$?

9. (10%) L is lower triangular and S is symmetric. Assume that they are invertible.

   (a) Which three cofactors of L are 0?

   (b) Which three pairs of cofactors of S are equal?

10. (10%) Find orthogonal vectors A, B, C by Gram-Schmidt from a, b, c, where

$$a = (1, -1, 0, 0), \quad b = (0, 1, -1, 0), \quad c = (0, 0, 1, -1).$$