線性代數期中考 I 2012.11.07

1. (10%) Suppose that $\mathbb{T}$ is the linear transformation in $\mathbb{R}^3$ that takes (u, v, w) to $(u + v + w, u + v, u)$. Describe what $\mathbb{T}^{-1}$ does to (x, y, z).

2. (10%) Let $\mathbf{A}$ be a square matrix.

   (a) Show that the nullspace of $\mathbf{A}^2$ *contains* the nullspace of $\mathbf{A}$.

   (b) Show that the column space of $\mathbf{A}^2$ *is contained in* the column space of $\mathbf{A}$.

3. (10%) Find the 4×3 matrix $\mathbf{A}$ that represents a *right shift*: (x_1, x_2, x_3) is transformed to $(0, x_1, x_2, x_3)$.

4. (10%) The adjacency matrix $\mathbf{M}$ of a graph has $M_{ij} = 1$ if node i and node j are connected by an edge, and $M_{ij} = 0$ otherwise. Draw of the graph with the following edge-node incident matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix},$$

   and find its adjacency matrix $\mathbf{M}$.

5. (10%) Find a 3×3 matrix $\mathbf{A}$ whose nullspace consists of all vectors in $\mathbb{R}^3$ such that

$$x_1 + 2x_2 + 4x_3 = 0.$$

6. (10%) If $\mathbf{A}$ is 5×4 with rank 4. Show that $\mathbf{A}\mathbf{x} = \mathbf{b}$ has no solution if the matrix $[\mathbf{A}\ \mathbf{b}]$ is invertible.

7. (10%) Find the solution set to the following system of linear equations

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

8. (10%) If $\mathbf{E}$ is 2×2 and it adds the first equation to the second, what are $\mathbf{E}^2$, $\mathbf{E}^8$, and $8\mathbf{E}$?

9. (10%) Find the symmetric factorization

$$\mathbf{A} = \mathbf{LDL}^T,$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 11 \end{bmatrix}.$$

10. (10%) For the system of equations

$$\begin{cases} x + y = 4 \\ 2x - 2y = 4 \end{cases}$$

draw the row picture and the column picture.

11. (10%) Find a basis for the subspace of $\mathbb{R}^4$ in which $x_1 + x_2 + x_3 = 0$ and $x_3 + x_4 = 0$.

12. (10%) Find the 4 fundamental subspaces for the matrix

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$