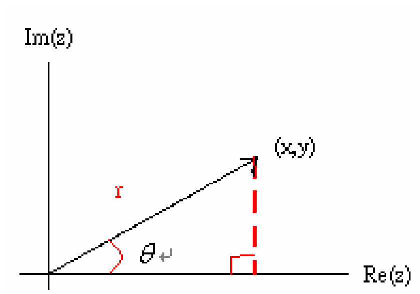
Chap3 複變分析(Complex Analysis)

◎3-1 基本觀念

在複數平面上定義一個複數 $z = x + iy$ 如圖所示，



其中:

z 稱複數

x 稱爲實數，記爲 Re(z)

y 稱爲虛數，記爲 Im(z)

由圖可得(x, y)與(r, θ)之關係可知

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \qquad \text{(極座標)} \end{aligned}$$

Ex. 1.

z= 2+3i

Re(z):2

Im(z):3

定義：一個複數的絕對值（長度）

$$|z| \equiv \sqrt{x^2 + y^2} \equiv r$$

定義：一個複數的共軛複數

$$\bar{z} = x + iy$$

複數加法定義：

若 $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

則 $z_1 + z_2 \equiv (x_1 + x_2) + i(y_1 + y_2)$

$\boxed{\textbf{Ex.}}$ 2.　加法

$$\begin{aligned} z_1 &= 1 + 2i \\ z_2 &= 4 + 5i \\ z_1 + z_2 &= (1+4) + i(2+5) \\ &= 5 + 7i \end{aligned}$$

複數乘法定義：

若 $z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

則 $z_1 z_2 \equiv (x_1 x_2 - y_1 y_2) + i(x_1 y_2 - x_2 y_1)$

$\boxed{\textbf{Ex.}}$ 3.　乘法

若 $z = x + iy$　求 $z\bar{z} = ?$

$$\begin{aligned} z\bar{z} &= (x^2 + y^2) + i(xy - xy) \\ &= x^2 + y^2 = |z|^2 \end{aligned}$$

$\boxed{\textbf{Ex.}}$ 4. 乘法

$$z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

複數除法的定義：

$$z_1 = x_1 + iy_1$$

$$z_2 = x_2 + iy_2$$

$$\frac{z_1}{z_2} \equiv \frac{x_1 + iy_1}{x_2 + iy_2}$$

$\boxed{\textbf{Ex.}}$ 5.

$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$= \cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)$$
$$= e^{i(\theta+2k\pi)}$$
$$k = 0, 1, 2, 3, \cdots$$

＊複數的 n 次方根:

若 $z = re^{i\theta}$

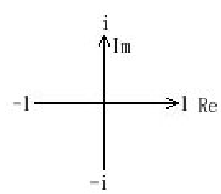
則 $z^{\frac{1}{n}} = ?$

由 $z = re^{i\theta} = re^{i(\theta+2k\pi)}$

得 $z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i(\frac{\theta+2k\pi}{n})}$

$k = 0,1,2,3,\ldots\ldots,n-1$

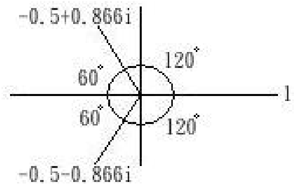
$\boxed{\textbf{Ex.}}$ 6. 已知　z=1 ,求 Z 的 4 次方根



$$z = re^{i\theta}, r = 1, \theta = 0$$
$$z^{\frac{1}{4}} = r^{\frac{1}{4}} e^{i(\theta+2k\pi)\frac{1}{4}}$$
k=0,1,2,3
$$k = 0, w_0 = e^{i(0)} = 1$$
$$k = 1, w_1 = e^{i(\frac{\pi}{2})} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = i$$
$$k = 2, w_2 = e^{i(\pi)} = -1$$
$$k = 3, w_3 = e^{i(\frac{6\pi}{4})} = -i$$

$\boxed{\textbf{Ex.}}$ 7.　已知 z =1,求 z 的 3 次方根?

$$z^{\frac{1}{3}} = r^{\frac{1}{3}} e^{i(\theta+2k\pi)\frac{1}{3}}$$
$$\mathrm{k} = 0,1,2$$
$$k = 0, w_0 = e^{i(0)} = 1$$
$$k = 1, w_1 = e^{i(\frac{2\pi}{3})} = -0.5 + 0.866i$$
$$k = 2, w_2 = e^{i(\frac{4\pi}{3})} = -0.50.866i$$

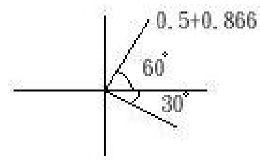


$\boxed{\textbf{Ex.}}$ 8. 已知 z =-1,求 z 的 3 次方根?

$$\theta = \pi$$
$$z^{\frac{1}{3}} = r^{\frac{1}{3}} e^{i(\theta+2k\pi)\frac{1}{3}}$$

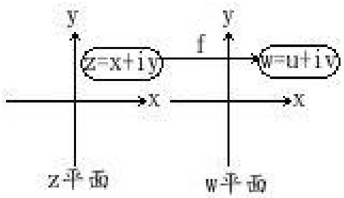
其中 k=0, 1, 2

$$k = 0 : w_0 = e^{i(\frac{\pi}{3})} = 0.5 + 0.866i$$
$$k = 1 : w_1 = e^{i(\frac{5\pi}{3})} = \frac{\sqrt{2}}{2} - 0.5i$$
$$k = 2 : w_2 = e^{i(\frac{7\pi}{3})} = 0.5 + 0.866i$$



◎3-2 複變函數

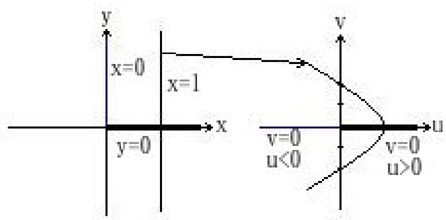
設 z 為複數平面一點，w 為複數平面 w 平面一點，若存在一個從 z 平面的對應關係，使得 w=f(z)則稱此對應關係 f 為複數 z 的函數



Ex.

計算 $f(z)=z^2$ 之實部與虛部

$$\begin{aligned} w &= f(z) \\ &= f(x+iy) \\ &= (x+iy)^2 \\ &= x^2-y^2+i2xy \\ &= u(x,y)+iv(x,y) \end{aligned}$$



則

實部： $u(x,y)=x^2-y^2$

虛部： $v(x,y)=2xy$

說明：

當 u=c 為常數時得 $x^2-y^2=c$ 。上式說明 z 平面上的雙曲線 $x^2-y^2=c$ 對應至 w 平面上的垂直線。同理 v = k 是常數時得 $2xy=k$ 。上式說明 z 平面上的雙曲線 $2xy=k$ 對應至 w 平面上的水平線 v = k。

複數函數的極限:

對於任意的一正數 ε >0，必存在另一正數 δ >0，使得當 $|z-z_0|<\delta$ 時，可以滿足

$$|f(z)-f(z_0)|<\varepsilon$$

記為:

$$\lim_{z\to z_0} f(z)=L$$

L 稱為當 Z 趨近於 Z_0 時 f(Z)之極限值

複數函數的連續

若 f(z)滿足下列條件

(1) z_0在 f 的定義域內

(2) $\lim_{z\to z_0} f(z) = L$ 存在

(3) $\lim_{z\to z_0} f(z) = f(z_0)$

則稱 $f(z)$ 在 $z = z_0$ 處連續。

複數函數的可微

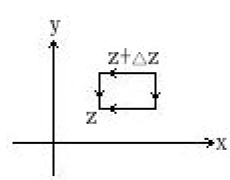
設 z_0 爲 f(z)的定義域內一點，若 $\lim_{\Delta z\to 0} \dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ 存在，則稱 f(z)在 z_0 爲可微分，且將極限值稱爲 f(z)在 z_0 的導數。

Ex. 試証 $f(z) = \bar{z}$ 不可微。

證明： 由已知得

$$\begin{aligned} f(z+\Delta z) &= \overline{z+\Delta z} \\ &= \overline{x+iy+\Delta x+i\Delta y} \\ &= \overline{x+\Delta x+i(y+\Delta y)} \\ &= x+\Delta x-i(y+\Delta y) \end{aligned}$$

$$\begin{aligned} f(z) &= \bar{z} \\ &= \overline{x+iy} \\ &= x-iy \end{aligned}$$



則 $f(z)$ 的導函數

$$\lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \frac{x_0+\Delta x-i(y_0+\Delta y)-(x_0-iy_0)}{\Delta x+i\Delta y}$$

(1) 沿著 $\Delta \mathrm{x} = 0. \Delta \mathrm{y} \to 0$ 的路徑逼近極限

則 $\lim_{\Delta z\to 0} \dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta \mathrm{y}\to 0} \dfrac{-i\Delta \mathrm{y}}{i\Delta \mathrm{y}} = -1$

(2) 沿著 $\Delta y = 0. \Delta y \to 0$ 的路徑逼近極限

則 $\lim_{\Delta z\to 0} \dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta x\to 0} \dfrac{\Delta x}{\Delta x} = 1$

因此$\lim\limits_{\Delta z \to 0} \dfrac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$之極限值不存在，故$f(z)=\bar{z}$不可微。

基本複數函數的定義

指數函數的定義:

$$
\begin{aligned}
e^Z &\equiv e^{x+iy} \\
&\equiv e^x(\cos y + i\sin y)
\end{aligned}
$$

即

$$e^Z = e^X\cos y + ie^x\sin y$$

實部與虛部：

$$u = e^x\cos y$$
$$v = e^x\sin y$$

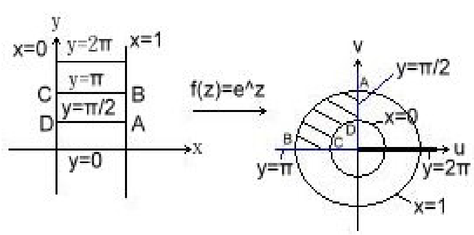
$\boxed{\textbf{Ex.}}$ 右圖中 z 平面上的 ABCD 在$f=e^z$映射之後得部分圓環

（1）x=0

$$\Rightarrow u = \cos y, \quad v = \sin y$$
$$\Rightarrow u^2 + v^2 = 1$$

（2） $y = 2\pi$

$$\Rightarrow u = e^x \quad v = 0$$



（3）x=1

$$u = e\cos y, \quad v = e\sin y$$
$$\Rightarrow u^2 + v^2 = e^2$$

（4）y=x

$$\Rightarrow u = 0 \quad v = -e^x$$

複數三角函數與雙曲線函數的定義：

$$\cos z \equiv \frac{1}{2}(e^{iz}+e^{-iz}) \qquad \cosh z \equiv \frac{1}{2}(e^z+e^{-z})$$

$$\sin z \equiv \frac{1}{2i}(e^{iz}-e^{-iz}) \qquad \sinh z \equiv \frac{1}{2}(e^z-e^{-z})$$

$\boxed{\textbf{Ex.}}$ 求$\cos z$之實部 u 與虛部 v （$\cos z = u+iv$）

$$
\begin{aligned}
\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\
&= \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) \\
&= \frac{1}{2}(e^{ix-y} + e^{-ix+y}) \\
&= \frac{1}{2}[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)] \\
&= \cos x(\frac{e^{y}+e^{-y}}{2}) + i\sin x(\frac{e^{-y}-e^{y}}{2}) \\
&= \cos x\cosh y - i\sin x\sinh y
\end{aligned}
$$

實部： $u = \cos x\cosh y$

虛部： $v = -\sin x\sinh y$

$\boxed{\textbf{Ex.}}$ 求 $\sin z$ 之實部 u 與虛部 v　$(\sin z = u + iv)$

$$
\begin{aligned}
\cos z &= \frac{1}{2i}(e^{iz} - e^{-iz}) = \frac{1}{2i}(e^{i(x+iy)} - e^{-i(x+iy)}) \\
&= \frac{1}{2i}(e^{ix-y} + e^{-ix+y}) = \frac{1}{2i}[e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)] \\
&= \frac{1}{2}[e^{-y}(-i\cos x + \sin x) - e^{y}(-i\cos x - \sin x)] \\
&= \sin x(\frac{e^{-y}+e^{y}}{2}) + i\cos x(\frac{e^{y}-e^{-y}}{2}) \\
&= \sin x\cosh y + i\cos x\sinh y
\end{aligned}
$$

實部： $u = \sin x\cosh y$

虛部： $v = \cos x\sinh y$

$\boxed{\textbf{Ex.}}$ 求 $\cosh z$ 之實部 u 與虛部 v

$$
\begin{aligned}
\cosh z &= \frac{1}{2}(e^{z} + e^{-z}) = \frac{1}{2}(e^{x+iy} + e^{-x-iy}) \\
&= \frac{1}{2}[e^{x}(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y)] \\
&= \cos y(\frac{e^{x}+e^{-x}}{2}) + i\sin y(\frac{e^{x}-e^{-x}}{2}) \\
&= \cos y\cosh x + i\sin y\sinh x
\end{aligned}
$$

實部： $u = \cos y\cosh x$

虛部： $v = \sin y\sinh x$

三角函數與雙曲線函數之定義：

$$\cos z \equiv \frac{1}{2}(e^{iz} + e^{-iz}) \qquad \cosh z \equiv \frac{1}{2}(e^{z} + e^{-z})$$

$$\sin z \equiv \frac{1}{2i}(e^{iz} - e^{-iz}) \qquad \sinh z \equiv \frac{1}{2}(e^{z} - e^{-z})$$

$\boxed{\textbf{Ex.}}$ 求 $\cos z$ 之實部 u 與虛部 v $(\cos z = u + iv)$

$$\begin{aligned}
\cos z &= \frac{1}{2}(e^{iz} + e^{-iz}) \\
&= \frac{1}{2}(e^{i(x+iy)} + e^{-i(x+iy)}) \\
&= \frac{1}{2}(e^{ix-y} + e^{-ix+y}) \\
&= \frac{1}{2}[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)] \\
&= \cos x(\frac{e^{y} + e^{-y}}{2}) + i\sin x(\frac{e^{-y} - e^{y}}{2}) \\
&= \cos x\cosh y - i\sin x\sinh y
\end{aligned}$$

實部： $u = \cos x\cosh y$

虛部： $v = -\sin x\sinh y$

$\boxed{\textbf{Ex.}}$ 求 $\sin z$ 之實部 u 與虛部 v $(\sin z = u + iv)$

$$\begin{aligned}
\sin z &= \frac{1}{2i}(e^{iz} - e^{-iz}) \\
&= \frac{1}{2i}(e^{i(x+iy)} - e^{-i(x+iy)}) \\
&= \frac{1}{2i}(e^{ix-y} + e^{-ix+y}) \\
&= \frac{1}{2i}[e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)] \\
&= \frac{1}{2}[e^{-y}(-i\cos x + \sin x) - e^{y}(-i\cos x - \sin x)] \\
&= \sin x(\frac{e^{-y} + e^{y}}{2}) + i\cos x(\frac{e^{y} - e^{-y}}{2}) \\
&= \sin x\cosh y + i\cos x\sinh y
\end{aligned}$$

實部： $u = \sin x\cosh y$

虛部： $v = \cos x\sinh y$

Ex. 求 $\cosh z$ 之實部 u 與虛部 v

$$
\begin{aligned}
\cosh z &= \frac{1}{2}(e^{z}+e^{-z}) = \frac{1}{2}(e^{x+iy}+e^{-x-iy}) \\
&= \frac{1}{2}[e^{x}(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y)] \\
&= \cos y(\frac{e^{x}+e^{-x}}{2}) + i\sin y(\frac{e^{x}-e^{-x}}{2}) \\
&= \cos y\cosh x + i\sin y\sinh x
\end{aligned}
$$

實部： $u = \cos y\cosh x$

虛部： $v = \sin y\sinh x$

Ex. 求 $\sin(ix) = ?$ $\cos(ix) = ?$ $\sinh(iy) = ?$ $\cosh(iy) = ?$

$$
\begin{aligned}
\sin(ix) &= \frac{1}{2i}(e^{i(ix)} - e^{-i(ix)}) \\
&= \frac{1}{2i}(e^{-x} - e^{x}) \\
&= i\sinh x
\end{aligned}
$$

$$
\begin{aligned}
\sinh(iy) &= \frac{1}{2}(e^{y} + e^{-y}) \\
&= \frac{1}{2}(\cos y + i\sin y - \cos y + i\sin y) \\
&= i\sin y
\end{aligned}
$$

$$
\begin{aligned}
\cos(ix) &= \frac{1}{2}(e^{i(ix)} + e^{-i(ix)}) \\
&= \frac{1}{2}(e^{-x} + e^{x}) \\
&= \cosh x
\end{aligned}
$$

$$
\begin{aligned}
\cosh(iy) &= \frac{1}{2}(e^{iy} + e^{-iy}) \\
&= \frac{1}{2}(\cos y + i\sin y + \cos y - i\sin y) \\
&= \cos y
\end{aligned}
$$

說明：所有實數的三角函數公式在複數中皆可使用

例如： $\sin^2 z + \cos^2 z = 1$

$1 + \tan^2 z = \sec^2 z$

Ex.

$$
\begin{aligned}
&\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \\
&\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha \\
&\cos(z) = \cos(x+iy) \\
&\qquad = \cos x\cos(iy) - \sin x\sin(iy) \\
&\qquad = \cos x\cosh y - i\sin x\sinh y
\end{aligned}
$$

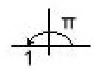
對數函數之定義：

$$\ln(z) \equiv \ln r + i\theta$$

其中$z = e^{i\theta}$，$\theta = \Theta + 2n\pi$　　　n=0,±1,±2,…..

Θ：稱主幅角，　$-\pi \le \Theta \le \pi$

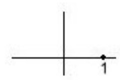
Ex. 計算 $\ln(-1) = ?$



$$-1 = re^{i\theta}, \begin{cases} r = 1 \\ \theta = \pi \pm 2n\pi \end{cases}$$

$$\begin{aligned} \ln(z) &= \ln r + i\theta = \ln 1 + i(\pi \pm 2n\pi) \\ &= 0 + i(\pi \pm 2n\pi) \quad n = 0, \pm 1, \pm 2, ... \end{aligned}$$

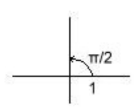
Ex. 計算 $\ln(1) = ?$



$$1 = re^{i\theta}, \begin{cases} r = 1 \\ \theta = 0 \pm 2n\pi \end{cases}$$

$$\begin{aligned} \ln(z) &= \ln r + i\theta = \ln 1 + i(2n\pi) \\ &= 0 + i(2n\pi) \quad n = 0, \pm 1, \pm 2, ... \end{aligned}$$

Ex. 計算 $\ln(i) = ?$



$$i = re^{i\theta}, \begin{cases} r = 1 \\ \theta = \dfrac{\pi}{2} \pm 2n\pi \end{cases}$$

$$\begin{aligned} \ln(z) &= \ln r + i\theta \\ &= \ln 1 + i(\frac{\pi}{2} \pm 2n\pi) \\ &= 0 + i(\frac{\pi}{2} \pm 2n\pi) \ . \end{aligned}$$

其中 $n = 0, \pm 1, \pm 2, ...$

Ex. 計算 $i^i = ?$

令 $w = i^i$ 　則 $\ln w = i \ln i$

$$i \ln i = i[\ln 1 \pm i(\frac{\pi}{2} \pm 2n\pi)]$$

$$= -(\frac{\pi}{2} \pm 2n\pi)$$

其中 $n = 0, \pm 1, \pm 2, ...$

$$i^i = e^{\ln w}$$

$$= e^{-(\frac{\pi}{2} \pm 2n\pi)}$$

Cauchy-Riemann Equation

$\boxed{\text{定理：}}$ 　若 $f(z) = u(x, y) + iv(x, y)$ 是可微函數

則 $\boxed{\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x}}$, $\boxed{\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}}$

證明：

$f(z)$ 是可微函數，因此 $f'(z)$ 存在，即

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

之極限存在

(1)沿 Δ y=0 路徑逼近

Δ x→0

$$f'(z) = \lim_{\Delta x \to 0} \frac{f(x + iy + \Delta x + i\Delta y) - f(x + iy)}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{f(x + \Delta x + iy) - f(x + iy)}{\Delta x}$$

$$= \frac{\partial f}{\partial x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{--------(1)}$$

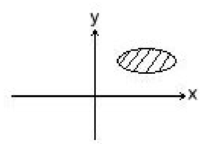
(2)沿 Δ x=0 路徑逼近

Δ y→0

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$
$$=\lim_{\Delta y\to 0}\frac{f(x+iy+\Delta x+i\Delta y)-f(x+iy)}{\Delta x+i\Delta y}$$
$$=\lim_{\Delta y\to 0}\frac{f(x+i(y+\Delta y))-f(x+iy)}{i\Delta y}$$
$$=\frac{\partial f}{i\partial y}$$
$$=\frac{1}{i}\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)$$
$$=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$
$$=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y} \quad \text{--------(2)}$$

因 $f'(z)$ 存在，所以不論由任何路徑逼近極限，其極限值都必相等，比較(1)，(2)式之實部、虛部可以得 **Cauchy-Riemann Equation**

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ 與 } \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$



定義：(解析)

f(z)在定義域(Domain)內可微，則 f(z)稱爲解析函數(Analytic function)

若 f(z)在 $z=z_0$ 的鄰域內可微，則稱 f(z)在 z_0 處解析。（解析$\Rightarrow$可微$\Rightarrow$Couchy Riemann Eq.）

定理： 若 $f(z)=u(x,y)+iv(x,y)$ ，其中 u, v 是實數函數，且 u, v 的一階偏導數連續，在定義域內滿足柯西黎曼式，則 f(z)是解析函數

Ex. $f(z)=x^2+ixy$ ，是否爲解析函數?

$u=x^2 \quad ,v=xy$

$\frac{\partial u}{\partial x}=2x,\frac{\partial v}{\partial x}=y$

$\frac{\partial u}{\partial y}=0 \quad ,\frac{\partial v}{\partial y}=x$

一階偏導數連續

$\frac{\partial u}{\partial x}\neq\frac{\partial v}{\partial y} \quad ,-\frac{\partial u}{\partial y}\neq\frac{\partial v}{\partial x}$

不解析

Ex. $f(z)=z^2$ ，是否爲解析函數?

$$f(z)=(x+iy)^2=x^2+2xyi-y^2=x^2-y^2+i2xy$$

$$u=x^2-y^2 \quad ,v=2xy$$

$$\frac{\partial u}{\partial x}=2x \quad ,\frac{\partial v}{\partial x}=2y$$

$$\frac{\partial u}{\partial y}=-2y \quad ,\frac{\partial v}{\partial y}=2x$$

一階偏導數連續　　　　　$f(z)$解析

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad ,-\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \qquad * f'(z)=2z$$

滿足柯西黎曼式

Ex. 若 $f(z)=u+iv$ 是解析函數，則必滿足 $\nabla^2 u=0$ 與 $\nabla^2 v=0$

由 f(z)是解析

$$得\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}$$

$$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}$$

$$\frac{\partial^2 u}{\partial x}+\frac{\partial^2 u}{\partial y}=0$$

即 $\nabla^2 u=0$

同理

$$\nabla^2 v=0$$

因 v 是連續函數，則 $\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 v}{\partial y\partial x}$ 。

◎3-3 複變函數的積分

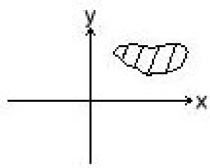
定理：

若 f(z)是解析函數，在一簡單連通區域內，則存在 F(z)
使得 $F'(z) = f(z)$

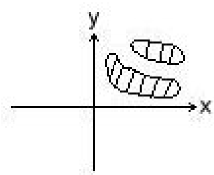
即 $\int f(z)dz = F(z)$

說明：簡單連通區域是此區域沒有分開成兩個或以上的區域

例如：



而如：



Ex. $\int_0^{1+i} z^2 dz$

$$= \frac{z^3}{3}\Big|_0^{1+i} = \frac{1}{3}(1+i)^3$$

Ex. $\int_{-i\pi}^{i\pi} \cos z dz$

$$= \sin z \Big|_{-i\pi}^{i\pi}$$
$$= \sin(i\pi) - \sin(-i\pi)$$
$$= 2\sin(i\pi)$$
$$= 2\sinh(\pi)$$

定理：

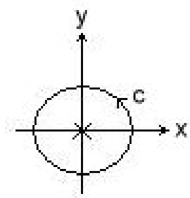
若 c 是一條，分段不滑曲線，可表示成 z=z(t)，且 f(z)在曲線 c 上是連續函數
則

$$\int_c f(z)dz = \int_a^b f(z(t))(\frac{dz}{dt})dt$$

$\boxed{\textbf{Ex.}}$ $\oint_c \frac{1}{z} dz$ **=?** 路徑 $c:|z|=1$ 逆時針方向

$$z = e^{i\theta} \quad ,0 \le \theta \le 2\pi$$
$$dz = ie^{i\theta}d\theta$$
$$\frac{1}{z} = e^{-i\theta}$$

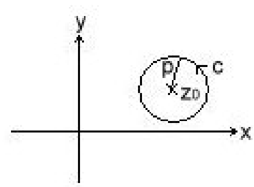
$$z = re^{i\theta}$$
$$|z| = r = 1$$



$$\text{原式} = \int_0^{2\pi} e^{-i\theta} ie^{i\theta} d\theta$$
$$= i\int_0^{2\pi} 1 d\theta$$
$$= i\,\theta\Big|_0^{2\pi} = 2\pi i$$

$\boxed{\textbf{Ex.}}$ $\oint_c \frac{1}{z-z_0} dz$ 路徑 c：$|z-z_0| = \rho$ ，逆時針方向

$$z - z_0 = \rho e^{i\theta}$$
$$dz = i\rho e^{i\theta} d\theta$$



$$\text{原式} = \int_0^{2\pi} \rho^{-1} e^{-i\theta} ie^{i\theta} d\theta$$
$$= i\int_0^{2\pi} 1 d\theta$$
$$= i\,\theta\Big|_0^{2\pi} = 2\pi i$$

$\boxed{\textbf{Ex.}}$ $\oint_c \frac{1}{(z-z_0)^m} dz$ ， $\begin{matrix} m \ne 1 \\ m = 2, 3, 4... \end{matrix}$ 路徑 c：$|z-z_0| = \rho$ ，逆時針方向

令$z - z_0 = \rho e^{i\theta}$

$dz = i\rho e^{i\theta} d\theta$

$c:$包住z_0

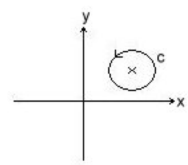
m是整數

$$
\begin{aligned}
\text{原式} &= \int_0^{2\pi} \rho^{-m} e^{-im\theta} i e^{i\theta} d\theta \\
&= \int_0^{2\pi} i\rho^{1-m} e^{i(1-m)\theta} d\theta \\
&= i\,\rho^{1-m} \int_0^{2\pi} e^{i(1-m)\theta} d\theta \\
&= i\,\rho^{1-m} \frac{1}{i(1-m)} e^{i(1-m)\theta} \Big|_0^{2\pi} \\
&= \frac{\rho^{1-m}}{1-m}\left(e^{i(1-m)(2\pi)} - 1\right) \\
&= 0
\end{aligned}
$$

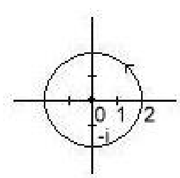
◎柯西積分公式

$$\oint_c \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0)$$

路徑 c：包住 $z = z_0$ 之逆時針封閉路徑



Ex. $\oint_c \frac{z^2 - 4z + 4}{z+i} dz$　　　c：$|z| = 2$ 逆時針方向



$$
\begin{aligned}
&= 2\pi i (z^2 - 4z + 4)\Big|_{z=-i} \\
&= 2\pi i(-1 + 4i + 4) \\
&= 2\pi i(3 + 4i) = 6\pi i - 8\pi
\end{aligned}
$$

$z_0 = -i$

$f(z) = z^2 - 4z + 4$

附註：

$$
\begin{aligned}
& e^{i(1-m)(2\pi)} \\
&= \cos[(1-m)(2\pi)] + i\sin[(1-m)(2\pi)] \\
&= 1
\end{aligned}
$$

結論：

$$\oint_c \frac{1}{(z-z_0)^m}dz = \begin{cases} 2\pi i, m=1 \\ 0, m \neq 1 \end{cases}$$

Ex. $\oint_c \frac{z}{z^2+9}dz$ c：$|z-2i|=4$ 逆時針方向

$$= \oint_c \frac{z}{(z+3i)(z-3i)}dz$$

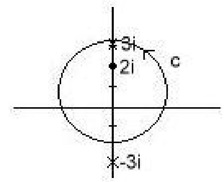
$$= 2\pi i \left(\frac{z}{z+3i}\right)\Big|_{z=3i}$$

$$= 2\pi i \left(\frac{3i}{3i+3i}\right)$$

$$= -\pi i$$

$z_0 = 3i$

$f(z) = \frac{z}{z+3i}$



定理： (柯西積分公式)

$\oint_c \frac{f(z)}{(z-z_0)}dz = 2\pi i f(z_0)$ 其中 n=1,2,3,…..

c：包住 $z = z_0$ 之逆時針封閉路徑

說明

由 $\oint_c \frac{f(z)}{(z-z_0)}dz = 2\pi i f(z_0)$ 等號兩邊對 z_0 作微分

$$\oint_c f(z)(z-z_0)^{-1}dz = 2\pi i f(z_0)$$

$$\oint_c f(z)(z-z_0)^{-2}dz = 2\pi i f'(z_0)$$

$$(-1)^3(-2)\oint_c f(z)(z-z_0)^{-3}dz = 2\pi i f''(z_0)$$

$$(-1)^4(-2)(-3)\oint_c \frac{f(z)}{(z-z_0)^4}dz = 2\pi i f'''(z_0)$$

.

.

$$\Rightarrow \oint_c \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^n(z_0)$$

上式 z 在 z_0 的極點(pole)

稱爲 n+1 階極點。

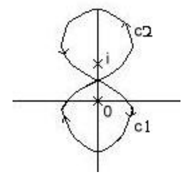
$\boxed{\textbf{Ex.}}$ 計算積分 $\oint_c \frac{z^2+3}{z(z-i)^2}dz$ 其中 c： c_1+c_2 c_1: 順時針, c_2: 逆時針

$z_0=0$

$f(z)=\frac{z^2+3}{(z-i)^2}$

$z_0=i$

$f(z)=\frac{z^2+3}{z}$



$$原式=\oint_{c_1}\frac{z^2+3}{z(z-i)^2}dz+\oint_{c_2}\frac{z^2+3}{z(z-i)^2}dz$$

$$其中\oint_{c_1}\frac{z^2+3}{z(z-i)^2}dz$$
$$=-2\pi i\left.\frac{z^2+3}{(z-i)^2}\right|_{z=0}$$
$$=6\pi i$$

$$其中\oint_{c_1}\frac{z^2+3}{z(z-i)^2}dz$$
$$=\frac{2\pi i}{1}f'(z_0)$$
$$=2\pi i\frac{(2z.z)-(z^2+3)}{z^2}$$
$$=2\pi i\left.\frac{z^2+3}{z^2}\right|_{z=i}=2\pi i(\frac{-1-3}{-1})=8\pi i$$

$$=6\pi i+8\pi i$$
$$=14\pi i$$

$\boxed{定理：}$ 若 f(z)是解析函數

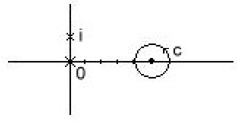
$$\oint_c f(z)dz=0$$

$\boxed{定理：}$ $\oint_c\frac{f(z)}{(z-z_0)^{n+1}}dz=0$

若路徑 c 沒有包住 z_0 ，則上式積分爲零

f(z)是解析函數

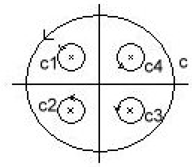
$\boxed{\textbf{Ex.}}$ $\oint_c\frac{z^2+3}{z(z-i)^2}dz=0$



$c=|z-5|=1$

$\boxed{\text{定理：}}$ $\oint_c f(z)dz$

其中路徑 c 如圖所示包住 k 個 pole



則 $\oint_c f(z)dz$

$$=\oint_{c_1} f(z)dz+\oint_{c_2} f(z)dz+\oint_{c_3} f(z)dz+\ldots+\oint_{c_k} f(z)dz$$

◎3-4 泰勒級數與羅倫級數(Taylor's Series & Laurent's Series)

Taylor's Series(泰勒級數)

公式：$f(z)=\sum_{n=0}^{\infty}\frac{f^n(z_0)}{n!}(z-z_0)^n$， f(z)是解析函數

說明：

$$f(z)=a_0+a_1(z-z_0)+a_2(z-z_0)^2+a_3(z-z_0)^3+a_4(z-z_0)^4+\ldots..$$
$$f(z_0)=a_0$$
$$f'(z)=a_1+2a_2(z-z_0)+3a_3(z-z_0)^2+4a_4(z-z_0)^3+\ldots..$$
$$f'(z_0)=a_1$$
$$f''(z)=(1)(2)a_2+(2)(3)a_3(z-z_0)^2+(3)(4)a_4(z-z_0)^3+\ldots..$$
$$f''(z_0)=2!a_2$$
$$f'''(z)=(1)(2)(3)a_3+(2)(3)(4)a_4(z-z_0)+\ldots..$$
$$f'''(z_0)=3!a_3$$
$$\boxed{a_3=\frac{f^3(z_0)}{3!}}\ldots..\boxed{a_n=\frac{f^n(z_0)}{n!}}$$
$$\oint_c (z-z_0)^n dz=0 \quad ,n=0,1,2,3,\ldots..$$
$$\oint_c \frac{1}{z-z_0}dz=2\pi i$$
$$\oint_c \frac{1}{(z-z_0)^m}dz=0 \quad ,m=2,3,4,5,\ldots..$$
c：包住z_0

$\boxed{\text{Ex.}}$ 求 $f(z)=\cos z$ 對 z=0 之 Taylor's Series，

$$f(0)=1$$

$$f'(z)=-\sin z \qquad f'(0)=0$$

$$f''(z)=-\cos z \qquad f''(0)=-1$$

$$f'''(z)=\sin z \qquad f'''(0)=0$$

$$f^{4}(z)=\cos z \qquad f^{4}(0)=1$$

$$f(z)=f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\frac{f'''(z_0)}{3!}(z-z_0)^3+.....$$

$$\cos z=1+0-\frac{1}{2!}z^2+0+\frac{1}{4!}z^4+0-\frac{1}{6!}z^6+0+.....$$

$$=1-\frac{1}{2}z^2+\frac{1}{4!}z^4-\frac{1}{6!}z^6-.....$$

Ex. $f(z)=\sin z$ 　對 z=0 之 Taylor's Series

$$f(0)=0$$

$$f'(z)=\cos z \qquad f'(0)=1$$

$$f''(z)=-\sin z \qquad f''(0)=0$$

$$f'''(z)=-\cos z \qquad f'''(0)=-1$$

$$f^{4}(z)=\sin z \qquad f^{4}(0)=0$$

$$f(z)=f(z_0)+\frac{f'(z_0)}{1!}(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\frac{f'''(z_0)}{3!}(z-z_0)^3+.....$$

$$\sin z=0+\frac{1}{1!}z+0-\frac{1}{3!}z^3+0+\frac{1}{5!}z^5+.....$$

$$=\frac{1}{1!}z-\frac{1}{3!}z^3+\frac{1}{5!}z^5+.....$$

Ex. $f(z)=e^{z}$ 　對 z=0 展開 Taylor's Series

$$f(0)=1$$

$$f'(z)=e^{z} \qquad f'(0)=1$$

$$f''(z)=e^{z} \qquad f''(0)=1$$

$$f'''(z)=e^{z} \qquad f'''(0)=1$$

$$f^{4}(z)=e^{z} \qquad f^{4}(0)=1$$

$$e^{z}=1+z+\frac{1}{2!}z^2+\frac{1}{3!}z^3+\frac{1}{4!}z^4+\frac{1}{5!}z^5+.....$$

說明：對 z=0 之 Taylor Series 又稱 Maclaurin Series。

$\boxed{\textbf{Ex.}}$

$$\cos z = 1 - \frac{1}{2}z^2 + \frac{1}{4!}z^4 - \frac{1}{6!}z^6 - .....$$

$$\sin z = \frac{1}{1!}z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + ...$$

$$e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \frac{1}{4!}z^4 + \frac{1}{5!}z^5 + ...$$

$$e^{iz} = 1 + iz - \frac{1}{2!}z^2 - i\frac{1}{3!}z^3 + \frac{1}{4!}z^4 + i\frac{1}{5!}z^5 + -\frac{1}{6!}z^6 - i\frac{1}{7!}z^7 + \frac{1}{8!}z^8 + ...$$

$\boxed{\textbf{Ex.}}$ $f(z) = \dfrac{1}{1-z}$ 對 z=0 展開之 Maclaurin Series

$$f'(z) = (-1)(1-z)^{-2}(-1) \qquad f'(0) = 1$$

$$f''(z) = (-1)(-2)(1-z)^{-3}(-1)^2 \qquad f''(0) = 2!$$

$$f'''(z) = (-1)(-2)(-3)(1-z)^{-4}(-1)^3 \qquad f'''(0) = 3!$$

$$f^4(z) = (-1)(-2)(-3)(-4)(1-z)^{-5}(-1)^4 \qquad f^4(0) = 4!$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + z^5 + ...$$

$f(-1) = \dfrac{1}{2} = 1-1+1-1+1-1+1-1+1-1+.....$爲發散級數

等比級數和公式推導：

$$Sn = a + ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1}$$

$$rSn = \quad ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1} + ar^n$$

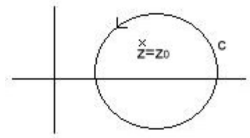
$$(1-r)Sn = a \qquad - ar^n$$

$$\mathrm{S}n = \frac{a(1-r^n)}{r}$$

若$|r|<1, a=1$　　則　$Sn = 1 + r + r^2 + r^3 + r^4 + ...$

則$Sn = \dfrac{1}{1-r}$　　　　　　$|r|=1$稱爲收斂半徑

◎3-5 羅倫級數(Laurent's Series)與留數定理



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 + ...$$
$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + ...$$

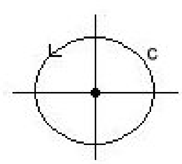
$$\oint_c f(z)dz = 2\pi i b_1$$

其中b_1稱爲留數。

Ex. $\oint_c \sin(z)dz = 0$

$\oint_c \cos(z)dz = 0$

Ex. $\oint_c \dfrac{\sin(z)}{z}dz$　其中路徑　$c:|z|=1$　逆時針方向



$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$$

$$\frac{\sin z}{z} = z - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + ...\qquad \oint_c \frac{\sin(z)}{z}dz = 0$$

Ex. $\oint_c \dfrac{\sin(z)}{z^2}dz$　其中路徑　$c:|z|=1$　逆時針方向

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...$$

$$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} - \frac{z^5}{7!} + ...$$

$$\oint_c \frac{\sin(z)}{z^2}dz = 2\pi i$$

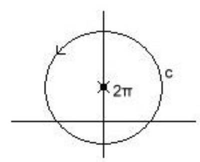
$\boxed{\textbf{Ex.}}$ $\oint_c \frac{\sin(z)}{z^5} dz = 0$

※積分公式

$$\oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$ 其中路徑 c：包住 $z = z_0$

$$\oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz = 0$$ 其中路徑 c：沒有包住 $z = z_0$

$\boxed{\textbf{Ex.}}$ $\oint_c \frac{\cos z}{(z-i\pi)^2} dz$　其中路徑 c：$|z-i\pi| = 1$　逆時針方向

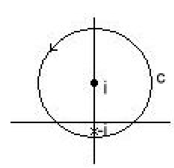


$$z_0 = i\pi$$
$$f(z) = \cos z$$
$$f'(z) = -\sin z$$

$$= 2\pi i[-\sin\Big|_{z=\pi i}]$$
$$= 2\pi i(-\sin \pi i)$$
$$= 2\pi i(-i\sinh \pi)$$

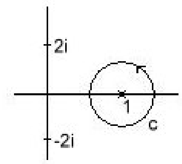
$\boxed{\textbf{Ex.}}$ $\oint_c \frac{z^4 - 3z^2 + 6}{(z-i)^3} dz$　其中路徑　c：$|z-i| = 3$　逆時針方向

$$z_0 = i$$
$$f(z) = z^4 - 3z^2 + 6$$
$$f'(z) = 4z^3 - 6z$$
$$f''(z) = 12z^2 - 6$$



$$= \frac{2\pi i}{2!}[12z - 6\Big|_{z=-i}]$$
$$= \frac{2\pi i}{2!}[12(-1) - 6]$$
$$= -18\pi i$$

$\boxed{\textbf{Ex.}}$ $\oint_c \frac{e^z}{(z-1)^2(z^2+4)}dz$ 其中路徑 c：$|z-1|=\frac{1}{2}$ 逆時針方向



$$z_0 = 1$$

$$f(z) = \frac{e^z}{z^2+4}$$

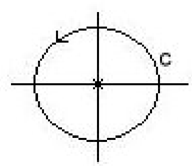
$$f'(z) = \frac{e^z(z^2+4)-e^z(2z)}{z^2+4}$$

$$\oint_c \frac{e^z}{(z-1)^2(z+2i)(z-2i)}dz$$

$$= (\frac{e^z}{z^2+4})'\Big|_{z=1}$$

$$= \frac{6\pi ie}{25}$$

$\boxed{\textbf{Ex.}}$ $\oint_c z^2 e^{\frac{1}{z}} dz$ 其中路徑 c：$|z|=1$ 逆時針方向



$$e^z = 1+\frac{1}{1!}\frac{1}{z}+\frac{1}{2!}(\frac{1}{z})^2+\frac{1}{3!}(\frac{1}{z})^3+\frac{1}{4!}(\frac{1}{z})^4+.....$$

$$z^2e^z = z^2+z+\frac{1}{2!}+\frac{1}{3!}\frac{1}{z}+\frac{1}{4!}\frac{1}{z^2}+.....$$

$$b_1 = \frac{1}{3!}$$

$$\oint_c z^2 e^{\frac{1}{z}} dz = 2\pi i b_1 = \frac{2\pi i}{3!} = \frac{\pi i}{3}$$

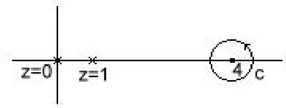
$\boxed{\textbf{Ex.}}$ 計算 $f(z) = \frac{1}{z^3-z^4}$ 對 z=0 之 Laurent’s Series 展開

$$f(z) = \frac{1}{z^3-z^4} \qquad |z|<1$$

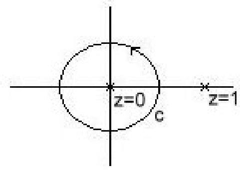
$$= \frac{1}{z^3}(1+z+z^2+z^3+z^4+.....)$$

$$= \frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+.....)$$

Ex. $\oint_c \frac{1}{z^3 - z^4} dz$ 其中路徑 c：$|z-4|=1$



Ex. $\oint_c \frac{1}{z^3 - z^4} dz$ 其中路徑 c：$|z|=\frac{1}{2}$



$$z_0 = 0$$

$$f(z) = \frac{1}{1-z}$$

$$f'(z) = \frac{1}{(1-z)^2}$$

$$f''(z) = \frac{(2)(3)}{(1-z)^3}$$

$$\oint_c \frac{1}{z^3(1-z)}$$

$$= 2\pi i [\frac{(2)(3)}{(1-z)^3}\Big|_{z=0}]$$

$$= 2\pi i(6)$$

$$= 12\pi i$$

●簡單奇點的留數

$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + ...$$

$$\oint_c f(z)dz = \oint_c \frac{b_1}{z-z_0} dz + \oint_c a_0 dz + \oint_c a_1(z-z_0)dz + \oint_c a_2(z-z_0)^2 dz + ...$$

$$= 2\pi i b_1 + 0 + 0 + 0 + ... \qquad c: \text{包住} z_0$$

因此

$$(z-z_0)f(z) = b_1 + a_0(z-z_0) + ...$$

定理：

$f(z)$ 在 $z = z_0$ 簡單奇點之留數爲

$$\lim_{z \to z_0} (z-z_0) f(z) = b_1$$

定理：

若 $f(z)$ 之型式為 $f(z)=\dfrac{p(z)}{q(z)}$ 其中 p(z),q(z)是解析函數，則在 $z=z_0$ 簡單奇點之留數為

$$\boxed{b_1 \equiv \operatorname*{Res}_{z\to z_0} f(z)=\left.\frac{p(z)}{q'(z)}\right|_{z=z_0}}$$

Ex. $f(z)=\dfrac{1}{z^4-z^3}$ 在 z=1 的留數

$$(1)\ \mathrm{Res}f(z)=(z-1)\left.\frac{1}{z^3(z-1)}\right|_{z=1}=1$$

$$(2)\ \mathrm{Res}f(z)=\left.\frac{1}{(z^4-z^3)'}\right|_{z=1}=\left.\frac{1}{4z^3-3z^2}\right|_{z=1}=1$$

※柯西積分公式

$$\oint\frac{g(z)}{(z-z_0)}dz=2\pi i g(z_0)$$

$$(z-z_0)\left.\frac{g(z)}{(z-z_0)}\right|_{z=z_0}=g(z_0)$$

$g(z_0)$ 為 $\dfrac{g(z)}{(z-z_0)}$ 之留數

●二階奇點

$$f(z)=\frac{b_2}{(z-z_0)^2}+\frac{b_1}{z-z_0}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\ldots..$$

$$\Rightarrow (z-z_0)^2 f(z)=b_2{}^2+b_1(z-z_0)+a_0(z-z_0)^2+a_1(z-z_0)^3+a_2(z-z_0)^4+\ldots..$$

$$\frac{d}{dz}[(z-z_0)^2 f(z)]=b_1+2a_0(z-z_0)+3a_1(z-z_0)^2+4a_2(z-z_0)^3+\ldots..$$

$$\left.\frac{d}{dz}[(z-z_0)^2 f(z)]\right|_{z=z_0}=b_1$$

※柯西積分公式

$$\oint\frac{g(z)}{(z-z_0)^2}dz=2\pi i g'(z_0)$$

$g'(z_0)$ 為 $\dfrac{g(z)}{(z-z_0)^2}$ 之留數

Ex. 計算 $\oint_c \dfrac{4-3z}{z^2-z}dz$ 條件如下：

<1> c 只包住 z=0

<2> c 只包住 z=1

<3> c 只包住 z=0,1

<4> c 沒有包住 z=0,1

<1>原式$=2\pi i \operatorname{Res} f(z) = z\dfrac{4-3z}{z(z-1)}\Big|_{z=0}(2\pi i) = -8\pi i$

<2>原式$=2\pi i \operatorname{Res} f(z) = z\dfrac{4-3z}{z(z-1)}\Big|_{z=1}(2\pi i) = 2\pi i$

<3>原式$=-8\pi i + 2\pi i = -6\pi i$

<4>原式=0

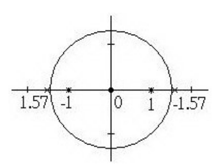
定理： $\oint_c f(z)dz = 2\pi i \sum_{n=1}^{k} \operatorname*{Res}_{z=z_0} f(z)$ 其中路徑 c 包住 k 個奇點。

Ex. 求 $f(z)=\dfrac{50z}{(z+4)(z-1)^2}$,在 z=1 的留數

$$\operatorname{Res} f(z) = \left(\frac{50z}{z+4}\right)'\Big|_{z=1} = \frac{50(z+4)-50z}{(z+4)^2}\Big|_{z=1} = 8$$

Ex. $\oint_c \dfrac{\tan z}{z^2-1}dz$ 其中路徑 $c:|z|=\dfrac{3}{2}$

$$\frac{\pi}{2}=\frac{3.1415926}{2}=1.57$$



$$\begin{aligned}\operatorname{Res} f(z) &= \left[\frac{\tan z}{(z^2-1)'}\Big|_{z=-1} + \frac{\tan z}{(z^2-1)'}\Big|_{z=1}\right](2\pi i)\\ &= \left[\frac{\tan z}{2z}\Big|_{z=-1} + \frac{\tan z}{2z}\Big|_{z=1}\right](2\pi i)\\ &= \left(\frac{\tan(-1)}{-2} + \frac{\tan 1}{2}\right)(2\pi i)\\ &= (2\pi i)\tan 1\end{aligned}$$

◎3-6 複數積分的應用

● **實數積分**

型式：$\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$

取

$$dz = ie^{i\theta} d\theta$$

$$d\theta = \frac{dz}{iz}$$

以及

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

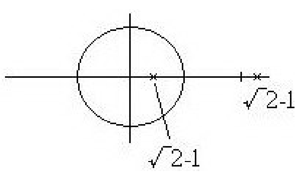
得

$$\begin{cases} \cos\theta = \frac{1}{2}(z + \frac{1}{z}) \\ \sin\theta = \frac{1}{2i}(z - \frac{1}{z}) \end{cases}$$

$$z = e^{i\theta}, 0 \le \theta \le 2\pi$$

則原式可以改寫爲 $\oint_c g(z, \frac{1}{z}) \frac{dz}{iz}$ 其中路徑 $c : |z| = 1$

Ex. $\int_0^{2\pi} \frac{1}{\sqrt{2} - \cos\theta} d\theta$



$$\text{原式} = \oint_c \frac{1}{\sqrt{2} - \frac{1}{2}(z + \frac{1}{z})} \frac{dz}{iz}$$

$$= \frac{1}{i} \oint_c \frac{1}{\sqrt{2}z - \frac{1}{2}z^2 - \frac{1}{2}} dz$$

$$= \frac{-2}{i}\oint_c \frac{1}{(z^2 - 2\sqrt{2}z + 1)'}dz$$

$$= \frac{-2}{i}\left[\frac{1}{2z - 2\sqrt{2}}\bigg|_{z = \sqrt{2}-1}\right](2\pi i)$$

$$= \frac{-2}{1}\frac{1}{-2}2\pi$$

$$= 2\pi$$

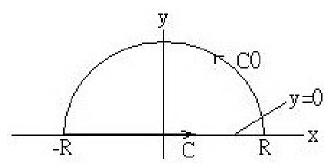
柯西主值積分

$$\int_{-\infty}^{\infty} f(x)dz = \lim_{\substack{R_2\to\infty \\ R_1\to-\infty}} \int_{R_1}^{R_2} f(z)dz$$

上式不是柯西主值積分不一定能夠收斂，而 $\lim_{R\to\infty}\int_{-R}^{R} f(x)dx$ 才是柯西主值積分

型式：$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$

條件：$|xf(x)| \to 0$ 當 $x \to \infty, -\infty$



$$\oint_c f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz$$
$$= \int_{-R}^{R} f(x)dx + \int_{c_2} f(z)dz$$

其中 $\lim_{R\to\infty}\int_{c_2} f(z)dz = 0$

因此

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \oint_c f(z)dz$$
$$= 2\pi i \sum \operatorname{Re} sf(z)$$

$\boxed{\textbf{Ex.}}$ $\int_0^{\infty} \frac{1}{x^4+1}dx$

首先求根 $z^4 = -1e^{i(\pi+2k\pi)}$

$$z = e^{\frac{i(\pi+2k\pi)}{4}}, k = 0,1,2,3$$

$$z_0 = e^{i\frac{\pi}{4}}$$

$$= \cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$z_1 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

原式等於

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx$$

$$= \frac{1}{2}\oint_c \frac{1}{z^4+1}dz$$

$$= \frac{1}{2}(2\pi i)[\frac{1}{4z^3}\bigg|_{z=z_0} + \frac{1}{4z^3}\bigg|_{z=z_1}]$$

$$= \pi i[\frac{1}{2\sqrt{2}i-2\sqrt{2}} - \frac{1}{2\sqrt{2}i-2\sqrt{2}}]$$

$$= \frac{\pi}{2\sqrt{2}}$$