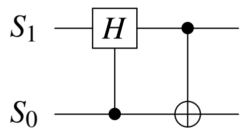
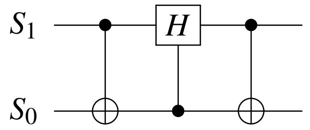
1 Clebsch-Gordan matrix circuit



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{bmatrix} = \begin{bmatrix} |00\rangle \\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |11\rangle \\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{bmatrix} = \begin{bmatrix} |J=1, M=1\rangle \\ |J=1, M=0\rangle \\ |J=1, M=-1\rangle \\ |J=0, M=0\rangle \end{bmatrix} \tag{1}$$

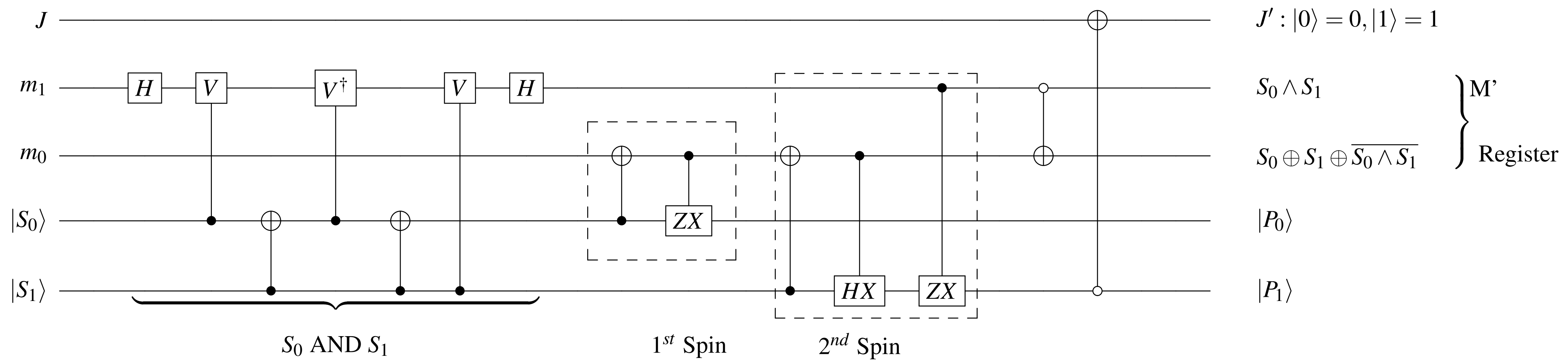
Circuit for Clebsch-Gordan transform.



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{bmatrix} = \begin{bmatrix} |00\rangle \\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \\ |11\rangle \end{bmatrix} = \begin{bmatrix} |J=1, M=1\rangle \\ |J=1, M=0\rangle \\ |J=0, M=0\rangle \\ |J=1, M=-1\rangle \end{bmatrix} \tag{2}$$

Adding a C-NOT corresponds to swapping the last two rows.

2 Registered 2 qubit Schur transform- 12 Two-qubit gates



Adding another CNOT allows you to encode the M' register using the same values for M'=0.

Spin values		Circuit output		M value
S_1	S_0	m_1	m_0	M
0	0	0	1	M=+1
0	1	0	0	M=0
1	0	0	0	M=0
1	1	1	0	M=-1

Figure 1: Table giving M register decoding

3 Clebsch-Gordan coefficients for 3 qubits

$J=0$	$S=\frac{1}{2}$
$\frac{1}{\sqrt{2}}(010-100)$	$M=\frac{1}{2}$
$\frac{1}{\sqrt{2}}(011-101)$	$M=-\frac{1}{2}$
$J=1$	$S=\frac{1}{2}$
$\sqrt{\frac{2}{3}}(001)-\sqrt{\frac{1}{6}}(010+100)$	$M=\frac{1}{2}$
$-\sqrt{\frac{2}{3}}(110)+\sqrt{\frac{1}{6}}(011+101)$	$M=-\frac{1}{2}$
$J=1$	$S=\frac{3}{2}$
000	$M=\frac{3}{2}$
$\sqrt{\frac{1}{3}}(001+010+100)$	$M=\frac{1}{2}$
$\sqrt{\frac{1}{3}}(110+011+101)$	$M=-\frac{1}{2}$
111	$M=-\frac{3}{2}$

Figure 2: J & M values for 3 qubits using encoding 0=spin up, 1=spin down

matrix for transform is:

$$\begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{6}} & 0 & -\sqrt{\frac{1}{6}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{1}{6}} & 0 & \sqrt{\frac{1}{6}} & -\sqrt{\frac{2}{3}} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{bmatrix} \mathit{Rearranging}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 000 \\ 001 \\ 010 \\ 100 \\ 011 \\ 101 \\ 110 \\ 111 \end{bmatrix} \tag{3}$$

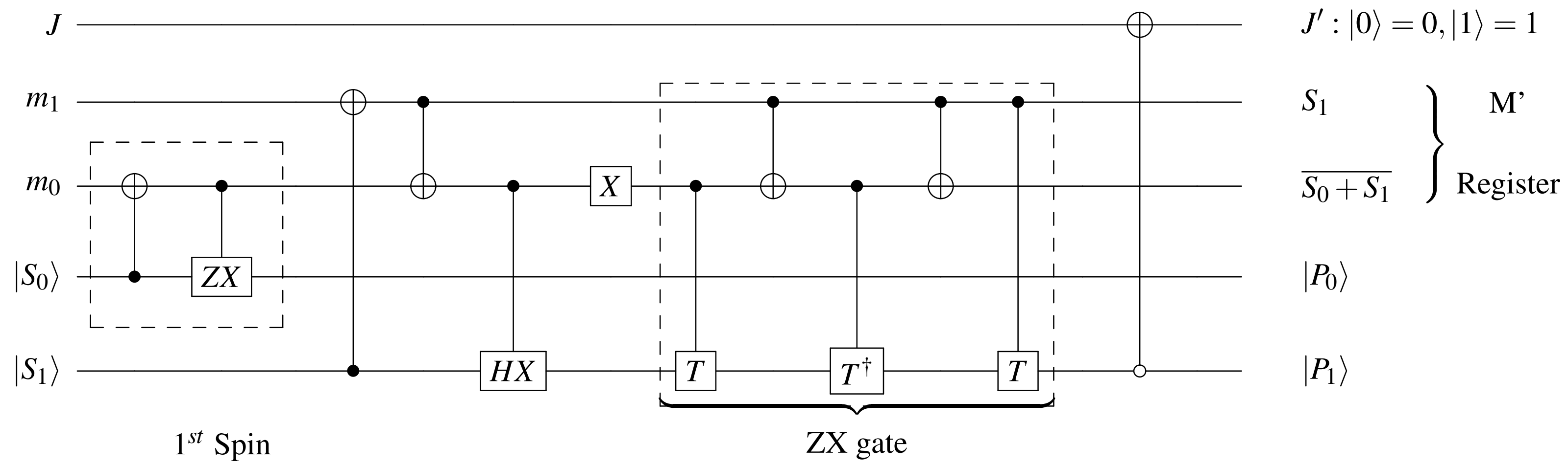
have to swap 011 & 100 to get block form.

$J=\frac{1}{2}, P=0$	$S=\frac{1}{2}$
$\frac{1}{\sqrt{3}}(e^{2\pi i/3}001+e^{4\pi i/3}010+100)$	$M=\frac{1}{2}$
$\frac{1}{\sqrt{3}}(e^{2\pi i/3}011+e^{4\pi i/3}101+110)$	$M=-\frac{1}{2}$
$J=\frac{1}{2}, P=1$	$S=\frac{1}{2}$
$\frac{1}{\sqrt{3}}(e^{2\pi i/3}001+e^{4\pi i/3}010+100)$	$M=\frac{1}{2}$
$\frac{1}{\sqrt{3}}(e^{4\pi i/3}011+e^{2\pi i/3}101+110)$	$M=-\frac{1}{2}$
$J=\frac{3}{2}$	$S=\frac{3}{2}$
000	$M=\frac{3}{2}$
$\sqrt{\frac{1}{3}}(001+010+100)$	$M=\frac{1}{2}$
$\sqrt{\frac{1}{3}}(110+011+101)$	$M=-\frac{1}{2}$
111	$M=-\frac{3}{2}$

Figure 3: Schur transform with Phase encoding?

$$\frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & e^{2\pi i/3} & e^{4\pi i/3} & 0 & 1 & 0 & 0 & 0 \\ 0 & e^{4\pi i/3} & e^{2\pi i/3} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2\pi i/3} & 0 & e^{4\pi i/3} & 1 & 0 \\ 0 & 0 & 0 & e^{4\pi i/3} & 0 & e^{2\pi i/3} & 1 & 0 \end{bmatrix} \begin{bmatrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{bmatrix} \mathit{Rearranging}, \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 000 \\ 001 \\ 010 \\ 100 \\ 011 \\ 101 \\ 110 \\ 111 \end{bmatrix} \tag{4}$$

4 Spatially Multiplexed Minimal gate explicit J & M recording- 11 Two-qubit gates



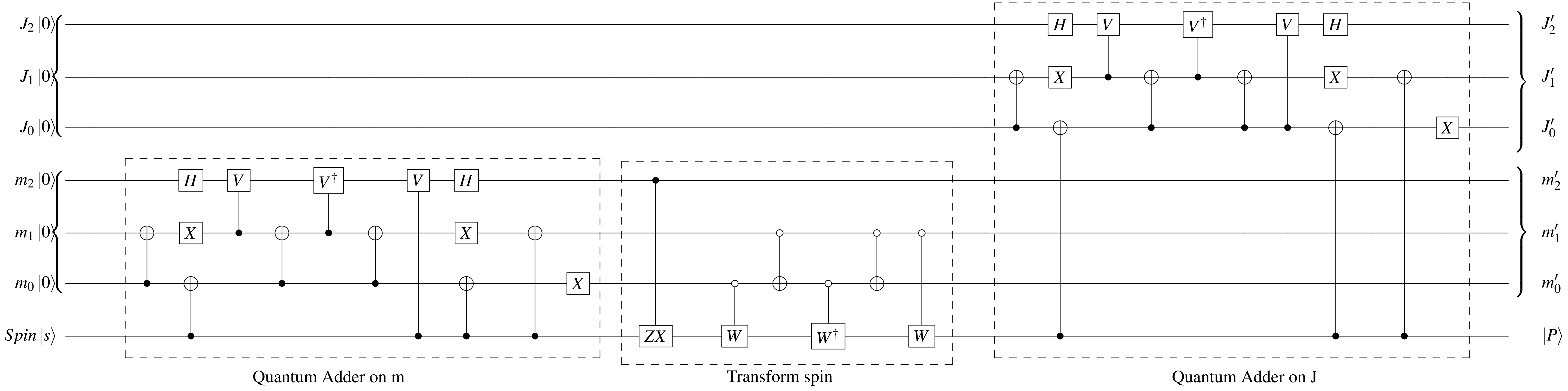
Circuit uses 11 two-qubit gates but only stores the final output values of J' & M'.

The HX gate triggers if $M = 0$ meaning $S_0 \neq S_1$ which is implemented using an XOR between S_0 & S_1. The other gate (T) is triggered when $M = -1$ meaning $S_0 = S_1 = 1$ which is done using an AND (Toffoli) gate between, $S_0 = S_1$ AND $S_1 = 1$ which is decomposed into 5 two-qubit gates. T^2 is the ZX gate, meaning $T^2 |S\rangle = XZ|S\rangle$.

Spin values		Circuit output		M value
S_1	S_0	S_1	$\overline{S_0 + S_1}$	M
0	0	0	1	M=+1
0	1	0	0	M=0
1	0	1	0	M=0
1	1	1	1	M=-1

Figure 4: Table giving M register decoding

5 Minimal general gate circuit for up to the 2 qubit Schur transform



J_2	J_1	J_0	J
0	0	0	0
0	0	1	$\frac{1}{2}$
0	1	0	1
0	1	1	$\frac{3}{2}$
1	0	0	-2
1	0	1	-$\frac{3}{2}$
1	1	0	-1
1	1	1	-$\frac{1}{2}$

m_2	m_1	m_0	M
0	0	0	0
0	0	1	$\frac{1}{2}$
0	1	0	1
0	1	1	$\frac{3}{2}$
1	0	0	-2
1	0	1	-$\frac{3}{2}$
1	1	0	-1
1	1	1	-$\frac{1}{2}$

Figure 5: Tables giving binary Two's complement encoding to spin values of the M and J registers

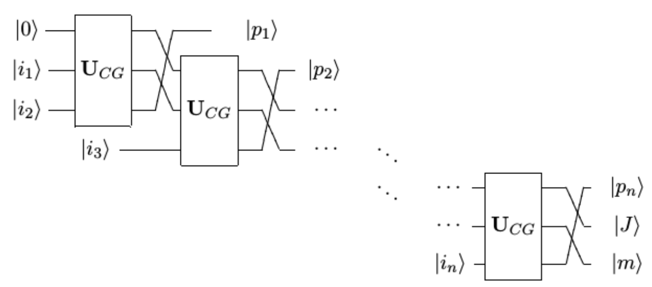


FIG. 2: Quantum circuit for the Schur transformation $\mathbf{U}_{\mathrm{Sch}}$, transforming between $|i_1 i_2 \cdots i_n\rangle$ and $|J,m,p\rangle$.

Figure 6: Taken from Bacon, Chaung, Harrow (2008) Arxiv /0407082v4

Circuit uses the encoding for $|S\rangle : |0\rangle \mapsto Spin = +\frac{1}{2}, |1\rangle \mapsto Spin = -\frac{1}{2}$ and the same for $|P\rangle$.

Where V is the phase gate, $V = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, $V^\dagger V = I$ and $V^2 = Z$. V is used here to expand the double controlled Toffoli gate into single control gates in the quantum adder subroutine.

The W gate, $W^2 = HX$ with $W^\dagger = I$, is used to expand the HX gate into single control gates in the spin transform region.

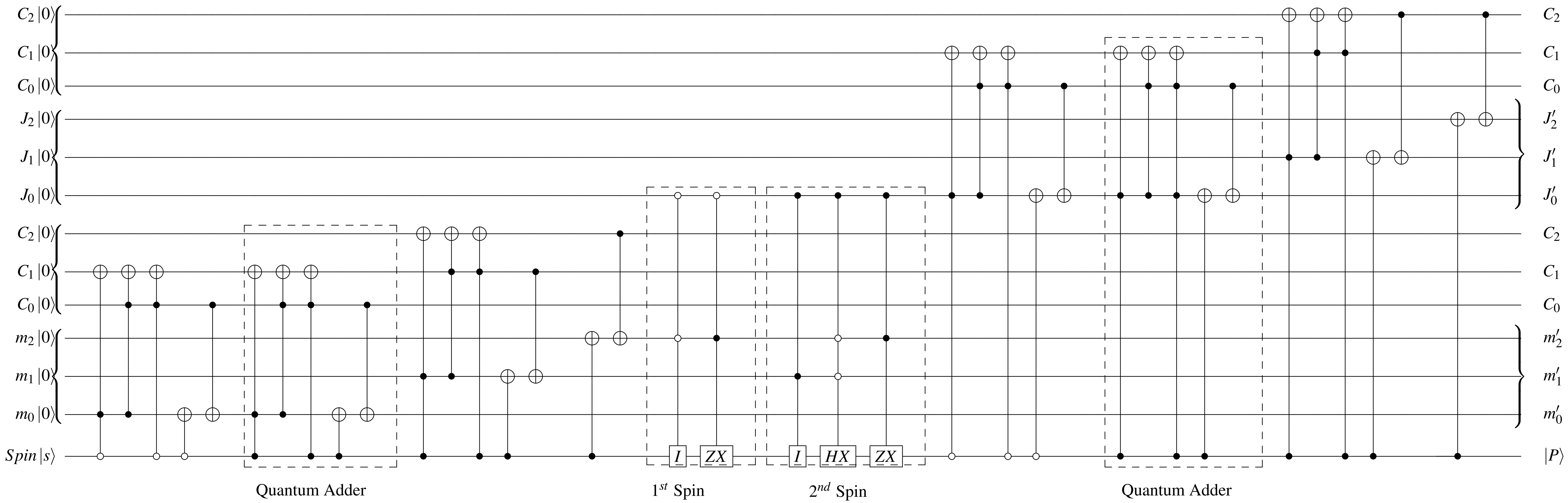
The circuit checks that if (m_1 XNOR m_0) AND (m_0 XOR S_0) and will then change m_2. Then m_1 is updated using $m_1 = m_0$ XOR S_0. m_0 is always incremented by 1, if $|S\rangle = |0\rangle$ increment only m_0 by 1 corresponding to adding $\frac{1}{2}$ to the M register. $|S\rangle = |1\rangle$ corresponds to subtracting $\frac{1}{2}$ from the M register by adding the string 111 bitwise to M.

For the most positive values of M the Identity is performed on the spin corresponding to the strings $M = 001(J = \frac{1}{2}, M' = \frac{1}{2})$ for the first spin and $M = 010(J = 1, M' = 1)$ for the second coupled in spins.

The most negative values of M performs $XZ\,|S\rangle$ corresponding to the strings $M = 111(J = \frac{1}{2}, M' = -\frac{1}{2})$ for the first spin and $M = 110(J = 1, M' = -1)$ for the second spin.

If $M = 000(J = 0, M' = 0)$ do $XH\,|S\rangle$.

6 Circuit for the Quantum Schur transform for up to 2 qubits ($|S\rangle$)



Circuit uses the encoding for $|S\rangle : |0\rangle \mapsto Spin = +\frac{1}{2}, |1\rangle \mapsto Spin = -\frac{1}{2}$ and the same for $|P\rangle$.

J_2	J_1	J_0	J
0	0	0	0
0	0	1	$\frac{1}{2}$
0	1	0	1
0	1	1	$\frac{3}{2}$
1	0	0	-2
1	0	1	-$\frac{3}{2}$
1	1	0	-1
1	1	1	-$\frac{1}{2}$

m_2	m_1	m_0	M
0	0	0	0
0	0	1	$\frac{1}{2}$
0	1	0	1
0	1	1	$\frac{3}{2}$
1	0	0	-2
1	0	1	-$\frac{3}{2}$
1	1	0	-1
1	1	1	-$\frac{1}{2}$

Figure 7: Tables giving binary Two's complement encoding to spin values of the M and J registers

The circuit adds the value of the spin to be added, $|S\rangle$, to the M register to calculate the M' register value. This is done by implementing the quantum reversible equivalent to the digital full adder.

The case where $|S\rangle = |0\rangle$ means the spin is $+\frac{1}{2}$ so to add $\frac{1}{2}$ to M one is added to the m_0 bit. The very first Quantum Adder (QAdd) uses Toffoli gates controlled on $|0\rangle$ on $|s\rangle$ (denoted by the white control circle) with the current m_0 value and C_0 (an ancilla carry) so that in the case $m_0 = 1$ and we try and add 1 to it, m_0 goes to 0 and m_1 is increased using the carry as $001 + 1 = 010$. The rest of the QAdd stages then just check the carry of the previous qubit to complete to M $+\frac{1}{2}$ addition as $|S\rangle = |0\rangle$ does not trigger any of the rest of the control gates.

The case where $|S\rangle = |1\rangle$ means the spin is $-\frac{1}{2}$ we do M $-\frac{1}{2}$ which is done by adding the binary string for $-\frac{1}{2}$ which is the all 1's string, 111. This time the very first Quantum Adder does not trigger and $|s\rangle$ is then added to all of the bits of M using C-NOT gates with carries to check for overflow.

The Unitary is then performed on $|S\rangle$ depending on the values of the newly calculated M' and J registers. The Identity is shown in the circuit for completeness on all the J and M' values. The J register is then updated to J' by adding the value of $|P\rangle$ to J using the QAdd sequence of gates.

To add the second qubit in the values of J' and M' are passed in as the initial register values. It is easy to extend this to many qubits being streamed in one at a time by carefully conditioning the controls on the unitaries, I think in the general case you need at most N controls for coupling up to N qubits in one at a time. The circuit written here has redundancy in the Identity and ZX gates appearing twice.