Spontaneous parametric down-conversion in waveguides: A backward Heisenberg picture approach

Zhenshan Yang, Marco Liscidini, and J. E. Sipe
Department of Physics and Institute for Optical Sciences, University of Toronto, 60 St. George Street, Ontario M5S 1A7, Canada

(Received 7 November 2007; published 5 March 2008)

We develop a fully quantum-mechanical approach for the description of spontaneous parametric down-conversion in waveguides. Our method allows us to write an expression for the quantum state easily and directly, even when multiple photon pairs are generated in the process. The role of material dispersion in the normalization of the modes of the system, usually neglected, is properly treated here. The example of an $Al_yO_x/Al_{0.5}Ga_{0.5}As$ ridge waveguide is discussed to illustrate the application of the formalism.

DOI: 10.1103/PhysRevA.77.033808 PACS number(s): 42.65.Lm, 42.65.Wi

I. INTRODUCTION

Advances in the development of quasi-phase-matched structures, and even more recently in the design of waveguide structures with phase-matched modes (see, e.g., Scaccabarozzi *et al.* [1]), will lead to a new generation of devices for efficient nonlinear wave mixing in artificially structured media [2]. While much of the initial focus has been on classical processes [3–5], such as second harmonic generation (SHG), a device that will lead to enhanced SHG will also lead to enhanced spontaneous parametric down conversion (SPDC), an inherently quantum process leading to the generation of entangled photons [6–9]. In this paper we present a treatment of SPDC in dielectric waveguides.

We have in mind a device such as a ridge waveguide with a core structure, as sketched in Fig. 1. In the linear regime, the modes of the structure, which we take to be specified by the electric displacement and magnetic fields $\mathbf{D}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, respectively, take the form

$$\mathbf{B}_{Ik}(\mathbf{r}) = \mathbf{b}_{Ik}(x,y)\frac{e^{ikz}}{\sqrt{2\pi}}, \quad \mathbf{D}_{Ik}(\mathbf{r}) = \mathbf{d}_{Ik}(x,y)\frac{e^{ikz}}{\sqrt{2\pi}}, \tag{1}$$

where z is the direction of propagation, and in general the wave number k can be positive or negative. Here I labels the "mode type" (e.g., the "lowest order *TE*-like mode") which we refer to as the "channel." In SHG typically one channel is used for the fundamental field ($I=F$) and a different channel for the second harmonic ($I=S$). Each channel has a dispersion relation, which can be specified as a wave-number-dependent frequency; we write these as ω_{Fk} and ω_{Sk}. They are determined both by the underlying dispersion in the material medium, and by modal effects due to the confinement of the field in the waveguide. In a structure phase matched for SHG at an incident wave number k_{F_0} we have $\omega_{S(2k_{F_0})} = 2\omega_{F(k_{F_0})}$, but our treatment will apply as well to structures where this is not satisfied. In the reverse SPDC process, two photons in channel F are generated from one pump photon in channel S.

We allow for the possibility of quasi-phase-matching, involving a periodic z dependence of the second-order nonlinear response coefficient χ_2, or even for more engineered variations in z of χ_2 along the lines suggested by Harris [10] for the generation of few-cycle biphotons. For structures composed of materials from many of the simple crystal classes, such a variation in χ_2 can be engineered without inducing a variation in the linear properties of the medium. For others materials, it would be accompanied by a periodic variation of the linear optical properties, say with period Λ, and instead of the form (1) the linear modes would be of Bloch form, with $(\mathbf{b}_{Ik}(x,y),\mathbf{d}_{Ik}(x,y))$ replaced by $(\mathbf{b}_{Ik}(\mathbf{r}),\mathbf{d}_{Ik}(\mathbf{r}))$ such that $(\mathbf{b}_{Ik}(\mathbf{r}+\hat{\mathbf{z}}\Lambda)=\mathbf{b}_{Ik}(\mathbf{r}),\mathbf{d}_{Ik}(\mathbf{r}+\hat{\mathbf{z}}\Lambda)=\mathbf{d}_{Ik}(\mathbf{r}))$. Much of what we do here could be generalized in a straightforward way to treat such a structure; as well, much of our treatment could be applied to treat SPDC in bulk crystals where the excitation geometry is essentially one dimensional. We leave these generalizations to future works where particular applications will be considered. In addition to providing a calculation of SPDC in the type of structures we consider, our treatment here goes beyond many of the usual calculations of SPDC in three important respects.

First, while usual treatments certainly take into account the dispersion relations ω_{Fk} and ω_{Sk} in their description of phase matching, the effects of material dispersion on the normalization of the modes have typically been neglected. When ω_{Sk} is close to the band gap of the material medium, these effects are not negligible. This problem is sometimes skirted by dealing with spectral mode operators that, in the Heisenberg picture, depend on (ω,z), rather than on the usual (k,t) (see, e.g., Huttner *et al.* [11], and for a recent application to microcavities see Raymer *et al.* [12]); the focus on energy flux leads to a natural inclusion of material dispersion in the momentum operator, which generates the evolution in space. However, this is usually implemented under the assumption of truly one-dimensional propagation, neglecting any structure of the modes in the (x,y) plane; we need to go

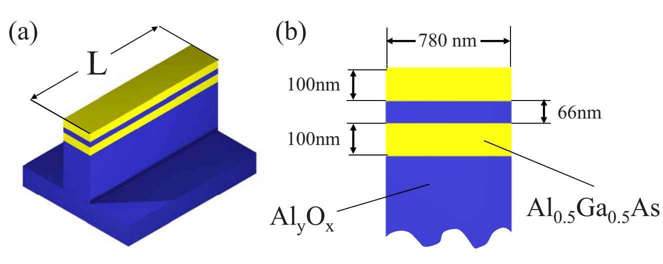

FIG. 1. (Color online) (a) Sketch of the waveguide. (b) Waveguide cross section and geometrical parameters

beyond this to treat the structured media of interest. And for extensions to propagation in two-dimensional (2D) geometries such as slabs, and indeed for complete treatments of 3D structures, a $(\mathbf{k},t)$ framework is essential. As well, it is only within the more usual $(\mathbf{k},t)$ framework that the inclusion of both material dispersion and absorption has been included in the quantization of the fields [13]. While in this paper we are interested in frequencies within the dispersive but transparent regime of all constituent materials, the generality of the $(\mathbf{k},t)$ framework leads us to adopt it here, especially since we can easily include dispersion effects on the calculation of the mode normalization, and do so here.

A second extension has to do with the ability to treat the production of many pairs of photons. In many calculations of SPDC the process is considered sufficiently weak that there is only a small probability that a pair of photons will emerge in the output channel. Within such an approximation the state of that channel is then of the form

$$|\psi\rangle = |\text{vac}\rangle + \beta C_{\text{II}}^{\dagger}|\text{vac}\rangle + \cdots, \tag{2}$$

where $|\text{vac}\rangle$ is the vacuum state and $C_{\text{II}}^{\dagger}$ is an operator involving the creating of two photons

$$C_{\text{II}}^{\dagger} = \frac{1}{\sqrt{2}}\int dk_1 dk_2 \phi(k_1,k_2) c_{k_1}^{\dagger} c_{k_2}^{\dagger}$$

in a notation that will be defined in detail below; $\phi(k_1,k_2)$ is the biphoton wave function, and in the limit that Eq. (2) is valid, $|\beta|^2 \ll 1$ is the probability that a pair of photons is generated. Clearly for many excitation scenarios, particularly as cw excitation is approached, this is too restrictive. Many workers [14–18] have shown—some explicitly and others implicitly—that the generalization of Eq. (2) to treat pump fields of amplitudes such that many photon pairs can be created is given by

$$|\psi\rangle = e^{\beta C_{\text{II}}^{\dagger} - \text{H.c.}}|\text{vac}\rangle, \tag{3}$$

as long as pump depletion can be neglected, where "H.c." denotes Hermitian conjugate. In both Eq. (3), and its limit (2) when $|\beta|^2 \ll 1$, β is proportional to the amplitude of a specified pump pulse. While discussions aimed at establishing Eq. (2) often employ the Schrödinger picture, most treatments that explicitly construct Eq. (3) rely on a Heisenberg picture calculation followed by a Bloch-Messiah reduction (see, e.g., the discussions in Bennink and Boyd [19] and Lvovsky *et al.* [20]), and are restricted to a classical description of the pump pulse. In our approach we aim at calculating Eq. (3) through a solution of Schrödinger's equation. Rather than relying on a formal transformation after the usual Heisenberg problem is solved, we convert the dynamical problem of solving Schrödinger's equation to a different Heisenberg problem. In this new Heisenberg problem, the solution for a pump field operator is all that is required, and it is trivially found in the undepleted pump approximation. Especially for extension to structures more complicated than ridge waveguides, we feel this results in a great simplification of the analysis; indeed, we have used our approach already to describe the output of SPDC in ring resonator structures [9], but have not had the opportunity to describe it fully. Even for simple ridge we feel the approach is easier than the usual strategies to obtain Eq. (3) or its equivalent, and will provide a natural route to generalization beyond the undepleted pump approximation.

Third, while in many treatments of SPDC the pump pulse is treated classically, we here treat it quantum mechanically. In the particular example we present we do for simplicity take the field to be a coherent state, but the possibility of describing input pulses of arbitrary form is an important generalization.

The outline of the paper is as follows. In Sec. II we discuss the linear and nonlinear contributions to the Hamiltonian, present the correct normalization of the modes including material dispersion, and write the nonlinear contribution in a form that will simplify later calculations. In Sec. III we present our approach for treating photon pair production, which relies on using asymptotic-in and -out states to eliminate the trivial effects of propagation due to the linear Hamiltonian. We can then formulate the evolution of quite general in-states to their corresponding out-states in terms of a "backward Heisenberg picture." In Sec. IV we specify a very general coherent state for the pump pulse, and solve for the final state of the system in the undepleted pump approximation. In Sec. V we then examine the state of the generated photons, which we find to be of the form (3). This state is characterized by the biphoton wave function and the amplitude parameter β. We evaluate these quantities in terms of the pump pulse parameters, giving us a complete specification of the generated photon state. In Sec. VI we present an example, and in Sec. VII we conclude. Some technical details are given in Appendix A, and in Appendix B we work out the SPDC in the interesting but unphysical special case of no group velocity dispersion in either channel, and perfect phase matching.

II. FIELDS AND HAMILTONIANS

In the absence of free charges and currents it is convenient to take the Schrödinger operators for the electric displacement and magnetic fields $\mathbf{D}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$, respectively, as the fundamental field operators [21]. We construct the usual type of mode expansion of these operators, and for structures of the form indicated in Fig. 1 the modes of the linear problem are identified by a channel index I and a wave number k, which can be positive or negative. We will be interested into two channels: One, which we label $I=S$, identifies the channel of the input pump field; the other, which we label $I=F$, is associated with the channel of the signal and idler photons generated. Then the expansions of the full fields $\mathbf{D}(\mathbf{r})$ and $\mathbf{B}(r)$, in terms of the modes of interest of the linear problem, are given by

$$\mathbf{B}(\mathbf{r}) = \left(\int dk \sqrt{\frac{\hbar\omega_{Sk}}{2}} a_{Sk}\mathbf{B}_{Sk}(\mathbf{r}) + \int dk \sqrt{\frac{\hbar\omega_{Fk}}{2}} a_{Fk}\mathbf{B}_{Fk}(\mathbf{r})\right) + \text{H.c.},$$

$$\mathbf{D}(\mathbf{r}) = \left(\int dk \sqrt{\frac{\hbar\omega_{Sk}}{2}} a_{Sk}\mathbf{D}_{Sk}(\mathbf{r}) + \int dk \sqrt{\frac{\hbar\omega_{Fk}}{2}} a_{Fk}\mathbf{D}_{Fk}(\mathbf{r})\right) + \text{H.c.} \tag{4}$$

if the mode amplitudes $\mathbf{B}_{Sk}(\mathbf{r})$ and $\mathbf{D}_{Sk}(\mathbf{r})$ are normalized correctly. Here

$$[a_{Ik},a_{I'k'}]=0, \quad [a_{Ik},a_{I'k'}^{\dagger}]=\delta_{II'}\delta(k-k'), \tag{5}$$

with I and I' ranging over S and F; putting for convenience

$$a_{Fk}\to c_k, \quad a_{Sk}\to b_k, \tag{6}$$

the linear Hamiltonian is given by

$$H_L=\int dk\hbar\omega_{Fk}c_k^{\dagger}c_k+\int dk\hbar\omega_{Sk}b_k^{\dagger}b_k, \tag{7}$$

where we neglect zero point energies and ω_{Ik} are the eigenfrequencies of the modes. Often the mode amplitudes are found neglecting any material dispersion, but the inclusion of material dispersion at the linear level is straightforward [22]. If at frequencies of interest the structure is characterized by a real-, position-, and frequency-dependent refractive index $n(x,y;\omega)$, then the mode amplitudes and eigenfrequencies are found by solving a self-consistent version of the usual "master equation"

$$\boldsymbol{\nabla}\times\left[\frac{\boldsymbol{\nabla}\times\mathbf{B}_{Ik}(\mathbf{r})}{n^2(x,y;\omega_{Ik})}\right]=\left(\frac{\omega_{Ik}}{c}\right)^2\mathbf{B}_{Ik}(\mathbf{r}),$$

where we take $\omega_{Ik}\geq 0$, and then constructing

$$\mathbf{D}_{Ik}(\mathbf{r})=\frac{i}{\mu_0\omega_{Ik}}\boldsymbol{\nabla}\times\mathbf{B}_{Ik}(\mathbf{r}).$$

The nature of the structures we consider (see Fig. 1) guarantees that the modes will be of the form (1), and the normalization condition for the modes [23] takes the form

$$\begin{aligned}&\int dxdy\frac{\mathbf{d}_{Ik}^{*}(x,y)\cdot\mathbf{d}_{Ik}(x,y)}{\varepsilon_0 n^2(x,y;\omega_{Ik})}\frac{v_p(x,y;\omega_{Ik})}{v_g(x,y;\omega_{Ik})}\\&\equiv\int dxdy\frac{\mathbf{d}_{Ik}^{*}(x,y)\cdot\mathbf{d}_{Ik}(x,y)}{\varepsilon_0}\left\{\frac{1}{n^2(x,y;\omega_{Ik})}\right.\\&\left.-\frac{\omega_{Ik}}{2}\left[\frac{\partial}{\partial\omega}\left(\frac{1}{n^2(x,y;\omega)}\right)\right]_{\omega=\omega_{Ik}}\right\}=1,\end{aligned} \tag{8}$$

where $v_p(x,y;\omega)$ and $v_g(x,y;\omega)$ are, respectively, the local phase and group velocities

$$v_p(x,y;\omega)=c/n(x,y;\omega),$$

$$v_g(x,y;\omega)=\frac{c/n(x,y;\omega)}{1+\frac{\omega}{n(x,y;\omega)}\frac{\partial n(x,y;\omega)}{\partial\omega}}.$$

This prescription, which harks back to pioneering work by Brillouin [24], has been used and discussed earlier by us [22] and others [25].

In constructing the nonlinear response, we first consider the Heisenberg picture for operators and consider the specification of the nonlinear polarization operator $\mathbf{P}_{\mathrm{NL}}(\mathbf{r},t)$. While we have included material dispersion in the linear response of our structure, since the nonlinear response is small it should be a good first approximation to neglect material dispersion in describing the nonlinear response. Doing this, and adopting the usual strategy of writing the nonlinear polarization as driven by the electric field, we would take

$$P_{\mathrm{NL}}^{i}(\mathbf{r},t)=\varepsilon_0\chi_2^{ijk}(\mathbf{r})E^j(\mathbf{r},t)E^k(\mathbf{r},t), \tag{9}$$

where superscripts denote Cartesian components and are to be summed over if repeated. However, with our choice of $\mathbf{D}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ as our fundamental fields, it is necessary to specify $\mathbf{P}_{\mathrm{NL}}(\mathbf{r},t)$ in terms of $\mathbf{D}(\mathbf{r},t)$ instead,

$$P_{\mathrm{NL}}^{i}(\mathbf{r},t)=\Gamma_2^{ijk}(\mathbf{r})D^j(\mathbf{r},t)D^k(\mathbf{r},t), \tag{10}$$

and we discuss the position dependence of $\Gamma_2^{ijk}(\mathbf{r})$ [and $\chi_2^{ijk}(\mathbf{r})$] below. In the absence of material dispersion and any nonlinear magneto-optical effects, both $\chi_2^{ijk}(\mathbf{r})$ and $\Gamma_2^{ijk}(\mathbf{r})$ are symmetric under all permutations of (ijk). The Schrödinger picture nonlinear Hamiltonian that should to added to H_L to construct the full Hamiltonian [26] is

$$H_{\mathrm{NL}}=-\frac{1}{3\varepsilon_0}\int d\mathbf{r}\Gamma_2^{ijk}(\mathbf{r})D^i(\mathbf{r})D^j(\mathbf{r})D^k(\mathbf{r}), \tag{11}$$

and using Eq. (4) in Eq. (11) we find

$$H_{\mathrm{NL}}=-\int dk_1dk_2dk\ \ S(k_1,k_2,k)c_{k_1}^{\dagger}c_{k_2}^{\dagger}b_k+\mathrm{H.c.}, \tag{12}$$

where we have only kept terms associated with down conversion of photons and

$$\begin{aligned}S(k_1,k_2,k)=&\frac{1}{\varepsilon_0}\sqrt{\frac{(\hbar\omega_{Fk_1})(\hbar\omega_{Fk_2})(\hbar\omega_{Sk_1})}{(4\pi)^3}}\int d\mathbf{r}\Gamma_2^{ijk}(\mathbf{r})\\&\times[d_{Fk_1}^{i}(x,y)]^{*}[d_{Fk_2}^{j}(x,y)]^{*}d_{Sk}^{k}(x,y)e^{i(k-k_1-k_2)z}.\end{aligned} \tag{13}$$

For ease in final calculations we will replace $\Gamma_2^{ijk}(\mathbf{r})$ by the more familiar $\chi_2^{ijk}(\mathbf{r})$. To lowest order in the nonlinear response, we can use the linear response to write $\mathbf{D}(\mathbf{r},t)$ in terms of $\mathbf{E}(\mathbf{r},t)$ in Eq. (10), and then by comparison with Eq. (9) relate $\Gamma_2^{ijk}(\mathbf{r})$ to $\chi_2^{ijk}(\mathbf{r})$. In the absence of material dispersion, taking the local index of refraction to be $n_0(x,y)$, we find

$$\Gamma_2^{ijk}(\mathbf{r})=\frac{\chi_2^{ijk}(\mathbf{r})}{\varepsilon_0 n_0^6(x,y)}. \tag{14}$$

Now in our formulation here, where material dispersion is included at the linear level but still neglected at the nonlinear level, the obvious question is which $n(x,y;\omega)$ to use in place of $n_0(x,y)$ when writing $\Gamma_2^{ijk}(\mathbf{r})$ in terms of $\chi_2^{ijk}(\mathbf{r})$ in Eq. (13). Given the frequencies involved in Eq. (13), the obvious choice is to take

$$\Gamma_2^{ijk}(\mathbf{r})\to\frac{\chi_2^{ijk}(\mathbf{r})}{\varepsilon_0 n^2(x,y;\omega_{Fk_1})n^2(x,y;\omega_{Fk_2})n^2(x,y;\omega_{Sk})} \tag{15}$$

there, and this we do; the difference between Eqs. (15) and (14), with $n_0(x,y)$ in the latter taken as any of the $n(x,y;\omega)$ in Eq. (15), can be taken as an indication of the error introduced by neglecting material dispersion on the Hamiltonian describing nonlinear response.

We now specify the position dependence of $\chi_2^{ijk}(\mathbf{r})$. In a structure completely translationally invariant in the z direction, $\chi_2^{ijk}(\mathbf{r})$ would depend only on x and y, $\chi_2^{ijk}(\mathbf{r})$

$=\chi_{2,\mathrm{ideal}}^{ijk}(x,y)$. However, a variation in $\chi_2^{ijk}(\mathbf{r})$ due to poling performed to achieve quasi-phase-matching, for example, would lead to a z dependence in $\chi_2^{ijk}(\mathbf{r})$. As well, to investigate the dependence of SPDC on the length of the nonlinear structure used we will assume that $\chi_2^{ijk}(\mathbf{r})$ vanishes for $z<-L/2$ or $z>L/2$. So we take

$$\chi_2^{ijk}(\mathbf{r}) = \chi_{2,\mathrm{ideal}}^{ijk}(x,y)s(z), \tag{16}$$

where in the absence of any poling $s(z)=1$ for $-L/2<z<L/2$, but in any case $s(z)=0$ for $z<-L/2$ or $z>L/2$. An extension to consider a more general spatial dependence of $\chi_2^{ijk}(\mathbf{r})$ than Eq. (16) is straightforward.

Using Eqs. (15) and (16) in Eq. (13) we can write

$$S(k_1,k_2,k) = \sqrt{\frac{(\hbar\omega_{Fk_1})(\hbar\omega_{Fk_2})(\hbar\omega_{Sk_1})}{(4\pi)^3\varepsilon_0}}\frac{\mathfrak{s}^*(k-k_1-k_2)\bar{\chi}_2}{\bar{n}^3} \times\frac{e^{-i\nu(k_1,k_2,k)}}{\sqrt{\mathcal{A}(k_1,k_2,k)}}, \tag{17}$$

where

$$\mathfrak{s}(k) \equiv \int dz\ \ s(z)e^{-ikz} \tag{18}$$

and

$$\frac{e^{i\nu(k_1,k_2,k)}}{\sqrt{\mathcal{A}(k_1,k_2,k)}} \equiv \int dxdy\frac{\bar{n}^3\chi_{2,\mathrm{ideal}}^{ijk}(x,y)}{\bar{\chi}_2} \times\frac{d_{Fk_1}^i(x,y)d_{Fk_2}^j(x,y)[d_{Sk}^k(x,y)]^*}{\varepsilon_0^{3/2}n^2(x,y;\omega_{Fk_1})n^2(x,y;\omega_{Fk_2})n^2(x,y;\omega_{Sk})}, \tag{19}$$

with $\nu(k_1,k_2,k)$ and $\mathcal{A}(k_1,k_2,k)$ taken to be real, and $\mathcal{A}(k_1,k_2,k)>0$. To help identify the physics of the terms in Eq. (17), we have introduced $\bar{\chi}_2$ as a typical size of one of the nonvanishing components of $\chi_{2,\mathrm{ideal}}^{ijl}(x,y)$ in the structure, and $\bar{n}$ as a typical value of the local refractive index. Then $\mathcal{A}(k_1,k_2,k)$ has units of area and can be understood as an effective coupling area between the modes involved in the spontaneous parametric down-conversion. Often codes (e.g., MODE SOLUTIONS [27]) for finding the linear mode profiles will directly generate not the displacement fields $\mathbf{d}_{Ik}(x,y)$, but the corresponding electric field amplitudes $\mathbf{e}_{Ik}(x,y)$,

$$\mathbf{d}_{Ik}(x,y) = \varepsilon_0 n^2(x,y;\omega_{Ik})\mathbf{e}_{Ik}(x,y). \tag{20}$$

We can use this in Eq. (19) to get an expression in terms of the $\mathbf{e}_{Ik}(x,y)$, and combining it with the normalization expressions (8), also written in terms of the $\mathbf{e}_{Ik}(x,y)$, we find

$$\frac{e^{i\nu(k_1,k_2,k)}}{\sqrt{\mathcal{A}(k_1,k_2,k)}} = \frac{\mathcal{N}_{FFS}(k_1,k_2,k)}{\sqrt{\mathcal{D}_F(k_1)\mathcal{D}_F(k_2)\mathcal{D}_S(k)}}, \tag{21}$$

where

$$\mathcal{N}_{FFS}(k_1,k_2,k) = \int dxdy\frac{\chi_{2,\mathrm{ideal}}^{ijk}(x,y)}{\bar{\chi}_2}e_{Fk_1}^i(x,y)e_{Fk_2}^j(x,y) \times[e_{Sk}^k(x,y)]^*,$$

$$\mathcal{D}_I(k) = \int dxdy\mathbf{e}_{Ik}^*(x,y)\cdot\mathbf{e}_{Ik}(x,y)\frac{n^2(x,y;\omega_{Ik})}{\bar{n}^2}\frac{v_p(x,y;\omega_{Ik})}{v_g(x,y;\omega_{Ik})}. \tag{22}$$

An advantage of expression (21) over the formally simpler Eq. (19) is that Eq. (21) can be used even with unnormalized mode fields $\mathbf{e}_{Ik}(x,y)$, as are likely to be generated from most codes [27].

III. ASYMPTOTIC STATES AND OPERATOR EQUATIONS

Our Hamiltonian is then

$$H = H_L + H_{\mathrm{NL}},$$

with H_L given by Eq. (7) and H_{NL} by Eq. (12). We imagine an initial, input state $|\psi(t_0)\rangle$ at $t_0\ll 0$ incident on the nonlinear region $(-L/2<z<L/2)$ with the electromagnetic field energy localized at $z\ll 0$ outside the nonlinear region, and we seek the final state $|\psi(t_1)\rangle$ for a time $t_1\gg 0$ such that the electromagnetic field energy is localized at $z\gg 0$,

$$|\psi(t_1)\rangle = e^{-iH(t_1-t_0)/\hbar}|\psi(t_0)\rangle.$$

We borrow a strategy from scattering theory, where typically one takes $t_0\to-\infty$ and $t_1\to\infty$, as we eventually do in part of the calculation below, and introduce asymptotic-in and -out states. Imagine that the initial state $|\psi(t_0)\rangle$ evolved from $t=t_0$ to $t=0$ according to H_L. The resulting state is defined as the asymptotic-in state $|\psi_{\mathrm{in}}\rangle$,

$$e^{-iH_L(0-t_0)/\hbar}|\psi(t_0)\rangle = e^{iH_L t_0/\hbar}|\psi(t_0)\rangle = |\psi_{\mathrm{in}}\rangle.$$

The asymptotic-out state $|\psi_{\mathrm{out}}\rangle$ is defined as the state at $t=0$ that would evolve into $|\psi(t_1)\rangle$ if the evolution occurred according to H_L,

$$e^{-iH_L(t_1-0)/\hbar}|\psi_{\mathrm{out}}\rangle = e^{-iH_L t_1/\hbar}|\psi_{\mathrm{out}}\rangle = |\psi(t_1)\rangle.$$

Then we can write

$$|\psi_{\mathrm{out}}\rangle = U(t_1,t_0)|\psi_{\mathrm{in}}\rangle,$$

where it is useful to define $U(t',t)$ not only for $t'=t_1$ and $t=t_0$, but for all t' and t, according to

$$U(t',t) = e^{iH_L t'/\hbar}e^{-iH(t'-t)/\hbar}e^{-iH_L t/\hbar}. \tag{23}$$

All the trivial evolution is eliminated by introducing $|\psi_{\mathrm{in}}\rangle$ and $|\psi_{\mathrm{out}}\rangle$, and by identifying $U(t_1,t_0)$ we focus on the effect of the nonlinearity. The evolution of any initial state $|\psi(t_0)\rangle$ to $|\psi_{\mathrm{in}}\rangle$ is easily calculated, and once $|\psi_{\mathrm{out}}\rangle$ has been determined the final state of interest $|\psi(t_1)\rangle$ can be easily found; these calculations involve only the simple Hamiltonian H_L.

We proceed by identifying the dynamics of $U(t',t)$, and using it to relate $|\psi_{\mathrm{out}}\rangle$ to $|\psi_{\mathrm{in}}\rangle$. From Eq. (23) we see that

$$U(t,t) = 1, \tag{24}$$

for all t, and by differentiating Eq. (23) we find

$$-i\hbar\frac{\partial U(t',t)}{\partial t} = U(t',t)V(t), \tag{25}$$

where

$$V(t) \equiv e^{iH_L t/\hbar} H_{\mathrm{NL}} e^{-iH_L t/\hbar}. \tag{26}$$

In this paper we consider asymptotic-in states of the form

$$|\psi_{\mathrm{in}}\rangle = e^{O}|\mathrm{vac}\rangle, \tag{27}$$

where $|\mathrm{vac}\rangle$ is a vacuum state, and O is some (Schrödinger) operator; generalizations to states of the form $f(O_1, O_2, \ldots)|\mathrm{vac}\rangle$, where $\{O_i\}$ is a set of Schrödinger operators, follow from the approach below. From Eq. (27) we have

$$|\psi_{\mathrm{out}}\rangle = U(t_1,t_0)e^{O}|\mathrm{vac}\rangle.$$

and since both H and H_L are simple enough that

$$H|\mathrm{vac}\rangle = H_L|\mathrm{vac}\rangle = 0,$$

because each of the terms in those Hamiltonians involves at least one lowering operator on the right, we have

$$U(t_1,t_0)|\mathrm{vac}\rangle = U^{\dagger}(t_1,t_0)|\mathrm{vac}\rangle = |\mathrm{vac}\rangle$$

and we can write

$$\begin{aligned}|\psi_{\mathrm{out}}\rangle &= U(t_1,t_0)e^{O}U^{\dagger}(t_1,t_0)|\mathrm{vac}\rangle = e^{U(t_1,t_0)OU^{\dagger}(t_1,t_0)}|\mathrm{vac}\rangle \\ &\equiv e^{\bar{O}(t_0)}|\mathrm{vac}\rangle,\end{aligned} \tag{28}$$

where

$$\bar{O}(t) = U(t_1,t)OU^{\dagger}(t_1,t). \tag{29}$$

The operator $\bar{O}(t)$ satisfies the "final" condition

$$\bar{O}(t_1) = O, \tag{30}$$

and from Eq. (29) we can derive a differential equation that it satisfies; using Eq. (25) we find

$$i\hbar\frac{d\bar{O}(t)}{dt} = [\bar{O}(t), \hat{V}(t)], \tag{31}$$

where

$$\hat{V}(t) \equiv U(t_1,t)V(t)U^{\dagger}(t_1,t) \tag{32}$$

and where we have used the fact that $V^{\dagger}(t) = V(t)$. So our goal is to integrate the Heisenberg Eq. (31) from $t=t_1$ back to $t=t_0$, subject to the final condition (30). This will give us $\bar{O}(t_0)$ in terms of Schrödinger operators, and we can then determine $|\psi_{\mathrm{out}}\rangle$. Using the nonlinear part Eq. (12) of our Hamiltonian in the definition (26) of $V(t)$, and then using that in the definition (32) of $\hat{V}(t)$, we find

$$\hat{V}(t) = -\int dk_1 dk_2 dk\ \ S(k_1,k_2,k;t)\bar{c}^{\dagger}_{k_1}(t)\bar{c}^{\dagger}_{k_2}(t)\bar{b}_k(t) + \mathrm{H.c.},$$

where

$$S(k_1,k_2,k;t) = S(k_1,k_2,k)e^{-i(\omega_{Sk}-\omega_{Fk_1}-\omega_{Fk_2})t}. \tag{33}$$

IV. THE UNDEPLETED PUMP APPROXIMATION

As an asymptotic-in state we want to consider a coherent state of the electromagnetic field associated with a very general waveform incident on the nonlinear region. As we confirm in Appendix A, we can do this by writing

$$|\psi_{\mathrm{in}}\rangle = e^{(\alpha A_P^{\dagger} - \mathrm{H.c.})}|\mathrm{vac}\rangle, \tag{34}$$

where

$$A_P^{\dagger} = \int dk \phi_P(k) b_k^{\dagger}, \tag{35}$$

and the function $\phi_P(k)$ is normalized according to

$$\int |\phi_P(k)|^2 dk = 1. \tag{36}$$

We show there that

$$\langle\psi_{\mathrm{in}}|\mathbf{D}(\mathbf{r})|\psi_{\mathrm{in}}\rangle = \alpha\int dk\sqrt{\frac{\hbar\omega_{Sk}}{4\pi}}\phi_P(k)\mathbf{d}_{Sk}(x,y)e^{ikz} + \mathrm{c.c.}, \tag{37}$$

with an expected number of photons in the input pulse given by $|\alpha|^2$. We take $\phi_P(k)$ to be peaked at some value $k_0 > 0$. Considering functions $\phi_P(k)$ with a range of widths in $|k - k_0|$, functions $\phi_P(k)$ described by wider and wider widths correspond to shorter and shorter pulses; as the width of $\phi_P(k)$ becomes very small we approach an incident cw beam. We plan to turn to more general input states in a later publication.

From the form (27) of our asymptotic-in state Eq. (34), we have an asymptotic-out state Eq. (28)

$$|\psi_{\mathrm{out}}\rangle = e^{(\alpha\bar{A}_P^{\dagger}(t_0) - \mathrm{H.c.})}|\mathrm{vac}\rangle, \tag{38}$$

where [recall Eqs. (29) and (35)]

$$\bar{A}_P^{\dagger}(t_0) = \int dk \phi_P(k)\bar{b}_k^{\dagger}(t_0). \tag{39}$$

From the Heisenberg Eqs. (31) follows a set of differential equations for the $\bar{b}_k^{\dagger}(t)$,

$$i\hbar\frac{d\bar{b}_k^{\dagger}(t)}{dt} = \int dk_1 dk_2\ \ S(k_1,k_2,k;t)\bar{c}^{\dagger}_{k_1}(t)\bar{c}^{\dagger}_{k_2}(t), \tag{40}$$

and a corresponding set of equations for the $\bar{c}_k^{\dagger}(t)$. We are integrating from t_1 back to t_0, subject to the final condition that for all barred operators $\bar{O}(t_1) = O$, the corresponding Schrödinger operator; the zeroth order solution of these equations is thus given by

$$[\bar{c}_k^{\dagger}(t)]^{(0)} = \bar{c}_k^{\dagger}(t_1) = c_k^{\dagger}, \quad [\bar{b}_k^{\dagger}(t)]^{(0)} = \bar{b}_k^{\dagger}(t_1) = b_k^{\dagger},$$

where the right-hand sides of course indicate the Schrödinger operators. So, to first order the solution for $\bar{b}_k^{\dagger}(t)$ is

$$\bar{b}_k^{\dagger}(t) = b_k^{\dagger} + \frac{1}{i\hbar}\int dk_1 dk_2\left[\int_{t_1}^{t} dt' S(k_1,k_2,k;t')\right]c^{\dagger}_{k_1}c^{\dagger}_{k_2}$$

or, after changing the range of integration

$$b_k^\dagger(t_0) = b_k^\dagger + \frac{i}{\hbar}\int dk_1 dk_2 \left[\int_{t_0}^{t_1} dt' S(k_1,k_2,k;t')\right] c_{k_1}^\dagger c_{k_2}^\dagger.$$

Using this in Eq. (39) and taking $t_0 \to -\infty$ and $t_1 \to \infty$, we have

$$\bar{A}_P^\dagger(-\infty) = A_P^\dagger + \frac{\beta}{\alpha} C_{\mathrm{II}}^\dagger, \tag{41}$$

where $A_P^\dagger$ is given by Eq. (35), and

$$C_{\mathrm{II}}^\dagger = \frac{1}{\sqrt{2}}\int dk_1 dk_2 \phi(k_1,k_2) c_{k_1}^\dagger c_{k_2}^\dagger, \tag{42}$$

with

$$\phi(k_1,k_2) = \frac{\sqrt{2}\alpha}{\beta}\frac{i}{\hbar}\int dk \phi_P(k)\int_{-\infty}^{\infty} dt \;\; S(k_1,k_2,k;t), \tag{43}$$

and the amplitude parameter β chosen so that $\phi(k_1,k_2) = \phi(k_2,k_1)$ is normalized according to

$$\int dk_1 dk_2 |\phi(k_1,k_2)|^2 = 1. \tag{44}$$

Using Eq. (41) in our expression (38) for the asymptotic-out state, we have

$$|\psi_{\mathrm{out}}\rangle = e^{(\alpha A_P^\dagger + \beta C_{\mathrm{II}}^\dagger) - \mathrm{H.c.}}|\mathrm{vac}\rangle = e^{\beta C_{\mathrm{II}}^\dagger - \mathrm{H.c.}} e^{\alpha A_P^\dagger - \mathrm{H.c.}}|\mathrm{vac}\rangle$$
$$= e^{\beta C_{\mathrm{II}}^\dagger - \mathrm{H.c.}}|\psi_{\mathrm{in}}\rangle,$$

since A_P and C_{II} commute. The operator $\exp(\beta C_{\mathrm{II}}^\dagger - \mathrm{H.c.})$ acting on $|\psi_{\mathrm{in}}\rangle$ describes the presence of the generated photons, since the operator $C_{\mathrm{II}}^\dagger$ involves the creation of photons in channel F. The fact that the state of the pump mode is unchanged within this calculation is of course a result of the undepleted pump nature of the approximation made in solving Eq. (40).

V. THE STATE OF GENERATED PHOTONS

Note that we can also write

$$|\psi_{\mathrm{out}}\rangle = e^{\alpha A_P^\dagger - \mathrm{H.c.}}|\psi_{\mathrm{out}}^F\rangle,$$

where

$$|\psi_{\mathrm{out}}^F\rangle = e^{\beta C_{\mathrm{II}}^\dagger - \mathrm{H.c.}}|\mathrm{vac}\rangle \tag{45}$$

describes the state of the generated photons. From the expression (43) for $\phi(k_1,k_2)$ and the normalization condition (44) it satisfies, we see that as the number of photons in the pump pulse becomes small ($|\alpha|^2 \to 0$), for a fixed pump pulse shape $\phi_P(k)$ we have $|\beta|^2 \to 0$. In the limit that $|\beta| \ll 1$ we can approximate

$$|\psi_{\mathrm{out}}^F\rangle = |\mathrm{vac}\rangle + \beta|\mathrm{II}\rangle + \cdots, \tag{46}$$

where

$$|\mathrm{II}\rangle = C_{\mathrm{II}}^\dagger|\mathrm{vac}\rangle = \frac{1}{\sqrt{2}}\int dk_1 dk_2 \phi(k_1,k_2) c_{k_1}^\dagger c_{k_2}^\dagger |\mathrm{vac}\rangle \tag{47}$$

is a normalized two-photon state

$$\int dk c_k^\dagger c_k |\mathrm{II}\rangle = 2|\mathrm{II}\rangle, \quad \langle \mathrm{II}|\mathrm{II}\rangle = 1. \tag{48}$$

Equations (48) follow from the normalization condition (44) and the commutation relations (5). Thus, in the limit $|\beta| \ll 1$, we see from Eq. (46) that $|\beta|^2$ is the probability that an entangled pair of photons will be generated. However, we stress that outside this limit, even if many pairs are generated, Eq. (45) gives the state of the generated photons, as long as the undepleted pump approximation is valid. The biphoton wave function thus has an applicability well beyond the limit where there is only a small probability that a pair of photons is generated. As long as the undepleted pump approximation is valid, the biphoton wave function $\phi(k_1,k_2)$, together with the amplitude parameter β, serve to fully specify the state (45) of the generated photons, although it is more complicated than Eq. (46). We now turn to the expressions for $\phi(k_1,k_2)$ and β.

Using the time dependence (33) of $S(k_1,k_2,k;t)$ in the result (43) for the biphoton wave function, that result reduces to

$$\phi(k_1,k_2) = \frac{2\sqrt{2}\pi\alpha}{\beta}\frac{i}{\hbar}\int dk \phi_P(k) S(k_1,k_2,k)$$
$$\times \delta(\omega_{Sk} - \omega_{Fk_1} - \omega_{Fk_2}). \tag{49}$$

We assume that the pump waveform function $\phi_P(k)$ is sufficiently peaked at a $k_0 > 0$ so that its value is negligible for $k < 0$; thus the integral in Eq. (49) can be restricted to the range from 0 to ∞. The combination of energy conservation, expressed by the Dirac delta function in Eq. (49) and the requirement of approximate momentum conservation, residing in the factor $\mathfrak{s}^*(k-k_1-k_2)$ of $S(k_1,k_2,k)$ (17) that restricts k_1+k_2 to approximately k, forces $\phi(k_1,k_2)$ to be non-negligible only for $k_1 > 0$ and $k_2 > 0$ if typical dispersion relations are assumed. That is, to good approximation all the photons generated by a pump propagating in the forward direction are themselves propagating in the forward direction. With this accepted, we can then rewrite our expressions (35) and (42) for $A_P^\dagger$ and $C_{\mathrm{II}}^\dagger$, respectively, as

$$A_P^\dagger = \int_0^\infty dk \phi_P(k) b_k^\dagger, \quad C_{\mathrm{II}}^\dagger = \frac{1}{\sqrt{2}}\int_0^\infty dk_1 \int_0^\infty dk_2 \phi(k_1,k_2) c_{k_1}^\dagger c_{k_2}^\dagger. \tag{50}$$

Specifying our dispersion relations now by the dependence of (positive) wave number on (positive) frequency, $k_F(\omega)$ and $k_S(\omega)$ with the channels indicated by subscripts, for reasonable dispersion relations there is a one-to-one relation between wave numbers and frequencies, and it is possible to label the modes (1), for $k > 0$, equivalently by their frequencies. Formally we can do this by introducing new operators

$$\tilde{b}_\omega \equiv \sqrt{\frac{dk_S(\omega)}{d\omega}} b_{k_S(\omega)}, \quad \tilde{c}_\omega \equiv \sqrt{\frac{dk_F(\omega)}{d\omega}} c_{k_F(\omega)},$$

where the prefactors are introduced so our new operators are of canonical form

$$[\tilde{b}_{\omega},\tilde{b}^{\dagger}_{\omega'}]=\delta(\omega-\omega'),\quad [\tilde{c}_{\omega},\tilde{c}^{\dagger}_{\omega'}]=\delta(\omega-\omega').$$

We also introduce new functions

$$\tilde{\phi}_P(\omega)=\sqrt{\frac{dk_S(\omega)}{d\omega}}\phi_P(k_S(\omega)), \tag{51}$$

$$\tilde{\phi}(\omega_1,\omega_2)=\sqrt{\frac{dk_F(\omega_1)}{d\omega_1}}\sqrt{\frac{dk_F(\omega_2)}{d\omega_2}}\phi(k_F(\omega_1),k_F(\omega_2)), \tag{52}$$

where we write $dk_F(\omega_1)/d\omega_1$ for $(dk_F(\omega)/d\omega)_{\omega=\omega_1}$, etc.; the prefactors are introduced so our new functions are appropriately normalized,

$$\int_0^{\infty}|\tilde{\phi}_P(\omega)|^2 d\omega=1, \tag{53}$$

$$\int_0^{\infty}d\omega_1\int_0^{\infty}d\omega_2|\tilde{\phi}(\omega_1,\omega_2)|^2=1, \tag{54}$$

where we have assumed the range of the wave numbers in the original integrals (36) and (44) can be restricted to positive values. Then in place of (50) we can write

$$A_P^{\dagger}=\int_0^{\infty}d\omega\tilde{\phi}_P(\omega)\tilde{b}^{\dagger}_{\omega},$$

$$C_{II}^{\dagger}=\frac{1}{\sqrt{2}}\int_0^{\infty}d\omega_1\int_0^{\infty}d\omega_2\tilde{\phi}(\omega_1,\omega_2)\tilde{c}^{\dagger}_{\omega_1}\tilde{c}^{\dagger}_{\omega_2}.$$

and use $\tilde{\phi}(\omega_1,\omega_2)$ to characterized the biphoton wave function; the original $\phi(k_1,k_2)$ can then always be recovered from Eq. (52).

From Eq. (49) we find, converting the integral over k to one over ω,

$$\begin{aligned}\tilde{\phi}(\omega_1,\omega_2)=&\frac{2\sqrt{2}\pi\alpha}{\beta}\frac{i}{\hbar}\sqrt{\frac{dk_F(\omega_1)}{d\omega_1}}\sqrt{\frac{dk_F(\omega_2)}{d\omega_2}}\int_0^{\infty}d\omega\\&\times\sqrt{\frac{dk_S(\omega)}{d\omega}}\tilde{\phi}_P(\omega)S(k_F(\omega_1),k_F(\omega_2),k_S(\omega))\\&\times\delta(\omega-\omega_1-\omega_2)\end{aligned} \tag{55}$$

and the integral over ω can be trivially done due to the Dirac delta function. Using the expression (17) for $S(k_1,k_2,k)$ we then use the normalization condition (54) to determine β, choosing for simplicity the phase of β to be the same as α. The result is

$$\beta=\frac{\alpha\bar{\chi}_2}{\bar{n}^3}\sqrt{\frac{K}{8\pi\varepsilon_0}}, \tag{56}$$

where

$$K=\int_0^{\infty}\omega_1\frac{dk_F(\omega_1)}{d\omega_1}\int_0^{\infty}\omega_2\frac{dk_F(\omega_2)}{d\omega_2}J(\omega_1,\omega_2)d\omega_2 d\omega_1 \tag{57}$$

with

$$J(\omega_1,\omega_2)=\left(\frac{dk_S(\omega)}{d\omega}\right)_{\omega=\omega_1+\omega_2}\frac{\hbar(\omega_1+\omega_2)|\tilde{\phi}_P(\omega_1+\omega_2)\mathfrak{s}(k_S(\omega_1+\omega_2)-k_F(\omega_1)-k_F(\omega_2))|^2}{\mathcal{A}(k_F(\omega_1),k_F(\omega_2),k_S(\omega_1+\omega_2))}, \tag{58}$$

and using this in Eq. (55) we have

$$\begin{aligned}\tilde{\phi}(\omega_1,\omega_2)=&i\sqrt{\frac{dk_F(\omega_1)}{d\omega_1}}\sqrt{\frac{dk_F(\omega_2)}{d\omega_2}}\left(\sqrt{\frac{dk_S(\omega)}{d\omega}}\right)_{\omega=\omega_1+\omega_2}\\&\times\sqrt{\frac{\omega_1\omega_2}{K}}\sqrt{\frac{\hbar(\omega_1+\omega_2)}{\mathcal{A}(k_F(\omega_1),k_F(\omega_2),k_S(\omega_1+\omega_2))}}\\&\times e^{-i\nu(k_F(\omega_1),k_F(\omega_2),k_S(\omega_1+\omega_2))}\tilde{\phi}_P(\omega_1+\omega_2)\mathfrak{s}^*(k_S(\omega_1\\&+\omega_2)-k_F(\omega_1)-k_F(\omega_2)),\end{aligned} \tag{59}$$

With explicit expressions for the amplitude parameter β and the biphoton wave function $\tilde{\phi}(\omega_1,\omega_2)$ in hand, we can construct the output state of the channel in which photons are generated (45) either using the ω representation, or returning to the k representation via Eqs. (52) and (59),

$$\begin{aligned}|\psi^F_{\rm out}\rangle&=\exp\left(\frac{\beta}{\sqrt{2}}\int_0^{\infty}d\omega_1\int_0^{\infty}d\omega_2\tilde{\phi}(\omega_1,\omega_2)\tilde{c}^{\dagger}_{\omega_1}\tilde{c}^{\dagger}_{\omega_2}-\text{H.c.}\right)|\text{vac}\rangle\\&=\exp\left(\frac{\beta}{\sqrt{2}}\int_0^{\infty}dk_1\int_0^{\infty}dk_2\phi(k_1,k_2)c^{\dagger}_{k_1}c^{\dagger}_{k_2}-\text{H.c.}\right)|\text{vac}\rangle.\end{aligned} \tag{60}$$

In Appendix B we work out these expressions for the interesting but artificial case of no group velocity dispersion in either channel and perfect phase matching. A more realistic sample calculation is given in the next section.

VI. EXAMPLE

In this section we use the formalism we have established to describe SPDC in a particular ridge waveguide. Our main focus is not the details of how this particular waveguide

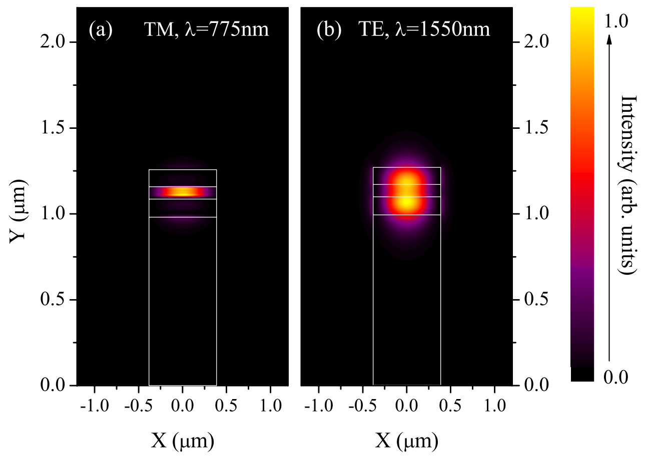

FIG. 2. (Color online) Electric field intensity for the lowest order (a) TM mode at $\lambda = 775$ nm and (b) TE mode at $\lambda = 1550$ nm.

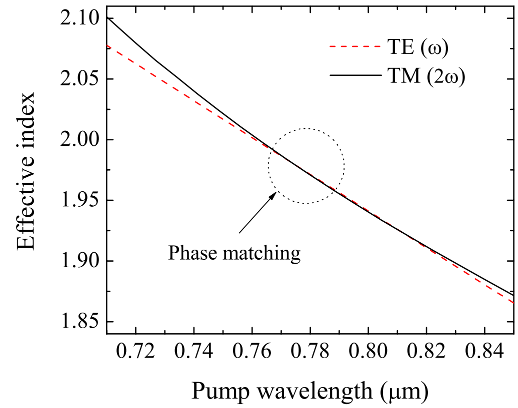

FIG. 3. (Color online) Effective index for TE (red solid line) and TM (black dashed line) as a function for the pump wavelength λ_s. Phase matching main region is indicated by the arrow.

could be used to generate entangled photons of interest, but rather the illustration of the formalism and the identification of a few crucial physical points that the discussion of a concrete example allows.

The structure we consider involves a ridge of width $w = 780$ nm and length L, as sketched in Fig. 1. The guide core is composed of a 66 nm Al_yO_x slot embedded between two 100 nm $Al_{0.5}Ga_{0.5}As$ layers on an Al_yO_x substrate. These parameters have been chosen to achieve phase matching between the lowest order transverse-electric (TE) mode in the neighborhood of $\lambda_{F_0} = 1.55$ μm and the lowest order transverse-magnetic (TM) mode in the neighborhood of $\lambda_{S_0} = 0.775$ μm. This design follows that proposed by Scaccabarozzi *et al.* [1], where form birefringence between TE and TM modes is exploited to achieve the phase matching condition [28]. This effect can be understood by looking at Fig. 2(a) and 2(b), where we show the mode field profiles, calculated using MODE SOLUTIONS [27]. The TM mode is strongly localized in the low-index slot, due to the discontinuity of the normal component of the electric field at the slot interfaces, and hence its effective index is sensibly reduced. On the contrary, the field components of the TE mode are tangential to the interfaces, and thus its confinement is only slightly affected by the slot presence [29,30]. Adopting the usual definition of the effective indices $n_{F,S}(\omega)$ of the modes, $n_{F,S}(\omega) \equiv ck_{F,S}(\omega)/\omega$, we plot the indices in Fig. 3 as a function of the pump wavelength λ_S, taking $\lambda_F = 2\lambda_S$; the phase-matching condition is realized close to $\lambda_{S_0} = 0.775$ μm. Material dispersion is included in the calculations of both the mode profiles and the effective indices.

Let us study the conversion efficiency and the nature of the photon pairs generated in the F channel. For this we need to calculate the amplitude parameter β in Eq. (56) and the biphoton wave function $\phi(k_1,k_2)$ or $\tilde{\phi}(\omega_1,\omega_2)$ [see Eqs. (52) and (59)], and we now identify the parameters and functions necessary to do that. Since no modulation of χ_2 is involved in this structure, the $s(z)$ in Eq. (16) is given by

$$s(z) = 1 \quad \text{for} \quad |z| < \frac{L}{2},$$

$$= 0 \quad \text{for} \quad |z| > \frac{L}{2} \tag{61}$$

and thus, according to Eq. (18),

$$\mathfrak{s}(k) = L\frac{\sin\left(\frac{kL}{2}\right)}{\left(\frac{kL}{2}\right)}. \tag{62}$$

For a sample calculation, we take the length of the waveguide to be

$$L = 600 \quad \mu\text{m}. \tag{63}$$

To specify the dispersion of the waveguide modes, we use a quadratic model around the phase-matching condition to match the dispersions obtained from numerical simulations and shown in Fig. 3,

$$k_S(\omega) = k_{S_0} + \frac{\omega - \omega_{S_0}}{v_{S_0}} + \Lambda_{S_0}(\omega - \omega_{S_0})^2,$$

$$k_F(\omega) = k_{F_0} + \frac{\omega - \omega_{F_0}}{v_{F_0}} + \Lambda_{F_0}(\omega - \omega_{F_0})^2, \tag{64}$$

where $k_{S_0,F_0} = k_{S,F}(\omega_{S_0,F_0})$ with $\omega_{S_0,F_0} = 2\pi c/\lambda_{S_0,F_0}$ and $\lambda_{F_0} = 2\lambda_{S_0}$, $1/v_{S_0,F_0} = [dk_{S,F}(\omega)/d\omega]_{\omega=\omega_{S_0,F_0}}$, and $\Lambda_{S_0,F_0} = [d^2k_{S,F}(\omega)/d\omega^2]_{\omega=\omega_{S_0,F_0}}/2$. At the phase-matching condition $\lambda_{S_0} = \lambda_{F_0}/2 = 0.775$ μm, we find

$$v_{F_0} = 95.1\frac{\mu\text{m}}{\text{ps}}, \quad v_{S_0} = 92.9\frac{\mu\text{m}}{\text{ps}}, \quad \Lambda_{S_0} = 2.7 \times 10^{-6}\frac{\text{ps}^2}{\mu\text{m}},$$

$$\Lambda_{F_0} = 3.5 \times 10^{-10} \frac{\text{ps}^2}{\mu\text{m}}. \tag{65}$$

The expansions (64) give a good description of both $k_{S,F}(\omega)$ over the frequency ranges of that arise below. In fact, the dispersion in this structure is sufficiently small that the frequency range of generated entangled photons is limited by the cutoff frequency ω_C of the fundamental mode, the minimum frequency at which this mode exists in the structure. Our numerical calculations identify the cutoff wavelength $\lambda_C = 2\pi c/\omega_C$ as about 1.70 μm; hence photons of longer wavelengths will not be generated in the waveguide. Turning to the other quantities that appear in the expressions (57)–(59), since the derivatives $dk_{S,F}(\omega)/d\omega$ change very slowly with ω, we can approximate them as constants $(dk_{F,S}(\omega)/d\omega \approx 1/v_{F_0,S_0})$ in Eqs. (57)–(59). Similarly we find from the simulation [27] that the coupling area $\mathcal{A}(k_F(\omega_1), k_F(\omega_2), k_S(\omega_1+\omega_2))$ is essentially constant over the frequencies of interest; we write it as $\mathcal{A}_0$. In calculating $\mathcal{A}(k_F(\omega_1), k_F(\omega_2), k_S(\omega_1+\omega_2))$ we take $\chi_{2,\text{ideal}}^{ijk}(x,y)=0$ in Al_yO_x and $\chi_{2,\text{ideal}}^{ijk}(x,y) = \bar{\chi}_2 \delta^{ijk}$ in $\text{Al}_{0.5}\text{Ga}_{0.5}\text{As}$, where appropriate for crystals of zinc-blende symmetry we have $\delta^{ijk}=1$ when $i \neq j \neq k$ and zero otherwise, with $i,j,k = x,y,z$.

With a value of $\bar{\chi}_2 = 100$ pm/V [31] appropriate for the GaAs alloys, and a nominal index $\bar{n}=3.0$ (the value of which will not affect any final results), we find $\mathcal{A}_0 = 2$ μm^2. This value is strongly affected by material dispersion. Were we to neglect the difference between the group and phase velocities in Eq. (22), we would underestimate $\mathcal{A}_0$ by a factor of about 4.

The dependence on ω_1 and ω_2 of the phase $\nu(k_F(\omega_1), k_F(\omega_2) k_S(\omega_1+\omega_2))$ appearing in Eq. (21) strictly depends, of course, on how the phases of the mode functions (1) are chosen to vary with k, and in principle this is arbitrary. Here, however, the physics provides a guide. We are using the functions $\phi_P(k)$ and $\phi(k_1,k_2)$ as surrogates for the electromagnetic field (4) or, more properly in the analysis of an actual experiment, as surrogates for the pump field as coupled into the waveguide, and for the fields associated with the entangled photons as coupled out of the waveguide. Thus we want fields with Fourier components that are slowly varying in k to be associated with functions $\phi_P(k)$ and $\phi(k_1,k_2)$ that are slowly varying functions of wave number. Since in typical waveguide structures the fields amplitudes $\mathbf{b}_{Ik}(x,y)$ and $\mathbf{d}_{Ik}(x,y)$ are vectors lying largely in the xy plane, and since in any case it is those components that will largely determine the coupling out of the waveguide structure, it is reasonable to define the amplitude $\mathbf{d}_{Ik}(x,y)$ so that, as a function of k, the integral of its transverse component over the xy plane has a phase independent of k. When this is done, we find that the phase $\nu(k_F(\omega_1), k_F(\omega_2) k_S(\omega_1+\omega_2))$ varies very little over the range of values of ω_1 and ω_2 of interest, and thus can be taken to be zero.

With these matters taken into account, the biphoton wave function $\tilde{\phi}(\omega_1,\omega_2)$ of Eq. (59) and the parameter K of Eq. (57) can be written as

$$\tilde{\phi}(\omega_1,\omega_2) = i \frac{\sqrt{\hbar} L}{v_{F_0}\sqrt{v_{S_0} K \mathcal{A}_0}} \sqrt{\omega_1\omega_2(\omega_1+\omega_2)}\, \tilde{\phi}_P(\omega_1+\omega_2) \times \frac{\sin(\Delta\theta_1 + \Delta\theta_2)}{\Delta\theta_1 + \Delta\theta_2} \tag{66}$$

and

$$K = \frac{\hbar L^2}{v_{F_0}^2 v_{S_0} \mathcal{A}_0} \int_{\omega_C}^{\infty} d\omega_1 d\omega_2 \omega_1 \omega_2 (\omega_1+\omega_2) |\tilde{\phi}_P(\omega_1+\omega_2)|^2 \times \left| \frac{\sin(\Delta\theta_1 + \Delta\theta_2)}{\Delta\theta_1 + \Delta\theta_2} \right|^2, \tag{67}$$

where

$$\Delta\theta_1 = \frac{\omega_1+\omega_2-\omega_{S_0}}{\delta v_p} \frac{L}{2}, \qquad \frac{1}{\delta v_p} \equiv \frac{1}{v_{S_0}} - \frac{1}{v_{F_0}},$$

$$\Delta\theta_2 = \left[\frac{\Lambda_{S_0} L}{2} - \frac{\Lambda_{F_0} L}{4} \right] (\omega_1+\omega_2-\omega_{S_0})^2 - \frac{\Lambda_{F_0} L}{4} (\omega_1-\omega_2)^2. \tag{68}$$

We have taken into account the fundamental mode cutoff at ω_C in Eq. (67), and because of that cutoff $\tilde{\phi}(\omega_1,\omega_2)$ should only be taken to be given by Eq. (66) for ω_1 and $\omega_2 > \omega_C$; otherwise it should be taken to vanish.

To evaluate Eq. (67) we change variables to $\omega_t = \omega_1+\omega_2$ and $\omega_r = (\omega_1-\omega_2)/2$, allowing us to write

$$K = \frac{\hbar L^2}{v_{F_0}^2 v_{S_0} \mathcal{A}_0} \int_{2\omega_C}^{\infty} d\omega_t \omega_t |\tilde{\phi}_P(\omega_t)|^2 \int_{-(1/2\omega_t-\omega_C)}^{1/2\omega_t-\omega_C} d\omega_r \left(\frac{1}{4}\omega_t^2 - \omega_r^2 \right) \times \left| \frac{\sin(\Delta\theta_1+\Delta\theta_2)}{\Delta\theta_1+\Delta\theta_2} \right|^2.$$

For a pump pulse spectrum narrowly centered at ω_{S_0}, in the first integral we can take

$$|\tilde{\phi}_P(\omega_t)|^2 \approx \delta(\omega_t - \omega_{S_0}) \tag{69}$$

and we find

$$K \approx K_0 = \frac{\hbar L^2}{v_{F_0}^2 v_{S_0} \mathcal{A}_0} \omega_{S_0} \int_{-(1/2\omega_{S_0}-\omega_C)}^{1/2\omega_{S_0}-\omega_C} d\omega_r \left(\frac{1}{4}\omega_{S_0}^2 - \omega_r^2 \right) \times \left| \frac{\sin \frac{\omega_r^2}{\Omega_{F_0}^2}}{\frac{\omega_r^2}{\Omega_{F_0}^2}} \right|^2, \tag{70}$$

where $\Omega_{F_0} \equiv |\Lambda_{F_0} L|^{-1/2} \approx 2\pi \times 350$ THz. Since $\Omega_{F_0} \gg \omega_{S_0}/2 - \omega_C \approx 2\pi \times 17$ THz, we see clearly that the spectral range of photon pairs produced is limited in this structure not by the group velocity dispersion, but by the mode cutoff; we can thus write

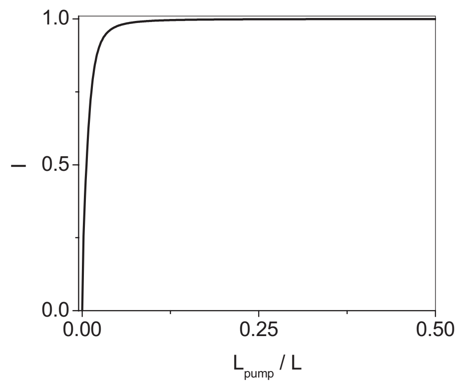

FIG. 4. Normalized conversion efficiency ($\mathcal{I}$) as a function of L_{pump}/L.

$$K_0 \approx \frac{\hbar L^2}{v_{F_0}^2 v_{S_0} \mathcal{A}_0} \omega_{S_0} \int_{-(1/2\omega_{S_0}-\omega_C)}^{1/2\omega_{S_0}-\omega_C} d\omega_r \left(\frac{1}{4}\omega_{S_0}^2 - \omega_r^2 \right)$$
$$= \frac{\hbar L^2 \omega_{S_0}(\omega_{S_0} - 2\omega_C)}{4 v_{F_0}^2 v_{S_0} \mathcal{A}_0} \left[\omega_{S_0}^2 - \frac{(\omega_{S_0} - 2\omega_C)^2}{3} \right].$$

For a broader pump spectrum where Eq. (69) is not valid, we find instead

$$K = K_0 \mathcal{I}, \tag{71}$$

where

$$\mathcal{I} = \frac{4}{\omega_{S_0}(\omega_{S_0} - 2\omega_C)\left[\omega_{S_0}^2 - \frac{(\omega_{S_0} - 2\omega_C)^2}{3} \right]} \times \int_{2\omega_C}^{\infty} d\omega_t \omega_t |\tilde{\phi}_P(\omega_t)|^2 \int_{-(1/2\omega_t - \omega_C)}^{1/2\omega_t - \omega_C} d\omega_r \left(\frac{1}{4}\omega_t^2 - \omega_r^2 \right) \times \left| \frac{\sin(\Delta\theta_1 + \Delta\theta_2)}{\Delta\theta_1 + \Delta\theta_2} \right|^2$$

approaches unity in the cw limit (69). For a pump pulse that is Gaussian,

$$\tilde{\phi}_P(\omega) = \left(\sqrt{\frac{\pi}{2}} \Delta_\omega \right)^{-1/2} e^{-[(\omega - \omega_{S_0})/\Delta_\omega]^2}, \tag{72}$$

for example, the quantity $\mathcal{I}$ depends on the ratio L_{pump}/L, where L_{pump} is the full-width-at-half-maximum spatial width of the pump pulse

$$L_{\text{pump}} = \frac{2\sqrt{2 \ln 2} v_{S_0}}{\Delta_\omega}$$

and rises quickly from zero at $L_{\text{pump}}/L=0$ to unity (see Fig. 4).

With K established by Eq. (71) the biphoton wave function (66) immediately follows. When the duration of the pump pulse is longer than a picosecond, the group velocity dispersion plays a small role, with the factor $\sin(\Delta\theta_1 + \Delta\theta_2)/(\Delta\theta_1 + \Delta\theta_2)$ appearing in Eq. (66) is essentially unity for ω_1, $\omega_2 > \omega_C$. As a result, $|\tilde{\phi}(\omega_1, \omega_2)|^2$ varies as $(\omega_1 + \omega_2 - \omega_{S_0})$ according to the dependence of $|\tilde{\phi}_P(\omega_1 + \omega_2)|^2$ [see, e.g., Eq. (72)], but vanishes when ω_1 or $\omega_2 < \omega_C$ and thus, given the dependence on $(\omega_1 + \omega_2 - \omega_{S_0})$ through the pump profile, whenever ω_1 or $\omega_2 \gtrsim \omega_{S_0} - \omega_C$. In the limit of a low probability of pair production by a pump pulse, $|\beta|^2$ gives the probability of pair production [see discussion after Eq. (46)], and the average number of photons generated per pump pulse is given by

$$N_D = 2|\beta|^2 = \frac{K \bar{\chi}_2^2}{4\pi \bar{n}^6 \varepsilon_0} |\alpha|^2,$$

where we have used Eq. (56). Since our input state is a coherent state $|\alpha|^2 = N_P$, where N_P is the expectation value of the number of pump photons, and we find

$$N_D = (2.2 \times 10^{-6}) \mathcal{I} N_P$$

in that limit for this structure.

VII. CONCLUSIONS

We have presented an approach to the calculation of spontaneous parametric down-conversion in waveguide structures. It is formulated specifically to take into account the nature of the waveguide modes involved, and to allow the use of the profiles of those modes to make realistic calculations of both biphoton wave functions and conversion efficiencies. However, many aspects of it will survive generalizations to other structures. We fully take into account the effect of both modal and material dispersion on the normalization of the linear modes. We need not restrict ourselves to classically described pump pulses. An asymptotic state formalism allows the difficult part of the calculation to be cleanly separated from the linear propagation. And our strategy of seeking a Schrödinger description of the quantum state allows us to easily calculate the full asymptotic-out state

$$|\psi_{\text{out}}^F\rangle = e^{(\beta/\sqrt{2})\int dk_1 dk_2 \phi(k_1,k_2) c_{k_1}^\dagger c_{k_2}^\dagger - \text{H.c.}} |\text{vac}\rangle, \tag{73}$$

within the undepleted pump approximation, using a route no more complicated than would be involved were there only a small probability a pair of photons were generated. We have given explicit expressions for the amplitude parameter β and the biphoton wave function $\phi(k_1, k_2)$ in the usual case of a pump pulse described by a coherent state of arbitrary wave form. These expressions involve simple integrals of the mode profiles over the distribution of the nonlinearity, and integrals over frequencies involving the channel dispersion relations. Thus once the linear mode profiles and their dispersion relations have been determined, the construction of Eq. (73) for SPDC in a ridge waveguide subject to a coherent state pump pulse is reduced to simple integration. With the expression (73) for $|\psi_{\text{out}}^F\rangle$ in hand, there are a number of ways to proceed to analyze the action of the generated entangled photons on other systems, perhaps most usefully a Schmidt decomposition of the biphoton wave function $\phi(k_1, k_2)$ (see, e.g., Lvovsky *et al.* [20]).

In the preceding section, by way of example, we looked at β and $\phi(k_1,k_2)$ for a ridge waveguide structure recently proposed for enhancing second harmonic generation. We defer to later publications the larger topic of using Eq. (73), and the expressions for β and $\phi(k_1,k_2)$, as a design tool for seeking structures and excitation geometries for generating pairs of entangled photons, or streams of pairs of entangled photons, with desired properties. We also plan to address the generalization of this approach to the treatment of SPDC and other nonlinear quantum optical processes in a wider range of artificially structured materials, including those in which the entangled photons appear in separate channels. We have already presented a preliminary study of its application to simple ring resonator structures [9]. As well, later work will consider the extension of this approach beyond the undepleted pump approximation.

ACKNOWLEDGMENTS

This work was supported by the Natural Sciences and Engineering Research Council of Canada. One of us (J.E.S.) acknowledges useful discussions with S. E. Harris and M. G. Raymer.

APPENDIX A

In this appendix we confirm that Eq. (34) indeed describes a coherent state and characterize its properties. Note first that the commutation relation

$$[A_P,A_P^\dagger]=1$$

follows immediately from the normalization condition (36) and the commutation relations that the b_k satisfy [recall Eqs. (5) and (6)], and so Eq. (34) is clearly a coherent state associated with the boson operator A_P

$$A_P|\psi_{\text{in}}\rangle=\alpha|\psi_{\text{in}}\rangle.$$

We temporarily relabel A_P and $\phi_P(k)$ as A_{P_0} and $\phi_{P_0}(k)$, respectively. To see the physical significance of the function $\phi_{P_0}(k)$, consider it together with a set of functions $\{\phi_{P_i}(k)\}_{i=1,2,\ldots}$ that are orthonormal and all orthogonal to $\phi_{P_0}(k)$,

$$\int \phi_{P_i}^*(k)\phi_{P_j}(k)dk=\delta_{ij}, \tag{A1}$$

such that the set $\{\phi_{P_i}(k)\}_{i=0,1,2,\ldots}$ forms a complete set of states. Then using the completeness condition

$$\delta(k-k')=\sum_{i=0}^{\infty}\phi_{P_i}(k)\phi_{P_i}^*(k'),$$

and defining the A_{P_i}, $i=1,2,\ldots$ following the pattern (35) of A_{P_0},

$$A_{P_i}=\int dk\phi_{P_i}^*(k)b_k, \tag{A2}$$

we find that

$$b_k=\sum_{i=0}^{\infty}\phi_{P_i}(k)A_{P_i}.$$

And since we have

$$[A_{P_i},A_{P_j}^\dagger]=\delta_{ij},$$

$$[A_{P_i},A_{P_j}]=0,$$

which follows from the definition (35) and (A2) and the orthonormalization conditions (A1), the operators A_{P_i} are thus independent boson annihilation operators, and we can write the contribution to $\mathbf{D}(r)$ from the S channel [see Eqs. (1), (4), and (6)] as

$$\mathbf{D}_S(\mathbf{r})=\int dk\sqrt{\frac{\hbar\omega_{Sk}}{4\pi}}b_k\mathbf{d}_{Sk}(x,y)e^{ikz}+\text{H.c.}=\sum_{i=0}\mathcal{D}_i(\mathbf{r})A_{P_i}{+\text{H.c.}}, \tag{A3}$$

where

$$\mathcal{D}_i(\mathbf{r})=\int dk\phi_{P_i}(k)\sqrt{\frac{\hbar\omega_{Sk}}{4\pi}}\mathbf{d}_{Sk}(x,y)e^{ikz},$$

and the $\mathcal{D}_i(\mathbf{r})$ are "generalized modes." Our particular $|\psi_{in}\rangle$ in Eq. (34) corresponds to the generalized mode $\mathcal{D}_0(\mathbf{r})$ excited to a coherent state, and all other generalized modes unexcited, as well as all of the modes in the F channel unexcited,

$$A_{P_i}|\psi_{\text{in}}\rangle=0,\ \ i\neq 0,\quad c_k|\psi_{\text{in}}\rangle=0,$$

where the second follows from the commutation of c_k with all the $b_{k'}$ and the $b_{k'}^\dagger$. Then from Eqs. (4), (1), and (A3) we find the expectation value (37) for the operator $\mathbf{D}(r)$. Finally, the photon number operator associated with channel S is

$$N_S=\int dk\ \ b_k^\dagger b_k=\sum_{i=0}^{\infty}A_{P_i}^\dagger A_{P_i}$$

and so the expected number of photons in the incident pulse is

$$\langle\psi_{\text{in}}|N_S|\psi_{\text{in}}\rangle=|\alpha|^2.$$

APPENDIX B

In this appendix we work out the expressions for the amplitude parameter β and the biphoton wave function in the limit of no group velocity dispersion in either channel and perfect phase matching. That is, we take

$$\omega_{Sk}=vk,\quad \omega_{Fk}=vk,$$

for the positive ω_{Ik} and k of interest, where v is a fixed number. To model modes of this sort, we simply consider a cross-sectional area $\mathcal{A}$ in the xy plane of a uniform medium of frequency independent refractive index $\bar{n}$ as our "structure," with plane waves propagating in the z direction. Then from the normalization condition (8) we see that

$$\mathbf{d}_{Ik}(x,y) = \hat{\mathbf{e}}_I \frac{\varepsilon_0^{1/2}\bar{n}}{\sqrt{\mathcal{A}}},$$

where $\hat{\mathbf{e}}_I$ is the polarization vector for the mode. Depending on the tensor properties of χ_2^{ijk}, we let $\bar{\chi}_2$ be chosen such that $\mathcal{A}(k_1,k_2,k)$ of Eqs. (19) and (21) is equal to $\mathcal{A}$, our chosen sectional-area, and of course we have $\nu(k_1,k_2,k)=0$.

From Eq. (58) we then find that

$$J(\omega_1,\omega_2) = \frac{\hbar(k_1+k_2)|\phi_P(k_1+k_2)|^2}{v}\frac{L^2}{\mathcal{A}},$$

since $\mathfrak{s}(k_S(\omega_1+\omega_2)-k_F(\omega_1)-k_F(\omega_2))=\mathfrak{s}(0)=L$, expressing the perfect phase matching, and where we have used Eq. (51) to return to functions of wave number rather than frequencies. Using this in Eq. (57) we then find

$$K = \frac{\hbar v L^2}{\mathcal{A}}\int_0^\infty\int_0^\infty k_1k_2(k_1+k_2)|\phi_P(k_1+k_2)|^2.$$

We can simplify the double integral by changing variables to

$$k_t = k_1+k_2, \quad k_r = \frac{1}{2}(k_1-k_2).$$

Clearly k_t can range from 0 to ∞, while k_r can only range from $-k_t/2$ to $k_t/2$ because of the restriction that both k_1 and k_2 are positive. We find

$$K = \frac{\hbar v L^2}{6\mathcal{A}}\int_0^\infty dk_t|\phi_P(k_t)|^2k_t^4 \equiv \frac{\hbar v L^2}{6\mathcal{A}}\bar{k}^4.$$

Note that if $|\phi_P(k_t)|^2$ is strongly peaked at its center wave number k_0 then $\bar{k}\approx k_0$. From Eq. (56) we then have

$$\beta = \sqrt{\frac{\hbar v}{48\pi\varepsilon_0}}\frac{\bar{\chi}_2}{\bar{n}^3}\frac{\bar{k}^2L}{\sqrt{\mathcal{A}}}\alpha, \tag{B1}$$

and using Eqs. (59), (51) and (52) we find

$$\phi(k_1,k_2) = i\frac{\phi_P(k_1+k_2)\sqrt{6k_1k_2(k_1+k_2)}}{\bar{k}^2}. \tag{B2}$$

Note that the main dependence is on k_1+k_2, as controlled by the pump pulse; the other factors will make $\phi(k_1,k_2)$ vanish as k_1 or k_2 approaches zero, but sufficiently slowly that one can question the limitation to positive k_1 and k_2 we imposed in the discussion after Eq. (49). A more careful calculation along the lines of the above, but including the possibility of negative k_1 and k_2, shows that $\phi(k_1,k_2)$ will indeed survive even if both arguments are not positive, but that if k_0 is much larger than the width of the function $\phi_P(k)$ the contribution to β of $\phi(k_1,k_2)$ in those regions will be negligible.

If the quantity $|\beta|^2$ if sufficiently small, it corresponds to the probability that a pair of photons is produced, and there is essentially no probability that more pairs are produced. In this limit we can put $|\beta|^2=N_D/2$, where N_D is the expected number ($\ll 1$) of photons generated. Using $|\alpha|^2=N_P$, where N_P is the expectation value of the number of pump photons, these expressions together with Eq. (B1) yield

$$N_D = \frac{\hbar v}{24\pi\varepsilon_0}\left(\frac{\bar{\chi}_2}{\bar{n}^3}\right)^2\frac{\bar{k}^4L^2}{\mathcal{A}}N_P \tag{B3}$$

in the limit $|\beta|^2\ll 1$.

To see the physics in this expression, we recall that earlier [3] in a discussion of second harmonic generation we introduced a characteristic power for a material

$$\mathcal{P} = \frac{4\varepsilon_0\bar{n}^6v_F^2v_S}{(\bar{\chi}_2)^2\omega_F\omega_S}, \tag{B4}$$

where ω_F and ω_S were, respectively, the fundamental and second harmonic frequencies, and v_F and v_S the group velocities for those waves. The physical significance of $\mathcal{P}$ is this: For perfectly phase-matched SHG over a length L through a cross-sectional area $\mathcal{A}$ in a uniform medium, in the undepleted pump approximation the second-harmonic power $\mathcal{P}_S$ generated is related to the fundamental power incident, $\mathcal{P}_F$, according to

$$\mathcal{P}_S = \frac{\mathcal{P}_F^2}{\mathcal{P}}\frac{L^2}{\mathcal{A}},$$

assuming the $\bar{\chi}_2$ in Eq. (B4) characterizes the second harmonic coefficient accessed. Equivalently, if in a certain time there are N_F fundamental photons incident, then in that time N_S second harmonic photons will be generated, where

$$2N_S = N_F\frac{\mathcal{P}_F}{\mathcal{P}}\frac{L^2}{A}, \tag{B5}$$

the extra factor of 2 reflecting the fact that a second harmonic photon has double the energy of a fundamental photon.

Now in our down-conversion problem in this appendix we have assumed that a single v can be used to characterize the phase and group velocity of all modes, so perfect phase matching is achieved for all pairs of frequencies ω_1 and ω_2 such that $\omega_1+\omega_2$ is the frequency of a pump photon. To characterize our material for down-conversion, in Eq. (B4) we put $\omega_S\approx vk_{S_0}$ for a pump photon, and $\omega_F\approx vk_{S_0}/2$ for a typical generated photon, to write in place of that equation

$$\mathcal{P} = \frac{8\varepsilon_0\bar{n}^6v}{(\bar{\chi}_2)^2k_{S_0}^2}.$$

Then also using $\omega_F=2\pi/\tau_F$ where τ_F is a typical generated photon period, we can write Eq. (B3) as

$$\frac{1}{2}N_D = N_P\left(\frac{\frac{4}{3}\frac{\hbar\omega_F}{\tau_F}}{\mathcal{P}}\right)\frac{L^2}{A}, \tag{B6}$$

where we have introduced a factor 1/2 because a typical generated photon has half the energy of a pump photon. Thus in the limit of an undepleted pump and perfect phase matching, as required by both Eqs. (B5) and (B6), we see that we can understand the initial scaling of SPDC much as the scaling of SHG, although the first is a spontaneous process and the second is not. In the spontaneous process one can think of a characteristic quantum fluctuating power of $4\hbar\omega_F/3\tau_F$ of signal mixing with the pump photons to generate idler photons, and the same characteristic power of idler mixing with pump photons to generate signal photons (B6); in SHG, of course, it is the incident fundamental power itself that mixes with the fundamental photons to yield second harmonic photons.

[1] L. Scaccabarozzi, M. M. Fejer, Y. Huo, S. Fan, X. Yu, and J. S. Harris, Opt. Lett. **31**, 3626 (2006).
[2] M. A. Foster, A. C. Turner, R. Salem, M. Lipson, and A. L. Gaeta, Opt. Express **15**, 12949 (2007).
[3] Z. Yang, P. Chak, A. D. Bristow, H. M. van Driel, R. Iyear, J. S. Aitchison, A. L. Smirl, and J. E. Sipe, Opt. Lett. **32**, 826 (2007).
[4] M. Liscidini, A. Locatelli, L. C. Andreani, and C. De Angelis, Phys. Rev. Lett. **99**, 053907 (2007).
[5] A. S. Helmy, B. Bijlani, and P. Abolghasem, Opt. Lett. **32**, 2399 (2007).
[6] B. A. U'Ren, R. K. Erdmann, M. de la Cruz-Gutierrez, and I. A. Walmsley, Phys. Rev. Lett. **97**, 223602 (2006).
[7] L. Lanco, S. Ducci, J.-P. Likforman, X. Marcadet, J. A. W. van Houwelingen, H. Zbinden, G. Leo, and V. Berger, Phys. Rev. Lett. **97**, 173901 (2006).
[8] L. Sciscione, M. Centini, C. Sibilia, M. Bertolotti, and M. Scalora, Phys. Rev. A **74**, 013815 (2006).
[9] Z. Yang and J. E. Sipe, Opt. Lett. **32**, 3296 (2007).
[10] S. E. Harris, Phys. Rev. Lett. **98**, 063602 (2007).
[11] B. Huttner, S. Serulnik, and Y. Ben-Aryeh, Phys. Rev. A **42**, 5594 (1990).
[12] M. G. Raymer, K. J. N. Banaszek, and I. A. Walmsley, Phys. Rev. A **72**, 023825 (2005).
[13] See, e.g., Ref. [22], and references cited therein.
[14] For a few examples, see Bennink and Boyd [19], Gatti and co-workers [15,16], Shapiro [17], Wong *et al.*, [18], Lvovsky *et al.* [20], and references cited therein.
[15] A. Gatti, E. Brambilla, and L. A. Lugiato, Phys. Rev. Lett. **90**, 133603 (2003).
[16] A. Gatti, R. Zambrini, M. San Miguel, and L. A. Lugiato, Phys. Rev. A **68**, 053807 (2003).
[17] H. J. Shapiro, Proc. SPIE **5111**, 382 (2003).
[18] F. N. C. Wong, J. H. Shapiro, and T. Kim, Laser Phys. **16**, 1517 (2006).
[19] R. S. Bennink and R. W. Boyd, Phys. Rev. A **66**, 053815 (2002).
[20] A. I. Lvovsky, W. Wasilewski, and K. Banaszek, J. Mod. Opt. **54**, 721 (2007).
[21] As far as we have been able to ascertain, this approach goes back to M. Born and L. Infeld, Proc. R. Soc. London, Ser. A **147**, 522 (1934). The motivation for this strategy is discussed in, e.g., Bhat and Sipe [22].
[22] N. A. R. Bhat and J. E. Sipe, Phys. Rev. A **73**, 063808 (2006).
[23] See [22]. Note that there the factors $\sqrt{\hbar\omega/2}$ were included in the normalization conditions; here we find it more convenient to include them in the expressions (4) for the fields themselves.
[24] L. Landau and E. Lifshitz, *Electrodynamics of Continuous Media*, 2nd ed. (Pergamon, New York, 1984).
[25] P. D. Drummond, Phys. Rev. A **42**, 6845 (1990).
[26] J. E. Sipe, N. A. R. Bhat, P. Chak, and S. Pereira, Phys. Rev. E **69**, 016604 (2004).
[27] MODE SOLUTIONS, version 2.0.3, Numerical Solution, Inc.
[28] A. Fiore, V. Berger, E. Rosencher, P. Bravetti, and J. Nagle, Nature (London) **391**, 463 (1998).
[29] V. R. Almeida, Q. Xu, C. A. Barrios, and M. Lipson, Opt. Lett. **29**, 1209 (2004).
[30] M. Galli, D. Gerace, A. Politi, M. Liscidini, M. Patrini, L. C. Andreani, A. Canino, M. Miritello, R. Lo Savio, A. Irrera, and F. Priolo, Appl. Phys. Lett. **89**, 241114 (2006).
[31] A. Yariv, *Quantum Electronics* (Wiley, New York, 1975).