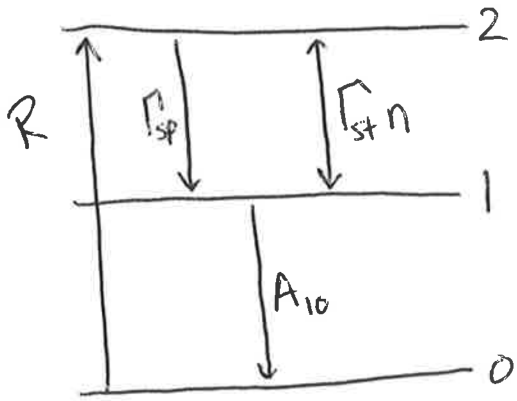
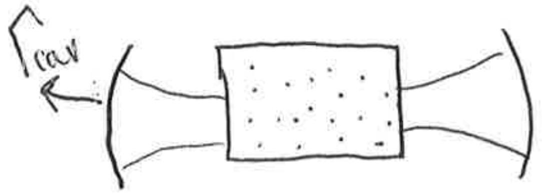
QO2 - Lecture 2

- The simple, phenomenological model we have seen teaches us a lot!
- We saw the effect of saturation on a two-level atom (no gain!) so let's apply the same technique to a 3-level atom.





A few notes/assumptions:

- the pumping rate is R. Assume the spontaneous emission rate from $2 \to 0$ is $\emptyset$. This is not so hard to imagine.
- Assume $A_{10} \gg R, \Gamma_{sp}$ & $\Gamma_{st} n$. In this way, level 1 may be efficiently depopulated. This guarantees $N_2 > N_1$ (population inversion!).
- Γ_{sp} is <u>spontaneous</u> emission rate into ALL MODES. Since all the modes except the cavity mode are lost, there is no stimulated emission in any mode but the cavity mode.
- The rate of <u>spontaneous</u> emission into the cavity mode is Γ_{st} (I know, confusing notation, but stay with me...)
  → This is why the stimulated emission rate into the cavity is $\Gamma_{st} n$. (recall the last lecture: $\frac{B_{21} U}{A_{21}} = n$)

Ok! Let's solve the rate equations!

$$\frac{dN_2}{dt} = N_0 R - N_2 \Gamma_{sp} - N_2 \Gamma_{st} n + N_1 \Gamma_{st} n$$

Remember our assumptions!

Since $A_{10} >> R, \Gamma_{st}, \Gamma_{st} n$, then $N_1 \approx 0$

We will also assume $N_0 \sim N$

Then $\frac{dN_2}{dt} = NR - N_2 \Gamma_{sp} - N_2 \Gamma_{st} n$

In steady state,

$$0 = NR - N_2 \Gamma_{sp} - N_2 \Gamma_{st} n$$

$$N_2 = \frac{NR}{\Gamma_{sp} + n\Gamma_{st}}$$

Now, we will do an unjustified thing, but it will make the calculation easier and won't greatly effect the outcome.

$$n \rightarrow \langle n \rangle$$

So $\boxed{N_2 \simeq \frac{NR}{\Gamma_{sp} + \Gamma_{st}\langle n \rangle}}$ **

What about the average number of photons in the cavity?

$$\frac{d\langle n \rangle}{dt} = \underbrace{N_2 \Gamma_{st}}_{\text{Spont. emission into cavity}} + \underbrace{N_2 \Gamma_{st} \langle n \rangle}_{\text{Stim. emission into cavity}} - \underbrace{\Gamma_{cav} \langle n \rangle}_{\text{Photons escaping through cavity mirrors}}$$

Grouping terms:

$$\frac{d\langle n\rangle}{dt} = N_2\Gamma_{st}(1+\langle n\rangle) - \Gamma_{cav}\langle n\rangle$$

Subbing in N_2 ...

$$\frac{d\langle n\rangle}{dt} = \frac{NR\Gamma_{st}}{\Gamma_{sp}}\left(\frac{1+\langle n\rangle}{1+\frac{\Gamma_{st}}{\Gamma_{sp}}\langle n\rangle}\right) - \Gamma_{cav}\langle n\rangle$$

In steady state...

$$\Gamma_{cav}\langle n\rangle = \frac{NR\Gamma_{st}}{\Gamma_{sp}}\left(\frac{1+\langle n\rangle}{1+\frac{\Gamma_{st}}{\Gamma_{sp}}\langle n\rangle}\right)$$

Now call $C = \dfrac{NR\Gamma_{st}}{\Gamma_{sp}\Gamma_{cav}}$ (the "cooperation parameter")

Then:

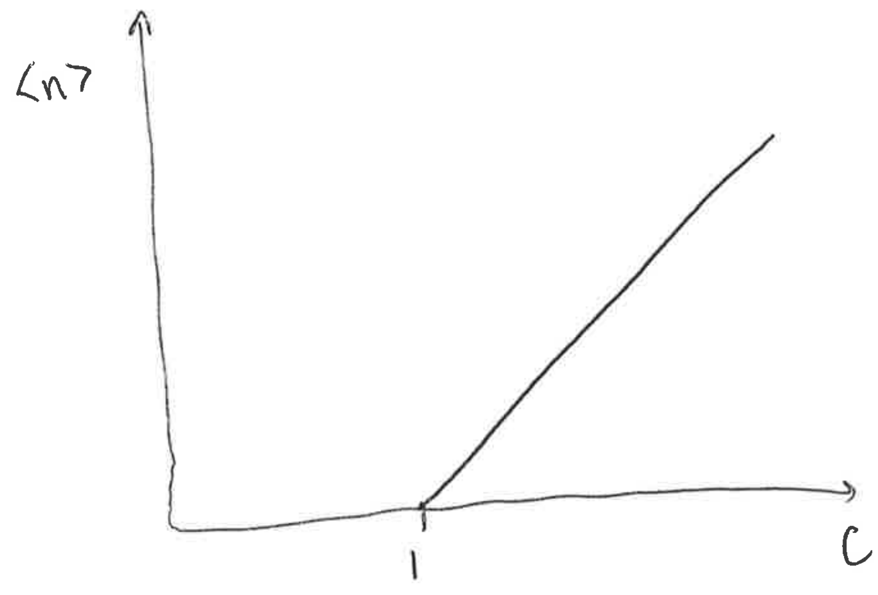
$$\langle n\rangle + \frac{\Gamma_{st}}{\Gamma_{sp}}\langle n\rangle^2 = C(1+\langle n\rangle)$$

Finally, denote $n_s = \dfrac{\Gamma_{sp}}{\Gamma_{st}}$

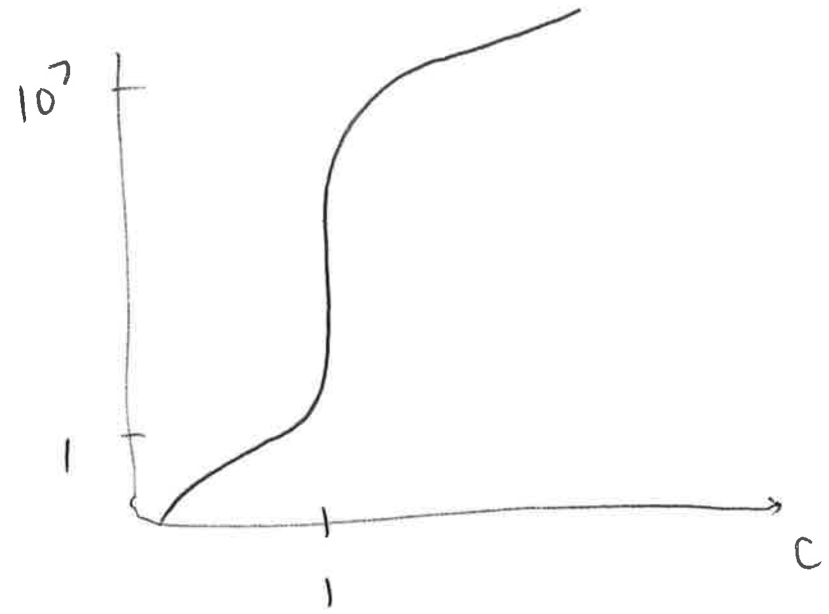
and $$\boxed{\langle n\rangle^2 - (C-1)n_s\langle n\rangle - Cn_s = 0}$$

The solutions of this quadratic equation yield the number of photons in the cavity (and thus the photons outside the cavity) as a function of C (which is itself linearly related to the pumping rate R).

The solution for $\Gamma_{sp} \sim 10^7 s^{-1}$ & $\Gamma_{st} \sim 1\ s^{-1}$ & $\Gamma_{cav} \sim 10^6 s^{-1}$



or on a log scale:

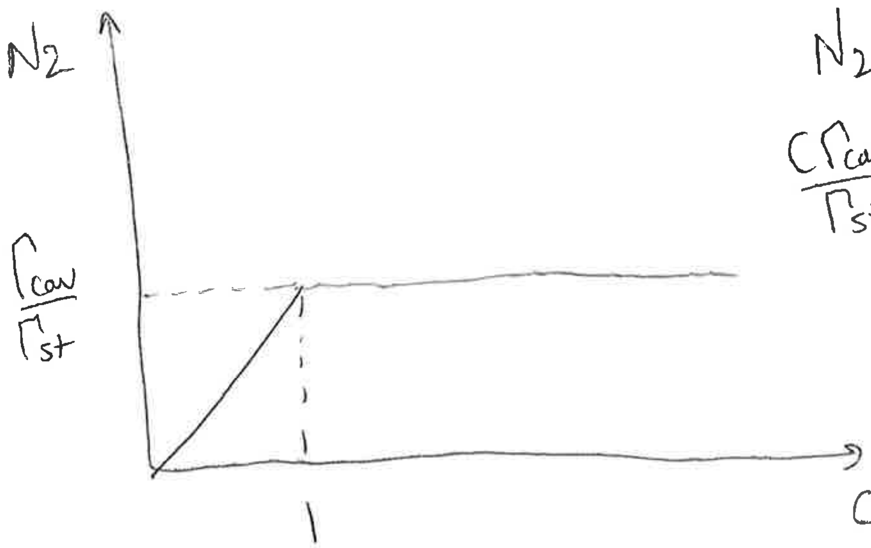


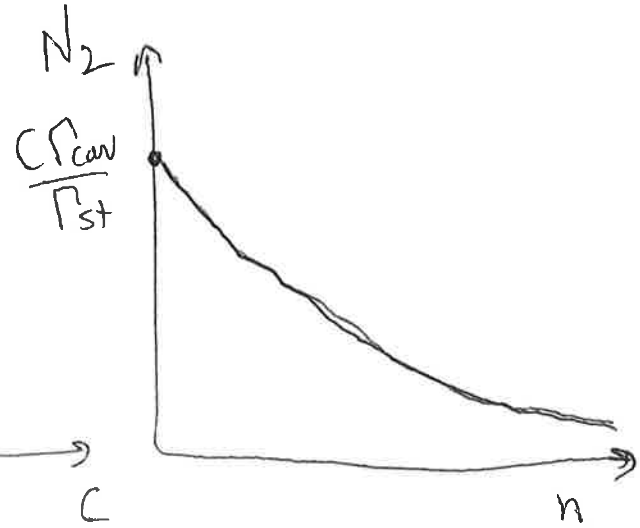
$C = 1$ is a "threshold"

This is (one of) the characteristic behaviours of a laser!

Also, $N_2 = \frac{\Gamma_{cav}}{\Gamma_{st}} \frac{C n_s}{n_s + \langle n \rangle}$

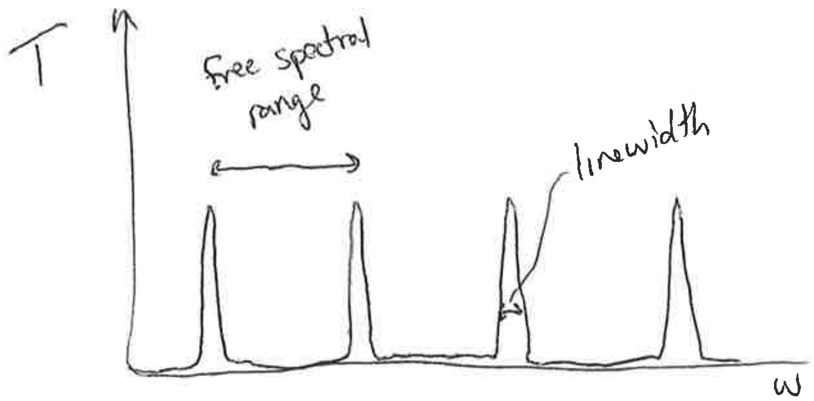
we can plug in $\langle n \rangle$ from above to get





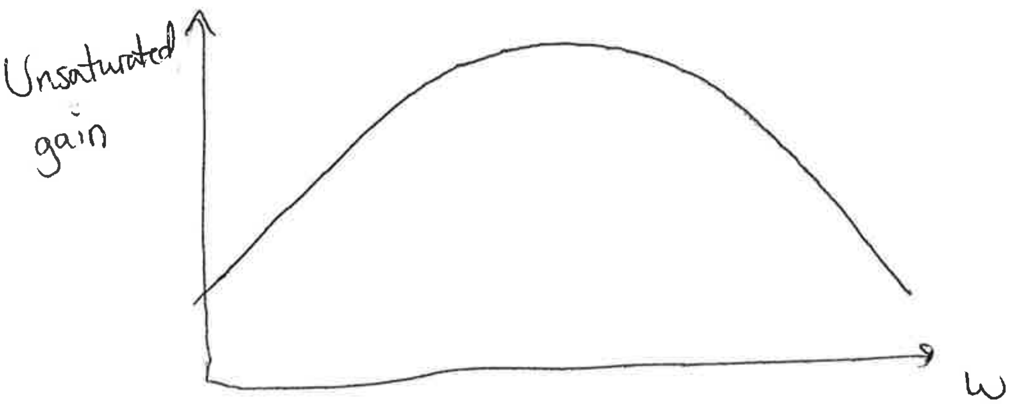
Remember that gain $\propto (N_2 - N_1)$. ie) the gain is saturated above threshold! This has implications!

We will learn later that the cavity has many equally spaced modes:



Similarly, the laser gain medium is frequency-dependant.
ie) there is some resonance condition.

This response is typically quite broad:

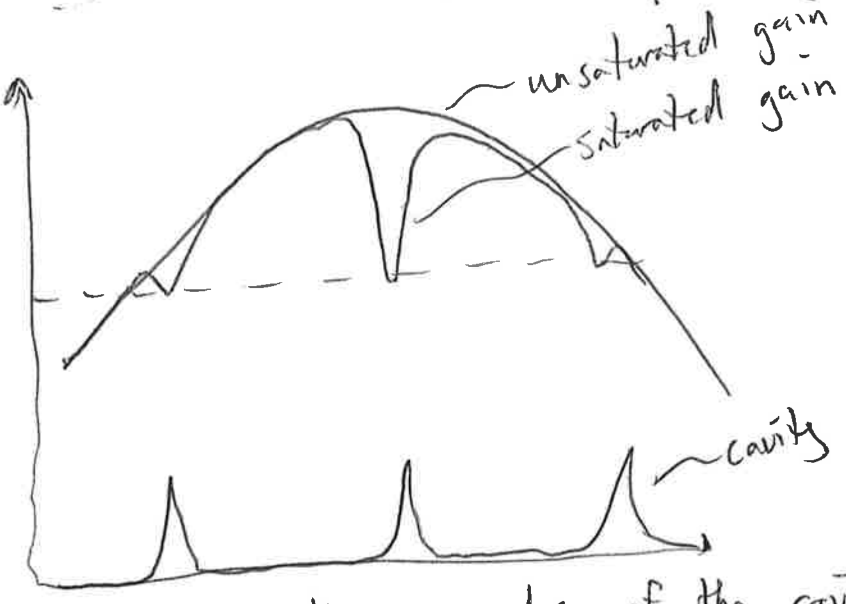


What will the spectrum of the laser look like?

<u>Homogeneous broadening</u>: All atoms have the same (broad) gain profile. They all saturate at the same frequency. The entire gain profile decreases until a single mode lases:

"single frequency laser"

<u>Inhomogeneous broadening</u>: Each atom has a <u>different</u> spectral response, the average of which is broad. Each atom saturates at a different frequency and many modes can lase!



In this drawing, all three modes of the cavity that fit inside the gain bandwidth will lase. ⟹ "Multimode laser"

$$\# \text{ modes} \approx \frac{\text{Gain BW}}{\text{FSR}}$$

The Schawlow - Townes Limit

We never observe single frequency lasers.

Why?

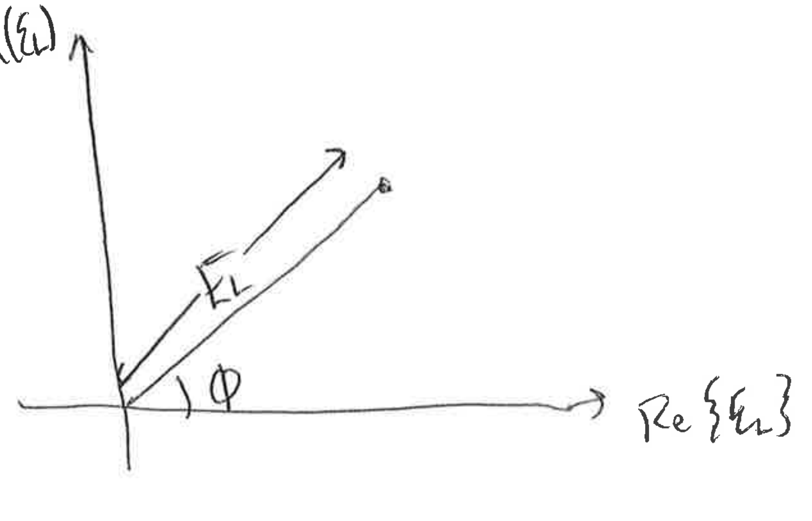
Two Reasons:

A) Technical Faults: The cavity is unstable. The temperature of the medium is unstable. The complex interplay between homogeneous - inhomogeneous effects is unstable.
ALL of these processes "broaden" the linewidth of the laser output.

B) Fundamental Limit: Even if all the above processes are controlled or eliminated (which, in principle, they could be) the laser will still have a finite linewidth.

Start with a single mode field oscillating; and write the field $\psi = \mathcal{E}_L e^{-i\omega t}$

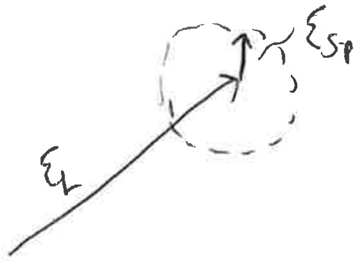
where $\mathcal{E}_L$ is a complex number that gives the amplitude and phase of the laser: $\mathcal{E}_L = \bar{E}_L e^{i\phi}$



If $\mathcal{E}_L$ moves, the field is (by construction) not monochromatic.

What could move $\mathcal{E}_L$? SPONTANEOUS EMISSION!

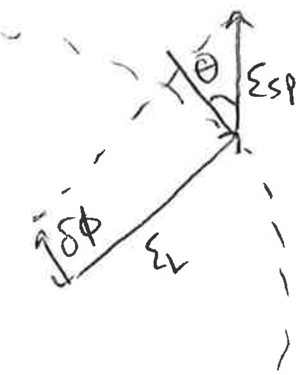
↳ This will add to $\mathcal{E}_L$, in a random direction:



$\mathcal{E}_{sp}$ can: 1) Change the amplitude (length) of $\mathcal{E}_L$. This will cause a momentary fluctuation in intensity which will rapidly stabilize

Remember $N_2 \propto \frac{1}{\langle n \rangle}$ so if $\langle n \rangle$ increases, N_2 decreases (and so does the gain) temporarily. This "restoring force" keeps n close to $\langle n \rangle$.

2) Change the phase of $\mathcal{E}_L$. There is no restoring force here, so ϕ will eventually become completely random!

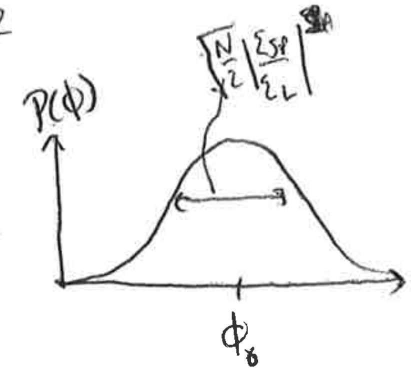


$$\delta\phi = \frac{|\mathcal{E}_{sp}|\cos\theta}{|\mathcal{E}_L|}, \quad \text{with } \theta = \text{Rand}[0, 2\pi]$$

$$\langle \delta\phi \rangle = \left|\frac{\mathcal{E}_{sp}}{\mathcal{E}_L}\right| \langle \cos\theta \rangle = 0$$

But $\langle (\delta\phi)^2 \rangle = \left|\frac{\mathcal{E}_{sp}}{\mathcal{E}_L}\right|^2 \langle \cos^2\theta \rangle = \frac{1}{2}\left|\frac{\mathcal{E}_{sp}}{\mathcal{E}_L}\right|^2$

And after N events $\langle (\delta\phi)^2 \rangle = \frac{N}{2}\left|\frac{\mathcal{E}_{sp}}{\mathcal{E}_L}\right|^2$;



How often does this happen?

Recall:

$$\frac{d\langle n\rangle}{dt} = N_2 \Gamma_{st}\left(\underbrace{1}_{\substack{\text{Spontaneous}\\ \text{emission into}\\ \text{cavity mode}}} + \underbrace{\langle n\rangle}_{\substack{\text{stimulated}\\ \text{emission}}}\right) - \Gamma_{cav}\langle n\rangle$$

Then for every $\langle n\rangle$ photons, 1 of them (on average) came from spontaneous emission.

Since the energy in the cavity is $\propto E_L^2$, we have:

$$\boxed{\langle n\rangle = \frac{E_L^2}{E_{sp}^2}}$$

How long do N stimulated emission events take?

$$\text{Rate} = \frac{N}{\tau} = N_2\Gamma_{st}$$

Then $\boxed{N = \tau N_2 \Gamma_{st}}$

Sub this in:

$$\sqrt{\langle(\delta\phi)^2\rangle} = \frac{1}{\sqrt{\langle n\rangle}}\,\frac{\sqrt{\tau\Gamma_{st}N_2}}{\sqrt{2}}$$

This is appreciable when $\sqrt{\langle(\delta\phi)^2\rangle} \sim 1$, so

$$\tau = \left(\frac{N_2\Gamma_{st}}{2\langle n\rangle}\right)^{-1}$$

But recall that in equilibrium (above threshold): $N_2 = \frac{\Gamma_{cav}}{\Gamma_{st}}$

then $\tau = \left(\frac{\Gamma_{cav}}{2\langle n\rangle}\right)^{-1}$ and $\boxed{\Delta\omega \sim \frac{\Gamma_{cav}}{\langle n\rangle}}$