QE-CDT QUANTUM OPTICS:
Exercises Left For Students

P.S.T.

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Introduction and background

Review of classical optics

Review of electromagnetism

- Remind yourself why, for any vector field $\boldsymbol{X}$, the vector calculus identity

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{X} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{X}) - \nabla^2 \boldsymbol{X}. \tag{1}$$

  is true.

- Show, following our discussion of the classical electric and magnetic fields in terms of the vector potential, that the cycle averaged energy of a single mode (labelled by momentum $\boldsymbol{k}$ and polarization ℓ, confined to a cavity of volume V) is given by

$$H_{\boldsymbol{k}\ell} = 2\epsilon_0 V \omega_{\boldsymbol{k}}^2 |\boldsymbol{A}_{\boldsymbol{k}\ell}|^2. \tag{2}$$

Review of the simple harmonic oscillator

- Prove, using the methods from our proof for the position operator, that the momentum operator is also Hermitian.

- Give simplified expressions for the commutators $[\hat{p}^n, \hat{x}]$ and $[\hat{x}^n, \hat{p}]$.

- Iterate the argument we used to show that $\hat{a}^\dagger |n\rangle$ is an eigenstate of the simple harmonic oscillator Hamiltonian with eigenvalue $E_n + \hbar\omega$ to convince yourself that its entire spectrum is evenly spaced in units of $\hbar\omega$.

Multiple particles and Fock space

Quantisation of EM modes

- Using our expressions for the electric and magnetic field operators, write the Hamiltonian operator in terms of $\hat{a}$ and $\hat{a}^\dagger$.

Quantum states of a single mode; number states

- Compute the number state expectation value of the square of the electric field operator, $\langle \hat{E}^2 \rangle = \langle n| \hat{E}\hat{E} |n\rangle$.

Quantum states of a single mode; coherent states

- Show that coherent states are not strictly orthogonal, $|\langle \alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$.
- Show that the coherent state expectation value of the square of the electric field operator is $E_0^2 \left(4|\alpha|^2 \sin^2(\boldsymbol{k}\cdot\boldsymbol{r} - \omega_{\boldsymbol{k}} t - \phi) + 1\right)$.

Quantum states of a single mode; mixed, thermal states

- Show that, in the notation of the lecture notes, the thermal state has maximum entropy when

$$p_n = \frac{e^{-bE_n/k_{\mathrm{B}}}}{e^{a/k_{\mathrm{B}}} - 1}. \tag{3}$$

- Compute the thermal state expectation value of the square of the number operator and show that the photon number statistics are super-Poissonian.

Quadratures of an EM mode

- Using the definitions of the quadrature operators, show that $[\hat{x}, \hat{p}] = i\hat{1}$.
- Show that the displacement operator takes vacuum states to coherent states (hint: use Baker-Campbell-Haussdorff).
- Show that the displacement operator takes raising and lower operators to displaced raising and lower operators

$$\hat{D}(\alpha)^\dagger \hat{a} \hat{D}(\alpha) = \hat{a} + \alpha\hat{1}, \tag{4}$$
$$\hat{D}(\alpha)^\dagger \hat{a}^\dagger \hat{D}(\alpha) = \hat{a}^\dagger + \bar{\alpha}\hat{1}. \tag{5}$$

- Check that the (phase) rotation operator gives

$$\hat{R}(\theta)^\dagger \hat{a} \hat{R}(\theta) = e^{-i\theta}\hat{a}, \tag{6}$$
$$\hat{R}(\theta)^\dagger \hat{a}^\dagger \hat{R}(\theta) = e^{i\theta}\hat{a}^\dagger. \tag{7}$$

  (Make sure I got my -1s and $\sqrt{2}$s right.)

Quantum states of a single mode; squeezed states

- (a) If the squeezing operator $\hat{S}$ is to be of the form $\hat{S} = e^{-i\hat{H}_S}$ for some squeezing Hamiltonian $\hat{H}_S$, what must be true of this Hamiltonian that is not true of the Hamiltonian $\hat{H}_D$ for the displacement operator? (The question is vague because making it more precise gives the answer away.)

  (b) See how far you can get deriving $\hat{H}_S$ without looking it up.

- Compute the squeezed state expectation value of the p-quadrature operator squared.
- Use the above to show that $\Delta\hat{p} = e^r/\sqrt{2}$.
- Use the line of reasoning in the lecture notes to derive a recursion relation for ξ_n and try to solve it before looking it up. You can use your result for $\hat{H}_S$ to reason about the odd n case.
- How does the quadrature variance of a squeezed coherent state compare to that of a coherent state and a squeezed state? Think about the quadrature space pictures we've been drawing to guess, then check by calculation.
- How does the number variance of a squeezed coherent state compare to that of a coherent state and a squeezed state? Check that the limits $\alpha = 0$ and $\xi = 0$ behave sanely.

Two or more modes

- What is the (phase) rotation operator's action on a coherent state?
- Check the wave plate action on $\hat{a}^\dagger$ given in the lecture notes, and give the action for $\hat{b}^\dagger$.
- Check that the beam splitter operator $\hat{B}$ given in the lecture notes yields the correct canonical transformations also given there.
- Put a two mode coherent state $|\alpha\rangle_a \otimes |\beta\rangle_b$ on a beam splitter and compute the output.
- Repeat the Mach-Zehnder interferometer calculation in the lecture notes for a coherent state input.

Coherence and correlation

- Show that an ideal classical wave $E(t) = E_0 e^{-i\omega t}$ is always first order coherent.
- Compute the second order coherence $g^{(2)}$ for number, coherent, and thermal states.

Two mode squeezing

- Derive the matrix representation in quadrature space for the two-mode squeezing operator given in the lecture notes.
- Show that the "centre of mass" quadratures defined in the notes behave analogously to single mode quadratures.

Wigner-Moyal picture

- Use your favourite mathematical software to draw the Wigner-Moyal functions for some number, coherent, squeezed and thermal states.
- Show that a Gaussian state is completely specified by it's covariance matrix, given in the lecture notes.