UNIVERSITY OF BRISTOL, QUANTUM ENGINEERING CDT

Quantum Optics: Assignment 1

Dylan Mahler

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PROBLEM 1: THE LIGHT-BULB LASER

Read Howard Wiseman's paper: "How many principles does it take to change a lightbulb...into a laser?"

1.1

Derive an expression for the average number of photons at frequency ω in a blackbody oven at temperature t. Plot it for some reasonable value of T.

1.2

Derive Howard's equation for the power per unit length of a lightbulb-laser: $P_{coll} = \frac{\pi}{12}\frac{(k_B T^2)^2}{\hbar}$. Make sure you understand how this derivation works, Howard skips quite a few conceptual steps in his derivation.

1.3

What does the approximation in equation 15 mean? Show all the steps between equation 17 and equation 18. Note that you will have to use the approximation you just justified!

1.4

Look at equation 20 and 21. How does the fraction of collected power (collimated and filtered) scale with the temperature? There are two parameters we are trying to control here:

1. The power per "laser-like" mode

2. The fraction of power collected from the total power

Do you see why these are at odds with each other? Explain!

Problem 2: The mode-locked laser

Here we will study pulsed lasers.

2.1

Consider a very multimode laser, that has N modes. These modes are equally spaced and have frequencies $\omega_k = \omega_0 + k\Delta$ and phases ϕ_k. Derive an expression for the electric field of these N modes (you will need to express it as a sum of the electric field due to each mode separately). You may assume the modes are monochromatic. Don't forget that each mode will have its own phase!

2.2

Show that the instantaneous intensity is the sum of two terms, one which osciallates very fast and one which oscillates with a frequency Δ.

2.3

Using the argument that your photodetector will be fast enough to see the oscillations at frequency Δ, but not fast enough to see the oscillations at the faster frequency, derive the expression for the time-averaged intensity seen by the detector:

$$I(t) = \frac{NE_0^2}{2} + E_0^2 \sum_{k,j>k} \cos[(j-k)\Delta t + \phi_j - \phi_k] \tag{2.1}$$

Plot this for $E_0 = 1$, $N = 10$, $\Delta = 1kHz$, and $\Phi = [0, 0.32, 0.54, 1.35, -0.07, 2.15, -1.89, -0.76, -1.33, 0.44]$.

2.4

Finally, imagine all of the phases have some fixed relationship (they are not random, but determined by some physical process). To make the math easy, let's assume all the phases are zero.

What does the expression for the intensity become? Hint: Equation 2.1 can be greatly simplified in this case by seeing it as a geometric series.

Plot this and compare this to the random phase case.

2.5

We know from lecture that each mode in a laser does not oscillate perfectly monochromatically. Technical broadening (as well as the Autler-Townes limit) broadens each mode. Does this effect the previous treatment? If so, how? If not, why?

2.6

The case where all the phases have a fixed relationship is called a *mode-locked laser.* There are numerous methods for obtaining this type of laser operation such as saturable absorbers, amplitude modulation, phase modulation, Kerr-lens mode-locking, etc. Read about one to tell us in the next lecture.

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Quantum Optics: Assignment 2

Dylan Mahler

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PROBLEM 1: SPONTANEOUS EMISSION

1.1

Consider a two-level atom initially prepared in the excited state. Semi-classically, this is an eigenstate of the system (since there is no electromagnetic field to interact with the atom) and so the atom will remain in the excited state for all time. We will later learn that in the full quantum mechanical description of this problem, the electromagnetic vacuum field causes the atom to transition to the ground state. Recall that in lecture, we learned about two possible behaviours: Rabi oscillation and exponential decay. Which of these behaviours do you expect the atom to exhibit and why? Please justify with energy level diagrams. Finally, state an expression for the probability to find the atom in the excited state as a function of time (for short times only).

PROBLEM 2: THE QUANTUM WATCHED POT (SOMETIMES) NEVER BOILS

2.1

There is a set of well known philosophical paradoxes (called Zeno's paradoxes) that can be summarized by the rather simplified "a watched pot never boils". Stated more eloquently:

> If everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless.
>
> —as recounted by Aristotle, Physics VI:9, 239b5

Now, consider the atom from problem 1 and the expression you derived for finding the atom in the excited state. Imagine, in time T, performing a measurement of which state the atom is in L times. Calculate the probability that the atom is still in the excited state at the end of the time T. Does this differ from the case if you had just left the atom alone?

2.2

Now, consider an atom making a transition to the ground state due to the presence of an electromagnetic field. We know from lecture that at small times the probability of being in the excited state goes something like

$$P_e = 1 - \alpha T^2 \tag{2.1}$$

Again, imagine measuring the state of the atom L times in time T and calculate the probability of remaining in the excited state. What happens as $L \to \infty$?

2.3

This effect of measurement-induced halting of evolution is called the Quantum Zeno Effect. The case of a quadratic Hamiltonian was demonstrated quite some time ago (1989, Wineland). The case of a halting the evolution of an *unstable* quantum system was not observed until 2001 (Raizan), due to the difficulty of engineering an unstable system that was governed by a quadratic evolution.

Problem 3: Two Photon Absorption

Consider a three level atom, with energy levels E_1, E_2, and E_3. Each of the levels are coupled by a sinusoidal perturbation, that is gradually turned on, given by $H_I = \hat{W} e^{\epsilon t} \cos \omega t$, where $\omega = (E_2 - E_1)/\hbar + \delta = (E_3 - E_2)/\hbar - \delta$, and $\langle i|\hat{W}|j\rangle = W$ for $i \neq j$. ϵ is the time it takes for the interaction to turn on, and should be very small, such that $T\epsilon \sim 1$ (ϵ multiplied by the final time is near unity). δ is called the "detuning".

3.1

Draw an energy level diagram, indicating the energy levels, $\hbar\omega$, and the detuning.

3.2

Calculate the first order transition probability from state 1 to state 3, as a function of time T and starting at time $t_0 = -\infty$. Evaluate this for realistic values of ω_{12}, ω_{32}, and the detuning.

3.3

Calculate the second order transition probability from state 1 to state 3, as a function of time T and starting at time $t_0 = -\infty$. Again, evaluate this for realistic values of ω_{12}, ω_{32}, and the detuning.

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Quantum Optics: Assignment 3

Dylan Mahler

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PROBLEM 1: SATURATION (AGAIN....)

1.1

Recall the optical bloch equations:

$$\frac{d}{dt}\sigma_x = \delta\sigma_y - \frac{\Gamma}{2}\sigma_x \tag{1.1}$$

$$\frac{d}{dt}\sigma_y = -\delta\sigma_x - \Omega_R\sigma_z - \frac{\Gamma}{2}\sigma_y \tag{1.2}$$

$$\frac{d}{dt}\sigma_z = \Omega_R\sigma_y - \Gamma(\sigma_z + 1) \tag{1.3}$$

Derive an explicit expression for the expectation values of the three Pauli operators in steady state.

1.2

Now, show that the probability of finding the atom in the excited state can be expressed as:

$$P_{ee} = \frac{S/2}{1+S} \tag{1.4}$$

What is S? Plot this function? Look familiar?

1.3

Now, consider the special case $\delta = 0$. Solve the OBE's exactly (hint: the equations are a set of inhomogeneous first-order differential equations), for an atom initially in the ground state. Plot the probability of being in the excited state as a function of time for several different values of the dimensionless parameter Γ/Ω_R.
Now answer the question we asked in class: which wins, the drive or the damping?

PROBLEM 2: FUN WITH THE OBE'S

2.1

Often in quantum optics, it is desirable to know the frequency of a laser very precisely. There are many different ways of achieving this, depending on how presicely the laser frequency needs to be known. For example, simply using the absorption profile of a warm gas with a known electronic level structure is enough to determine the frequency of a laser to some MHz. Using more complicated techniques (for a REALLY interesting read, look up "Pound-Drever-Hall") this can be brought down to something like the linewidth of the laser. But, as we know, atomic clocks achieve precision much greater than this. Here, the frequency reference is again atoms, but we use a slightly different method.
Recall the OBE's for an atomic subject to a constant drive (see question 1), but set $\Gamma = 0$ for simplicity. Imagine turning this drive on at time $t = 0$, and off again at time $t = \pi/\Omega_R$.
Solve these equations exactly (its much easier to solve these now, since there is no damping).

2.2

Plot the probability of being in the excited state after at time t as a function of the detuning δ, in units of Ω_R.

2.3

What is the width of this function? (or: using P_e as your observable, what is the precision with which you can determine δ)

PROBLEM 3: MORE FUN WITH THE OBE'S

2.1

We learned in the last question that Rabi oscillations can be used as a frequency reference. This suffers from a few problems, however. Hopefully, you realized that your ability to determine the frequency of the drive is limited by the time that the atom interacts with the laser. This time cannot be arbitrarily extended, however, because the drive has a finite linewidth and thus the beam is only phase coherent for a certain amount of time. Instead, consider a different situation:

- Apply a drive for time $\tau = \pi/2\Omega_R$
- Wait for a time T.
- Apply a drive for time $\tau = \pi/2\Omega_R$

Again, solve the OBE's and calculate the probability of being in the excited state. NB: this is a complicated series of solutions. It is highly suggested that you solve it step by step, making simplifications as you go along. Finally, express this in units of Ω_R, as you did in problem 2, and plot it as a function of δ for different values of T. What is the width of this function? (this might be difficult to determine, but by plotting for several different values of T you should be ableto eyeball it)

2.2

This is called "Ramsey Interferometry" and is used in atomic clocks. The probability of being in the excited state is used as a measurement of the detuning, and can be used to establish a very accurate frequency reference. By increasing T, the precision of this measurement can be increased nearly arbitrarily.

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Quantum Optics: Assignment 4

Dylan Mahler

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PROBLEM 1: THE LANDAU-ZENER CRITERION

In this problem, we will derive the Landau-Zener criterion (perturbatively), and then make some well-founded guesses about the full-solution. Recall in class that we stated that for an atom initially in the ground state, with the laser far red-detuned, the atom will adiabatically transition to the excited state as the laser frequency is swept as long as:

$$\frac{d}{dt}\delta \ll \Omega_R^2 \tag{1.1}$$

1.1

Let's begin with the time-dependent atom-field Hamiltonian:

$$\hat{H} = \frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\Omega_R\cos(\omega t)\hat{\sigma}_x \tag{1.2}$$

Before, we went into the rotating frame by effecting the transformation:

$$|\Psi\rangle \to \hat{R}_z(\omega t)|\Psi'\rangle \tag{1.3}$$

where $\hat{R}_z(\omega t) = \mathrm{e}^{-i\omega t\hat{\sigma}_z/2}$.
We must be careful now, since ω will now be a function of time. Instead, go into the rotating frame with $\hat{R}_z(\phi) = \mathrm{e}^{-i\phi\hat{\sigma}_z/2}$ and remember that ϕ is a function of time.
Show that

$$\hat{H}' = \frac{\hbar(\omega_0 - \dot{\phi})}{2}\sigma_z + \hbar\Omega_R\cos(\omega t)\{\hat{R}_z^\dagger(\phi)\sigma_x\hat{R}_z(\phi)\} \tag{1.4}$$

1.2

Imagine a situation where we are linearly sweeping the detuning. Think about what $\dot{\phi}$ should be (remembering the expression for ϕ from Lecture 5 stated above). Integrate this to find ϕ in terms of the atomic transition frequency, ω_0, and the rate at which we sweep the detuning, which we can call α.

1.3

Show that, using the rotating wave approximation, the second term (which we call $\hat{V}$ here) in the expression for $\hat{H}'$ evaluates to:

$$\hat{V} = \frac{\hbar\Omega_R}{2}\begin{pmatrix} 0 & \mathrm{e}^{i(\phi-\omega t)} \\ \mathrm{e}^{-i(\phi-\omega t)} & 0 \end{pmatrix} \tag{1.5}$$

This is not particularly easy, but you can use the tricks we learned in lecture 5.

1.4

Now, treat $\hat{V}$ like a perturbation and (going all the way back to lecture 3), calculate the probability for ending up in the excited state at $t = +\infty$ given that you began in the ground state at time $t = -\infty$ (HINT1: You can go back to lecture 3 and copy the formula that we used directly, no need to derive it again. You will notice that a term that depends on the energy level splitting cancels out nicely! HINT2:you have to integrate some expression, and you will have to look up the solution to this integral unless it happens to be fresh in your mind).

1.5

It is customary to write this probability:

$$P_e = 2\pi\Gamma \tag{1.6}$$

What is Γ? Remember our earlier guess for the adiabatic criterion. Was it correct?

Problem 2: Rabi Revival

2.1

We derived an expression for the state of an atom, initially in the excited state, interacting with a fully quantized field. In class, we assumed the field was very intense, and thus the coherent state amplitude $\alpha \gg 1$. Let's not make this assumption here. Instead, go back to the equation we derived for the state of the field and atom as a function of time, and calculate the probability of the atom being in the excited state *regardless* of the state of the field.

2.2

Plot this, for $\Omega_R = 1$ and $\alpha = 4$, up to time $t = 20$. Please note that this must be done numerically (and approximately) since the series expression will not have an analytic solution. Think carefully about how many terms in the expansion you should keep to get a reasonably accurate result.

2.3

The behavior you see is the real behavior of the system, no semi-classical approximations made. There are a range of frequencies (the dominant one being the Rabi frequency). These different frequency terms will eventually be out of phase, causing the collapse behavior. Let's estimate the time it takes for the system to decay ie. the "dephasing time".
Classically, oscillators with a range of frequencies $\Delta\Omega$ will dephase after a time $(\Delta\Omega)^{-1}$. What is the range of frequencies that contribute to the dynamics here? ie. look at the equation for the wavefunction and note that the expansion coefficients are all small, except for some values of n. Calculate the range of important frequencies in this problem (Hint: they are a direct result of the terms in the coherent state with $n = \bar{n} \pm \sqrt{\bar{n}}$ where $\bar{n} = |\alpha|^2$ is the average number of photons) and thus calculate the dephasing time.

2.4

Your answer for the dephasing time should not depend on $\bar{n}$. But from our treatment in class, when $\bar{n} \gg 1$ we see the semi-classical Rabi oscillations (with no collapse). So what's going on here?! (Hint: if you plot the bahavior for different values of $\bar{n}$, the relevant time scales will also be a function of $\bar{n}$ so the x-axis scale should change!) (BIG Hint: Think if you were actually trying to do an experiment and observe Rabi oscillations. If all you knew was a semi-classical world, would you care what happens after a few Rabi oscillations?)