Quantum Circuits for the Schur Transform

Oliver Thomas

Quantum Engineering CDT
University of Bristol

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1 The Schur Transform

background.

There are two distinct ways of performing the Schur transform on n qubits, it can either be built up from coupling all n qubits together in a single iteration which we call the spatial multiplexed approach. The other approach is performing Clebsch-Gordan (CG) transforms on the n qubits one at a time which we call the temporal multiplexed approach.
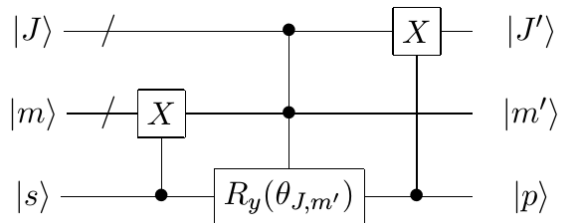
2 Streaming Scheme

The vanilla transform directly maps the computation basis states to a labeling of J and M values. The vanilla approach of calculating the CG coefficients and finding a gate decomposition does not scale well. A better approach is to use a scheme explicitly storing the values of J & M in registers.

The streaming scheme uses temporal multiplexing to perform the Schur transform in polynomial time if a recursive streaming scheme is used [1].
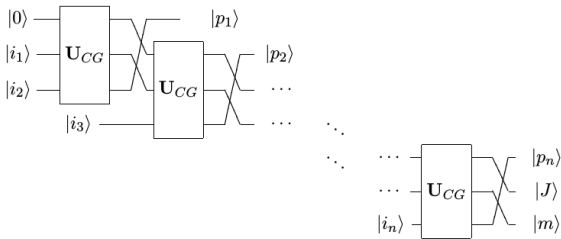
(a) Streaming structure where the Schur transform is built up from consecutive Clebsch-Gordan transforms [2].

(b) U_{CG} block, Qadder, controlled rotation, Qadder [2].

Figure 1: Streaming scheme [2].

The Rotation matrix, $R_y(\theta_{J,m'})$ [Fig. 1](#) is given by,

$$R_y(\theta_{J,m'}) = \begin{bmatrix} \cos(\theta_{J,m'}) & -\sin(\theta_{J,m'}) \\ \sin(\theta_{J,m'}) & \cos(\theta_{J,m'}) \end{bmatrix} = \frac{1}{\sqrt{2J+1}} \begin{bmatrix} \sqrt{J+\frac{1}{2}+m'} & -\sqrt{J+\frac{1}{2}-m'} \\ \sqrt{J+\frac{1}{2}-m'} & \sqrt{J+\frac{1}{2}+m'} \end{bmatrix} \quad (1)$$

Where primed variables means after the angular momentum addition so J is the total J that the spin is coupling to, the system will have J' total angular momentum after the coupling. m is the z component of the system before and m' is the total z component after the coupling.

To build this circuit the rotation matrix, $R_y(\theta_{J,m'})$ needs to be calculated using [Eq. 1](#), values here [Eq. 11e](#), and a function to update the $|m\rangle$ and $|J\rangle$ registers is needed.

3 Spatially multiplexed Clebsch-Gordan transform

The Clebsch-Gordan transform is a basis transformation into the Schur basis. The transform for 2 qubits is given by,

$$|J = 1, M = +1\rangle = |00\rangle$$
$$|J = 1, M = \ \ 0\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad (2a)$$
$$|J = 1, M = -1\rangle = |11\rangle$$
$$|J = 0, M = \ \ 0\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \quad (2b)$$

Throughout the encoding $|0\rangle = +1/2, |1\rangle = -1/2$ is used unless stated otherwise.
The transform expressed as a matrix is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{bmatrix} = \text{(spin labeling)} \begin{bmatrix} |00\rangle \\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |11\rangle \\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{bmatrix} = \begin{bmatrix} |J = 1, M = 1\rangle \\ |J = 1, M = 0\rangle \\ |J = 1, M = -1\rangle \\ |J = 0, M = 0\rangle \end{bmatrix} \quad (3)$$

Which can be implemented in a circuit as,

$$(4)$$

Figure 2: Schur transform for 2 qubits

Circuit for Clebsch-Gordan transform [Fig. 2](#) contains 2 gates. As two-qubit (entangling) gates are much more expensive to perform compared to single qubit gates, the cost of the circuits discussed here will all be given in terms of the number of two-qubit gates.

3.1 Clebsch-Gordan coefficients for 3 qubits

The CG coefficients for three qubits are no multiplicities of 4 for J=3/2 and 2 multiplicities of 2 for J=1/2 Eq. 6c. The multiplicities, P are defined as $J' - J$ the new J value minus the previous J value, the number of 1s in a P string is the number of multiplicities for that J value.

The matrix for the transform which takes the computational basis to the spin basis is,

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{6}} & 0 & -\sqrt{\frac{1}{6}} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{1}{6}} & 0 & \sqrt{\frac{1}{6}} & -\sqrt{\frac{2}{3}} & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111
\end{bmatrix}
=
\begin{bmatrix}
|J=3/2, M=3/2\rangle \\
|J=3/2, M=1/2\rangle \\
|J=3/2, M=-1/2\rangle \\
|J=3/2, M=-3/2\rangle \\
|J=1/2, M=1/2, P=0\rangle \\
|J=1/2, M=-1/2, P=0\rangle \\
|J=1/2, M=1/2, P=1\rangle \\
|J=1/2, M=-1/2, P=1\rangle
\end{bmatrix}
\tag{5}
$$

The decomposition scheme for the n-qubit case could take at most $2^{n-1}(2^n - 1)$ $C^{n-1}U$ gates [3], where $C^{n-1}U$ means a unitary acting on 1 qubit controlled on the other n-1 qubits. For 3 qubits this upper bound is 128 C^2U gates. It has been shown that in terms of gate count, $C^nU \sim 5C^{n-1}V$ where $U\&V$ are unitaries [4]. This means the maximum two-qubit gates needed would be 140 CU gates.

The decomposition of the 3 qubit CG transform was performed using the Givens rotation method for unitary decomposition into a gate-set. The matrix Eq. 5 can be expressed as a product of 19 C^2U gates (control-control-unitaries) which is $\sim 80\ CU$ gates.

There are multiple ways of writing the spin basis, there is the traditional CG coefficients and there is also what is referred to here as the phase encoding Eq. 9. The phase encoded transform matrix will have a different decomposition as the shape of the matrix is different to the regular encoding.

3.2 Circuit for 3 qubit transform

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See online [5] for Fortran code which implements the Givens rotation method to give the 19 C^2U gate decomposition. The majority of the gates are CNOT gates. This is mainly due to the re-ordering of the basis and is similar to the quantum Fourier transform (QFT). The QFT produces the output in reverse qubit order the actual number of gates required to do the transform is massively reduced. The overhead calculated here is due to the rearranging of the basis. This means that depending on what the transform is used the transform could be computed with less gates. For example, if the transform was only used to check if the state was in a particular J block but didn't need to know the specific M value the order afterwards wouldn't be as important reducing the CNOTs needed.

3.3 4 Qubit CG coefficients

The CG coefficients for four qubits contains 16 terms, 5 for J=2, 3 multiplicities of 3 for J=1, 2 multiplicities of 1 for J=0. The equations are given in Eq. 10f. In the J=1 case there are 3 acceptable bit strings, 0001, 0010, 0100 meaning there are 3 multiplicities present.

The decomposition scheme for the 4 qubit case could take at most $2^{n-1}(2^n - 1) \ C^{n-1}U$ gates [3], where n is 4 and. For 4 qubits this upper bound is 120 C^3U gates which could be up to $\sim 3000 \ CU$ gates. In reality it will be much fewer gates as the matrix is sparse, however this suggests a different approach was needed.

3.4 Registering J & M explicitly Fig. 3, Fig. 4

The registered circuit, which takes in two qubits can be constructed from 11 gates if the m register is not compressed. The spatially multiplexed minimal gate explicit J & M 2 qubit transform is shown in Fig. 4 which contains 11 two-qubit gates. Another CNOT can be added to compress the M' register, which means using the same values for M'=0, see Fig. 3 for the 12 two-qubit gate circuit.

This spatial multiplexing approach only stores the final output values of the J & M registers.

4 General circuit for the Quantum Schur transform Fig. 6

Circuit uses the encoding for $|S\rangle : |0\rangle \mapsto Spin = +\frac{1}{2}, |1\rangle \mapsto Spin = -\frac{1}{2}$ and the same for $|P\rangle$.

The circuit adds the value of the spin to be added, $|S\rangle$, to the M register to calculate the M' register value. This is done by implementing the quantum reversible equivalent to the digital full adder.

The case where $|S\rangle = |0\rangle$ means the spin is $+\frac{1}{2}$ so to add $\frac{1}{2}$ to M, 1 is added to the m_0 qubit. The first Quantum Adder (QAdd) uses Toffoli gates controlled on $|s\rangle = |0\rangle$ (denoted by the white control circle) with the current m_0 value and C_0 (an ancilla carry). This ensures that the case when $m_0 = 1$ and 1 is added to it, m_0 goes to 0 and m_1 is increased using the carry as $001 + 1 = 010$. The rest of the QAdd stages then just check the carry of the previous qubit to complete to M $+\frac{1}{2}$ addition as $|S\rangle = |0\rangle$ does not trigger any of the rest of the control gates.

The case where $|S\rangle = |1\rangle$ means the spin is $-\frac{1}{2}$ we do M $-\frac{1}{2}$ which is done by adding the binary string for $-\frac{1}{2}$ which is the all 1's string, 111. This time the very first QAdd does not trigger and $|s\rangle$ is then added to all of the bits of M using C-NOT gates with carries to check for overflow.

The Unitary is then performed on $|S\rangle$ depending on the values of the newly calculated M' and J registers using $R_y(\theta_{J,m'})$ Eq. 1. The J register is then updated to J' by adding the value of $|P\rangle$ to J using the QAdd sequence of gates.

To add the second qubit in the values of J' and M' are passed in as the initial register values. It is easy to extend this to many qubits being streamed in one at a time by carefully conditioning the controls on the unitaries, I think in the general case you need at most N controls for coupling up to N qubits in one at a time. The circuit written here has redundancy in the Identity and ZX gates appearing twice which is shown in the circuit for completeness Fig. 6.

J_2	J_1	J_0	J
0	0	0	0
0	0	1	$\frac{1}{2}$
0	1	0	1
0	1	1	$\frac{3}{2}$
1	0	0	-2
1	0	1	$-\frac{3}{2}$
1	1	0	-1
1	1	1	$-\frac{1}{2}$

m_2	m_1	m_0	M
0	0	0	0
0	0	1	$\frac{1}{2}$
0	1	0	1
0	1	1	$\frac{3}{2}$
1	0	0	-2
1	0	1	$-\frac{3}{2}$
1	1	0	-1
1	1	1	$-\frac{1}{2}$

Table 1: Tables giving binary Two's complement encoding to spin values of the M and J registers

4

5 Reduced general gate circuit for up to the 2 qubit Schur transform Fig. 5

Where V is the phase gate, $V = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, $V^\dagger V = I$ and $V^2 = Z$. V is used here to expand the double controlled Toffoli gate into single control gates in the quantum adder subroutine.

The W gate, $W^2 = HX$ with $W^\dagger = I$, is used to expand the HX gate into single control gates in the spin transform region.

The circuit checks that if (m_1 XNOR m_0) AND (m_0 XOR S_0) and will then change m_2. Then m_1 is updated using $m_1 = m_0$ XOR S_0. m_0 is always incremented by 1, if $|S\rangle = |0\rangle$ increment only m_0 by 1 corresponding to adding $\frac{1}{2}$ to the M register. $|S\rangle = |1\rangle$ corresponds to subtracting $\frac{1}{2}$ from the M register by adding the string 111 bitwise to M.

For the most positive values of M the Identity is performed on the spin corresponding to the strings $M = 001(J = \frac{1}{2}, M' = \frac{1}{2})$ for the first spin and $M = 010(J = 1, M' = 1)$ for the second coupled in spins.

The most negative values of M performs $XZ|S\rangle$ corresponding to the strings $M = 111(J = \frac{1}{2}, M' = -\frac{1}{2})$ for the first spin and $M = 110(J = 1, M' = -1)$ for the second spin.

If $M = 000(J = 0, M' = 0)$ do $XH|S\rangle$ Fig. 5.

References

[1] Dave Bacon, Isaac L Chuang, and Aram W Harrow. The quantum schur and clebsch-gordan transforms: I. efficient qudit circuits. In *Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 1235–1244. Society for Industrial and Applied Mathematics, 2007.

[2] Dave Bacon, Isaac L Chuang, and Aram W Harrow. Efficient quantum circuits for schur and clebsch-gordan transforms. *Physical review letters*, 97(17):170502, 2006.

[3] Chi-Kwong Li, Rebecca Roberts, and Xiaoyan Yin. Decomposition of unitary matrices and quantum gates. *International Journal of Quantum Information*, 11(01):1350015, 2013.

[4] Adriano Barenco, Charles H Bennett, Richard Cleve, David P DiVincenzo, Norman Margolus, Peter Shor, Tycho Sleator, John A Smolin, and Harald Weinfurter. Elementary gates for quantum computation. *Physical review A*, 52(5):3457, 1995.

[5] https://github.com/ot561/schurtransform/blob/master/matrixmul.f90.

A Appendix: Maths

A.1 3 Qubit transformation

This is the J=3/2 block

$$|J = 3/2, M = +3/2, P = 000\rangle = |000\rangle$$

$$|J = 3/2, M = +1/2, P = 000\rangle = \sqrt{\frac{1}{3}}(|001\rangle + |010\rangle + |100\rangle)$$

$$|J = 3/2, M = -1/2, P = 000\rangle = \sqrt{\frac{1}{3}}(|110\rangle + |011\rangle + |101\rangle)$$

$$|J = 3/2, M = -3/2, P = 000\rangle = |111\rangle$$

(6a)

This is the J=1/2 block from J=1, multiplicity zero

$$|J = 1/2, M = +1/2, P = 001\rangle = +\sqrt{\frac{2}{3}}|001\rangle - \sqrt{\frac{1}{6}}(|010\rangle + |100\rangle)$$

$$|J = 1/2, M = -1/2, P = 001\rangle = -\sqrt{\frac{2}{3}}|110\rangle + \sqrt{\frac{1}{6}}(|011\rangle + |101\rangle)$$

(6b)

This is the J=1/2 block from J=0, multiplicity one

$$|J = 1/2, M = +1/2, P = 010\rangle = \frac{1}{\sqrt{2}}(|010\rangle - |100\rangle)$$

$$|J = 1/2, M = -1/2, P = 010\rangle = \frac{1}{\sqrt{2}}(|011\rangle - |101\rangle)$$

(6c)

The CG transform for 3 qubits Eq. 5 can be rearranged to a block diagonal form which looks like it could be implemented in a circuit.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} & 0 & 0 & 0 & 0 \\
0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\
0 & 0 & 0 & 0 & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & -\sqrt{\frac{2}{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
000 \\
001 \\
010 \\
100 \\
011 \\
101 \\
110 \\
111
\end{bmatrix}
$$

(7)

A.2 3 Qubit phase encoding

This is the J=3/2 block

$$|J = 3/2, M = +3/2, P = 000\rangle = |000\rangle$$

$$|J = 3/2, M = +1/2, P = 000\rangle = \sqrt{\frac{1}{3}}(|001\rangle + |010\rangle + |100\rangle)$$

$$|J = 3/2, M = -1/2, P = 000\rangle = \sqrt{\frac{1}{3}}(|110\rangle + |011\rangle + |101\rangle)$$

$$|J = 3/2, M = -3/2, P = 000\rangle = |111\rangle$$

(8a)

This is the J=1/2 block from J=1, multiplicity zero

$$|J = 1/2, M = +1/2, P = 001\rangle = \frac{1}{\sqrt{3}}(|001\rangle + e^{2\pi i/3}|100\rangle + e^{4\pi i/3}|010\rangle)$$

$$|J = 1/2, M = -1/2, P = 001\rangle = \frac{1}{\sqrt{3}}(|011\rangle + e^{2\pi i/3}|101\rangle + e^{4\pi i/3}|110\rangle)$$

(8b)

This is the J=1/2 block from J=0, multiplicity one

$$|J = 1/2, M = +1/2, P = 010\rangle = \frac{1}{\sqrt{3}}(|001\rangle + e^{4\pi i/3}|100\rangle + e^{2\pi i/3}|010\rangle)$$

$$|J = 1/2, M = -1/2, P = 010\rangle = \frac{1}{\sqrt{3}}(|011\rangle + e^{4\pi i/3}|101\rangle + e^{2\pi i/3}|110\rangle)$$

(8c)

The phase encoding matrix is given by,

$$\frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & e^{2\pi i/3} & e^{4\pi i/3} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2\pi i/3} & 0 & e^{4\pi i/3} & 1 & 0 \\ 0 & e^{4\pi i/3} & e^{2\pi i/3} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4\pi i/3} & 0 & e^{2\pi i/3} & 1 & 0 \end{bmatrix} \begin{bmatrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{bmatrix} = \begin{bmatrix} |J = 3/2, M = 3/2\rangle \\ |J = 3/2, M = 1/2\rangle \\ |J = 3/2, M = -1/2\rangle \\ |J = 3/2, M = -3/2\rangle \\ |J = 1/2, M = 1/2, P = 0\rangle \\ |J = 1/2, M = -1/2, P = 0\rangle \\ |J = 1/2, M = 1/2, P = 1\rangle \\ |J = 1/2, M = -1/2, P = 1\rangle \end{bmatrix}$$

(9)

Where this is a different form to the other basis for 3 qubits Eq. 5.

A.3   4 Qubit CG coefficients

The J=2 block, P=0000, (J=1/2, J=1, J=3/2, J=2)

$$|J = 2, M = +2, P = 0000\rangle = |0000\rangle$$

$$|J = 2, M = +1, P = 0000\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$$

$$|J = 2, M = \;\;\,0, P = 0000\rangle = \sqrt{\frac{1}{6}}(|0011\rangle + |0101\rangle + |1001\rangle + |1100\rangle + |1010\rangle + |0110\rangle) \qquad (10a)$$

$$|J = 2, M = -1, P = 0000\rangle = \frac{1}{2}(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle)$$

$$|J = 2, M = -2, P = 0000\rangle = |1111\rangle$$

The J=1 (0) block, P=0001, (J=1/2, J=1, J=3/2, J=1)

$$|J = 1, M = +1, P = 0001\rangle = +\sqrt{\frac{3}{4}}|0001\rangle - \sqrt{\frac{1}{12}}(|0010\rangle + |0100\rangle + |1000\rangle)$$

$$|J = 1, M = \;\;\,0, P = 0001\rangle = \sqrt{\frac{1}{6}}(|0011\rangle + |0101\rangle + |1001\rangle - |1100\rangle - |1010\rangle - |0110\rangle) \qquad (10b)$$

$$|J = 1, M = -1, P = 0001\rangle = -\sqrt{\frac{3}{4}}|1110\rangle + \sqrt{\frac{1}{12}}(|1101\rangle + |1011\rangle + |0111\rangle)$$

The J=1 (1) block, P=0010, (J=1/2, J=1, J=1/2, J=1)

$$|J = 1, M = +1, P = 0010\rangle = +\sqrt{\frac{2}{3}}|0010\rangle - \sqrt{\frac{1}{6}}(|0100\rangle + |1000\rangle)$$

$$|J = 1, M = \;\;\,0, P = 0010\rangle = \sqrt{\frac{1}{3}}(|0011\rangle - |1100\rangle) + \sqrt{\frac{1}{12}}(|0110\rangle + |1010\rangle - |0101\rangle - |1001\rangle)$$

$$|J = 1, M = -1, P = 0010\rangle = -\sqrt{\frac{2}{3}}|1101\rangle + \sqrt{\frac{1}{6}}(|1011\rangle + |0111\rangle)$$

$$(10c)$$

The J=1 (2) block, P=0100, (J=1/2, J=0, J=1/2, J=1)

$$|J = 1, M = +1, P = 0100\rangle = +\sqrt{\frac{1}{2}}(|0100\rangle - |1000\rangle)$$

$$|J = 1, M = \;\;\,0, P = 0100\rangle = \frac{1}{2}(|0101\rangle - |1001\rangle + |0110\rangle - |1010\rangle) \qquad (10d)$$

$$|J = 1, M = -1, P = 0100\rangle = -\sqrt{\frac{1}{2}}(|0111\rangle - |1011\rangle)$$

The J=0 block, P=0011, (J=1/2, J=1, J=1/2, J=0)

$$|J=0,M=0,P=0011\rangle = \sqrt{\frac{1}{3}}(|0011\rangle + |1100\rangle) - \sqrt{\frac{1}{12}}(|0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle) \quad (10e)$$

The J=0 block, P=0101, (J=1/2, J=0, J=1/2, J=0)

$$|J=0,M=0,P=0101\rangle = \frac{1}{2}(|0101\rangle - |1001\rangle - |0110\rangle + |1010\rangle) \quad (10f)$$

A.4 Rotation matrix for J & M values

J=0 values

$$\begin{aligned}
|J=0,M'=+1/2\rangle &= R = I \\
|J=0,M'=-1/2\rangle &= R = XZ
\end{aligned} \quad (11a)$$

J=1/2 values

$$\begin{aligned}
|J=1/2,M'=+1\rangle &= R = I \\
|J=1/2,M'=\ \ 0\rangle &= R = XH \\
|J=1/2,M'=-1\rangle &= R = XZ
\end{aligned} \quad (11b)$$

J=1 values

$$\begin{aligned}
|J=1,M'=+3/2\rangle &= R = I \\
|J=1,M'=+1/2\rangle &= R = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{bmatrix} \\
|J=1,M'=-1/2\rangle &= R = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \\
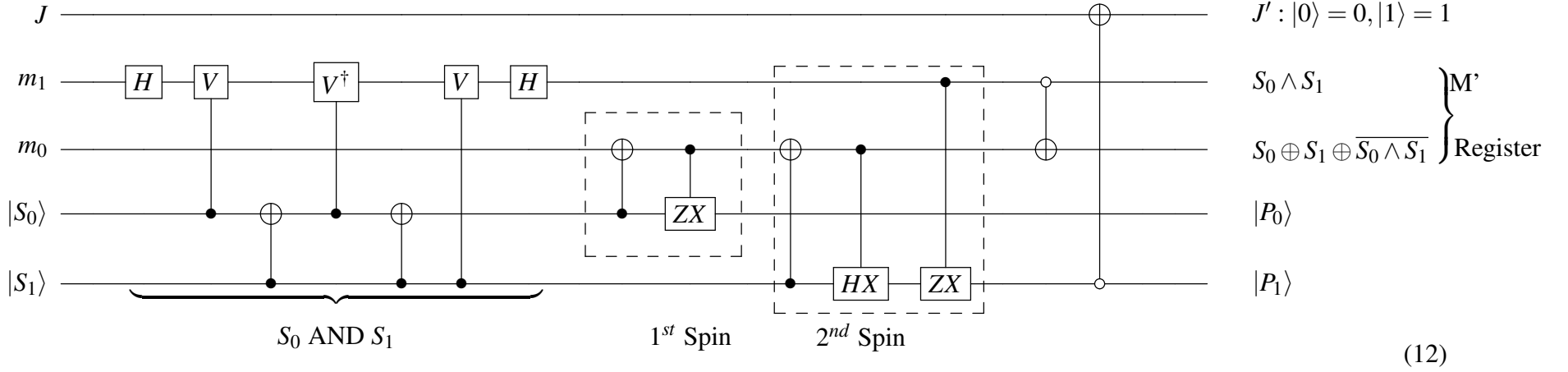|J=1,M'=-3/2\rangle &= R = XZ
\end{aligned} \quad (11c)$$

J=3/2 values

$$\begin{aligned}
|J=3/2,M'=+2\rangle &= R = I \\
|J=3/2,M'=+1\rangle &= R = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \\
|J=3/2,M'=\ \ 0\rangle &= R = XH \\
|J=3/2,M'=-1\rangle &= R = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \\
|J=3/2,M'=-2\rangle &= R = XZ
\end{aligned} \quad (11d)$$

J=2 values

$$\left|J=2,M'=+5/2\right\rangle = R = I$$

$$\left|J=2,M'=+3/2\right\rangle = R = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\left|J=2,M'=+1/2\right\rangle = R = \frac{1}{\sqrt{5}}\begin{bmatrix} \sqrt{3} & -\sqrt{2} \\ \sqrt{2} & \sqrt{3} \end{bmatrix}$$

$$\left|J=2,M'=-1/2\right\rangle = R = \frac{1}{\sqrt{5}}\begin{bmatrix} \sqrt{2} & -\sqrt{3} \\ \sqrt{3} & \sqrt{2} \end{bmatrix}$$

$$\left|J=2,M'=-3/2\right\rangle = R = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

$$\left|J=2,M'=-5/2\right\rangle = R = XZ$$

(11e)

A Appendix- Circuits

$$ (12) $$

Figure 3: Spatial multiplexed 2 XOR

HX gate triggers if $M = 0$ meaning $S_0 \neq S_1$ which is implemented using an XOR between S_0 & S_1. The other gate (T) is triggered when $M = -1$ meaning $S_0 = S_1 = 1$ which is done using an AND (Toffoli) gate between, $S_0 = S_1$ AND $S_1 = 1$ which is decomposed into 5 two-qubit gates. T^2 is the ZX gate, meaning $T^2 |S\rangle = XZ |S\rangle$.

Spin values		Circuit output		M value
S_1	S_0	m_1	m_0	M
0	0	0	1	M=+1
0	1	0	0	M=0
1	0	0	0	M=0
1	1	1	0	M=-1

Table 2: Table giving M register decoding for 2 qubit spatial multiplexing

$$\tag{13}$$
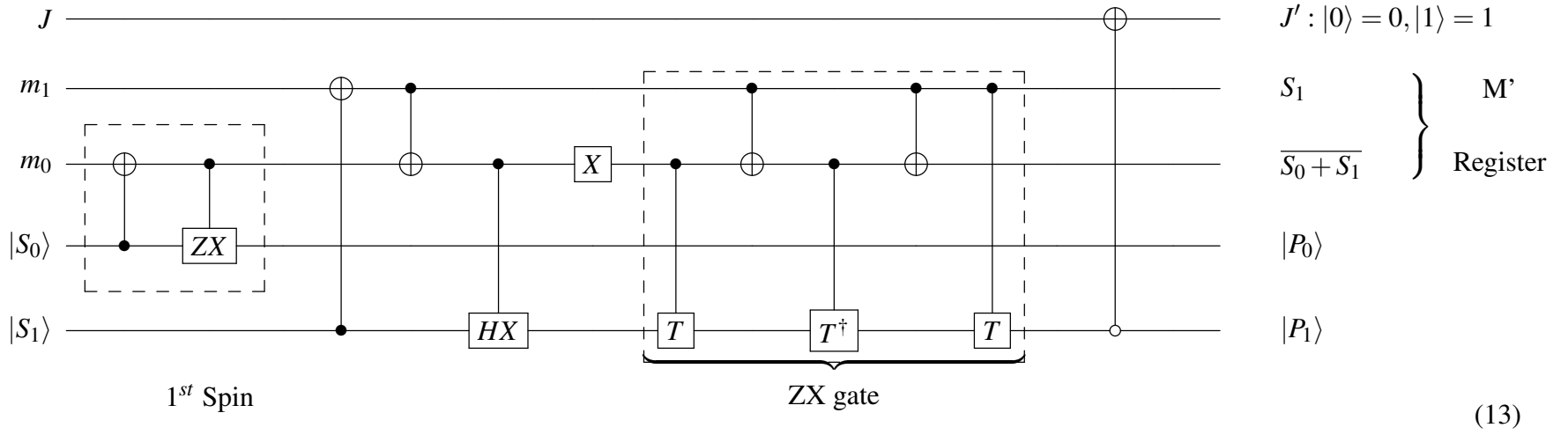
Figure 4: minimal gate spatial multiplexing

Spin values		Circuit output		M value
S_1	S_0	S_1	$\overline{S_0 + S_1}$	M
0	0	0	1	M=+1
0	1	0	0	M=0
1	0	1	0	M=0
1	1	1	1	M=-1

Table 3: Table giving M register decoding for minimal gate number

Figure 5: temporal multiplexed streaming

(14)

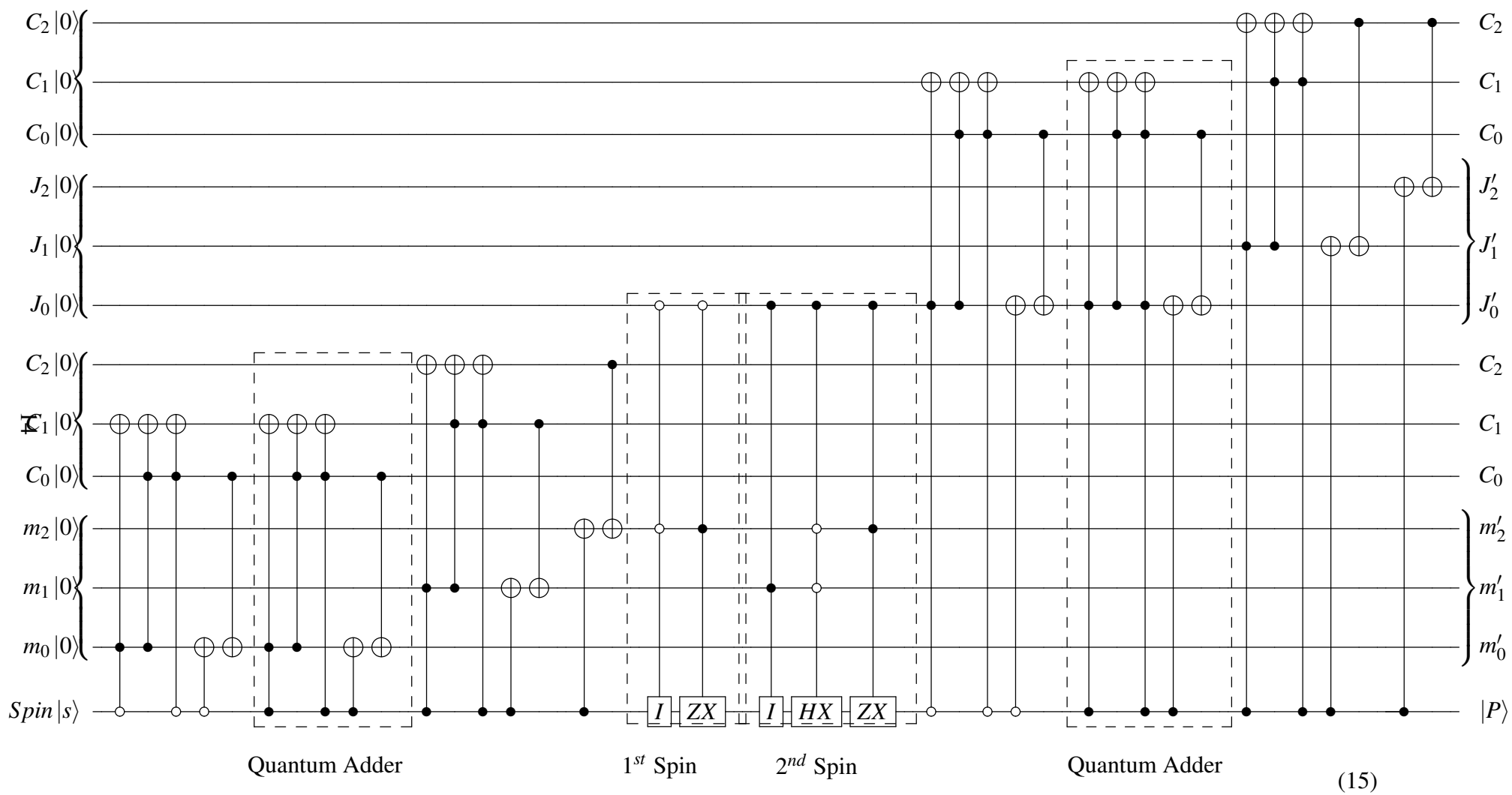| J_2	J_1	J_0	J		m_2	m_1	m_0	M
0	0	0	0		0	0	0	0
0	0	1	$\frac{1}{2}$		0	0	1	$\frac{1}{2}$
0	1	0	1		0	1	0	1
0	1	1	$\frac{3}{2}$		0	1	1	$\frac{3}{2}$
1	0	0	-2		1	0	0	-2
1	0	1	$-\frac{3}{2}$		1	0	1	$-\frac{3}{2}$
1	1	0	-1		1	1	0	-1
1	1	1	$-\frac{1}{2}$		1	1	1	$-\frac{1}{2}$

Table 4: Tables giving binary Two's complement encoding to spin values of the M and J registers

Figure 6: general streaming circuit

$$(15)$$