COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL. VII, 649-673 (1954)

# On the Exponential Solution of Differential Equations for a Linear Operator*

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## Introduction and Summary

The present investigation was stimulated by a recent paper of K. O. Friedrichs [1], who arrived at some purely algebraic problems in connection with the theory of linear operators in quantum mechanics. In particular, Friedrichs used a theorem by which the Lie elements in a free associative ring can be characterized. This theorem is proved in Section II of the present paper together with some applications which concern the addition theorem of the exponential function for non-commuting variables, the so-called Baker-Hausdorff formula. Section I contains some algebraic preliminaries. It is of a purely expository character and so is part of Section III. Otherwise, Section III deals with the following problem, also considered by Friedrichs: Let $A(t)$ be a linear operator depending on a real variable $t$. Let $Y(t)$ be a second operator satisfying the differential equation

$$dY/dt = AY \tag{1}$$

and the initial condition $Y(0) = I$, where $I$ denotes the identity operator. The problem is to define, in terms of $A$, an operator $\Omega(t)$ such that $Y = \exp \Omega$. Feynman [2], using a symbolic interpretation of

$$\exp \int_0^t A \, dt \tag{2}$$

has derived a solution of (1) in the infinite series form obtained when (1) is solved by iteration. The expression for $\Omega$ obtained in the present paper is also an infinite series but it satisfies the condition that the partial sums of this series become Hermitian after multiplication by $i$ if $iA$ is a Hermitian operator. This formula for $\Omega$ is the continuous analogue of the Baker-Hausdorff formula. All of these results are essentially algebraic; they are supplemented in Section IV by a proof of Zassenhaus' formula, which may be described as the dual of Hausdorff's formula.

The simplest instance of an equation of type (1) is given by a finite system of linear differential equations. In this case, $A(t)$ is the matrix of the coefficients of the system, and the convergence of the infinite series for $\Omega(t)$ can be discussed.

*This research was supported in part by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command.

This is done in Section VI. In a special case, which is treated in Section VII, necessary and sufficient conditions are derived for the existence and regularity of $\Omega(t)$ for all values of $t$. Section V supplements a recent investigation by H. B. Keller and J. B. Keller [3], who resumed and continued the work by H. F. Baker [4] on systems of ordinary linear differential equations. In [3], the investigation starts with the assumption that the matrix $A(t)$ can be diagonalized. In Section V we show that the continuous analogue of the Baker-Hausdorff formula provides non-trivial sufficient conditions for the existence of elementary solutions of (1) in cases where $A(t)$ cannot be diagonalized. Here the term "elementary" refers to algebraic operations and applications of a finite number of quadratures to $A(t)$.

# FIRST PART: FORMAL ALGEBRA

## 1. *Preliminaries*

A free associative ring $R_0$ with $n$ free generators $x_1, \cdots, x_n$ and an identity 1 will be defined by the following four axioms:

(a) The elements of a field $f_0$ of characteristic zero (for instance the field of real numbers) are in $R_0$, and the unit element 1 of $f_0$ is the identity of $R_0$. The field $f_0$ belongs to the center of $R_0$, i.e., all its elements commute with all the elements of $R$. We shall call $f_0$ the *field of the coefficients.*

(b) The addition in $R_0$ is commutative and associative, and the multiplication is associative.

(c) There exist no relations between the elements of $R$ except those which follow from (a) and (b).

(d) Every element of $R_0$ can be obtained from the elements of $f_0$ and from $x_1, \cdots, x_n$ by carrying out a finite number of additions and multiplications.

It follows from the axioms (a)–(d) that the identity 1 and the products of any number of "factors" $x_\nu^\alpha$ i.e., the products

$$x_{\nu_1}^{\alpha_1} x_{\nu_2}^{\alpha_2} \cdots x_{\nu_m}^{\alpha_m}, \qquad (\alpha_1, \alpha_2, \cdots, \alpha_m = 1, 2, 3, \cdots), \tag{1.1}$$

with

$$\nu_1 \neq \nu_2 \neq \nu_3 \cdots \nu_{m-1} \neq \nu_m, \; \nu_\mu = 1, 2, \cdots n \text{ for } \mu = 1, \cdots, m \tag{1.2}$$

form a basis of linearly independent elements of $R_0$ with respect to $f_0$.

We may extend $R_0$ to a ring $R$, which consists of all the formal power series with coefficients $c_0$, $c(\nu_1, \cdots, \nu_m, \alpha_1, \cdots, \alpha_m)$ in $f_0$, the generic element $A$ of $R$ being

$$A = c_0 + \sum c(\nu_1, \cdots, \nu_m; \alpha_1, \cdots, \alpha_m) x_{\nu_1}^{\alpha_1} \cdots x_{\nu_m}^{\alpha_m}. \tag{1.3}$$

The sum in (1.3) is taken over all possible combinations of integers $\nu_1, \cdots, \nu_m, \alpha_1, \cdots, \alpha_m$ satisfying (1.1) and (1.2) for $m = 1, 2, 3, \cdots$. A power series of the type described by (1.3) will also be called a *function* of $x_1, \cdots, x_n$ and will be denoted by $F(x_1, \cdots, x_n)$. Problems of convergence do not play a role; if, for instance, $f_0$ is the field of rational numbers, both

$$\sum_{n=0}^{\infty} n!x_1^n$$

and its square

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} r!(n-r)!x_1^n$$

belong to $R$. Here we have used the natural notation according to which $x_\nu^0 = 1$ for any $\nu$.

As an example which will be used later we may consider the case of two free generators $x$, $y$; we take for $f_0$ the field of real numbers. The exponential function is defined by

$$e^x = \sum_{n=0}^{\infty} x^n/n! \tag{1.4}$$

and we have

$$e^x e^y = \sum_{n,m=0}^{\infty} x^n y^m/(n!m!). \tag{1.5}$$

We can find a function $z$ of $x$ and $y$ such that

$$e^x e^y = e^z \tag{1.6}$$

and

$$z = u - \tfrac{1}{2}u^2 + \tfrac{1}{3}u^3 \mp \cdots, \tag{1.7}$$

where

$$u = (e^x e^y - 1) = x + y + \frac{x^2}{2!} + \frac{xy}{1!1!} + \frac{y^2}{2!} + \cdots. \tag{1.8}$$

If we substitute for $u$ its value from (1.8), it is easily shown that the series in (1.7) leads to an element of $R$. For this purpose, we shall call

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n$$

the *degree* (in $x_1, \cdots, x_n$) of the basis element (1.1); in (1.5) the terms of degree $l$ (in $x$, $y$) will then consist of the sum

$$\frac{1}{l!} \sum_{r=0}^{l} \binom{l}{r} x^r y^{l-r}. \tag{1.9}$$

Only a finite number of powers of $u$ will involve terms of a given degree in $x$, $y$, since the terms of lowest degree in $u^k$ are of degree $k$ in $x$ and $y$. From this we derive easily that $z$ becomes a power series in $x$, $y$ with rational coefficients, the first terms being

$$z = x + y + \tfrac{1}{2}xy - \tfrac{1}{2}yx + \cdots. \tag{1.10}$$

Therefore, the mere existence of an element $z$ of $R$ satisfying (6) is almost obvious. Also, it would be easy to prove that $z$ is uniquely determined by (6). But it is much more difficult to show that $z$ has a certain algebraic property which will be described presently, and to provide a method by which $z$ can be computed and expressed in a form which exhibits this property. These results were obtained first and independently by Baker [5] and Hausdorff [6]; in order to formulate them, we need the following definitions:

Let $u$, $v$ be any elements of $R$. Then the *bracket-product* or *Lie-product* $[u, v]$ of $u$ and $v$ is defined by

$$[u, v] = uv - vu. \tag{1.11}$$

Using Lie-multiplication, we can define recursively a *Lie-element* of $R$. We shall call $x$ and $y$ (or, in the general case, the free generators $x_\nu$) Lie-elements of degree one. Any linear combination of Lie-elements of degree one with coefficients from $f_0$ and any Lie-product of Lie-elements shall also be called a Lie-element. The total set of Lie-elements obtained in this manner will be called $\Lambda$. For properties of this set see [7], [8], and [9]. In the case of two free generators $x$, $y$ the general Lie-element involving terms of a degree $\leqq 3$ is

$$\begin{aligned} c_1x + c_2y + c_{12}(xy - yx) + c_{121}((xy - yx)x - x(xy - yx)) \\ + c_{122}((xy - yx)y - y(xy - yx)), \end{aligned} \tag{1.12}$$

where $c_1, \cdots, c_{122}$ are elements of the field $f_0$ (e.g., rational numbers).

We shall call the following statement the *Baker-Hausdorff theorem*: *let $z$ be the element of $R$ defined by $\exp x \exp y = \exp z$. Then $z$ is a Lie-element.* A new proof of this theorem will be given in the next section. The explicit expression of $z$ in terms of Lie-elements of $R$ shall be called the *Baker-Hausdorff formula.* Methods for finding recursively the terms of a degree $\leqq n$ for $n = 1, 2, 3, \cdots$ of this formula will be discussed in Sections III and IV.

## II. *A Theorem of Friedrichs**

In a discussion of the theory of operators of quantum mechanics, Friedrichs [1] found a characterization of Lie-elements and proved it in a particular case by using a theory of representation of these operators. We shall formulate and prove Friedrichs' theorem in the case where the ring $R$ has two free generators $x$, $y$. But the proof for a free ring with any number of generators is almost literally the same.

We construct first an isomorphic replica $R'$ of $R$ which has two free generators $x'$, $y'$. Then we construct the direct product $\tilde{R}$ of $R$ and $R'$, identifying the elements of the underlying isomorphic fields of coefficients in $R$ and $R'$. The

*A proof different from the one given here and based on a lemma due to Birkhoff and Witt [8] has been communicated to the author by Dr. P. M. Cohn of Manchester University.

ring $\tilde{R}$ is generated by $x$, $y$, $x'$, $y'$, where the generators are not entirely free but satisfy the relation

$$(2.1) \qquad xx' = x'x, \qquad yy' = y'y, \qquad xy' = y'x, \qquad x'y = yx'.$$

We may consider any element of $R$ as an element of $\tilde{R}$, which contains $R$. Now we can state Friedrichs' theorem:

THEOREM I: *Let $F(x, y)$ be a function of $x$ and $y$. Then $F$ is a Lie-element of $R$ if and only if*

$$(2.2) \qquad F(x + x', y + y') = F(x, y) + F(x', y').$$

To prove this theorem, we may confine ourselves to the case where $F$ is homogeneous in $x$, $y$, that is where $F$ is a sum of terms of a fixed degree. It is easily seen that (2.2) holds if $F$ is a Lie-element. In this case, $F$ must be a sum of terms which have been derived from the generators by a repeated application of Lie-multiplication. Consider a term $w$ involved in $F$, which has been obtained from two Lie-elements $u$ and $v$ of $R$ by Lie-multiplication:

$$w = [u, v].$$

Then $u$ and $v$ are of lesser degree than $w$ and we may assume that, as functions of $x$ and $y$, they satisfy (2.2). Since it follows from (2.1) that the bracket product of a factor depending on $x$, $y$ and a factor depending on $x'$, $y'$ is always zero, we find that $w$ also satisfies (2.2).

To show that only Lie-elements satisfy (2.2) we introduce first the following definition:

Any $r$ elements ($r = 1, 2, 3, \cdots$) of $R$ are called *algebraically independent* if they generate a subring of $R$ which is isomorphic with a free ring of $r$ free generators. More specifically, let $u_1, \cdots, u_r$ be the $r$ elements of $R$ and let $y_1, \cdots, y_r$ be free generators of a free ring $R^*$. Then we shall call the $u_\rho$, ($\rho = 1, \cdots, r$) algebraically independent if the mapping

$$y_\rho \rightarrow u_\rho \qquad (\rho = 1, \cdots, r)$$

determines an isomorphic (one-to-one) correspondence between $R^*$ and the smallest subring of $R$ which contains the $u_\rho$. (We define the maps of sums and products in the natural manner.)

Next, we need the following two lemmas, which have been proved elsewhere (see [7]):

LEMMA 1: *Let $u_1, \cdots, u_r$ and $v$ be $r + 1$ algebraically independent elements of a free ring $R^*$. For $l = 1, 2, 3, \cdots$ and $\rho = 1, 2, \cdots, r$, let*

$$(2.3) \qquad u_\rho^{(l)} = [[\cdots [[u_\rho v]v] \cdots ]v],$$

$$(2.4) \qquad u_\rho^{(0)} = u_\rho .$$

*The right hand side in (3) is the result of an $l$-fold bracket multiplication of $u_\rho$ by $v$ from the right. Then all the $u_\rho^{(l)}$ are algebraically independent (for $l = 0, 1, 2, \cdots$).*

Lemma 1 shows that there exist any number of algebraically independent elements (and in particular of Lie-elements) in $R$; if we take, for instance, $r = 1$, $u_1 = x$, $v = y$, the resulting elements $u_1^{(l)}$ are all Lie-elements. With the same assumptions as in Lemma 1 we have:

LEMMA 2: *Every function $H(u_1, \cdots, u_r, v)$ can be written in a unique way in the form*

$$H = H_0 + H_1 v + H_2 v^2 + \cdots, \tag{2.5}$$

*where $H_0, H_1, H_2, \cdots$ are functions of the $u_\rho^{(l)}$, $(l = 0, 1, 2, \cdots)$.*

We shall apply these lemmas to the proof of Friedrichs' theorem. Let $F(x, y)$ be a function which satisfies (2) and is of degree $d$ in $x$, $y$. $F$ is then expressed in terms of Lie-elements of the first degree.

Assume that $F$ could also be written as a function $H(u_1, \cdots, u_r, v)$ of algebraically *independent Lie-elements* $u_1, u_2, \cdots, u_r, v$ of degrees

$$l_1 \geqq l_2 \geqq \cdots \geqq l_r \geqq l_0,$$

where $l_0$ is the degree of $v$ in $x$, $y$ and $l_\rho$ denotes the degree of $u_\rho$. Then we can prove

LEMMA 3: *If $H$ is expanded according to Lemma 2, then either $H = H_0(u_\rho^{(l)})$ or $H = H_0(u_\rho^{(l)}) + hv$, where $h$ is a constant (that is where $h$ is an element of the field of coefficients $f_0$).*

Proof: We have now to use the fact that, in terms of the Lie-elements $u_1, u_2, \cdots, u_r, v$,

$$F(x, y) = H(u_1, u_2, \cdots, u_r; v), \tag{2.6}$$

and to show that if $F$ has the property (2.2) either

$$F(x, y) = H_0(u_\rho^{(l)}) \tag{2.7a}$$

or

$$F(s, y) = H_0(u_\rho^{(l)}) + hv \tag{2.7b}$$

holds. Since the "variables" $u_1, \cdots, u_r, v$ are Lie-elements of the ring $R$ generated by $x$, $y$, we have

$$w(x + x', y + y') = w(x, y) + w(x', y'), \tag{2.8}$$

where $w$ stands for any one of the elements $u_1, \cdots, u_r, v$. We shall write $w'$ for $w(x', y')$ and, correspondingly, $u_\rho'$, $v'$ for $u_\rho(x', y')$, $v(x', y')$. Now we have from (2.2), (2.6) and (2.8)

$$F(x + x', y + y') = H(u_1 + u_1', \cdots, u_r + u_r', v + v'). \tag{2.9}$$

Applying Lemma 2 to $H$, we find

$$\begin{aligned} H(u_1 + u_1', \cdots, u_r', v + v') &= H_0(u_\rho^{(l)} + u_\rho^{(l)\prime}) \\ &+ H_1(u_\rho^{(l)} + u_\rho^{(l)\prime})(v + v') + H_2(u_\rho^{(l)} + u^{(l)\prime})(v + v')^2 + \cdots, \end{aligned} \tag{2.10}$$

where the $u_\rho^{(l)\prime}$ are derived from the $u_\rho^{(l)}$ by the transition from $x$, $y$ to $x'$, $y'$. Now condition (2.2) gives

$$
\begin{aligned}
H(u_1 + u_1', \cdots, u_r + u_r', v + v') \\
&= H(u_1, \cdots, u_r, v) + H(u_1', \cdots, u_r', v') \\
&= H_0(u_\rho^{(l)}) + H_0(u_\rho^{(l)\prime}) + H_1(u_\rho^{(l)})v + H_1(u_\rho^{(l)\prime})v' \\
&\quad + H_2(u_\rho^{(l)})v^2 + H_2(u_\rho^{(l)\prime})v'^2 + \cdots .
\end{aligned} \tag{2.11}
$$

According to Lemma 2, the coefficients of the products $v^k v'^{k'}$ on the right hand sides of (2.10) and (2.11) must be the same. From this we have

$$H_0(u_\rho^{(l)} + u_\rho^{(l)\prime}) = H_0(u_\rho^{(l)}) + H_0(u_\rho^{(l)\prime}), \tag{2.12}$$

$$H_1(u_\rho^{(l)} + u_\rho^{(l)\prime}) = H_1(u_\rho^{(l)}) = H_1(u_\rho^{(l)\prime}) = h, \tag{2.13}$$

$$H_2(u_\rho^{(l)} + u_\rho^{(l)\prime}) = H_2(u_\rho^{(l)}) = H_2(u_\rho^{(l)\prime}) = 0, \tag{2.14}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots .$$

Equations (2.12), (2.13), (2.14) contain the proof of Lemma 3. Now we can prove Theorem I by using the following argument: Since $F$ is homogeneous with respect to $x$, $y$, the case where $h \neq 0$ in (2.13) can arise only if $l_0 = l_1 = \cdots = l_r = d$ where $d$ is the degree of $F$ with respect to $x$, $y$. In this case, $F$ is a linear combination of the Lie-elements $u_1, \cdots, u_r, v$ and therefore is itself a Lie-element as we wanted it to be. If $d > l_0$, the constant $h$ in (2.13) must vanish. But in this case we can apply our Lemma 3 to $H_0(u_\rho^{(l)})$. According to Lemma 1, the $u_\rho^{(l)}$ are again algebraically independent Lie-elements, their degrees with respect to $x$, $y$ are $l_\rho + ll_0$, therefore the number of Lie-elements of lowest degree $l_0$ involved in $H_0$ has diminished by one, and we see that a repeated application of Lemma 3 leads to a proof of Theorem I.

Friedrichs used Theorem I for a proof of the following results, both of which had been derived by Hausdorff in a different manner:

THEOREM II: *Let $x$, $y$ be free generators of an associative ring $R$. Let $z$ and $w$ be the elements defined by*

$$e^x e^y = e^z, \tag{2.15}$$

$$e^{-x} y e^x = w. \tag{2.16}$$

*Then $z$ and $w$ are Lie-elements in $R$.*

The proof follows immediately from Theorem I if we observe that $e^{x+x'} = e^x \cdot e^{x'}$ and $xx' = x'x$. Baker [5] and Hausdorff [6] proved Theorem II by a recursive construction of $z$ and by an explicit formula for $w$. Once one knows that $w$ is a Lie-element, it is easy to determine it explicitly. Since it is of first degree in $y$, it must be expressible in terms of the Lie-elements

$$\{y, x^l\} = [\cdots[[y, x]x]\cdots x] \tag{2.17}$$

which are obtained from $y$ by an $l$-fold bracket multiplication by $x$. The result is

$$w = \sum_{l=0}^{\infty} \frac{1}{l!} \{y, x^l\}, \tag{2.18}$$

where $\{y, x^0\}$ stands for $y$ itself.

In a remarkably simple manner, D. Finkelstein has proved that Lie-elements can also be characterised by the following property: Let $F(x, y)$, $x$, $y$, $x'$, $y'$ be defined as in Theorem I. Let $\Delta x = x - x'$, $\Delta y = y - y'$, and define $\Delta F$ in such a manner that for a sum $F = C_1F_1 + C_2F_2$ with constant $C_1$, $C_2$

$$\Delta F = C_1\Delta F_1 + C_2\Delta F_2 .$$

Then $\Delta F$ is defined for all $F$ if $\Delta F_0$ is defined for all products $F_0$ of the generators. For any such product $F_0$, let $\tilde{F}_0$ denote the product of the same factors in the inverse order (for example $(\widetilde{xy}) = yx$) and define

$$\Delta F_0(x, y) = F_0(x, y) - \tilde{F}_0(x', y').$$

Then $F(x, y)$ is a Lie element if and only if

$$\Delta F(x, y) = F(x - x', y - y'). \tag{2.19}$$

It can be shown that the properties (2.2) and (2.19) are equivalent by observing that they are equivalent for any $F$ of type (4.4). D. Finkelstein's results, which include also a new and simple derivation of Theorem II and of formulas (3.16), (3.17), are presented in a forthcoming publication.

## III. *Differentiation and Differential Operators*

Let $R$ be a free associative ring with free generators $x$, $y$, $x_1$, $y_1$ $\cdots$ . Let $F(x, y)$ be an element of $R$ and let $\lambda$ be a parameter, i.e., an arbitrary number from the field $f_0$ (for instance an arbitrary real number). Then we can expand $F(x + \lambda x_1, y)$ in a series of powers of $\lambda$:

$$F(x + \lambda x_1, y) = F(x, y) + \lambda F_1(x, x_1, y) + \cdots . \tag{3.1}$$

We shall call the coefficient of $\lambda$ a derivative and write

$$F_1(x, x_1, y) = \left(x_1 \frac{\partial}{\partial x}\right)F(x, y). \tag{3.2}$$

The word "derivative" was introduced by Hausdorff. Another more customary term for this coefficient is "*polar*," and the process by which $F_1$ is obtained from $F$ is also called "polarization." The definition of a derivative used here is related to but different from the one used by Falk [10]. If $F$ is a monomial (that is, an element of the type (1.1)), polarization consists of first replacing one factor $x$ by $x_1$ in every possible way and then adding all the resulting terms afterwards. The polar of a sum of terms is the sum of the polars of the terms. If $F$ does not involve $x$, its polar with respect to $x$ is zero. It is not necessary that $x_1$ be different from $x$; we can define $(x \cdot \partial/\partial x)F$ by forming $(x \cdot \partial/\partial x)F$ and substituting $x$ for $x_1$ after the polarization. We may also substitute any element $u$ of $R$ for $x_1$ after polarization; the result will be denoted by $(u \cdot \partial/\partial x)F$.

We extend the notation introduced by (2.17) and (2.18) to

$$\{y, P(x)\} = \sum_{l=0}^{\infty} p_l \{y, x^l\} \tag{3.3}$$

where

$$P(x) = \sum_{l=0}^{\infty} p_l x^l \tag{3.4}$$

with constant coefficients $p_l$. With this notation, we have, according to Hausdorff [6], the formulas

$$e^{-x}\left(y \frac{\partial}{\partial x}\right) e^{x} = \left\{y, \frac{e^{x} - 1}{x}\right\}, \qquad \left(\left(y \frac{\partial}{\partial x}\right) e^{x}\right) e^{-x} = \left\{y, \frac{1 - e^{-x}}{x}\right\} \tag{3.5}$$

and the following theorem:

LEMMA 4: *Let $P(x)$ and $Q(x)$ be two power series in $x$ which satisfy*

$$P(x)Q(x) = 1. \tag{3.6}$$

*Then each of the equations*

$$\{y, P(x)\} = u, \qquad y = \{u, Q(x)\} \tag{3.7}$$

*is a consequence of the other.*

The proof consists of a simple straightforward computation, using the relations between the coefficients of $P(x)$ and $Q(x)$ which result from (3.6) and the identity

$$\{\{y, x^n\}, x^m\} = \{y, x^{n+m}\} \qquad (n, m = 0, 1, 2, \cdots). \tag{3.8}$$

The equations (3.5) and Lemma 4 lead to Hausdorff's representation of the function $z(x, y)$ defined by $\exp x \exp y = \exp z$. We have from (3.5)

$$e^{-z}\left(u \frac{\partial}{\partial x}\right) e^{z} = \left\{\zeta, \frac{e^{z} - 1}{z}\right\} \tag{3.9}$$

where $u$ is any indeterminate quantity and where

$$\zeta = \left(u \frac{\partial}{\partial x}\right) z. \tag{3.10}$$

On the other hand, we have from the definition of $z$ and from (3.5)

$$e^{-z}\left(u \frac{\partial}{\partial x}\right) e^{z} = e^{-y}e^{-x}\left(u \frac{\partial}{\partial x}\right) e^{x}e^{y} = e^{-y}\left\{u, \frac{e^{x} - 1}{x}\right\}e^{y}. \tag{3.11}$$

Similarly, if we put

$$\zeta^* = \left(y \frac{\partial}{\partial y}\right) z \tag{3.12}$$

we have from (3.5) and the definition of $z$:

$$e^{-z}\left(y \frac{\partial}{\partial y}\right) e^{z} = \left\{\zeta^*, \frac{e^z - 1}{z}\right\} = e^{-y}e^{-x}\left(y \frac{\partial}{\partial y}\right) e^x e^y = e^{-y}\left(y \frac{\partial}{\partial y}\right) e^y = y \tag{3.13}$$

Now we shall replace $u$ in (3.9) and (3.11) by

$$\omega = \left\{y, \frac{x}{e^x - 1}\right\}. \tag{3.14}$$

Then we see from Lemma 4 that the last term in (3.11) becomes simply $y$, and by substituting this value of the first term in (3.11) for the left hand side of (3.9), we find from (3.9), Lemma 4 and (3.12)

$$\left(y \frac{\partial}{\partial y}\right) z = \left(\omega \frac{\partial}{\partial x}\right) z = \left\{y, \frac{z}{e^z - 1}\right\}. \tag{3.15}$$

This is a partial differential equation for $z$ which can be solved by the method of power series expansion and coefficient comparison. Indeed, if we write

$$z = x + z_1 + z_2 + z_3 + \cdots . \tag{3.16}$$

where $z_n$ contains exactly these terms of $z$ which are of degree $n$ with respect to $y$, we find first $(y \cdot \partial/\partial y)\, z_n = nz_n$ and then from (3.15)

$$z_1 = \left(\omega \frac{\partial}{\partial x}\right) x = \omega, \quad z_2 = \frac{1}{2}\left(\omega \frac{\partial}{\partial x}\right) z_1 , \; \cdots , \; z_{n+1} = \frac{1}{n+1}\left(\omega \frac{\partial}{\partial x}\right) z_n , \; \cdots \tag{3.17}$$

since $\omega$ is linear in $y$. It is obvious that (3.17) gives recurrence formulas for the computation of $z$ in terms of Lie-elements. The first terms are

$$\begin{aligned} z = {} & x + y + \frac{1}{2}[x, y] + \frac{1}{12}\{x, y^2\} + \frac{1}{12}\{y, x^2\} \\ & - \frac{1}{24}[\{x, y^2\}, x] - \frac{1}{720}\{x, y^4\} - \frac{1}{720}\{y, x^4\} \\ & + \frac{1}{180}[[\Delta_1 , x]y] - \frac{1}{180}\{\Delta_1 , y^2\} \\ & - \frac{1}{120}[\Delta_1 , \Delta] - \frac{1}{360}[\Delta_2 , \Delta] + \cdots \end{aligned} \tag{3.19}$$

where

$$\Delta = [x, y], \qquad \Delta_1 = [\Delta, x], \qquad \Delta_2 = [\Delta, y]. \tag{3.20}$$

We shall consider now an associative ring $R$ the elements of which are differentiable functions of a real parameter $t$. As an example we may take a ring of finite or infinite matrices with real or complex elements which are differentiable functions of $t$. For our purposes, two types of properties of these ring elements are required, formal properties and properties dealing with convergence.

The formal properties needed are:

(a) $A(t)$ is an element of $R$ for all real values of $t$. For any sufficiently small $\epsilon$,

$$A(t + \epsilon) = A(t) + \epsilon A_1(t) + \epsilon^2 A_2(t) + \cdots, \tag{3.21}$$

where the terms on the right hand side of (3.10) are in $R$ and where their sums are to be defined in $R$. We shall define $dA/dt$ to be $A_1(t)$.

(b) The formal laws of differentiation for a sum and a product hold.

(c) If $P(x)$ is a power series in $x$ and if $P(A(t)) = B(t)$ exists in $R$, then

$$\frac{dB}{dt} = \left(y \frac{\partial}{\partial x}\right) P(x), \qquad y = \frac{dA}{dt}, \qquad x = A(t). \tag{3.22}$$

(d) Given $A(t)$, there exists in $R$ a uniquely determined function $A^*(t)$ such that

$$\frac{dA^*}{dt} = A(t), \qquad A^*(0) = 0. \tag{3.23}$$

We shall write

$$\int_0^t A(\tau)\, d\tau = A^*(t). \tag{3.24}$$

The second type of property, which is more difficult to describe, concerns problems of convergence. For the present purposes it will be necessary to assume that $\exp A$ exists for all functions under consideration and is differentiable in the sense described under (a). Also, it will be necessary to assume that certain repeated integrals (where integration is defined by (3.23), (3.24)) exist.

Furthermore, we must be able to define certain infinite sums. If the elements of $R$ are linear operators acting on a Hilbert space, definitions are readily available. A different type of example of a ring $R$ in which the constructions of this section are permissible can be obtained as follows: Let $u_n$ be a sequence of elements of a free associative ring of the type defined in Section I. Let $\mu_n$ be the minimum of the degrees of terms involved in $u_n$; if $u_n = 0$, we define $\mu_n$ to be $-\infty$. We shall call the sequence of the $u_n$ a null sequence if

$$\lim_{n\to\infty} \mu_n^{-1} = 0.$$

Now we define $R$ as the ring of all power series in a real variable $t$ of the type

$$A(t) = \sum_{n=0}^{\infty} u_n t^n$$

where the $u_n$ form a null sequence and where $t$ commutes with all other quantities.

The case where the $A(t)$ are finite matrices with elements depending on $t$ will be considered in Sections V and VI.

We can state

THEOREM III: *Let $A(t)$ be a known function of $t$ in an associative ring $R$, and let $U(t)$ be an unknown function satisfying*

$$\frac{dU}{dt} = AU, \qquad U(0) = 1. \tag{3.25}$$

*Then, if certain unspecified conditions of convergence are satisfied, $U(t)$ can be written in the form*

$$U(t) = \exp \Omega(t) \tag{3.26}$$

*where*

$$\begin{aligned} \frac{d\Omega}{dt} &= \left\{A, \frac{\Omega}{1 - e^{-\Omega}}\right\} = \sum_{n=0}^{\infty} \beta_n\{A, \Omega^n\} \\ &= A + \frac{1}{2}[A, \Omega] + \frac{1}{12}\{A, \Omega^2\} \mp \cdots . \end{aligned} \tag{3.27}$$

*The $\beta_n$ vanish for $n = 3, 5, 7, \cdots$, and $\beta_{2m} = (-1)^{m-1} B_{2m}/(2m)!$, where the $B_{2m}$ (for $m = 1, 2, 3, \cdots$) are the Bernoulli numbers. Integration of* (3.27) *by iteration leads to an infinite series for $\Omega$ the first terms of which (up to terms involving three integrations) are*

$$\begin{aligned} \Omega = \int_0^t A(\tau)\, d\tau &+ \frac{1}{2}\int_0^t \left[A(\tau), \int_0^\tau A(\sigma)\, d\sigma\right] d\tau \\ &+ \frac{1}{4}\int_0^t \left[A(\tau), \int_0^\tau \left[A(\sigma), \int_0^\sigma A(\rho)\, d\rho\right] d\sigma\right] d\tau \\ &+ \frac{1}{12}\int_0^t \left[\left[A(\tau), \int_0^\tau A(\sigma)\, d\sigma\right] \int_0^\tau A(\sigma)\, d\sigma\right] d\tau \\ &+ \cdots . \end{aligned} \tag{3.28}$$

Formula (3.28) is the continuous analogue of the Baker-Hausdorff formula by which $z$ is expressed in (3.19) as a Lie-element of $x$, $y$. We can prove Theorem III in the following manner: If $U = \exp \Omega$, then, according to (3.22) and (3.5),

$$\frac{dU}{dt} = \left(\frac{d\Omega}{dt}\frac{\partial}{\partial\Omega}\right)e^{\Omega} = \left\{\frac{d\Omega}{dt}, \frac{1 - e^{-\Omega}}{\Omega}\right\}e^{\Omega}. \tag{3.29}$$

Therefore we find from (3.25) and from Lemma 4

$$A = \left\{\frac{d\Omega}{dt}, \frac{1 - e^{-\Omega}}{\Omega}\right\}, \qquad \frac{d\Omega}{dt} = \left\{A, \frac{\Omega}{1 - e^{-\Omega}}\right\}. \tag{3.30}$$

This proves (3.27). The proof of (3.28) is carried out by defining

$$\Omega_0 = 0, \qquad \Omega_1 = \int_0^t A(\tau)\, d\tau,$$

(3.31)

$$\Omega_n = \int_0^t \left( A(\tau) + \frac{1}{2}[A,\, \Omega_{n-1}] + \frac{1}{12}[[A,\, \Omega_{n-1}]\Omega_{n-1}] + \cdots \right) d\tau$$

and putting $\Omega = \lim_{n\to\infty} \Omega_n$. In the case where $A(\tau)$ is a finite matrix with bounded elements it can be shown by standard methods that (3.31) actually leads to a function $\Omega$ satisfying (3.26) for sufficiently small values of $t$.

A different method of deriving (3.28) can be based on (3.19). If we set

$$t = n\delta, \qquad A(\nu\delta) = A_\nu \qquad (\nu = 1, 2, \cdots, n)$$

and if we integrate (3.25) by substituting for $A(\tau)$, $0 \leqq \tau \leqq t$ a piecewise constant function with values $A_\nu$, we find for $U(t)$ the approximate value

$$\exp A_n\delta \exp A_{n-1}\delta \cdots \exp A_2\delta \exp A_1\delta. \tag{3.20}$$

Repeated application of (3.19) and passage to the limit $n \to \infty$ gives easily the first two terms of the right hand side of (3.28). But the complexity of the calculations increases rapidly with the number of terms, and the convergence difficulties involved in the application of (3.19) are considerable even if $x$, $y$ are finite matrices.

## IV. *The Zassenhaus Formula*

Let $R$ be the free ring with two generators $x$, $y$ and with rational coefficients. It has been observed by Zassenhaus [11] that there exists a formula which may be called the dual of Hausdorff's formula. We may state his result as follows:

There exist uniquely determined Lie-elements $C_n$ $(n = 2, 3, 4, \cdots)$ in $R$ which are exactly of degree $n$ in $x$, $y$ such that

$$e^{x+y} = e^x e^y e^{C_2} e^{C_3} \cdots e^{C_n} \cdots . \tag{4.1}$$

The existence of a formula of type (1) is an immediate consequence of Hausdorff's theorem.

In fact, we find successively that $\exp(-x)\exp(x + y) = \exp(y + C)$, where $C$ involves Lie-elements of a degree $> 1$, that $\exp(-y)\exp(y + C) = \exp(C_2 + C^*)$ where $C^*$ involves Lie-elements of a degree $> 2$ and so on. But the computation of $C_n$ becomes rather difficult if it is based on Hausdorff's complicated formula. A simpler method for the calculation of the $C_n$ can be derived from a result due to Dynkin [12], Specht [13], and Wever [14]. The method employed here has already been used by Dynkin to derive the coefficients of the terms of degree $n$ in Hausdorff's formula without the use of the coefficients of terms of lower degree.

For every element $F(x, y)$ in $R$ we define a corresponding Lie-element $F$, where the "curly bracket operator" $\{\ \ \}$ has the following properties:

(a) For any element $C$ in the field of coefficients,

$$\{CF\} = C\{F\}. \tag{4.2}$$

(b) For any two elements $F_1$, $F_2$ of $R$

$$\{F_1 + F_2\} = \{F_1\} + \{F_2\}. \tag{4.3}$$

(c) Let $x_\nu$, for $\nu = 1, 2, \cdots, n$, be any one of the generators. Then for any monomial $x_1 x_2 \cdots x_n$ we define

$$\{x_1\, x_2 \cdots x_n\} = [[\cdots[[x_1\, x_2]x_3] \cdots ]x_n]; \qquad \{x_\nu\} = x_\nu \tag{4.4}$$

and for the identity we define

$$\{1\} = 0.$$

It is clear that the operator $\{\ \ \}$ is defined uniquely for all $F$ in $R$ by the rules (a), (b), (c). In [12], [13], [14], the following theorem is proved: Let $G$ be a homogeneous Lie-element in $R$ which is of degree $n$; then

$$\{G\} = nG. \tag{4.5}$$

From Wever's paper [14] we can easily derive

LEMMA 5: *If $G$ is a homogeneous Lie-element and $F$ is any element of $R$, then*

$$\{G^2F\} = 0. \tag{4.6}$$

We expand both sides of (1) in power series and apply the operator $\{\ \ \}$. According to Lemma 5 we find

$$\begin{aligned}\{e^{x+y}\} &= \{1\} + \{x + y\} + \frac{1}{2!}\{(x + y)^2\} + \frac{1}{3!}\{(x + y)^3\} + \cdots \\ &= x + y\end{aligned} \tag{4.7}$$

since $\{(x + y)^n\} = 0$ if $n > 1$. In the same manner we find for the right hand side in (1)

$$\begin{aligned}\{e^x e^y e^{C_2} e^{C_3} \cdots\} &= \left\{\sum \frac{x^n}{n!} \sum \frac{y^n}{n!} \sum \frac{C_2^n}{n!} \cdots\right\} \\ &= x + y + \{xy\} + \{C_2\} + \frac{\{xy^2\}}{2!} + \{xC_2\} + \{yC_2\} + \{C_3\} + \cdots,\end{aligned} \tag{4.8}$$

where the omitted terms are of a degree greater than three. By comparing terms of the same degree in (7) and (8) we find

$$\{C_2\} + \{xy\} = 0,$$

$$\{C_3\} + \{xC_2\} + \{yC_2\} + \tfrac{1}{2}\{xy^2\} = 0$$

and therefore, because of (4.5),

$$
\begin{aligned}
C_2 &= -\tfrac{1}{2}[x, y], \\
C_3 &= -\tfrac{1}{6}[[x, y]y] + \tfrac{1}{6}\{(x + y)(xy - yx)\} \\
&= -\tfrac{1}{3}[[x, y]y] - \tfrac{1}{6}[[x, y]x].
\end{aligned}
$$

It is clear that by this method we may also compute $C_n$ for any $n > 3$ by recurrence formulas.

## SECOND PART: MATRICES

### V. *Integration of Systems of Ordinary Differential Equations by Elementary Formulas*

Let $A(t)$ and $Y(t)$ be $n$ by $n$ matrices the elements of which depend on a parameter $t$. We consider the system of linear homogeneous differential equations

$$dY/dt = AY \tag{5.1}$$

subject to the initial conditions

$$Y(0) = I, \tag{5.2}$$

where $I$ denotes the unit matrix.

From well-known general theorems we know that (5.1) always has a uniquely determined solution $Y(t)$ which is continuous and has a continuous first derivative in any interval in which $A(t)$ is continuous. The elements of the $k$-th column of $Y(t)$ are the solutions $y_\nu$ of the system of linear differential equations

$$\frac{dy_\nu}{dt} = \sum_{\mu=1}^{n} a_{\nu,\mu} y_\mu , \qquad (\nu = 1, 2, \cdots, n) \tag{5.3}$$

subject to the initial conditions

$$y_\nu(0) = 0 \text{ if } \nu \neq k, \qquad y_k(0) = 1. \tag{5.4}$$

The $a_{\nu,\mu}$ are, of course, the elements of $A$, and they are functions of $t$. The determinant of $Y$ is always different from zero; its value is given by

$$| Y | = \exp\left(\int_0^t \sum_{\nu=1}^{n} a_{\nu\nu}(s)\, ds\right). \tag{5.5}$$

We wish to apply Theorem III and in particular formula (3.28) to equation (5.1), assuming that $Y$ can be written in the form $Y = \exp \Omega$. In general, the use of Theorem III involves difficulties of convergence; some of these will be discussed in Sections VI, VII. But there is one case in which (3.28) clearly determines $\Omega$ for all values of $t$, namely, when the series in the right hand side of (3.28) terminates. This will happen, for instance, if

$$A(t)\left(\int_0^t A(\tau)\,d\tau\right) - \left(\int_0^t A(\tau)\,d\tau\right)A(t) \equiv 0 \tag{5.6}$$

identically for all values of $t$. If (5.6) is true, $\Omega$ becomes simply

$$\int_0^t A(\tau)\,d\tau \tag{5.7}$$

and $Y = \exp \Omega$ satisfies (5.1).

In order to state a concise result, we introduce the following

*Definition of a Lie-integral Functional.* Let $A(t)$ be an integrable function of $t$ (in the ordinary sense). We define a Lie-integral functional $\Phi_n$ of weight $n$ of $A$ recursively as follows:

(i) The functional of weight 1 is any multiple of

$$\int_0^t A(s)\,ds. \tag{5.8}$$

(ii) Let $\Phi_\lambda, \Phi_\mu, \cdots, \Phi_\nu$ be any functionals of weight $\lambda, \mu, \cdots, \nu$ such that

$$\lambda + \mu + \cdots + \nu + \rho = n - 1. \tag{5.9}$$

Then a functional of weight $n$ is defined as any linear combination of terms of the type

$$\int_c^t [[\cdots[[A(s), \Phi_\lambda]\Phi_\mu]\cdots]\Phi_\nu]\Phi_\rho]\,ds, \tag{5.10}$$

where $\Phi_\lambda, \cdots, \Phi_\rho$ are written as functions of the independent variable $s$.

Apparently, the terms involving $1, 2, \cdots, n$ integrations in (3.28) are functionals of the type described above; we shall call them the *Baker-Hausdorff functionals* $B_n$ of $A(t)$ (for $c = 0$), and we shall write (3.28) in the form

$$\Omega = B_1 + B_2 + B_3 + \cdots, \tag{5.11}$$

where the $B_n$ will be written as

$$B_n(A, t, c) \tag{5.12}$$

if the matrix $A$, the variable $t$, and the constant $c$ are to be exhibited. In (3.28), we assumed that $c = 0$. Now we can state

THEOREM IV: *If all Lie-integral functionals of $A$ of a weight $m$ vanish* ($n < m \leqq 2n - 1$), *then the solution of* (5.1) *with initial conditions* (5.2) *is given by $Y = exp\ \Omega$, where*

$$\Omega = \sum_{\nu=1}^{n} B_\nu(A, t, 0). \tag{5.13}$$

The $B_\nu$ are the Baker-Hausdorff functions defined by (3.28) and (5.11).

A sufficient (but not a necessary) condition for the vanishing of Lie-integral functionals of weight greater than $n$ is that

(5.14) $$[[\cdots[[A(s_1), A(s_2)]A(s_3)]\cdots]A(s_{n+1})] = 0$$

for any choice of $s_1, \cdots, s_{n+1}$.

Clearly, the fact that all functionals of type (5.10) vanish for weight $m$ between $n$ and $2n - 1$ implies that but a finite number of linearly independent functionals must vanish identically.

In order to prove Theorem IV we need first

LEMMA 6: *If all Lie-integral functionals of a weight $m$ vanish, where $n < m \leqq 2n - 1$, then all Lie-integral functionals of any weight $m > n$ also vanish.*

Proof: We shall consider a functional of type (5.10), assuming now that $m = 1 + \lambda + \mu + \cdots + \rho$ and that $m > n$. Since our Lemma is trivial for $n = 1$, we may assume that $n > 1$. Now we shall apply induction with respect to $m$, assuming that $m \geqq 2n$. If one of the weights $\lambda, \mu, \cdots, \rho$ is greater than $n$, the corresponding $\Phi$ vanishes identically and the Lemma holds. But suppose all of the weights $\lambda, \mu, \cdots, \rho$ are $\leqq n$; then we consider the first number $S$ greater than $n$ in the sequence

$$1,\ 1 + \lambda,\ 1 + \lambda + \mu,\ \cdots,\ 1 + \lambda + \mu + \cdots + \nu,\ 1 + \lambda + \cdots + \nu + \rho.$$

This number is necessarily at most equal to $2n$. If $S < 2n$, and, if $S = 1 + \lambda + \cdots + \nu$, then

(5.15) $$[[\cdots[[A, \Phi_\lambda]\Phi_\mu]\cdots]\Phi_\nu]$$

is the derivative of a functional of degree $S$, where $n < S < 2n$, and therefore not only this functional but also (5.10) vanishes identically. Hence for this case our lemma is true. There remains the case $S = 2n$ which can take place only if the last term $\nu$ in $S = 1 + \lambda + \cdots + \nu$ is equal to $n$. Now we use a lemma proved by Wever [14] according to which a Lie-product (5.15) can also be written as a sum of Lie-products in which $\Phi_\nu$ always appears in the first place, but in which the factors and the arrangement of the brackets are the same as in (5.15). This lemma follows without difficulty from the Dynkin-Wever-Specht formula (4.5). Consider now any Lie-product of type (5.15) in which the first factor is $\Phi_\nu$. The second factor is either $A$ or one of the other factors, which may be called $\Psi$. Now we merely have to show that

(5.16) $$[\Phi_\nu, A] = 0, \qquad [\Phi_\nu, \Psi] = 0.$$

If (5.16) holds, any product involving the left hand sides in (5.16) also vanishes and therefore the product in (5.15) vanishes. Now (5.16) is true because $[\Phi_\nu, A] = -[A, \Phi_\nu]$ is the derivative of a vanishing functional of weight $n + 1$. Similarly, $[\Phi_\nu, \Phi_\mu] = 0$ since

(5.17) $$\frac{d}{dt}[\Phi_\nu, \Psi] = \left[\frac{d\Phi_\nu}{dt}, \Psi\right] - \left[\frac{d\Psi}{dt}, \Phi_\nu\right].$$

Both $d\Phi_\nu/dt$ and $d\Psi/dt$ are sums of terms of the type

$$[\cdots [[A, \Psi_1]\Psi_2] \cdots ],$$

where $\Psi_1, \Psi_2, \cdots$ are Lie-integral functionals. Therefore, the right hand side of (5.17) vanishes since the individual terms are derivatives of Lie-integral functionals of a weight $k$ $(n + 1 \leqq k \leqq 2n - 1)$. The inequalities for $k$ follow from the fact that the weight of $\Phi_\nu$ equals $n$ and the weight of $\Psi$ is less than $n$ and at least equal to unity.

This finishes the proof of Lemma 6. Probably, a better result could be obtained for any given $n$; for example, for $n = 2$ it is easily shown that the vanishing of all functionals of weight 3 implies the vanishing of all functionals of weight 4. For $n = 1$, it is trivial that all functionals of a weight $\geqq 2$ vanish if those of weight 2 vanish.

To prove Theorem IV we observe that all steps in the formal proof of (3.28) now involve only a finite number of terms. First, the solution of (3.27) by iteration gives a finite sum of functionals which we denote by $B_\nu$ $(\nu = 1, \cdots, n)$ in accordance with (5.11). Secondly, it follows directly that the Lie-product

$$\left[\left[\cdots\left[\left[\frac{d\Omega}{dt}, \Omega\right]\Omega\right]\cdots\right]\Omega\right], \tag{5.18}$$

of at least $m + 1$ factors vanishes identically if $m \geqq n$. Now we consider the following equation which is equivalent to (3.29):

$$\left(\frac{d}{dt} e^{\Omega}\right) e^{-\Omega} = \frac{d\Omega}{dt} - \frac{1}{2!}\left[\frac{d\Omega}{dt}, \Omega\right] + \frac{1}{3!}\left[\left[\frac{d\Omega}{dt}, \Omega\right]\Omega\right] \mp \cdots . \tag{5.19}$$

This formula makes sense for any differentiable finite matrix $\Omega(t)$ since it can be shown to be an absolutely and uniformly convergent rearrangement of the series obtained by differentiating exp $\Omega$ directly term by term and multiplying by exp $(-\Omega)$ afterwards. From (5.18) it follows that the right hand side in (5.19) is a terminating series. Calling its sum $B$, we find directly from (5.18), (5.19) that

$$\frac{d\Omega}{dt} = \sum_{\nu=0}^{n-1} \beta_\nu\{B, \Omega^\nu\}, \tag{5.20}$$

where the $\beta_\nu$ are explained in Theorem III. On the other hand, $\Omega$ could also have been derived from

$$\frac{d\Omega}{dt} = \sum_{\nu=0}^{n-1} \beta_\nu\{A, \Omega^\nu\} \tag{5.21}$$

since the higher terms in (3.27) do not contribute to $d\Omega/dt$. Putting $B - A = C$ we merely have to show that the equation

$$\sum_{\nu=0}^{n-1} \beta_\nu\{C, \Omega^\nu\} = 0 \tag{5.22}$$

cannot have a solution $C$ which does not vanish identically and which is such that $\{C, \Omega^m\} = 0$ if $m \geqq n$. This can be shown by applying bracket multiplication to (5.22) $k = n - 1, n - 2, \cdots, 1$ times. Then we find that

$$\sum_{\nu=0}^{n-1} \beta_\nu \{C, \Omega^{\nu+k}\} = \sum_{\nu=0}^{n-k-1} \beta_\nu \{C, \Omega^{\nu+k}\} = 0, \tag{5.23}$$

and this gives recursively

$$\beta_0\{C, \Omega^{n-1}\} = \beta_0\{C, \Omega^{n-2}\} = \cdots = \beta_0[C, \Omega] = 0. \tag{5.24}$$

Since $\beta_0 \neq 0$, combining (5.22) and (5.24) we find that $C = 0$, and this finishes the proof of the first part of Theorem IV.

The statement in Theorem IV that (5.14) is a sufficient condition for (5.13) is almost trivial. To show that it is not a necessary condition we take $n = 1$ and

$$\begin{aligned} A(t) &= (\cos t - \cos 2t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for } 0 \leqq t \leqq 2\pi, \\ A(t) &= \begin{pmatrix} (t-2\pi)^2 & 0 \\ 0 & (t-2\pi)^3 \end{pmatrix} \quad \text{for } t \geqq 2\pi. \end{aligned} \tag{5.25}$$

Clearly,

$$\left[A(t), \int_0^t A(s)\, ds\right] = 0 \tag{5.26}$$

but if $0 < s_1 < \pi/4$ and if $s_2 > 2\pi$, then

$$[A(s_1), A(s_2)] \neq 0. \tag{5.27}$$

It can be shown that even in the case $n = 1$ and even if the elements of $A(t)$ are polynomials in $t$, there exist infinitely many examples involving an $A(t)$ which satisfies (5.26) but not (5.27). The implications of (5.26) will be investigated in detail in a forthcoming paper by M. Hellman.

## VI. *Conditions for Existence of a Solution $Y = \exp \Omega$ for $\dot{Y} = AY$*

In this section, we shall derive some results about the existence in the large of solutions of (5.1).

We consider again a system of linear differential equations of the first order

$$dY/dt = A(t)Y(t), \tag{6.1}$$

where $Y$ and $A$ are $n$ by $n$ matrices the elements of which are functions of parameter $t$. We assume again that $Y(0)$ is the identity $I$ and that $A(t)$ is continuous in $t$, although it is well known that the latter condition could be weakened.

If we wish to represent the solution of (6.1) in the form $Y = \exp \Omega$, the technique based on (3.27) and (3.28) will work for sufficiently small values of $| t |$. Also, it is well known that any preassigned constant matrix $Y$ can be written in the form $\exp \Omega$, if the determinant $| Y |$ of $Y$ is different from zero. From Section V we know that $Y$ is finite and $| Y | \neq 0$ everywhere. Nevertheless, if the function $\Omega(t)$ is assumed to be differentiable, it may not exist everywhere. This can be shown by the following considerations. We may start from $t = 0$ and arrive at the value for the solution of (6.1) for $t = t_0$:

$$Y_0 = Y(t_0) = \exp \Omega_0 , \ \Omega_0 = (t_0). \tag{6.2}$$

Let us consider $Y_0$ and $\Omega_0$ as points of $2n^2$-dimensional spaces $S_y$ and $S_\omega$, respectively, where the Cartesian coordinates $\eta_\nu$ and $\omega_\nu$ of these spaces consist of the real and imaginary parts of the $n^2$ elements of an arbitrary matrix $Y$ or $\Omega$. The formula

$$Y = \exp \Omega \tag{6.3}$$

defines a mapping of $S_\omega$ into $S_y$ such that the coordinates in $S_y$ become entire analytic functions of the coordinates of $S_\omega$. The functional determinant

$$\Delta = | \partial\eta_\nu/\partial\omega_\mu | \qquad (\nu, \mu = 1, \cdots, 2n^2) \tag{6.4}$$

of this mapping is an analytic function of the $\omega_\mu$. If $\Delta$ does not vanish at a certain point $\Omega_0$ of $S_\omega$, then a neighborhood of $\Omega_0$ is mapped continuously with a one-to-one correspondence upon a full neighborhood of $Y_0$. In this case, the solution $Y = \exp \Omega$ of (6.1) can be continued beyond the value $t = t_0$ to a value $t_1 > t_0$. If, however, $\Delta = 0$ at $Y = Y_0$, then $dY/dt = AY$ may point towards a part of $S_y$ which is not covered by the map of $S_\omega$ in $S_y$. In this case, $d\Omega/dt$ cannot exist at $\Omega = \Omega_0$. Actually, the question reduces to the problem of solving

$$\left\{\frac{d\Omega}{dt}, \frac{e^{\Omega} - 1}{\Omega}\right\} = A \tag{6.5}$$

with respect to $d\Omega/dt$. If this is possible for any $A$ in the neighborhood (in $S_\omega$) of a point $\Omega = \Omega_0$, and if the result is of the type

$$d\Omega/dt = F(A, \Omega), \tag{6.6}$$

where the right-hand side in (6.6) is a matrix depending analytically on the elements of $\Omega$, then (1) has a solution of type (6.3) in the neighborhood of $Y_0 = \exp \Omega_0 = \exp \Omega(t_0)$. We shall prove the following result:

THEOREM V: *The functional determinant* $\Delta$ *defined by* (6.4) *does not vanish, and* (6.5) *can be solved by an expression* (6.6) *in the neighborhood of any point* $\Omega_0$ *in* $S_\omega$ *for an arbitrary* $A$ *if and only if none of the differences between any two of the eigenvalues of* $\Omega_0$ *equals* $2m\pi i$, *where* $m = \pm 1, \pm 2, \cdots, m \neq 0$.

Proof: If (6.3) holds, the determinant $| Y |$ of $Y$ is different from zero. Setting

$$dY \cdot Y^{-1} = dZ, \tag{6.7}$$

we may compute the determinant which connects the elements of $dZ$ and $d\Omega$. It will differ from $\Delta$ only by a power of $|Y|^{-1}$ since the elements of each row of $dZ$ are obtained from the corresponding row of $dY$ by a linear substitution, the matrix of which is the transpose of $Y^{-1}$. From (3.24) we have

$$dZ = d\Omega - \frac{1}{2!}[d\Omega, \Omega] + \frac{1}{3!}[[d\Omega, \Omega]\Omega] \mp \cdots . \tag{6.8}$$

Let

$$dZ = (dz_{\nu.\mu}), \qquad d\Omega = (d\omega_{\nu.\mu}) \qquad (\nu, \mu = 1, \cdots, n) \tag{6.9}$$

and assume that

$$\Omega = \Lambda = |\lambda_1, \cdots, \lambda_n| \tag{6.10}$$

is a diagonal matrix with the numbers $\lambda_1, \cdots, \lambda_n$ in the main diagonal. In this case we have from (6.8), (6.9), and (6.10)

$$dz_{\nu.\mu} = \frac{\exp(\lambda_\nu - \lambda_\mu) - 1}{\lambda_\nu - \lambda_\mu} d\omega_{\nu.\mu} . \tag{6.11}$$

The $n^2$ quantities $dz_{\nu.\mu}$ are linear functions of the $n^2$ quantities $d\omega_{\nu.\mu}$, and the determinant of (6.11) is

$$\Delta^* = \prod_{\nu \neq \mu} \frac{\exp(\lambda_\nu - \lambda_\mu) - 1}{\lambda_\nu - \lambda_\mu} \tag{6.12}$$

where the product is extended over $\nu, \mu = 1, \cdots, n$, with $\nu \neq \mu$. Let

$$\begin{aligned}(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)& \\ &= x^n - s_1 x^{n-1} + s_2 x^{n-2} + \cdots + (-1)^n s_n ,\end{aligned} \tag{6.13}$$

where the $s_\nu$ $(\nu = 1, \cdots, n)$ are the elementary symmetric functions of the $\lambda_\nu$. Then $\Delta^*$ becomes a function of the $s_\nu$ which is analytic and entire in each $s_\nu$. Next, let

$$\Omega = C\Lambda C^{-1}, \tag{6.14}$$

where $C$ is a matrix the determinant of which equals unity. If we introduce

$$dZ^* = C^{-1} \cdot dZ \cdot C, \qquad d\Omega^* = C^{-1} \cdot d\Omega \cdot C, \tag{6.15}$$

equation (6.8) becomes

$$dZ^* = d\Omega^* - \frac{1}{2!}[d\Omega^*, \Lambda] + \frac{1}{3!}[[d\Omega^*, \Lambda]\Lambda] \mp \cdots \tag{6.16}$$

and instead of (6.11) we have

$$dz^*_{\nu.\mu} = \frac{\exp(\lambda_\nu - \lambda_\mu) - 1}{\lambda_\nu - \lambda_\mu} d\omega^*_{\nu.\mu} . \tag{6.17}$$

Now

$$(dz^*_{\nu,\mu}) = C^*(dz_{\nu,\mu}), \qquad (d\omega^*_{\nu,\mu}) = C^*(d\omega_{\nu,\mu}),$$

where $C^* = C^{-1} \otimes C'$ is the Kronecker product of $C^{-1}$ and the transpose $C'$ of $C$ and where $(dz^*_{\nu,\mu})$ stands for the vector with $n^2$ components $dz^*_{\nu,\mu}$. Since the determinant of $C^*$ equals 1, the $dz_{\nu,\mu}$ are again linear functions of the $d\omega_{\nu,\mu}$, where the determinant of the relation is $\Delta^*$. This shows that

$$\Delta^* = | \partial z_{\nu,\mu}/\partial \omega_{\nu,\mu} | \tag{6.18}$$

whenever $\Omega$ can be transformed into diagonal form. Now $\Delta^*$ in (6.12) is a function of the $s_\nu$, which are the coefficients of the characteristic equation for $\Omega$. Since $\Delta^*$ must be a continuous function of the $\omega_{\nu,\mu}$, we shall write $\Delta^*$ as a function of the $s_\nu$. This gives an expression for $\Delta^*$ which is valid if $\Omega$ has different eigenvalues. But in $S_\omega$ the neighborhood of every point $\Omega_0$ contains points corresponding to matrices which have different eigenvalues. Therefore the $\Delta^*$ in (6.18) is given by (6.12) for all $\Omega$. This proves Theorem V, since $\Delta^*$ will vanish if and only if the conditions of Theorem V are satisfied.

## VII. *Example of the Ordinary Differential Equation of Second Order*

In this section, the simplest non-trivial case of a differential equation of type (6.1) will be studied. It will be shown explicitly that, in general, a solution of the type $Y = \exp \Omega$ does not exist in the large.

The equation

$$y'' + Q(t)y = 0 \tag{7.1}$$

may be written as

$$y_1' = y_2, \qquad y_2' = -Qy_1 \tag{7.2}$$

and leads to the matrix equation

$$\frac{dY}{dt} = AY, \qquad Y = \begin{pmatrix} y_1 & \eta_1 \\ y_2 & \eta_2 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -Q & 0 \end{pmatrix}, \tag{7.3}$$

where $y_1$, $\eta_1$ denote two linearly independent solutions of (7.1). If we choose these in such a way that

$$\begin{aligned} y_1(0) &= 1, & y_2(0) &= y_1'(0) = 0, \\ \eta_1(0) &= 0, & \eta_2(0) &= \eta_1'(0) = 1, \end{aligned} \tag{7.4}$$

then $Y(0) = I$ and the determinant $| Y |$ of $Y$ equals unity for all values of $t$. Therefore, if

$$Y = \exp \Omega,$$

the trace of $\Omega$ vanishes and we may put $\Omega$ into the form

$$\Omega = \begin{pmatrix} \omega & \phi \\ \Psi & -\omega \end{pmatrix}. \tag{7.5}$$

If we introduce

$$\Delta = \sqrt{\omega^2 + \phi\Psi}, \tag{7.6}$$

we find after some calculation

$$Y = \frac{(\sinh \Delta)}{\Delta}\, \Omega + (\cosh \Delta) I. \tag{7.7}$$

Let

$$\theta = y_1 + \eta_2 \tag{7.8}$$

be the trace of $Y$. Then we have from (7.7)

$$\theta = 2 \cosh \Delta \tag{7.9}$$

(since the trace of $\Omega$ vanishes). Actually, $2\Delta$ is the difference between the eigenvalues $\lambda_1$, $\lambda_2$ of $\Omega$. If this difference is a multiple of $2\pi i$ but different from zero, then $\lambda_1 \neq \lambda_2$ since $\lambda_1 + \lambda_2 = 0$. In this case, $\Omega$ can be transformed into the diagonal form. But then, although both the eigenvalues of $Y$ equal $+1$ or $-1$, it is possible that $Y$ is not a multiple of the unit matrix. Instead, if we combine (7.7) and (7.9) we find

$$\Omega = (2Y - \theta I)(\Delta/\sqrt{\theta^2 - 4}) \tag{7.10}$$

where we have $\theta^2 = 4$ if $\Delta = 2n\pi i$; therefore if $\Delta \neq 0$ and $\Omega$ is finite, $2Y - \theta I = 0$. In order to discuss at least one case completely we prove

THEOREM VI: *Let $Q(t) > 0$ for all $t \geqq 0$. Then a solution $Y$ of* (7.3), *with the initial condition $Y(0) = I$, has a representation*

$$Y(t) = \exp \Omega(t), \qquad \Omega(0) = 0,$$

*for $t \geqq 0$ with a two times differentiable $\Omega(t)$, if and only if*

$$(\text{trace } Y)^2 \leqq 4 \tag{7.11}$$

*for $t \geqq 0$.*

Proof: We know from (7.10) that $2Y = \theta I$ whenever $\theta^2 = 4$ and $\Delta \neq 0$ (which implies $\Omega \neq 0$). Also, we know that $Y = I$ for $t = 0$, $\theta = 2$, $\Delta = 0$. Now we shall consider a value of $t = t_0$ for which $\theta^2 = 4$, $Y = \frac{1}{2}\theta I$. We find at this point

$$\frac{d\theta}{dt} = y_1' + \eta_2' = y_1' - Q\eta_1 = 0, \tag{7.12}$$

$$\frac{d^2\theta}{dt^2} = y_1'' - Q\eta_2 - Q'\eta_1 = -\theta Q = \mp 2Q. \tag{7.13}$$

Therefore, if $Q > 0$, then $\theta^2 < 4$ for $t = t_0 + \epsilon$ if $\epsilon$ is positive and sufficiently small.

Next, we observe that $\theta' \neq 0$ and $\Delta' \neq 0$ in any interval $t_0 < t < t_1$ in which $\theta^2 < 4$. Since $| Y | = y_1\eta_1' - y_1'\eta_1 = 1$ we find for $\theta' = 0$ from (7.13) that $y_1\eta_1' = 1 + Q\eta_1^2$. Therefore, $y_1$ and $\eta_1'$ have the same sign and are different from zero. However,

$$(7.14) \qquad (y_1 - \eta_1')^2 = \theta^2 - 4Y_1\eta_1' = \theta^2 - 4 - Q\eta_1^2 < 0$$

if $\theta^2 - 4 < 0$, and this is a contradiction. Therefore $\theta' \neq 0$. We find from (7.9) by differentiation that $\Delta'$ can vanish only if $\theta' = 0$ or if $\sin h\ \Delta = 0$; but then $\theta^2 = 4$ which had been excluded.

We consider now the behaviour of $\theta(t)$ and $\Delta(t)$ for $t > 0$, starting at $t = 0$. We have shown that there exists a smallest positive number $t_1$ (which may be $\infty$) such that $\theta^2 < 4$ in $0 < t < t_1$. For $0 < t < t_1$, $\Delta$ must be purely imaginary according to (7.9). We put $\Delta = iD$. We find from (7.9) that

$$(7.15) \qquad \theta' = -2D' \sin D.$$

Therefore, $D$ either increases in $0 < t < t_1$ or decreases monotonically, and if $t_1$ is finite and $\theta(t_1) = -2$, then $D(t_1) = \pi$ or $-\pi$. If $t_1 = \infty$, we see from (7.10) and (7.9) that Theorem VI is true. Therefore, assume that $t_1$ is finite. Then beyond this point, say for $t = t_1 + \epsilon$, $\theta(t)$ must increase again since it follows from (7.10) that $Y(t_1) = -I$. If we differentiate (7.9) two times, we find for $t = t_1$ from $\sin D(t_1) = 0$ and from (7.13) that

$$(7.16) \qquad \tfrac{1}{2}\theta'' = -D'^2 \cos D = D'^2(t_1) > 0.$$

Therefore, $D'$ cannot even vanish at $t = t_1$. Going from $t = t_1$ to the next point $t_2 > t_1$ for which $\theta^2 = 4$, etc., we find that $D'$ must be always real and positive or always real and negative since we shall never come back to $D = 0$ for $t > 0$. Therefore, $D(t)$ is a monotonic function of $t$ for $t > 0$, and this proves that $\theta^2 < 4$ is a necessary condition for the existence of $\Omega$. That it is also a sufficient condition can be seen as follows: If $\Delta \neq \pm in\pi$, $n = 1, 2, 3, \cdots$, then $\Omega$ is completely determined by postulating that $D(t)$ is monotonic and by using (7.9) and (7.10). In order to find $\Omega(t)$ for the exceptional values $t = t_n$ for which $\Delta = in\pi$ (or $\Delta = -in\pi$, consistently with the same sign), we multiply both sides of (7.10) by $\sqrt{\theta^2 - 4}$ and then differentiate with respect to $t$. We find

$$(7.17) \qquad 2(\Delta' \cosh \Delta)\Omega + 2(\sinh \Delta)\Omega' = (2Y - 2\theta I)\Delta' + (2Y' - \theta' I)\Delta.$$

Since $\sinh \Delta = 0$, $\cosh \Delta = (-1)^n$ and $2Y - \theta I = 0$, $\theta' = 0$, $Y' = AY$ for $t = t_n$, (7.17) becomes

$$(7.18) \qquad (-1)^n\Delta'(t_n)\Omega(t_n) = AY\Delta = A(-1)^n\Delta(t_n).$$

Since $\Delta(t_n) = \epsilon\, in\pi$ where $\epsilon$ is independent of $n$ and $\epsilon = \pm 1$, it is easily seen that (7.18), (7.6), (7.5) and (7.3) suffice to determine $\Omega$ uniquely for $t = t_n$. The result is independent of $\epsilon$ if we define consistently $\sqrt{\theta^2 - 4} = 2 \sinh \Delta$.

Also, it is easily seen that the condition (7.11) will not be satisfied generally for a differential equation (7.1) or (7.2). As an example, we may take $Q = \exp 2t$. Then

$$y_1 = \frac{J_0(e^t)}{J_0(1)},$$

$$\eta_1 = \frac{[Y_0(e^t) - Y_0(1)y_1(t)]}{Y_0'(1)}$$

where $J_0$, $Y_0$ denote the Bessel functions of the first and second kind respectively. The asymptotic expansions for the Bessel functions and their derivatives show that $\theta^2 > 4$ for infinitely many values of $t$.

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