QO2 - Lecture 9

So far, we have only paid attention to the atoms. What about the light? How do we relate population levels to things like the index of refraction?

Recall that:

$$\vec{P} = \varepsilon_0 \chi \frac{\vec{E}_0}{2} e^{-i(\omega t + \phi)} + c.c.$$

$\vec{P}$ is the polarization induced in an atomic sample resulting from the imposition of an electric field $\vec{E}_0 \cos(\omega t + \phi)$.

Assumptions:

1. Medium is isotropic ($\vec{P}$ is in the same direction as $\vec{E}$)
2. Medium is linear ($\vec{P} \propto \vec{E}$)

Hopefully, you remember from E+M that this medium has a complex index of refraction given by

$$n^2 = 1 + \chi(\omega)$$

And if $\chi(\omega) \ll 1$, then

$$n \simeq 1 + \frac{\chi'}{2} + i\frac{\chi''}{2}$$

and the wavevector is:

$$k = \frac{n\omega}{c} = k' + ik''$$

where $$k' \simeq \left(1 + \frac{\chi'_{\varepsilon}}{2}\right)\frac{\omega}{c}$$

$$k'' \simeq \frac{\chi''}{2}\frac{\omega}{c}$$

Finally, the field evolves like:

$$\vec{E}(z,t) = \frac{\vec{E}_0}{2} e^{i(kz-\omega t)} + c.c.$$

$$\boxed{\vec{E}(z,t) = \vec{E}_0 e^{-k''z} \cos(\omega t - k'z)}$$

So, k' gives us dispersion and k'' gives us absorption.

So how do we figure out χ (and thus k' & k'') ?!

We calculate the dipole moment $\langle \hat{D} \rangle$ of each individual atom, the multiply by the density of the medium to get the maroscopic polarization!

Remember density matrix theory:

$$\langle \hat{D} \rangle = Tr(\hat{\rho}\hat{D})$$

and remember that $D_{ii} = \langle i | \hat{D} | i \rangle = 0$

so we have:

$$\langle \hat{D} \rangle = \sum_{j,k} \rho_{jk} D_{kj}$$

For a two level system, we have:

$$\langle \hat{D} \rangle = \rho_{ge} D_{eg} + \rho_{eg} D_{ge}$$

But we have up until now assumed $\langle e|\hat{D}|g\rangle = \langle g|\hat{D}|e\rangle = d$,

so then

$$\boxed{\langle \hat{D} \rangle = d(\rho_{ge} + \rho_{eg})}$$

Now, recall from homework that for:

- atoms initially in ground state
- in the rotating frame
- in steady state.

$$\langle \sigma_x \rangle = \frac{\Omega_R \delta}{\delta^2 + \Omega_R^2/2 + (\Gamma/2)^2}$$

$$\langle \sigma_y \rangle = \frac{\Omega_R \Gamma/2}{\delta^2 + \Omega_R^2/2 + (\Gamma/2)^2}$$

$$\langle \sigma_z \rangle = - \frac{\delta^2 + (\Gamma/2)^2}{\delta^2 + \Omega_R^2/2 + (\Gamma/2)^2}$$

And also recall (from lecture 6):

$$\rho = \frac{1}{2}\begin{pmatrix} 1 + \langle \sigma_z \rangle & \langle \sigma_x \rangle - i\langle \sigma_y \rangle \\ \langle \sigma_x \rangle + i\langle \sigma_y \rangle & 1 - \langle \sigma_z \rangle \end{pmatrix}$$

Then we should be able to substitute the steady state values into the equation for $\langle \hat{D} \rangle$, compare this to $\ddot{P}$ and find χ!

But <u>WE FORGOT</u>: these steady state values are in the <u>rotating frame</u>.

Recall that to go into the rotating frame, we made the transformation:

$$|\psi\rangle = R_z(\omega t)|\psi'\rangle$$

then solved for $|\psi'\rangle$.

To get the lab frame picture back, we must now use the transformation:

$$|\psi\rangle = R_z(\omega t)|\psi'\rangle$$

or rather

$$\rho = R_z(\omega t)\,\rho'\,R_z^{\dagger}(\omega t)$$

Given that $R_z(\omega t) = e^{-i\omega t\cdot\sigma_z}$

$$= \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}$$

and the previous expression for ρ', it is easy to see that:

$$\rho = R_z(\omega t)\rho' R_z(\omega t) = \frac{1}{2}\begin{pmatrix} 1+\langle\sigma_z\rangle & e^{-i\omega t}(\langle\sigma_x\rangle - i\langle\sigma_y\rangle) \\ e^{i\omega t}(\langle\sigma_x\rangle + i\langle\sigma_y\rangle) & 1-\langle\sigma_z\rangle \end{pmatrix}$$

Substituting this into the equation for $\langle\hat{D}\rangle$, we have:

$$\langle\hat{D}\rangle = e^{-i\omega t}\frac{d}{2}\left(\langle\sigma_x\rangle - i\langle\sigma_y\rangle\right) + \text{c.c.}$$

$$= \frac{e^{-i\omega t}\, d\,\Omega_R}{2\left(\delta^2 + \Omega_R^2/2 + (\Gamma/2)^2\right)}\left(\delta - i\,\Gamma/2\right) + \text{c.c.}$$

But remember, $\hbar\Omega_R = -dE_0$, so

$$\langle\hat{D}\rangle = \frac{d^2 e^{-i\omega t} E_0}{2\hbar\left(\delta^2 + \frac{\Omega_R^2}{2} + (\Gamma/2)^2\right)}\left(-\delta + i\,\Gamma/2\right) + \text{c.c.}$$

This is the steady state dipole moment for each individual atom. To get the polarization of the medium, we must multiply by the density $\left(\frac{N}{V}\right)$.

$$P = \left(\frac{N}{V}\right)\left(\frac{d^2}{\hbar}\right)\left(\frac{-\delta + i\Gamma/2}{\delta^2 + \frac{\Omega_R^2}{2} + (\Gamma/2)^2}\right)\frac{E_0}{2}e^{-i\omega t} + c.c.$$

$$= \varepsilon_0 \chi \frac{E_0}{2} e^{-i\omega t} + c.c.$$

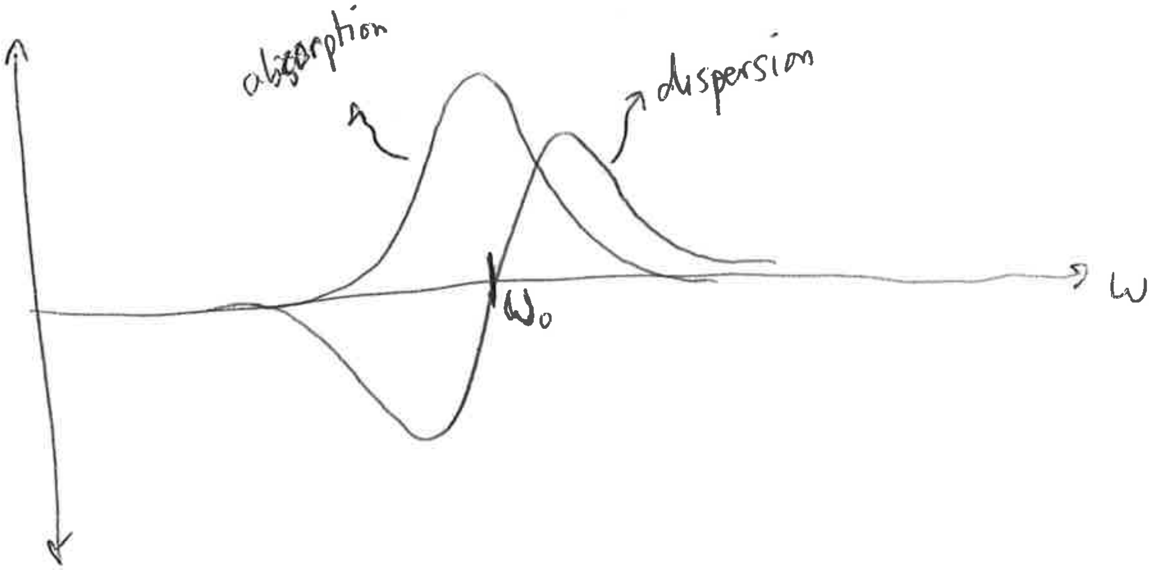
$$\therefore \quad \chi = \frac{N}{V}\frac{d^2}{\varepsilon_0 \hbar}\left(\frac{-\delta + i\Gamma/2}{\delta^2 + \frac{\Omega_R^2}{2} + (\Gamma/2)^2}\right)$$

The real part (dispersion) is:

$$\chi' = \frac{N}{V}\frac{d^2}{\varepsilon_0 \hbar}\left(\frac{\omega_0 - \omega}{\delta^2 + \frac{\Omega_R^2}{2} + (\Gamma/2)^2}\right)$$

and the imaginary part (absorption) is:

$$\chi'' = \frac{N}{V}\frac{d^2}{\varepsilon_0 \hbar}\left(\frac{\Gamma/2}{\delta^2 + \frac{\Omega_R^2}{2} + (\Gamma/2)^2}\right)$$



Note 1: Linewidths

The linewidths of these functions are $\propto \sqrt{\Omega_R^2/2 + (\Gamma/2)^2}$

- The first term $\left(\frac{\Omega_R^2}{2}\right)$ is called POWER BROADENING
  ↳ turning the laser up broadens the transition
- The second term $\left((\Gamma/2)^2\right)$ is called the NATURAL LINEWIDTH.
  ↳ spontaneous emission broadens the transition
  (the faster the level decays, the broader the transition).

Note 2: Saturation.

If we define $$s = \frac{\Omega_R^2/2}{\delta^2 + (\Gamma/2)^2}$$

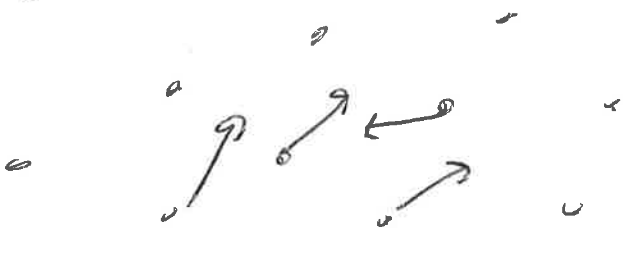
and $$\chi_1 = \frac{N}{V}\frac{d^2}{\varepsilon_0 \hbar}\left(\frac{-\delta + i\Gamma/2}{\delta^2 + (\Gamma/2)^2}\right) \Rightarrow \text{"linear susceptibility"}$$

then $$\boxed{\chi = \frac{\chi_1}{1+s}}$$

- the susceptibility tends to 0 as $\Omega_R \to \infty$.
- the transition is "saturated"

Note 3: Other broadening mechanisms

Homogeneous: Collisional Broadening



- Atoms collide with each other and can cause different effects. Non-radiative transitions can occur (mostly in solids) causing a modification of Γ (ie: $\Gamma \to \Gamma + \Gamma_{coll}$).

This broadens the curve (though in a somewhat trivial manner).

- During collisions, ω_0 will fluctuate causing <u>dephasing effects</u>.

This is a non-trivial effect: the Bloch equations are modified!

$$\frac{d\langle\sigma_x\rangle}{dt} = \delta\langle\sigma_y\rangle - \gamma\langle\sigma_x\rangle$$

$$\frac{d\langle\sigma_y\rangle}{dt} = -\delta\langle\sigma_x\rangle - \Omega_R\langle\sigma_z\rangle - \gamma\langle\sigma_y\rangle$$

$$\frac{d\langle\sigma_z\rangle}{dt} = \Omega_R\langle\sigma_y\rangle - \Gamma(\langle\sigma_z\rangle + 1)$$

Note the Γ & γ's!

With no dephasing, $\gamma = \Gamma/2$.

Otherwise $\gamma = \Gamma/2 + \Gamma_{deph}$.

The absorption is modified:

$$\chi'' = \frac{N}{V}\frac{d^2}{\varepsilon_0\hbar}\left(\frac{\gamma}{\delta^2 + \gamma^2 + \Omega_R^2\left(\frac{\gamma}{\Gamma}\right)}\right)$$

Inhomogeneous: Doppler Broadening

In a gas, the particles are moving at different speeds, with velocity distribution (in the x-direction):

$$f(v_x) = \frac{1}{\bar{v}\sqrt{2\pi}} e^{-\frac{v_x^2}{2\bar{v}^2}} \quad , \quad \bar{v} = \sqrt{\frac{2k_B T}{m}}$$

At rest, the atoms see a field:

$$E(r,t) = E_0 \cos(\omega t - k\cdot r)$$

But the atoms have trajectories like $r = r_0 + vt$

and therefore experience a field

$$E = E_0 \cos\left[(\omega - k\cdot v)t - k\cdot r_0\right]$$

ie) the resonance condition becomes

$$\hbar(\omega - k\cdot v) = \hbar\omega_0$$

or for $k = k_x \hat{x}$ and $|k| = \frac{\omega}{c}$,

$$\boxed{\frac{v_x}{c} = \frac{\omega - \omega_0}{\omega}}$$

The absorption profile is modified. Each "velocity class" has a different resonance frequency and the total absorption is just the weighted average:

$$\chi(\omega) = \int_{-\infty}^{\infty} \underbrace{f(v_x)}_{\text{Maxwell distribution}} \underbrace{\chi_s\left(\omega\cdot\left(1+\frac{v_x}{c}\right)\right)}_{\text{the "stationary atom" susceptibility.}} dv_x$$

The profile is a mixture between the old LORENTZIAN lineshape, and a GAUSSIAN LINESHAPE introduced by the doppler effect. It is broader than each lineshape alone.

Called the "VOIGT PROFILE"