Q02 - Lecture 8

Interaction Between Atom and Quantized Field

Now, we will solve the fully quantum problem and see if any of the behaviour we have observed (Rabi oscillations, light shift) changes!

The total Hamiltonian for the field + atom system can be written as:

$$\hat{H} = \underbrace{\frac{1}{2m}(\hat{p} - q\hat{A}_\perp)^2 + \hat{V}_{coul}}_{\text{From Lecture 5}} + \underbrace{\hat{H}_F}_{\text{from Q01}}$$

The first two terms are simply from lecture 5, except we have put a hat on $\hat{A}$, to indicate it is an operator.

The second term you studied in Q01:

$$\hat{H}_F = \frac{\varepsilon_0}{2}\int d^3r\,(\hat{E}^2 + c^2\hat{B}^2) = \sum_l \hbar\omega_l \hat{a}_l^\dagger \hat{a}_l$$

with $[\hat{a}_l, a_{l'}^\dagger] = \delta_{ll'}$ [l indicates the mode]

Recall from lecture 5 that the first two terms can be written as (using the Coulomb guage and the long-wavelength approximation):

$$= \hat{H}_0 + \hat{H}_I$$

where in the two level atom :

$$\hat{H}_0 = \frac{1}{2}\hbar\omega_0 \left(|e\rangle\langle e| - |g\rangle\langle g| \right)$$

$$\hat{H}_I = -\frac{q}{m}\hat{p}\cdot\hat{A}_\perp(r_0,t) + \frac{q^2}{2m}\hat{A}_\perp^2(r_0,t)$$

Now remember from Q01 :

$$\hat{A}_\perp(r) = \sum_l \frac{\varepsilon_l}{\omega_l}\sqrt{\frac{\hbar\omega_l}{2\varepsilon_0 L^3}}\left(\hat{a}_l e^{ik_l\cdot r} + \hat{a}_l^\dagger e^{-ik_l\cdot r}\right)$$

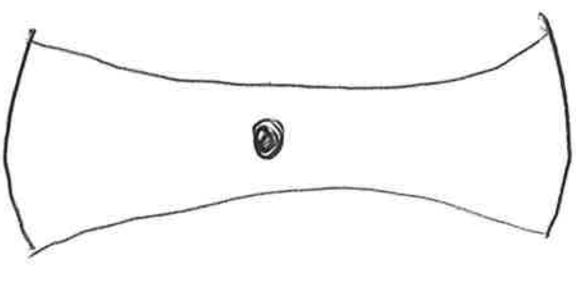
Then setting $r_0 = 0$:

$$-\frac{q}{m}\hat{p}\cdot\hat{A}_\perp(r_0,t) = -\frac{q}{m}\sum_l \sqrt{\frac{\hbar}{2\varepsilon_0\omega_l L^3}}\;\hat{p}\cdot\varepsilon_l\left(\hat{a}_l + \hat{a}_l^\dagger\right) = H_{I1}$$

$$\frac{q^2}{2m}A_\perp^2(r_0,t) = \frac{q^2}{2m}\left(\frac{\hbar}{2\varepsilon_0 L^3}\right)\sum_j\sum_l \frac{\varepsilon_j\cdot\varepsilon_l}{\sqrt{\omega_j\omega_l}}\left(a_j a_l^\dagger + a_j^\dagger a_l + a_j a_l + a_j^\dagger a_l^\dagger\right)$$

$$= H_{I2}$$

Phew! This is complicated. We are ready to simplify matters. Consider an atom interacting with just a single mode of the electromagnetic field (for example, an atom between two highly reflective mirrors):



Now we can write (as before):

$$\hat{H} = \underbrace{\hat{H}_0}_{\text{atom}} + \underbrace{\hat{H}_F}_{\text{field}} + \underbrace{\hat{H}_I}_{\text{interaction}}$$

$$\hat{H}_0 = \frac{1}{2}\hbar\omega_0 \left(|e\rangle\langle e| - |g\rangle\langle g| \right)$$

$$\hat{H}_F = \hbar\omega\, a^{+} a$$

$$\hat{H}_I = \hat{H}_{I1} + \hat{H}_{I2}$$

$$\hat{H}_{I1} = -\frac{q}{m}\sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}}\, \hat{p}\cdot\varepsilon\, (a^{+} + a)$$

$$\hat{H}_{I2} = \frac{q^2}{2m}\frac{\hbar}{2\varepsilon_0 \omega V}(a^{+} + a)^2$$

Note that $\hat{H}_{I2}$ only operates on field variables, and thus is only responsible for a energy level shift. This shift is small (for low-intensity fields) and here we will ignore it.

Thus, we set $\hat{H}_{I2} = 0$.

Recall that $\langle e|p\cdot\varepsilon|e\rangle = 0 = \langle g|p\cdot\varepsilon|g\rangle$.

Then we can write:

$$\hat{H}_I = \hat{H}_{I1} = \frac{\hbar\Omega_R^{(0)}}{2}\left(|e\rangle\langle g| + |g\rangle\langle e| \right)(a^{+} + a)$$

where $\Omega_R^{(0)} = -\frac{2q}{m}\frac{1}{\sqrt{2\hbar\varepsilon_0 \omega V}}\langle e|\hat{p}\cdot\varepsilon|g\rangle$ is the "vacuum Rabi frequency"

But how do we solve this to get the dynamics?

Start by considering the uncoupled Hamiltonian:

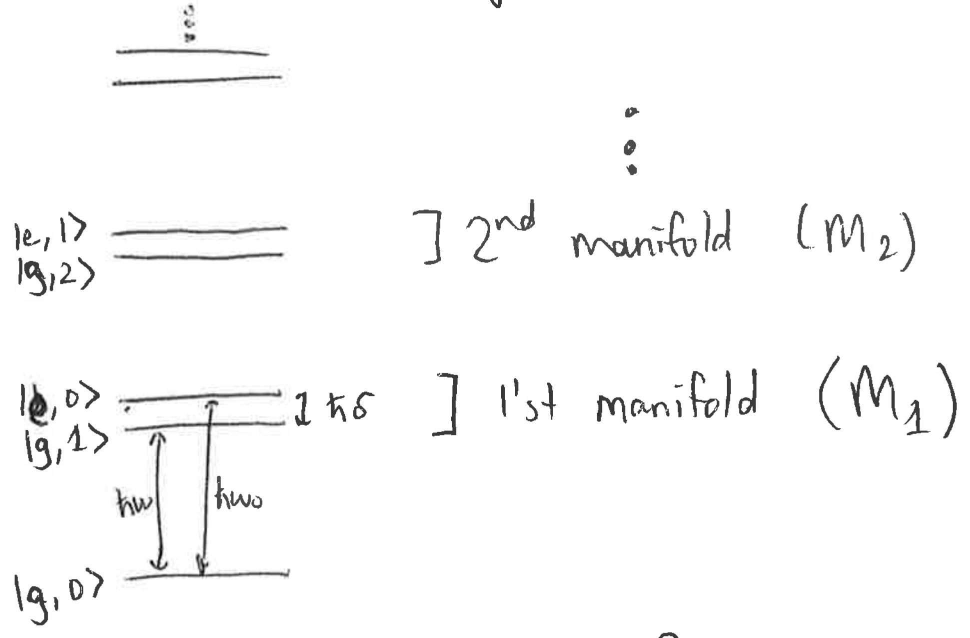
$$\hat{H}_u = \hat{H}_0 + \hat{H}_F$$

The eigenstates of this Hamiltonian are:

$$(\hat{H}_0 + \hat{H}_F)\,|e,n\rangle = \hbar\left(\frac{\omega_0}{2} + n\omega\right)|e,n\rangle$$

$$(\hat{H}_0 + \hat{H}_F)\,|g,n\rangle = \hbar\left(-\frac{\omega_0}{2} + n\omega\right)|g,n\rangle$$

These states are arranged in a ladder:



What about the interaction term?

$$\langle i,n|\hat{H}_I|i',n'\rangle = \frac{\hbar\Omega_R^{(0)}}{2}\left\{\langle i|a\times b|i'\rangle + \langle i|b\times a|i'\rangle\right\}\langle n|a+a^\dagger|n'\rangle$$

This is only non-zero if:

$i = g$ & $i' = e$ OR $i = e$ & $i' = g$

with $n' = n \pm 1$

Thus, this couples the two states in the same manifold M_n:

$$\{|g,n\rangle, |e,n-1\rangle\}$$

as well as two states that are two manifolds apart:

$$\{|g,n\rangle, |e,n+1\rangle\}$$

These second two states differ in energy by:

$$\Delta E = \hbar(\omega + \omega_0)$$

REMEMBER THE THIRD LECTURE!

This <u>time-independent</u> coupling will only produce appreciable state transfer when the interaction time is sufficiently short:

$$\Delta t \lesssim \frac{\hbar}{2\Delta E} = \frac{1}{2(\omega+\omega_0)}$$

Since $\omega+\omega_0$ is <u>very</u> large (think 10^{14}), t must be very short!

This is the same approximation we have been making all along, the quasi-resonant approximation, or...

THE ROTATING WAVE APPROXIMATION

This means we can write our Hamiltonian in block diagonal form, where the blocks only couple within a single manifold!

Within the n^{th} manifold:

$$\hat{H} = \hbar \begin{pmatrix} n\omega & \frac{\Omega_R^{(0)}\sqrt{n}}{2} \\ \frac{\Omega_R^{(0)}\sqrt{n}}{2} & n\omega - \delta \end{pmatrix}$$

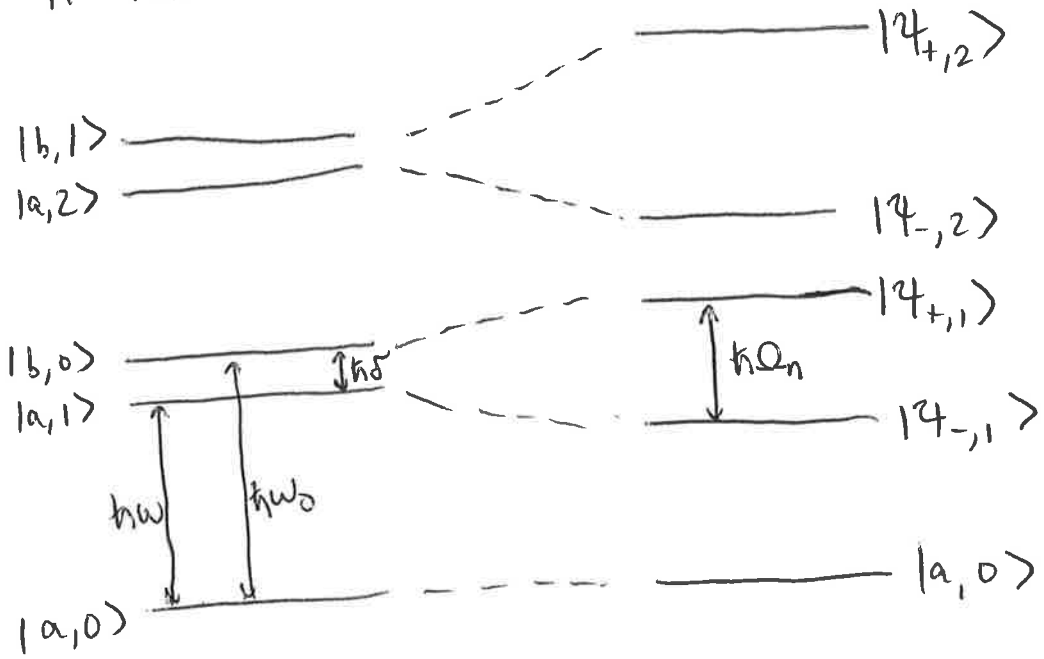
<u>Note</u>: $\langle g,n|H_I|e,n-1\rangle = \frac{\hbar\Omega_R^{(0)}}{2}\langle n|a+a^\dagger|n-1\rangle$

$$= \frac{\hbar\Omega_R^{(0)}}{2}\sqrt{n}$$

We have studied this Hamiltonian before. We will call the eigenstates $|\psi_{+,n}\rangle$ & $|\psi_{-,n}\rangle$:

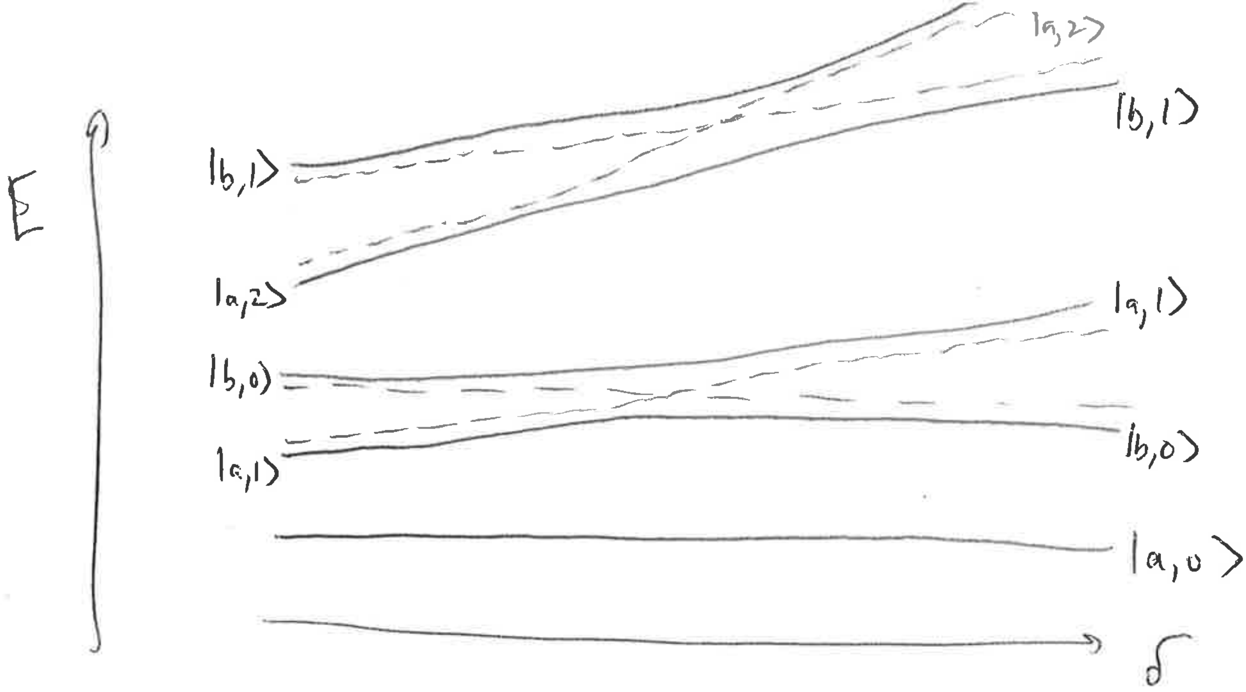
$$\boxed{\begin{aligned} |\psi_{+,n}\rangle &= \cos\theta_n |g,n\rangle + \sin\theta_n |e,n-1\rangle \\ |\psi_{-,n}\rangle &= -\sin\theta_n |g,n\rangle + \cos\theta_n |e,n-1\rangle \\ \tan 2\theta_n &= \frac{\Omega_R^{(0)}\sqrt{n}}{\delta} \\ E_{\pm,n} &= \hbar\left(n\omega - \frac{\delta}{2} \pm \frac{1}{2}\sqrt{n\Omega_R^{(0)2} + \delta^2}\right) \end{aligned}}$$

This looks different from what we know. But drawing it makes it clear:



where $\Omega_n = \dfrac{(E_{+,n} - E_{-,n})}{\hbar} = \sqrt{n\Omega_R^{(0)2} + \delta^2}$

We can draw $E_{\pm,n}$ vs. δ as we have done before:



Notes

- When $\delta = 0$, inter manifold separation is $\sqrt{n}\,\hbar\Omega_R$
- the asymptotic states (when $\delta \to \pm\infty$) are similar to the semi-classical ones we studied before
- the intra manifold spacing is two

Example 1: Excited atom in cavity.

Consider an excited atom in a cavity.
If the cavity is resonant with the atom, $\delta = 0$ and

$$|\psi(0)\rangle = |e,0\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{+,1}\rangle + |\psi_{-,1}\rangle\right)$$

Then

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{+,1}\rangle e^{-iE_{+,1}t/\hbar} + |\psi_{-,1}\rangle e^{-iE_{-,1}t/\hbar}\right)$$

$$= e^{-i\omega t}\left(-i|g,1\rangle \sin\left(\frac{\Omega_R^{(0)} t}{2}\right) + |e,0\rangle \cos\left(\frac{\Omega_R^{(0)} t}{2}\right)\right)$$

And finally, $P_e = \sum_n |\langle b,n|\psi(t)\rangle|^2 = \cos^2\left(\frac{\Omega_R^{(0)} t}{2}\right)$

The atom undergoes Rabi oscillations!

This spontaneous emission is very different from the spontaneous emission we studied before (exponential decay), because this atom is only interacting with a single mode.

When $\delta \neq 0$, one can show:

$$P_e(t) = 1 - \frac{\Omega_R^{(0)2}}{\Omega_R^2 + \delta^2} \sin^2\left(\sqrt{\Omega_R^{(0)2} + \delta^2}\ t/2\right)$$

For $|\delta| \gg 0$ (for detuned cavity)

$$\boxed{P_e(t) = 1}$$

Spontaneous emission is suppressed by an off-resonant cavity. This is yet another demonstration that spontaneous emission is BOTH A PROPERTY OF THE EMITTER AND THE ENVIRONMENT.

Example 2: Field initially in an intense coherent state.

Consider an atom initially in its excited state interacting with a coherent state $|\alpha\rangle$ with $\delta = 0$.

Recall: $|\alpha\rangle = \sum C_n |n\rangle = \sum e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

Then

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \sum C_{n-1} \left(|\psi_{+,n}\rangle + |\psi_{-,n}\rangle\right)$$

and $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\sum C_{n-1} e^{-in\omega t}\left(|\psi_{+,n}\rangle e^{-i\Omega_R^{(0)}\sqrt{n}t/2} + |\psi_{-,n}\rangle e^{+i\Omega_R^{(0)}\sqrt{n}t/2}\right)$

$$= \sum C_{n-1} e^{-in\omega t}\left[-i|g,n\rangle \sin\left(\frac{\Omega_R^{(0)}\sqrt{n}t}{2}\right) + |e,n-1\rangle \cos\left(\frac{\Omega_R^{(0)}\sqrt{n}t}{2}\right)\right]$$

This looks like complicated dynamics!

Simplify, since $n >>> 1$,

$$\sqrt{n} \approx \sqrt{\bar{n}} + (n-\bar{n})\frac{1}{2\sqrt{\bar{n}}}$$

When $|n-\bar{n}| < \sqrt{\bar{n}}$, the second term is <1, so $\sqrt{n} \sim \sqrt{\bar{n}}$. Since we know for a coherent state that

$|C_n|^2_{n-\bar{n} > \sqrt{\bar{n}}} << 1$ we can safely assume that the Rabi frequency is constant over the range of n that matter (ie that $\sqrt{n} \sim \sqrt{\bar{n}}$ over the range of n that matter).

Thus

$$|\psi(t)\rangle \approx -i\left(\sum C_{n-1} e^{-in\omega t}|g,n\rangle\right)\sin\left(\frac{\sqrt{\bar{n}}\,\Omega_R^{(0)}}{2}t\right)$$
$$+\left(\sum C_{n-1} e^{-in\omega t}|e,n-1\rangle\right)\cos\left(\frac{\sqrt{\bar{n}}\,\Omega_R^{(0)}}{2}t\right)$$

Finally, we will make the approximation

$$C_{n-1} \approx C_n$$

Since $\frac{C_{n-1}}{C_n} = \frac{\alpha}{\sqrt{n}} = \frac{\sqrt{\bar{n}}}{\sqrt{n}} \approx 1$, this is OK.

Finally, we have

$$|\psi(t)\rangle \approx \left(-i|g\rangle \sin\left(\frac{\sqrt{\bar{n}}\,\Omega_R}{2}t\right) + |e\rangle e^{-i\omega t} \cos\left(\frac{\sqrt{\bar{n}}\,\Omega_R}{2}t\right)\right) \otimes |\alpha e^{-i\omega t}\rangle$$

Notes:

- this is a separable state: the atom and field aren't entangled.
- the coherent state evolves the same as if the atom didn't exist.
- the atom evolves just like the semi-classical model (since $\sqrt{n}\,\Omega_R^{(0)} \approx \Omega_R$)