Q02 - Lecture 6

DENSITY MATRICES (REVIEW)

Recall that for a pure state $|\psi\rangle$ we can define a density operator $\hat{\rho} = |\psi\rangle\langle\psi|$.

We can also describe a statistical mixture of pure states $\{|\psi_i\rangle\}$ with probabilities $\{p_i\}$ by

$$\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

$\hat{\rho}$ has some useful properties:

$$tr(\hat{\rho}) = 1$$

$$\langle\hat{O}\rangle = tr(\hat{O}\hat{\rho}) \quad \text{(measurement)}$$

Purity $\quad P(\hat{\rho}) = tr(\hat{\rho}^2) \geq 0$

↳ 1 when $p_1 = 1$ + $p_{i \neq 1} = 0$

↳ < 1 otherwise.

Also, we can describe the time evolution of $\hat{\rho}$ the same as any other operator:

commutator

$$\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar}[\hat{H}, \hat{\rho}]$$

Finally, for a two-level system, $\hat{\rho}$ takes a particularly simple form, in terms of observable quantities:

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1 + \langle\sigma_z\rangle & \langle\sigma_x\rangle - i\langle\sigma_y\rangle \\ \langle\sigma_x\rangle + i\langle\sigma_y\rangle & 1 - \langle\sigma_z\rangle \end{pmatrix}$$

where $\langle\sigma_i\rangle = tr(\sigma_i \rho)$

Optical Bloch Equations

Recall from the last lecture the, for an atom interacting with an electromagnetic field in the rotating frame (and in the rotating wave approximation) the Hamiltonian is:

$$\hat{H} = \frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_R \\ \Omega_R & \delta \end{pmatrix}$$

Then $$\frac{d\hat{\rho}}{dt} = \frac{1}{i\hbar}[H, \hat{\rho}]$$

$$= \frac{1}{2}\begin{pmatrix} \Omega_R\langle\sigma_y\rangle & \delta\langle\sigma_y\rangle + i(\delta\langle\sigma_x\rangle + \Omega_R\langle\sigma_z\rangle) \\ \delta\langle\sigma_y\rangle - i(\delta\langle\sigma_x\rangle + \Omega_R\langle\sigma_z\rangle) & -\Omega_R\langle\sigma_y\rangle \end{pmatrix}$$

But $$\frac{d\rho}{dt} = \frac{1}{2}\begin{pmatrix} \frac{d\langle\sigma_z\rangle}{dt} & \frac{d\langle\sigma_x\rangle}{dt} - i\frac{d\langle\sigma_y\rangle}{dt} \\ \frac{d\langle\sigma_x\rangle}{dt} + i\frac{d\langle\sigma_y\rangle}{dt} & -\frac{d\langle\sigma_z\rangle}{dt} \end{pmatrix}$$

This leads us to the equations:

$$\boxed{\begin{aligned} \frac{d\langle\sigma_x\rangle}{dt} &= \delta\langle\sigma_y\rangle \\ \frac{d\langle\sigma_y\rangle}{dt} &= -\delta\langle\sigma_x\rangle - \Omega_R\langle\sigma_z\rangle \\ \frac{d\langle\sigma_z\rangle}{dt} &= +\Omega_R\langle\sigma_y\rangle \end{aligned}}$$ $\ast$

These can neatly be represented by:

$$\boxed{\frac{d}{dt}\langle\vec{\sigma}\rangle = \vec{\Omega}\times\langle\vec{\sigma}\rangle, \quad \text{where} \quad \begin{aligned} \vec{\Omega} &= (\Omega_R, 0, -\delta) \\ \langle\vec{\sigma}\rangle &= (\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle) \end{aligned}}$$

The Geometric Picture

Remember that $\langle \sigma_i \rangle$ are not just expectation values, we can use them to plot rho on the Bloch sphere!

Since $\langle \vec{\sigma} \rangle$ is the Bloch vector, what does the equation

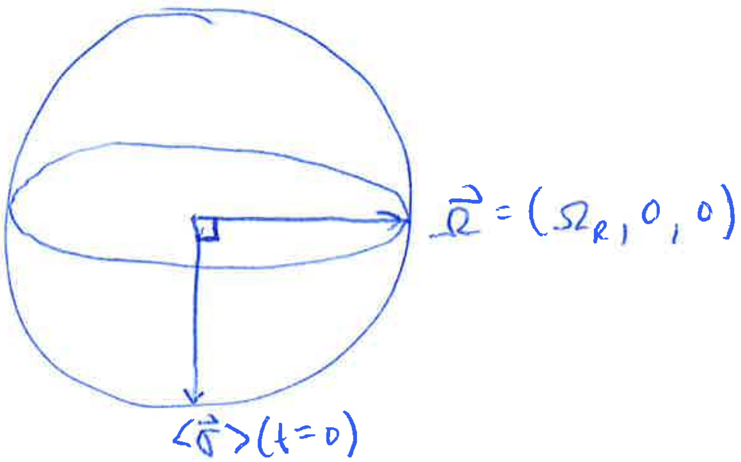
$$\frac{d\langle \vec{\sigma} \rangle}{dt} = \vec{\Omega} \times \langle \vec{\sigma} \rangle \quad \text{imply???}$$

⋮

This is a rotation! (recall: $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$)

So then the Bloch vector rotates around the "DRIVE VECTOR" $\vec{\Omega} = (\Omega_R, 0, -\delta)$ at an angular velocity $|\vec{\Omega}| = \sqrt{\Omega_R^2 + \delta^2}$ (the generalized Rabi frequency!)

On resonance:



This gives you a simple way to explain the dynamics of a two level atom in a classical field!

So what about spontaneous emission?
As it turns out, a full (quantum) treatment is outside the scope of this course (maybe, depending on time).

See Grynberg, Aspect, Fabre, Ch. 6.

But we can pull the same tricks we have this entire time!
(ie. introduce the concept <u>phenomenologically</u>)

<u>Time Evolution of Density Matrices</u>

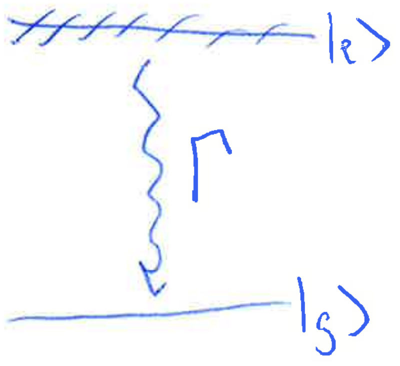
$$\frac{d\hat{\rho}}{dt} = (\text{coherent}) \quad + \quad (\text{incoherent})$$

$$= \frac{1}{i\hbar}[\hat{H}, \hat{\rho}]$$

$$\frac{d\rho_{ii}}{dt} = -\left(\sum_{j\neq i} \Gamma_{i\to j}\right)\rho_{ii} \quad \text{pop. loss}$$

$$+ \sum_{j\neq i} \Gamma_{j\to i}\,\rho_{jj} \quad \text{pop. feed.}$$

$$\frac{d\rho_{ij}}{dt} = -\gamma_{ij}\,\rho_{ij}$$



<u>For 2-level atom, incoherent terms are:</u>

$$\dot{\rho}_{ee} = -\Gamma \rho_{ee}$$
$$\dot{\rho}_{gg} = +\Gamma \rho_{ee}$$
$$\dot{\rho}_{eg} = -\gamma \rho_{eg}$$

one can show that $\Gamma \geq \gamma/2$.

How?!

Density matrix must be physical.

Let's translate these to the Bloch picture.

Recalling: $$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\langle\sigma_z\rangle & \langle\sigma_x\rangle - i\langle\sigma_y\rangle \\ \langle\sigma_x\rangle + i\langle\sigma_y\rangle & 1-\langle\sigma_z\rangle \end{pmatrix}$$

So $\dfrac{d\rho_{ee}}{dt} = -\Gamma\rho_{ee}$ implies:

$$\frac{d}{dt}\frac{\langle\sigma_z\rangle}{2} = -\Gamma\left(\frac{\langle\sigma_z\rangle+1}{2}\right) \Rightarrow \frac{d}{dt}\langle\sigma_z\rangle = -\Gamma(\langle\sigma_z\rangle+1)$$

Similarly, you can show that:

$$\frac{d}{dt}\langle\sigma_x\rangle = -\frac{\Gamma}{2}\langle\sigma_x\rangle$$

$$\frac{d}{dt}\langle\sigma_y\rangle = -\frac{\Gamma}{2}\langle\sigma_y\rangle$$

Taking the coherent terms too we get:

$$\boxed{\begin{aligned} \frac{d}{dt}\langle\sigma_x\rangle &= \delta\langle\sigma_y\rangle - \frac{\Gamma}{2}\langle\sigma_x\rangle \\ \frac{d}{dt}\langle\sigma_y\rangle &= -\delta\langle\sigma_x\rangle - \Omega_R\langle\sigma_z\rangle - \frac{\Gamma}{2}\langle\sigma_y\rangle \\ \frac{d}{dt}\langle\sigma_z\rangle &= \Omega_R\langle\sigma_y\rangle - \Gamma(\langle\sigma_z\rangle+1) \end{aligned}}$$

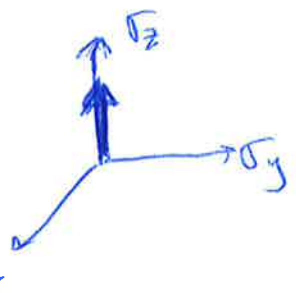
OPTICAL *
BLOCH *
EQUATIONS *

The OBEs describe the evolution of the Bloch vector in the $\{|e\rangle, |g\rangle\}$ basis, in a frame rotating at ω, the drive.

So examples:

Case 1a: Start in $|e\rangle$, no drive.

Initially, $\langle\sigma_z\rangle = 1$, $\langle\sigma_y\rangle = 0 = \langle\sigma_x\rangle$



The only non-zero rate is:

$$\frac{d}{dt}\langle\sigma_z\rangle = -\Gamma(\langle\sigma_z\rangle + 1)$$

which yields a decay until $\langle\sigma_z\rangle = -1 \Rightarrow$ ground state.

Case 1b: Start in $|g\rangle$, no drive

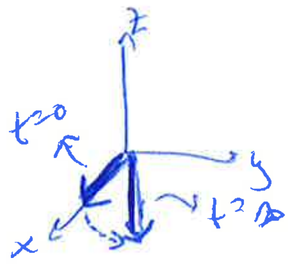
Nothing happens!

Case 1c: Start in $|\psi\rangle = \frac{|e\rangle + |g\rangle}{\sqrt{2}}$ or $\rho = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

then $\langle\sigma_z\rangle = 0$, $\langle\sigma_y\rangle = 0$, $\langle\sigma_x\rangle = 1$

Now $\frac{d}{dt}\langle\sigma_x\rangle = -\frac{\Gamma}{2}\langle\sigma_x\rangle$

$\frac{d}{dt}\langle\sigma_z\rangle = -\Gamma(\langle\sigma_z\rangle + 1)$

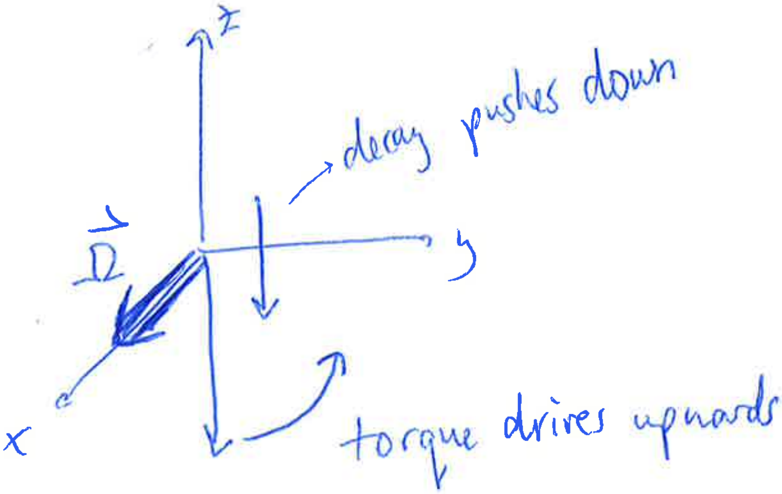


Note: This vector is spinning in the lab frame @ ω_0!

Case II: Drive on resonance.

$$\vec{\Omega} = (\Omega_R, 0, 0)$$

Start in $|g\rangle$. What happens?

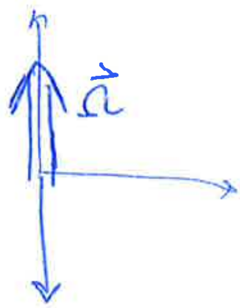


Which wins? See assignment 3!

Case III: Drive far from resonance: $|\delta| \gg \Omega_R$

Choose $\delta < 0$ so $\vec{\Omega} = (0, 0, -\delta)$

Start in $|g\rangle$:



Nothing happens

Start in $\frac{|e\rangle + |g\rangle}{\sqrt{2}}$:

State precesses @ δ

Eventually decays.