# QO2 - Lecture 4

Recall from last time:

$$P_{a\to b}(T) = \frac{4W^2}{\hbar^2\Omega^2}\sin^2\left(\frac{\Omega T}{2}\right)$$ (exact solution for constant perturbation)

How does this compare to perturbation theory?

1st order: $$P_{i\to k}(T) = \frac{W^2T^2}{\hbar^2}\frac{\sin^2(ET/2\hbar)}{(ET/2\hbar)^2}$$

if $T \ll 2\hbar/E$, then $P_{i\to k} \simeq \frac{W^2}{\hbar^2}T^2$

From the exact result,

$$P_{a\to b}(T) = \frac{4W^2}{\hbar^2\Omega^2}\sin^2\frac{\Omega T}{2} \approx \frac{W^2}{\hbar^2}T^2, \text{ for } T \ll \frac{2}{\Omega} = \frac{2\hbar}{\sqrt{E^2+4W^2}}$$

So both the result from p.t. and the exact result have the same behaviour at small times (independent of $E_a - E_b$)

What about longer times?

$$P_{a\to b}(T) = \frac{W^2T^2}{\hbar^2}\frac{\sin^2(\Omega T/2)}{(\Omega T/2)^2}$$ vs. same but with $\Omega \to E/\hbar$

p.t. predicts $\Omega = \frac{1}{\hbar}(E_a - E_b)$

instead of $\Omega = \frac{\sqrt{(E_a-E_b)^2+4W^2}}{\hbar}$

So instantaneous probability is wrong after a sufficient amount of time.

Maybe this is still ok, as long as $W \ll \Delta E$ ?

Nope!

$$\Omega T = \frac{1}{\hbar}\sqrt{E^2+4W^2}\; T$$

$$\simeq \frac{E}{\hbar}\left(1+2\,W^2/E^2\right) T$$

$$= \frac{ET}{\hbar} + \frac{2W^2T}{E\hbar}$$

So even if $\Omega \simeq E$, after a sufficient amount of time p.t. will still be wrong. The phase accumulated after long $T$: $\Omega T - ET \geq \pi$ when $T \gtrsim \frac{\pi\hbar|E_a-E_b|}{W_{ab}^2}$

This tells us that our condition for p.t.'s validity was necessary but not sufficient!

The <u>accumulated phase</u> must be small. ie)

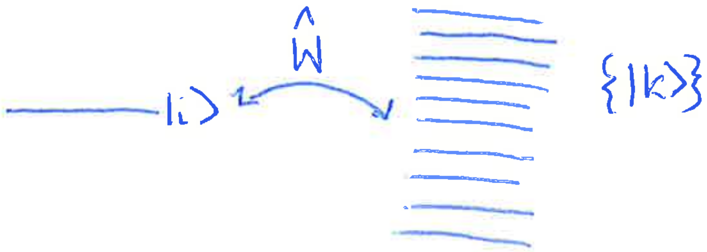
$$\frac{2W^2T}{E\hbar} \ll 1$$

or rather $\boxed{\frac{ET}{\hbar} \ll \frac{|E_a-E_b|^2}{2W_{ab}^2}}$

This is the real validity criterion for perturbation theory!

# Transitions to Continuum: Fermi's Golden Rule

Consider a single level $|i\rangle$ coupled to a continuum of states $\{|k\rangle\}$




Actually, easier to consider a quasi-continuum where the spacing is finite, but smaller than any other energy scale. For instance, recall the particle in a box, with momentum eigenstates with

$$E_k = \frac{\hbar^2\pi^2}{2mL^2}k^2 \, , \qquad \Delta E = k\,\frac{\hbar^2\pi^2}{mL^2} = \frac{2\pi\hbar}{\sqrt{2m}}\,\frac{\sqrt{E_k}}{L}$$

As $L \to \infty$, $\Delta E \to 0$ and this will give us our continuum.

BUT, for the actual math we will simply use $E_k = \varepsilon k$ (uniform spacing) for the range of levels of interest.

The coupling operator $\hat{W}$ will, as before, only have off diagonal elements

$$\langle k|\hat{W}|i\rangle = W$$
$$\langle i|\hat{W}|i\rangle = 0$$

## Short-Time Behaviour

Start in $|i\rangle$, and use first order p.t. to calculate the rate at which population moves into the continuum.

$$P_i = 1 - \sum_k P_k$$

prob. to be in state $|i\rangle$

prob. to be in state $|k\rangle$

Now remember the result from 1st order p.t.:

$$P_{i\to k}(T) = \frac{|W_{ki}|^2}{\hbar^2} \frac{\sin^2(ET/2\hbar)}{(ET/2\hbar)^2} T^2$$

Write this as:

$$P_{i\to k}(T) = T\frac{2\pi}{\hbar}|W_{fi}|^2 \; \delta_T(E_k - E_i)$$

where $$\delta_T(E) = \frac{2\hbar \sin^2(ET/2\hbar)}{\pi T E^2}$$

This $\delta_T$ approximates a Dirac delta function!

- Max at $E=0$ is $T/2\pi\hbar$
- Area $=1$
- Width $\simeq \frac{2\pi\hbar}{T}$

Now, going back to our continuum example, the equation for $P_i$ has a sum over $k$ which can be replaced by an integral when the level spacing is small (remember level spacing is $\varepsilon$):

$$\sum_k \to \frac{1}{\varepsilon}\int_{-\infty}^{\infty} dE$$

Then $$P_i(T) = 1 - \frac{1}{\varepsilon}\int dE \, \frac{2\pi T}{\hbar} W^2 \, \delta_T(\underbrace{k\varepsilon}_{E})$$

$$\boxed{P_i(T) = 1 - T\left(\frac{2\pi W^2}{\hbar\varepsilon}\right)}$$

We can identify $\Gamma = \frac{2\pi W^2}{\hbar \varepsilon}$ as the rate of departure per unit time.

This is a linear decay! (as opposed to quadratic seen before).

# Long-Time Behavior: The Wigner-Weisskopf Solution

Let's try to work through the exact solution. Recall our earlier work:

$$|\psi(t)\rangle = \gamma_i(t)|i\rangle + \sum_k \gamma_k(t) e^{-ik\varepsilon t/\hbar}|k\rangle$$

this is because we have set $E_i = 0$ & $E_k = k\varepsilon$

We found that for any state:

$$i\hbar\dot{\gamma}_n = \sum_m \langle n|\hat{H}_1(t)|m\rangle e^{i(E_n - E_m)t/\hbar}\gamma_m(t)$$

To apply this here, set $\hat{H}_1(t) = \hat{W}$, $E_i = 0$, & $E_k = k\varepsilon$:

$$\begin{cases} i\hbar\dot{\gamma}_i(t) = \sum_k W e^{-ik\varepsilon t/\hbar}\gamma_k(t) \\ i\hbar\dot{\gamma}_k(t) = W e^{ik\varepsilon t/\hbar}\gamma_i(t) \end{cases}$$

Take $\gamma_i(0) = 1$, $\gamma_k(0) = 0$, and integrate the second equation:

$$\gamma_k(t) = \frac{W}{i\hbar}\int_0^t e^{ik\varepsilon t'/\hbar}\,dt'\,\gamma_i(t') + \underbrace{\gamma_k(0)}_{=0}$$

Now substitute this into 1st equation:

$$\frac{d}{dt}\gamma_i(t) = -\frac{W^2}{\hbar^2}\int_0^t dt'\,\gamma_i(t')\sum_k e^{ik\varepsilon(t'-t)/\hbar}$$

Now we replace the sum with an integral, as before:

$$\sum_k \to \frac{1}{\varepsilon}\int dE$$

and realize that $\displaystyle\int_{-\infty}^{\infty} dE\, e^{iE(t'-t)/\hbar} = 2\pi\hbar\,\underbrace{\delta(t'-t)}_{\text{Dirac delta function!}}$

thus, $$\frac{d}{dt}\gamma_i(t) = -\underbrace{\frac{2\pi t_i W^2}{\sum t_i^2}}_{\Gamma} \int_0^t dt'\, \gamma_i(t')\,\delta(t'-t)$$

This integral looks easy, but be careful: the limits are $[0,t]$, not $[-t,t]$.

Switch variables: $\tau \equiv t'-t$, so

$$\frac{d}{dt}\gamma_i(t) = -\Gamma \int_{-t}^{0} d\tau\, \gamma_i(t+\tau)\,\delta(\tau)$$

Now recall: $$\gamma_i(t) = \int_{-t}^{t} \gamma_i(t+\tau)\,\delta(\tau)\,d\tau = \int_{-t}^{0} \gamma_i(t+\tau)\,\delta(\tau)\,d\tau + \int_0^t \gamma_i(t+\tau)\,\delta(\tau)\,d\tau$$

To be rigorous, one must solve this by taking the delta function as the limit of a narrow funtion as the width $\to 0$.

Since $\delta(\tau)$ is even, however, we will guess that each term is equal.

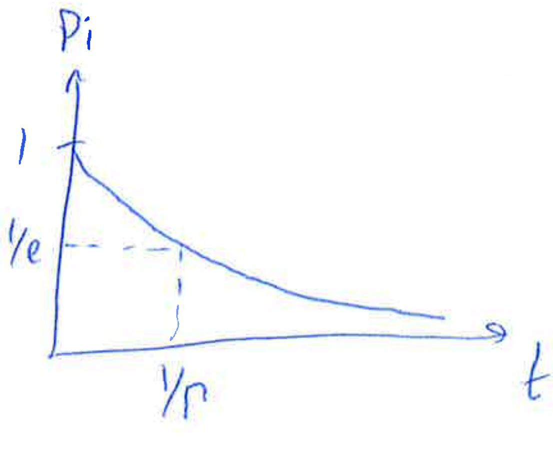
$$\therefore \int_{-t}^{0} d\tau\, \gamma_i(t+\tau)\,\delta(\tau) = \frac{\gamma_i(t)}{2}$$

So we have $$\frac{d}{dt}\gamma_i(t) = -\frac{\Gamma}{2}\gamma_i(t)$$

and finally: $$\gamma_i(t) = \exp(-\Gamma t/2)$$

Then $$\boxed{P_i(T) = \exp(-\Gamma t)}$$

exponential decay!



# Energy distribution of final states

As $t \to \infty$, what energy distribution of final states do we have?

You should be able to guess! The lifetime of the initial state is $\Gamma^{-1}$ so a time-energy uncertainty principle gives: $\Delta E \Delta t \gtrsim \hbar$

$$\boxed{\therefore \Delta E \sim \hbar \Gamma}$$

Rigorously: $\gamma_k(t) = \frac{W}{i\hbar} \int_0^t dt' \gamma_i(t') e^{ik\epsilon t'/\hbar}$

substitute in $\gamma_i(t') = \exp\left(-\frac{\Gamma t'}{2}\right)$

$$\gamma_k(t) = \frac{W}{i\hbar} \int_0^t dt' \exp(-\Gamma t'/2) e^{ik\epsilon t'/\hbar} = \frac{W}{i\hbar} \left(\frac{1}{-\Gamma/2 + ik\epsilon/\hbar}\right) \left(e^{-(\mathrm{const})t}\right)\Big|_0^t$$

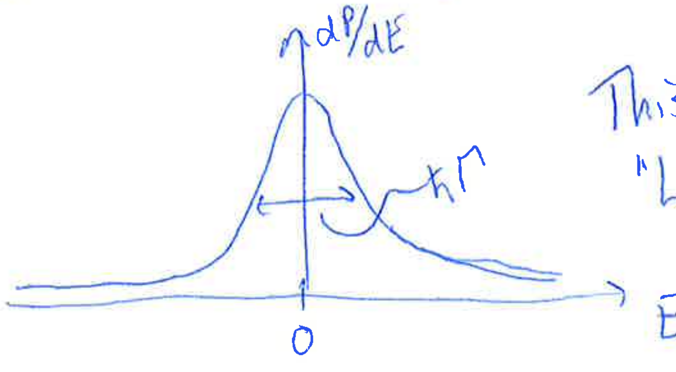
As $t \to \infty$, $\gamma_k(t) = \frac{W}{i\hbar} \left(\frac{1}{-\Gamma/2 + ik\epsilon/\hbar}\right)$

and $P_k(t \to \infty) = \frac{W^2}{(k\epsilon)^2 + \hbar^2\Gamma^2/4}$ $\Rightarrow$ This is just for one state

In the continuum limit, our answer must be independent of $W$ & $\epsilon$, so write:

$$\frac{dP}{dE} = \frac{1}{\epsilon} P_k = \frac{W^2}{\epsilon}\left[E^2 + \hbar^2\Gamma^2/4\right]^{-1} = \frac{\hbar\Gamma}{2\pi}\left(\frac{1}{E^2 + \hbar^2\Gamma^2/4}\right)$$

density of states




This is called a "Lorentzian"

# Fermi's Golden Rule

We've done all the difficult math already. What we want to do is generalize the treatment of the previous section without doing all that math again.

Recall from p.t.:

$$P_i(T) = 1 - \sum_k T \frac{2\pi}{\hbar} |W_{ki}|^2 \delta_T(E_k - E_i)$$

Fermi generalized the way we change this sum into an integral, using the density of states:

$$\sum_k \longrightarrow \int \rho(E_k)\, dE$$

Continuing as before we will get the same results, except now

$$\boxed{\Gamma = \frac{2\pi}{\hbar} |W_{fi}|^2 \rho(E_f)}$$

This, in all its glory, is Fermi's Golden Rule.

It tells us: that transition rates are proportional to:

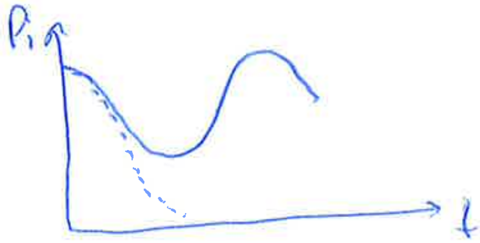
1) Coupling strength squared
2) Density of states

## Conclusions

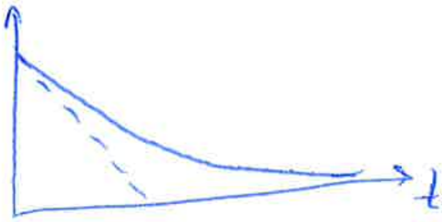
We have used perturbation theory and exact solutions to explain the dynamics of quantum systems with time-dependent interactions.

We saw two behaviours:

(a) ISOLATED FINAL STATE showed $t^2$ evolution at short times, oscillatory at longer times.




(b) ENSEMBLE of FINAL STATES showed $t$ evolution at short times and exponential decay at long times




We saw the quantization of light-matter interaction emerge WITHOUT any quantized drive field.

In quasi-resonant approximation,

$$P_{i\to k}(T) = T\,\frac{|W_{ki}|^2}{4}\left(\frac{2\pi}{\hbar}\right)\delta_T(E_k - E_i \pm \hbar\omega)$$

where the (+) sign is for $E_k > E_i$ (absorption)
and the (−) sign is for $E_k < E_i$ (emission)

These treatments are general. We could apply them to photons, electrons, atoms, etc. You will see all of these implementations as a quantum engineer!