Quantum Circuits for the Schur Transform

Oliver Thomas

Quantum Engineering CDT
University of Bristol

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1 Introduction

background.

2004 Bacon, Chuang & Harrow proposed a scheme for implementing the Schur transform in Poly time.

There are two distinct ways of performing the Schur transform on n qubits, it can either be built up from coupling all n qubits together in a single iteration which we call the spatial multiplexed approach. The other approach is performing Clebsch-Gordan (CG) transforms on the n qubits one at a time which we call the temporal multiplexed approach.

This report is structured as follows,

2 Streaming Scheme

Information processing tasks, such as classical compression gain a huge advantage implementing the process using a streaming scheme. Rather than start the compression on all the data to be sent and wait for it all to be compressed then send the data, as compression can be performed sequentially, compress part of the message and send it while compressing the next part. The Schur transform can also be thought of in this way. However, with compression it can still be useful to only compress part of the message, it is meaningless to perform only part of the Schur transform which suggests there may be a more optimal scheme opposed to streaming for implementing the Schur transform.

The streaming Schur transform is described in Fig. 1,

For the streaming scheme the U_{CG} block can be chosen so that it contains all of the gates for upto the n-th qubit meaning the same block can be repeated. Where the U_{CG} is the Clebsch-Gordan transform between the J & M registers and the k-th qubit $|i_k\rangle$.

The controlled Rotation matrix, $R_y(\theta_{J,m'})$ Fig. 1 is the Clebsch-Gordan coefficients for coupling 1

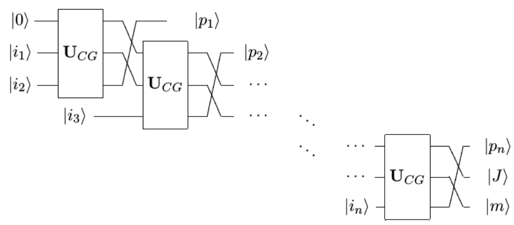


Figure 1: Streaming structure where the Schur transform is built up from consecutive Clebsch-Gordan transforms [1].

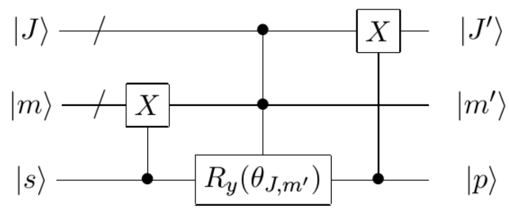


Figure 2: U_{CG} block, Qadder, controlled rotation, Qadder [1].

qubit sequentially, given by,

$$R_y(\theta_{J,m'}) = \begin{bmatrix} \cos(\theta_{J,m'}) & -\sin(\theta_{J,m'}) \\ \sin(\theta_{J,m'}) & \cos(\theta_{J,m'}) \end{bmatrix} = \frac{1}{\sqrt{2J+1}} \begin{bmatrix} \sqrt{J+\frac{1}{2}+m'} & -\sqrt{J+\frac{1}{2}-m'} \\ \sqrt{J+\frac{1}{2}-m'} & \sqrt{J+\frac{1}{2}+m'} \end{bmatrix} \tag{1}$$

Where primed variables means after the angular momentum addition so J is the total J that the spin is coupling to, the system will have J' total angular momentum after the coupling. m is the z component of the system before and m' is the total z component after the coupling.

To build this circuit the rotation matrix, $R_y(\theta_{J,m'})$ needs to be calculated using Eq. 1, values here Eq. 11e, and a function to update the $|m\rangle$ and $|J\rangle$ registers is needed. Updating the registers can be implemented relatively simply using the coherent (meaning the registers are allowed to be in superpositions) equivalent of the digital full adders and subtractors. The complexity now has been reduced to implementing the Clebsch-Gordan transform, U_{CG}.

The aim of this report is to look at whether it is possible to achieve a log decrease in time by instead of coupling 1 qubit in at a time to the J & M registers, to pairing the couplings up in a **binary tree?** technique. To investigate the problem we look at how the complexity of performing the Clebsch-Gordan transform scales moving from coupling a single qubit to an arbitrary J&M to coupling arbitrary J&M registers together. looking at the pairing approach there is a symmetry in the fact that the max range of J&M values will be the same within each pairing which will help simplify the problem compared to any arbitrary J&M values.

3 Implementing the Clebsch-Gordan Transform

The vanilla transform directly maps the computation basis states to a labeling of J and M values. The vanilla approach of calculating the CG coefficients and finding a gate decomposition does not scale well. A better approach is to use a scheme explicitly storing the values of J & M in a register.

3.0.1 Clebsch-Gordan transform for 2 qubits

The Clebsch-Gordan transform is a basis transformation into the Schur basis. The transform for 2 qubits is given by,

$$
\begin{aligned}
|J=1, M=+1\rangle &= |00\rangle \\
|J=1, M=\ \ 0\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|J=1, M=-1\rangle &= |11\rangle
\end{aligned} \tag{2a}
$$

$$
|J=0, M=\ \ 0\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{2b}
$$

Throughout the encoding $|0\rangle = +1/2$, $|1\rangle = -1/2$ is used unless stated otherwise.

The transform expressed as a matrix is,

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{bmatrix} = \text{(spin labeling)} \begin{bmatrix} |00\rangle \\ \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\ |11\rangle \\ \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \end{bmatrix} = \begin{bmatrix} |J=1, M=1\rangle \\ |J=1, M=0\rangle \\ |J=1, M=-1\rangle \\ |J=0, M=0\rangle \end{bmatrix} \tag{3}
$$

3.0.2 Clebsch-Gordan coefficients for 3 qubits

The CG coefficients for three qubits are no multiplicities of 4 for J=3/2 and 2 multiplicities of 2 for J=1/2 Eq. 5c. The multiplicities, P are defined as $J' - J$ the new J value minus the previous J value, the number of 1s in a P string is the number of multiplicities for that J value.

The matrix for the transform which takes the computational basis to the spin basis is, There are multiple ways of writing the spin basis, there is the traditional CG coefficients and there is also what is referred to here as the phase encoding Eq. 9. The phase encoded transform matrix will have a different decomposition as the shape of the matrix is different to the regular encoding.

3.0.3 4 Qubit CG coefficients

The CG coefficients for four qubits contains 16 terms, 5 for J=2, 3 multiplicities of 3 for J=1, 2 multiplicities of 1 for J=0. The equations are given in Eq. 10f. In the J=1 case there are 3 acceptable bit strings, 0001, 0010, 0100 meaning there are 3 multiplicities present.

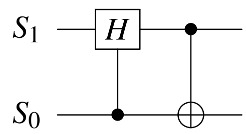
3.1 Glebsch-Gordan circuits

As two-qubit (entangling) gates are much more expensive to perform compared to single qubit gates, the cost of the circuits discussed here will all be given in terms of the number of two-qubit gates. The decomposition scheme for the n-qubit case could take at most $2^{n-1}(2^n - 1)$ $C^{n-1}U$ gates [2], where $C^{n-1}U$ means a unitary acting on 1 qubit controlled on the other n-1 qubits.

3.1.1 2 qubit circuit

Which can be implemented in a circuit as,

Circuit for Clebsch-Gordan transform Fig. 3 contains 2 gates.



(4)

Figure 3: Schur transform for 2 qubits

3.1.2 Circuit for 3 qubit transform

For 3 qubits this upper bound is 28 C^2U gates. It has been shown that in terms of gate count, $C^nU \sim 5C^{n-1}V$ where $U\&V$ are unitaries [3]. This means the maximum two-qubit gates needed would be 140 CU gates.

The decomposition of the 3 qubit CG transform was performed using the Givens rotation method for unitary decomposition into a gate-set. The matrix Eq. 6 can be expressed as a product of 19 C^2U gates (control-control-unitaries) which is $\sim$80 CU gates.

See online [4] for Fortran code which implements the Givens rotation method to give the 19 C^2U gate decomposition. The majority of the gates are CNOT gates. This is mainly due to the re-ordering of the basis and is similar to the quantum Fourier transform (QFT). The QFT produces the output in reverse qubit order the actual number of gates required to do the transform is massively reduced. The overhead calculated here is due to the rearranging of the basis. This means that depending on what the transform is used the transform could be computed with less gates. For example, if the transform was only used to check if the state was in a particular J block but didn't need to know the specific M value the order afterwards wouldn't be as important reducing the CNOTs needed.

3.1.3 Circuit for 4 qubit transform

The decomposition scheme for the 4 qubit case could take at most $2^{n-1}(2^n-1)\ C^{n-1}U$ gates [2], where n is 4 and. For 4 qubits this upper bound is 120 C^3U gates which could be up to $\sim$ 3000 CU gates. In reality it will be much fewer gates as the matrix is sparse, however this suggests a different approach was needed.

The 4 qubit Clebsch-Gordan transform requires 72 C^3U gates which is equal to $\sim$ 1800CU gates.

4 General circuit for the Quantum Schur transform Fig. 7

In the section above we have investigated the cost of implementing directly the Clebsch-gordan unitary matrix using a unitary gate decomposition scheme. We note that the majority, over half of the gates are CNOT or higher dimensional equivalents, the 2 qubit case 1 out of 2 gates are CNOTs, 3 qubits, 14 out of 15 are C^2NOTs and 4 qubits, 50 out of 72 are C^3-NOT gates.

4.1 Registering J & M explicitly Fig. 4, Fig. 5

This spatial multiplexing approach only stores the final output values of the J & M registers.

Circuit uses the encoding for $|S\rangle : |0\rangle \mapsto Spin = +\frac{1}{2}, |1\rangle \mapsto Spin = -\frac{1}{2}$ and the same for $|P\rangle$.

The circuit adds the value of the spin to be added, $|S\rangle$, to the M register to calculate the M' register value. This is done by implementing the quantum reversible equivalent to the digital full adder.

The streaming scheme uses temporal multiplexing to perform the Schur transform in polynomial time if a recursive streaming scheme is used [5].

J_2	J_1	J_0	J
0	0	0	0
0	0	1	$\frac{1}{2}$
0	1	0	1
0	1	1	$\frac{3}{2}$
1	0	0	-2
1	0	1	$-\frac{3}{2}$
1	1	0	-1
1	1	1	$-\frac{1}{2}$

Table 1: Binary Two's complement encoding to spin values of the J register

5 Reduced general gate circuit for up to the 2 qubit Schur transform Fig. 6

We note that it is possible to reduce the gate count of the general streaming scheme by changing the encoding. Here we remove the intermediate values and hence can reduce both the number of qubits required and the gate count. This scheme works by coupling in two qubits at a time as opposed to one.

6 Conclusion

There is a lot of freedom in the choice of basis used. We have chosen to use Two's complement encoding for the registers. The encoding for the multiplicities remains consistent throughout althought it is more complex. For the general circuits the encoding remains consistent throughout. In order to achieve the optimal number of gates for the Schur transform on a specific number of qubits the encoding is changed on a case by case basis.

Clebsch-Gordan for higher qubit numbers has redundancy in. depending on application of schur transform and input states it could be reasonable to approximate the transform when acting on highly symmetric states to just doing hadamard gates.

References

[1] Dave Bacon, Isaac L Chuang, and Aram W Harrow. Efficient quantum circuits for schur and clebsch-gordan transforms. *Physical review letters*, 97(17):170502, 2006.

[2] Chi-Kwong Li, Rebecca Roberts, and Xiaoyan Yin. Decomposition of unitary matrices and quantum gates. *International Journal of Quantum Information*, 11(01):1350015, 2013.

[3] Adriano Barenco, Charles H Bennett, Richard Cleve, David P DiVincenzo, Norman Margolus, Peter Shor, Tycho Sleator, John A Smolin, and Harald Weinfurter. Elementary gates for quantum computation. *Physical review A*, 52(5):3457, 1995.

[4] https://github.com/ot561/schurtransform/blob/master/matrixmul.f90.

[5] Dave Bacon, Isaac L Chuang, and Aram W Harrow. The quantum schur and clebsch-gordan transforms: I. efficient qudit circuits. In *Proceedings of the eighteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 1235–1244. Society for Industrial and Applied Mathematics, 2007.

A Appendix: Maths

A.1 3 Qubit transformation

This is the J=3/2 block

$$\begin{aligned}
|J=3/2, M=+3/2, P=000\rangle &= |000\rangle \\
|J=3/2, M=+1/2, P=000\rangle &= \sqrt{\frac{1}{3}}(|001\rangle + |010\rangle + |100\rangle) \\
|J=3/2, M=-1/2, P=000\rangle &= \sqrt{\frac{1}{3}}(|110\rangle + |011\rangle + |101\rangle) \\
|J=3/2, M=-3/2, P=000\rangle &= |111\rangle
\end{aligned} \tag{5a}$$

This is the J=1/2 block from J=1, multiplicity zero

$$\begin{aligned}
|J=1/2, M=+1/2, P=001\rangle &= +\sqrt{\frac{2}{3}}|001\rangle - \sqrt{\frac{1}{6}}(|010\rangle + |100\rangle) \\
|J=1/2, M=-1/2, P=001\rangle &= -\sqrt{\frac{2}{3}}|110\rangle + \sqrt{\frac{1}{6}}(|011\rangle + |101\rangle)
\end{aligned} \tag{5b}$$

This is the J=1/2 block from J=0, multiplicity one

$$\begin{aligned}
|J=1/2, M=+1/2, P=010\rangle &= \frac{1}{\sqrt{2}}(|010\rangle - |100\rangle) \\
|J=1/2, M=-1/2, P=010\rangle &= \frac{1}{\sqrt{2}}(|011\rangle - |101\rangle)
\end{aligned} \tag{5c}$$

this is in matrix form,

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{1}{3}} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{1}{3}} & 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{6}} & 0 & -\sqrt{\frac{1}{6}} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\frac{1}{6}} & 0 & \sqrt{\frac{1}{6}} & -\sqrt{\frac{2}{3}} & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0
\end{bmatrix}
\begin{bmatrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{bmatrix}
=
\left[\begin{array}{c}
|J=3/2, M=3/2\rangle \\
|J=3/2, M=1/2\rangle \\
|J=3/2, M=-1/2\rangle \\
|J=3/2, M=-3/2\rangle \\ \hline
|J=1/2, M=1/2, P=0\rangle \\
|J=1/2, M=-1/2, P=0\rangle \\ \hline
|J=1/2, M=1/2, P=1\rangle \\
|J=1/2, M=-1/2, P=1\rangle
\end{array}\right] \tag{6}$$

The CG transform for 3 qubits Eq. 6 can be rearranged to a block diagonal form which looks like it could be implemented in a circuit.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{6}} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 000 \\ 001 \\ 010 \\ 100 \\ 011 \\ 101 \\ 110 \\ 111 \end{bmatrix} \tag{7}$$

A.2 3 Qubit phase encoding

This is the J=3/2 block

$$\begin{aligned} |J=3/2, M=+3/2, P=000\rangle &= |000\rangle \\ |J=3/2, M=+1/2, P=000\rangle &= \sqrt{\frac{1}{3}}(|001\rangle + |010\rangle + |100\rangle) \\ |J=3/2, M=-1/2, P=000\rangle &= \sqrt{\frac{1}{3}}(|110\rangle + |011\rangle + |101\rangle) \\ |J=3/2, M=-3/2, P=000\rangle &= |111\rangle \end{aligned} \tag{8a}$$

This is the J=1/2 block from J=1, multiplicity zero

$$\begin{aligned} |J=1/2, M=+1/2, P=001\rangle &= \frac{1}{\sqrt{3}}(|001\rangle + e^{2\pi i/3}|100\rangle + e^{4\pi i/3}|010\rangle) \\ |J=1/2, M=-1/2, P=001\rangle &= \frac{1}{\sqrt{3}}(|011\rangle + e^{2\pi i/3}|101\rangle + e^{4\pi i/3}|110\rangle) \end{aligned} \tag{8b}$$

This is the J=1/2 block from J=0, multiplicity one

$$\begin{aligned} |J=1/2, M=+1/2, P=010\rangle &= \frac{1}{\sqrt{3}}(|001\rangle + e^{4\pi i/3}|100\rangle + e^{2\pi i/3}|010\rangle) \\ |J=1/2, M=-1/2, P=010\rangle &= \frac{1}{\sqrt{3}}(|011\rangle + e^{4\pi i/3}|101\rangle + e^{2\pi i/3}|110\rangle) \end{aligned} \tag{8c}$$

The phase encoding matrix is given by,

$$
\frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3} \\ 0 & e^{2\pi i/3} & e^{4\pi i/3} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2\pi i/3} & 0 & e^{4\pi i/3} & 1 & 0 \\ 0 & e^{4\pi i/3} & e^{2\pi i/3} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{4\pi i/3} & 0 & e^{2\pi i/3} & 1 & 0 \end{bmatrix}\begin{bmatrix} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{bmatrix} = \left[\begin{array}{c} |J=3/2, M=3/2\rangle \\ |J=3/2, M=1/2\rangle \\ |J=3/2, M=-1/2\rangle \\ |J=3/2, M=-3/2\rangle \\ \hline |J=1/2, M=1/2, P=0\rangle \\ |J=1/2, M=-1/2, P=0\rangle \\ \hline |J=1/2, M=1/2, P=1\rangle \\ |J=1/2, M=-1/2, P=1\rangle \end{array}\right] \tag{9}
$$

Where this is a different form to the other basis for 3 qubits Eq. 6.

A.3 4 Qubit CG coefficients

The J=2 block, P=0000, (J=1/2, J=1, J=3/2, J=2)

$$
\begin{aligned}
|J=2, M=+2, P=0000\rangle &= |0000\rangle \\
|J=2, M=+1, P=0000\rangle &= \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \\
|J=2, M=\ \ 0, P=0000\rangle &= \sqrt{\frac{1}{6}}(|0011\rangle + |0101\rangle + |1001\rangle + |1100\rangle + |1010\rangle + |0110\rangle) \\
|J=2, M=-1, P=0000\rangle &= \frac{1}{2}(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle) \\
|J=2, M=-2, P=0000\rangle &= |1111\rangle
\end{aligned} \tag{10a}
$$

The J=1 (0) block, P=0001, (J=1/2, J=1, J=3/2, J=1)

$$
\begin{aligned}
|J=1, M=+1, P=0001\rangle &= +\sqrt{\frac{3}{4}}|0001\rangle - \sqrt{\frac{1}{12}}(|0010\rangle + |0100\rangle + |1000\rangle) \\
|J=1, M=\ \ 0, P=0001\rangle &= \sqrt{\frac{1}{6}}(|0011\rangle + |0101\rangle + |1001\rangle - |1100\rangle - |1010\rangle - |0110\rangle) \\
|J=1, M=-1, P=0001\rangle &= -\sqrt{\frac{3}{4}}|1110\rangle + \sqrt{\frac{1}{12}}(|1101\rangle + |1011\rangle + |0111\rangle)
\end{aligned} \tag{10b}
$$

The J=1 (1) block, P=0010, (J=1/2, J=1, J=1/2, J=1)

$$
\begin{aligned}
|J=1, M=+1, P=0010\rangle &= +\sqrt{\frac{2}{3}}\,|0010\rangle - \sqrt{\frac{1}{6}}(|0100\rangle + |1000\rangle) \\
|J=1, M=\quad 0, P=0010\rangle &= \sqrt{\frac{1}{3}}(|0011\rangle - |1100\rangle) + \sqrt{\frac{1}{12}}(|0110\rangle + |1010\rangle - |0101\rangle - |1001\rangle) \\
|J=1, M=-1, P=0010\rangle &= -\sqrt{\frac{2}{3}}\,|1101\rangle + \sqrt{\frac{1}{6}}(|1011\rangle + |0111\rangle)
\end{aligned}
\tag{10c}
$$

The J=1 (2) block, P=0100, (J=1/2, J=0, J=1/2, J=1)

$$
\begin{aligned}
|J=1, M=+1, P=0100\rangle &= +\sqrt{\frac{1}{2}}(|0100\rangle - |1000\rangle) \\
|J=1, M=\quad 0, P=0100\rangle &= \frac{1}{2}(|0101\rangle - |1001\rangle + |0110\rangle - |1010\rangle) \\
|J=1, M=-1, P=0100\rangle &= -\sqrt{\frac{1}{2}}(|0111\rangle - |1011\rangle)
\end{aligned}
\tag{10d}
$$

The J=0 block, P=0011, (J=1/2, J=1, J=1/2, J=0)

$$
|J=0, M=0, P=0011\rangle = \sqrt{\frac{1}{3}}(|0011\rangle + |1100\rangle) - \sqrt{\frac{1}{12}}(|0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle) \tag{10e}
$$

The J=0 block, P=0101, (J=1/2, J=0, J=1/2, J=0)

$$
|J=0, M=0, P=0101\rangle = \frac{1}{2}(|0101\rangle - |1001\rangle - |0110\rangle + |1010\rangle) \tag{10f}
$$

A.4 Rotation matrix for J & M values

J=0 values

$$
\begin{aligned}
\left|J=0, M'=+1/2\right\rangle &= R = I \\
\left|J=0, M'=-1/2\right\rangle &= R = XZ
\end{aligned}
\tag{11a}
$$

J=1/2 values

$$
\begin{aligned}
\left|J=1/2, M'=+1\right\rangle &= R = I \\
\left|J=1/2, M'=\quad 0\right\rangle &= R = XH \\
\left|J=1/2, M'=-1\right\rangle &= R = XZ
\end{aligned}
\tag{11b}
$$

J=1 values

$$
\begin{aligned}
\left|J=1, M'=+3/2\right\rangle &= R = I \\
\left|J=1, M'=+1/2\right\rangle &= R = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} & -1 \\ 1 & \sqrt{2} \end{bmatrix} \\
\left|J=1, M'=-1/2\right\rangle &= R = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix} \\
\left|J=1, M'=-3/2\right\rangle &= R = XZ
\end{aligned}
\tag{11c}
$$

J=3/2 values

$$
\begin{aligned}
\left|J=3/2, M'=+2\right\rangle &= R = I \\
\left|J=3/2, M'=+1\right\rangle &= R = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \\
\left|J=3/2, M'=\ \ 0\right\rangle &= R = XH \\
\left|J=3/2, M'=-1\right\rangle &= R = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \\
\left|J=3/2, M'=-2\right\rangle &= R = XZ
\end{aligned}
\tag{11d}
$$

J=2 values

$$
\begin{aligned}
\left|J=2, M'=+5/2\right\rangle &= R = I \\
\left|J=2, M'=+3/2\right\rangle &= R = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \\
\left|J=2, M'=+1/2\right\rangle &= R = \frac{1}{\sqrt{5}}\begin{bmatrix} \sqrt{3} & -\sqrt{2} \\ \sqrt{2} & \sqrt{3} \end{bmatrix} \\
\left|J=2, M'=-1/2\right\rangle &= R = \frac{1}{\sqrt{5}}\begin{bmatrix} \sqrt{2} & -\sqrt{3} \\ \sqrt{3} & \sqrt{2} \end{bmatrix} \\
\left|J=2, M'=-3/2\right\rangle &= R = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \\
\left|J=2, M'=-5/2\right\rangle &= R = XZ
\end{aligned}
\tag{11e}
$$

A Appendix- Circuits

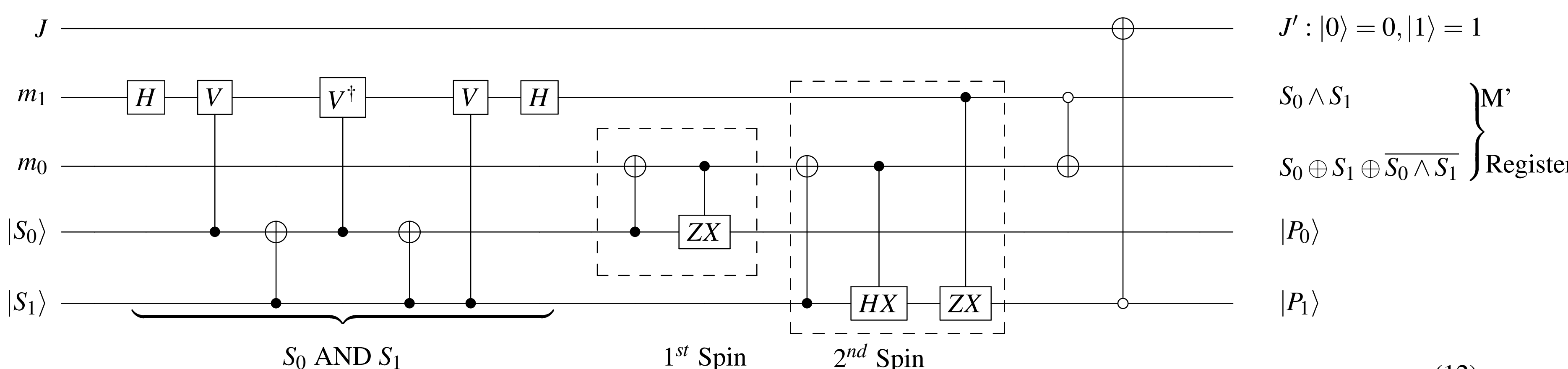
The case where $|s\rangle = |0\rangle$ means the spin is $+\frac{1}{2}$ so to add $\frac{1}{2}$ to M, 1 is added to the m_0 qubit. The first Quantum Adder (QAdd) uses Toffoli gates controlled on $|s\rangle = |0\rangle$ (denoted by the white control circle) with the current m_0 value and C_0 (an ancilla carry). This ensures that the case when $m_0 = 1$ and 1 is added to it, m_0 goes to 0 and m_1 is increased using the carry as $001+1=010$. The rest of the QAdd stages then just check the carry of the previous qubit to complete to M $+\frac{1}{2}$ addition as $|S\rangle = |0\rangle$ does not trigger any of the rest of the control gates.

The case where $|S\rangle = |1\rangle$ means the spin is $-\frac{1}{2}$ we do M $-\frac{1}{2}$ which is done by adding the binary string for $-\frac{1}{2}$ which is the all 1's string, 111. This time the very first QAdd does not trigger and $|s\rangle$ is then added to all of the bits of M using C-NOT gates with carries to check for overflow.

The Unitary is then performed on $|S\rangle$ depending on the values of the newly calculated M' and J registers using $R_y(\theta_{J,m'})$ Eq. 1. The J register is then updated to J' by adding the value of $|P\rangle$ to J using the QAdd sequence of gates.

To add the second qubit in the values of J' and M' are passed in as the initial register values. It is easy to extend this to many qubits being streamed in one at a time by carefully conditioning the controls on the unitaries, I think in the general case you need at most N controls for coupling up to N qubits in one at a time. The circuit written here has redundancy in the Identity and ZX gates appearing twice which is shown in the circuit for completenessFig. 7. egisters.

The registered circuit, which takes in two qubits can be constructed from 11 gates if the m register is not compressed. The spatially multiplexed minimal gate explicit J & M 2 qubit transform is shown in Fig. 5 which contains 11 two-qubit gates. Another CNOT can be added to compress the M' register, which means using the same values for M'=0, see Fig. 4 for the 12 two-qubit gate circuit.



(12)

Figure 4: Spatial multiplexed 2 qubit

HX gate triggers if $M = 0$ meaning $S_0 \neq S_1$ which is implemented using an XOR between S_0 & S_1. The other gate (T) is triggered when $M = -1$ meaning $S_0 = S_1 = 1$ which is done using an AND (Toffoli) gate between, $S_0 = S_1$ AND $S_1 = 1$ which is decomposed into 5 two-qubit gates. T^2 is the ZX gate, meaning $T^2 |S\rangle = XZ|S\rangle$.

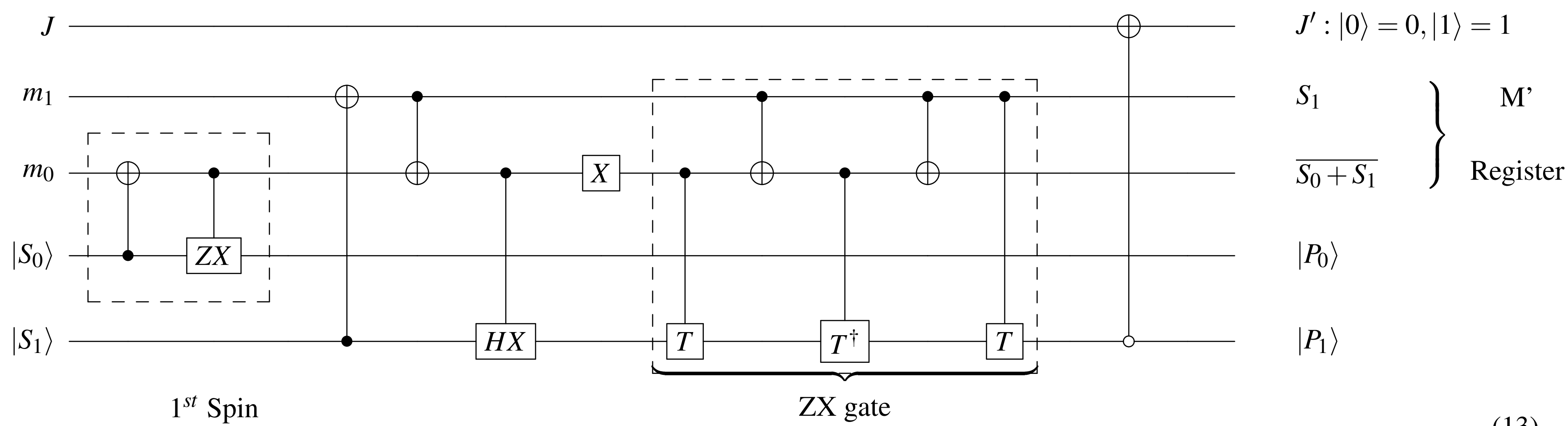
The registered circuit, which takes in two qubits can be constructed from 11 gates if the m register is not compressed. The spatially multiplexed minimal gate explicit J & M 2 qubit transform is shown in Fig. 5 which contains 11 two-qubit gates.

Spin values		Circuit output		M value
S_1	S_0	m_1	m_0	M
0	0	0	1	M=+1
0	1	0	0	M=0
1	0	0	0	M=0
1	1	1	0	M=-1

Table 2: Table giving M register decoding for 2 qubit spatial multiplexing

Another CNOT can be added to compress the M' register, which means using the same values for M'=0, see Fig. 4 for the 12 two-qubit gate circuit.

Where V is the phase gate, $V = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$, $V^\dagger V = I$ and $V^2 = Z$. V is used here to expand the double controlled Toffoli gate into single



(13)

Figure 5: minimal gate spatial multiplexing

Spin values		Circuit output		M value
S_1	S_0	S_1	$\overline{S_0+S_1}$	M
0	0	0	1	M=+1
0	1	0	0	M=0
1	0	1	0	M=0
1	1	1	1	M=-1

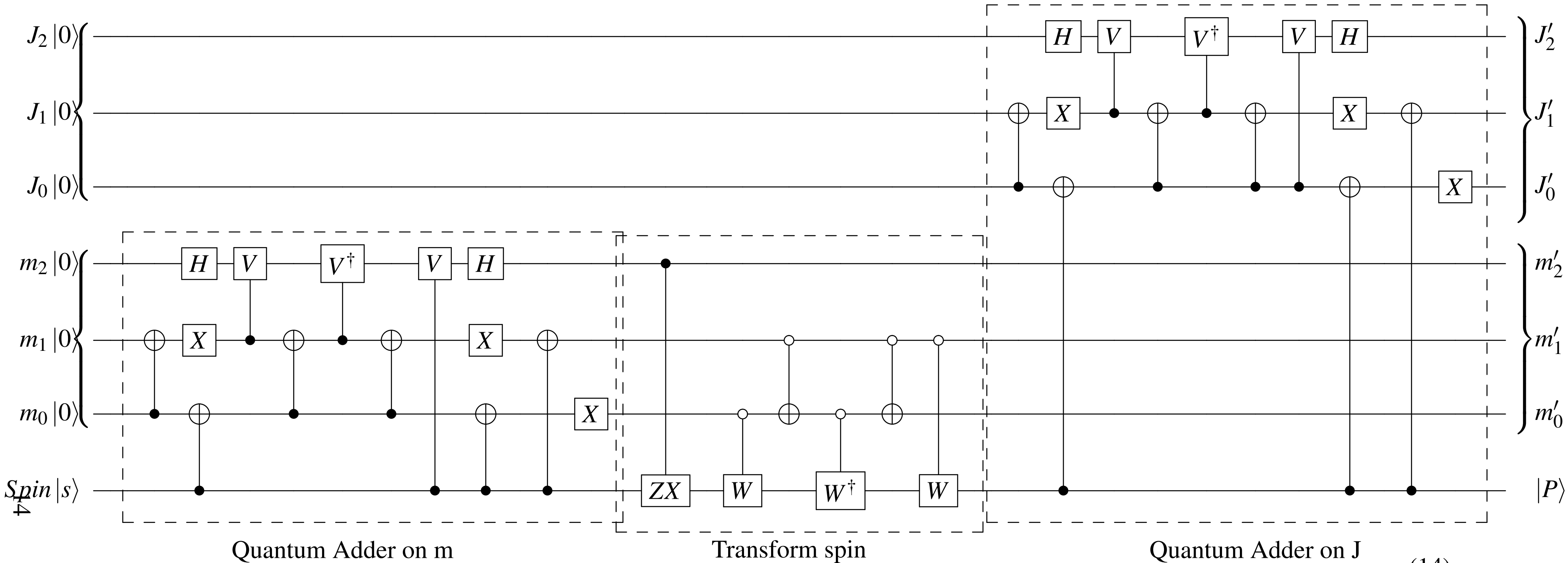
Table 3: Table giving M register decoding for minimal gate number

control gates in the quantum adder subroutine.

The W gate, $W^2 = HX$ with $W^\dagger = I$, is used to expand the HX gate into single control gates in the spin transform region.

The circuit checks that if (m_1 XNOR m_0) AND (m_0 XOR S_0) and will then change m_2. Then m_1 is updated using $m_1 = m_0$ XOR S_0. m_0 is always incremented by 1, if $|S\rangle = |0\rangle$ increment only m_0 by 1 corresponding to adding $\frac{1}{2}$ to the M register. $|S\rangle = |1\rangle$ corresponds to subtracting $\frac{1}{2}$ from the M register by adding the string 111 bitwise to M.

For the most positive values of M the Identity is performed on the spin corresponding to the strings $M = 001(J = \frac{1}{2}, M' = \frac{1}{2})$ for the first spin and $M = 010(J = 1, M' = 1)$ for the second coupled in spins.



(14)

Figure 6: temporal multiplexed streaming

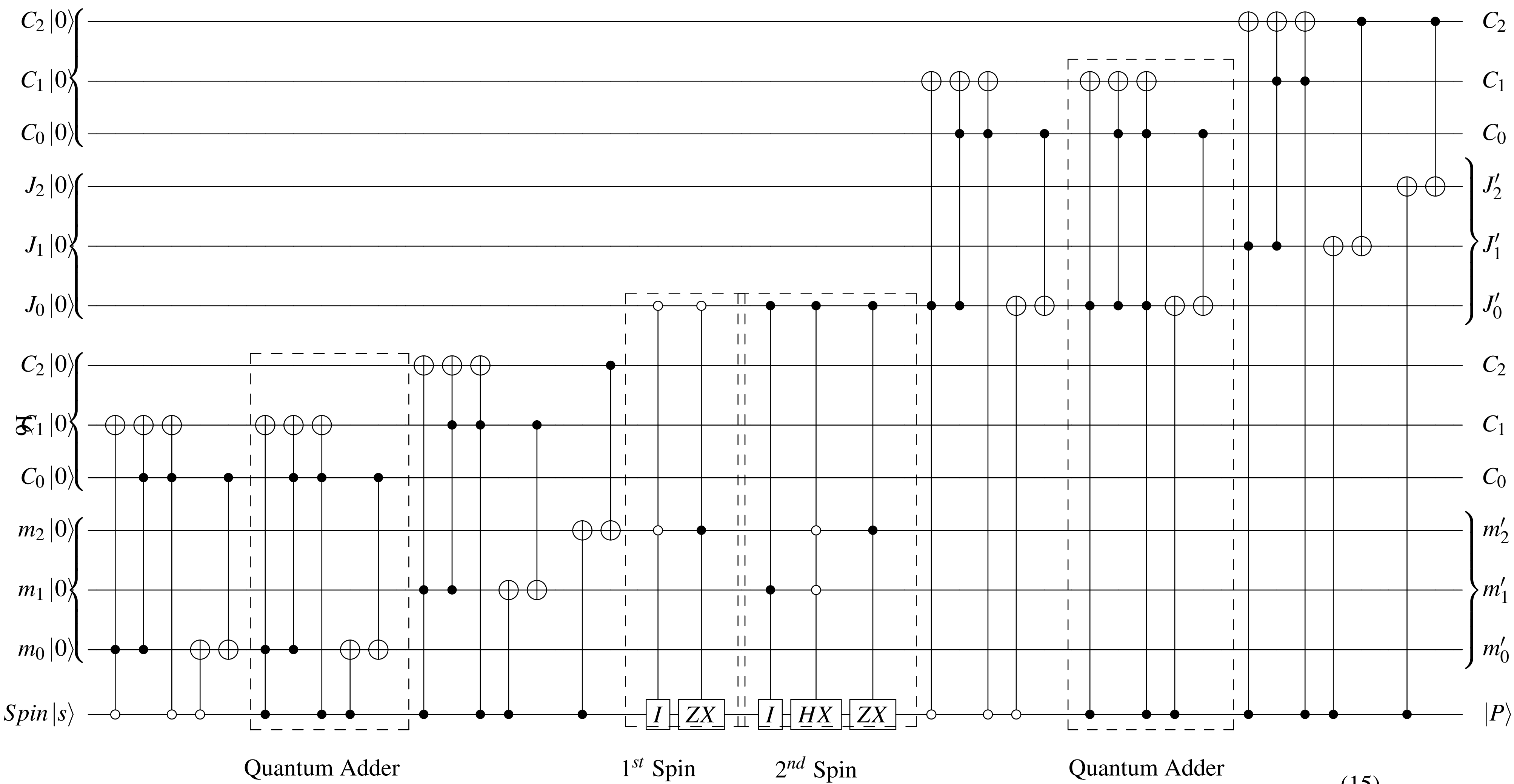
The most negative values of M performs $XZ\,|S\rangle$ corresponding to the strings $M = 111(J = \frac{1}{2}, M' = -\frac{1}{2})$ for the first spin and $M = 110(J = 1, M' = -1)$ for the second spin.

If $M = 000(J = 0, M' = 0)$ do $XH\,|S\rangle$ Fig. 6.

J_2	J_1	J_0	J
0	0	0	0
0	0	1	$\frac{1}{2}$
0	1	0	1
0	1	1	$\frac{3}{2}$
1	0	0	-2
1	0	1	$-\frac{3}{2}$
1	1	0	-1
1	1	1	$-\frac{1}{2}$

m_2	m_1	m_0	M
0	0	0	0
0	0	1	$\frac{1}{2}$
0	1	0	1
0	1	1	$\frac{3}{2}$
1	0	0	-2
1	0	1	$-\frac{3}{2}$
1	1	0	-1
1	1	1	$-\frac{1}{2}$

Table 4: Tables giving binary Two's complement encoding to spin values of the M and J registers



(15)

Figure 7: general streaming circuit

A Appendix- Fortran code

```
!Oliver Thomas 2018 Bristol
program matrixmul
implicit none

integer, parameter :: dp=selected_real_kind(15,300)
integer :: n, qubits, numofdecomp, i, j, counter
integer, allocatable, dimension(:) :: p, gatenum

real(kind=dp), parameter :: invr2=1/sqrt(real(2,kind=dp)), invr3=1/sqrt(real(3,kind=dp)), 
real(kind=dp), parameter :: r2=sqrt(real(2,kind=dp))

real(kind=dp), allocatable, dimension(:,:) :: unitary, ident, uprod
real(kind=dp), allocatable, dimension(:,:,:)  :: u, gateseq

counter=1
print*, 'Enter number of qubits, 2, 3 or 4'
read*, qubits
n= 2**qubits
numofdecomp=int(n*(n-1)/2.0_dp)

allocate(ident(n,n))
allocate(unitary(n,n))
allocate(u(n,n,n*n))
allocate(uprod(n,n))
allocate(p(n))
allocate(gateseq(n,n,n*n))
allocate(gatenum(numofdecomp))


ident=0.0_dp
unitary=0.0_dp
u=0.0_dp
gateseq=0.0_dp
gatenum=1

!#make identity
ident=identity(n)

!#make u's ident
do i=1, size(u,3)
  u(:,:,i)=identity(n)
end do
gateseq=u
```

```
!#init uprod as ident

!#make unitary
if (qubits==2) then
  p(1:n)=(/1,2,4,3/)

  unitary(1:n,1)=(/1.0_dp, 0.0_dp, 0.0_dp, 0.0_dp/)
  unitary(1:n,2)=(/0.0_dp, invr2, invr2, 0.0_dp/)
  unitary(1:n,3)=(/0.0_dp, 0.0_dp, 0.0_dp, 1.0_dp/)
  unitary(1:n,4)=(/0.0_dp, invr2, -invr2,0.0_dp/)

else if (qubits==3) then

  p(1:n)=(/1,2,4,3,7,8,6,5/)
!# col,row
  unitary(1,1)=1.0_dp

  unitary(2,2)=invr3
  unitary(2,5)=r2*invr3

  unitary(3,2)=invr3
  unitary(3,5)=-invr6
  unitary(3,7)=invr2

  unitary(4,3)=invr3
  unitary(4,6)=invr6
  unitary(4,8)=invr2

  unitary(5,2)=invr3
  unitary(5,5)=-invr6
  unitary(5,7)=-invr2

  unitary(6,3)=invr3
  unitary(6,6)=invr6
  unitary(6,8)=-invr2

  unitary(7,3)=invr3
  unitary(7,6)=-r2*invr3

  unitary(8,4)=1.0_dp

else if (qubits==4) then

  p(1:n)=(/1,2,4,3,7,8,6,5,13,9,10,12,11,15,16,14/)
  !# col,row
```

```
!0000
unitary(1,1)=1.0_dp
!0001
unitary(2,2)=0.5_dp
unitary(2,6)=sqrt(0.75_dp)
!0010
unitary(3,2)=0.5_dp
unitary(3,6)=-sqrt(1.0_dp/12.0_dp)
unitary(3,9)=sqrt(2.0_dp/3.0_dp)
!0011
unitary(4,3)=sqrt(1.0_dp/6.0_dp)
unitary(4,7)=sqrt(1.0_dp/6.0_dp)
unitary(4,10)=sqrt(1.0_dp/3.0_dp)
unitary(4,15)=sqrt(1.0_dp/3.0_dp)
!0100
unitary(5,2)=0.5_dp
unitary(5,6)=-sqrt(1.0_dp/12.0_dp)
unitary(5,9)=-sqrt(1.0_dp/6.0_dp)
unitary(5,12)=sqrt(1.0_dp/2.0_dp)
!0101
unitary(6,3)=sqrt(1.0_dp/6.0_dp)
unitary(6,7)=sqrt(1.0_dp/6.0_dp)
unitary(6,10)=-sqrt(1.0_dp/12.0_dp)
unitary(6,13)=0.5_dp
unitary(6,15)=-sqrt(1.0_dp/12.0_dp)
unitary(6,16)=0.5_dp
!0110
unitary(7,3)=sqrt(1.0_dp/6.0_dp)
unitary(7,7)=-sqrt(1.0_dp/6.0_dp)
unitary(7,10)=sqrt(1.0_dp/12.0_dp)
unitary(7,13)=0.5_dp
unitary(7,15)=-sqrt(1.0_dp/12.0_dp)
unitary(7,16)=-0.5_dp
!0111
unitary(8,4)=0.5_dp
unitary(8,8)=sqrt(1.0_dp/12.0_dp)
unitary(8,11)=sqrt(1.0_dp/6.0_dp)
unitary(8,14)=-sqrt(0.5_dp)
!1000
unitary(9,2)=0.5_dp
unitary(9,6)=-sqrt(1.0_dp/12.0_dp)
unitary(9,9)=-sqrt(1.0_dp/6.0_dp)
unitary(9,12)=-sqrt(0.5_dp)
!1001
unitary(10,3)=sqrt(1.0_dp/6.0_dp)
```

```
  unitary(10,7)=sqrt(1.0_dp/6.0_dp)
  unitary(10,10)=-sqrt(1.0_dp/12.0_dp)
  unitary(10,13)=-0.5_dp
  unitary(10,15)=-sqrt(1.0_dp/12.0_dp)
  unitary(10,16)=-0.5_dp
  !1010
  unitary(11,3)=sqrt(1.0_dp/6.0_dp)
  unitary(11,7)=-sqrt(1.0_dp/6.0_dp)
  unitary(11,10)=sqrt(1.0_dp/12.0_dp)
  unitary(11,13)=-0.5_dp
  unitary(11,15)=-sqrt(1.0_dp/12.0_dp)
  unitary(11,16)=0.5_dp
  !1011
  unitary(12,4)=0.5_dp
  unitary(12,8)=sqrt(1.0_dp/12.0_dp)
  unitary(12,11)=sqrt(1.0_dp/6.0_dp)
  unitary(12,14)=sqrt(0.5)
  !1100
  unitary(13,3)=sqrt(1.0_dp/6.0_dp)
  unitary(13,7)=-sqrt(1.0_dp/6.0_dp)
  unitary(13,10)=-sqrt(1.0_dp/3.0_dp)
  unitary(13,15)=sqrt(1.0_dp/3.0_dp)
  !1101
  unitary(14,4)=0.5_dp
  unitary(14,8)=sqrt(1.0_dp/12.0_dp)
  unitary(14,11)=-sqrt(2.0_dp/3.0_dp)
  !1110
  unitary(15,4)=0.5_dp
  unitary(15,8)=-sqrt(3.0_dp/4.0_dp)
  !1111
  unitary(16,5)=1.0_dp

end if

21 format ( 16F7.3)
write(*,21) unitary
!#!!! make unitary gates

print*,
write(*,21) matmul(unitary,transpose(unitary))

uprod=unitary
do i=1,n !#col
  do j=1,n-1 !#row
    if(p(n-j+1).ne.p(i)) then
```

```
      call makeunitary(p(n-j),p(n-j+1),p(i), uprod, u(:,:,(i-1)*n+j))
    end if
  end do
end do

print*,
print*, 'unitaries'

uprod=unitary
do i=1, n*n
  if (icheck(u(:,:,i))==0) then
    uprod=matmul(uprod(:,:),u(:,:,i))
  end if
end do

  print*, 'THIS IS UNITARY'
  call invert(u,gateseq,counter)
  call gateset(gateseq)

do i=1, counter
  uprod=matmul(uprod,gateseq(:,:,i))
end do

print*, '-----------------------------------------------------------------'
print*, 'Unitary matrix from', counter-1, 'gates'
print*, '-----------------------------------------------------------------'

print*,

deallocate(ident)
deallocate(unitary)
deallocate(u)
deallocate(uprod)
deallocate(p)
deallocate(gateseq)
deallocate(gatenum)

!#!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!!
contains

!# print non identity elements
subroutine gateset(matrix)
  real(kind=dp), dimension(:,:,:), intent(in) :: matrix
  integer :: n, i
  n=size(matrix,3)
```

```
  do i=1, n
    if (icheck(matrix(:,:,i))==0) then
    end if
  end do
end subroutine gateset

!# transpose and invert array
subroutine invert(matrix,inverted,count)
  real(kind=dp), dimension(:,:,:), intent(inout) :: inverted
  real(kind=dp), dimension(:,:,:), intent(in) :: matrix
  integer :: i, n, count
  n=size(matrix,3)
  count=1
  do i=1,n
    if (icheck(matrix(:,:,n-i+1))==0) then
      inverted(:,:,count) = transpose(matrix(:,:,n-i+1))
      count=count+1
    end if
  end do

end subroutine invert

!#!!!! Check product gives identity
function icheck(uni)
  real(kind=dp) :: icheck
  real(kind=dp), dimension(:,:), intent(in) :: uni
  integer :: i, j

icheck=1
!#check ident
itest:do i=1, size(uni,1)
  do j=1, size(uni,1)
    if (i.ne.j) then
      if (abs(uni(i,j))>=1e-10) then
        icheck=0
        exit itest
      end if
    else if (i.eq.j) then
      if (abs(abs(uni(i,j))-1)>=1e-10) then
        icheck=0
        exit itest
      end if
    end if
  end do
end do itest
```

```
end function icheck

!#!!!! Check product gives identity
function unitarycheck(umatrices, uni)
  real(kind=dp) :: unitarycheck
  real(kind=dp), dimension(:,:,:), intent(in) :: umatrices
  real(kind=dp), dimension(:,:), intent(in) :: uni
  real(kind=dp), dimension(:,:), allocatable :: uprod
  integer :: i, j

unitarycheck=1
uprod=uni
!#do u_n*u_n-1*...*u1*Unitary=Ident
do i=1, size(umatrices,3)-1
  if (icheck(umatrices(:,:,n*n-i))==0) then
    uprod=matmul(uprod(:,:), umatrices(:,:,i))
  end if
end do

unitarycheck=icheck(uprod)

end function unitarycheck

!#!!! find type of u
subroutine makeunitary(row1,row2,col,ucurrent, ugate)
  real(kind=dp) :: c,s,r
  real(kind=dp), dimension(:,:), intent(inout) :: ucurrent, ugate
  integer, intent(in) :: row1, row2, col

  ugate=identity(size(ucurrent,1))
  c=0.0
  s=0.0
  r=0.0

call givensrot(ucurrent(col,row1), ucurrent(col,row2), c,s,r)
  ugate(row1,row1)=c
  ugate(row1,row2)=-s
  ugate(row2,row1)=s
  ugate(row2,row2)=c
ucurrent=matmul(ucurrent,ugate)
end subroutine makeunitary

!#!!! Calc givens rotation
subroutine givensrot(a, b, c, s, r)
  real(kind=dp) :: a, b, c, s, r, h, d
```

```
h=0.0
d=0.0
 !#write(*,*) 'a',a,'b',b,'c',c,'s',s
if (abs(b)>=1e-1) then
  h=hypot(a,b)
  d=1.0_dp/h
  c=abs(a)*d
  s=sign(d,a)*b
  r=sign(1.0_dp,a)*h
else
  c=1.0_dp
  s=0.0_dp
  r=a

end if
end subroutine givensrot


!#!!!!! make identiy matrix dim n
function identity(n)
  real(kind=dp), dimension(n,n) :: identity
  integer :: n, i

identity=0.0_dp
do i=1, n
  identity(i,i) =1.0_dp
end do
end function identity

end program matrixmul
```