

Generalized parity in multi-photon Rabi model

Bartłomiej Gardas¹ and Jerzy Dajka^{a)}

*Institute of Physics, University of Silesia, PL-40-007 Katowice,
Poland^{b)}*

Quantum multi-photon spin-boson model is considered. We solve an operator Riccati equation associated with that model and present a candidate for a generalized parity operator allowing to transform spin-boson Hamiltonian to a block diagonal form what indicates an existence of the related symmetry of the model.

PACS numbers: 03.65.Yz, 03.67.-a

^{a)}Electronic mail: jerzy.dajka@us.edu.pl

^{b)}Electronic mail: bartek.gardas@gmail.com

I. INTRODUCTION

Phenomenological modeling of interacting matter and (quantized) light in physics has a long and interesting history¹. It is of quantum optical origin² but is present in wide range of other branches of physics such as condensed matter³ or involving mechanical oscillators⁴. The Rabi model⁵, describing a qubit coupled to a single mode electromagnetic field, is the one which has attracted continuous attention for almost a century. More⁶ and less⁷ recent studies on its integrability have inspired increasingly growing research.

An existence of a symmetry of any quantum model is directly related to a quality of our understanding of its properties⁸. A ‘sufficient’ (in certain sense) symmetry can result in an integrability of the model⁹. That is why seeking for any underlying symmetry of quantum models is always of great interest and often of great importance. In this paper we present our contribution to this activity. We consider a family of generalized single-mode Rabi models¹⁰:

$$H = \alpha\sigma_z + \omega a^\dagger a + \sigma_x (g^* a^k + g(a^\dagger)^k), \quad (1)$$

where σ_z and σ_x are the Pauli matrices, α and ω correspond to the energy gap of the spin and boson, respectively, whereas a and a^\dagger are the annihilation and creation operators of quantized mode of light satisfying canonical commutation relation, $[a, a^\dagger] = \mathbb{I}$. It is assumed that the coupling between the qubit and the field, controlled by the strength constant g , incorporates $k > 0$ photons.

In this paper, by solving an operator Riccati equation associated with Eq. (1), we construct an operator exhibiting significant similarities to the parity operator acting on the bosonic space. This operator, the generalized parity, can be used to simplify multi-photon Rabi model (1) and transform it to a block-diagonal form. Our work is a complementary expansion of certain results obtained in Ref.¹⁰ for $k = 1$ and $k = 2$ in the context of approximate methods of solving the Rabi model.

The paper is organized as follows: In Sec. II we present operator Riccati equation associated with (1) serving as a main tool applied in our studies. Next, in Sec. III the known results concerning the $k = 1, 2$ cases are reviewed. Sec. IV has been devoted to the construction of the generalized parity and contains main results of our work. Finally, in Sec. V, followed by conclusions, we apply the general construction to a simple example.

II. A TOOL: RICCATI EQUATION

Multi-photon Rabi model considered here belongs to a general class of qubit–environment composite systems described by Hamiltonian

$$H_{QE} = H_Q \otimes \mathbb{I}_E + \mathbb{I}_Q \otimes H_E + H_{\text{int}} \sim \begin{bmatrix} H_+ & V \\ V^\dagger & H_- \end{bmatrix} \equiv \mathbf{H}_{QE}, \quad (2)$$

where H_Q (H_E) is the Hamiltonian of the system (environment). H_{int} is the interaction of the qubit with its surroundings. \mathbb{I}_Q and \mathbb{I}_E are identities acting on corresponding Hilbert spaces \mathbb{C}^2 and \mathcal{H}_E . The total Hamiltonian H_{QE} acts on $\mathbb{C}^2 \otimes \mathcal{H}_E$ and the symbol \sim should be understood as ‘*it corresponds to*’ in the sense of block operator matrix representation of operators. This correspondence is established via the isomorphism $\mathbb{C}^2 \otimes \mathcal{H}_E \sim \mathcal{H}_E \oplus \mathcal{H}_E$. Finally, the form of remaining operators H_\pm and V depends upon how H_Q , H_E and H_{int} are defined.

Any steps toward diagonalization of \mathbf{H}_{QE} is valuable as it can be followed by variety of different approximation schemes¹⁰. There is often an additional benefit emerging from such transformations which can help to exhibit useful symmetry properties being often obscured by an ‘improper choice’ of a basis. As it is pointed out below it is also the case of the multi-photon Rabi model (1) discussed in this paper.

Our idea originates from an observation that Hamiltonian \mathbf{H}_{QE} can be converted to a block-diagonal form

$$\mathbf{S}^{-1} \mathbf{H}_{QE} \mathbf{S} = \begin{bmatrix} H_+ + VX & 0 \\ 0 & H_- - (VX)^\dagger \end{bmatrix}, \quad \text{with} \quad \mathbf{S} = \begin{bmatrix} \mathbb{I}_E & -X^\dagger \\ X & \mathbb{I}_E \end{bmatrix}, \quad (3)$$

provided that X satisfies an operator Riccati equation

$$XVX + XH_+ - H_-X - V^\dagger = 0. \quad (4)$$

For general considerations regarding an operator Riccati equation we refer the reader to^{11,12}. This equation provides valuable tool allowing to study the exact diagonalization¹³, stationary states¹⁴ and in general, the dynamics¹⁵ of two level open quantum systems¹⁶. From the decomposition (3) it is evident that the dynamics of a qubit–environment quantum system is actually governed by the Riccati (4) and pair of *uncoupled* Schrödinger equations.

For the k -photon Rabi model studied in our paper

$$H_{\pm} = \omega a^{\dagger} a \pm (g^* a^k + g(a^{\dagger})^k), \quad V = \alpha \mathbb{I}_{\mathcal{H}_B} \quad (5)$$

and the corresponding Riccati equation reads as follows:

$$\alpha X^2 + XH_+ - H_-X - \alpha = 0. \quad (6)$$

Its mathematical properties has already been addressed in literature¹⁷. In Eq. (6), H_{\pm} are operators acting on the bosonic Fock space \mathcal{H}_B , α is a real constant, whereas X is a solution to be found. If it does not lead to a confusion, we write α rather than $\alpha \mathbb{I}_{\mathcal{H}_B}$, with $\mathbb{I}_{\mathcal{H}_B}$ being the identity on \mathcal{H}_B .

III. KNOWN SOLUTIONS: $k = 1, 2$.

For the sake of self-consistency, we begin with reviewing known solutions and their properties for the two particular cases, where $k = 1, 2$. For the simplest possible case, $k = 1$ the solution of the Riccati equation (6) was found in¹³ to be the bosonic parity operator

$$P = \sum_{n \in \mathbb{N}} e^{i\pi n} |n\rangle \langle n| = \sum_{n \in \mathbb{N}} (-1)^n |n\rangle \langle n|, \quad (7)$$

which can also be written in a more compact form as $P = \exp(i\pi a^{\dagger} a)$, where $\{|n\rangle\}_{n \in \mathbb{N}}$ is the Fock basis, *i.e.*, $a^{\dagger} a |n\rangle = n |n\rangle$. Such operator is both hermitian and unitary, hence it is an involution ($P^2 = \mathbb{I}_{\mathcal{H}_B}$). Interestingly, it solves Eq. (6) for both $\alpha = 0$ (dephasing¹⁸) and $\alpha \neq 0$ (exchange energy between the systems is present) cases, although they reflect quite different physical processes.

In the context of RWA-type approximation methods the two-photon ($k = 2$) Rabi model was studied in details within¹⁰. The two-photon parity operator

$$T := \exp \left[i \frac{\pi}{2} a^{\dagger} a (a^{\dagger} a - 1) \right], \quad (8)$$

was introduced therein. It has not been stated explicitly in¹⁰ but the parity T is, as will be shown below, a solution of the Riccati equation (6) for $k = 2$.

IV. GENERAL CASE: $k > 0$

In what follows we show how to construct a solution of the Riccati equation (6) with coefficients H_{\pm} provided by (5) in the general case $k > 0$. Before we start let us emphasize

that the parity operator P (T) introduced in the preceding section solves Eq. (6) not only for $k = 1$ ($k = 2$) but also for all odd $k = 2n + 1$ (even, of the form $k = 2n + 4$) cases. This has already been noticed in¹⁰. Here we will not only fill the remaining gap $k = 2n + 2$ but also present unified approach allowing to obtain a linear solution for arbitrary k . As a first step toward constructing this solution, we define a family of orthogonal projectors

$$P_l := \sum_{n=0}^{\infty} |n, l\rangle \langle n, l|, \quad \text{with} \quad |n, l\rangle := |kn + l - 1\rangle, \quad (9)$$

for $n \in \mathbb{N}$ and $1 \leq l \leq k$. The states $|n, l\rangle$ satisfy the following orthogonality condition:

$$\langle i, l | j, l \rangle = \delta_{kn+i-1, km+j-1} = \delta_{ij} \delta_{nm}, \quad (10)$$

where δ_{xy} is the Kronecker delta. The first equality in Eq. (10) comes from the orthogonality of the Fock basis. The second one can be justified as follows.

When $i = j$ both sides of (10) reduce to δ_{nm} since $\delta_{kn+i-1, km+i-1} = \delta_{nm}$. If $i \neq j$ (say $i > j$) the right hand side is zero. The left hand side also vanishes, as one gets either $m = n$ or $m \neq n$ in this case. Indeed, if $m = n$ then, to get nonzero left hand side, one would expect $i - j = 0$ what is impossible. Finally, for $m \neq n$ ($m > n$, say) one would expect that $k(m - n) = j - i$ (in order to keep the left side nonzero) what also does not occur as $k > i - j$ and it cannot divide $i - j$.

For a given family of orthogonal projectors one can split the space \mathcal{H}_B into k subspaces so that

$$\mathcal{H}_B = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_{k-1} \oplus \mathcal{H}_k = \bigoplus_{l=1}^k \mathcal{H}_l, \quad (11)$$

where $\mathcal{H}_l := P_l(\mathcal{H}_B)$. The symbol \oplus indicates the (orthogonal) direct sum of Hilbert spaces. Hereafter, we use it interchangeably with $+$ when it refers to the sum of operators.

The decomposition (11) allows us to think of operators H_{\pm} as of $k \times k$ block operator matrices $[H_{\pm}^{lm}]_{k \times k}$ such that

$$H_{\pm}^{lm} := P_l(H_{\pm})P_m : \mathcal{H}_l \rightarrow \mathcal{H}_m. \quad (12)$$

Obviously, H_{\pm}^{ll} act within the space \mathcal{H}_l , and they may be considered as operators H_{\pm} restricted to the space \mathcal{H}_l , *i.e.*, $H_{\pm}^{ll} := (H_{\pm})|_{\mathcal{H}_l}$. The off-diagonal elements H_{\pm}^{lm} act between the subspaces \mathcal{H}_l , \mathcal{H}_m and therefore transform state from one space into the other. In this simple picture instead of speaking of compositions, sums and any other operations involving

two or more operators acting on \mathcal{H}_B , we operate with corresponding $k \times k$ matrices. The block operator matrix representation of H_\pm is useful provided that it has a relatively simple form. As an example, an occurrence of a block diagonal structure ($H_\pm^{lm} = \delta_{lm} H_\pm^{(l)}$) would be an ideal situation. As it will be shortly seen this is indeed the case here.

Since (10) holds, it is a matter of straightforward calculations to show that $\langle i, l | a^\dagger a | j, m \rangle = \xi_{jm} \delta_{ij} \delta_{lm}$ and $\langle i, l | a^k | j, m \rangle = \eta_{jm} \delta_{i, j-1} \delta_{lm}$, for a given k , where

$$\xi_{jm} = kj + m - 1 \quad \text{and} \quad \eta_{jm} = \sqrt{\frac{(kj + m - 1)!}{[k(j - 1) + m - 1]!}}. \quad (13)$$

By making use of the above equations we obtain

$$\begin{aligned} P_l(a^\dagger a)P_m &= \sum_{i,j=0}^{\infty} (\langle i, l | a^\dagger a | j, m \rangle) |i, l\rangle \langle j, m| = \left(\sum_{j=0}^{\infty} \xi_{jm} |j, l\rangle \langle j, m| \right) \delta_{lm} \\ &\equiv (kn_l + (l - 1)\mathbb{I}_{\mathcal{H}_l}) \delta_{lm} \equiv N_l \delta_{lm}, \end{aligned} \quad (14)$$

where we have introduced N_l - the number operator restricted to the subspace \mathcal{H}_l , $N_l := (a^\dagger a)_{|\mathcal{H}_l}$:

$$N_l = kn_l + (l - 1)\mathbb{I}_{\mathcal{H}_l}, \quad n_l = \sum_{n=0}^{\infty} n |n, l\rangle \langle n, l|, \quad (15)$$

with $\mathbb{I}_{\mathcal{H}_l}$ being the identity on \mathcal{H}_l . In a similar fashion we have

$$P_l(a^k)P_m = \sum_{i,j=0}^{\infty} (\langle i, l | a^k | j, m \rangle) |i, l\rangle \langle j, m| = \left(\sum_{j=0}^{\infty} \eta_{jm} |j - 1, l\rangle \langle j, m| \right) \delta_{lm}, \quad (16)$$

from which $A_l := \sum_{j=0}^{\infty} \eta_{jl} |j - 1, l\rangle \langle j, l|$ is nothing but $(a^k)_{|\mathcal{H}_l}$ and plays on \mathcal{H}_l a role of annihilation operator.

Combining all the above results into a single equation we finally obtain the block matrix representation of H_\pm with respect to the decomposition (11). It has a diagonal structure indeed:

$$H_\pm^{lm} = \left[\omega N_l \pm (g^* A_l + g A_l^\dagger) \right] \delta_{lm} \equiv H_\pm^{(l)} \delta_{lm}, \quad (17)$$

for $l, m \leq k$.

Having (17) in place, we can formulate the result of this paper:

Proposition 1. *The solution to the Riccati equation (6) is given by the generalized parity operator*

$$X_k = \sum_{l=1}^k \sum_{n=0}^{\infty} (-1)^n |n, l\rangle \langle n, l|, \quad \text{for } k > 0. \quad (18)$$

Proof. We begin by defining *partial parities* J_l to be

$$J_l := e^{i\pi n_l} = \sum_{n=0}^{\infty} (-1)^n |n, l\rangle \langle n, l|, \quad l \leq k. \quad (19)$$

It is justified to refer to J_l as a parity since it possesses all the desired properties required from the parity operator on \mathcal{H}_l , namely

$$J_l^2 = \mathbb{I}_{\mathcal{H}_l}, \quad [N_l, J_l] = 0, \quad \text{and} \quad J_l A_l J_l = -A_l. \quad (20)$$

As an immediate consequence of this conditions $J_l H_+^{(l)} J_l = H_-^{(l)}$, what in the block operator matrix terminology developed in the preceding section leads to

$$X_k H_+ X_k \sim \text{diag}[J_1 H_+^{(1)} J_1, \dots, J_k H_+^{(k)} J_k] \sim H_-, \quad (21)$$

or simply $X_k H_+ = H_- X_k$. Therefore, to prove (6) holds true, it is sufficient to show that X_k is an involution. It can be established in a following way:

$$X_k^2 \sim \text{diag}[J_1^2, \dots, J_k^2] \sim \bigoplus_{l=1}^k \mathbb{I}_{\mathcal{H}_l} = \mathbb{I}_{\mathcal{H}_B}. \quad (22)$$

Both in Eqs. (21) and (22) we have taken into account the correspondence $X_k \sim \text{diag}[J_1, \dots, J_k]$. \square

V. EXAMPLES

It is interesting to see how the solutions which have been found in the previous section can be used to recover the known results for $k = 1, 2$. When $k = 1$ there is only one subspace of \mathcal{H}_B , namely \mathcal{H}_B itself, and X_1 is equal to the bosonic parity operator (7). For $k = 2$ there are only two projection within the family of operators (9), *i.e.*,

$$P_1 = \sum_{n=0}^{\infty} |2n\rangle \langle 2n|, \quad P_2 = \sum_{n=0}^{\infty} |2n+1\rangle \langle 2n+1|, \quad (23)$$

or in a compact form $P_{1,2} = \frac{1}{2}(\mathbb{I}_{\mathcal{H}_B} \pm P)$, which according to (11) split the bosonic Hilbert space into two subspaces. The first one consists only of odd, while the second one of even Fock states:

$$\mathcal{H}_B = \text{span}\{|2n\rangle : n \in \mathbb{N}\} \oplus \text{span}\{|2n+1\rangle : n \in \mathbb{N}\}. \quad (24)$$

The block operator matrix representation of H_{\pm} with respect to (24) reads

$$H_{\pm} \sim \begin{bmatrix} H_{\pm}^{(1)} & 0 \\ 0 & H_{\pm}^{(2)} \end{bmatrix} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} H_{\mp}^{(1)} & 0 \\ 0 & H_{\mp}^{(2)} \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \sim H_{\mp}, \quad (25)$$

where diagonal entries are explicitly given by (17). To see that the two-photon parity T is indeed the same as X_2 one only needs to invoke a simple fact, $(-1)^{n(2n-1)} = (-1)^{n(2n+1)} = (-1)^n$, then

$$T = \sum_{n=0}^{\infty} (-1)^{\frac{n(n-1)}{2}} |n\rangle\langle n| = \sum_{n=0}^{\infty} (-1)^n (|2n\rangle\langle 2n| + |2n+1\rangle\langle 2n+1|). \quad (26)$$

Eq. (25) also allows us to identify $X_2 \sim \text{diag}[J_1, J_2]$.

VI. CONCLUSION

In summary, we have found the solution of the operator Riccati equation (6) associated with the multi-photon Rabi model (1) indicating certain symmetry of the original model. This solution is a natural candidate for a parity operator as it not only reduces to well known one¹³ and two-photon¹⁰ parities but also its properties are that of a typical parity operator.

We have also proved by an explicit construction that the solution of (6) with coefficients given by (5) exist for every $k > 0$. We have excluded from our analysis the case where $k = 0$ only because it does not reflect any relevant physical phenomena. Since there is no interaction between the systems they evolve in time separately.

The symmetry associated with the generalized parity Eq. (18) allows to convert Rabi model into a block-diagonal form. Such a transformation decouples the original qubit-boson eigenproblem into a pair of bosonic Schrödinger equations. Despite certain mathematical subtleties of the multi-photon Rabi models¹⁹ we hope that the results presented in this paper can serve as a starting point for useful approximation methods.

Finally, let us notice that any involution J for which $JH_+ = H_-J$ is a solution of (6) for $k > 0$. Unfortunately, the question whether all the solution to (6) are of that special kind still remains open.

VII. ACKNOWLEDGMENTS

This work was supported by the Polish Ministry of Science and Higher Education under project **Iuventus Plus**, No. 0135/IP3/2011/71 (B. G) and NCN Grant N202 052940 (J. D)

REFERENCES

- ¹H. Haken, *Light: Waves, photons, atoms* (North-Holland Pub. Co., 1985)
- ²V. Vedral, *Modern Foundations of Quantum Optics* (Imperial College Press, London, 2006)
- ³T. Holstein, *Annals of Physics* **8**, 325 (1959) I. Chiorescu, *et al*, *Nature (London)* **431**, 159 (2004) A. Wallraff, *et al*, *ibid.* **431**, 162 (2004) K. Hennessy, *et al*, *ibid.* **445**, 896 (2007)
- ⁴E. K. Irish and K. Schwab, *Phys. Rev. B* **68**, 155311 (Oct 2003) K. C. Schwab and M. L. Roukes, *Physics Today* **58**, 36 (2005)
- ⁵I. I. Rabi, *Phys. Rev.* **49**, 324 (Feb 1936) **51**, 652 (Apr 1937)
- ⁶D. Braak, *Phys. Rev. Lett.* **107**, 100401 (Aug 2011) F. A. Wolf, M. Kollar, and D. Braak, *Phys. Rev. A* **85**, 053817 (May 2012) Q.-H. Chen, C. Wang, S. He, T. Liu, and K.-L. Wang, **86**, 023822 (Aug 2012) V. V. Albert, *Phys. Rev. Lett.* **108**, 180401 (May 2012)
- ⁷M. Fannes, B. Nachtergaele, and A. Verbeure, *Commun. Math. Phys.* **114**, 537 (December 1988) H. Spohn, **123**, 277 (June 1989)
- ⁸E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press, New York, 1959)
- ⁹M. Olshanetsky and A. Perelomov, *Physics Reports* **94**, 313 (1983), ISSN 0370-1573
- ¹⁰V. V. Albert, G. D. Scholes, and P. Brumer, *Phys. Rev. A* **84**, 042110 (Oct 2011)
- ¹¹V. Kotsykin, K. A. Makarov, and A. K. Motovilov, *Contemporary Mathematics* **327**, 181 (2003) V. Adamjan, H. Langer, and C. Tretter, *J. Funct. Anal.* **179**, 448 (2001)
- ¹²A. I. Egoriv, *Riccati Equations* (Pensoft Publishers, Bulgaria, 2007)
- ¹³K. Fujii and H. Oike, *Int. J. Geom. Methods Mod. Phys.* **7**, 1437 (2010) B. Gardas, *J. Phys. A: Math. Theor.* **44**, 195301 (April 2011)
- ¹⁴B. Gardas and Z. Puchała, *J. Phys. A: Math. Theor.* **44**, 215306 (May 2011)
- ¹⁵B. Gardas, *Phys. Rev. A* **82**, 042115 (Oct 2010)
- ¹⁶H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002) R. Alicki and K. Lendi, *Quantum dynamical semigroups and applications 2en ed.*, Springer Lecture Notes in Physics 717 (Springer, Berlin, 2007)
- ¹⁷B. Gardas, *J. Math. Phys.* **51**, 062103 (June 2010) **52**, 042104 (April 2011)
- ¹⁸R. Alicki, *Open Sys. & Inf. Dyn.* **11**, 53 (March 2004) J. Dajka, M. Mierzejewski, J. Łuczka, and P. Hänggi, *Physica E* **42**, 374 (2010)
- ¹⁹C. F. Lo, K. L. Liu and K. M. Ng, *Europhys. Lett.* **42**, 1 (1998)