Modelling Challenge: Fluids on membranes

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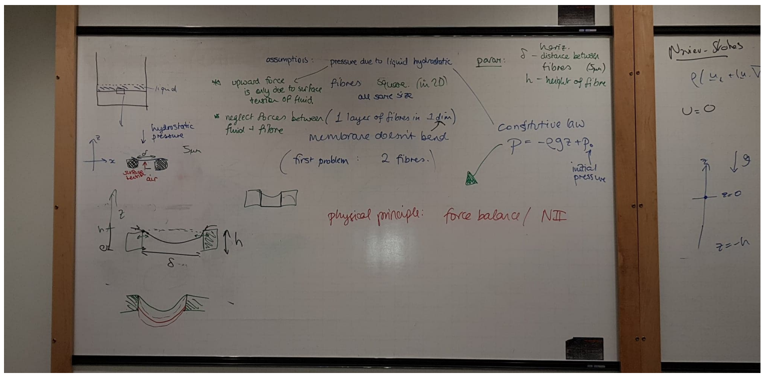

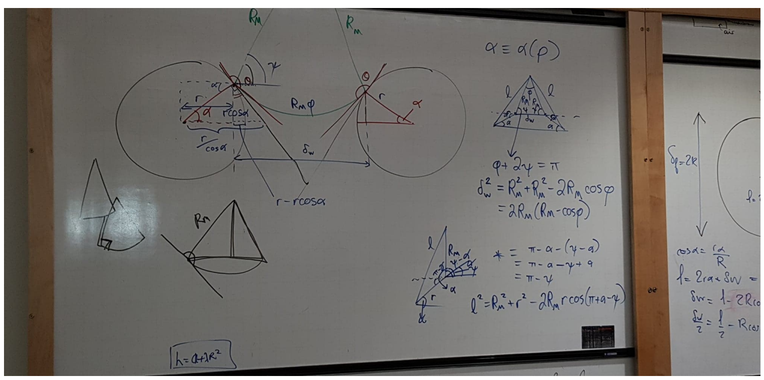

Assumptions

- The **pressure** due to the liquid is *hydrostatic* (so there is **no fluid flow**)
- The upward force is only due to the surface tension of fluid
- All the fibres are **square in cross-section** and of the *same size*
- There is only one layer of fibres and these are in 1 dimension
- The membrane **does not bend**
- The pressure is constant over the widthscale of the fibre
- The flow is incompressible and irrotational

Physical law

We consider **Newton's Second Law**, in other words, balancing forces.

At each point on the surface, pressure balance results in (at each point on the interface)

$$\rho g h = \kappa \gamma$$

where ρ is the density of water, κ is the curvature of the interface, γ is a constant, and $h = h(x)$ is the vertical distance of the meniuscus at point x from the bottom edge of one of the fibres. In particular (from page 13 of the *Topics in Fluids* notes), κ is the curvature of the meniscus, given by

$$\kappa = \frac{h_{xx}}{(1 + h_x^2)^{\frac{3}{2}}}.$$

However, in the simplest possible case we can assume that this is a constant, that is, that the meniuscus is a circular arc. So we can use $\kappa = \dfrac{1}{R_M}$ where R_M is the radius of this circular arc. All that remains is to find R_M in terms of the other parameters, which can be done using trigonometry.

Constitutive law

We want to find the steady state of the membrane. To start with we assume that the pressure is hydrostatic, i.e. $h = h(x)$.

Variables

- α is the angle (in *radians*) between each point of contact of the meniscus and fibre, and the horizontal axis pointing towards the middle of the two fibres. If these are different for each of the fibres, we shall call these α_1 and α_2.
- R_M is the (signed) radius of curvature of the meniuscus (in *metres*) when it is in its steady state. If $R_M > 0$ then the meniscus is $\cup$-shaped; if $R_M < 0$ then the meniscus is $\cap$-shaped.

Parameters

- d_w is the distance between the fibres (in *metres*)
- R is the radius of each of the fibres (in *metres*). If the radii of the two fibres are different, we shall call these r_1 and r_2.
- θ is the angle of contact (in *radians*) between the meniscus and the fibre (i.e. if there is a point where the meniscus touches the fibre, this is the angle between the tangent to the circle at that point, and the tangent of the meniscus at that point).
- ℓ is the distance (in *metres*) between the centres of two fibres.

Constitutive law

We want to find the steady state of the membrane. To start with we assume that this is a circular arc, i.e. R_M.

Solution

We can use trigonometry to derive the formula

$$R_M = \frac{\dfrac{\ell}{2} - R\cos\alpha}{-\cos(\alpha - \theta)}.$$

This specifies the function R_M in terms of α.

There is a value of α, which we will call α_{crit}, for which the curvature changes sign (so the meniscus changes from being $\cap$-shaped to $\cup$-shaped). In between these cases, the meniscus is completely flat, which happens as $|R_M| \to \infty$. This happens when $\cos(\alpha - \theta) = 0$, i.e. when $\alpha - \theta = \frac{\pi}{2}$

so

$$\alpha_{\text{crit}} = \frac{\pi}{2} + \theta.$$