[IrisToolbox] for Macroeconomic Modeling

Simulating nonlinear models

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System of nonlinear equations with model-consistent expectations

System of n nonlinear conditional-expectations equations

$$\mathrm{E}_t \Big[f_1 \big(x_{t-1}, x_t, x_{t+1}, \epsilon_t \mid \theta \big) \Big] = 0$$

$$\vdots$$

$$\mathrm{E}_t \Big[f_n \big(x_{t-1}, x_t, x_{t+1}, \epsilon_t \mid \theta \big) \Big] = 0$$

- Vector of n variables: $x_t = \left[x_t^1, \ldots, x_t^n \right]'$
- Vector of s shocks: $\epsilon_t = \left[\epsilon_t^1, \ldots, \epsilon_t^s \right]'$
- Vector of p parameters: $\theta_t = \left[\theta_t^1, \ldots, \theta_t^p \right]'$
- Conditional expectations of shocks: $\mathrm{E}_{t-1}[x_t] = \mathrm{E}_{t-2}[x_t] = \cdots = 0$
- Conditional higher moments: $\mathrm{E}_{t-1}[\epsilon_t \, \epsilon_t{}'] = \mathrm{E}_{t-2}[\epsilon_t \, \epsilon_t{}'] = \cdots = \Omega, \ldots$

Methods for nonlinear simulations

Characteristics	Local approximation	Global approximation	Stacked time
Solution form	Function	Function	Sequence
Explicit terminal	✖	✖	✔
Global nonlinearities	✖	✔	✔
Stochastic nonlinearities	✔	✔	✖
Automated design	✔	✖	✔
Large scale models	✔	✖	✔
Computational load	Increasing	Large	Manageable

Local approximation methods

Non-stochastic steady state

- A "fixed point" calculated under the following "non-stochastic" assumptions

$$\epsilon_t = 0$$

$$\mathrm{E}_{t-k}[\epsilon_t] = 0$$

$$\mathrm{E}_{t-k}\left[\epsilon_t \, {\epsilon_t}'\right] = 0$$

$$k = 1, \ldots, \infty$$

- Stationary steady state: characterized by a single number, $\bar x$

$$\bar x_t = \bar x$$

- Steady growth path with a constant difference: characterized by two numbers, $\bar x_t$ and $\Delta \bar x$, at a particular yet arbitrary snapshot along the path

$$\bar x_t = \bar x_{t-1} + \Delta \bar x$$

- Steady growth path with a constant rate of change: characterized by $\bar x_t$ and $\delta \bar x$, after logarithm, conceptually the same as the constant difference case

$$\bar x_t = \bar x_{t-1} \cdot \delta \bar x$$

$$\log \bar x_t = \log \bar x_{t-1} + \log \delta \bar x$$

- A noteworth special case: unit root process with zero difference/rate of change – flat in steady state but not stationary (not pinned down to a fixed number)

$$x_t = x_{t-1}$$

Local approximation methods

Deviations from non-stochastic steady state

Vector of deviations from steady path

$$\hat{x}_t = \left[\hat{x}_t^1, \ldots, \hat{x}_t^n\right]'$$

$$\hat{x}_t^i = x_t^i - \bar{x}_t^i \qquad \text{or} \qquad \hat{x}_t^i = \log x_t^i - \log \bar{x}_t^i$$

Find a function approximated around the nonstochastic steady state by terms up to a desired order, with coefficient matrices (solution matrices) A_0, A_1, A_2, $\ldots$, B

$$\hat{x}_t = A_0 + A_1\, \hat{x}_{t-1} + \hat{x}_{t-1}'\, A_2\, \hat{x}_{t-1} + \ \cdots \ + B_1\, \epsilon_t + \epsilon_t'\, B_2\, \epsilon_t + \ \cdots$$

that are consistent with the original system of equations up to a desired order

The coefficient matrices $A_0, A_1,\ A_2,\ A_3,\ \ldots, B_1, B_2,\ \ldots$ dependent on

- the 1st, 2nd, ..., k-th order Taylor expansions of the original functions $f_1,\ \ldots,\ f_k$
- model parameters θ

The higher-order coefficient matrices $A_2, A_3, \ldots, B_2, B_3\ \ldots$ also dependent on

- the higher moments of shocks $\Omega, \ldots$

Sequential calculation of local approximate solutions

1. Calculate non-stochastic steady state
2. Use generalized Schur decomposition to determine the first-order solution matrices
3. Based on steps 1 and 2, calculate second-order solution matrices
4. Based on steps 1, 2, and 3, calculate third-order solution matrices

Global approximation

Find a parametric policy ("solution") function g

$$x_t = g\left(x_{t-1}, \epsilon_t \mid \theta, \Omega, \ldots\right)$$

consistent with the original system taking into account the expectations operator

$$\mathrm{E}_t\Big[f_1\big(x_{t-1}, g\left(x_{t-1}, \epsilon_t\right), g\left(g(x_{t-1}, \epsilon_t), \epsilon_{t+1}\right)\right) \mid \theta\big)\Big] = 0$$

$$\vdots$$

The function g is a parameterized global approximation of the true function, e.g. parameterized sum of polynominals, function over a discrete grid of points, etc.

Policy function method versus parametrized expectations method

Stacked time

Find a sequence of numbers, $x_1, \ldots, x_T$ that comply with the original system of equations stacked T times underneath each other **dropping** the expectations operator

$$f_1\left(x_{-1}, x_1, x_2, \epsilon_1 \mid \theta\right) = 0$$

$$\vdots$$

$$f_k\left(x_{t-1}, x_t, x_2, \epsilon_1 \mid \theta\right) = 0$$

$$\vdots$$

$$\vdots$$

$$f_1\left(x_{T-1}, x_T, x_{T+1}, \epsilon_T \mid \theta\right) = 0$$

$$\vdots$$

$$f_k\left(x_{T-1}, x_T, x_{T+1}, \epsilon_T \mid \theta\right) = 0$$

Initial condition x_{-1} given

Terminal condition x_{T+1} needs to be determined

Combining anticipated and unanticipated shocks in stacked time

By design, all shocks included within one particular simulation run are known/seen/anticipated throughout the simulation range

Simulating a combination of anticipated and unanticipated shocks means

- split the simulation range into sub-ranges by the occurrence of unanticipated shocks
- run each sub-range as a separate simulation, taking the end-points of the previous sub-range simulation as initial condition
- make sure you run a sufficient number of periods in each sub-simulation