Appendix

Elchin

June 2021

1 Appendix

1.1 Full Non-Linear Model

$$\varepsilon_t^p(c_t - h_c c_{t-1})^{-\sigma_c} - \beta h_c \varepsilon_{t+1}^p (c_{t+1} - h_c c_t)^{-\sigma_c} = \lambda_t \tag{1.1}$$

$$1 = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{r_t}{\pi_{t+1}} \tag{1.2}$$

$$\phi_t = \exp\left[-\phi_b^1 \left(\frac{b_t^f}{gdp_t}\right) + \phi_b^2 \left(\left\{\frac{e_{t+1}}{e_t}\right\} - 1\right) + \log(\varepsilon_t^{rp})\right] \tag{1.3}$$

$$1 = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{e_{t+1}}{e_t} \frac{\phi_t r_t^*}{\pi_{t+1}} \tag{1.4}$$

$$U_{m,t} = \epsilon_t^P \epsilon_t^{mD} m_t^{-\sigma_m} \tag{1.5}$$

$$U_{m,t} = \lambda_t - \beta \frac{\lambda_{t+1}}{\pi_{t+1}} \tag{1.6}$$

$$x_t = \frac{i_t}{i_{t-1}} \tag{1.7}$$

$$s_t = \frac{\phi^x}{2}(x_t - 1)^2 \tag{1.8}$$

$$s_t' = \phi^x (x_t - 1) \tag{1.9}$$

$$k_t = (1-\delta)k_{t-1} + \{1 - s_t(\cdot)\} i_t \tag{1.10}$$

$$1 = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{r_{t+1}^k u_{t+1} - a_{t+1}(u_{t+1}) + (1-\delta) q_{t+1}}{q_t} \tag{1.11}$$

$$1 = q_t \{1 - s_t(\cdot) - s_t'(\cdot) x_t\} + \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} s_{t+1}'(\cdot) x_{t+1}^2 \tag{1.12}$$

$$r_t^k = a'(u_t) \tag{1.13}$$

$$f_t^1 = \frac{\eta - 1}{\eta} w_t^* \lambda_t \left(\frac{w_t^*}{w_t}\right)^{-\eta} h_t^d + (\beta \xi_w) \left(\frac{\pi_t^{\gamma_w}}{\pi_{t+1}}\right)^{1-\eta} \left(\frac{w_t^*}{w_{t+1}^*}\right)^{1-\eta} f_{t+1}^1 \tag{1.14}$$

$$f_t^2 = \omega \epsilon_t^P \epsilon_t^h \left(\frac{w_t^*}{w_t} \right)^{-\eta(1+\kappa)} {h_t^d}^{1+\kappa} + (\beta \xi_w) \left(\frac{\pi_t^{\gamma_w}}{\pi_{t+1}} \right) \left(\frac{w_t^*}{w_{t+1}^*} \right)^{-\eta(1+\kappa)} f_{t+1}^2 \tag{1.15}$$

$$f_t^1 = f_t^2 \tag{1.16}$$

$$1 = \xi_w \left(\frac{\pi_{t-1}^{\gamma_w}}{\pi_t} \right)^{1-\eta} \left(\frac{w_{t-1}}{w_t} \right)^{1-\eta} + (1 - \xi_w) \left(\frac{w_t^*}{w_t} \right)^{1-\eta} \tag{1.17}$$

$$\vartheta_t^w = \xi_w \left(\frac{w_{t-1}}{w_t} \frac{\pi_{t-1}^{\gamma_w}}{\pi_t} \right)^{-\eta} \vartheta_{t-1}^w + (1 - \xi_w) \left(\frac{w_t^*}{w_t} \right)^{-\eta} \tag{1.18}$$

$$h_t^d = \frac{h_t}{\vartheta_t^w} \tag{1.19}$$

$$Y_t^N = A_t^N {k_t^N}^{\alpha^N} \left(A_t h_t^N \right)^{1-\alpha^N} \tag{1.20}$$

$$\frac{W_t}{P_t} = w_t = \nu_t^N (1 - \alpha^N) \frac{Y_t^N}{h_t^N} \tag{1.21}$$

$$\frac{R_t^k}{P_t} = r_t^k = \nu_t^N (\alpha^N) \frac{Y_t^N}{k_t^N} \tag{1.22}$$

$$Y_t^T = A_t^T {k_t^T}^{\alpha^T} \left(A_t h_t^T \right)^{1-\alpha^T} \tag{1.23}$$

$$\frac{W_t}{P_t} = w_t = \nu_t^T (1 - \alpha^T) \frac{Y_t^T}{h_t^T} \tag{1.24}$$

$$\frac{R_t^k}{P_t} = r_t^k = \nu_t^T (\alpha^T) \frac{Y_t^T}{k_t^T} \tag{1.25}$$

$$u_t k_{t-1} = k_t^T + k_t^N \tag{1.26}$$

$$h_t^d = h_t^T + h_t^N \tag{1.27}$$

Non-Tradables:

$$j_{1,t}^N = \lambda_t y_t^N \nu_t^N + \beta \xi^N j_{1,t+1}^N \left(\pi_{t+1}^N \right)^{\zeta^N} \tag{1.28}$$

$$j_{2,t}^N = \lambda_t \frac{P_t^N}{P_t} y_t^N + \beta \xi^N j_{2,t+1}^N \left(\pi_{t+1}^N \right)^{\zeta^N - 1} \tag{1.29}$$

$$1 = \xi^N {\pi_t^N}^{\xi^N - 1} + (1 - \xi^N) \left(\frac{\zeta^N}{\zeta^N - 1} \frac{j_{1,t}^N}{j_{2,t}^N} \right)^{1-\zeta^N} \tag{1.30}$$

$$\vartheta_t^N = \xi^N \vartheta_{t-1}^N \left(\pi_t^N \right)^{\zeta^N} + (1 - \xi^N) \left(\frac{\zeta^N}{\zeta^N - 1} \frac{j_{1,t}^N}{j_{2,t}^N} \right)^{-\zeta^N} \tag{1.31}$$

Domestic Tradables:

$$j_{1,t}^{Td} = \lambda_t y_t^{Td} \nu_t^T + \beta \xi^T j_{1,t+1}^{Td} \left(\pi_{t+1}^{Td} \right)^{\zeta^T} \tag{1.32}$$

$$j_{2,t}^{Td} = \lambda_t \frac{P_t^{Td}}{P_t} y_t^{Td} + \beta \xi^T j_{2,t+1}^{Td} \left(\pi_{t+1}^{Td}\right)^{\zeta^T - 1} \tag{1.33}$$

$$1 = \zeta^T \pi_t^{Td^{\zeta^T-1}} + (1-\zeta^T) \left(\frac{\zeta^T}{\zeta^T - 1} \frac{j_{1,t}^{Td}}{j_{2,t}^{Td}} \right)^{1-\zeta^T} \tag{1.34}$$

$$\vartheta_t^{Td} = \zeta^T \vartheta_{t-1}^{Td} \left(\pi_t^{Td}\right)^{\zeta^T} + (1-\zeta^T) \left(\frac{\zeta^T}{\zeta^T - 1} \frac{j_{1,t}^{Td}}{j_{2,t}^{Td}} \right)^{-\zeta^T} \tag{1.35}$$

Tradable Exports:

$$j_{1,t}^{Tx} = \lambda_t y_t^{Tx} \nu_t^T + \beta \xi^T j_{1,t+1}^{Tx} \left(\pi_{t+1}^{Tx}\right)^{\zeta^T} \tag{1.36}$$

$$j_{2,t}^{Tx} = \lambda_t \frac{e_t P_t^{Tx}}{P_t} y_t^{Tx} + \beta \xi^T j_{2,t+1}^{Tx} \left(\pi_{t+1}^{Tx}\right)^{\zeta^T - 1} \tag{1.37}$$

$$1 = \zeta^T \pi_t^{Tx^{\zeta^T-1}} + (1-\zeta^T) \left(\frac{\zeta^T}{\zeta^T - 1} \frac{j_{1,t}^{Tx}}{j_{2,t}^{Tx}} \right)^{1-\zeta^T} \tag{1.38}$$

$$\vartheta_t^{Tx} = \zeta^T \vartheta_{t-1}^{Tx} \left(\pi_t^{Tx}\right)^{\zeta^T} + (1-\zeta^T) \left(\frac{\zeta^T}{\zeta^T - 1} \frac{j_{1,t}^{Tx}}{j_{2,t}^{Tx}} \right)^{-\zeta^T} \tag{1.39}$$

Importing firms:

$$j_{1,t}^{Tm} = \lambda_t y_t^{Tm} rer_t + \beta \xi^{Tm} j_{1,t+1}^{Tm} \left(\pi_{t+1}^{Tm}\right)^{\zeta^{Tm}} \tag{1.40}$$

$$j_{2,t}^{Tm} = \lambda_t \{ \frac{P_t^{Tm}}{P_t} \} y_t^{Tm} + \beta \xi^{Tm} j_{2,t+1}^{Tm} \left(\pi_{t+1}^{Tm}\right)^{\zeta^{Tm} - 1} \tag{1.41}$$

$$1 = \xi^{Tm} \pi_t^{Tm^{\zeta^{Tm}-1}} + (1-\xi^{Tm}) \left(\frac{\zeta^{Tm}}{\zeta^{Tm} - 1} \frac{j_{1,t}^{Tm}}{j_{2,t}^{Tm}} \right)^{1-\zeta^{Tm}} \tag{1.42}$$

$$\vartheta_t^{Tm} = \xi^{Tm} \vartheta_{t-1}^{Tm} \left(\pi_t^{Tm}\right)^{\zeta^{Tm}} + (1-\xi^{Tm}) \left(\frac{\zeta^{Tm}}{\zeta^{Tm} - 1} \frac{j_{1,t}^{Tm}}{j_{2,t}^{Tm}} \right)^{-\zeta^{Tm}} \tag{1.43}$$

Demands:

$$z_t = c_t + g_t + i_t + a_t(u) k_{t-1} \tag{1.44}$$

$$y_t^d = (1-\gamma_m) \left(\frac{P_t^d}{P_t} \right)^{-\mu_m} z_t \tag{1.45}$$

$$y_t^{Tm} = \gamma_m \left(\frac{P_t^{Tm}}{P_t} \right)^{-\mu_m} z_t \tag{1.46}$$

$$y_t^N = (1-\gamma_d) \left(\frac{P_t^N}{P_t^d} \right)^{-\mu_d} y_t^d \tag{1.47}$$

$$y_t^{Td} = \gamma_d \left(\frac{P_t^{Td}}{P_t^d} \right)^{-\mu_d} y_t^d \tag{1.48}$$

$$y_t^{Tx} = \left(\frac{P_t^{Tx}}{P_t^{T*}}\right)^{-\mu_x^*} y_t^* \tag{1.49}$$

Market clearing in goods Market:

$$Y_t^N = \vartheta_t^N y_t^N \tag{1.50}$$

$$Y_t^T = \vartheta_t^{Td} y_t^{Td} + \vartheta_t^{Tx} y_t^{Tx} \tag{1.51}$$

Prices:

$$\frac{e_t P_t^{Tx}}{P_t} = \Psi_t^{Tx} \frac{P_t^{Td}}{P_t} \tag{1.52}$$

$$\frac{\Psi_t^{Tx}}{\Psi_{t-1}^{Tx}} = \frac{e_t}{e_{t-1}} \frac{\pi_t^{Tx}}{\pi_t^{Td}} \tag{1.53}$$

$$\frac{rer_t}{rer_{t-1}} = \frac{e_t}{e_{t-1}} \frac{\pi_t^{T*}}{\pi_t} \tag{1.54}$$

$$\frac{P_t^{Tx}}{P_t^{T*}} = \frac{\frac{e_t P_t^{Tx}}{P_t}}{rer_t} \tag{1.55}$$

Terms of Trade:

$$\mathcal{T}_t = \frac{P_t^{Tm}}{P_t^d} \tag{1.56}$$

Sectoral price differential:

$$\mathcal{T}_t^{TN} = \frac{P_t^{Td}}{P_t^N} \tag{1.57}$$

$$\frac{P_t^N}{P_t^d} = \frac{1}{\left[1 - \gamma_d + \gamma_d \left(\mathcal{T}_t^{TN}\right)^{1-\mu_d}\right]^{\frac{1}{1-\mu_d}}} \tag{1.58}$$

$$\frac{P_t^{Td}}{P_t^d} = \frac{1}{\left[\gamma_d + (1 - \gamma_d) \left(\mathcal{T}_t^{TN}\right)^{\mu_d - 1}\right]^{\frac{1}{1-\mu_d}}} \tag{1.59}$$

$$\frac{P_t^d}{P_t} = \frac{1}{\left[1 - \gamma_m + \gamma_m \left(\mathcal{T}_t\right)^{1-\mu_m}\right]^{\frac{1}{1-\mu_m}}} \tag{1.60}$$

$$\frac{P_t^{Tm}}{P_t} = \frac{1}{\left[\gamma_m + (1 - \gamma_m) \left(\mathcal{T}_t\right)^{\mu_m - 1}\right]^{\frac{1}{1-\mu_m}}} \tag{1.61}$$

$$\frac{P_t^N}{P_t} = \frac{P_t^N}{P_t^d} \frac{P_t^d}{P_t} \tag{1.62}$$

$$\frac{P_t^{Td}}{P_t} = \frac{P_t^{Td}}{P_t^d} \frac{P_t^d}{P_t} \tag{1.63}$$

$$\pi_t^d = \left[(1-\gamma_d) \left(\pi_t^N \left\{ \frac{P_{t-1}^N}{P_{t-1}^d} \right\} \right)^{1-\mu_d} + \gamma^d \left(\pi_t^{Td} \left\{ \frac{P_{t-1}^{Td}}{P_{t-1}^d} \right\} \right)^{1-\mu_d} \right]^{\frac{1}{1-\mu_d}} \tag{1.64}$$

$$\pi_t = \left[(1-\gamma_m) \left(\pi_t^d \left\{ \frac{P_{t-1}^d}{P_{t-1}} \right\} \right)^{1-\mu_m} + \gamma_m \left(\pi_t^m \left\{ \frac{P_{t-1}^{Tm}}{P_{t-1}} \right\} \right)^{1-\mu_m} \right]^{\frac{1}{1-\mu_m}} \tag{1.65}$$

Monetary Policy:

$$\log\left(\frac{r_t}{\bar{r}}\right) = \rho_r \log\left(\frac{r_{t-1}}{\bar{r}}\right) + (1-\rho_r)\left[\theta^\pi \log\left(\frac{\pi_t}{\bar{\pi}}\right) + \theta^{gdp} \log\left(\frac{gdp_t}{\bar{gdp}}\right)\right] + \varepsilon_t^M \tag{1.66}$$

Fiscal policy:

$$g_t = \overbrace{\frac{T_t}{P_t}}^{tax_t} + oT_t + m_t - \frac{m_{t-1}}{\pi_t} \tag{1.67}$$

$$gdp_t = z_t + nx_t \tag{1.68}$$

$$gdp_t = \frac{P_t^N}{P_t} y_t^N + \frac{P_t^{Td}}{P_t} y_t^{Td} + \frac{e_t P_t^{Tx}}{P_t} y_t^{Tx} \tag{1.69}$$

$$nx_t + oT_t = \left(\frac{b_t^f}{\phi_t r_t^*} - \frac{e_t}{e_{t-1}} \frac{b_{t-1}^f}{\pi_t} \right) \tag{1.70}$$

$$gdp_t^O = rer_t p_t^O y_t^O \tag{1.71}$$

$$oT_t = \tau_t^{oT} gdp_t^O \tag{1.72}$$

Exogenous Variables:

$$\log\left(\frac{A_t}{\bar{A}}\right) = \rho^A \log\left(\frac{A_{t-1}}{\bar{A}}\right) + \varepsilon_t^A \tag{1.73}$$

$$\log\left(\frac{A_t^N}{\bar{A}^N}\right) = \rho^{A^N} \log\left(\frac{A_{t-1}^N}{\bar{A}^N}\right) + \varepsilon_t^{A^N} \tag{1.74}$$

$$\log\left(\frac{A_t^T}{\bar{A}^T}\right) = \rho^{A^T} \log\left(\frac{A_{t-1}^T}{\bar{A}^T}\right) + \varepsilon_t^{A^T} \tag{1.75}$$

$$\log\left(\frac{\epsilon_t^P}{\bar{\epsilon}^P}\right) = \rho^{\epsilon^P} \log\left(\frac{\epsilon_{t-1}^P}{\bar{\epsilon}^P}\right) + \varepsilon_t^{\epsilon^P} \tag{1.76}$$

$$\log\left(\frac{\epsilon_t^h}{\bar{\epsilon}^h}\right) = \rho^{\epsilon^h} \log\left(\frac{\epsilon_{t-1}^h}{\bar{\epsilon}^h}\right) + \varepsilon_t^{\epsilon^h} \tag{1.77}$$

$$\log\left(\frac{\epsilon_t^{RP}}{\bar{\epsilon}^{RP}}\right) = \rho^{\epsilon^{RP}} \log\left(\frac{\epsilon_{t-1}^{RP}}{\bar{\epsilon}^h}\right) + \varepsilon_t^{\epsilon^{RP}} \tag{1.78}$$

$$\log\left(\frac{\epsilon_t^{mD}}{\bar{\epsilon}^{mD}}\right) = \rho^{\epsilon^{mD}} \log\left(\frac{\epsilon_{t-1}^{mD}}{\bar{\epsilon}^{mD}}\right) + \varepsilon_t^{\epsilon^{mD}} \tag{1.79}$$

$$\log\left(\frac{y_t^*}{\bar{y}^*}\right) = \rho^{y^*}\log\left(\frac{y_{t-1}^*}{\bar{y}^*}\right) + \varepsilon_t^{y^*} \tag{1.80}$$

$$\log\left(\frac{\pi_t^{T*}}{\bar{\pi}^{T*}}\right) = \rho^{\pi^{T*}}\log\left(\frac{\pi_{t-1}^{T*}}{\bar{\pi}^{T*}}\right) + \varepsilon_t^{\pi^{T*}} \tag{1.81}$$

$$\log\left(\frac{r_t^*}{\bar{r}^*}\right) = \rho^{r^*}\log\left(\frac{r_{t-1}^*}{\bar{r}^*}\right) + \varepsilon_t^{r^*} \tag{1.82}$$

$$\log\left(\frac{g_t}{\bar{g}}\right) = \rho_g\log\left(\frac{g_{t-1}}{\bar{g}}\right) + \epsilon_t^{g} \tag{1.83}$$

$$\log\left(\frac{y_t^O}{\bar{y}_t^O}\right) = \rho^{y^O}\log\left(\frac{y_{t-1}^O}{\bar{y}_t^O}\right) + \varepsilon_t^{y^O} \tag{1.84}$$

$$\log\left(\frac{p_t^O}{\bar{p}_t^O}\right) = \rho^{p^O}\log\left(\frac{p_{t-1}^O}{\bar{p}_t^O}\right) + \varepsilon_t^{p^O} \tag{1.85}$$

$$\log\left(\frac{\tau_t^{oT}}{\bar{\tau}^{oT}}\right) = \rho^{oT}\log\left(\frac{\tau_{t-1}^{oT}}{\bar{\tau}^{oT}}\right) + \varepsilon_t^{oT} \tag{1.86}$$

1.2 Households

For the ease of notation we dropped household' index i for all variables apart from the labour so that it is explicit enough to introduce the wage stickiness. The utility function is given by,

$$U_t = \sum_{t=0}^{\infty} \beta^t \epsilon_t^p \left[\frac{(c_t - h_c c_{t-1})^{1-\sigma_c}}{1-\sigma_c} + \frac{\epsilon_t^{mD}}{1-\sigma_m} \left(\frac{M_t}{P_t} \right)^{1-\sigma_m} - \epsilon_t^h \omega \frac{h_t(i)^{1+\kappa}}{1+\kappa} \right] \tag{1.87}$$

subject to the budget constraint,

$$P_t c_t + P_t i_t + \frac{B_t^d}{r_t} + \frac{e_t B_t^f}{\phi_t r_t^*} + M_t = W_t(i) h_t(i) + \left(R_t^k u_t - P_t a(u_t) \right) k_{t-1} + B_{t-1}^d + e_t B_{t-1}^f + M_{t-1} + T_t + \Phi_t \tag{1.88}$$

here we implicitly assumed that the utilization cost is paid in terms of the final consumption. R_t^k refers to the nominal rental rate of the capital. We also assumed that we have the same the price for investment and consumption.

And,

$$k_t = (1-\delta) k_{t-1} + (1 - s\left(\frac{i_t}{i_{t-1}} \right)) i_t \tag{1.89}$$

First we formulate the problem with the nominal budget constraint:

$$\begin{aligned} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Bigg\{ \epsilon_t^p \left[\frac{(c_t - h_c c_{t-1})^{1-\sigma_c}}{1-\sigma_c} + \frac{\epsilon_t^{mD}}{1-\sigma_m} \left(\frac{M_t}{P_t} \right)^{1-\sigma_m} - \epsilon_t^h \omega \frac{h_t(i)^{1+\kappa}}{1+\kappa} \right] - \Lambda_t \Bigg(P_t c_t + P_t i_t + \frac{B_t^d}{r_t} \\ + e_t \frac{B_t^f}{\phi_t r_t^*} + M_t - W_t(i) h_t(i) - \left(R_t^k u_t - P_t a(u_t) \right) k_{t-1} - B_{t-1}^d - e_t B_{t-1}^f - M_{t-1} - T_t - \Phi_t \Bigg) \\ + Q_t \left[(1-\delta) k_{t-1} + \left(1 - s\left(\frac{i_t}{i_{t-1}} \right) \right) i_t - k_t \right] \Bigg\} \end{aligned} \tag{1.90}$$

The first order condition w.r.t. to consumption is given as,

$$\frac{\partial \mathcal{L}}{\partial c_t} = \varepsilon_t^p (c_t - h_c c_{t-1})^{-\sigma_c} - \beta h_c \varepsilon_{t+1}^p (c_{t+1} - h_c c_t)^{-\sigma_c} - \underbrace{\Lambda_t P_t}_{\lambda_t} = 0 \tag{1.91}$$

Define $\lambda_t = \Lambda_t P_t$, or $\Lambda_t = \frac{\lambda_t}{P_t}$. This can be written in terms the Lagrangian multiplier of the real B.C.:

$$\varepsilon_t^p (c_t - h_c c_{t-1})^{-\sigma_c} - \beta h_c \varepsilon_{t+1}^p (c_{t+1} - h_c c_t)^{-\sigma_c} = \lambda_t \tag{1.92}$$

The optimality condition w.r.t. nominal money balances is given by,

$$\frac{\partial \mathcal{L}}{\partial M_t} = \epsilon_t^P \epsilon_t^{mD} \left(\frac{M_t}{P_t} \right)^{-\sigma_m} \frac{1}{P_t} - \Lambda_t + \beta \Lambda_{t+1} = 0 \tag{1.93}$$

Again replacing Λ_t by λ_t, and multiplying both sides by P_t allows to express this in terms of inflation:

$$\epsilon_t^P \epsilon_t^{mD} \left(\frac{M_t}{P_t} \right)^{-\sigma_m} - \lambda_t + \beta \lambda_{t+1} \frac{P_t}{P_{t+1}} = 0 \tag{1.94}$$

The first order condition w.r.t. nominal domestic bonds is given by,

$$\frac{\partial \mathcal{L}}{\partial B_t^d} = -\frac{\Lambda_t}{r_t} + \beta \Lambda_{t+1} = 0 \tag{1.95}$$

Which can be expressed in terms of the inflation to gives the familiar Euler equation:

$$\frac{\lambda_t}{P_t} = \beta \frac{\lambda_{t+1}}{P_{t+1}} r_t \tag{1.96}$$

w.r.t. foreign bonds,

$$\frac{\partial \mathcal{L}}{\partial B_t^f} = -\Lambda_t \frac{e_t}{\phi_t r_t^*} + \beta e_{t+1} \Lambda_{t+1} = 0 \tag{1.97}$$

In terms of the real Lagrangian multiplier,

$$\frac{\lambda_t}{P_t} \frac{e_t}{\phi_t r_t^*} = \beta e_{t+1} \frac{\lambda_{t+1}}{P_{t+1}} \tag{1.98}$$

which simply is,

$$1 = \beta \frac{\lambda_{t+1}}{\lambda_t} \frac{e_{t+1}}{e_t} \frac{\phi_t r_t^*}{\pi_{t+1}} \tag{1.99}$$

And with respect to capital we have,

$$\frac{\partial \mathcal{L}}{\partial k_t} = -Q_t + \beta \Lambda_{t+1} (R_{t+1}^k u_{t+1} - P_{t+1} a_{t+1}(u_{t+1})) + \beta Q_{t+1}(1-\delta) = 0 \tag{1.100}$$

$$Q_t = \beta \lambda_{t+1} (r_{t+1}^k u_{t+1} - a_{t+1}(u_{t+1})) + \beta Q_{t+1}(1-\delta) \tag{1.101}$$

Now divide by λ_t and denote $q_t = \frac{Q_t}{\lambda_t}$,

$$q_t = \beta \left[\frac{\lambda_{t+1}}{\lambda_t} \left(r_{t+1}^k u_{t+1} - a_{t+1}(u_{t+1}) \right) + \frac{Q_{t+1}}{\lambda_t} \frac{\lambda_{t+1}}{\lambda_{t+1}} \right] \tag{1.102}$$

which we can write as follows,

$$q_t = \beta \frac{\lambda_{t+1}}{\lambda_t} \left[r_{t+1}^k u_{t+1} - a_{t+1}(u_{t+1}) + (1-\delta) q_{t+1} \right] \tag{1.103}$$

and with respect to investment

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial i} &= -\Lambda_t P_t + Q_t \left(1 - s\left\{ \frac{i_t}{i_{t-1}} \right\} - s' \left\{ \frac{i_t}{i_{t-1}} \right\} \frac{i_t}{i_{t-1}} \right) + \beta Q_{t+1} s' \left\{ \frac{i_{t+1}}{i_t} \right\} \left(\frac{i_{t+1}}{i_t} \right)^2 \\ &= 0\end{aligned} \tag{1.104}$$

$$1 = q_t \left(1 - s\left\{ \frac{i_t}{i_{t-1}} \right\} - s' \left\{ \frac{i_t}{i_{t-1}} \right\} \frac{i_t}{i_{t-1}} \right) + \beta \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} s' \left\{ \frac{i_{t+1}}{i_t} \right\} \left(\frac{i_{t+1}}{i_t} \right)^2 \tag{1.105}$$

The f.o.c. w.r.t. utilization is given by

$$r_t^k = a'(u_t) \tag{1.106}$$

1.3 Labour Market

Labour aggregation technology:

$$h_t^d = \left(\int_0^1 h_t(i)^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}} \tag{1.107}$$

The problem of the Labour Packer:

$$\max_{h_t(i)} w_t h_t^d - \int_0^1 w_t(i) h_t(i) di \tag{1.108}$$

Solution: Demand for each type of the labour,

$$h_t(i) = \left(\frac{w_t(i)}{w_t}\right)^{-\eta} h_t^d \tag{1.109}$$

The aggregate wage index,

$$w_t = \left(\int_0^1 w_t(i)^{1-\eta}\right)^{\frac{1}{1-\eta}} \tag{1.110}$$

We assume Calvo-Yun arrangement in the labour market. In each period of the time only $1 - \xi_w$ proportion of households (or labour markets) can optimally reset the price of the labour. All other labour markets the wage is partially indexed to the past inflation by thee parameter γ_w. The nominal wage at the next period ($W_{t+1}(i) = P_{t+1}w_{t+1}(i)$) is given by

$$\begin{aligned} P_{t+1}w_{t+1}(i) &= \pi^{\gamma_w} P_t w_t(i) \\ w_{t+1}(i) &= \frac{\pi_t^{\gamma_w}}{\pi_{t+1}} w_t(i) \end{aligned} \tag{1.111}$$

Similarly, 2 periods later the nominal wage is given by,

$$\begin{aligned} P_{t+2}w_{t+2}(i) &= \pi_{t+1}{}^{\gamma_w} P_{t+1} w_{t+1}(i) \\ w_{t+2}(i) &= \frac{\pi_{t+1}^{\gamma_w}}{\pi_{t+2}} \frac{\pi_t^{\gamma_w}}{\pi_{t+1}} w_t(i) \end{aligned} \tag{1.112}$$

After τ periods in markets which cannot optimise the real wage is given by,

$$w_{t+\tau}(i) = \prod_{s=1}^{\tau} \frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}} w_t(i) \tag{1.113}$$

In the Calvo-Yun set-up this means ,

$$w_{t+\tau}(i)h_{t+\tau}(i)di = (1-\xi_w)w_{t+\tau}^* h_{t+\tau}(i) + \xi_w \prod_{s=1}^{\tau} \frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}} w_t(i) h_{t+\tau}(i) \tag{1.114}$$

Where in $(1 - \xi_w)$ fraction of the labour market sectors the wage will be optimised therefore it drops from the problem of the labor packer. We assume that labour unions are owned by households that is why they take into account households' utility function into account when deciding the wage.

$$\max_{w_t(i)} \sum_{\tau=0}^{\infty} (\beta\xi_w)^{\tau} \left\{ \epsilon_{t+\tau}^p \left[U_{t+\tau}(\cdot, \cdot, h_{t+\tau}(i))\right] - \lambda_{t+\tau} \left(... - \prod_{s=1}^{\tau} \frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}} w_t(i) h_t(i) \right) \right\} \tag{1.115}$$

s.t.

$$h_{t+\tau}(i) = \left(\frac{\prod_{s=1}^{\tau} \frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}} w_t(i)}{w_{t+\tau}} \right)^{-\eta} h_{t+\tau}^d \tag{1.116}$$

The unconstrained problem is given by,

$$\begin{aligned} \max_{w_t(i)} \sum_{\tau=0}^{\infty} (\beta\xi_w)^{\tau} \Bigg\{ \epsilon_{t+\tau}^p \Bigg[U_{t+\tau} \Bigg(\cdot, \cdot, \left(\frac{\prod_{s=1}^{\tau} \frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}} w_t(i)}{w_{t+\tau}} \right)^{-\eta} h_{t+\tau}^d \Bigg) \Bigg] \\ - \lambda_{t+\tau} \Bigg(... - \prod_{s=1}^{\tau} \frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}} w_t(i) \left(\frac{\prod_{s=1}^{\tau} \frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}} w_t(i)}{w_{t+\tau}} \right)^{-\eta} h_{t+\tau}^d \Bigg) \Bigg\} \end{aligned} \tag{1.117}$$

or we can simplify it as,

$$\max_{w_t(i)} \sum_{\tau=0}^{\infty} (\beta\xi_w)^\tau \left\{ \epsilon^p_{t+\tau} \left[U_{t+\tau}\left(\cdot, \cdot, \left(\frac{\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} w_t(i)}{w_{t+\tau}} \right)^{-\eta} h^d_{t+\tau} \right) \right] \right.$$
$$\left. - \lambda_{t+\tau}\left(... - \left(\frac{\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} w_t(i)}{w_{t+\tau}} \right)^{1-\eta} w_{t+\tau} h^d_{t+\tau} \right) \right\} \tag{1.118}$$

The solution:

$$\sum_{\tau=0}^{\infty} (\beta\xi_w)^\tau \left\{ \epsilon^P_{t+\tau} U_{h(i),t+\tau} \frac{\partial h_{t+\tau}(i)}{\partial w_t(i)} + \right.$$
$$\left. \lambda_{t+\tau}(1-\eta) \left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{w_t(i)}{w_{t+\tau}} \right)^{-\eta} \left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{1}{w_{t+\tau}} \right) w_{t+\tau} h^d_{t+\tau} \right\} = 0 \tag{1.119}$$

In a more compact form,

$$\sum_{\tau=0}^{\infty} (\beta\xi_w)^\tau \left\{ \epsilon^P_{t+\tau} U_{h(i),t+\tau} \frac{\partial h_{t+\tau}(i)}{\partial w_t(i)} + \lambda_{t+\tau}(1-\eta) \left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{w_t(i)}{w_{t+\tau}} \right)^{1-\eta} \frac{w_{t+\tau}}{w_t(i)} h^d_{t+\tau} \right\} = 0 \tag{1.120}$$

Here,

$$\frac{\partial h_{t+\tau}(i)}{\partial w_t(i)} = -\eta \left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{w_t(i)}{w_{t+\tau}} \right)^{-\eta-1} \left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{1}{w_{t+\tau}} \right) h^d_{t+\tau}$$
$$= -\eta \left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{w_t(i)}{w_{t+\tau}} \right)^{-\eta} \frac{h^d_{t+\tau}}{w_t(i)} \tag{1.121}$$

$U_{h,t+\tau}(i)$ is given by,

$$U_{h(i),t+\tau} = \omega \epsilon^h_{t+\tau} \left[\left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{w_t(i)}{w_{t+\tau}} \right)^{-\eta} h^d_{t+\tau} \right]^\kappa \tag{1.122}$$

Thus, we can express the f.o.c. for the optimal wage as follows,

$$\sum_{\tau=0}^{\infty} (\beta\xi_w)^\tau \left\{ \omega \epsilon^P_{t+\tau} \epsilon^h_{t+\tau} \left[\left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{w_t(i)}{w_{t+\tau}} \right)^{-\eta} h^d_{t+\tau} \right]^\kappa \left(-\eta \left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{w_t(i)}{w_{t+\tau}} \right)^{-\eta} \frac{h^d_{t+\tau}}{w_t(i)} \right) \right.$$
$$\left. + \lambda_{t+\tau}(1-\eta) \left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{w_t(i)}{w_{t+\tau}} \right)^{1-\eta} \frac{w_{t+\tau}}{w_t(i)} h^d_{t+\tau} \right\} = 0 \tag{1.123}$$

$$\sum_{\tau=0}^{\infty} (\beta\xi_w)^\tau \left\{ -\eta\omega \epsilon^P_{t+\tau} \epsilon^h_{t+\tau} \left[\left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{w_t(i)}{w_{t+\tau}} \right)^{-\eta} \frac{h^d_{t+\tau}}{w_t(i)} \right]^{\kappa+1} w_t(i)^\kappa + \right.$$
$$\left. + \lambda_{t+\tau}(1-\eta) \left(\prod_{s=1}^{\tau} \frac{\pi^{\gamma_w}_{t+s-1}}{\pi_{t+s}} \frac{w_t(i)}{w_{t+\tau}} \right)^{1-\eta} \frac{w_{t+\tau}}{w_t(i)} h^d_{t+\tau} \right\} = 0 \tag{1.124}$$

which we can further simplify as,

$$\sum_{\tau=0}^{\infty}(\beta\xi_w)^{\tau}\left\{-\eta\omega\epsilon_{t+\tau}^{P}\epsilon_{t+\tau}^{h}\left(\prod_{s=1}^{\tau}\frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}}\frac{w_t(i)}{w_{t+\tau}}\right)^{-\eta(1+\kappa)}{h_{t+\tau}^{d}}^{1+\kappa}w_t(i)^{-1}\right.$$
$$\left.+\lambda_{t+\tau}(1-\eta)\left(\prod_{s=1}^{\tau}\frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}}\right)^{1-\eta}\left(\frac{w_t(i)}{w_{t+\tau}}\right)^{-\eta}h_{t+\tau}^{d}\right\}=0 \quad (1.125)$$

Hence the optimal wage at time t is given by,

$$\sum_{\tau=0}^{\infty}(\beta\xi_w)^{\tau}\left\{\omega\epsilon_{t+\tau}^{P}\epsilon_{t+\tau}^{h}\left(\prod_{s=1}^{\tau}\frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}}\frac{w_t^*}{w_{t+\tau}}\right)^{-\eta(1+\kappa)}{h_{t+\tau}^{d}}^{1+\kappa}\right\}$$
$$=\frac{\eta-1}{\eta}w_t^*\sum_{\tau=0}^{\infty}(\beta\xi_w)^{\tau}\left\{\lambda_{t+\tau}\left(\prod_{s=1}^{\tau}\frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}}\right)^{1-\eta}\left(\frac{w_t^*}{w_{t+\tau}}\right)^{-\eta}h_{t+\tau}^{d}\right\} \quad (1.126)$$

To express the problem recursively define two auxiliary variables,

$$f_t^1=\frac{\eta-1}{\eta}w_t^*\sum_{\tau=0}^{\infty}(\beta\xi_w)^{\tau}\left\{\lambda_{t+\tau}\left(\prod_{s=1}^{\tau}\frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}}\right)^{1-\eta}\left(\frac{w_t^*}{w_{t+\tau}}\right)^{-\eta}h_{t+\tau}^{d}\right\} \quad (1.127)$$

and,

$$f_t^2=\sum_{\tau=0}^{\infty}(\beta\xi_w)^{\tau}\left\{\omega\epsilon_{t+\tau}^{P}\epsilon_{t+\tau}^{h}\left(\prod_{s=1}^{\tau}\frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}}\frac{w_t^*}{w_{t+\tau}}\right)^{-\eta(1+\kappa)}{h_{t+\tau}^{d}}^{1+\kappa}\right\} \quad (1.128)$$

Let us consider the first sum,

$$f_t^1=\frac{\eta-1}{\eta}w_t^*\lambda_t\left(\frac{w_t^*}{w_t}\right)^{-\eta}h_t^d$$
$$+\frac{\eta-1}{\eta}w_t^*\sum_{\tau=0}^{\infty}(\beta\xi_w)^{\tau+1}\left\{\lambda_{t+\tau+1}\left(\prod_{s=1}^{\tau+1}\frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}}\right)^{1-\eta}{w_t^*}^{-\eta}w_{t+\tau+1}^{\eta}\left(\frac{w_{t+1}^*}{w_{t+1}^*}\right)^{1-\eta}h_{t+\tau+1}^{d}\right\} \quad (1.129)$$

where we multiplied the term inside the sum by $\left(\frac{w_{t+1}^*}{w_{t+1}^*}\right)^{1-\eta}$. Now take the term $(\beta\xi_w)\left(\frac{\pi_t^{\gamma_w}}{\pi_{t+1}}\right)^{1-\eta}{w_t^*}^{-\eta}{w_{t+1}^*}^{\eta-1}$ outside of the sum.

$$f_t^1=\frac{\eta-1}{\eta}w_t^*\lambda_t\left(\frac{w_t^*}{w_t}\right)^{-\eta}h_t^d \quad (1.130)$$
$$+(\beta\xi_w)\left(\frac{\pi_t^{\gamma_w}}{\pi_{t+1}}\right)^{1-\eta}\left(\frac{w_t^*}{w_{t+1}^*}\right)^{1-\eta}\underbrace{\frac{\eta-1}{\eta}w_{t+1}^*\sum_{\tau=0}^{\infty}(\beta\xi_w)^{\tau}\left\{\lambda_{t+\tau}\left(\prod_{s=1}^{\tau}\frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}}\right)^{1-\eta}\left(\frac{w_{t+1}^*}{w_{t+\tau+1}}\right)^{-\eta}h_{t+\tau+1}^{d}\right\}}_{f_{t+1}^1}$$

Hence we can simply write the first sum as,

$$f_t^1=\frac{\eta-1}{\eta}w_t^*\lambda_t\left(\frac{w_t^*}{w_t}\right)^{-\eta}h_t^d+(\beta\xi_w)\left(\frac{\pi_t^{\gamma_w}}{\pi_{t+1}}\right)^{1-\eta}\left(\frac{w_t^*}{w_{t+1}^*}\right)^{1-\eta}f_{t+1}^1 \quad (1.131)$$

Now consider the second sum,

$$f_t^2=\omega\epsilon_t^P\epsilon_t^h\left(\frac{w_t^*}{w_t}\right)^{-\eta(1+\kappa)}{h_t^d}^{1+\kappa}$$
$$+\sum_{\tau=0}^{\infty}(\beta\xi_w)^{\tau+1}\left\{\omega\epsilon_{t+\tau+1}^{P}\epsilon_{t+\tau+1}^{h}\left(\prod_{s=1}^{\tau+1}\frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}}\right)^{-\eta(1+\kappa)}\left(\frac{w_t^*}{w_{t+\tau+1}}\right)^{-\eta(1+\kappa)}{h_{t+\tau+1}^{d}}^{1+\kappa}\right\} \quad (1.132)$$

Multiply the term inside the sum by $\left(\frac{w_{t+1}^*}{w_{t+1}^*}\right)^{-\eta(1+\kappa)}$. Hence we can rewrite the sum as,

$$\begin{aligned}f_t^2 &= \omega\epsilon_t^P\epsilon_t^h\left(\frac{w_t^*}{w_t}\right)^{-\eta(1+\kappa)} {h_t^d}^{1+\kappa} \\ &+ (\beta\xi_w)\left(\frac{\pi_t^{\gamma_w}}{\pi_{t+1}}\right)\left(\frac{w_t^*}{w_{t+1}^*}\right)^{-\eta(1+\kappa)} \underbrace{\sum_{\tau=0}^{\infty}(\beta\xi_w)^\tau\left\{\omega\epsilon_{t+\tau+1}^P\epsilon_{t+\tau+1}^h\left(\prod_{s=1}^{\tau}\frac{\pi_{t+s-1}^{\gamma_w}}{\pi_{t+s}}\right)^{-\eta(1+\kappa)} {h_{t+\tau+1}^d}^{1+\kappa}\right\}}_{f_{t+1}^2}\end{aligned} \tag{1.133}$$

Or simply,

$$f_t^2 = \omega\epsilon_t^P\epsilon_t^h\left(\frac{w_t^*}{w_t}\right)^{-\eta(1+\kappa)} {h_t^d}^{1+\kappa} + (\beta\xi_w)\left(\frac{\pi_t^{\gamma_w}}{\pi_{t+1}}\right)\left(\frac{w_t^*}{w_{t+1}^*}\right)^{-\eta(1+\kappa)} f_{t+1}^2 \tag{1.134}$$

And the f.o.c. is given by

$$f_t^1 = f_t^2 \tag{1.135}$$

Now we describe the law of motion for the aggregate wage. To do so we use the definition of the aggregate wage index.

$$w_t^{1-\eta} = \int_0^1 w_t(i)^{1-\eta}di \tag{1.136}$$

Owing to the Calvo-Yun set up and Law of Large Numbers the aggregate wage can be described as,

$$w_t^{1-\eta} = \xi_w\left(\frac{\pi_{t-1}^{\gamma_w}}{\pi_t}\right)^{1-\eta} w_{t-1}^{1-\eta} + (1-\xi_w)\left(w_t^*\right)^{1-\eta} \tag{1.137}$$

Dividing by $w_t^{1-\eta}$ allows to express this as,

$$1 = \xi_w\left(\frac{\pi_{t-1}^{\gamma_w}}{\pi_t}\right)^{1-\eta}\left(\frac{w_{t-1}}{w_t}\right)^{1-\eta} + (1-\xi_w)\left(\frac{w_t^*}{w_t}\right)^{1-\eta} \tag{1.138}$$

The wage dispersion is given by,

$$\vartheta_t^w = \int_0^1\left(\frac{w_t(i)}{w_t}\right)di = \xi_w\left(\frac{w_{t-1}}{w_t}\frac{\pi_{t-1}^{\gamma_w}}{\pi_t}\right)^{-\eta}\vartheta_{t-1}^w + (1-\xi_w)\left(\frac{w_t^*}{w_t}\right)^{-\eta} \tag{1.139}$$

1.4 Firms' demand i^{th} labor type

Firms' choose to hire labor inputs from each labor market to minimize the cost of total wage bills subject to the aggregation technology.

$$\min_{h_t(i)}\int_0^1 w_t(i)h_t(i)di$$

Subject to:

$$h_t^d = \left(\int_0^1 h_t(i)^{\frac{\eta-1}{\eta}}di\right)^{\frac{\eta}{\eta-1}}$$

The Lagrangian is given by,

$$\mathcal{L} = \int_0^1 w_t(i)h_t(i)di - \lambda_t^w\left(\left(\left(\int_0^1 h_t(i)^{\frac{\eta-1}{\eta}}di\right)^{\frac{\eta}{\eta-1}} - h_t^d\right)\right) \tag{1.140}$$

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial h_t(i)} = w_t(i) - \lambda_{t,i} \left(\int_0^1 h_t(i)^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}-1} h_t(i)^{\frac{\eta-1}{\eta}-1} &= 0 \\
w_t(i) &= \lambda_{t,i} \left(\int_0^1 h_t(i)^{\frac{\eta-1}{\eta}} \right)^{\frac{1}{\eta-1}} \\
w_t(i) &= \lambda_t^w {h_t^d}^{\frac{1}{\eta}} h_t(i)^{-\frac{1}{\eta}} \\
h_t(i)^{-\frac{1}{\eta}} &= \frac{w_t(i)}{\lambda_t^w {h_t^d}^{1/\eta}} \\
h_t(i) &= \left(\frac{w_t(i)}{\lambda_t^w} \right)^{-\eta} h_t^d
\end{aligned}
$$

The FOC is formulated by plugging this back into the definition of the aggregate demand:

$$
\begin{aligned}
h_t^d &= \left(\int_0^1 \left(\left(\frac{w_t(i)}{\lambda_t^w} \right)^{-\eta} h_t^d \right)^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}} \\
h_t^d &= \left(\int_0^1 \left(\frac{w_t(i)}{\lambda_t^w} \right)^{1-\eta} di \right)^{\frac{\eta}{\eta-1}} h_t^d \\
1 &= \left(\frac{1}{\lambda_t^w} \right)^{-\eta} \left(\int_0^1 w_t(i)^{1-\eta} di \right)^{\frac{1}{1-\eta}(-\eta)} \\
{\lambda_t^w}^{-\eta} &= w_t^{-\eta} \\
\lambda_t^w &= w_t
\end{aligned}
$$

Thus, the Lagrangian multiplier associated with budget constraint equals the aggregate wage index. Therefore, the demand for the labour type i can be written as

$$
h_t(i) = \left(\frac{w_t(i)}{w_t} \right)^{-\eta} h_t^d \tag{1.141}
$$

1.5 Firms

1.5.1 Non-Tradable

$$\begin{aligned}\mathcal{L}=\sum_{t=\tau}^{\infty}(\beta\xi^N)^{\tau}\frac{\lambda_{t+\tau}}{\lambda_t}&\frac{P_t^N(j)\left(\frac{P_t^N(j)}{P_{t+\tau}^N}\right)^{-\zeta^N}y_{t+\tau}^N-W_{t+\tau}h_t^N(j)-R_{t+\tau}^k k_{t+\tau-1}^N(j)}{P_{t+\tau}}+\\&+\nu_t(j)^N\left[A_{t+\tau}^N\left(k_{t+\tau}^N(j)\right)^{\alpha^N}\left(A_t h_{t+\tau}^N(j)\right)^{1-\alpha^N}-\left(\frac{P_t^N(j)}{P_{t+\tau}^N}\right)^{-\zeta^N}y_{t+\tau}^N\right]\end{aligned}\tag{1.142}$$

which we can rewrite as follows,

$$\begin{aligned}\mathcal{L}=\sum_{t=\tau}^{\infty}(\beta\xi^N)^{\tau}\frac{\lambda_{t+\tau}}{\lambda_t}&\left[\frac{P_{t+\tau}^N}{P_{t+\tau}}\left(\frac{P_t^N(j)}{P_{t+\tau}^N}\right)^{1-\zeta^N}y_{t+\tau}^N-\frac{W_{t+\tau}}{P_{t+\tau}}h_{t+\tau}^N(j)-\frac{R_{t+\tau}^k}{P_{t+\tau}}k_{t+\tau-1}^N(j)\right]+\\&+\nu_{t+\tau}(j)\left[A_{t+\tau}^N k_{t+\tau-1}^N(j)^{\alpha^N}h_{t+\tau}^N(j)^{1-\alpha^N}-\left(\frac{P_t^N(j)}{P_{t+\tau}^N}\right)^{-\zeta^N}y_{t+\tau}^N\right]\end{aligned}\tag{1.143}$$

where we used $P_t^N(j)\left(\frac{P_t^N(j)}{P_{t+\tau}^N}\right)^{-\zeta^N}y_{t+\tau}^N=\left(\frac{P_t^N(j)}{P_{t+\tau}^N}\right)^{1-\zeta^N}P_{t+\tau}^N y_{t+\tau}^N$. The FOCs w.r.t. production factors are simple and given by,

$$\frac{\partial\mathcal{L}}{\partial h_t^N(j)}=0:\quad \frac{W_t}{P_t}=\nu_t(j)^N(1-\alpha^N)\frac{y_t^N(j)}{h_t^N(j)}\tag{1.144}$$

$$\frac{\partial\mathcal{L}}{\partial k_t^N(j)}=0:\quad \frac{R_t^K}{P_t}=\nu_t(j)^N(\alpha^N)\frac{y_t^N(j)}{k_t^N(j)}\tag{1.145}$$

The FOC w.r.t optimal reset price is relatively involved and given as,

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial P_t^N(j)}=0:&\\ &\sum_{t=\tau}^{\infty}(\beta\xi^N)^{\tau}\frac{\lambda_{t+\tau}}{\lambda_t}\left[\frac{(1-\zeta^N)}{P_{t+\tau}^N}\left(\frac{P_t^N(j)}{P_{t+\tau}^N}\right)^{-\zeta^N}\frac{P_{t+\tau}^N}{P_{t+\tau}}y_{t+\tau}^N\right.\\&\qquad\left.+\frac{\zeta^N}{P_{t+\tau}^N}\left(\frac{P_t^N(j)}{P_{t+\tau}^N}\right)^{-\zeta^N-1}y_{t+\tau}^N\nu_{t+\tau}(j)^N\right]=0\end{aligned}\tag{1.146}$$

Which is can be simplified as,

$$\sum_{t\,=\tau}^{\infty}(\beta\xi^N)^{\tau}\frac{\lambda_{t+\tau}}{\lambda_t}\left[(\zeta^N-1)\left(\frac{P_t^N(j)}{P_{t+\tau}^N}\right)^{-\zeta^N}\frac{y_{t+\tau}^N}{P_{t+\tau}}-\frac{\zeta^N}{P_{t+\tau}^N}\left(\frac{P_t^N(j)}{P_{t+\tau}^N}\right)^{-\zeta^N-1}y_{t+\tau}^N\nu_{t+\tau}^N(j)\right]=0\tag{1.147}$$

or,

$$\sum_{t\,=\tau}^{\infty}(\beta\xi^N)^{\tau}\frac{\lambda_{t+\tau}}{\lambda_t}\left[(\zeta-1)\left(P_t^N(j)^{-\zeta^N}{P_{t+\tau}^N}^{\zeta^N}\right)\frac{y_{t+\tau}^N}{P_{t+\tau}}-\frac{\zeta^N}{P_{t+\tau}^N}\left(P_t^N(j)^{-\zeta^N-1}{P_{t+\tau}^N}^{\zeta^N+1}\right)y_{t+\tau}^N\nu_{t+\tau}^N(j)\right]=0\tag{1.148}$$

Now dividing by $\frac{P_t^N(j)^{-\zeta^N}}{\lambda_t}$ gives,

$$\sum_{t\,=\tau}^{\infty}(\beta\xi^N)^{\tau}\frac{\lambda_{t+\tau}}{\lambda_t}\left[(\zeta^N-1)\left(P_{t+\tau}^N\right)^{\zeta^N-1}\left(\frac{P_{t+\tau}^N}{P_{t+\tau}}\right)y_{t+\tau}^N-\zeta^N P_t^N(j)^{-1}{P_{t+\tau}^N}^{\zeta^N}y_{t+\tau}^N\nu_{t+\tau}^N(j)\right]=0\tag{1.149}$$

which solves for the optimal reset price,

$$\tilde{P}^N_t(j) = \frac{\zeta^N}{\zeta^N - 1} \frac{\sum_{t=\tau}^{\infty} (\beta\xi^N)^\tau \lambda_{t+\tau} y^N_{t+\tau} \left(P^N_{t+\tau}\right)^{\zeta^N} \nu^N_{t+\tau}(j)}{\sum_{t=\tau}^{\infty} (\beta\xi^N)^\tau \lambda_{t+\tau} \frac{P^N_{t+\tau}}{P_{t+\tau}} y^N_{t+\tau} \left(P^N_{t+\tau}\right)^{\zeta^N - 1}} \tag{1.150}$$

To express infinite sums recursively define following auxiliary variables

$$J^N_{1,t} = \sum_{t=\tau}^{\infty} (\beta\xi^N)^\tau \lambda_{t+\tau} y^N_{t+\tau} \left(P^N_{t+\tau}\right)^{\zeta^N} \nu^N_{t+\tau} \tag{1.151}$$

and

$$J^N_{2,t} = \sum_{t=\tau}^{\infty} (\beta\xi^N)^\tau \lambda_{t+\tau} \frac{P^N_{t+\tau}}{P_{t+\tau}} y^N_{t+\tau} \left(P^N_{t+\tau}\right)^{\zeta^N - 1} \tag{1.152}$$

The first sum can be written as

$$J^N_{1,t} = \lambda_t y^N_t \left(P^N_t\right)^{\zeta^N} \nu^N_t + \sum_{t=\tau}^{\infty} (\beta\xi^N)^{\tau+1} \lambda_{t+1+\tau} y^N_{t+1+\tau} \left(P^N_{t+1+\tau}\right)^{\zeta^N} \nu^N_{t+1+\tau} \tag{1.153}$$

which is simply

$$J^N_{1,t} = \lambda_t y^N_t \left(P^N_t\right)^{\zeta^N} \nu^N_t + (\beta\xi^N) J^N_{1,t+1} \tag{1.154}$$

dividing by $\left(P^N_t\right)^\zeta$ gives,

$$\frac{J^N_{1,t}}{\left(P^N_t\right)^{\zeta^N}} = \lambda_t y^N_t \nu^N_t + (\beta\xi^N) \frac{J^N_{1,t+1}}{\left(P^N_{t+1}\right)^{\zeta^N}} \frac{\left(P^N_{t+1}\right)^{\zeta^N}}{\left(P^N_t\right)^{\zeta^N}} \tag{1.155}$$

where we multiplied and divided the last term with $\left(P^N_{t+1}\right)^{\zeta^N}$. Now define $j^N_{1,t} = \frac{J^N_{1,t}}{\left(P^N_t\right)^{\zeta^N}}$ to express the sum as,

$$j^N_{1,t} = \lambda_t y^N_t \nu^N_t + \beta\xi^N j^N_{1,t+1} \left(\pi^N_{t+1}\right)^{\zeta^N} \tag{1.156}$$

Similarly the second sum can be expressed as,

$$j^N_{2,t} = \lambda_t \frac{P^N_t}{P_t} y^N_t + \beta\xi^N j^N_{2,t+1} \left(\pi^N_{t+1}\right)^{\zeta^N - 1} \tag{1.157}$$

where $j^N_{2,t} = \frac{J^N_{2,t}}{\left(P^N_t\right)^{\zeta^N - 1}}$

The optimal reset price is given as

$$\tilde{P}^N_t = \frac{\zeta^N}{\zeta^N - 1} \frac{J^N_{1,t}}{J^N_{2,t}} = \frac{j^N_{1,t} \left(P^N_t\right)^{\zeta^N}}{j^N_{2,t} \left(P^N_t\right)^{\zeta^N - 1}} \tag{1.158}$$

$$\frac{\tilde{P}^N_t}{P^N_t} = \frac{j^N_{1,t}}{j^N_{2,t}} \tag{1.159}$$

The price of composite non-tradable good is given by,

$${P^N_t}^{1-\zeta^N} = \int_0^1 P^N_t(j)^{1-\zeta^N} dj \tag{1.160}$$

The Calvo price contract implies that the aggregate non-tradable price index is given

$${P^N_t}^{1-\zeta^N} = \int_0^{\xi^N} P^N_{t-1}(j)^{1-\zeta^N} dj + \int_0^{1-\xi^N} {\tilde{P}^N_t}^{1-\zeta^N} dj \tag{1.161}$$

Due to the law of large numbers

$$P_t^{N\,1-\zeta^N} = \xi^N P_{t-1}^{N\;\;1-\zeta^N} + (1-\xi)\tilde{P}_t^{N\,1-\zeta^N} \tag{1.162}$$

Dividing by $P_t^{N\,1-\zeta^N}$ allow to write this in terms of the inflation rate

$$1 = \xi^N \pi_{t-1}^{N\;\;\zeta^N-1} + (1-\xi^N)\left(\frac{\tilde{P}_t^N}{P_t}\right)^{1-\zeta^N} \tag{1.163}$$

Similarly the price dispersion

$$\int_0^1 \left(\frac{P_t^N(j)}{P_t^N}\right)^{-\zeta^N} dj = \int_0^{\xi^N} \left(\frac{P_{t-1}^N(j)}{P_t^N}\frac{P_{t-1}^N}{P_{t-1}^N}\right)^{-\zeta^N} dj + \int_0^{1-\xi^N} \left(\frac{\tilde{P}_t^N}{P_t}\right)^{-\zeta^N} dj \tag{1.164}$$

which can be written is

$$\vartheta_t = \xi^N \vartheta_{t-1}\pi_t^{N\zeta^N} + (1-\xi^N)\left(\frac{\tilde{P}_t^N}{P_t}\right)^{-\zeta^N} \tag{1.165}$$

1.5.2 Tradable sector

The Lagrangian of Traded goods firm can be written

$$\begin{aligned}\mathcal{L} = \sum_{t=\tau}^{\infty}(\beta\xi^T)^\tau \frac{\lambda_{t+\tau}}{\lambda_t P_{t+\tau}} \Bigg[& P_t^{Td}(j)\left(\frac{P_t^{Td}(j)}{P_{t+\tau}^{Td}}\right)^{-\zeta^T} y_{t+\tau}^{Td} + e_{t+\tau}P_t^{Tx}(j)\left(\frac{P_t^{Tx}(j)}{P_{t+\tau}^{Tx}}\right)^{-\zeta^T} y_{t+\tau}^{Tx} - W_{t+\tau}h_{t+\tau}^T(j) \\ & - R_{t+\tau}^k k_{t+\tau-1}^T(j)\Bigg] \\ & + \nu_t^T(j)\left[A_t^T k_t^T(j)^{\alpha^T}\left(A_t h_t^T(j)\right)^{1-\alpha^T} - \left(\frac{P_t^{Td}(j)}{P_{t+\tau}^{Td}}\right)^{-\zeta^T} y_{t+\tau}^{Td} - \left(\frac{P_t^{Tx}(j)}{P_{t+\tau}^{Tx}}\right)^{-\zeta^T} y_{t+\tau}^{Tx}\right]\end{aligned} \tag{1.166}$$

The FOCs are,

$$\frac{W_t}{P_t} = \nu_t(j)^T(1-\alpha^T)\frac{y_t^T(j)}{h_t^T(j)} \tag{1.167}$$

$$\frac{R_t^K}{P_t} = \nu_t(j)^T(\alpha^T)\frac{y_t^T(j)}{k_t^T(j)} \tag{1.168}$$

Since all derivations are similar to non-tradable firms we skip steps for the Tradable domestic market where the optimal reset price is,

$$\tilde{P}_t^{Td} = \frac{\zeta^T}{\zeta^T-1}\frac{\sum_{\tau=0}^{\infty}(\beta\xi^T)^\tau \lambda_{t+\tau}P_{t+\tau}^{Td\;\zeta^T} y_{t+\tau}^{Td}\nu_{t+\tau}^T}{\sum_{\tau=0}^{\infty}(\beta\xi^T)^\tau \lambda_{t+\tau}P_{t+\tau}^{Td\;\zeta^T-1}\frac{P_{t+\tau}^{Td}}{P_{t+\tau}}y_{t+\tau}^{Td}} \tag{1.169}$$

Although it is conceptually identical we repeat the derivation for the export market:

$$\frac{\partial}{\partial P_t^{Tx}(j)}\left\{\sum_{\tau=0}^{\infty}(\beta\xi^T)^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\left[\left(\frac{P_t^{Tx}(j)}{P_{t+\tau}^{Tx}}\right)^{1-\zeta^T}\frac{e_{t+\tau}P_{t+\tau}^{Tx}}{P_{t+\tau}}y_{t+\tau}^{Tx} - \left(\frac{P_t^{Tx}(j)}{P_{t+\tau}^{Tx}}\right)^{-\zeta^T} y_{t+\tau}^{Tx}\nu_{t+\tau}^T(j)\right]\right\} = 0 \tag{1.170}$$

which means that

$$\sum_{\tau=0}^{\infty}(\beta\xi^T)^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\left[(1-\zeta^T)\left(\frac{P_t^{Tx}(j)}{P_{t+\tau}^{Tx}}\right)^{-\zeta^T}\frac{e_{t+\tau}y_{t+\tau}^{Tx}}{P_{t+\tau}} + \zeta^T\left(\frac{P_t^{Td}(j)}{P_{t+\tau}^{Tx}}\right)^{-\zeta^T-1} y_{t+\tau}^{Tx}\frac{\nu_{t+\tau}^T(j)}{P_{t+\tau}^{Tx}}\right] = 0 \tag{1.171}$$

Divide by, $\frac{P_t^{Tx}(j)^{-\zeta^T}}{\lambda_t}$

$$\sum_{\tau=0}^{\infty}(\beta\xi^T)^\tau \lambda_{t+\tau}\left[(1-\zeta^T){P_{t+\tau}^{Tx}}^{\zeta^T}\frac{e_{t+\tau}y_{t+\tau}^{Tx}}{P_{t+\tau}}+\zeta^T {P_{t+\tau}^{Tx}}^{\zeta^T+1}P_t^{Tx}(j)^{-1}y_{t+\tau}^{Tx}\frac{\nu_{t+\tau}^T(j)}{P_{t+\tau}^{Tx}}\right]=0 \quad (1.172)$$

$$\sum_{\tau=0}^{\infty}(\beta\xi^T)^\tau \lambda_{t+\tau}\left[(1-\zeta^T){P_{t+\tau}^{Tx}}^{\zeta^T-1}\frac{e_{t+\tau}P_{t+\tau}^{Tx}}{P_{t+\tau}}y_{t+\tau}^{Tx}+\zeta^T {P_{t+\tau}^{Tx}}^{\zeta^T}P_t^{Tx}(j)^{-1}y_{t+\tau}^{Tx}\nu_{t+\tau}^T(j)\right]=0 \quad (1.173)$$

$$\tilde{P}_t^{Tx}(j)=\frac{\zeta^T}{\zeta^T-1}\frac{\sum_{\tau=0}^{\infty}(\beta\xi^T)^\tau\lambda_{t+\tau}{P_{t+\tau}^{Tx}}^{\zeta^T}y_{t+\tau}^{Tx}\nu_{t+\tau}^T(j)}{\sum_{\tau=0}^{\infty}(\beta\xi^T)^\tau\lambda_{t+\tau}{P_{t+\tau}^{Tx}}^{\zeta^T-1}\frac{e_{t+\tau}P_{t+\tau}^{Tx}}{P_{t+\tau}}y_{t+\tau}^{Tx}} \quad (1.174)$$

1.5.3 Importing firms

Profits of Importing Firms are given by,

$$\phi_t^m(j)=P_t^{Tm}(j)y_t^{Tm}(j)-e_tP_t^{T*}y_t^{Tm}(j) \quad (1.175)$$

Importing firms solve the following problem

$$\max_{P_t^{Tm}(j)}\sum_{\tau=0}^{\infty}\left(\beta\xi^{Tm}\right)^\tau\frac{y_{t+\tau}^{Tm}\lambda_{t+\tau}}{P_{t+\tau}\lambda_t}\left[\left(\frac{P_t^{Tm}(j)}{P_{t+\tau}^{Tm}}\right)^{-\zeta^{Tm}}P_t^{Tm}(j)-\left(\frac{P_t^{Tm}(j)}{P_{t+\tau}^{Tm}}\right)^{-\zeta^{Tm}}e_{t+\tau}P_{t+\tau}^{T*}\right] \quad (1.176)$$

Which we can rewrite as,

where we multiplied the first term inside the bracket with $\frac{P_{t+\tau}^{Tm}}{P_{t+\tau}^{Tm}}$ and factored out $P_{t+\tau}^{Tm}$

$$\sum_{\tau=0}^{\infty}\left(\beta\xi^{Tm}\right)^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\left[\frac{P_{t+\tau}^{Tm}y_{t+\tau}^{Tm}}{P_{t+\tau}}\left(\frac{P_t^{Tm}(j)}{P_{t+\tau}^{Tm}}\right)^{1-\zeta^{Tm}}-\left(\frac{P_t^{Tm}(j)}{P_{t+\tau}^{Tm}}\right)^{-\zeta^{Tm}}\frac{e_{t+\tau}P_{t+\tau}^{T*}}{P_{t+\tau}^{Tm}}\frac{P_{t+\tau}^{Tm}y_{t+\tau}^{Tm}}{P_{t+\tau}}\right] \quad (1.177)$$

which is simplified as follows

$$\sum_{\tau=0}^{\infty}\left(\beta\xi^{Tm}\right)^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\left[\frac{P_{t+\tau}^{Tm}y_{t+\tau}^{Tm}}{P_{t+\tau}}\left(\frac{P_t^{Tm}(j)}{P_{t+\tau}^{Tm}}\right)^{1-\zeta^{Tm}}-\left(\frac{P_t^{Tm}(j)}{P_{t+\tau}^{Tm}}\right)^{-\zeta^{Tm}}\frac{e_{t+\tau}P_{t+\tau}^{T*}}{P_{t+\tau}}y_{t+\tau}^{Tm}\right] \quad (1.178)$$

The first-order condition is

$$\begin{aligned}\sum_{\tau=0}^{\infty}\left(\beta\xi^{Tm}\right)^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\Bigg[&(1-\zeta^{Tm})\frac{P_{t+\tau}^{Tm}y_{t+\tau}^{Tm}}{P_{t+\tau}}\left(\frac{P_t^{Tm}(j)}{P_{t+\tau}^{Tm}}\right)^{-\zeta^{Tm}}\left(P_{t+\tau}^{Tm}\right)^{-1}\\&+\zeta^{Tm}\left(\frac{P_t^{Tm}(j)}{P_{t+\tau}^{Tm}}\right)^{-\zeta^{Tm}-1}\left(P_{t+\tau}^{Tm}\right)^{-1}\frac{e_{t+\tau}P_{t+\tau}^{T*}}{P_{t+\tau}}y_{t+\tau}^{Tm}\Bigg]=0\end{aligned} \quad (1.179)$$

Dividing by $\frac{P^{Tm}(j)^{-\zeta^{Tm}}}{\lambda_t}$ gives the following simplification

$$\sum_{\tau=0}^{\infty}\left(\beta\xi^{Tm}\right)^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\left[(1-\zeta^{Tm})\frac{P_{t+\tau}^{Tm}y_{t+\tau}^{Tm}}{P_{t+\tau}}\left(P_{t+\tau}^{Tm}\right)^{\zeta^{Tm}-1}+\zeta^{Tm}\left(P_t^{Tm}(j)\right)^{-1}\left(P_{t+\tau}^{Tm}\right)^{\zeta^{Tm}}\frac{e_{t+\tau}P_{t+\tau}^{T*}}{P_{t+\tau}}y_{t+\tau}^{Tm}\right]=0 \quad (1.180)$$

where we can solve $\tilde{P}_t^{Tm}$

$$\tilde{P}_t^{Tm}=\frac{\zeta^{Tm}}{\zeta^{Tm}-1}\frac{\sum_{\tau=0}^{\infty}\left(\beta\xi^{Tm}\right)^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\left(P_{t+\tau}^{Tm}\right)^{\zeta^{Tm}}\frac{e_{t+\tau}P_{t+\tau}^{T*}}{P_{t+\tau}}y_{t+\tau}^{Tm}}{\sum_{\tau=0}^{\infty}\left(\beta\xi^{Tm}\right)^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\frac{P_{t+\tau}^{Tm}y_{t+\tau}^{Tm}}{P_{t+\tau}}} \quad (1.181)$$

Alternatively the solution can be expressed in terms of the import LOOP gap

$$\tilde{P}_t^{Tm}=\frac{\zeta^{Tm}}{\zeta^{Tm}-1}\frac{\sum_{\tau=0}^{\infty}\left(\beta\xi^{Tm}\right)^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\left(P_{t+\tau}^{Tm}\right)^{\zeta^{Tm}}\overbrace{\frac{e_{t+\tau}P_{t+\tau}^{T*}}{P_{t+\tau}^{Tm}}}^{\text{Import LOOP Gap}}\frac{P_{t+\tau}^{Tm}}{P_{t+\tau}}y_{t+\tau}^{Tm}}{\sum_{\tau=0}^{\infty}\left(\beta\xi^{Tm}\right)^\tau\frac{\lambda_{t+\tau}}{\lambda_t}\frac{P_{t+\tau}^{Tm}y_{t+\tau}^{Tm}}{P_{t+\tau}}} \quad (1.182)$$

1.6 Steady State

We choose a deterministic steady state with no risk premium. Which implies that the net non oil export together with oil transfers balances the total trade balance and thus the net non-oil export is negative.

The UIP condition is given by

$$\phi_t \frac{e_{t+1}}{e_t} \frac{r_t^*}{r_t} = 1 \tag{1.183}$$

which implies that for $r^* = r$ and and when the nominal does not appreciate (depreciate) in the steady state we have

$$\phi = 1 \tag{1.184}$$

On the other hand the risk premium takes following empirical form in our set up

$$\phi_t = \exp\left[-\phi_b^1\left(\frac{b_t^f}{gdp_t}\right) + \phi_b^2\left(\left\{\frac{e_{t+1}}{e_t}\frac{e_t}{e_{t-1}}\right\} - 1\right) + \log(\varepsilon_t^{rp})\right] \tag{1.185}$$

Taking the logs in the steady state.

$$log(\phi) = -\phi_b^1\left(\frac{b^f}{gdp}\right) : \qquad b^f = 0 \tag{1.186}$$

Thus according to our normalization the foreign assets is 0 in the steady state.

Now we also take into account the evolution of net foreign assets

$$nx_t + o_t = \left(\frac{b_t^f}{\phi_t r_t^*} - \left\{\frac{e_t}{e_{t-1}}\right\}\frac{b_{t-1}^f}{\pi_t}\right) \tag{1.187}$$

Which implies that $nx + o = 0$ should also be true in the steady state. Alternatively the risk premium in the steady can be adjusted in the steady state to allow for more general cases. The idea is to pin down net non-oil exports in the steady state to solve for the rest of variables. Good candidates for normalization would be to make the steady-state non-oil exports proportional to the non-oil GDP in the steady state. Now with the normalization $nx + o = 0$ in mind we proceed to following steps.

The aim is to find $\frac{Y^T}{Y^N}$. The idea is to to express everything in terms of a single variable (y^d) so that this variable will drop in this ratio. YT has an export component (y^{Tx}) that does not directly involve z. To do this we need express the export in terms of imports. Since, the trade balance is pinned down in the steady state using the UIP condition, it should be very straightforward task.

Definition of the gdp (All goods produced domestically):

$$gdp = \frac{P^{Td}}{P}y^{Td} + \frac{eP^{Tx}}{P}y^{Tx} + \frac{P^N}{P}y^N \tag{1.188}$$

$$gdp = z + nx = c + i + g + nx \tag{1.189}$$

$$nx + o = 0 : \qquad o = -nx \tag{1.190}$$

$$\frac{eP^{Tx}}{P}y^{Tx} - \frac{P^{Tm}}{P}y^{Tm} = -o \tag{1.191}$$

The real value of exports become

$$\frac{eP^{Tx}}{P}y^{Tx} = \frac{P^{Tm}}{P}y^{Tm} - o \tag{1.192}$$

where

$$y^{Tm} = \gamma_m\left(\frac{P^{Tm}}{P}\right)^{-\mu_m} z \tag{1.193}$$

Normalizations we make:

$$g = \tau_1 gdp \qquad o = \tau_2 g = \tau_1\tau_2 gdp \tag{1.194}$$

$$gdp = z - o: \qquad gdp = z - \tau_1\tau_2 gdp \tag{1.195}$$

$$(1+\tau_1\tau_2)gdp = z \tag{1.196}$$

$$gdp = \frac{z}{1+\tau_1\tau_2} \tag{1.197}$$

Thus,

$$\frac{eP^{Tx}}{P}y^{Tx} = \frac{P^{Tm}}{P}\gamma_m\left(\frac{P^{Tm}}{P}\right)^{-\mu_m} z - o \tag{1.198}$$

Noting the relation between o and z which becomes

$$\frac{eP^{Tx}}{P}y^{Tx} = \frac{P^{Tm}}{P}\gamma_m\left(\frac{P^{Tm}}{P}\right)^{-\mu_m} z - \frac{\tau_1\tau_2 z}{1+\tau_1\tau_2} \tag{1.199}$$

Hence, y^{Tx} becomes

$$y^{Tx} = \frac{\frac{P^{Tm}}{P}\gamma_m\left(\frac{P^{Tm}}{P}\right)^{-\mu_m} - \frac{\tau_1\tau_2}{1+\tau_1\tau_2}}{\frac{eP^{Tx}}{P}} z \tag{1.200}$$

Tradable (domestically consumed) output:

$$y^{Td} = \gamma_d\left(\frac{P^{Td}}{P^d}\right)^{-\mu_d} y^d \tag{1.201}$$

Non Tradable

$$y^N = (1-\gamma_d)\left(\frac{P^N}{P^d}\right)^{-\mu_d} y^d \tag{1.202}$$

$$y^d = (1-\gamma_m)\left(\frac{P^d}{P}\right)^{-\mu_m} z \tag{1.203}$$

z in terms of y^d

$$z = \frac{y^d}{(1-\gamma_m)\left(\frac{P^d}{P}\right)^{-\mu_m}} \tag{1.204}$$

Now note that

$$Y^T = \vartheta^{Tx}y^{Tx} + \vartheta^{Tx}y^{Td} \tag{1.205}$$

Here we assumed that tradable exporters and tradable domestic producers are the firms such that the price dispersion is the same in both sectors (At least in the steady state).

Hence

$$y^T = \vartheta^T\left[y^{Tx} + y^{Td}\right] \tag{1.206}$$

Plugging in relative demands we have

$$Y^T = \vartheta^T\left[\frac{\frac{P^{Tm}}{P}\gamma_m\left(\frac{P^{Tm}}{P}\right)^{-\mu_m} - \frac{\tau_1\tau_2}{1+\tau_1\tau_2}}{\frac{eP^{Tx}}{P}} z + \gamma_d\left(\frac{P^{Td}}{P^d}\right)^{-\mu_d} y^d\right] \tag{1.207}$$

Now express everything in terms of y^d

$$Y^T = \vartheta^T\left[\frac{\frac{P^{Tm}}{P}\gamma_m\left(\frac{P^{Tm}}{P}\right)^{-\mu_m} - \frac{\tau_1\tau_2}{1+\tau_1\tau_2}}{\frac{eP^{Tx}}{P}} \frac{y^d}{(1-\gamma_m)\left(\frac{P^d}{P}\right)^{-\mu_m}} + \gamma_d\left(\frac{P^{Td}}{P^d}\right)^{-\mu_d} y^d\right] \tag{1.208}$$

$$Y^N = \vartheta^N y^N = \vartheta^N (1-\gamma_d) \left(\frac{P^N}{P^d}\right)^{-\mu_d} y^d \tag{1.209}$$

$$\frac{Y^T}{Y^N} = \frac{\vartheta^T}{\vartheta^N} \frac{\left[\frac{\frac{P^{Tm}}{P}\gamma_m \left(\frac{P^{Tm}}{P}\right)^{-\mu_m} - \frac{\tau_1 \tau_2}{1+\tau_1\tau_2}}{\frac{eP^{Tx}}{P}} \frac{1}{(1-\gamma_m)\left(\frac{P^d}{P}\right)^{-\mu_m}} + \gamma_d \left(\frac{P^{Td}}{P^d}\right)^{-\mu_d} \right]}{(1-\gamma_d)\left(\frac{P^N}{P^d}\right)^{-\mu_d}} \tag{1.210}$$

Now we will try to find sector specific variables from the ratio above.
Here we have standard outcomes form the model regarding productions and factor prices,

$$Y^N = A^N {k^N}^{\alpha^N} (Ah^N)^{1-\alpha^N} \tag{1.211}$$

$$Y^T = A^T {k^T}^{\alpha^T} (Ah^T)^{1-\alpha^T} \tag{1.212}$$

$$r^k = \alpha^T \nu^T \frac{Y^T}{k^T} \tag{1.213}$$

$$w = (1-\alpha^T)\nu^T \frac{Y^T}{h^T} \tag{1.214}$$

$$r^k = \alpha^N \nu^N \frac{Y^N}{k^N} \tag{1.215}$$

$$w = (1-\alpha^N)\nu^N \frac{Y^N}{h^N} \tag{1.216}$$

Because of the perfect factor mobility across sectors factor prices are the same across sectors. Now we have the following outcome,

$$\frac{h^T}{h^N} = \frac{\nu^T}{\nu^N} \frac{1-\alpha^T}{1-\alpha^N} \frac{Y^T}{Y^N} \tag{1.217}$$

which depends only on parameters. Since we also normalized the total labour supply to $(\bar{h})$ $\frac{1}{3}$ in the steady state we can trace back individual labour supplies in each sector.

$$h^T + h^N = \bar{h} : \qquad \frac{h^T}{h^N} + 1 = \frac{\bar{h}}{h^N} \tag{1.218}$$

Thus

$$h^N = \frac{\bar{h}}{\frac{h^T}{h^N} + 1} \tag{1.219}$$

$$h^T = \bar{h} - h^T \tag{1.220}$$

Now it is a good time to find individual capital demands for each sector. For this purpose we utilise the arbitrage condition of households between investment and risk-less bonds. This allows us to find the steady state value for the rental rate of the capital in terms of parameters.

$$r^k + 1 - \delta = \frac{1}{\beta} : \qquad r^k = \frac{1}{\beta} - 1 + \delta \tag{1.221}$$

Which is implies the following relation in terms of parameters,

$$\frac{Y^T}{k^T} = \frac{r^k}{\alpha^T \nu^T} \tag{1.222}$$

We divide the production to the capital in each sector,

$$\frac{Y^T}{k^T} = A^T k^{T^{\alpha^T - 1}} (Ah^T)^{1-\alpha^T} = A^T \left(\frac{k^T}{Ah^T} \right)^{1-\alpha^T} \tag{1.223}$$

Then the capital to labour ratio can be expressed in terms of parameters,

$$\frac{k^T}{h^T} = \left(\frac{Y^T}{k^T} \right)^{\frac{1}{1-\alpha^T}} A \left(A^T \right)^{\frac{1}{\alpha^T - 1}} \tag{1.224}$$

Since we have already pinned down relative labour supplies in each sector we can also find the capital in each sector.

$$k^T = \frac{k^T}{h^T} h^T \tag{1.225}$$

In the similar fashion we can pin down $\frac{k^N}{h^N}$ and k^N. Which obviously solves for Y^T and Y^N in the steady state. We also get the aggregate capital stock for the economy

$$k = k^N + k^T \tag{1.226}$$

The investment in the steady state (from the capital accumulation equation) is given by

$$i = \delta k \tag{1.227}$$

Now we have to pin down the gdp in the steady state. Which means we have to decompose Y^T into y^{Td} and y^{Tx}. The idea is to get the gdp components from Y^T and Y^N.

For convenience we repeat the definition of the gdp here once again

$$gdp = \frac{P^{Td}}{P} y^{Td} + \frac{eP^{Tx}}{P} y^{Tx} + \frac{P^N}{P} y^N \tag{1.228}$$

$$Y^T = \left[y^{Tx} + y^{Td} \right] \vartheta^T \tag{1.229}$$

$$y^{Tx} = \frac{\frac{P^{Tm}}{P} \gamma_m \left(\frac{P^{Tm}}{P} \right)^{-\mu_m} - \frac{\tau_1 \tau_2}{1+\tau_1 \tau_2}}{\frac{eP^{Tx}}{P}} z \tag{1.230}$$

$$y^{Td} = \gamma_d \left(\frac{P^{Td}}{P^d} \right)^{-\mu_d} (1 - \gamma_m) \left(\frac{P^d}{P} \right)^{-\mu_m} z \tag{1.231}$$

Now we define a price index of tradable goods ($\frac{P^T}{P}$) (domestically consumed and exported tradables) that is missing in the model. Since the price of z is P we conclude that the price index should have proportional components as follows.

$$\frac{P^T}{P} = \frac{\frac{eP^{Tx}}{P} y^{Tx} + \frac{P^{Td}}{P} y^{Td}}{y^{Tx} + y^{Td}} \tag{1.232}$$

Since y^{Tx} and y^{Td} can be expressed only in terms of 'parameters' and z, z drops from the ratio:

$$\frac{P^T}{P} = \frac{\left\{ \frac{\frac{P^{Tm}}{P} \gamma_m \left(\frac{P^{Tm}}{P} \right)^{-\mu_m} - \frac{\tau_1 \tau_2}{1+\tau_1 \tau_2}}{\frac{eP^{Tx}}{P}} \right\} \frac{eP^{Tx}}{P} + \left\{ \gamma_d \left(\frac{P^{Td}}{P^d} \right)^{-\mu_d} (1 - \gamma_m) \left(\frac{P^d}{P} \right)^{-\mu_m} \right\} \frac{P^{Td}}{P}}{\left\{ \frac{\frac{P^{Tm}}{P} \gamma_m \left(\frac{P^{Tm}}{P} \right)^{-\mu_m} - \frac{\tau_1 \tau_2}{1+\tau_1 \tau_2}}{\frac{eP^{Tx}}{P}} \right\} + \left\{ \gamma_d \left(\frac{P^{Td}}{P^d} \right)^{-\mu_d} (1 - \gamma_m) \left(\frac{P^d}{P} \right)^{-\mu_m} \right\}} \tag{1.233}$$

In the symmetric case (where constraints of the tradable domestic and exporters are the same), simply;

$$\begin{aligned} \frac{P^T}{P} &= \frac{eP^{Tx}}{P} \\ &= \frac{P^{Td}}{P} \end{aligned} \tag{1.234}$$

$$\frac{eP^{Tx}}{P}y^{Tx}+\frac{P^{Td}}{P}y^{Td}=\frac{P^{T}}{P}y^{T}=\frac{P^{T}}{P}\frac{Y^{T}}{\vartheta^{T}} \tag{1.235}$$

Hence we were able to express the gdp in terms of parameters

$$gdp=\frac{P^{T}}{P}\frac{Y^{T}}{\vartheta^{T}}+\frac{P^{N}}{P}\frac{Y^{N}}{\vartheta^{N}} \tag{1.236}$$

Now the cosnumption becomes

$$c=gdp-i-\underbrace{\tau_1 gdp}_{g}-nx \tag{1.237}$$