OG! INMO 2026 Solutions

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30 January 2026

§1 Introduction

§2 P1

Problem. Let $x_1, x_2, x_3, \dots$ be a sequence of positive integers defined as follows: $x_1 = 1$ and for each $n \geq 1$ we have

$$x_{n+1} = x_n + \lfloor \sqrt{x_n} \rfloor.$$

Determine all positive integers m for which $x_n = m^2$ for some $n \geq 1$. (Here $\lfloor x \rfloor$ denotes the greatest integer less or equal to x for every real number x.)

Proposed by Anant Mudgal, Mrudul Thatte and Siddhartha Choppara

¶ Solution(Sanskar): The answer all m such that $m = 2^k$ for some non negative integer k.

Claim — If m^2 occurs in the sequence, then the next perfect square which appears is $(2m)^2$

Proof. Suppose $a_n = m$, now let the next perfect square after a_n be a_{n+k}. Hence, there are no perfect squares in between them.
Using the recurrence relation mentioned:

$$a_{n+k} = a_n + km + \sum_{i=0}^{k-2} \lfloor \frac{i}{2} \rfloor = m^2 + km + \lfloor \frac{(k-2)^2}{4} \rfloor$$

If $k < 2m + 1$, then

$$(m + \frac{k-1}{2})^2 < a_{n+k} < (m + \frac{k}{2})^2$$

But, clearly this means a_{n+k} can't be a perfect square; hence $k \geq 2m + 1$. Based on this we conclude that none of

$$a_{m+1} \dots a_{2m}$$

are perfect squares.
Now, using the recurrence relation we get,

$$a_{n+(2m+1)} = a_n + (2m+1)m + \sum_{i=0}^{2m-1} \lfloor \frac{2m+1}{2} \rfloor = 4m^2 = (2m)^2$$

Thus, a_{n+2m+1} is a perfect square and hence $k = 2m + 1$, so the next perfect square is indeed $(2m)^2$ as desired.
Since $a_1 = (2^0)^2$ is the first perfect square in the sequence; the claim directly implies only and all perfect squares of the form $(2^k)^2$ occur in the sequence, as desired.

§3 P2

Problem. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following condition: for each $k > 2026$, the number $f(k)$ equals the maximum number of times a number appears in the list $f(1), f(2), \ldots, f(k-1)$. Prove that

$$f(n) = f(n + f(n))$$

for infinitely many $n \in \mathbb{N}$.

(Here $\mathbb{N}$ denotes the set $\{1, 2, 3, \ldots\}$ of positive integers.)
Proposed by Anant Mudgal

¶ Solution 1(Sanskar): Firstly, it is clear that f is a non-decreasing function. Assume there are only finitely many such n. Thus, there exists an $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $f(n + f(n)) > f(n)$

Claim — $f(n)$ is unbounded.

Proof. Assume $f(n)$ is bounded for the sake of contradiction.
Let the maximal value f can take be M. Thus, $1 \leq f(n) \leq M$ for all $n \in \mathbb{N}$
Now, $f(n)$ can only take finitely many values; thus, by PHP, there exists some $k \in \mathbb{N}$ that occurs infinitely many times.
If this is the case then for some r; the number of times k appears in the list

$$f(1), f(2) \ldots f(r)$$

must be greater than M, but then $f(r + 1) > M$ is a contradiction!

Now, since $f(n)$ is unbounded , take a $m > N_0$ such that $f(m) > f(i)\ \forall\ i \leq m-1$.

Claim — $f(m + j) = f(m) \forall \quad 0 \leq j \ \leq f(m)$

Proof. We shall induct on j, with base case of $j = 0$ being trivial.
Assume the claim is true for $j = p$, where $p \leq f(m) - 1$.
In the list,

$$f(1), f(2) \ldots f(m-1), f(m) \ldots f(m+p)$$

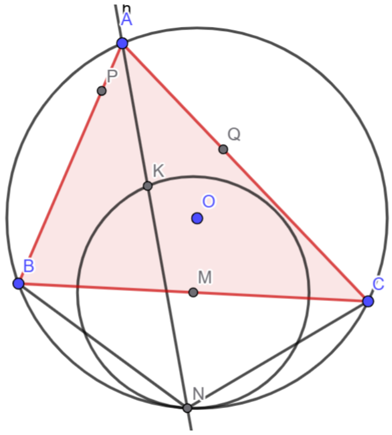
It is clear that maximum no. of times a number occurs is $f(m)$ since,
$f(m) > f(i) \ \ \forall \ \ i \leq m - 1.$ Hence $f(m + p + 1) = f(m)$, completing the induction.
The claim directly implies $f(m + f(m)) = f(m)$ for $m > N_0$, which is a contradiction!

§4 P3

Problem. Let ABC be an acute-angled scalene triangle with circumcircle Γ. Let M be the midpoint of BC and N be the midpoint of the minor arc BC of Γ. Points P and Q lie on segments AB and AC respectively such that $BP = BN$ and $CQ = CN$. Point $K \neq N$ lies on line AN with $MK = MN$. Prove that $\angle PKQ = 90°$.

Proposed by Anant Mudgal

¶ Solution 1(Sanskar): We will complex bash this.



Let the circumcenter of $\triangle ABC$ be O. Let M be the origin.
Let the complex number denoting B and C be $-i$ and i. Let $\angle ABN = \theta$.
Now, let a, b, c, p, q, k, m be complex numbers at A, B, C, P, Q, K, M respectively.
Also, note that AN is the angle bisector of $\angle BAC$ since N is the midpoint of the minor arc BC.
Let complex number at N be n, where $n \in R$ Now,

$$\frac{p-b}{n-b} = e^{i\theta} \iff p = (n+i)e^{i\theta} - i$$

Similarly,

$$\frac{q-c}{c-n} = e^{i(\pi - \angle ACN)} \implies q = (i-n)e^{i\theta} + i$$

. Now, by angle chasing, see that $\angle KMN = 2\pi - 2\theta$. So,

$$\frac{k-m}{m-n} = e^{i(2\theta - 2\pi)} \iff k = -ne^{2i\theta}$$

.

Now,

$$\frac{p-k}{k-q} = \frac{ne^{i\theta} - ne^{2i\theta} + ie^{i\theta} - i}{ne^{2i\theta} + ne^{i\theta} - ie^{i\theta} - i} = \frac{(1-e^{i\theta})(ne^{i\theta} - i)}{(1+e^{i\theta})(ne^{i\theta} - i)} = \frac{1-e^{i\theta}}{1+e^{i\theta}}$$

.
Now, $\frac{1-e^{i\theta}}{1+e^{i\theta}} \in i\mathbb{R}$ because $|1| = |e^{i\theta}|$, $1 + e^{i\theta}; 1 - e^{i\theta}$ will be along the internal and external angle bisector of the lines joining M and 1 and M and $e^{i\theta}$.Thus, they are perpendicular.
Hence,

$$\frac{p-k}{k-q} \in i\mathbb{R} \implies \angle PKQ = 90°$$

§5 P4

Problem. Two integers a and b are called companions if every prime number p either divides both or none of a, b. Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(0) = 0$ and the numbers $f(m) + n$ and $f(n) + m$ are companions for all $m, n \in \mathbb{N}_0$.

(Here $\mathbb{N}_0$ denotes the set of all non-negative integers.)

Proposed by Anant Mudgal

¶ **Solution(Sanskar):** putting $m = 0, n = 1$, we get $f(1) = 1$.

Claim — $f(p) = p$, for all odd primes

Proof. putting $n = 0, m = p$ we get $f(p), p$ are companions thus, $f(p) = p^k$ for some $k \geq 1$. Now putting, $n = 1, m = p$, we get:
$p^k + 1, p + 1$ are companions.

- If k is even, then
$$p^k + 1 \equiv (-1)^k + 1 \equiv 2 \pmod{p+1}$$
. Thus, $\gcd(p+1, p^k+1) = 2$. Thus, 2 must be the only prime dividing $p^k+1, p+1$. Hence, $p^k + 1 = 2^m,\ p + 1 = 2^n$ where $m > n$.

  But this means, $p + 1 \mid p^k + 1$, which means $p + 1 = \gcd(p + 1, p^k + 1) = 2$. This is clearly impossible.

- If k is odd, then $p + 1 \mid p^k + 1$, if $k > 1$,let q be a prime dividing k. Now,
$$p + 1 \mid p^q + 1 | p^k + 1$$
, since $p + 1, p^k + 1$, are companions, $p + 1, p^q + 1$ must also be, let r be an odd prime diving $p + 1, p^q + 1$. Then by lifting the exponent
$$v_r(p^q + 1) = v_r(p + 1) + v_r(q)$$
Thus,
  a) for all odd primes $r \neq q$, $v_r(p^q + 1) = v_r(p + 1)$,
  whilst for $r = q$, $v_r(p^q + 1) = v_r(p + 1) = 0$ or $v_r(p^q + 1) = v_r(p + 1) + 1$.
  b) $v_2(\frac{p^q+1}{p+1}) = v_2(p - 1) - 1$ by lifting the exponent lemma.

  Thus, $\frac{p^q+1}{p+1} = 2^m$ or $q.2^m$, for some $m \leq v_2(p - 1) - 1$. Thus,
$$\frac{p^q + 1}{p + 1} \leq \frac{(p - 1)q}{2} \implies 2(p^q + 1) \leq (p^2 - 1)q \implies 2p^q < p^2q \implies 2p^{q-2} < q$$
  Now, since $p \geq 3$, we must have $3^{q-2} < q$, where q is a odd prime. This is clearly impossible.

  Hence $k = 1$

> **Remark.** Note that in this k=odd case, we could have trivially finished by **zsigmondy's theorem** and immediately get $k = 1$, but we wont do that as it feels unethical. Moreover the thing we did above implies we proved the **weak zsigmondy's** theorem for a particular case.

Since $k = 1$ is the only possibility, we get $f(p) = p$.
Now, put $(m, n) = (p, x)$ in the equation to get $(p + x, p + f(x))$ are companions for all $x \in \mathbb{N}$.
Let q be a prime dividing $p + x$, then

$$q \mid (f(x) + p) \implies q \mid (f(x) + p) - (x + p) \implies q \mid f(x) - x$$

Now, We take a large prime $r > |f(x) - x|, x$ and we set p to be a prime $\equiv -x \pmod r$ (such a prime exists by **dirchlet theorem**).
Now, we have $r > |f(x) - x|$ and that

$$r \mid p + x \implies r \mid f(x) - x$$

. This, means $f(x) - x = 0$ and hence $f(x) = x \quad \forall\ x \in \mathbb{N}$. Now, it can be trivially checked that $f(x) = x$ works.

§6 P5

Problem.

§7 P6

Problem. Two decks $\mathcal{A}$ and $\mathcal{B}$ of 40 cards each are placed on a table at noon. Every minute thereafter, we pick the top cards $a \in \mathcal{A}$ and $b \in \mathcal{B}$ and perform a duel.

For any two cards $a \in \mathcal{A}$ and $b \in \mathcal{B}$, each time a and b duel, the outcome remains the same and is independent of all other duels. A duel has three possible outcomes:

• If a card wins, it is placed back at the top of its deck and the losing card is placed at the bottom of its deck.
• If a and b are evenly matched, they are both removed from their respective decks.
• If a and b do not interact with each other, then both are placed at the bottom of their respective decks.

The process ends when both decks are empty. A process is called a game if it ends. Prove that the maximum time a game can last equals 356 hours.

Proposed by Anant Mudgal and Navilarekallu Tejaswi.

¶ Solution(Sanskar): . We reformulate the problem as follows:

Two wheels $\mathcal{A}$ and $\mathcal{B}$ of n cards equally spaced in a circle are there.
Every minute thereafter, we pick the topmost cards in the wheel $a \in \mathcal{A}$ and $b \in \mathcal{B}$ and perform a duel.

For any two cards $a \in \mathcal{A}$ and $b \in \mathcal{B}$, each time a and b duel, the outcome remains the same and is independent of all other duels. A duel has three possible outcomes:

1. **Type 1**:
   If a card wins, the winning wheel remains unchanged and the losing wheel is rotated **clockwise** 1 units such that the next card adjacent takes the top position .

2. **Type 2**:
   If a and b are evenly matched, they are both removed from the wheels, and the remaining $n-1$ cards in each wheel are rearranged in the same order as they were before, with the topmost card being the one to the left of a and b.

3. **Type 3**:
   If a and b do not interact with each other, both wheels are rotated **clockwise** 1 units such that the next card adjacent takes the top position at the bottom of its deck.

The process ends when both wheels are empty. A process is called a game if it ends.
We now obtain an upper bound on the maximum time a game can last.

Let $f(n)$ be the maximal time a game can last. Now, call a pair $a \in \mathcal{A}$, $b \in \mathcal{B}$ orz if the result of their duel is Type 2, else call the pair $un-orz$.
If a process ends, there clearly must be at least n orz pairs and hence at most, n^2-n $un-orz$ pairs,

Claim — Let the first duel in which Type-2 occurs be the $k-th$ duel. Then, we must have $k \leq n^2 - n + 1$

Proof. Assume $k > n^2 - n + 1$, then in the first $n^2 - n + 1$ duel, all moves must have outcome of Type-1 or Type-3.
Thus, in the first $n^2 - n + 1$ duels all ,pairs dueling must be un-orz but there are at most $n^2 - n$.
Hence, at least 1 un-orz pair must have dueled twice.

Let that pair be $(a, b) \in \mathcal{A} \times \mathcal{B}$
Since the outcome of each pair is fixed, this would mean the pair (a, b) dueling leads to itself dueling again, without any card being eliminated.

So the same set of duels will be repeated infinitely and the game will never end. A contradiction! Now, the claim implies in the first $n^2 - n + 1$ duels, we must have a **Type -2** duel. Thus, after the first $n^2 - n + 1$ duels, both wheels have at most $n - 1$ cards remaining. Thus, it takes at most $f(n-1)$ duels for the remaining game to end.
Thus, we obtain:

$$f(n) \leq n^2 - n + 1 + f(n-1) \implies \sum_{n=2}^{i}(f(n) - f(n-1)) \leq \sum_{n=2}^{i}(n^2 - n + 1)$$

$$\implies f(i) \leq \frac{i(i^2+2)}{3}$$

Now, we prove that $f(n) = \frac{n(n^2+2)}{3}$ is indeed achievable.

Construction:
We start labeling the cards on both wheels in **anti-clockwise** order as $1, 2, \ldots n$ in order.
Now, the outcome of a duel for a pair of card $(a, b) \in \mathcal{A} \times \mathcal{B}$ is as follows:

- If $a + b = n + 1$, the outcome shall be $Type - 2$
- If $a + b = n$, the outcome shall be $Type - 3$
- In all other cases, the outcome is $Type - 1$

We shall prove that for this construction, the process ends in exactly $\frac{n(n^2+2)}{3}$ duels.
For this we shall induct on n, with the base case of $n = 1$ being trivial.
Assume this construction works for $n - 1$, now for n, after performing the duels; it is clear that after exactly $n^2 - n + 1$ duels, both tables will have cards arranged as follows:

- wheel A has cards in the order $1, 2, 3, \ldots n - 1$ anti-clock wise with card 1 being the topmost one.
- wheel B has cards in the order $2, 3, \ldots n$ anti-clockwise with card 2 being the topmost one

Now suppose we renumber the cards in wheel B, assigning the number $b' = b - 1$ for each card in table B.
So, for example, 2 will have the new number 1, n will be newly numbered as $n - 1$.
Now, the original construction is equivalent to, for a pair of card (a, b'), the outcome of the duel is as follows:

- If $a + b' = (n - 1) + 1$, the outcome shall be $Type - 2$
- If $a + b = n - 1$, the outcome shall be $Type - 3$
- In all other cases, the outcome is $Type - 1$.

Clearly, this is equivalent to our construction for $n - 1$ and thus by inductive hypothesis, this takes exactly $\frac{(n-1)((n-1)^2+2)}{3}$ duels to end from here.
Thus, the process ends in exactly $(n^2 - n + 1) + \frac{(n-1)((n-1)^2+2)}{3} = \frac{n(n^2+2)}{3}$.

Hence, our induction is complete and indeed $\frac{n(n^2+2)}{3}$ duels is maximum a game can last, hence the maximum time a game take is

$$\frac{n(n^2+2)}{3}\textbf{minutes}$$

. For $n = 40$, this turns out to be 534*40 minutes or equivalently 356 hours.