OG! Multiplicative functions

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§1 Introduction

OG!

§2 Definitions

Definition 2.1. A function f is called arithmetic if and only if its domain is N and range is a subset of c. Essentially $f : N \mapsto C$

Definition 2.2. An arithmetic function f is called multiplicative if and only if $f(m).f(n) = f(mn)$ for all co prime (m, n)

Definition 2.3. An arithmetic function f is called completely multiplicative if and only if $f(m).f(n) = f(mn)$ for all (m, n)

Definition 2.4. $\phi(n)$(known as euler's totient function) denotes the number of integers coprime to n less than or equal to n.

Definition 2.5. $d(n)$ denotes the no. of divisors of n.

Definition 2.6. $\tau(n) = \sum_{d|n} d$

Definition 2.7. Mobius function is defined as

$$\mu(n) = 0(\text{if } n \text{ is not square free })$$
$$\mu(n) = (-1)^k(\text{if } n \text{ is square free and } >1),\ k \text{ denotes the number of primes dividing } n$$
$$\mu(n) = 1(\text{if } n = 1\ )$$

.

Definition 2.8. Dirchlet delta function is defined as $\delta(n) = 1(n = 1), = 0(n > 1)$

§3 Some Results

Prove it yourself!!!

- $$\phi(n) = n \prod_{p|n,,p\in P} (\frac{p-1}{p})$$

  for $n > 1$.

- $$d(n) = \prod_{p\in P} (v_p(n) + 1)$$

- $$\tau(n) = \prod_{p|n,,p\in P} \frac{p^{v_p(n)+1} - 1}{p - 1}$$

Lemma

OG! If f is a multiplicative function, then $\sum_{d|n} f(d) = q(n)$ is also multiplicative

Proof. Let D_k denote the set of divisors of k Let m, n be 2 co-prime integers. We have the following main claim:

Claim — Any divisor of mn can be uniquely represented as product of 2 numbers one of which is an element of d_m and the other is an element of d_n.

OG! We claim more specifically that for any $d|mn$, $d = xy$ where $x = gcd(d, m)$, $y = gcd(d, n)$ proof for this is left to the reader. on the other hand we see that $x|m, y|n \implies xy|mn$.
Hence, We can easily say that $D_{mn} = D_m \times D_n$, $\times$ denotes the set of the product of the elements of the elements of the cartesian product.

So, we can say

$$q(m)q(n) = (\sum_{d|n} f(d))((\sum_{d|m} f(d)) = \sum_{d_1|n,d_2|m} f(d_1d_2) = \sum_{d_1,d_2\in D_m\times D_n} f(d_1d_2) = \sum_{d|mn} f(d) = q(mn)$$

as desired. □

Question 3.1. Can you use this lemma to prove that τ is multiplicative?

Proposition

$\phi, d, \delta, \tau, \mu$ all are multiplicative functions.

Proof. I will give the proof for ϕ, d rest are exercises.

For d: I will present 2 proofs:
1. let $(m, n) = 1; m = p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3} \cdots p_k^{\alpha_k}; n = q_1^{\beta_1}.q_2^{\beta_2}.q_3^{\beta_3} \cdots q_r^{\beta_r}$

since m, n are co-prime, the prime factorisation of mn= $p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}\cdots p_k^{\alpha_k}.q_1^{\beta_1}.q_2^{\beta_2}.q_3^{\beta_3}\cdots q_r^{\beta_r}$

Thus, $d(mn) = (\alpha_1+1)(\alpha_2+1)\ldots.(\alpha_k+1)(\beta_1+1)(\beta_2+1)\ldots.(\beta_r+1) = d(m)d(n)$ ■

2. Define $f(n) = 1$ for all positive integers n.Obviously, $f(n)$ is multiplicative.

$d(n) = \sum_{d|n} f(n)$. Since f is multiplicative, by our previous mentioned lemma we do have that d is multiplicative as desired.

For ϕ: Consider 2 positive integerm, n such that $(m, n) = 1$ a number is co-prime to mn if and only if it is co- prime to both m, n.
A number is co-prime to m if and only if it is among the $\phi(m)$ coprime residues $\pmod m$.
A number is co-prime to m if and only if it is among the $\phi(n)$ coprime residues $\pmod n$.
Thus, by Chinese remainder theorem, A number is co-prime to mn if and only if it is among the $\phi(m)\phi(n)$ residues $\pmod{mn}$.

Thus, in the range $[1, mn]$ exactly $\phi(m)\phi(n)$ numbers are co-prime to mn. In other words, $\phi(mn) = \phi(m)\phi(n)$

□

§4 Dirchlet Convolution and Mobius Inversion

OG! For any 2 arithmetic functions, f, g, the dirchlet convolution $f * g$ is defined as:

$$f * g(n) = \sum_{d|n} f(d)g(\frac{n}{d})$$

.
So dirchlet convolution is an operation on 2 arithmetic functions, just like $+, -, \times, \div$ are operations on real numbers.

Question 4.1. Can you see that the dirchlet convolution is commutative?
i.e $f * g = g * f$

Lemma (OG!)
Dirchlet Convolution is associative i.e.

$$(f * g) * h = f * (g * h)$$

Proof. observe that both sides are equal to

$$\sum_{xyz=n; x,y,z \in N} f(x)g(y)h(z)$$

Proposition (Relation of function on the basis of dirchlet convolution)

- $\mu * 1 = \delta$
- $\phi * 1 = id$
- $id * 1 = \tau$
- $1 * 1 = d$
- $n * d = \tau * 1$
- $f * \delta = f$(i.e dirchlet delta function is the identity of dirchlet convolution)

Example 4.2

Prove that the dirchlet convolution of 2 multiplicative functions is multiplicative.

Let f, g be 2 multiplicative functions. and let (m, n) be 2 co-prime naturals.
We have that,

$$f * g(mn) = \sum_{d|mn} f(d)g(\frac{mn}{d}) = \sum_{d_1|m, d_2|n} f(d_1d_2)g(\frac{m}{d_1} \cdot \frac{n}{d_2}) =$$

$$\sum_{d_1|m, d_2|n} f(d_1)f(d_2)g(\frac{m}{d_1})g(\frac{n}{d_2}) = \sum_{d_1|m} \sum_{d_2|n} f(d_1)f(d_2)g(\frac{m}{d_1})g(\frac{n}{d_2})) =$$

$$\sum_{d_1|m} f(d_1)g(\frac{m}{d_1}) \sum_{d_2|n} g(\frac{n}{d_2})f(d_2) = f * g(n) \sum_{d_1|m} f(d_1)g(\frac{m}{d_1}) =$$

$$(f * g(n)) \cdot (f * g(m))$$

Now, as we deal with dirchlet convolution, it becomes clear that it is an operator between two arithmetic functions.
But, if there is an operator then we must have a way to get the original function back.
Or in other words the question is to find f in terms of g and h such that
$f * g = h$. We find it as follows: $(h * g^{-1}) = (f * g) * g^{-1} = f * (g * g^{-1}) = f$
Where g^{-1} is a function such that $g^{-1}g = \delta$. If $g = 1$, then $g^{-1} = \mu$.

So if we have that $f * 1 = g$, then $g * \mu = f * (1 * \mu) = f * (\delta) = f$.

This is called **Mobius Inversion**.

§5 Practice Problems:

Problem 5.1. (IMOSL 2004/N2) The function f from the set $\mathbb{N}$ of positive integers into itself is defined by the equality

$$f(n) = \sum_{k=1}^{n} \gcd(k, n), \qquad n \in \mathbb{N}.$$

- Prove that $f(mn) = f(m)f(n)$ for every two relatively prime $m, n \in \mathbb{N}$.
- Prove that for each $a \in \mathbb{N}$ the equation $f(x) = ax$ has a solution.
- Find all $a \in \mathbb{N}$ such that the equation $f(x) = ax$ has a unique solution.

Problem 5.2. For any $n \in \mathbb{N}$, the sum of the nth primitive roots of unity is $\mu(n)$. In other words, prove that

$$\mu(n) = \sum_{1 \le k \le n \, \gcd(k,n)=1} e^{\frac{2\pi i k}{n}}$$

Problem. (USA TST 2010/5) Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = 1$ and, for $n > 1$,

$$a_n = a_{\lfloor n/2 \rfloor} + a_{\lfloor n/3 \rfloor} + \ldots + a_{\lfloor n/n \rfloor} + 1.$$

Prove that there are infinitely many n such that $a_n \equiv n \pmod{2^{2010}}$

Problem. (Turkey NMO(Round 2) 2018/3) A sequence $a_1, a_2, \ldots$ satisfy

$$\sum_{i=1}^{n} a_{\lfloor \frac{n}{i} \rfloor} = n^{10},$$

for every $n \in \mathbb{N}$. Let c be a positive integer. Prove that, for every positive integer n,

$$\frac{c^{a_n} - c^{a_{n-1}}}{n}$$

is an integer.

Define the sequence a_n by

$$\sum_{d|n} a_d = 2^n.$$

Prove that $n \mid a_n$.

Problem. (Sanskar) Consider any $n \in \mathbb{N}$.

$$a_1, a_2 \ldots a_{\phi(n)}$$

are the natural numbers less than or equal to n, which are co-prime to n. Try to find a formula for

$$a_1^m + a_2^m + \cdots + a_{\phi(n)}^m$$

, for a natural number m. The formula could be in a non-closed form.

Problem. (China TST 2024/3) Given a positive integer M. For any $n \in \mathbb{N}_+$, let $h(n)$ be the number of elements in $[n]$ that are coprime to M. Define $\beta := \frac{h(M)}{M}$. Proof: there are at least $\frac{M}{3}$ elements n in $[M]$, satisfy

$$|h(n) - \beta n| \leq \sqrt{\beta \cdot 2^{\omega(M)-3}} + 1.$$

Here $[n] := \{1, 2, \ldots, n\}$ for all positive integer n..
This one is an exceptionally hard problem, so try at your own risk!