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• Démo 1 : def des coeff

$$MSE = \|s - s'\|^2 = (s-s')^T(s-s') = s^Ts - s^Ts' - s'^Ts + s'^Ts'$$
$$= s^Ts - s^T b.\,weff - weff^T b^T s + weff^T b^T b\, weff$$

$$\frac{\partial MSE}{\partial weff} = 0 - (s^Tb)^T - b^Ts + b^Tb\, weff + (weff^T b^Tb)^T = 0$$

$$\Leftrightarrow \; -2b^Ts + 2b^Tb\, weff = 0 \quad \Leftrightarrow \quad weff = (b^Tb)^{-1} b^T s$$

• Démo 2: $C_s = C_{weff}$

$$C_s = E(ss^T) = \begin{bmatrix} \sigma_s^2(1) & \cdots & \\ \vdots & & \\ & & \sigma_s^2(N) \end{bmatrix} \qquad C_{weff} = E(weff\, weff^T) = (b^T s s^T b) = b^T C_s b.$$

$$trace(C_s) = trace(C_{weff})$$

$$C_{weff}' = \begin{bmatrix} \sigma^2_{weff}(1) & \cdots & & \\ & & \sigma^2_{weff}(L) & \\ & & & 0 \end{bmatrix} \Rightarrow trace(C_{weff}') = trace\, C_{weff} - \sigma^2_{weff}(L+1) - \dots \sigma^2_{weff}(N)$$
$$= trace(C_s) - \sum_{k=L+1}^{N^2} \sigma^2_{weff}(k)$$

• Démo 3: $C_s b = b\Lambda$

$$E(MSE) = E\left((s-s')^T(s-s')\right) = E(s^Ts) + E(s'^Ts') - E(s^Ts') - E(s'^Ts)$$
$$= tr(C_s) + tr(C_{weff}') - tr(C_{weff}') - tr(C_{weff}')$$
$$= tr(C_s) - tr(C_{weff}')$$
$$= tr(C_s) - tr(C_s) + \sum_{k=L+1}^{N^2} \sigma^2_{weff}(k)$$
$$= \sum_{k=L+1}^{N^2} \sigma^2_{weff}(k)$$

$$\frac{\partial MSE}{\partial weff} = 0 \Rightarrow \sum_{k=1}^{L} \sigma^2 weff \text{ doit être max.}$$

$$\text{si } L = 1 \rightarrow \begin{cases} \sigma^2_{weff}(1) = C_{weff}(1,1) = b_1^T C_s b_1 \\ b_1^T b_1 = 1 \end{cases} \Rightarrow \frac{d}{db_1}\left(b_1^T C_s b_1 - \lambda(b_1^Tb_1 - 1)\right) = 0$$
$$\frac{d}{db_1}\left(b_1^T(C_s - \lambda I) b_1\right) = 0$$

$$\left[b_1^T(C_s - \lambda I)\right]^T + \left[(C_s - \lambda I) b_1\right] = 0$$

$$2(C_s b_1 - \lambda b_1) = 0$$

$C_s b_1 = \lambda b_1 \rightarrow$ max si λ_1 max

$C_s b = \underline{b}\Lambda$ with $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_L \end{bmatrix}$ with $\lambda_1 \geq \lambda_2 \geq \dots \lambda_L$.

et comme C_s diag avec σ_s^2 sur diag $\Rightarrow \Lambda$ doit être matrice des variances

$C_{weff} = b^T C_s b = b^{-1} b \Lambda = \Lambda$ as $b^T = b^{-1}$ for orthonormal basis.

+ démo propriétés de Fourier

· Démo 4 : $I_t' = \frac{I_t}{s}$

$I_t \in [-T, T]$ pour $f(t)$ alors $f'(t) := f(st) \Rightarrow -T \le st \le T$

$-T \le t \le T$ $\qquad \Rightarrow -\frac{T}{s} \le t \le \frac{T}{s}$

$I_\omega \in [\omega_{min} ; \omega_{max}]$ pour $F(\omega)$ alors $F'(\omega) = \int_{-T}^{T} f'(t)\, e^{-j\omega t}\, dt$

$$= \int_{-T}^{T} f(st)\, e^{-j\omega t}\, dt, \quad u = st$$

$$= \int_{-sT}^{sT} f(u)\, e^{-j\omega \frac{u}{s}} \frac{du}{s}$$

$$= \frac{1}{s} F\left(\frac{\omega}{s}\right)$$

$\omega_{min} \le \omega \le \omega_{max}$

$\omega_{min} \le \frac{\omega}{s} \le \omega_{max}$

$s\,\omega_{min} \le \omega \le s\,\omega_{max}$

· Démo 5 : normalization factor

$$\int |\psi(t)|^2 dt = \int |\psi_{a,0}(t)|^2 dt$$

$$\int |\psi_{a,0}(t)|^2 = \int \frac{1}{\sqrt{a}} \psi\left(\frac{t}{a}\right) \frac{1}{\sqrt{a}} \psi^*\left(\frac{t}{a}\right) dt = \frac{1}{a} \int \psi\left(\frac{t}{a}\right) \psi^*\left(\frac{t}{a}\right) dt$$

$u = \frac{t}{a}$ $\qquad = \frac{1}{a} \int \psi(u)\, \psi^*(u)\, a\, du = \int \psi(u)\, \psi^*(u)\, du = \int |\psi(t)|^2 dt$

· Démo 6 : uncertainty principle

resolution in time $\Delta t^2 = \dfrac{\int t^2 |\psi(t)|^2 dt}{\int |\psi(t)|^2 dt}$

resolution in frequency $\Delta \omega^2 = \dfrac{\int \omega^2 |\Psi(\omega)|^2 d\omega}{\int |\Psi(\omega)|^2 d\omega}$

uncertainty principle : $\Delta t^2 \cdot \Delta \omega^2 \geq \frac{1}{4}$

$$\int t^2 |\psi(t)|^2 dt \cdot \int \omega^2 |\Psi(\omega)|^2 d\omega \geq \frac{1}{4} \int |\psi(t)|^2 dt \int |\Psi(\omega)|^2 d\omega$$

par Parseval : $\int |\psi(t)|^2 dt = \frac{1}{2\pi} \int |\Psi(\omega)|^2 dt$

par rel. de Laplace : $\psi'(t) = -j\omega\, \Psi(\omega) \Rightarrow |\psi'(t)|^2 = -\omega^2 |\Psi(\omega)|^2$

$$\int t^2 |\psi(t)|^2 dt \cdot \cancel{2\pi} \int |\psi'(t)|^2 dt \geq \frac{\cancel{2\pi}}{4} \left(\int |\psi(t)|^2 dt\right)^2 \qquad (1)$$

Cauchy-Schwartz inequality : $\left(\int |f(t)|^2 dt\right) \left(\int |g(t)|^2 dt\right) \geq \left| \int f(t) g(t) \right|^2$

$$\int t^2 |\psi(t)|^2 dt \int |\psi'(t)|^2 dt \geq \left(\int t\, |\psi(t)|\, |\psi'(t)|\, dt \right)^2 =: I^2 \quad (2)$$

$$I = \int \underbrace{t\,|\psi(t)|}_{f} \underbrace{|\psi'(t)|}_{g'} dt = t\,\psi^2(t) \Big|_{-\infty}^{+\infty} - \int \left(\psi(t) + t\,\psi'(t)\right) \psi(t)\, dt = 0 - \int \psi^2(t) - I$$

$\int f g' = fg - \int f' g$

$$I = \frac{1}{2} \int \psi^2(t)\, dt \Rightarrow I^2 = \frac{1}{4} \left(\int \psi^2(t)\, dt \right)^2$$

$\Rightarrow$ equality for $\psi'(t) = k t\, \psi(t)$, $k = -2\alpha$, $\int \psi^2(t) = 1 \Rightarrow \psi = \sqrt[4]{\frac{2\alpha}{\pi}}\, e^{-\alpha t^2}$

- Demo 7: shift property of CWT

$$g(t) = f(t-\tau') \rightarrow CWT_g(a,\tau) = CWT_f(a,\tau-\tau')$$

$$CWT_g(a,\tau) = \frac{1}{\sqrt{a}}\int g(t)\,\psi^*\left(\frac{t-\tau}{a}\right)dt = \frac{1}{\sqrt{a}}\int f(t-\tau')\,\psi^*\left(\frac{t-\tau}{a}\right)dt$$

$u = t-\tau'$
$du = dt$

$$= \frac{1}{\sqrt{a}}\int f(u)\,\psi^*\left(\frac{u+\tau'-\tau}{a}\right)du = \frac{1}{\sqrt{a}}\int f(u)\,\psi^*\left(\frac{u-(\tau-\tau')}{a}\right)du$$

$$= CWT_f(a,\tau-\tau')$$

- Demo 8: scaling property of CWT

$$g(t) = \frac{1}{\sqrt{s}} f\left(\frac{t}{s}\right) \rightarrow CWT_g(a,\tau) = CWT_f\left(\frac{a}{s},\frac{\tau}{s}\right)$$

$$CWT_g(a,\tau) = \frac{1}{\sqrt{a}}\int g(t)\,\psi^*\left(\frac{t-\tau}{a}\right)dt = \frac{1}{\sqrt{a}}\int \frac{1}{\sqrt{s}} f\left(\frac{t}{s}\right)\psi^*\left(\frac{t-\tau}{a}\right)dt$$

$u = \frac{t}{s}$ $\quad du = \frac{dt}{s}$

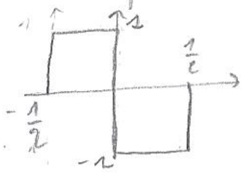
$$= \frac{1}{\sqrt{a}}\frac{1}{\sqrt{s}}\int f(u)\,\psi^*\left(\frac{su-\tau}{a}\right)s\,du$$

$$= \sqrt{\frac{s}{a}}\int f(u)\,\psi^*\left(\frac{u-\frac{\tau}{s}}{\frac{a}{s}}\right)du = CWT_f\left(\frac{\tau}{s},\frac{a}{s}\right)$$

- Demo 9: time localisation of CWT

CWT Haar wavelet of Dirac pulse: $CWT_\delta(a,\tau) = \frac{1}{\sqrt{a}}\psi^*\left(\frac{t_0-\tau}{a}\right)$

$$\text{Haar } \psi(t) = \begin{cases} 1 & -\frac{1}{2}\le t<0 \\ -1 & 0\le t<\frac{1}{2} \\ 0 & \text{else}\end{cases}$$



$\rightarrow \frac{t_0-\tau}{a} = -\frac{1}{2} \Rightarrow t_0-\tau = -\frac{a}{2}$

$\tau = t_0+\frac{a}{2}$

$\rightarrow \frac{t_0-\tau}{a} = \frac{1}{2} \Rightarrow t_0-\tau = \frac{a}{2}$

$\tau = t_0-\frac{a}{2}$

$\rightarrow \frac{t_0-\tau}{a} = 0 \Rightarrow \tau = t_0$

$$CWT\psi_{a,\tau}(t) = \begin{cases} \frac{1}{\sqrt{a}} & t_0\le\tau\le t_0+\frac{a}{2} \\ \frac{-1}{\sqrt{a}} & t_0-\frac{a}{2}\le\tau<t_0 \end{cases}$$

CWT Haar wavelet of step : $CWT_\psi(a,\tau) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} g(t)\, \psi^*\left(\frac{t-\tau}{a}\right) dt$

$\frac{t-\tau}{a} = -\frac{1}{2} \Leftrightarrow t = -\frac{a}{2} + \tau$

$\frac{t-\tau}{a} = 0 \Leftrightarrow t = \tau$

$\frac{t-\tau}{a} = \frac{1}{2} \Leftrightarrow t = \frac{a}{2} + \tau$

$= \frac{1}{\sqrt{a}} \int_{t_0}^{+\infty} \psi^*\left(\frac{t-\tau}{a}\right) dt$

· Demo 10 : frequency localization of CWT

$\psi(\omega) = \int_{-\infty}^{+\infty} \psi(t)\, e^{-j\omega t}\, dt$ donc $\hat{\psi}(\omega) = \int_{-\infty}^{+\infty} \psi\left(\frac{t}{a}\right) e^{-j\omega t}\, dt$

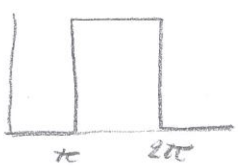
on pose $t_1 = \frac{t}{a} \Leftrightarrow t = a t_1$ et $t\Big|_{-\infty}^{\infty} \Rightarrow t_1\Big|_{-\infty}^{+\infty}\ a>0$

$t_1\Big|_{+\infty}^{-\infty}\ a<0$

donc $\hat{\psi}(\omega) = \frac{|a|}{\cancel{a}} \int_{-\infty}^{+\infty} \psi(t_1)\, e^{-j\omega a t_1}\, \cancel{a}\, dt_1$

donc $\hat{\psi}(\omega) = |a|\, \psi(a\omega)$ $\quad \left(\frac{a}{\sqrt{a}} = \sqrt{a}\right)$

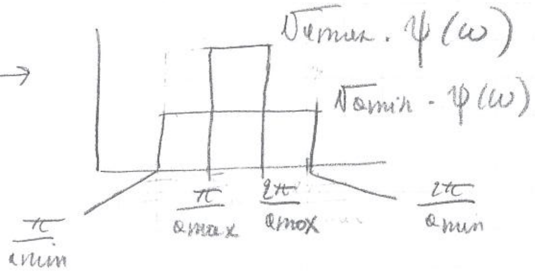
donc $\frac{1}{\sqrt{a}} \psi\left(\frac{t}{a}\right) \xrightarrow{FT} \sqrt{a}\, \psi(a\omega)$

$\psi(\omega) \rightarrow$ 

$\rightarrow \pi \leq \omega \leq 2\pi \Rightarrow \hat{\psi}(\omega) \rightarrow \pi \leq \underset{a\omega}{\omega} \leq 2\pi$

$\downarrow$

$\frac{\pi}{a} \leq \omega \leq \frac{2\pi}{a}$

$\hat{\psi}(\omega) \rightarrow$ 

↳ intéressant pour être localisé en ω_0

$\downarrow$

$\frac{\pi}{a} \leq \omega_0 \leq \frac{2\pi}{a}$

$\Downarrow$

$\frac{\pi}{\omega_0} \leq a \leq \frac{2\pi}{\omega_0}$

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• Demo 11: $\psi'(t) = j\omega\, \Psi(\omega)$

$$F(\psi(t)) = \int \psi(t)\, e^{-j\omega t} = \Psi(\omega) \quad (1)$$

$$\psi(t) = \frac{1}{2\pi} \int \Psi(\omega)\, e^{j\omega t}\, d\omega \quad (2)$$

from (2): $\psi'(t) = \frac{d\psi(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int \Psi(\omega)\, e^{j\omega t}\, d\omega\right)$

$$= \frac{1}{2\pi} \int \Psi(\omega)\, j\omega\, e^{j\omega t}\, d\omega$$

$$= \mathcal{F}^{-1}\left(j\omega\, \Psi(\omega)\right)$$

$$\mathcal{F}(\psi'(t)) = \mathcal{F}\left(\mathcal{F}^{-1}(j\omega\, \Psi(\omega))\right) = j\omega\, \Psi(\omega)$$

• Demo 12: $(f * \psi_s^1)(x,y) = s \frac{d}{dx}(f * \theta_s)(x,y)$

1D: $\psi(x) = \frac{d}{dx}\theta(x)$ $\qquad \psi_s(x) = \frac{1}{s}\psi\left(\frac{x}{s}\right) = \frac{1}{s}\frac{d}{d(x/s)}\theta\left(\frac{x}{s}\right)$

$f = \frac{x}{s}$, $g = \theta(x)$, $g \circ f = \theta\left(\frac{x}{s}\right)$ $\qquad = f'(g' \circ f)$

$\qquad = (g \circ f)'$

$\theta_s(x) = \frac{1}{s}\theta\left(\frac{x}{s}\right)$ $\qquad = \frac{d}{dx}\left(\theta\left(\frac{x}{s}\right)\right)$

$\qquad = \frac{d}{dx}\left(s \cdot \theta_s(x)\right)$

2D: $(f * \psi_s^1)(x,y) = \left(f * s \cdot \frac{d}{dx}\left(\theta_s(x,y)\right)\right) = s \cdot \left(f * \frac{d}{dx}\theta_s\right)(x,y)$

$= s \cdot \frac{d}{dx}(f * \theta_s)(x,y)$ by property of convolution:

$$\frac{\partial}{\partial x}(f * g) = \frac{\partial f}{\partial x} * g = f * \frac{\partial g}{\partial x}$$

• Demo 13: $\mathcal{Z}\{x[n] * y[m]\} = X(z)\, Y(z)$

$$X(z)\, Y(z) = \left(\sum_{n=0}^{\infty} x[n]\, z^{-n}\right)\left(\sum_{m=0}^{\infty} y[m]\, z^{-m}\right) = \sum_n \sum_m x[n]\, y[m]\, z^{-n-m}$$

$n + m = p$

$$= \sum_p \sum_m x[p-m]\, y[m]\, z^{-p} = \sum_p \Bigl(\underbrace{\sum_m x[p-m]\, y[m]}_{=:\, x[p] * y[p] = s[p]}\Bigr) z^{-p}$$

$$= \sum_p s[p]\, z^{-p} = \mathcal{Z}(s[p]) = S(z)$$

• Démo 14 : $y[n] = x[2n] \longrightarrow \mathcal{Z}(y[n]) = Y(z) = \frac{1}{2}\left[X(z^{1/2}) + (-z^{1/2})\right]$

$X_e(z) = \sum_n x[2n]\, z^{-n}$

$X_o(z) = \sum x[2n+1]\, z^{-n}$

$X(z) = \sum x[n]\, z^{-n} = \sum_m x[2m]\, z^{-2m} + \sum_m x[2m+1]\, z^{-2m}$

$= X_e(z^2) + X_o(z^2)\, z^{-1} \quad (1)$

$X(-z) = X_e(z^2) - X_o(z^2)\, z^{-1} \quad (2)$

$$Y(z) = X_e(z) = \frac{(1)+(2)}{2} = \frac{1}{2}\left(X(z^{1/2}) + X(-z^{1/2})\right)$$

• Démo 15 : $y[2n] = x[n],\ y[2n+1] = 0 \rightarrow \mathcal{Z}(y[n]) = Y(z) = X(z^2)$

$$Y(z) = \sum_n y[n]\, z^{-n} = \sum_m y[2m]\, z^{-2m} + \sum_m y[2m+1]\, z^{-2m-1}$$

$$= \sum x[m]\, z^{-2m} + 0$$

$$= X(z^2)$$

• Démo 16 : perfect reconstruction condition

$$\left.\begin{aligned} Y_H(z) &= \tfrac{1}{2}\left[H(z^{1/2})\, X(z^{1/2}) + H(-z^{1/2})\, X(-z^{1/2})\right] \\ Y_G(z) &= \tfrac{1}{2}\left[G(z^{1/2})\, X(z^{1/2}) + H(-z^{1/2})\, X(-z^{1/2})\right] \end{aligned}\right\} \text{ANALYSIS}$$

$$\left.\begin{aligned} \hat{Y}_H(z) &= \tilde{H}(z)\, Y_H(z^2) = \tfrac{1}{2}\tilde{H}(z)\left[H(z)\, X(z) + H(-z)\, X(-z)\right] \\ \hat{Y}_G(z) &= \tilde{G}(z)\, Y_G(z^2) = \tfrac{1}{2}\tilde{G}(z)\left[G(z)\, X(z) + G(-z)\, X(-z)\right] \end{aligned}\right\} \text{SYNTHESIS}$$

$$\hat{X}(z) = \hat{Y}_H(z) + \hat{Y}_G(z) = \frac{1}{2}\begin{pmatrix}\tilde{H}(z) & \tilde{G}(z)\end{pmatrix} \overbrace{\begin{pmatrix} H(z) & H(-z) \\ G(z) & G(-z)\end{pmatrix}}^{H_m(z)} \begin{pmatrix} X(z) \\ X(-z)\end{pmatrix}$$

$$\hat{X}(z) = X(z) \Rightarrow \begin{cases} \tilde{H}(z)\, H(z) + \tilde{G}(z)\, G(z) = 2 \ (I) \Rightarrow \begin{pmatrix}\tilde{H}(z) & \tilde{G}(z)\end{pmatrix} H_m(z) = \begin{pmatrix} 2 & 0\end{pmatrix} \\ \tilde{H}(z)\, H(-z) + G(z)\, G(-z) = 0 \end{cases}$$

$$\begin{pmatrix}\tilde{H}(z) \\ \tilde{G}(z)\end{pmatrix} = \frac{2}{\det H_m(z)} \begin{pmatrix} G(-z) \\ -H(-z)\end{pmatrix}$$

$$\left.\begin{aligned} P(z) &= \tilde{H}(z)\, H(z) = \frac{2}{\det H_m(z)}\, G(-z)\, H(z) \\ P(-z) &= \tilde{G}(z)\, G(z) = \frac{2}{\det H_m(z)}\, -H(-z)\, G(z) \end{aligned}\right\} (I) = P(z) + P(-z) = 2$$

Demo 17: 2 channel filter banks: verification of perfect reconstruction

$$h[n] = \left\{ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \,\middle|\, n=0,1 \right\} \qquad \tilde{h}[n] = \left\{ \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \,\middle|\, n=0,-1 \right\}$$

$$g[n] = \left\{ \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \,\middle|\, n=0,1 \right\} \qquad \tilde{g}[n] = \left\{ \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \,\middle|\, n=0,-1 \right\}$$

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = \frac{\sqrt{2}}{2}\left(z^{-1}+1\right)$$

$$\tilde{H}(z) = \sum_{n=0}^{\infty} \tilde{h}[n] z^{-n} = \frac{\sqrt{2}}{2}\left(1+z\right)$$

$$G(z) = \sum_{n=0}^{\infty} g[n] z^{-n} = \frac{\sqrt{2}}{2}\left(1-z^{-1}\right)$$

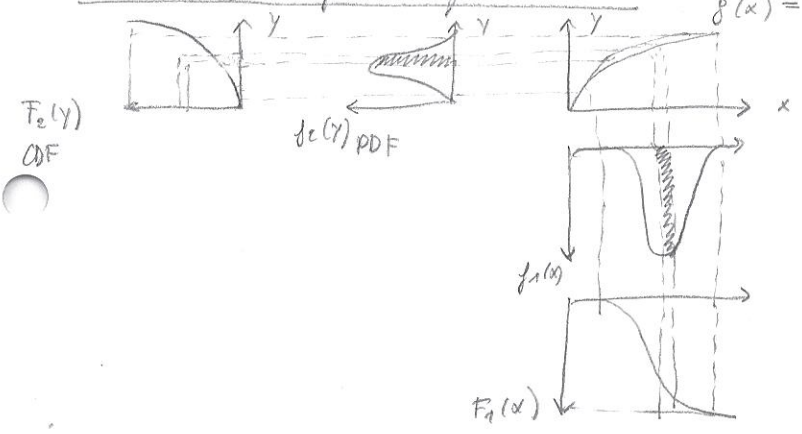
$$\tilde{G}(z) = \sum_{n=0}^{\infty} \tilde{g}[n] z^{-n} = \frac{\sqrt{2}}{2}\left(1-z\right)$$

$$P(z)+P(-z)=2 \Leftrightarrow \tilde{H}(z)H(z)+\tilde{G}(z)G(z)=2 \Leftrightarrow \frac{1}{2}\left(\left(z^{-1}+1\right)(1+z)+\left(1-z^{-1}\right)(1-z)\right)=2$$

$$\Leftrightarrow \frac{1}{2}\left(\cancel{z^{-1}}+1+1+\cancel{z}+1-\cancel{z}-\cancel{z^{-1}}+1\right)=2$$

$$\Leftrightarrow \frac{1}{2}\cdot 4 = 2 \quad \text{CQFD}$$

Demo 18: histogram equalisation



$$f_2(y)\,dy = f_1(x)\,dx$$

$$\forall y: \; F_2(y) = P(Y<y) = P\left(X \le g^{-1}(y)\right) = F_1\left(g^{-1}(y)\right)$$

$$f_2(y) = \frac{dF_2(y)}{dy} = \frac{dF_1\left(g^{-1}(y)\right)}{dg^{-1}(y)}\,\frac{dg^{-1}(y)}{dy} = f_1\left(g^{-1}(y)\right)\frac{dg^{-1}(y)}{dy} = f_1(x)\frac{dx}{dy}$$

If $f_2(Y)$ = uniform $\Rightarrow g = ?$

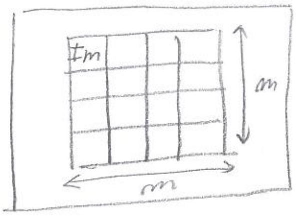
$$\begin{cases} 0 \le y \le 1 \rightarrow f_2(y) = 1 \\ \text{else} \quad f_2(y) = 0 \end{cases}$$

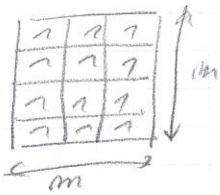
remember $\begin{cases} f_2(y) = f_1(x)\dfrac{dx}{dy} \\ f_2(y) = 1 \end{cases} \Rightarrow dy = f_1(x)\,dx$

$$y = \int_0^x f_1(x)\,dx$$

$$Y = F_1(x) = g(x)$$

· Demo 18: averaging reduces noise variance



$\frac{1}{n^2}$ 

$I_n = s_n + n_n$

noisy pixel   pixel   noise

Average patch: $A = \frac{1}{n^2} \sum_{i=1}^{n^2} I_n$

$$E(A) = \frac{1}{n^2} \sum_{i=1}^{n^2} E(s_n + n_n) = \frac{1}{n^2} \sum_{i=1}^{n^2} s_n \quad \text{for } n_n \sim G(0, \sigma^2)$$

$$Var(A) = E\left\{\left(\frac{1}{n^2} \sum_{i=1}^{n^2} (s_n + n_n) - \frac{1}{n^2} \sum_{i=1}^{n^2} s_n\right)^2\right\}$$

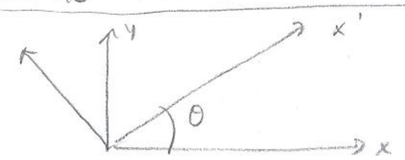
$$= E\left\{\left(\frac{1}{n^2} \sum_{i=1}^{n^2} n_n\right)^2\right\} = E\left\{\left(\frac{1}{n^2} \sum_{i=1}^{n^2} n_n\right)\left(\frac{1}{n^2} \sum_{i=1}^{n^2} n_m\right)\right\}$$

$$= E\left\{\frac{1}{n^2 n^2} \sum_{i=1}^{n^2} \sum_{i=1}^{n^2} \underbrace{n_n \, n_m}_{\rho_n = \begin{cases} \sigma^2 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}}\right\}$$

$$= E\left\{\frac{1}{n^4} \sum_{i=1}^{n^2} \sigma^2\right\}$$

$$= \frac{\sigma^2 n^2}{n^4} = \frac{\sigma^2}{n^2}$$

Demo 18: rotation invariance of gradient



$$x = x' \cos\theta - y' \sin\theta$$

$$y = x' \sin\theta + y' \cos\theta$$

$$\frac{\partial s}{\partial x'} = \frac{\partial s}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial x'} = \frac{\partial s}{\partial x}\cos\theta + \frac{\partial s}{\partial y}\sin\theta$$

$$\frac{\partial s}{\partial y'} = \frac{\partial s}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial s}{\partial y}\frac{\partial y}{\partial y'} = -\frac{\partial s}{\partial x}\sin\theta + \frac{\partial s}{\partial y}\cos\theta$$

$$\left(\frac{\partial s}{\partial x'}\right)^2 + \left(\frac{\partial s}{\partial y'}\right)^2 = \left(\frac{\partial s}{\partial x}\right)^2 \cos^2\theta + \left(\frac{\partial s}{\partial y}\right)^2 \sin^2\theta + \left(\frac{\partial s}{\partial x}\right)^2 \sin^2\theta + \left(\frac{\partial s}{\partial y}\right)^2 \cos^2\theta$$

$$= \left[\left(\frac{\partial s}{\partial x}\right)^2 + \left(\frac{\partial s}{\partial y}\right)^2\right] \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}$$

Démo 20: rotation invariance of laplacian

$$\frac{\partial^2 s}{\partial x'^2} = \left(\frac{\partial s}{\partial x}\right)^2 \cos^2\theta + \frac{\partial s^2}{\partial x \partial y}\cos\theta\sin\theta + \frac{\partial s^2}{\partial y \partial x}\sin\theta\cos\theta + \left(\frac{\partial s}{\partial y}\right)^2 \sin^2\theta$$

Démo 21: SNR

$$SNR = \frac{A \left| \int_{-w}^{0} f(x)\,dx \right|}{n_0 \sqrt{\int_{-w}^{w} f(x)^2\,dx}}$$

$$G(x) + n(x) \longrightarrow \boxed{f} \longrightarrow H(x)$$

By linearity, compute output of $G(x)$ and of $n(x)$ separately:

- $H_G(x) = G(x) * f(x) = \int_{-\infty}^{+\infty} G(x-u) f(u)\,du = A \int_{-\infty}^{0} f(u)\,du$

  $S_{H_G}(x) = |H_G(x)| = A \left| \int_{-\infty}^{0} f(u)\,du \right|$

- $S_{Hn}(\omega) = \mathcal{F}(\rho_{HnHn}(\tau))$ by W-K

  $= S_n(\omega) \cdot |F(\omega)|^2$ for LTI system

$$\rho_{HnHn}(\tau) = \rho_{nn}(\tau) \cdot \mathcal{F}^{-1}\Big(\underbrace{\underbrace{|F(\omega)|^2}_{F(\omega) \cdot F(\omega)^*}}_{f(\tau) * f(-\tau)}\Big)$$

$$= \rho_{nn}(\tau) * f(\tau) * f(-\tau)$$

$$= \sigma^2 * f(0) * f(0) = \sigma^2 \int_{-\infty}^{+\infty} f(u)^2\,du$$

$$S_{Hn}(x) = n_0 \sqrt{\int_{-\infty}^{+\infty} f^2(u)\,du}$$

$$SNR = \frac{A \left| \int_{-w}^{0} f(u)\,du \right|}{n_0 \sqrt{\int_{-\infty}^{+\infty} f^2(u)\,du}}$$

Demo 22: LOC

No noise: $H_G(0)$ max and $H_G'(0) = 0$

Noise: $H_G'(x_0) + H_n'(x_0) = 0 \iff H_G'(x_0) = -H_n'(x_0)$

Taylor: $H_G'(x_0) = \underbrace{H_G'(0)}_{=0} + x_0 H_G''(0)$

$= x_0 H_G''(0)$

$$x_0 = \frac{H_G'(x_0)}{H_G''(0)} = \frac{-H_n'(x_0)}{H_G''(0)}$$

$$\begin{cases} E[H_n'(x_0)^2] = n_0^2 \int_{-\infty}^{+\infty} f'^2(x)\,dx \\ H_G''(0) = \left(\int_{-\infty}^{+\infty} G(-x) f'(x)\,dx\right)^2 \end{cases}$$

$$E(x_0^2) = \frac{n_0^2 \int_{-\infty}^{+\infty} f'^2(x)\,dx}{\left(\int_{-\infty}^{+\infty} G'(-x) f'(x)\,dx\right)^2} \rightarrow LOC = \frac{1}{E(x_0)^2}$$

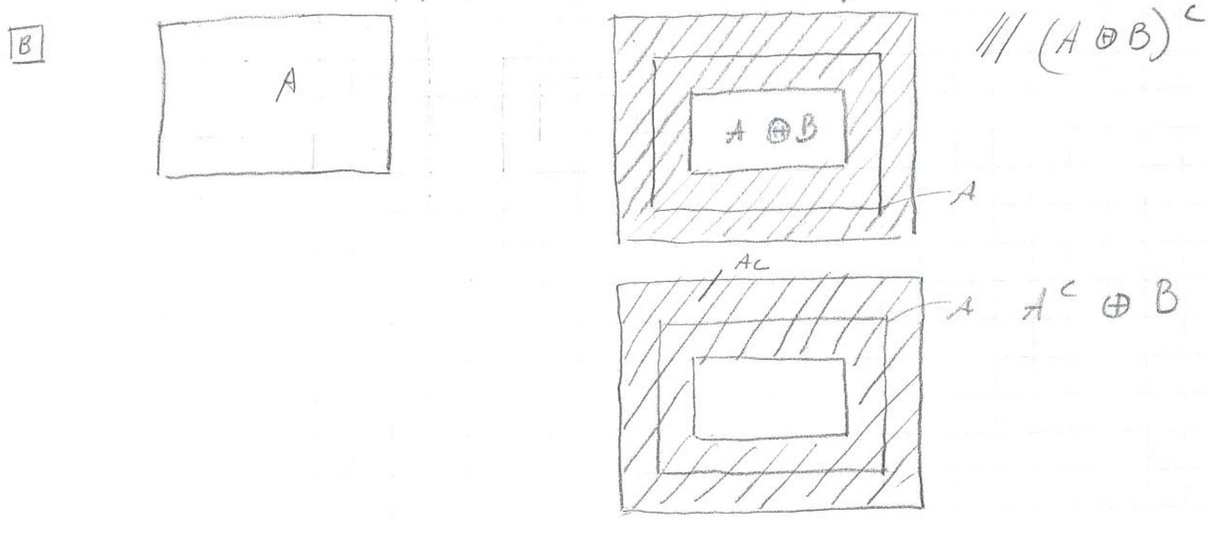
Demo 23: $(A \ominus B)^c = (A^c \oplus B)$

$(A \ominus B)^c = \{x : B_x \subseteq A\}^c$

if $B_x \subseteq A \Rightarrow B_x \cap A = \phi \Rightarrow (A \ominus B) = \{x : B_x \cap A^c = \phi\}^c$

complement of $(B_x \cap A^c = \phi) = (B_x \cap A^c \neq \phi)$

$(A \ominus B)^c = \{x : B_x \cap A^c = \phi\}^c = \{x : B_x \cap A^c \neq \phi\} = A^c \oplus B$



+ démo thm loc de CWT de Sophie avec les log!

De Smet
Engineers & Contractors

· Demo 24: properties of tilts

$g(t) = f(t)\, e^{j\omega_0 t}$

$$\mathcal{F}(g(t)) = \int g(t)\, e^{-j\omega t}\, dt = \int f(t)\, e^{j\omega_0 t}\, e^{-j\omega t}\, dt$$

$$= \int f(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

$$= F(\omega-\omega_0)$$