

1 Global Transforms

1.1 General Model

Represent an image s (with pixel values $s(i,j)$) as a weighted sum of basis images

$$b_k$$

with pixel values

$$b_k(i, j).$$

s : spatial domain (pixel domain); $coeff$: transformed domain (frequency domain)

- Matrix component equation:
 $s(i, j)$, $coeff_k$, $b_k(i,j)$: scalars, $i = 1, \dots, N$; $j = 1, \dots, M$; $L \leq NM$
 L should be equal to $M \times N$ for a lossless transform

$$s(i, j) = \sum_{k=1}^L coeff(k) b_k(i, j) \quad (1)$$

- Matrix component equation with the coefficients ordered as matrix:
 $s(i, j)$, $coeff(k, l)$, $b_{kl}(i,j)$: scalars, $i = 1, \dots, N$; $j = 1, \dots, M$;

$$s(i, j) = \sum_{k=1}^N \sum_{l=1}^M coeff(k, l) b_{kl}(i, j) \quad (2)$$

- forward transform: tool that brings us from the spatial domain to the transformed domain. coefficients used to represent the input
- inverse transform: tool that brings us from the transformed domain to the spatial domain.

1.1.1 Vector Notation

Recipe for matrix to vector conversion:

$$\begin{aligned} s(n, m) &= s(k) \\ k &= m + (n - 1)M \end{aligned} \quad (3)$$

inverse:

$$\begin{aligned} s(k) &= s(n, m) \\ m &= \text{mod}_M(k) \\ n &= 1 + \lfloor \frac{k}{M} \rfloor \end{aligned} \quad (4)$$

vector/matrix notation:

$$s = (b_1, b_2, \dots, b_L)(coeff_1, coeff_2, \dots, coeff_L)^T = bcoeff \quad (5)$$

dimensions:

$$s : [P, 1], coeff : [L, 1], b_k : [P, 1], b : [P, L] \quad (6)$$

$$s = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(NM) \end{bmatrix} = \begin{bmatrix} b_1(1) & b_2(1) & \dots & b_L(1) \\ b_1(2) & b_2(2) & \dots & b_L(2) \\ \vdots & \vdots & \ddots & \vdots \\ b_1(P) & b_2(P) & \dots & b_L(P) \end{bmatrix} \begin{bmatrix} coeff_1 \\ coeff_2 \\ \vdots \\ coeff_L \end{bmatrix} \quad (7)$$

1.1.2 Questions

- How to choose the basis images?
Every transform has its own basis images. The basis images are chosen to be orthogonal to each other.
- How to choose the value of L?
L should be equal to M x N for a losless transform
- What are the physical properties of the coefficients?
(their correlation, their meaning?)
- Numerical problems?
 - calculation of basis images
 - calculation of coefficients for given image
 - storage and speed requirements?
- Under what conditions is the transform invertible?
L should be equal to M x N for a losless transform
- What can we do with such transforms?

1.1.3 Physical meaning of the expansion in basis images

$$s = (b_1, b_2, \dots, b_L)(coeff_1, coeff_2, \dots, coeff_L)^T = b \times coeff \quad (8)$$

example: let s be a 3D vector

$$s = (s_1, s_2, s_3)^T \quad (9)$$

Projection s' of s on a vector b_1 is given by:

$$s' = b_1 \times coeff_1 \quad (10)$$

($coeff_1$ = projection coefficient)

example in 3D: Projection s' of s on a pkane defined by two vectors b_1 and b_2 is given by:

$$s' = b_1 \times coeff_1 + b_2 \times coeff_2 \quad (11)$$

($coeff_1$ and $coeff_2$ = projection coefficients)

to find the coefficients: if b_1 and b_2 are orthogonal, then:

$$coeff_i = b_i^T \times s' \quad (12)$$

which is scalar product of b_i and s' symbol of scalar product operator: \cdot To recover s from s': we need a number of linearly independent vectors equal to the dimension of s.

1.1.4 How to find the coefficients?

Easiest way: minimize the least square error between the original image and the projected image.

$$\frac{\partial SE}{\partial coeff} = 0 \quad (13)$$

$$\frac{\partial SE}{\partial coeff} = \begin{bmatrix} \frac{\partial SE}{\partial coeff_1} \\ \frac{\partial SE}{\partial coeff_2} \\ \vdots \\ \frac{\partial SE}{\partial coeff_L} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (14)$$

Set of L linear equations. SE:

$$s^T s - s^T b coeff - coeff^T b^T s + coeff^T b^T b coeff \quad (15)$$

in general:

$$\frac{\partial(Acoeff)}{\partial coeff} = A^T \quad (16)$$

and

$$\frac{\partial(coeff^T A)}{\partial coeff} = A \quad (17)$$

thus:

$$\frac{\partial SE}{\partial coeff} = 2b^T b coeff - 2b^T s \quad (18)$$

is zero at the solution:

$$\begin{aligned} (b^T b)coeff &= b^T s \\ coeff &= (b^T b)^{-1} b^T s \end{aligned} \quad (19)$$

$(b^T b)^{-1} b^T$ is called the pseudo-inverse of b.
incase vectors bk are orthonormal, then:

$$coeff = b^T s \quad (20)$$

1.1.5 Specializations of the general model: Separable transforms

$$s(i, j) = \sum_{k=1}^N \sum_{l=1}^M coeff(k, l) b_{kl}(i, j) \quad (21)$$

with:

$$\begin{aligned} b_{kl}(i, j) &= b_k(i) c_l(j) && \text{separable} \\ b_{kl}(i, j) &= b_k(i) b_l(j) && \text{symmetric separable} \end{aligned} \quad (22)$$

dimensions:

- $s(i, j)$ $coeff(k, l)$ $b_{kl}(i, j)$: scalars
- $i = 1, \dots, N$; $j = 1, \dots, M$;

$$\begin{aligned}
s(i, j) &= \sum_{k=1}^N \sum_{l=1}^M \text{coeff}(k, l) b_k(i) c_l(j) \\
&= \sum_{k=1}^N \left(b_k(i) \sum_{l=1}^M \text{coeff}(k, l) c_l(j) \right) \\
&= \sum_{k=1}^N b_k(i) z(k, j)
\end{aligned} \tag{23}$$

$s = bz$; $z = \text{coeff } c^T$

$s = b \text{coeff } c^T$

write forward transform as:

$$\text{coeff} = a^T s d \tag{24}$$

write inverse transform as:

$$s = b \text{coeff } c^T \tag{25}$$

From these two it follows that the forward transform followed by the inverse transform yields:

$$s = b a^T s d c^T = (b a^T) s (d c^T) \tag{26}$$

the perfect reconstruction condition requires that:

$$b = (a^T)^{-1} \quad \text{and} \quad c = (d^T)^{-1} \tag{27}$$

in this case, the direct (forward) transform is given by:

$$\text{coeff} = a^T s d = (b)^{-1} s (c^T)^{-1} \tag{28}$$

if $b \neq (a^T)^{-1}$ and $d \neq (c^T)^{-1}$, then the forward-inverse transformation yields an approximation of s:

$$s' = b a^T s d c \cong s \tag{29}$$

orthonormal case: in case of Orthonormality:

$a = b$ and $d = c$; also $b^T = b^{-1}$ and $c^T = c^{-1}$

The direct transform is given by:

$$\text{coeff} = b^T s c = b^{-1} s c \tag{30}$$

1.2 Discrete Karhunen Loeve Transform (KLT)

$(s_1, s_2, \dots, s_k, \dots, s_K)$ sample of an ensemble of $N \times M$ images.

Calculate L basis images such that for each $L \leq P$, the expected "difference" between s and s' is minimized.

$$E(\|s - s'\|^2) \quad \text{is minimal} \tag{31}$$

1.2.1 Notations

- Estimated Mean:

$$m_s = \frac{1}{K} \sum_{k=1}^K s_k \quad m_s, s_k \text{ are vectors} \tag{32}$$

dimension $[NM, 1]$, which is estimation of the mean vector $E\{s\}$

- Estimated Covariance matrix:

$$C_s = \frac{1}{K-1} \sum_{k=1}^K (s_k - m_s)(s_k - m_s)^T \quad (33)$$

dimension $[NM, NM]$, which is estimation of the covariance matrix $E\{(s - E\{s\})(s - E\{s\})^T\}$

variance-covariance matrix expresses how much the different elements of the vector are correlated.

1.2.2 Definition

- b_k are eigenvectors of C_s
- λ_k are eigenvalues of these eigenvectors
- eigenvalues ordered in decreasing order: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_L \geq \dots \geq \lambda_P$
- KLT transformation matrix is given by:

$$b = (b_1, b_2, \dots, b_P) \quad (34)$$

- DKLT is defined as:

$$coeff = b^T(s - m_s) \quad (35)$$

1.2.3 Properties

- $m_{coeff} = 0$
- $C_{coeff} = diag(\lambda_1, \lambda_2, \dots, \lambda_P)$
- The elements of coeff are uncorrelated
- λ_i is equal to the variance of $coeff_i$ along the eigenvector b_i
- Since C_s is a real symmetric matrix, it is always possible to find a set of orthogonal eigenvectors. It therefore follows that s can be reconstructed like this:

$$s = bcoeff + m_s \quad (36)$$