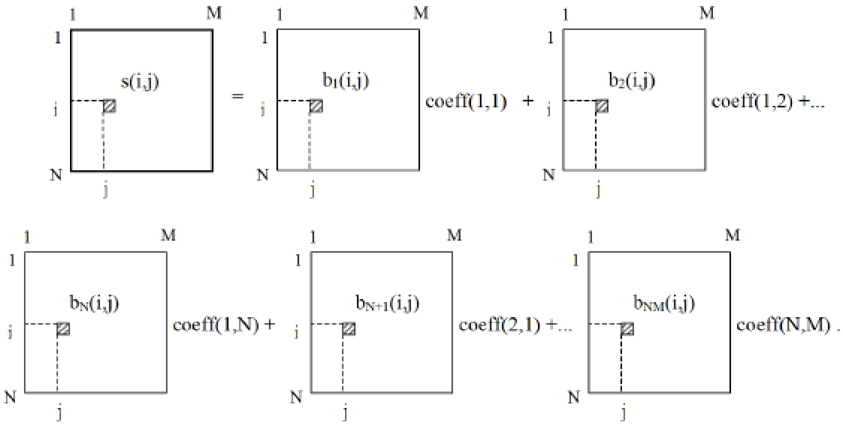
1 Global Transforms

1.1 General Model

Represent an image s (with pixel values s (i,j)) as a weighted sum of basis images b_k with pixel values

$$b_k(i, j).$$



s: spatial domain (pixel domain); coeff: transformed domain (frequency domain)

- Matrix component equation:
  s(i, j) , $coeff_k$, b_k(i,j): scalars, i = 1,. . . , N; j = 1,. . . , M; $L <= NM$
  L should be equal to M x N for a losless transform

$$s(i, j) = \sum_{k=1}^{L} coeff(k) b_k(i, j) \tag{1}$$

- Matrix component equation with the coefficients ordered as matrix:
  s(i, j) , $coeff(k, l)$, b_{kl}(i,j): scalars, i = 1,. . . , N; j = 1,. . . , M;

$$s(i, j) = \sum_{k=1}^{N} \sum_{l=1}^{M} coeff(k, l) b_{kl}(i, j) \tag{2}$$

- forward transform: tool that brings us from the spatial domain to the transformed domain. coefficients used to represent the input
- inverse transform: tool that brings us from the transformed domain to the spatial domain.

1.1.1 Vector Notation

Recipe for matrix to vector conversion:

$$\begin{aligned} s(n,m) &= s(k) \\ k &= m + (n-1)M \end{aligned} \tag{3}$$

inverse:

$$\begin{aligned} s(k) &= s(n,m) \\ m &= mod_M(k) \\ n &= 1 + \lfloor \frac{k}{M} \rfloor \end{aligned} \tag{4}$$

vector/matrix notation:

$$s = (b_1, b_2, \ldots, b_L)(coeff_1, coeff_2, \ldots, coeff_L)^T = bcoeff \tag{5}$$

dimensions:

$$s : [P,1], coeff : [L,1], b_k : [P,1], b : [P,L] \tag{6}$$

$$s = \begin{bmatrix} s(1) \\ s(2) \\ \vdots \\ s(NM) \end{bmatrix} = \begin{bmatrix} b_1(1) & b_2(1) & \ldots & b_L(1) \\ b_1(2) & b_2(2) & \ldots & b_L(2) \\ \vdots & \vdots & \ddots & \vdots \\ b_1(P) & b_2(P) & \ldots & b_L(P) \end{bmatrix} \begin{bmatrix} coeff_1 \\ coeff_2 \\ \vdots \\ coeff_L \end{bmatrix} \tag{7}$$

1.1.2 Questions

- How to choose the basis images?
  Every transform has its own basis images. The basis images are chosen to be orthogonal to each other.
- How to choose the value of L?
  L should be equal to M x N for a losless transform
- What are the physicical properties of the coefficients?
  (their correlation, their meaning?)
- Numerical problems?
  - calculation of basis images
  - calculation of coefficients for given image
  - storage and speed requirements?
- Under what conditions is the transform invertible?
  L should be equal to M x N for a losless transform
- What can we do with such transforms?

1.1.3 Physical meaning of the expansion in basis images

$$s = (b_1, b_2, \ldots, b_L)(coeff_1, coeff_2, \ldots, coeff_L)^T = b \times coeff \tag{8}$$

example: let s be a 3D vector

$$s = (s_1, s_2, s_3)^T \tag{9}$$

Projection s' of s on a vector b_1 is given by:

$$s' = b_1 \times coeff_1 \tag{10}$$

($coeff_1$ = projection coefficient)
example in 3D: Projection s' of s on a plane defined by two vectors b_1 and b_2 is given by:

$$s' = b_1 \times coeff_1 + b_2 \times coeff_2 \tag{11}$$

($coeff_1$ and $coeff_2$ = projection coefficients)
to find the coefficients: if b_1 and b_2 are orthogonal, then:

$$coeff_i = b_i^T \times s' \tag{12}$$

which is scalar product of b_i and s' symbol of scalar product operator: $\cdot$ To recover s from s': we need a number of linearly independent vectors equal to the dimension of s.

1.1.4 How to find the coefficients?

Easiest way: minimize the least square error between the original image and the projected image.

$$\frac{\partial SE}{\partial coeff} = 0 \tag{13}$$

$$\frac{\partial SE}{\partial coeff} = \begin{bmatrix} \frac{\partial SE}{\partial coeff_1} \\ \frac{\partial SE}{\partial coeff_2} \\ \vdots \\ \frac{\partial SE}{\partial coeff_L} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{14}$$

Set of L linear equations. SE:

$$s^T s - s^t bcoeff - coeff^T b^T s + coeff^T b^T bcoeff \tag{15}$$

in general:

$$\frac{\partial (Acoeff)}{\partial coeff} = A^T \tag{16}$$

and

$$\frac{\partial (coeff^T A)}{\partial coeff} = A \tag{17}$$

thus:

$$\frac{\partial SE}{\partial coeff} = 2b^T bcoeff - 2b^T s \tag{18}$$

is zero at the solution:

$$\begin{aligned} (b^T b) coeff &= b^T s \\ coeff &= (b^T b)^{-1} b^T s \end{aligned} \tag{19}$$

$(b^T b)^{-1} b^T$ is called the pseudo-inverse of b.
incase vectors bk are orthonormal, then:

$$coeff = b^T s \tag{20}$$

1.1.5 Specializations of the general model: Separable transforms

$$s(i,j) = \sum_{k=1}^{N} \sum_{l=1}^{M} coeff(k,l) b_{kl}(i,j) \tag{21}$$

with:

$$\begin{aligned} b_{kl}(i,j) &= b_k(i)c_l(j) \quad \text{separable} \\ b_{kl}(i,j) &= b_k(i)b_l(j) \quad \text{symmetric separable} \end{aligned} \tag{22}$$

dimensions:

- s(i,j) coeff(k,l) b_{kl}(i,j) : scalars
- i = 1, ..., N; j = 1, ..., M;

$$\begin{aligned} s(i,j) &= \sum_{k=1}^{N} \sum_{l=1}^{M} coeff(k,l) b_k(i) c_l(j) \\ . &= \sum_{k=1}^{N} \left(b_k(i) \sum_{l=1}^{M} coeff(k,l) c_l(j) \right) \\ . &= \sum_{k=1}^{N} b_k(i) z(k,j) \end{aligned} \tag{23}$$

s = bz; z = coeff c^T
s = bcoeff c^T
write forward transform as:

$$coeff = a^T sd \tag{24}$$

write inverse transform as:

$$s = bcoeffc^T \tag{25}$$

From these two it follows that the forward transform followed by the inverse transform yields:

$$s = ba^T sdc^T = \left(ba^T\right) s \left(dc^T\right) \tag{26}$$

the perfect reconstruction condition requires that:

$$b = (a^T)^{-1} \quad \text{and} \quad c = (d^T)^{-1} \tag{27}$$

in this case, the direct (forward) transform is given by:

$$coeff = a^T sd = (b)^{-1} s (c^T)^{-1} \tag{28}$$

if b $\neq (a^T)^{-1}$ and d $\neq (c^T)^{-1}$, then the forward-inverse transformation yields an approximation of s:

$$s' = ba^T sdc \approxeq s \tag{29}$$

orthonormal case: in case of Orthonormality:

a = b and d = c; also $b^T = b^{-1}$ and $c^T = c^{-1}$

The direct transform is given by:

$$coeff = b^T sc = b^{-1} sc \tag{30}$$

1.2 Discrete Karhunen Loeve Transform (KLT)

(s_1, s_2, ..., s_k, ..., s_K) sample of an ensemble of NxM images.
Calculate L basis images such that for each L <= P, the expected "difference" between s and s' is minimized.

$$E\left(\|s - s'\|^2\right) \quad \text{is minimal} \tag{31}$$

1.2.1 Notations

- Estimated Mean:

$$m_s = \frac{1}{K}\sum_{k=1}^{K} s_k \quad m_s,\ s_k \text{ are vectors} \tag{32}$$

  dimension [NM, 1], which is estimation of the mean vector E{s}

- Estimated Covariance matrix:

$$C_s = \frac{1}{K-1}\sum_{k=1}^{K}(s_k - m_s)(s_k - m_s)^T \tag{33}$$

  dimension [NM, NM], which is estimation of the covariance matrix E{(s - E{s})(s - E{s})^T}

variance-covariance matrix expresses how much the different elements of the vector are correlated.

1.2.2 Definition

- b_k are eigenvectors of C_s
- λ_k are eigenvalues of these eigenvectors
- eigenvalues ordered in decreasing order: $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_P$
- KLT transformation matrix is given by:

$$b = (b_1, b_2, ..., b_P) \tag{34}$$

- DKLT is defined as:

$$coeff = b^T(s - m_s) \tag{35}$$

1.2.3 Properties

- $m_{coeff} = 0$
- $C_{coeff} = diag(\lambda_1, \lambda_2, ..., \lambda_P)$ –> the elements are uncorrelated (dklt decorrelates the input)
- The elements of coeff are uncorrelated
- λ_i is equal to the variance of $coeff_i$ along the eigenvector b_i
- Since C_s is a real symmetric matrix, it is always possible to find a set of orthogonal eigenvectors. It therefore follows that s can be reconstructed like this:

$$s = bcoeff + m_s \tag{36}$$

- if we make L smaller than P

$$s \approxeq b_L coeff + m_s \\ b_L = (b_1, b_2, ..., b_L, 0, 0, 0) \tag{37}$$

- Mean Square Error (MSE) is given by:

$$MSE = \sum_{j=1}^{P} \lambda_j - \sum_{j=1}^{L} \lambda_j = \sum_{j=L+1}^{P} \lambda_j \tag{38}$$

  The DKLT is optimal in the least square sense.

1.2.4 Proof and Construction of KLT basis images

Assumptions:

- s is a square image: [N^2, 1] s': [L, 1] L<=N^2
  Hypothesis: E{s} = 0 E{s'} = 0
- Impose that E{$\|s - s'\|^2$} is minimal ∀ L
- orthonormal basis b=(b_1, b_2, ..., b_{N^2})
  b^Tb = I

forward transform: coeff=b^{T}s
C_s = E{ss^T} correlation of first element with itself, first element with second element, ...

$$\begin{bmatrix} \sigma_s^2(1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_s^2(N^2) \end{bmatrix} \tag{39}$$

C_{coeff} = E{coeff coeff^T} correlation of first element with itself, first element with second element, ...

$$\begin{bmatrix} \sigma_{coeff}^2(1) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \sigma_{coeff}^2(N^2) \end{bmatrix} \tag{40}$$

$C_{coeff} = E\{b^T ss^T b\} = b^T E\{ss^T\} b = b^T C_s b$ −> Link between C_s and C_{coeff}
tr{C_s} = tr{C_{coeff}} = $\sum_{i=1}^{N^2} \sigma_{coeff}^2(i) = \sum_{i=1}^{N^2} \sigma_s^2(i)$

inverse transform

- **L =** N^2 s = bcoeff

$$b = \begin{bmatrix} b_{1,1} & \dots & b_{1,N^2} \\ \vdots & \ddots & \vdots \\ b_{N^2,1} & \dots & b_{N^2,N^2} \end{bmatrix} = \begin{bmatrix} b_1 & \dots & b_{N^2} \end{bmatrix} \tag{41}$$

- **L** < N^2 s' = bcoeff

$$b' = \begin{bmatrix} b_{1,1} & \dots & b_{1,L} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{N^2,1} & \dots & b_{N^2,L} & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} b_1 & \dots & b_L & 0 & \dots & 0 \end{bmatrix} \tag{42}$$

  or s' = bcoeff', where:

$$coeff'(i) = \begin{cases} coeff(i) & \text{if } i = 1, \dots, L \\ 0 & \text{if } i = L+1, \dots, N^2 \end{cases} \tag{43}$$

Variance-covariance matrices of approximate set of coefficients (L $<= N^2$)

$$
\begin{aligned}
C_{coeff'} &= \begin{bmatrix} \sigma^2_{coeff}(1) & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \sigma^2_{coeff}(L) & \\ 0 & \dots & \dots & 0 \end{bmatrix} \\
tr\{C_{coeff'}\} &= tr\{C_s\} - (\sigma^2_{coeff}(L+1) + \dots + \sigma^2_{coeff}(N^2))
\end{aligned}
\tag{44}
$$