Signal::

→ Signal is the function of any independent param like time, space, co-ordinates which conveys th information of any physical phenomenal

eg.

$$x(t) = t^2 + 3t \rightarrow \text{one dimensional}$$

$$f(x,y) = x^2 + 3xy \rightarrow \text{Two-dimensional}$$

→ Signal may contain no information

eg: noise

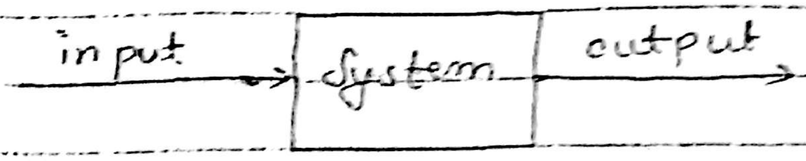
System:

→ System is a physical device which manipulates th input signal and yields a new output

→ Software (not physical) is also a system.

→ not new output is also a system (eg. bufferin

→ A system is always associated with generat of signal and another system is associated wit extraction of information from the signal.



Application of Digital Signal Processing :-

- Communication :-
  - Telecommunication :-
    - input → voice
    - output → estimated voice
  - Radar (Radio Detection and Ranging)
  - SONAR (Sound Navigation and Ranging)
  - Astronoutics :-
    - → Radar image
    - → Infrared image
  - Acoustics
  - Biomedical
    - ECG
    - EEG
    - EMG
    - X-ray (gives the image of object containing calcium & phosphorous)
    - C-T scan
  - Speech Processing
  - Seisomology
  - Simulation

1. Real Valued & Complex Valued :-

$\downarrow$ $\downarrow$

$x(t) = t^2 + 3t^2$ $x(t) = e^{j\omega t}$

$= \cos \omega t + j \sin \omega t$

2. Multidimensional & Multichannel :

A signal is said to be multidimensional if it depends on more than one independent variable.

Some signals are generated by a no. of sources or sensors and represented in a vector form.

for eg:-

- Colour Television picture is generated by a colour intensity of Red, Green and Blue & Represented as ;

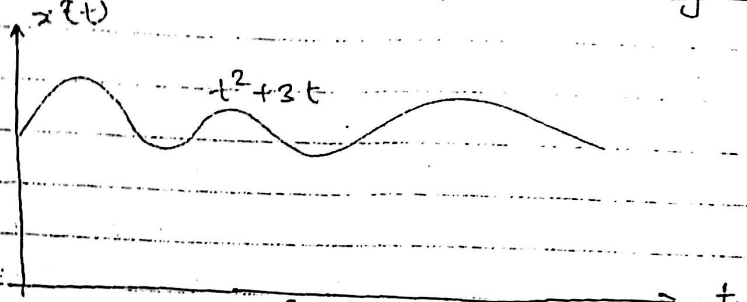
$$[I] = \begin{bmatrix} I_R \\ I_G \\ I_B \end{bmatrix}$$

- Colour TV picture is a function of X-axis, Y-axis and time 't'. Hence it is 3-dimensional & 3-channel signal.

$$[I(x,y,t)] = \begin{bmatrix} I_R(x,y,t) \\ I_G(x,y,t) \\ I_B(x,y,t) \end{bmatrix}$$

3. Continuous time and Discrete time Signal :-

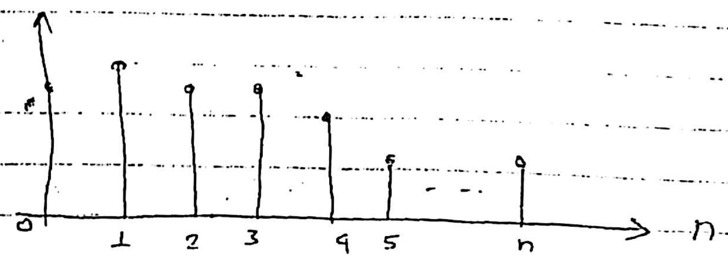
→ If the signal possess a value for each and every value of time 't' with in a range then it is called <u>continuous time signal</u>.



$x(t) = t^2 + 3t$, $a < t < b$

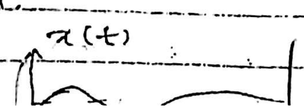
→ If the signal posses a value only for some finite time instant, it is called <u>Discrete time signal</u>.

eg: Price of gold everyday.

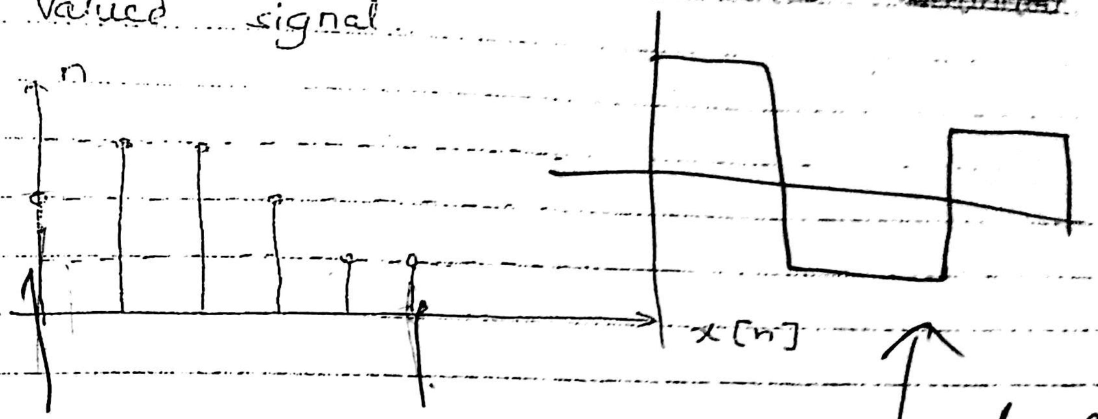


4. Continuous Valued & Discrete Valued Signal :-

→ If signal possess infinite set of values/amplitu within a range then it is called continuous valued signal.



If signal possess finite set of values / amplitude with in a range ~~of~~ then it is called ~~continuous~~ discrete valued signal.



discrete valued continuous signal

☺ Digital signal :-
↳ Discrete in time
↳ Discrete in amplitude / Value

5. Deterministic and Random Signal:
→ Deterministic signal are represented by a mathematical expression, graphical representation or by a set of rules.

→ Random signals are non-deterministic
eg: noise, speech
- The study and analysis of random signal is done by Statistical Analysis / technique.

2072-2-19

6. Odd and Even Signal:-
→ A signal $x[n]$ is said to be odd if $x[n] = -x[-n]$ for all n.
eg: $x[n] = \sin \omega_0 n$

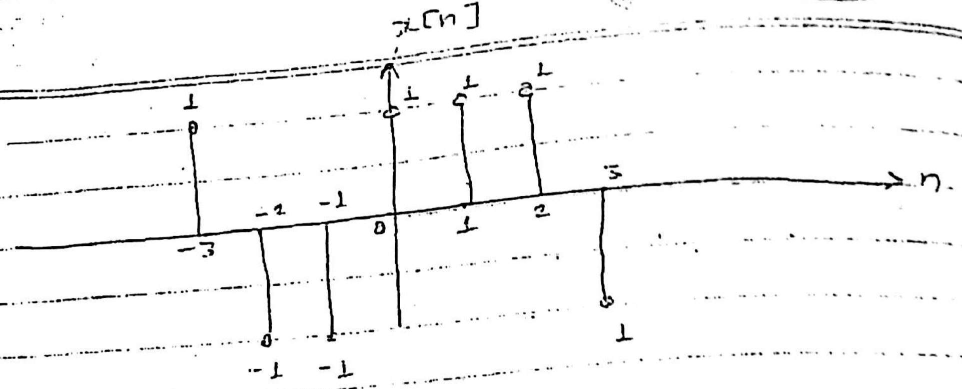


Fig: odd signal.

→ A signal is said to be even if $x[n] = x[-n]$ for all n.

eg: $x[n] = \cos \omega_0 n$

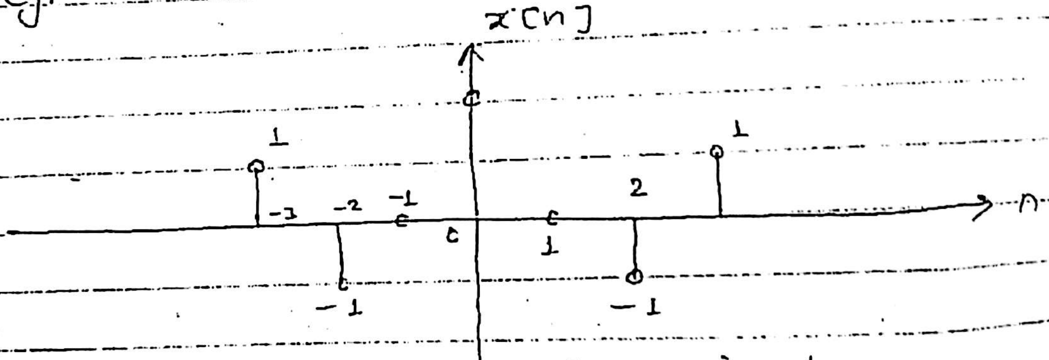


Fig: Even signal

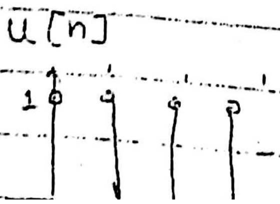
$$x[n] = x_o[n] + x_e[n]$$

$$x_o[n] = \frac{x[n] - x[-n]}{2} \quad \frac{x[n] + x[n]}{2}$$

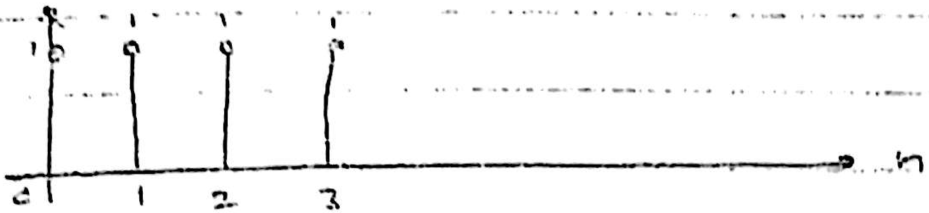
$$x_e[n] = \frac{x[n] + x[-n]}{2}$$

H.W. $x[n] = u[n] - u[n-4]$
Draw it's odd and even component.

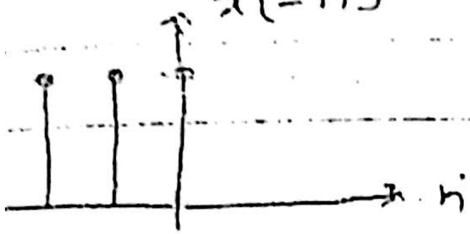
Sol^n



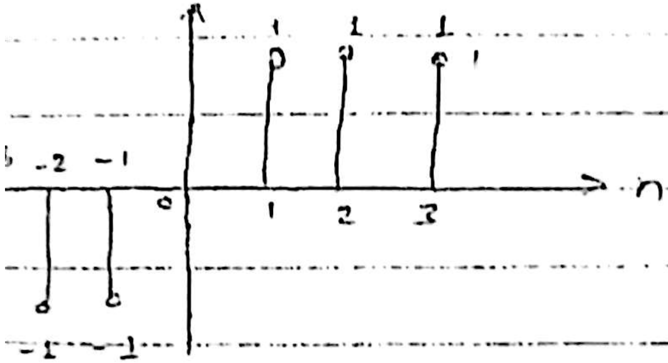
$x[n] = u[n] - u[n-4]$

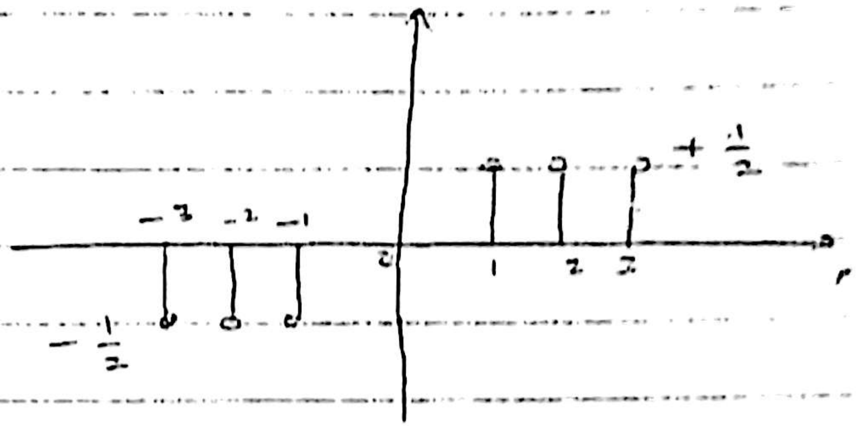


$x[-n]$

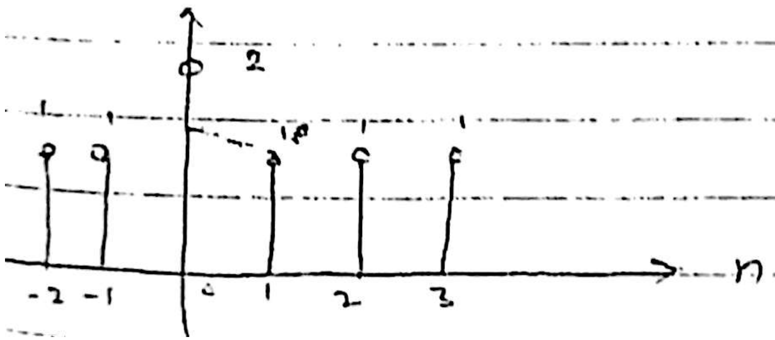


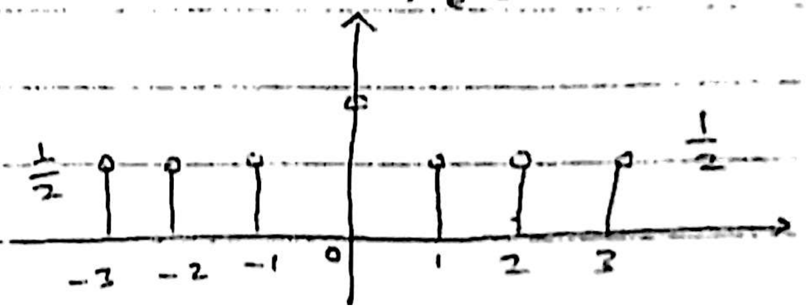
$x[n] - x[-n]$ $\Rightarrow$ $x_o[n]$





$x[n] + x[-n]$ $\Rightarrow$ $x_e[n]$





$\omega_0 = \frac{2\pi}{N} = 2\pi f_0 \rightarrow$ angular forquency.

$\rightarrow$ A signal which doesn't satifies above condition is called aperiodic signal.

Examples:

* $x[n] = \cos \frac{3\pi}{7} n$

$\Rightarrow$

$x[n] = \cos(\omega_0 n + \phi)$

Comparing:

$A=1, \ \phi = 0; \quad \omega_0 = \frac{3\pi}{7}$

$\Rightarrow \frac{2\pi}{N} = \frac{3\pi}{7}$

$\Rightarrow N = \frac{14}{3}$ (rational)

$\rightarrow$ Periodic with $N = 14$.

$f = \frac{3}{14}$

* $x[n] = \cos \sqrt{2}\pi n$

$\Rightarrow$

Comparing with $x[n] = A \cos(\omega_0 n + \phi)$

$A = 1, \ \phi = 0, \quad \omega_0 = \sqrt{2}\pi$

$\Rightarrow \frac{2\pi}{N} = \sqrt{2}\pi$

$\Rightarrow N = \sqrt{2}$ (irrational)

$\rightarrow$ Aperiodic.

* $x[n] = \sin \frac{3\pi}{5} n + \cos \frac{4\pi}{7} n$

$\Rightarrow$

$\sin \frac{3\pi}{5} n \Rightarrow \quad \omega_0 = \frac{3\pi}{5} = \frac{2\pi}{N_1}$

$\Rightarrow N_1 = \frac{10}{3}$

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$\cos \frac{4\pi}{7} n \Rightarrow \quad \omega_0 = \frac{4\pi}{7} \Rightarrow N_2 = \frac{7}{2}$

Since:

$$\frac{N_1}{N_2} = \frac{10/3}{7/2} = \frac{20}{21} \text{ (rational)}$$

$\therefore$ Periodic with $N = 21 * N_1 = 20 * N_2$

$$= 21 * \frac{10}{3} = 20 * \frac{7}{2}$$

$$= 70$$

i. $x[n] = \cos \frac{2\pi}{9} n + \sin \frac{3\pi}{11} n.$

sol^n

$\cos \frac{2\pi}{9} n \Rightarrow \omega_0 = \frac{2\pi}{9} = \frac{2\pi}{N_1}$

$$\Rightarrow N_1 = 9$$

$\sin \frac{3\pi}{19} n \Rightarrow \omega_0 = \frac{3\pi}{11} = \frac{2\pi}{N_2}$ $\boxed{N = 22N_1 = 27N_2}$

$\Rightarrow N_2 = \frac{22}{3}$ $N = 22 \times 9.$

Since:

$$\frac{N_1}{N_2} = \frac{9}{22/3} = \frac{27}{22} \text{ (rational)}$$

$\frac{N_1}{N_2} = \frac{27}{22}.$

$22N_1 = 27N_2 =$

$\therefore$ Periodic with $N = 22N_1 = 27N_2$

$$= 22 * 9 = 27 * \frac{22}{3}$$

$$= 198 \quad = 198$$

$1 \times 27 > 3$

ii. $x[n] = \cos \pi n + \sin \pi n$

sol^n

$\cos \pi n \Rightarrow \omega_0 = \pi = \frac{2\pi}{N_1}$

$$\Rightarrow N_1 = 2$$

5

$\sin \pi n \Rightarrow \omega_0 = \frac{2\pi}{N_2} = \pi$

$\Rightarrow N_2 = \cancel{4}\ 2$

Since, $\frac{N_1}{N_2} = \frac{\cancel{2}}{\cancel{2}} = 1$

$\therefore$ Periodic with $N = N_1 = N_2$

$= 2$ Ans

8. Power and Energy type signal :-

for given $x[n]$;

Total energy of $x[n]$;

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

for aperiodic signal:

Average Power; $P_{avg} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$

For Periodic signal;

Average power in one fundamental Period;

$$P_{avg.} = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

→ A signal is said to be energy type if it energy is finite and average power is zero.

i.e. $0 < E < \infty$ and $P_{avg.} = 0$.

eg: most of aperiodic signal

→ A signal is said to be power type if its average power is finite and energy is infinite.

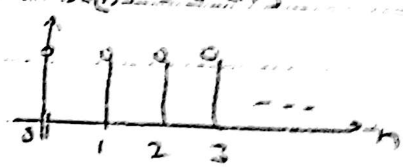
i.e. $0 < P_{avg} < \infty$ & $E \to \infty$

for eg :- Most of periodic signal.

Examples :

* $x[n] = u[n]$

⇒ $$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=-\infty}^{\infty} |u[n]|^2$$

$$= \sum_{n=0}^{\infty} 1$$

$$= \infty$$

$$P_{avg} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |u[n]|^2$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1$$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \times (N+1)$$

$$= \lim_{N \to \infty} \frac{1 + \frac{1}{N}}{2 + \frac{1}{N}}$$

$$= \frac{1}{2}$$

Since, Avg. Power is finite and $E \to \infty$. Hence this is power type signal.

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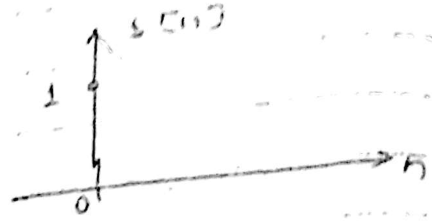
*. $x[n] = \delta[n]$.



Sol: $\delta[n] = \begin{cases} 1, & \text{for } n=0 \\ 0, & \text{for } n \neq 0 \end{cases}$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=-\infty}^{\infty} |\delta[n]|^2$$

$$= 1$$

$$P_{avg} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |\delta[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \cdot 1$$

$$= \frac{1}{\infty}$$

$$= 0$$

Since, Energy is finite & P_{avg} is zero. Hence It is Energy type signal

*. $x[n] = e^{j\frac{3\pi}{4}n}$

⇒ Normal exp. signal;

$$x[n] = A\, e^{j(\omega_0 n + \phi)}$$

$$A=1, \ \phi = 0, \ \omega_0 = \frac{2\pi}{N} = \frac{3\pi}{4}$$

$$\Rightarrow N = \frac{8}{3} \text{ (rational)}$$

Periodic with $N = 8$.

Energy of periodic signal : $E = \infty$

Average Power of one fundamental Period is :

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

$$= \frac{1}{8} \sum_{n=0}^{N-1} \left| e^{j\frac{3\pi}{4}n} \right|^2$$

$$= \frac{1}{8} \sum_{n=0}^{7} 1$$

$$= \frac{1}{8} \times 8$$

$$= 1$$

Since, P_{avg} is finite & $E \to \infty$. Hence it is power type.

w. $x[n] = 3e^{j\frac{5\pi}{3}n}$

$\Rightarrow$ Comparing, $A = 3$, $\phi = 0$, $\omega_0 = \frac{5\pi}{3} = \frac{2\pi}{N}$

$$\Rightarrow N = \frac{6}{5} \text{ (rational)}$$

Periodic signal with period $N = 6$

Energy of Periodic signal : $E \to \infty$

Avg. Power, $P_{avg} = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$

$$= \frac{1}{6} \sum_{n=0}^{5} \left| 3 \cdot e^{j5\pi/3 n} \right|^2$$

7

$$= \frac{1}{6} \sum_{n=0}^{5} 3^{2 \cdot 1}$$

$$= \frac{1}{6} \sum_{n=0}^{5} 9$$

$$= \frac{1}{6} * 6 \times 9$$

$$= 9 \text{ finite.}$$

Hence, this is power type signal. <u>Ans</u>

<u>2072-02-31</u>

<u>System:-</u>

$$\underset{x[n]}{\text{Input}} \longrightarrow \boxed{\text{System}} \longrightarrow \underset{y[n]}{\text{output}}$$

Type of e system (input - output relationship):

1. Linear System:-
   If input $x_1[n] \longrightarrow$ output $y_1[n] = T\{x_1[n]\}$
   input, $x_2[n] \longrightarrow$ output $y_2[n] = T\{x_2[n]\}$.
   Then a refaxed time ~~system~~ is called linear time system iff

input: $a_1 x_1[n] + a_2 x_2[n] \longrightarrow$ output $y_3[n]$

$$= T\{a_1 x_1[n] + a_2 x_2[n]\}$$

$$= a y_1[n] + a_2 y_2[n]$$

The system having superposition symmetry is called linear.

example.

① $y[n] = x[n^2]$

$y_1[n] = T\{x_1[n]\} = x_1[n^2]$

$y_2[n] = T\{x_2[n]\} = x_2[n^2]$

$y_3[n] = T\{a_1 x_1[n] + a_2 x_2[n]\}$

$= a_1 x_1[n^2] + a_2 x_2[n^2]$

$y_4[n] = a_1 y_1[n] + a_2 y_2[n]$

$= a_1 x_1[n^2] + a_2 x_2[n^2]$

Since $y_3[n] = y_4[n]$, System is linear.

② $y[n] = x^2[n]$

$y_1[n] = T\{x_1[n]\} = x_1^2[n]$

$y_2[n] = T\{x_2[n]\} = x_2^2[n]$

$y_3[n] = T\{a_1 x_1[n] + a_2 x_2[n]\}$

$= (a_1 x_1[n] + a_2 x_2[n])^2$

$y_4[n] = a_1 y_1[n] + a_2 y_2[n]$

$= a_1 x_1^2[n] + a_2 x_2^2[n]$

Since $y_3[n] \neq y_4[n]$ System is Non-linear.

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$y[n] = Ax[n] + B$

$y_1[n] = Ax_1[n] + B = T\{x_1[n]\}$

$y_2[n] = T\{x_2[n]\} = Ax_2[n] + B$

$y_3[n] = T\{a_1 x_1[n] + a_2 x_2[n]\}$

$= A\{a_1 x_1[n] + a_2 x_2[n]\} + B$

$= a_1 A x_1[n] + a_2 A x_2[n] + B$

$y_4[n] = a_1 y_1[n] + a_2 y_2[n]$

$= a_1(Ax_1[n] + B) + a_2 A x_2[n] + a_2$

$= a_1 A x_1[n] + a_1 B + a_2 A x_2[n] + a_2$

Though the system doesnot satisfy the condition of linearity, it can not be said non linear. Because,

for $x[n] = 0$, $y[n] = B$

But it can be said linear if $B = 0$

2072-02

ii. Memory and Memoryless System:

→ A system is said to be memoryless if its $y[n]$ at $n = n_0$ only depends upon the input $x[n]$ at $n = n_0$

→ A system is said to be memory type if its $y[n]$ at $n = n_0$ also depends upon the input at $n \neq n_0$

eg: $y[n] = kx[n]$ ← This is memory systems.

for $n = 2$; $y[2] = kx[2]$

⊛ $y[n] = x[n] + 2x[n-1]$

↳ This is memory system.

iii. Causal & Non-Causal System :-

→ A system is said to be causal if it's o/p $y[n]$ at $n = n_0$ only depends upon the i/p $x[n]$ for $n \le n_0$.

→ A system is said to be Non-causal if it's o/p $y[n]$ at $n = n_0$ also depends upon the input $x[n]$ at n

eg:-

⊛. $y[n] = x[n] + 2x[n-1]$ → memory & Causal

⊛. $y[n] = x[n] + x[n+1]$ → memory & Non-causal

⊛. $y[n] = \sum_{k=-\infty}^{n} x[k]$ → memory & Causal

⊛. $y[n] = x[-n]$ → memory & Non-Causal

iv. Time variant & Time invariant System:

→ A system is said to be time invariant if it's characteristics and behaviour is fixed over time

Mathematically:-

$x[n] \longrightarrow$ System $\longrightarrow y[n]$

If input $x[n] \longrightarrow$ output:

$y[n] = T\{x[n]\}$

If input : $x[n-k] \longrightarrow$ output;

$\underline{y[n,k]} = T\{x[n-k]\}$

Then, this system is called time invariant,

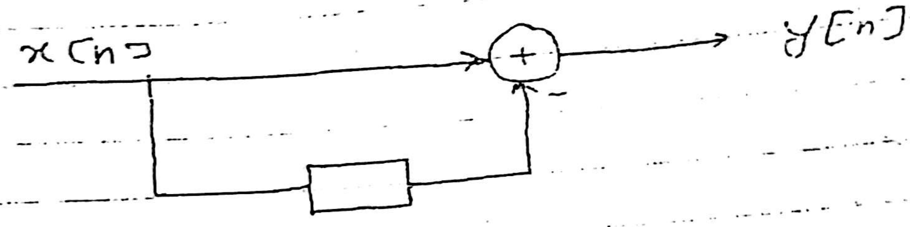
iff: $\underline{y[n,k] = y[n-k]}$

Otherwise, System will be time Variant.

example :-

$$y[n] = x[n] - x[n-1]$$

$$y[n,k] = T\{x[n-k]\}$$



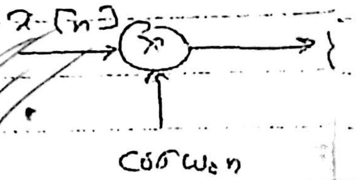
$$y[n,k] = x[n-k] - x[n-k-1]$$

$$y[n-k] = x[n-k] - x[n-k-1]$$

Since, $y[n,k] = y[n-k]$, system is time invari

Ex $y[n] = \cos\omega_0 n \cdot x[n]$

$$y[n,k] = T\{x[n-k]\}$$
$$= \cos\omega_0 n \cdot x[n-k]$$



$$y[n-k] = \cos\omega_0(n-k) \cdot x[n-k]$$

Since, $y[n,k] \neq y[n-k]$. System is not time invariant.

(v) Stable and Unstable System :

A system is said to be bounded input bounded output (BIBO) stable if bounded input $x[n]$ produces a bounded output

If; $|x[n]| < k_1 < \infty$

then $|y[n]| < k_2 < \infty$ stable

otherwise system will be unstable.

examples– ⓐ $y[n] = x[n] - x[n-1]$

let:

$$x[n] = u[n]$$

$$|x[n]| \leq 1 < \infty$$

$$x[\infty] = u[\infty] = 1$$

$$y[n] = u[n] - u[n-1]$$

$$y[\infty] = u[\infty] - u[\infty-1]$$

$$= 1 - 1$$

$$= 0 < \infty$$

So, the system is BIBO stable.

Linear Time Invariant (LTI) system:

- A system having superposition symmetry and fixed characterstic and behaviour over time then it is called Linear time Invariant (LTI) system. An LTI system is completely characterized by its impuse response.

- Convolution sum:

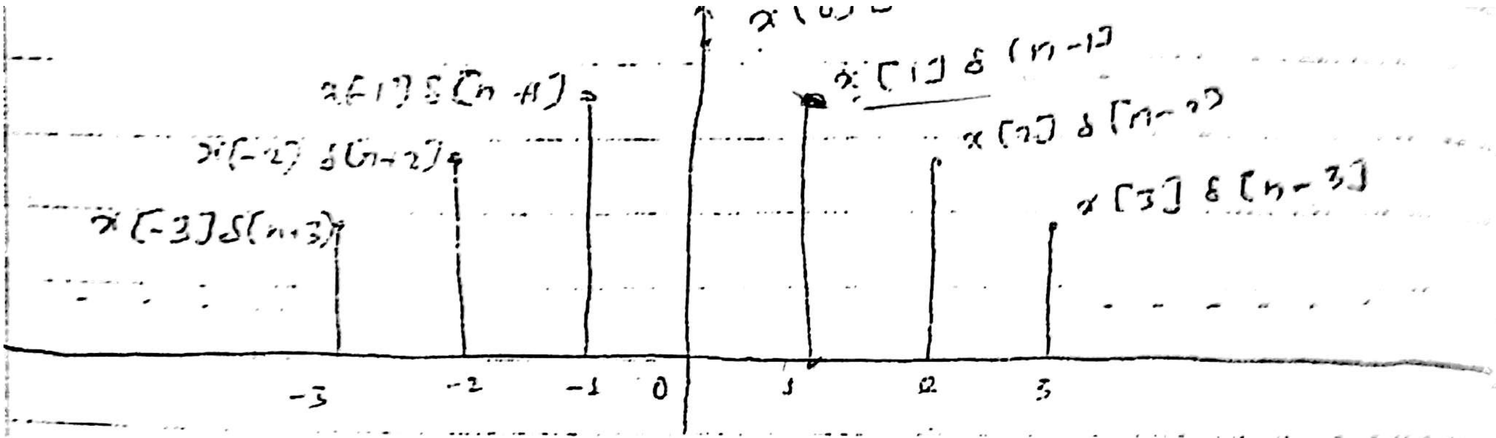
Let $x[n]$ be the discrete time input sequence and can be represented by superposition symmetry as;

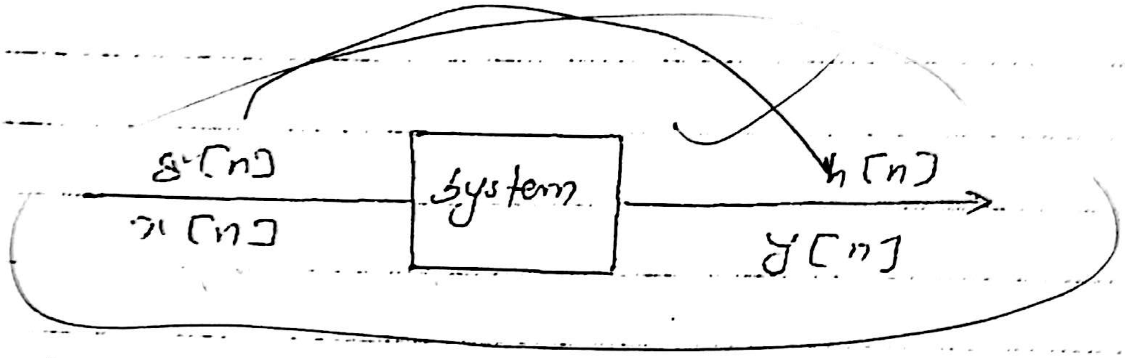
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\, \delta[n-k]$$

$\delta[n]$

$$= \cdots + x[-2]\, \delta[n+2] + x[-1]\, \delta[n+1] + x[0]\, \delta[n] + x[1]\, \delta[n-1] + \cdots$$

where, $x[k]$ is coefficient & $\delta[n-k]$ is impulse





Impulse response of system :

$$h[n] = y[n]\Big|_{x[n] = \delta[n]}$$

Output due to the input $x[n]$,

$$y[n] = \sum_{K=-\infty}^{\infty} x[K]\, h[n, k]$$

where; $h[n,k]$ is response of system due to $\delta[n-k]$

Since, $h[n]$ is the response of the system due to $\delta[n]$ and given system is time invariant

$$h[n,k] = h[n-k]$$

So,

$$y[n] = \sum_{K=-\infty}^{\infty} x[K] \cdot h[n-k]$$

$$= x[n] * h[n]$$

The output of LTI system is the convolution between the input signal & its impulse respo[nse]

Example :- Find the o/p of LTI System having input signal $x[n] = \{1, 2, -3, 0.5\}$ and impulse response $h[n] = \{-1, 2, 3, 1.5\}$

Sol: O/p of LTI system;

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$$= \cdots + x[-1] \cdot h[n+1] + x[0] \cdot \ldots + x[1] \cdot h[n-1] + x[2] \cdot h[n-2] + \cdots$$

$$= 1 \cdot h[n+1] + 2\, h[n] - 3\, h[n-1] + 0.5\, h[\ldots$$

For $n = -1$;

$$y[-1] = h[0] + 2\, h[-1] - 3\, h[-2] + 0.5\, h[-3]$$

$$= -1 + 2 * 0 - 3 * 0 + 0.5 * 0$$

$$= -1$$

For $n = -2$;

$$y[-2] = h[+1] + 2\, h[0] - 3\, h[-1] + 0.5\, h[-2]$$

$$= 2 + 2 * (-1)$$

$$= 0$$

For $n = 1$;

$$y[1] = h[2] + 2h[1] - 3h[0]$$
$$= 3 + 2*2 - 3*(-1)$$
$$= 3 + 4 + 3$$
$$= 10$$

For $n = 2$;

$$y[2] = h[3] + 2h[2] - 3h[1] + 0.5h[0]$$
$$= 1.5 + 2*3 - 3*2 + 0.5*(-1)$$
$$= 1.5 + 6 - 6 - 0.5$$
$$= 1$$

For $n = 3$;

$$y[3] = h[4] + 2h[3] - 3h[2] + 0.5h$$
$$= 0 + 2*1.5 - 3*3 + 0.5*2$$
$$= 3 - 9 + 1$$
$$= -5$$

for $n = 4$;

$$y[4] = h[5] + 2h[4] - 3h[3] + 0.5h[$$
$$= 0 + 0 - 3*(1.5) + 0.5*3$$
$$= -4.5 + 1.5$$
$$= -3.0$$

for $n = 5$;

$$y[5] = h[6] + 2h[5] - 3h[4] + 0.5h[3]$$
$$= 0.5*1.5$$
$$= 0.75$$

$y[n] = \{-1, 0, 10, 1, -5, -3, ...\}$

$x[n]$	1 $x(-1)$	2 $x[0]$	-3 $x[1]$	0.5 $x[2]$
$h[0]$ -1	-1	-2	3	-0.5
$h[1]$ 2	2	4	-6	1
$h[2]$ 3	3	6	-9	1.5
$h[3]$ 1.5	1.5	3	-4.5	0.75

$y[n] = \{-1, 0, 10, 1, -5, -3, 0.75\}$

H.W.

$x[n] = \{1, 2, 3, -1, 2\}$

$h[n] = \{2, -0.5, 1, -1, 1.5\}$

9. Find the output of LTI system having input: $x[n] = \left(\frac{1}{2}\right)^n u[n]$ & impulse response $h[n] = u[n] - u[n-4]$.

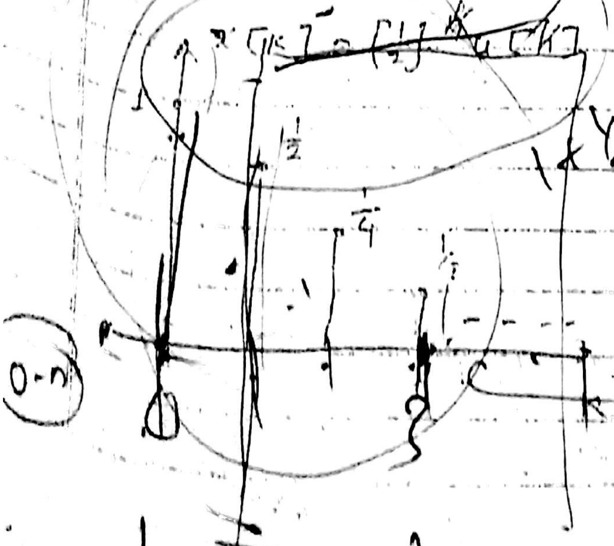
Sol. O/p of LTI system:

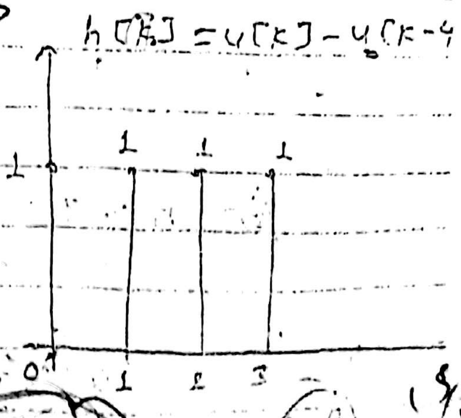
$y[n] = x[n] * h[n]$

$= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$

$x[k] = \left(\frac{1}{2}\right)^k u[k]$

$h[k] = u[k] - u[k-4]$

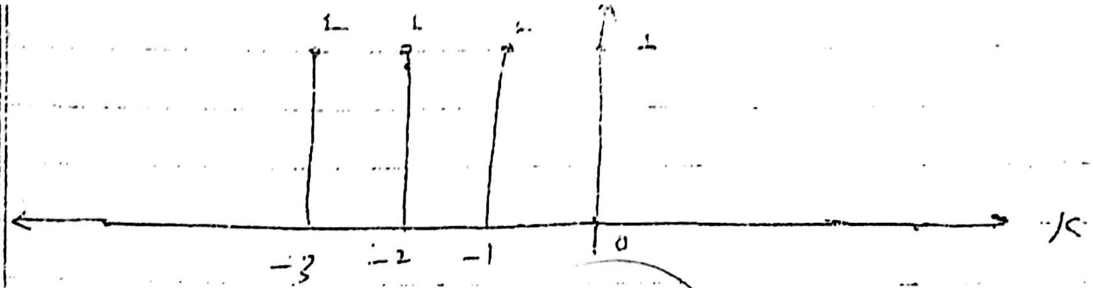




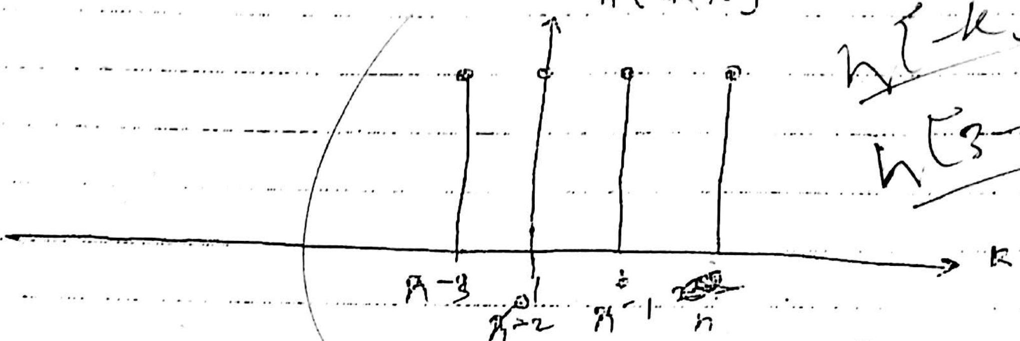
$h[n-k]$

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

$\frac{8+4+2+1}{8}$



$h[n-k]$



$h[-k]$

$h[3-k]$

For $n < 0$, $x[k] \cdot h[n-k] = 0$

$y[n] = 0$

For $0 \le n \le 3$:

$x[k] \cdot h[n-k] = \left(\frac{1}{2}\right)^k, \; (0 \le k \le n)$

$h[n-k]$

$$y[n] = \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$$

$$= \frac{1\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)}{1 - \frac{1}{2}}$$

$$= 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)$$

for $n > 3$:

$x[k] \cdot h[n-k] = \left(\frac{1}{2}\right)^k ; \; n-3 \le$

$$y[n] = \sum_{k=n-3}^{n} \left(\frac{1}{2}\right)^k$$

$$n < 0 = \left(\frac{1}{2}\right)^{n-3} \frac{\left(1 - \left(\frac{1}{2}\right)^4\right)}{1 - \frac{1}{2}}$$

$n > 0$

$$= 2\left(\frac{1}{2}\right)^{n-3}\left(1 - \frac{1}{16}\right)$$

$$= \frac{15}{8}\left(\frac{1}{2}\right)^{n-3}$$

$$= 15\left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{n-3}$$

$$= 15 \cdot \left(\frac{1}{2}\right)^n$$

$n - k \rightarrow \{0, 1, \dots 3\}$

$0 \le n-k \le 3$

output:

$$y[n] = \begin{cases} \frac{15}{8}\left(\frac{1}{2}\right)^{n-3} = 15\left(\frac{1}{2}\right)^n & \text{for } n > 3 \\ 2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right), & \text{for } 0 \le n \le 3 \\ 0, & \text{for } n < 0 \end{cases}$$

$n \le k \le 0$

$h(n-k) = 1$

$0 \le n-k \le 3$

0	(−3, 0)
1	(−2, 1)
2	(−1, 2)
3	(0, 3)

H.W.:

1. $x[n] = \left[\frac{1}{3}\right]^n u[n]$

and;

$h[n] = 2u[n+1] - u[n-3]$

$y[n] = x[n] * h[n]$.

$h[n-k]$

2. $h[n] = \left(\frac{1}{3}\right)^n u[n]$

$x[n] = \left[\frac{1}{2}\right]^n \cdot u[n]$

3 $y[n] = \sum x[k]\, h[n-k]$ $-n \le k \le 0$

$\left(\frac{1}{2}\right)^k$

$n \ge k \ge 0$

2. Find the output of the LTI system:

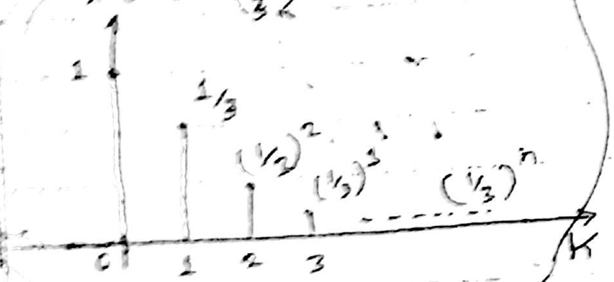
$x[n] = \left[\frac{1}{3}\right]^n . u[n]$

& $h[n] = 2u[n+1] - u[n-3]$

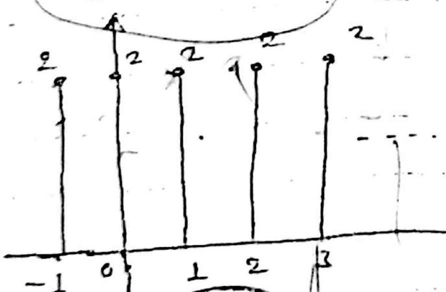
sol: we know that, output of LTI system;

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

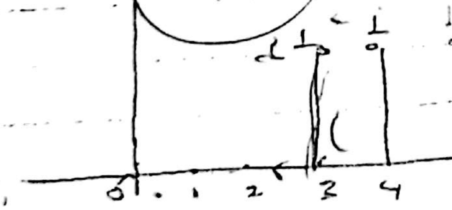
$x[k] = \left(\frac{1}{3}\right)^k u[k]$



$2u[k+1]$

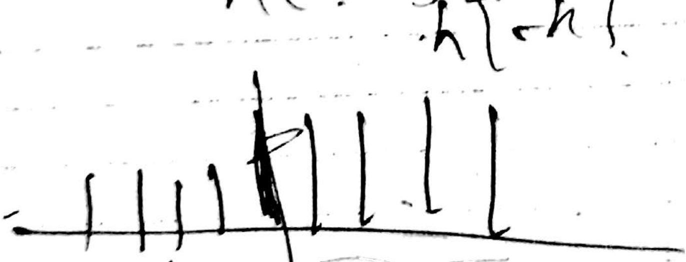


$u[k-3]$



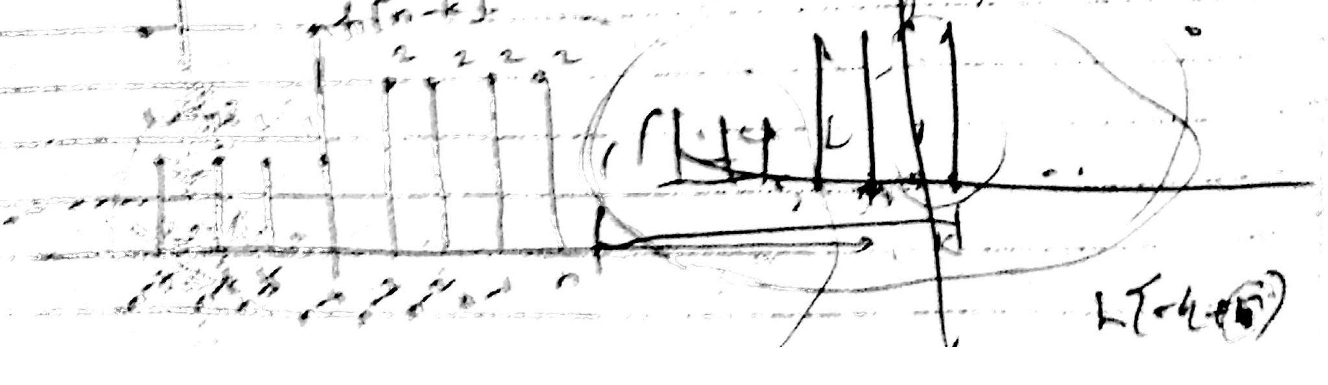
$h[k]$

$h[-k+n]$ $h[-k]$



$h[n+k]$

$h[n-k]$



$h[-k+n]$

For $n < 0$; $\quad x[k] \cdot h[n-k] = 0$

$\therefore y[n] = 0$

For $0 \leq n \leq 3$; $\quad x[k] \cdot h[n-k] = \left(\frac{1}{3}\right)^k * 2 , \quad 0 \leq k \leq 1$

$$\therefore y[n] = \sum_{k=0}^{n} 2\left(\frac{1}{3}\right)^k$$

$$= 2 \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k$$

$$= 2 \left\{ \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{1 - \frac{1}{3}} \right\}$$

$$= \frac{2 \left\{ 1 - \left(\frac{1}{3}\right)^{n+1} \right\}}{2/3}$$

$$= 3 \left\{ 1 - \left(\frac{1}{3}\right)^{n+1} \right\}$$

For $3 < n$;

$$x[k] \cdot h[n-k] = \sum_{k=n-3}^{n} 2 \cdot \left(\frac{1}{3}\right)^k + \sum_{k=0}^{n-4} \left(\frac{1}{3}\right)^k$$

$$= 2 \cdot \left\{ \frac{\left(\frac{1}{3}\right)^{n-3} \left(1 - \left(\frac{1}{3}\right)^4\right)}{1 - 1/3} \right\} + 1 * \left\{ \frac{1 - \left(\frac{1}{3}\right)^{n-3}}{1 - \frac{1}{3}} \right\}$$

$$= 3 \left(\frac{1}{3}\right)^{n-3} \left(1 - \frac{1}{81}\right) + \frac{3}{2} \left(1 - \left(\frac{1}{3}\right)^{n-3}\right)$$

$$= \frac{80}{27} \left(\frac{1}{3}\right)^{n-3} + \frac{3}{2} \left\{ 1 - \left(\frac{1}{3}\right)^{n-3} \right\}$$

$$= 80 \left(\frac{1}{3}\right)^{n+0} + \frac{3}{2} \left\{ 1 - \left(\frac{1}{3}\right)^{n-3} \right\}$$

$$= 80 \left(\frac{1}{3}\right)^{n} + \frac{3}{2} \left\{ 1 - \left(\frac{1}{3}\right)^{n-3} \right\}$$

$$y[n] = \begin{cases} 0 & , \text{ for } n < 0 \\ 3\left\{1 - \left(\frac{1}{3}\right)^{n+1}\right\} & , \text{ for } 0 \le n \le 3 \\ 80\left(\frac{1}{3}\right)^n + \frac{3}{2}\left(1 - \left(\frac{1}{3}\right)^{n-3}\right) & , \text{ for } n > 3 \end{cases}$$

Ans

3. $h[n] = \left[\frac{1}{3}\right]^n u[n]$

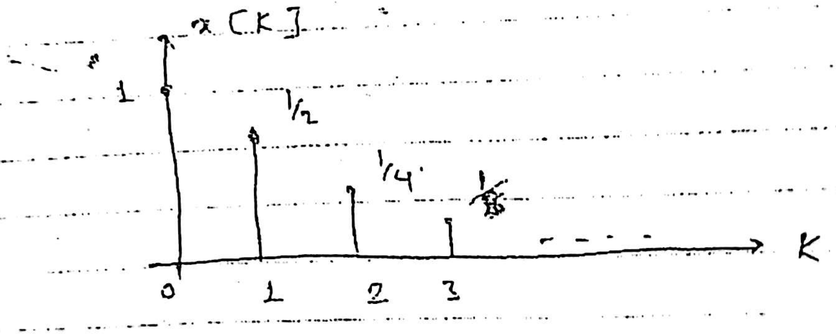
$x[n] = \left[\frac{1}{2}\right]^n u[n]$

Sol: The o/p of LTI system is;

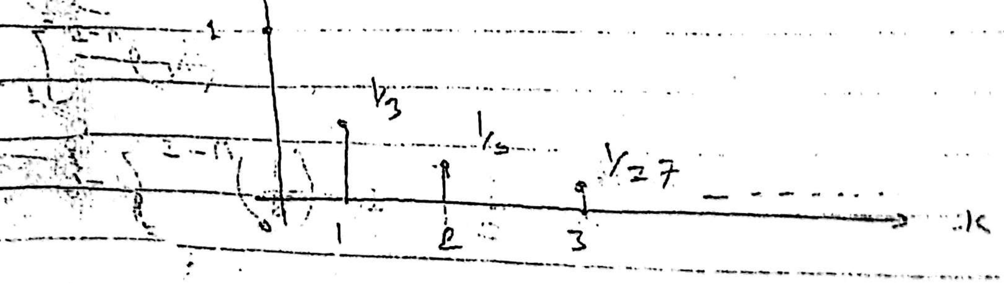
$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

Here,

$x[k] = \left[\frac{1}{2}\right]^k u[k]$



$h[k] = \left[\frac{1}{3}\right]^k u[k]$

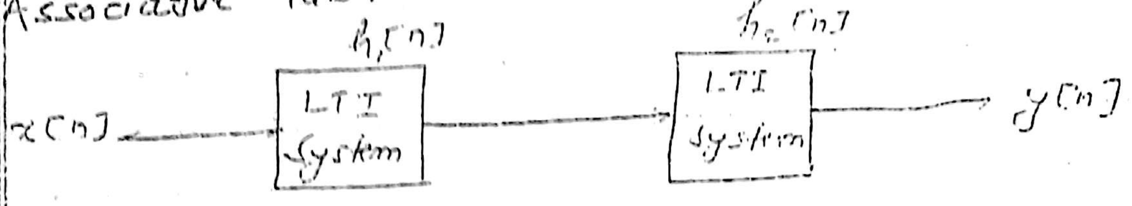


Characteristics (Properties) of LTI Systems:

1. Commutative law:

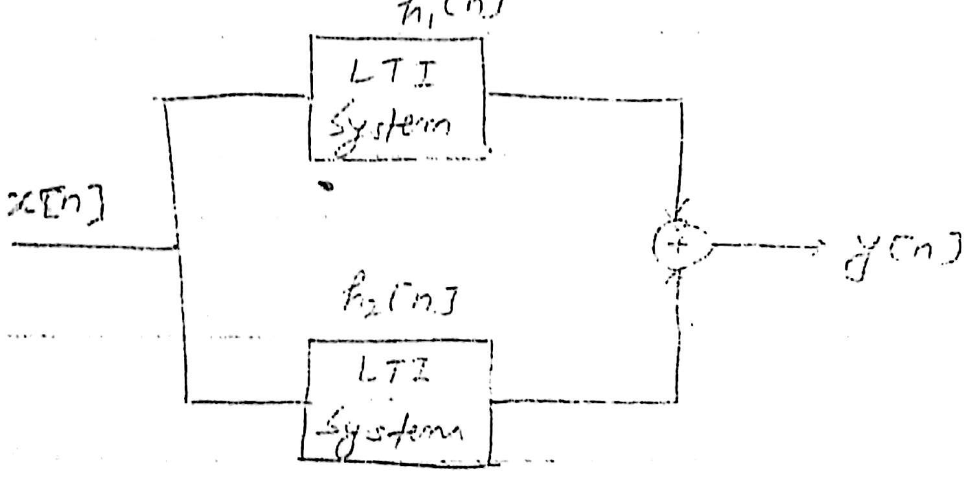
$$y[n] = x[n] * h[n] = h[n] * x[n]$$
$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k] \cdot x[n-k]$$

2. Associative law:

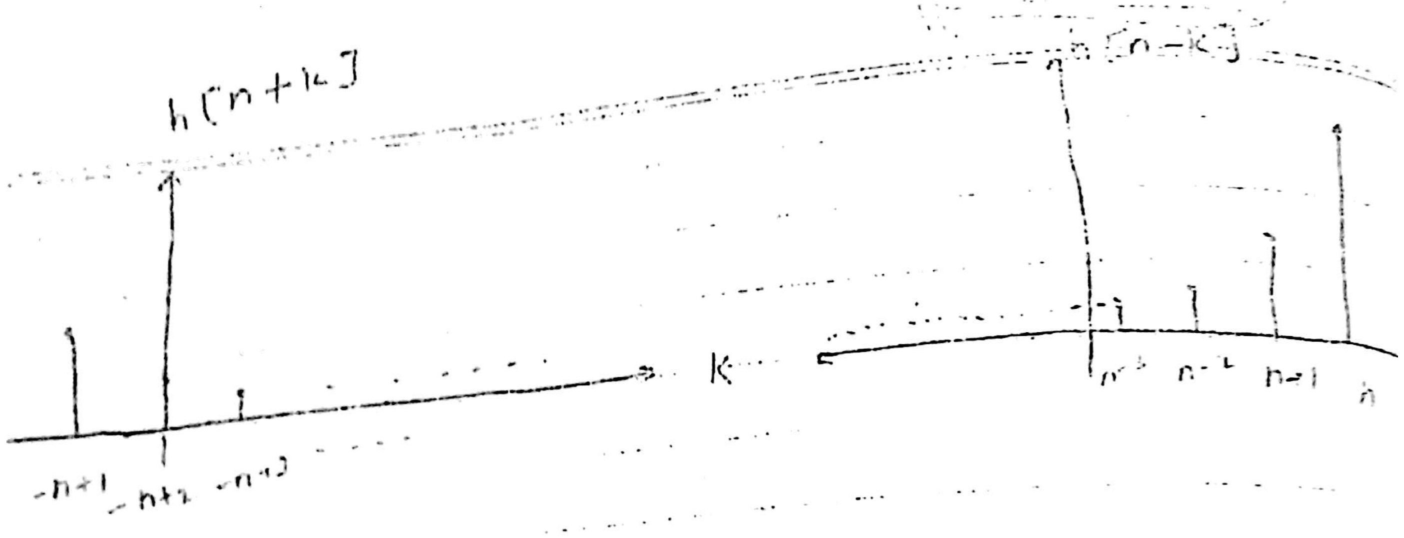


$$y[n] = (x[n] * h_1[n]) * h_2[n]$$
$$= x[n] * \{h_1[n] * h_2[n]\}$$

3. Distributive law:



$$y[n] = x[n] * h_1[n] + x[n] * h_2[n]$$
$$= x[n] * \{h_1[n] + h_2[n]\}$$



For $n < 0$; $x[n] \cdot h[n-k] = 0$

$\therefore y[n] = 0$

For $n > 0$; $x[n] \cdot h[n-k] = x[n] \cdot \left(\frac{1}{3}\right)^n u[n-k]$

$$= \left[\frac{1}{2}\right]^n \cdot \left(\frac{1}{3}\right)^n u[n] \cdot u[$$

$$= \left[\frac{1}{6}\right]^n \cdot u[n-k];$$

— Non-causal.

4. Causality of LTI System:

$$x[n] \longrightarrow \underset{h[n]}{\boxed{\text{LTI System}}} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n] = h[n] * x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[k]\, x[n-k]$$

For a system to be a causal;

$$y[n] = \sum_{k=-\infty}^{n} x[k] \cdot h[n-k] = \sum_{k=0}^{\infty} h[k] \cdot x[n-k]$$

Hence, LTI system will be causal iff impulse response $h[n] = 0$ for $n<0$.

5. Stability of LTI System:

$$x[n] \longrightarrow \underset{h[n]}{\boxed{\text{LTI System}}} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{\infty} h[n]\, x[n-k]$$

For a system to be BIBO stable,
if input $|x[n]| \leq k_1 < \infty$

$$y[n] = \left| \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] \right| < \infty$$

$$= \sum_{k=-\infty}^{\infty} h[k] \, |x[n-k]|$$

Hence LTI System will be stable iff

$$\left| \sum h[k] \right| < \infty$$

if impulse response

$$\left| \sum_{n=-\infty}^{\infty} h[n] \right| < \infty$$

⊛ $h[n] = \{1, \underset{\uparrow}{0.5}, 2, 3\}$

→ It has finite sum, so stable

→ Non-causal (h has value for -1)

⊛ $2^n u[n]$

→ unstable

→ Causal

⊛ $2^n u[n-1]$

→ Non causal

→ Stable (finite sum

⊛ $\left(\frac{1}{3}\right)^n u[n]$

→ causal

→ stable

⊛ $h[n] = \{\underset{\uparrow}{2}, 8, -1, 2\}$

→ Causal

→ Stable

⊛ Recursive System :-

There are many system which present output d upon the number of past outputs, such system is called Recursive and its output can be represented as:

$$y[n] = F\{y[n-1], y[n-2], y[n-3], \ldots y[n-N], x[n], x[n-1], \ldots x[n-M]\}$$

⊛ Non-Recursive System :

A system is said to be Non-recursive its present output only depends upon the p and past inputs. & can be represented

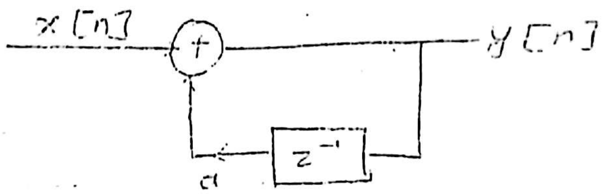
$$\therefore y[n] = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k\, x[n-k] - \sum_{k=1}^{N} a_k\, y[n-k]\right]$$

where, M & N denotes order of difference equation.

For example:

Let us take a first order difference equat

$$y[n] = a\,y[n-1] + x[n]$$



For $n=0$;

$$y[0] = a\,y[-1] + x[0]$$

For $n=1$;

$$y[+1] = a\,y[0] + x[1]$$
$$= a^2 y[-1] + a x[0] + x[1]$$

For $n=2$;

$$y[2] = a\,y[1] + x[2]$$
$$= a^3 y[-1] + a^2 x[0] + a_d\, x[1] + x[2]$$

⋮

$$y[n] = a^{n+1} y[-1] + a^n x[0] + a^{n-1} x[1] + \cdots + x[n].$$

$$= a^{n+1} y[-1] + \sum_{k=0}^{n} a^k x[n-k]$$

$$y[n] = F\{x[n], x[n-1], x[n-2], \ldots, x[n\ldots$$

example:

Moving Average System:

$$y[n] = \frac{1}{n+1} \sum_{k=0}^{n} x[k]$$

↳ Only depends on i/p. So non-Recursive system

→ Another way;

For $n-1$;

$$y[n-1] = \frac{1}{n-1+1} \sum_{k=0}^{n-1} x[k]$$

$$= \frac{1}{n} \sum_{k=0}^{n-1} x[k]$$

$$y[n] = \frac{1}{n+1} \left[\sum_{k=0}^{n-1} x[k] + x[n] \right]$$

$$\Rightarrow y[n] = \frac{1}{n+1} \left[n\, y[n-1] + x[n] \right]$$

$$\Rightarrow y[n] = \frac{n}{n+1} y[n-1] + \frac{1}{n+1} x[n]$$

↳ This is Recursive system.

Difference Equation:

A general form of constant coefficient diffe... equation for discrete time LTI system c... be represented as;

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$a_1\, a_0 y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Case - I : System is relaxed.
i.e. $y[-1] = 0$

Such response is called zero-state or forced respons

$$y_{zs}[n] = \sum_{k=0}^{n} a^k x[n-k]$$

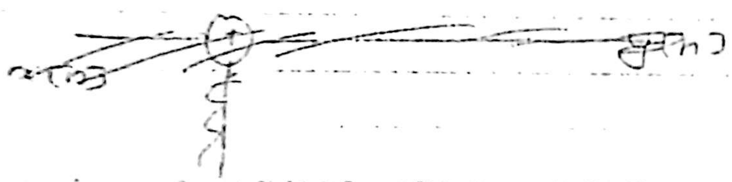
Case - II :

If input applied to the system is zero i.e. $x[n]$ for $n \geq 0$, then such response is called zero-input or natural response :

$$y_{zi}[n] = a^{n+1} y[-1]$$

Total Response ;

$$y[n] = y_{zi}[n] + y_{zs}[n]$$



FIR and IIR system :

Infinite Impulse Response (IIR) :

If the length of the impulse response of a system is infinite then it is called infinite impulse response system. IIR systems are mostly feedback system and recursive system.

For eg :-

$$h[n] = \left[\frac{1}{2}\right]^n u[n]$$

Finite Impulse Response (FIR)

If the length of the impulse response of the system is finite or countable then is called finite impulse response system.

FIR systems are mostly non-feedback sy and non-recursive system.

for eg: $h[n] = \{2, -1, \underset{\uparrow}{3}, 0.5, 2, 3\}$

3. Z- Transform

2072-5-26

Z- Transform is a power series expansion which converts a discrete time domain sequence x(n) into an equivalent complex variable domain 'j' and represented as ;

Bilateral or Two sided Z- transform $\longrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$

where. z is complex variable

$$z = re^{j\omega}$$

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

↳ Unisided or one sided Z- transform

If $x[n]$ is causal i.e. $x[n]=0$, for $n<0$ then both unilateral and Bilateral z- transform will be same.

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

Since, z- transform is a power series expansion it must converge for existance of z- transform.

Region of Convergence (ROC) :

The range of values of z, for which X(z) exists is called the region of convergence.

example:

$$x[n] = a^{n} u[n]$$

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z transform of $x[n]$ is given by:

$$X[z] = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n$$

$$= 1 + a z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

Using infinite G.S. sum:

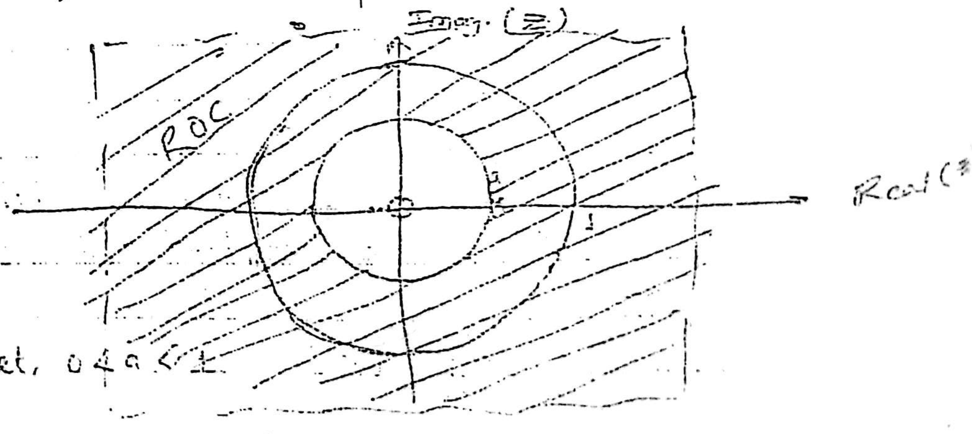
$$X(z) = \frac{1}{1 - a z^{-1}} , \quad |a z^{-1}| < 1$$

$$= \frac{z}{z-a} , \quad \left|\frac{a}{z}\right| < 1$$

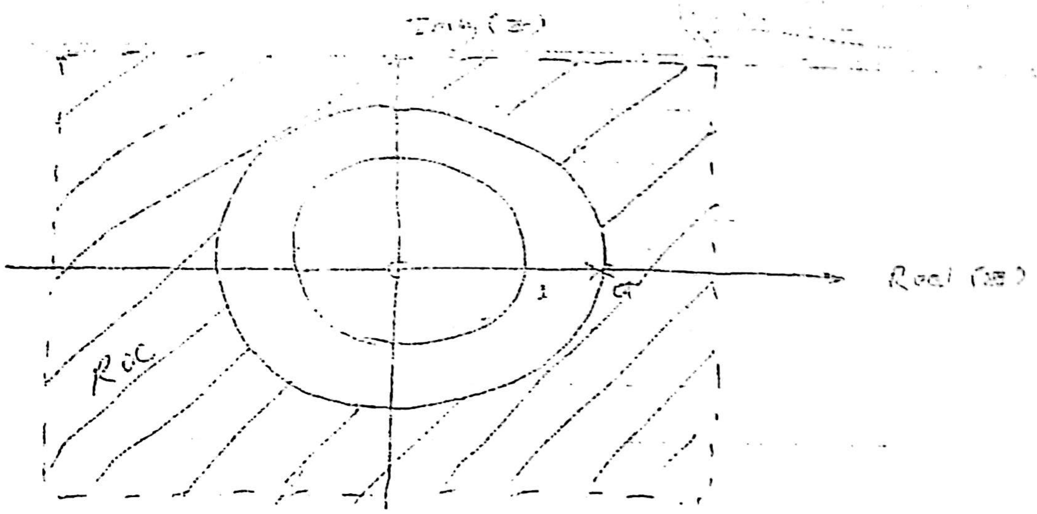
$$= \frac{z}{z-a} , \quad |z| > |a|$$

$X(z)$ has a zero at $z = 0$

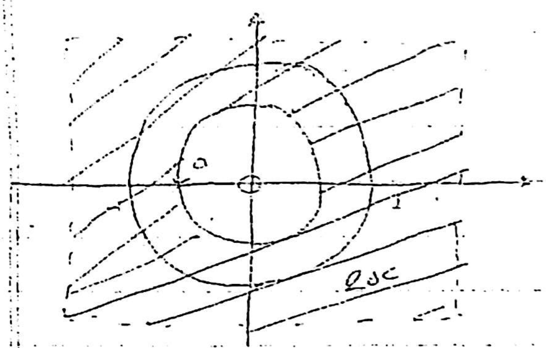
$X(z)$ has a pole at $z = a$



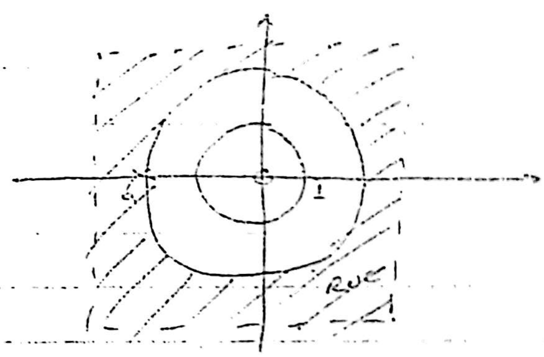
let, $0 < a < 1$



$a > 1$



$-1 < a < 0$



$a < -1$

example: $x[n] = -a^n u[-n-1]$

z - transform of $x[n]$ is given by:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n\, u[-n-1] \cdot z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n\, z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= -\left[a^{-1}z + a^{-2}z^2 + a^{-3}z^3 + \cdots\right]$$

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Using infinite G.S. Sum;

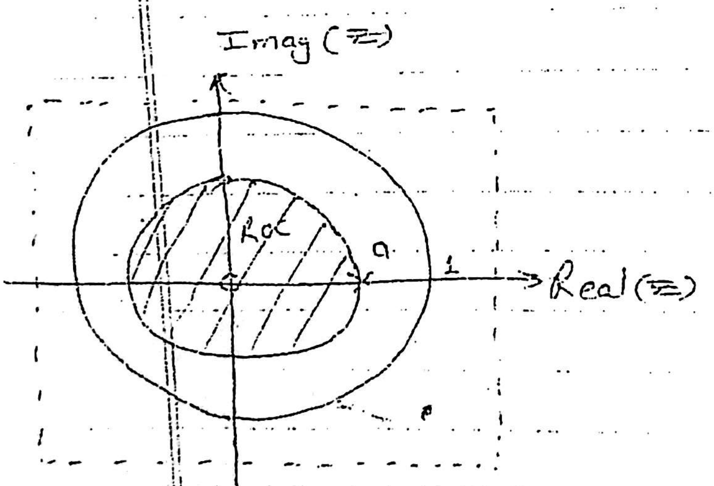
$$X(z) = -\frac{a^{-1}z}{1-a^{-1}z}, \quad |a^{-1}z| < 1$$

$$= -\frac{z/a}{1-z/a}, \quad \left|\frac{z}{a}\right| < 1$$

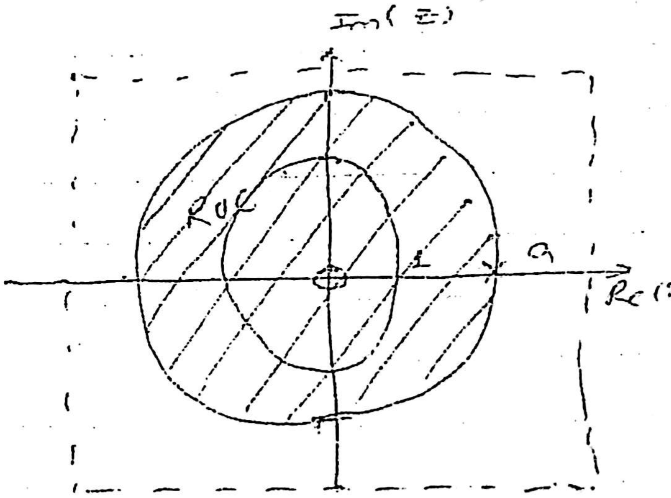
$$= \frac{z}{z-a}, \quad |z| < |a|$$

$X(z)$ has a zero at $z=0$

$X(z)$ has a pole at $z=a$.



$0<a<1$



$a>1$

Properties of Region of Convergence (ROC)

1. ROC doesnot contain any poles.

2. If $x[n]$ is finite length and right hand sided (for +ve of n only) sequence and its z-transform $X(z)$ exists for some values of z then ROC will be entire Z-plane except $z = 0$.

eg:- $x[n] = \{2, 1, -5, 3\}$

Z-transform of $x[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-1] z^{1} + x[0] + x[1] z^{-1} + x[2] z^{-2} + x[3] z^{-3} + \cdots$$

$$= 2 + z^{-1} - 5 z^{-2} + 3 z^{-3}$$

ROC: All z except $z = 0$.

3. If $x[n]$ is finite length and left sided sequence and its Z-transform $X(z)$ exists for some values of z then ROC will be entire Z-plane except $z = \infty$.

eg:-

$$x[n] = \{2, 1, -5, 3, \underset{\uparrow}{0}\}$$

z-transform of $x[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-4] z^{4} + x[-3] z^{3} + x[-2] z^{2} + x[-1] z^{1} + x[0] + x[1] z^{-1} + \cdots$$

$$= 2z^{4} + z^{3} - 5z^{2} + 3z$$

(12)

4. If $x[n]$ is finite length and both sided sequence and its z-transform $X(z)$ exists for some value of z then ROC will be entire z plane except $z=0$ & $z=\infty$.

eg:- $x[n] = \{2, 1, \underset{\uparrow}{-5}, 3\}$

z-transform of $x[n]$:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \cdots + x[-2]z^{2} + x[-1]z + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots$$

$$= 2z^{2} + z - 5 + 3z^{-1}$$

ROC: All z except $z=0$ & $z=\infty$

5. If $x[n]$ is infinite length & right sided sequence and its z-transform $X(z)$ exists for some value of z, then ROC has the form $|z| > r_{max}$ where r_{max} is the largest magnitude of any of poles of $X(z)$

i.e. ROC will be exterior of circle with radius r_{max}.

eg:- $x[n] = a^{n} u[n]$

z-transform of $x[n]$:

$$X[z] = \sum_{k=-\infty}^{\infty} x[n] z^{-n}$$

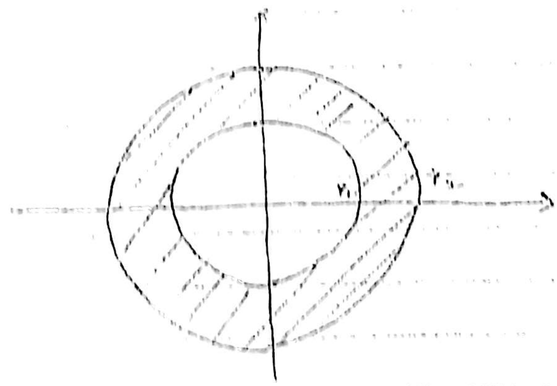
$$= \frac{z}{z-a}, \quad |z| > |a|$$

6. If $x[n]$ is infinite length & left sided sequence and it's z-transform $X(z)$ exists for some value of z, then ROC has the form $|z| < r_{min}$, where, r_{min} is the smallest magnitude of all of poles of $X(z)$

i.e. ROC will be interior of circle with radius r_{min}.

7. If $x[n]$ is infinite length & both sided sequence and it's z-transform $X(z)$ exists for some value of z, then ROC has the form $r_1 < |z| < r_2$ where r_1 & r_2 are two poles of $X(z)$.

i.e. ROC will be annular ring betn the circles with the radius r_1 & r_2



eg: i) $x[n] = n a^n u[n]$

ii) $x[n] = (n+1) a^n u[n]$

i) $x[n] = n a^n u[n]$

we know,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$X(z) = \sum_{n=1}^{\infty} n \cdot a^n z^{-n}$$

$$= a^1 z^{-1} + 2a^2 z^{-2} + 3a^3 z^{-3} + 4a^4 z^{-4} + \dots$$

$$= a z^{-1} (1 + 2a z^{-1} + 3a^2 z^{-2} + 4a^3 z^{-3} + \dots)$$

$$= a z^{-1} (1 - a z^{-1})^{-2}$$

$$= \frac{a z^{-1}}{(1 - a z^{-1})^2}$$

$$= \frac{a z^2 \cdot z^{-1}}{(z-a)^2}$$

$$= \frac{a z}{(z-a)^2}, \quad |z| > |a|$$

Ans

ii) Sol:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} (n+1) a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (n+1) a^n z^{-n}$$

$$= 1 + 2a z^{-1} + 3a^2 z^{-2} + 4a^3 z^{-3} + \dots$$

$$= \frac{1}{(1 - a z^{-1})^2}$$

$$= \frac{z^2}{(z-a)^2}$$

$$= \left(\frac{z}{z-a}\right)^2 ; \quad |z| > |a|$$

Ans

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Some standard z - transform pairs :

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	for all z.
$u[n]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}, \frac{z}{z-1}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 if ... or if,
$a^n u[n]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}, \frac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}, \frac{az}{(z-a)^2}$	$\lvert z\rvert < \lvert a\rvert$
$(n+1)a^n u[n]$	$\frac{1}{(1-az^{-1})^2}, \left(\frac{z}{z-a}\right)^2$	$\lvert z\rvert > \lvert a\rvert$
$\cos\omega_0 n \; u[n]$	$\frac{z^2 - \cos\omega_0 z}{z^2 - 2\cos\omega_0 z + 1}$	$\lvert z\rvert > 1$
$\sin\omega_0 n \; u[n]$	$\frac{\sin\omega_0 z}{z^2 - 2\cos\omega_0 z + 1}$	$\lvert z\rvert > 1$

(21)

$r^n \cos\omega_0 n\, u[n]$	$\dfrac{z^2 - r\cos\omega_0 z}{z^2 - 2r\cos\omega_0 z + r^2}$	$\lvert z\rvert > r$
$r^n \sin\omega_0 n\, u[n]$	$\dfrac{r\sin\omega_0 z}{z^2 - 2r\cos\omega_0 z + r^2}$	$\lvert z\rvert > r$
$x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{Otherwise} \end{cases}$	$\dfrac{1 - a^N z^{N}}{1 - az^{-1}}$	$\lvert z\rvert > 0$

Inverse Z-transform :-

$$x[n] = Z^{-1}\{X(z)\}$$

Inversion of z-transform to find the discrete time domain sequence $x[n]$ from its z-transfrom $X(z)$ is called inverse Z-transform.

1. Inversion formula:

$$x[n] = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

where c is counter-clockwise contour of integration enclosing the origin.

2. Use of tables of z-transform pairs:
In this method, X(z) is expressed as:

$$X(z) = X_1(z) + X_2(z) + \dots + X_K(z)$$

where, $X_1(z)$, $X_2(z)$, ..., $X_K(z)$ are known z-transform pairs.
So,

$$x[n] = x_1[n] + x_2[n] + \dots + x_K[n].$$

3. Power Series expansion (Direct-Division Method)
Expressing z-transform as a power series

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$= \dots + x[-1]\, z^{+1} + x[0] + x[1]\, z^{-1} + x[2]\, z^{-2} + \dots$$

Any particular value of sequence can be determined by finding the coefficient of the appropriate power of z^{-1}. This approach may not provide approach form solution but very useful for finite length sequence.
For infinite length sequence, long division method is required.

4. Partial fraction expansion method:-
X(z) is rational function of z.

$$X(z) = \frac{N(z)}{D(z)} = \frac{K(z-z_1)\dots(z-z_m)}{(z-p_1)(z-p_2)\dots(z-p_n)}$$

a) For $n \geq m$ and all poles p_n are simple then $X(z)$ can be represented as:

$$\frac{X(z)}{z} = \frac{c_0}{z} + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \frac{c_3}{z-p_3} + \cdots + \frac{c_n}{z-p_n}$$

where;

$$c_0 = X(z)\Big|_{z=0}$$

$$c_k = (z-p_k)\frac{X(z)}{z}\Big|_{z=p_k}$$

b) for $m > n$, improper rational function partial fraction is made by making it proper rational function and represented as:

$$X(z) = \sum_{q=0}^{m-n} b_q z^{-q} + \sum_{i=1}^{n} \frac{c_k}{1-p_k z^{-1}}$$

c) For multiple poles, $(z-p_i)^r$ then partial fraction form will be;

$$\frac{A_1}{(z-p_i)} + \frac{A_2}{(z-p_i)^2} + \cdots + \frac{A_r}{(z-p_i)^r}$$

where;

$$A_{r-k} = \frac{1}{k!}\frac{d^k}{dz^k}(z-p_i)^r \frac{X(z)}{z}\Big|_{z=p_i}$$

Properties of Z - transform :-

Notation : $x[n] \xleftrightarrow{z} X(z)$ ROC : R

1. Linearity

If $x_1[n] \xleftrightarrow{z} X_1(z)$, ROC : R_1

$x_2[n] \xleftrightarrow{z} X_2(z)$, ROC : R_2

then: $a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{z} a_1 X_1(z) + a_2 X_2(z)$

ROC : $R_1 \cap R_2$

where: a_1 & a_2 are arbitrary constant.

2. Time shifting

If $x[n] \xleftrightarrow{z} X(z)$, ROC : R

then; $x[n-m] \xleftrightarrow{z} z^{-m} X(z)$, ROC: $R \cap |z| > 0$ if r

$R \cap |z| < \infty$ if m

Proof :

$$Z\{x[n-m]\} = \sum_{n=-\infty}^{\infty} x[n-m] z^{-n}$$

put $n-m = k$

$$= \sum_{k=-\infty}^{\infty} x[k] z^{-m-k}$$

$$= z^{-m} \sum_{k=-\infty}^{\infty} x[k] z^{-k}$$

$$= z^{-m} X(z)$$

$x[n-1] \xleftrightarrow{z} z^{-1} X(z)$, ROC : $R \cap |z| > 0$

Unit delay

3. Multiplying by z_0^n

If $x[n] \xleftrightarrow{z} X(z)$, ROC : R

then:

$z_0^n x[n] \xleftrightarrow{z} X\left(\frac{z}{z_0}\right)$, ROC : $|z_0| R$

A pole or zero ~~at~~ $z = z_k$ moves to at $z = z_0 z_k$ after multiplication by z_0^n in time domain and region of convergence expands or contracts by the factor $|z_0|$.

Proof:

$$Z\{z_0^n x[n]\} = \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left(\frac{z}{z_0}\right)^{-n}$$

$$= X\left(\frac{z}{z_0}\right) \quad \#$$

For: $e^{j\omega_0 n} x[n] \xleftrightarrow{z} X\left(\frac{z}{e^{j\omega_0}}\right)$. ROC: $R|e^{j\omega_0}| = R$

4. Time ~~Scaling~~ Reversal :

If $x[n] \xleftrightarrow{z} X(z)$. ROC : R

then:

$x[-n] \xleftrightarrow{z} X\left(\frac{1}{z}\right)$: ROC : $\frac{1}{R}$

Therefore, a pole or zero in $X(z)$ at $z = z_k$ moves to $z = \frac{1}{z_k}$ after time reversal, and ROC will be reversed.

Proof:

$$Z\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]\, z^{-n}$$

put $-n = k$

$$= \sum_{k=-\infty}^{\infty} x[k]\, z^{+k}$$

$$= \sum_{K=-\infty}^{\infty} x[k] \left(\frac{1}{z}\right)^{-k}$$

$$= X\left(\frac{1}{z}\right).$$

5. Multiplication by n (differentation in z)

If $x[n] \xleftrightarrow{z} X(z)$, ROC : R

then;

$$n x[n] \xleftrightarrow{z} -z \frac{d}{dz} X(z), \quad \text{ROC} : R$$

Proof:

we have;

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

Differentiating on both sides w.r. to z;

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x[n] \left(-n z^{-n-1}\right)$$

or, $$-z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} n\, x[n]\, z^{-n}$$

So,

$$Z\{n x[n]\} = -z \frac{d}{dz} X(z)$$

6. Convolution Property :

If: $x_1[n] \xrightarrow{z} X_1(z)$, ROC R_1

$x_2[n] \xleftrightarrow{z} X_2(z)$, ROC R_2

then:

$x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z)\, X_2(z)$, ROC

<u>Proof</u>:

let: $y[n] = x_1[n] * x_2[n]$

$$= \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

$$Z\{y[n]\} = \sum_{n=-\infty}^{\infty} y[n]\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \cdot \sum_{k=-\infty}^{\infty} x_1[k]\; x_2[n-k]$$

$$= \sum_{k=-\infty}^{\infty} x_1[k] \cdot \sum_{n=-\infty}^{\infty} x_2[n-k]$$

Using time shifting property of Z-transform

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \cdot z^{-k}\, X_2(z)$$

$$= X_1(z) \cdot X_2(z)$$

, ROC : $R_1 \cap R_2$

7. Accumulation (summation):

If $x[n] \xleftrightarrow{z} X(z)$, ROC : R

then:

$$\sum_{k=-\infty}^{n} x[k] \xleftrightarrow{z} \frac{1}{1-z^{-1}} X(z) = \frac{z}{z-1} X(z),$$

ROC : $R \cap |z| > 1$.

Proof:

let, $$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, u[n-k]$$

$$= x[n] * u[n]$$

using convolution property of z-transform, we g

$$Y(z) = X(z) \cdot \frac{1}{1-z^{-1}}$$

$$= \frac{z}{z-1} X(z) \qquad ROC: R \cap |z| > 1$$

8. Complex Conjugate:

If $x[n] \xleftrightarrow{z} X(z)$, ROC : R

then:

$$x^*[n] \xleftrightarrow{z} X^*(z^*), \quad ROC : R$$

$$Real\{x[n]\} \xleftrightarrow{z} \frac{1}{2}[X(z) + X^*(z^*)]$$

and includes Roc

$$Imag\{x[n]\} \xleftrightarrow{z} \frac{1}{2}[X(z) - X^*(z^*)]$$

and include ROC

Parseval's relation:-

If $x_1[n] \xleftrightarrow{Z} X_1(z)$, $r_{1l} < |z| <$

$x_2[n] \xleftrightarrow{Z} X_2(z)$, $r_{2l} < |z| < r_{2u}$

Where, $x_1[n]$ & $x_2[n]$ are complex valued sequence;

then;

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n] = \frac{1}{2\pi j} \oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv$$

provided that $r_{1l}\, r_{2l} < 1 < r_{1u}\, r_{2u}$

2072-03-08

Unilateral Z-transform (One sided):-

$$X(z) = \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

Most of properties of unilateral z-transform is similar to as of Bilateral z-transform.

Time shifting:-

a) Time delay:

If $x[n] \xleftrightarrow{Z} X(z)$

then: $x[n-m] \xleftrightarrow{Z} z^{-m}\left[X(z) + \sum_{i=1}^{m} x[-i]\, z^{i}\right]$

Proof:

let, $y[n] = x[n-1]$

then;

$$Y(z) = \sum_{n=0}^{\infty} y[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} x[n-1]\, z^{-n}$$

$$= \sum_{n=1}^{\infty} x[n-1]\, z^{-n} + x[-1]$$

Now put $n-1 = k$

$$= \sum_{k=0}^{\infty} x[k]\, z^{-k-1} + x[-1]$$

$$= z^{-1} \sum_{k=0}^{\infty} x[k]\, z^{-k} + x[-1]$$

$$= z^{-1} X(z) + x[-1]$$

for, $y = x[n-2]$;

then: $$Y(z) = \sum_{n=0}^{\infty} y[n]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} x[n-2]\, z^{-n}$$

$$= \sum_{n=2}^{\infty} x[n-2]\, z^{-n} + x[-2] + x[-1]\, z^{-1}$$

Now put $n-2 = k$

$$= \sum_{k=0}^{\infty} x[k]\, z^{-k-2} + x[-2] + x[-1]\, z^{-1}$$

$$= z^{-2} \sum_{k=0}^{\infty} x[k]\, z^{-k} + x[-2] + x[-1]\, z^{-1}$$

$$= z^{-2} X(z) + x[-2] + x[-1]\, z^{-1}$$

For $y[n] = x[n-m]$

then;

$$Y(z) = z^{-m}X(z) + x[-m] + x[-m+1]z^{-1} + \cdots + x[-1]z^{-m+1}$$

$$= z^{-m}X(z) + \sum_{k=1}^{m} x[k-1-m]\, z^{-k+1}$$

put $k-1-m = l$;

$$Y(z) = z^{-m}X(z) + \sum_{l=-1}^{-m} x[l]\, z^{l-m}$$

$$= z^{-m}X(z) + \sum_{l=1}^{m} x[-l]\, z^{l-m}$$

$$= z^{-m}\left[X[z] + \sum_{l=1}^{m} x[-l]\, z^{l}\right]$$

b) Time Advance:

If $x[n] \xleftrightarrow{z} X(z)$

then;

$$x[n+m] \xleftrightarrow{z} z^{m}X(z) - \sum_{l=0}^{m-1} x[l]\, z^{m-l}$$

\+ Find ~~z-transform~~:

i. $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n]$

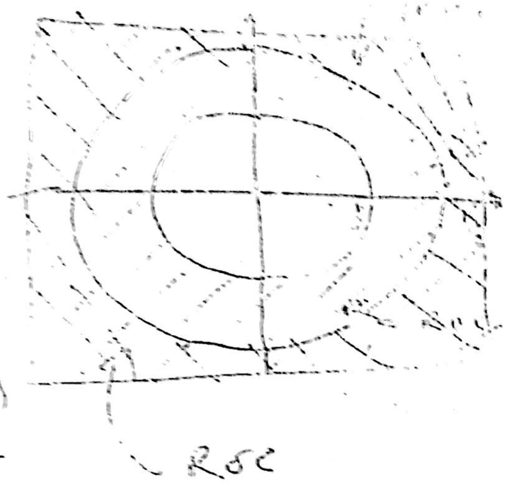
sol

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| > |a|$$

$$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{z}{z-\frac{1}{2}}, \quad |z| > \frac{1}{2}$$

$$\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{z}{z-\frac{1}{3}}, \quad |z| > \frac{1}{3}$$

Since there is a common ROC betn the z-transform of above two sequence.



$$Y(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{3}}, \quad |z| > \frac{1}{2}$$

$$= \frac{z\left(z-\frac{1}{3}\right) + z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

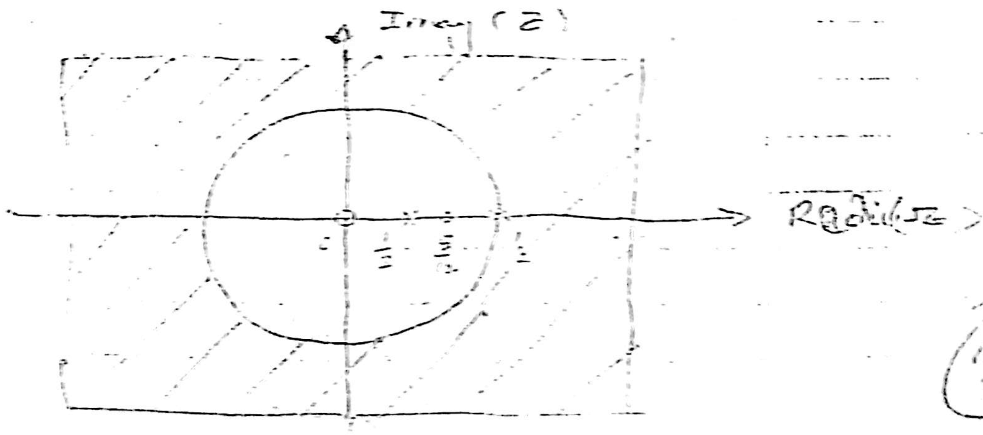
$$= \frac{z^2 - \frac{1}{3}z + z^2 - \frac{1}{2}z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{2z^2 - \frac{5}{6}z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$$= \frac{2z\left(z - \frac{5}{12}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

$Y(z)$ has two zeros at $z = 0$ and $z = 5/12$

$Y(z)$ has two poles at $z = 1/2$ & $z = 1/3$.



$x[n] = \left(\frac{1}{3}\right)^n u[n] + \left(\frac{1}{2}\right)^n u[-n-1]$

we have;

$$-a^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| < |a|$$

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| > |a|$$

$$\left(\frac{1}{2}\right)^n u[-n-1] \xleftrightarrow{z} -\frac{z}{z-\frac{1}{2}}, \quad |z| < \frac{1}{2}$$

$$\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{z} \frac{z}{z-\frac{1}{3}}, \quad |z| > \frac{1}{3}$$

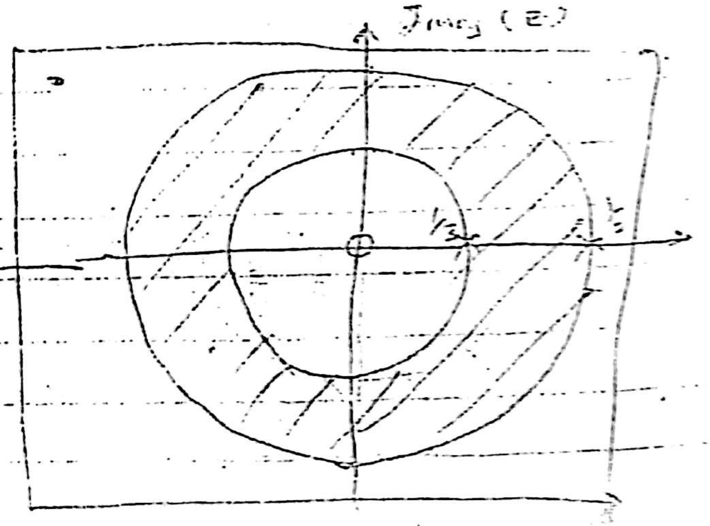
Since there is the common Roc both the z-transform of above two sequences of the

$$X(z) = -\frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{3}}, \quad \frac{1}{3} < |z| < \frac{1}{2}$$

$$= \frac{-z\left(z-\frac{1}{3}\right) + z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)} = \frac{-1}{6}\frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$

X(z) has a zero at $z = 0$.

X(z) has a pole at $z = \frac{1}{2}$ & $z = \frac{1}{3}$



iii) $x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[-n-1]$

we have,

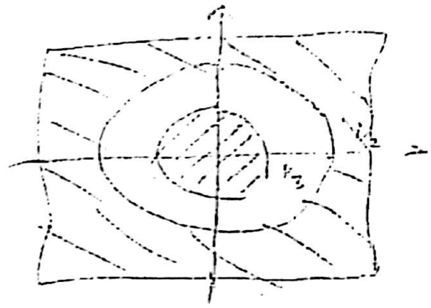
$-a^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-a}$, $|z| < |a|$

$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}$, $|z| > |a|$

$\left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} \frac{z}{z-\frac{1}{2}}$, $|z| > \frac{1}{2}$

$\left(\frac{1}{3}\right)^n u[-n-1] \xleftrightarrow{z} \frac{-z}{z-\frac{1}{3}}$, $|z| < \frac{1}{3}$

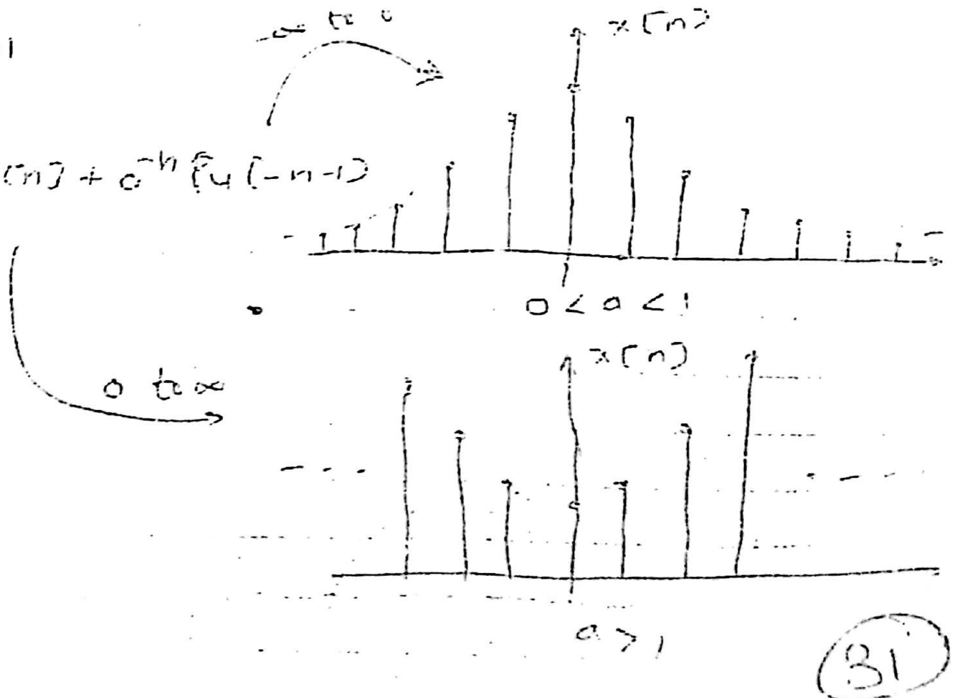
Since, there is no common ROC both the z-transform of above two sequences, Z-transform $X(z)$ doesnot exist.



iv. $x[n] = a^{|n|}$ for all n.

sol: $x(n) = a^{|n|}$

$x(n) = a^n u(n) + a^{-n} \{u(-n-1)\}$



$$a^n u[n] \xleftrightarrow{Z} \frac{z}{z-a} \quad |z| > |a|$$

$$-a^n u[-n-1] \xleftrightarrow{Z} \frac{z}{z-a} \quad |z| < |a|$$

$$-\left(\frac{1}{a}\right)^n u[-n-1] \xleftrightarrow{Z} \frac{z}{z-\frac{1}{a}} \quad |z| < \frac{1}{|a|}$$

Case I if $|a| > 1$, there is no common RoC both the Z-transform of above two sequences. Hence, Z-transform $X(z)$ doesn't exist.

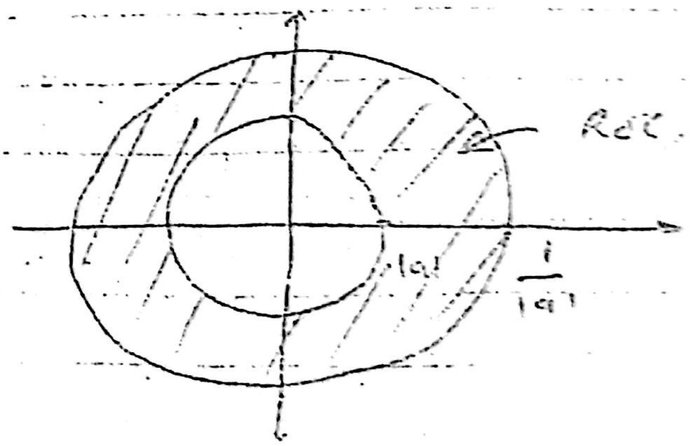
Case II: if $|a| < 1$, there is common RoC both the ~~above~~ Z-transform of above two sequences.

$$X(z) = \frac{z}{z-a} - \frac{z}{z-\frac{1}{a}}, \quad |a| < |z| < \frac{1}{|a|}$$

$$= \frac{z\left(z-\frac{1}{a}\right)-(z-a)z}{(z-a)\left(z-\frac{1}{a}\right)}$$

$$= \frac{z\left(a-\frac{1}{a}\right)}{(z-a)\left(z-\frac{1}{a}\right)} = \frac{z\left(\frac{a^2-1}{a}\right)}{(z-a)\left(z-\frac{1}{a}\right)}$$

$$|a| < |z| < \frac{1}{|a|}$$



1. Find Inverse z-transform of given :

1. $X(z) = z^2(1 - \frac{1}{2}z^{-1})(1 - z^{-1})(1 + 2z^{-1})$ $0 < |z| < \infty$

Sol. $X(z) = z^2\left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}\right)\left(1 + 2z^{-1}\right)$

$= z^2\left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} + 2z^{-1} - 3z^{-2} + z^{-3}\right)$

$= z^2 + \frac{1}{2}z - \frac{5}{2} + z^{-1}$

Comparing with $X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$

$= \cdots + x[-2]\, z^{+2} + x[-1]\, z^{1} + x[0]$
$+ x[1]\, z^{-1} + x[2]\, z^{-2}$

we get;

$x[n] = \{1, \frac{1}{2}, -\frac{5}{2}, 1\}$
$\qquad\qquad\quad\uparrow$

Ans.

2. $X(z) = \dfrac{z}{2z^2 - 3z + 1}$, $|z| > 1$.

Sol. Using Division method;
Since, ROC is $|z| > 1$, $x[n]$ must be right sided infinite length sequence, so we must divide to obtain a series in power of z^{-1}

$2z^2 - 3z + 1 \;\overline{)\; z \;}\; \left(\frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \right.$

$\underline{-z \mp \frac{3}{2} \pm \frac{1}{2}z^{-1}}$

$\frac{3}{2} - \frac{1}{2}z^{-1}$

$\underline{\frac{3}{2} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}}$
$\quad - \quad + \qquad -$

$\frac{7}{4}z^{-1} - \frac{3}{4}z^{-2}$

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$X(z) = \frac{1}{2} z^{-1} + \frac{3}{4} z^{-2} + \frac{7}{8} z^{-3} + \frac{15}{16} z^{-4} \cdots$

Comparing with $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

we get

$x[+1] = \frac{1}{2}$, $x[+2] = 3/4$, $x[3] =$

$x[n] = 0$, for $n < 0$, $x[0] = 0$.

$\therefore x[n] = \{ 0, \frac{1}{2}, \frac{3}{4}, 7/8, \frac{15}{16}, \cdots \}$

$\therefore x[n] = \left(1 - \left(\frac{1}{2}\right)^n\right) u[n]$ Ans

iii) $X(z) = \frac{z}{2z^2 - 3z + 1}$, $|z| < \frac{1}{2}$

Using Division method.

sol: Since ROC is $|z| < \frac{1}{2}$, $x[n]$ must be left sided infinite length sequence. So we must divide to obtain the series in power of z.

$$
\begin{array}{r|l}
1 - 3z + 2z^2 & z \quad\quad\quad \quad z + 3z^2 + 7z^3 + 15z^4 + \cdots \\
 & -z - 3z^2 + 2z^3 \\
 & 3z^2 - 2z^3 \\
 & 3z^2 - 9z^3 + 6z^4 \\
 & 7z^3 - 6z^4 \\
 & 7z^3 - 21z^4 + 14z^5 \\
 & 15z^4 - 14z^5 \\
 & 15z^4 - 45z^5 + 30z^6 \\
 & 31z^5 - 30z^6
\end{array}
$$

$$C_2 = \left(z-\tfrac{1}{2}\right) \cdot \frac{X(z)}{z}\bigg|_{z=\frac{1}{2}} = \left(z-\tfrac{1}{2}\right) \frac{1}{z(z-1)(z-\frac{1}{2})}\bigg|_{z=\frac{1}{2}}$$

$$= -1$$

$$\therefore X(z) = \frac{z}{z-1} - \frac{z}{z-\frac{1}{2}}$$

① ROC : $|z| > 1$:

we have;

$$a^n u[n] \xleftrightarrow{Z} \frac{z}{z-a}, \quad |z| > |a|$$

Now, taking inverse z-transform; using table, we get

$$x[n] = u[n] - \left(\frac{1}{2}\right)^n u[n]$$

$$= \left[1 - \left(\frac{1}{2}\right)^n\right] u[n].$$

Ans

ROC : $|z| < \frac{1}{2}$.

we have,

$$-a^n u[-n-1] \xleftrightarrow{Z} \frac{z}{z-a}, \quad |z| < |a|$$

Now, taking inverse z-transform using table, we get;

$$x[n] = -u[-n-1] + \left(\frac{1}{2}\right)^n u[-n-1]$$

$$= \left[\left(\frac{1}{2}\right)^n - 1\right] u[-n-1]$$

Ans

$X(z) = \cdots + 31z^5 + 15z^4 + 7z^3 + 3z^2 \ldots$

Comparing with $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

we get,

$x[-1] = 1, \quad x[-2] = 3, \quad x[-3] = 7, \ldots$

$x[-5] = 31, \ldots$

$x[n] = 0 \quad \text{for } n \geq 0.$

$\therefore x[n] = \{ \ldots 31, 15, 7, 3, 1, \underset{\uparrow}{0} \}$

$x[n] = \left(\left(\frac{1}{2}\right)^n - 1 \right) u[-n-1].$

iv) $X(z) = \dfrac{z}{2z^2 - 3z + 1}$,

if. a. ROC : $|z| > 1$

b. ROC : $|z| < \frac{1}{2}$

c. ROC : $\frac{1}{2} < |z| < 1$.

Soln

$$X(z) = \frac{z}{2z^2 - 3z + 1}$$

$$= \frac{z}{2(z-1)(z-\frac{1}{2})}$$

$$\frac{X(z)}{z} = \frac{1}{2(z-1)(z-\frac{1}{2})} = \frac{c_1}{(z-1)} \ldots$$

where,

$$c_1 = (z-1) \cdot \frac{X(z)}{z}\Big|_{z=1} = \frac{1}{2(z-\frac{1}{2})}$$

$= 1$

iii) Roc : $\frac{1}{2} < |z| < 1$.

Sol. we have;

$$-a^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| < |a|. \quad 1.$$

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| > |a|. \quad 1/2$$

Now, taking inverse z-transform using table, we get;

$$x[n] = -u[-n-1] - \left(\frac{1}{2}\right)^n u[n]$$

H.W. Q. $X(z) = \frac{2z+3}{z^2-z-6}$ → $1/z$

$$X(z) = \frac{1}{(z-2)(z-3)}$$

Sol. If (i) Roc : $|z| > 3$
(ii) Roc : $|z| < 2$
(iii) Roc : $2 < |z| < 3$

$$\frac{A}{z-2} + \frac{B}{z-3}.$$

Sol. Given:

$$X(z) = \frac{z}{z^2-z-6}$$

$$= \frac{z}{(z-3)(z+2)}$$

$$\frac{X(z)}{z} = \frac{c_1}{(z-3)} + \frac{c_2}{(z+2)}$$

when,

$$c_1 = (z-3)\left.\frac{X(z)}{z}\right|_{z=3} \qquad c_2 = (z+2)\left.\frac{X(z)}{z}\right|_{z=-2}$$

$$= \left.\frac{1}{z+2}\right|_{z=3} \qquad = \left.\frac{1}{z-3}\right|_{z=-2}$$

$$= \frac{1}{5} \qquad = -\frac{1}{5}$$

(3(i))

$$\therefore \frac{X(z)}{z} = \frac{1/5}{z-3} + \frac{(-\frac{1}{5})}{z+2}$$

$$\text{or, } X(z) = \frac{1}{5}\left(\frac{z}{z-3} - \frac{z}{z-(-2)}\right)$$

$(3, -2)$

i) ROC : $|z| > 3$

We have;

$$a^n u[n] \longleftrightarrow \frac{z}{z-a} \; ; \; |z| > |a|$$

Now taking inverse z-transform ; using table

$$\therefore x[n] = \frac{1}{5}\left[3^n u[n] - (-2)^n u[n]\right]$$

$$= \frac{1}{5}\left[3^n - (-2)^n\right] u[n]$$

Ans

ii) ROC : $|z| < 2$

We have:

$$-a^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-a} \; , \; |z| < |a|$$

Now, taking inverse z-transform, using table gets

$$x[n] = \frac{1}{5}\left[-(3)^n u[-n-1] + (-2)^n u[-n-1]\right]$$

$$= \frac{1}{5}\left[(-2)^n - (3)^n\right] u[-n-1]$$

Ans

$|z| < 3, \quad z = 3$

iii) $2 < |z| < 3$

we have:

$$-a^n u[-n-1] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| < |a|$$

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| > |a|$$

Now, taking inverse z-transform using table we get,

$$x[n] = \frac{1}{5}\left[-(3)^n u[-n-1] - (2)^n u[n]\right]$$

$|z| < 3 \qquad\qquad |z| > 2$

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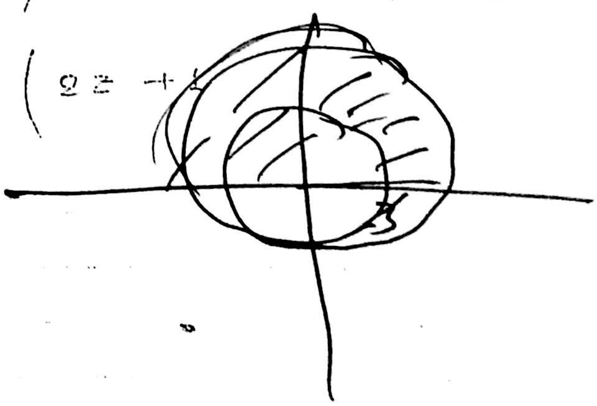
$*$ $$X(z) = \frac{2z^3 - 5z^2 + z + 3}{z^2 - 3z + 2}, \quad |z| < 1$$

(Power of Numerator is greater than Power of denominator)

Sol: First, make $X(z)$ a proper rational function

$$z^2 - 3z + 2 \overline{\big)\, 2z^3 - 5z^2 + z + 3 \,}\big(2z + 1$$
$$\underline{-2z^3 \mp 6z^2 \pm 4z}$$
$$z^2 - 3z + 3$$
$$\underline{-z^2 \mp 3z \pm 2}$$
$$1$$



$$\therefore X(z) = 2z + 1 + \frac{1}{z^2 - 3z + 2}$$

let $$X_1(z) = \frac{1}{z^2 - 3z + 2}$$

$$x[n] = 2\delta[n+1] + \delta[n] + \frac{1}{2}\delta[n] + \frac{1}{2}\left(-2^n u[-n-1]\right) + u[-n-1].$$

$$= 2\delta[n+1] + \frac{3}{2}\delta[n] - \frac{1}{2}\, 2^n u[-n-1] + u[-n-1]$$

Ans

Q: 1. $\dfrac{2z^4 - 5z^3 + 2z^2 + 5z + 1}{z^2 - 4z + 3}$, $|z| < 1$

(7) $X(z) = \dfrac{z}{(z-1)(z-2)^2}$, $|z| > 2$

Sol: $\dfrac{X(z)}{z} = \dfrac{1}{(z-1)(z-2)^2}$

$$\frac{X(z)}{z} = \frac{1}{(z-1)(z-2)^2} = \frac{C_1}{(z-1)} + \frac{A_1}{(z-2)} + \frac{A_2}{(z-2)^2}$$

where;

$$C_1 = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{1}{(z-2)^2} = 1$$

$$A_2 = \frac{1}{0!} \cdot \frac{d^0}{dz^0}(z-2)^2 \frac{X(z)}{z}\bigg|_{z=2}$$

$$= 1 \cdot (z-2)^2 \cdot \frac{1}{(z-1)(z-2)^2}\bigg|_{z=2}$$

$$= \frac{1}{(2-1)}$$

$$= 1.$$

$$A_2 = \frac{1}{1!}\frac{d}{dz}\left((z-2)^2 \cdot \frac{X(z)}{z}\right)\Big|_{z=2}$$

$$= \frac{d}{dz}(z-2)^2 \cdot \frac{1}{(z-1)(z-2)^2}\Big|_{z=2}$$

$$= \frac{d}{dz}\left(\frac{1}{z-1}\right)\Big|_{z=2}$$

$$= -(z-1)^{-2}\Big|_{z=2}$$

$$= -\frac{1}{(2-1)^2}$$

$$= -1.$$

~~We have;~~

$$X(z) = \frac{z}{z-1} - \frac{z}{z-2} + \frac{z}{(z-2)^2}, \quad |z|>2$$

we have;

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z|>|a|$$

$$n\, a^n u[n] \xleftrightarrow{z} \frac{az}{(z-a)^2}, \quad |z|>|a|$$

Now taking inverse z-transform using table,

$$x[n] = u[n] - 2^n u[n] + \frac{1}{2} n\, 2^n u[n]$$

$$= (1 - 2^n + n\, 2^{n-1})\, u[n]$$

Ans.

1. $u[n]$

Q. 92 $X(z) =$ [illegible]

H.W.

Q. 93 [illegible]

* Find the output of LTI system having input $x[n] = \alpha^n u[n]$ and impulse response $h[n] = \beta^n u[n]$. Also find the o/p. if $\alpha = \beta$.

<u>Sol:</u>

We have:

$$a^n u[n] \xleftrightarrow{z} \frac{z}{z-a}, \quad |z| > |a|$$

$$x[n] = \alpha^n u[n] \xleftrightarrow{z} X(z) = \frac{z}{z-\alpha}, \quad |z| > |\alpha|$$

$$h[n] = \beta^n u[n] \xleftrightarrow{z} H(z) = \frac{z}{z-\beta}, \quad |z| > |\beta|$$

o/p of LTI system:

$$y[n] = x[n] * h[n]$$

$$\therefore Y(z) = X(z) \cdot H(z) \qquad ; \text{ ROC} : R_1 \cap R_2$$

$$= \frac{z}{z-\alpha} \cdot \frac{z}{z-\beta} \qquad ; \text{ ROC} : |z| > \max\{|\alpha|, |\beta|\}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-\alpha)(z-\beta)}$$

$$= \frac{C_1}{z-\alpha} + \frac{C_2}{z-\beta}$$

where, $C_1 = (z-\alpha)\dfrac{Y(z)}{z}\Big|_{z=\alpha}$

$$= (z-\alpha)\cdot\frac{z}{(z-\alpha)\cdot(z-\beta)}\Big|_{z=\alpha}$$

$$= \frac{\alpha}{\alpha-\beta}$$

$$C_2 = (z-\beta)\frac{X(z)}{z}\Big|_{z=\beta}$$

$$= \frac{z}{z-\alpha}\Big|_{z=\beta} = \frac{\beta}{\beta-\alpha} = -\frac{\beta}{\alpha-\beta}$$

$$\therefore Y(z) = \frac{\alpha}{\alpha-\beta}\cdot\frac{z}{z-\alpha} - \frac{\beta}{\alpha-\beta}\cdot\frac{z}{z-\beta}.$$

Now; taking inverse z-transform using table, we gets | ROC: $|z| > \max\{|\alpha|$

$$y[n] = \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n]$$

$$= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha-\beta}\cdot u[n]$$

For $\alpha = \beta$;

$$Y(z) = \left(\frac{z}{z-\alpha}\right)^2 \quad |z|>$$

we have;

$$(n+1)\,\alpha^n u[n] \xleftrightarrow{z} \left(\frac{z}{z-\alpha}\right)^2$$

Taking inverse z-transform using table,

$$y[n] = [n+1]\,\alpha^{n}\,u[n] \qquad ROC: |z| >$$

Ans

H.W.

4. $x[n] = \alpha^{n} u[n]$, $h[n] = u[n]$

5. $x[n] = \left(\frac{1}{2}\right)^{n} u[n]$, $h[n] = \left(\frac{1}{3}\right)^{n} u[n]$

Find $y[n] = x[n] * h[n]$.

Causality of LTI System :

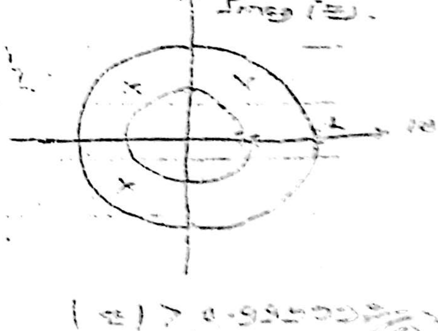
LTI system will be causal if $h[n] = 0$ for $n < 0$
i.e. $h[n]$ must be right sided sequence.

If $H(z)$ exists, ROC will be $|z| > r_{max}$.

Stability of LTI System :

LTI system will be stable if $\left|\sum_{n=-\infty}^{\infty} h[n]\right| < \infty$
If $H(z)$ exists, then ROC must include $|z| = 1$

To be a LTI system, Both causal & stable :-
All poles of $H(z)$ must lie inside the unit circle



$|z| > 0.99999$

* $h[n] = 2^n u[n] \xleftrightarrow{z} H(z) = \dfrac{z}{z-2}$, $|z|$

↳ Causal

↳ Unstable

* $h[n] = \left(\frac{1}{3}\right)^n u[-n-1] \xleftrightarrow{z} H(z) = -\dfrac{z}{z-\frac{1}{3}}$

↳ Non-causal

↳ Unstable

* $h[n] = 2^n u[-n-1] \xleftrightarrow{z} H(z) = -\dfrac{z}{z-2}$, $|z|$

↳ Non-causal

↳ ~~Causal~~ Stable

* $h[n] = \left(\frac{1}{2}\right)^n u[n] \xleftrightarrow{z} H(z) = \dfrac{z}{z-\frac{1}{2}}$, $|z|$

↳ Causal

↳ Stable.

Response of LTI system due to complex exp

O/p of LTI system;

$$y[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k] \cdot x[-n-k]$$

I/input; $x[n] = A e^{j\omega_0 n}$, $-\infty < n < \infty$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot A e^{j\omega(n-k)}$$

$$= A \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\omega_0 k}\, e^{j\omega_0 n}$$

$$= A\ H(e^{j\omega_0}) \cdot e^{j\omega_0 n}$$

$$= x[n] \cdot H(e^{j\omega_0})$$

↑ fourier transform of $h[n]$.

Also;

$$H(e^{j\omega_0}) = |H(e^{j\omega_0})| \cdot e^{j\theta_0}$$

$$\therefore y[n] = A|H(e^{j\omega_0})|\, e^{j\theta_0} \cdot e^{j\omega_0 n}$$

$$= A\,|H(e^{j\omega_0})|\, e^{j(\omega_0 n + \theta_0)}$$

iii) For input $x[n] = Ae^{-j\omega_0 n}$, $-\infty < n < \infty$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \cdot Ae^{-j\omega_0 [n-k]}$$

$$= A \cdot \sum_{k=-\infty}^{\infty} h[k] \cdot e^{j\omega_0 k} \cdot e^{-j\omega_0 n}$$

$$= A\ H(e^{-j\omega_0}) \cdot e^{-j\omega_0 n}$$

$$= x[n] \cdot H(e^{-j\omega_0})$$

2d 9.0 (He)

Also;

$$H(e^{-j\omega_0}) = |H(e^{+j\omega_0})| \cdot e^{-j\theta_0}$$

$$\therefore y[n] = A\,|H(e^{j\omega_0})|\, e^{-j\theta_0} \cdot e^{-j\omega_0 n}$$

$$= A\,|H(e^{j\omega_0})| \cdot e^{-j(\omega_0 n + \theta_0)}$$

Response ... LTI system due to sinusoidal ... system:

$\Rightarrow$ O/p ... system:

$y[n] = x[n] * h[n]$

$= \sum_{k=-\infty}^{\infty} h[n] \cdot x[n-k]$

(i) For input $x[n] = A\cos\omega_0 n, \quad -\infty < n < \infty$

$= \frac{A}{2} e^{j\omega_0 n} + \frac{A}{2} e^{-j\omega_0 n}, \quad -\infty < n <$

O/p will be;

$y[n] = \frac{A}{2} \cdot |H(e^{j\omega_0})| e^{j(\omega_0 n + \theta)} + \frac{A}{2} |H(e^{j\omega_0})| e^{-j\ldots}$

$= A |H(e^{j\omega_0})| \cos(\omega_0 n + \theta_0)$ #

(ii) For input $x[n] = A\sin\omega_0 n, \quad -\infty < n < \infty$

$= \frac{A}{2j} e^{j\omega_0 n} - \frac{A}{2j} e^{-j\omega_0 n}, \quad -\infty \ldots$

O/p will be;

$y[n] = \frac{A}{2j} |H(e^{j\omega_0})| e^{j(\omega_0 n + \theta)} - \frac{A}{2j} |H(e^{j\omega_0})| \ldots$

$= A |H(e^{j\omega_0})| \sin(\omega_0 n + \theta)$ #

Steady State Response :-
The response of the system observed at then i/p signal complex exponential of sinu is applied at the input of the syst for $n = \infty$ is called steady state resp

Transient State Response :-
When complex exponential or sinusoidal i/ signal is applied to the system, th response observed at the output for some finite time instant contains bot steady state & transient state Respons Natural response.

for example :-
system is $y[n] = a\,y[n-1] + x[n]$
and input signal $x[n] = Ae^{j\omega n}$ applied at $n \geq 0$.

System can be also written as;

$$y[n] = a^{n+1} y[-1] + \sum_{k=0}^{n} a^k x[n-k]$$

$$= a^{n+1} y[-1] + \sum_{k=0}^{n} a^k A e^{j\omega(n-k)}$$

$$= a^{n+1} y[-1] + A \sum_{k=0}^{n} (a e^{-j\omega})^k e^{j\omega n}$$

$$= a^{n+1} y[-1] + A \left(\frac{1(1-(a e^{-j\omega})^{n+1})}{1 - ae^{-j\omega}} \right) e^{j\omega n}$$

This system will be be BIBO stable iff $|a e^{-j\omega}|$

i) $x[n] = A e^{j\frac{\pi}{2}n}$ $\qquad h[n] = ($

$y[n] = x[n] * h[n]$

ii) $x[n] = 10 - 5\sin\frac{\pi}{2}n - 20\cos\pi n$

$h[n] = \left(\frac{1}{2}\right)^n u[n]$.

$y[n] = x[n] * h[n]$.

$$y[n] = a^{n+1} y[-1] + A \frac{1}{1-ae^{-j\omega}} e^{j\omega n} - A \frac{a^{n+1} e^{-j\omega(n+1)}}{1-ae^{-j\omega}} e^{j\omega n}$$

BIBO stable iff $|a| < 1$.

As n increases, a^{n+1} decreases.
$n \longrightarrow \infty$, $a^{n+1} \longrightarrow 0$

steady state response;

$$y_{ss}[n] = \lim_{n \to \infty} y[n]$$

$$= A \cdot \frac{1}{1-ae^{-j\omega}} e^{j\omega n}$$

Remaining terms are transient-state response:

$$y_{ts}[n] = a^{n+1} y[-1] - \frac{A\, a^{n+1} e^{-j\omega(n+1)}}{1-ae^{-j\omega}} e^{j\omega n}$$

As $n \to \infty$, $y_{tr}[n] \longrightarrow 0$.

General form of constant coefficient difference equation for discrete time LTI system is given by:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

Taking z-transform using properties, we get:

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

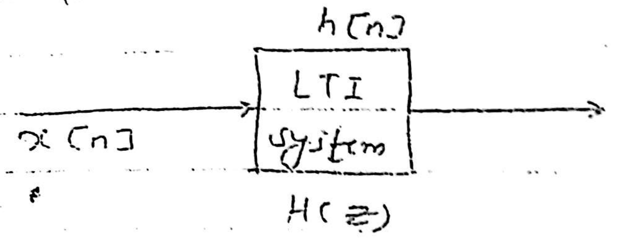
Transfer function $H(z)$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

or, $$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots + a_N z^{-N}}$$

$$= \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})} = \frac{N(z)}{D(z)}$$

system contains N number of poles at $z = p_k$



Output of system:

$$Y(z) = H(z) \cdot X(z)$$

$X(z)$ is also a rational function of z

$$X(z) = \frac{B(z)}{A(z)}$$

$A(z)$ contains L number of poles

$$A(z) = (1 - q_1 z^{-1}) \cdots (1 - q_L z^{-1})$$

(a) i) $y[n] = 0.5y[n-1] + x[n]$
system,
inputs $x[n] = 10 \cos \frac{\pi}{4} n$

Find steady state state and transient response.

$$= \frac{N(z)}{D(z)} \cdot \frac{B(z)}{A(z)}$$

After partial fraction expansion, we get;

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{B_k}{1 - q_k z^{-1}}$$

where, A_k & B_k are coefficients.

Taking inverse z-transform, we get;

$$y[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n] + \sum_{k=1}^{L} B_k (q_k)^n u[n]$$

If $|p_k|$ and $|q_k| > 1$, then output will be exponentially growing as n-increases.

If $|p_k|$ and $|q_k| < 1$, then output will be exponentially decaying as n-increases.

If $|p_k| = |q_k| = 1$, then o/p remains constant.

If $|p_k|$ or $|q_k| = 1$, then response will be steady state.

If $|p_k|$ or $|q_k| \neq 1$, then response will be transient state response.

4. Discrete Fourier Transform (DFT)

Periodic

$$x_p(t) = \sum_{K=-\infty}^{\infty} c_K e^{j\omega_0 K t} \quad , \omega_0 = 2\pi$$

$$c_k = \frac{1}{T} \int_T x_p(t) e^{-j\omega_0 t} dt$$

$$x_p[n] = \sum_{K=0}^{N-1} c_k e^{jk\omega_0 n}$$

&

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n] e^{-jk\omega_0 n}$$

aperiodic

$x(t)$

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$$

and

Why need DFT?

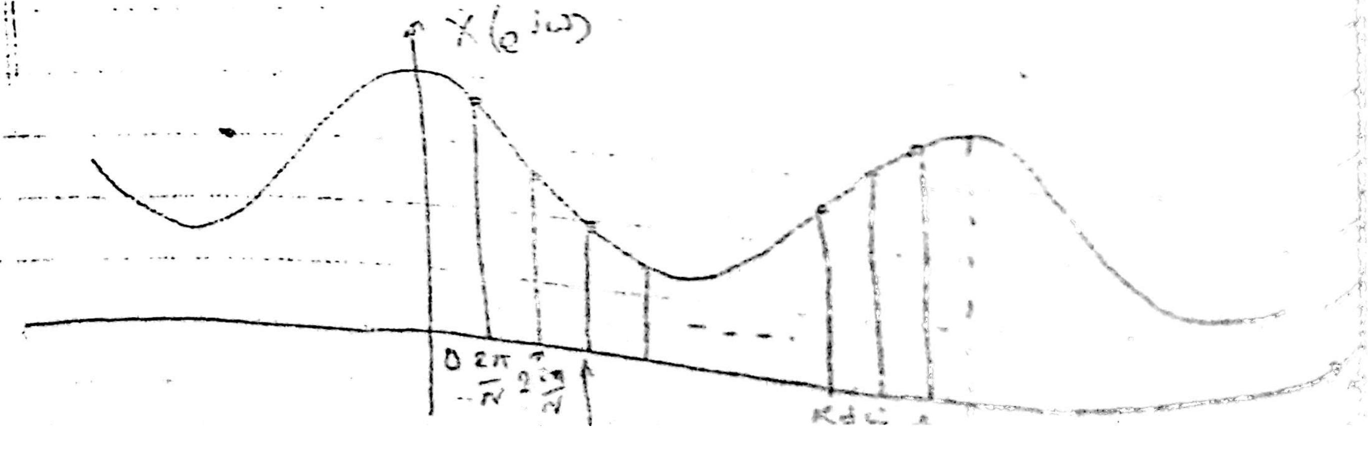
In many digital signal processing ... consists of analysis of discrete time fourier transform of ... often performed. Since fourier transform is a function of frequency, $x[n]$ is a continuous function of ... not a computationally ... ω from 0 to 2π. It is not a convinient representation of discrete time sequence in frequency domain. So, it is represented by samples of fourier transform $X(e^{j\omega})$.

Such a frequency domain representation is called <u>Discrete Fourier Transform</u> (DFT). DFT is a very useful and powerful computational tool for performing frequency analysis of sequence in ~~frequency~~ digital domain.

Frequency domain Sampling:

Let $x[n]$ be the discrete time finite or aperiodic sequence & its fourier transform is given by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$



$$Y(k) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x[n-lN]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x[n-lN]\, e^{-jk\frac{2\pi}{N}n} \quad ----$$

The sequence, $x_p[n] = \sum_{l=-\infty}^{\infty} x[n-lN]$ can be obtained by periodic repetition of $x[n]$ for every N samples.

Periodic signals can be represented by fourier series as,

$$x_p[n] = \sum_{k=0}^{N-1} c_k\, e^{jk\frac{2\pi}{N}n} \quad --- \text{(II)}$$

for $n = 0, 1, \ldots N-1$

and,

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p[n]\, e^{-jk\frac{2\pi}{N}n} \quad --- \text{(III)}$$

for $k = 0, 1, \ldots, N-1$.

from eq^n (I) & (III), we get

~~$X(k)=$~~ $c_k = \frac{1}{N} X(k)$

put the value of c_k in equation (II).

$$x_p[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{jk\frac{2\pi}{N}n}$$

for ~~k~~ $n = 0, 1, 2, \ldots, N-1$

This eq^n implies that periodic sequence $x_p[n]$

Let us take N - equidistance frequency samples from $\omega = 0$ to 2π with a spacing of $\delta\omega = \frac{2\pi}{N}$ between successive samples.

Since, fourier transform is periodic of $\omega = 0$ to 2π, It is only necessary to take the samples in fundamental range.

Put; $\omega = k\,\delta\omega = k \cdot \frac{2\pi}{N}$ for $k = 0, 1, 2, \cdots, N-1$

then;

$$X\left(e^{jk\frac{2\pi}{N}}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

for $k = 0, 1, \cdots, N-1$

$$X(k) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$= \cdots + \sum_{n=-N}^{-1} x[n]\, e^{-jk\frac{2\pi}{N}n} + \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

$$+ \sum_{n=N}^{2N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} + \cdots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x[n]\, e^{-jk\frac{2\pi}{N}n}$$

By changing the index in inner summation ($\sum_{n=lN}$) from $n - lN$ to n & interchanging the order of summation, we obtain the result;

$$X(k) = \sum_{n=0}^{N-1} \sum_{n=-\infty}^{\infty} x[n - lN]\, e^{-jk\frac{2\pi}{N}(n-lN)}$$

$$
\begin{array}{c}
x[n] \\
\Big\downarrow \text{FT} \\
X(e^{j\omega}) \longrightarrow \text{Continuous / periodic} \\
\Big\downarrow \text{FDS} \\
X(k) \\
\Big\downarrow \\
x_p[n]
\end{array}
$$

\# can be obtained from samples of fourier transform but it doesnot implies that $x[n]$ or $X(e^{j\omega})$ can be obtained from it's frequency sample $X(k)$.

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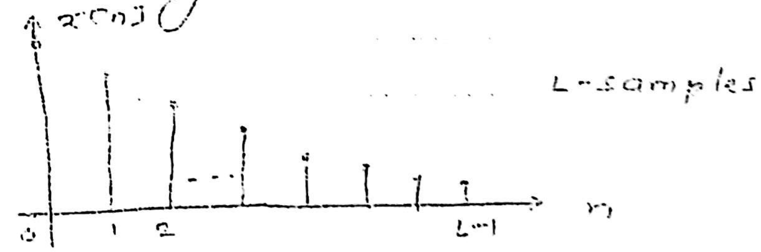
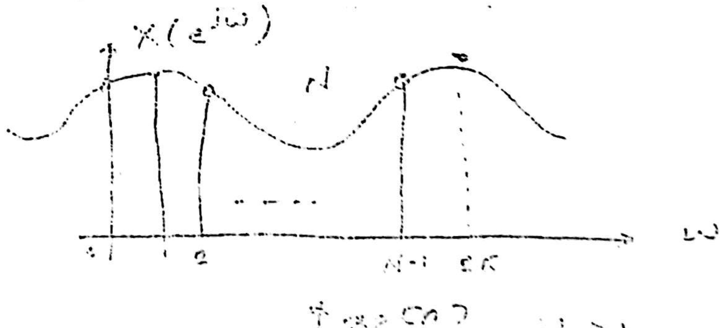
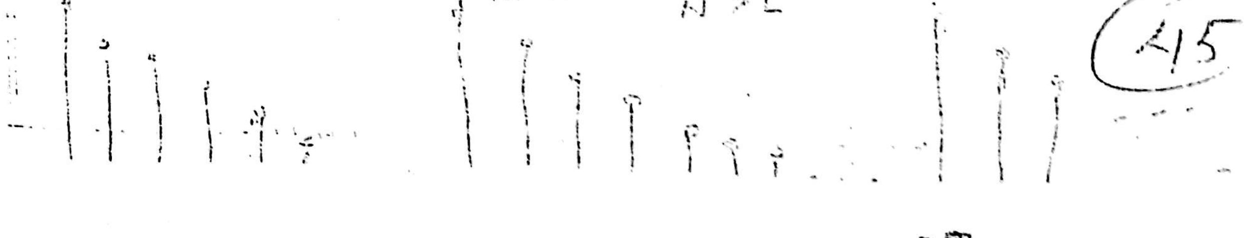
Since $x_p[n]$ is periodic repeatation of $x[n]$ over fundamental period N. It is clear that $x[n]$ can be recovered if there is no time domain alliasing.







Case - I   If $N < L$:

then there will be time domain aliasing

$$x[n] \neq x_p[n]$$

Original sequence $x[n]$ can not be recovered its frequency samples.

Case - II :   If $N \geq L$.

then $x[n] = x_p[n]$; $0 \leq n \leq N-1$

There is no time domain aliasing.

Original sequence $x[n]$ can be recovered its frequency samples.

So,

$$x[n] = x_p[n] , \quad 0 \leq n \leq N-1$$

$$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{jk\frac{2\pi}{N}n}}$$

for $n = 0, 1, \ldots, N$

Called Inverse discrete Fourier trans

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n}$$

for $k = 0, 1, \ldots, N-1$

Called Discrete Fourier Trans

Also:

$$x_p[n] = \begin{cases} x[n] & , \quad 0 \le n \le L-1 \\ 0 & , \quad L \le n \le N-1 \end{cases}$$

Padding the sequence $x[n]$ with $N-L$ zeros and computing N-point DFT but the display will be better.

Linear Transformation of DFT:

DFT of $x[n]$;

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-jk\frac{2\pi}{N}n} \quad \text{for } k=0,1,\ldots N-1$$

$$= \sum_{n=0}^{N-1} x[n]\, \omega_N^{kn}$$

where,

$$\omega_N = e^{-j\frac{2\pi}{N}}$$

↳ called N^{th} root of unity or Twiddle factor.

Let us represent, sequence $x[n]$ for $n=0,1,2,3,\ldots N-1$ as matrix form $(x)_N$

DFT values $X(k)$ for $k=0,1,2,\ldots,N-1$ as matrix form $(X)_N$ and (ω_N) as $N \times N$ matrix

$$(x)_N = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix} \qquad (X)_N = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \vdots \\ X(N-1) \end{bmatrix}$$

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$$(W_N) = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

In this way, DFT of $x[n]$ can be represe
in matrix form as:

$$(X)_N = (W_N)\,(x)_N$$

Similarly Inverse DFT can also be written

$$(x)_N = (W_N)^{-1}\,(X)_N$$

$$= \frac{1}{N}\,(W_N)^{*}\,(X)_N$$

<u>example</u>:

Compute four point DFT of the sequenc
$x[n] = \{2, -1, 3, 0.5\}$

<u>sol</u>

DFT of $x[n]$:

$$X(k) = \sum_{n=0}^{N-1} x[n]\cdot e^{-jk\frac{2\pi}{N}n}, \quad \text{for } k = 0,1\ldots$$

$$= \sum_{n=0}^{3} x[n]\cdot e^{-jk\frac{2\pi}{4}n}, \quad \text{for } k = 0,1\ldots$$

$$= x[0]\cdot e^{0} + x[1]\, e^{-jk\frac{2\pi}{4}} + x[2]\cdot e^{-jk\ldots} + x[3]\, e^{-jk\frac{\ldots}{4}}$$

$$= 2 - (-1)\, e^{-jk\frac{\pi}{2}} + 3\, e^{-jk\pi} + 0.5\, e^{-jk\frac{3\pi}{2}}$$

For $k = 0$;

$$Y(0) = 2 - 1 + 3 - 0.5 = 7.5$$

For $k = 1$;

$$Y(1) = 2 - e^{-j\pi/2} + 3e^{-j\pi} + 0.5\, e^{-j3\pi/2}$$

$$= 2 - \left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) + 3\left(\cos\pi - j\sin\pi\right)$$

$$+ 0.5\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right)$$

$$= 2 + j + 3\,(-1 - 0) + 0.5\,(0 - (-1)j)$$

$$= 2 + j - 3 + 0.5j$$

$$= -1 + 1.5j$$

For $k = 2$;

$$Y(2) = 2 - e^{-j\pi} + 3e^{-j2\pi} + 0.5\, e^{-j3\pi}$$

$$= 2 - (\cos\pi - j\sin\pi) + 3(\cos 2\pi - j\sin 2\pi)$$

$$+ 0.5(\cos 3\pi - j\sin 3\pi)$$

$$= 2 + 1 + 3 - 0.5$$

$$= 5.5$$

For $k = 3$;

$$Y(3) = 2 - e^{-j3\pi/2} + 3e^{-j3\pi} + 0.5\, e^{-j\frac{9\pi}{2}}$$

$$= 2 - \left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 3\left(\cos 3\pi - j\sin 3\pi\right)$$

$$+ 0.5\left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$$

$X[k] = \{4 + \ldots, \ldots\}$ 5.5, -1 [illegible]

(iii)

$x[n] = \{2, -1, 3, 0.5\}$

$$(x)_4 = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0.5 \end{bmatrix}$$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix}$$

where,
$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = \left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right)$$
$$= -j$$

So,
$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

We have DFT of $x[n]$:
$$(X)_4 = (W_4)(x)_4$$

$$[X]_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 1 + 3 + 0.5 \\ 2 + j - 3 + j0.5 \\ 2 + 1 + 3 - 0.5 \\ 2 - j - 3 - 0.5j \end{bmatrix} = \begin{bmatrix} 4.5 \\ -1 + 1.5j \\ 5.5 \\ -1 - 1.5j \end{bmatrix}$$

$\therefore X(k) = \{4.5,\ -1+1.5j,\ 5.5,\ -1-1.5j\}$

H.W. → Compute 4-point DFT of $x[n] = \{1, 2, -3\}$

Properties of Discrete fourier transform (DFT):

DFT of $x[n]$:

$$X(k) = \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n}, \quad \text{for } k = 0, 1, \ldots, N-1$$

and,

IDFT of $X(k)$:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{jk\frac{2\pi}{N}n} \quad \text{for } n = 0, 1, \ldots, N-1.$$

Notation:

$$x[n] \xleftrightarrow[N]{DFT} X(k)$$

1. Linearity :

If $x_1[n] \xleftrightarrow[N]{DFT} X_1[k]$

$x_2[n] \xleftrightarrow[N]{DFT} X_2[k]$

then, $a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow[N]{DFT} a_1 X_1(k) + a_2 X_2(k)$

where a_1 & a_2 are arbitrary constants.

2. Time Reversal :

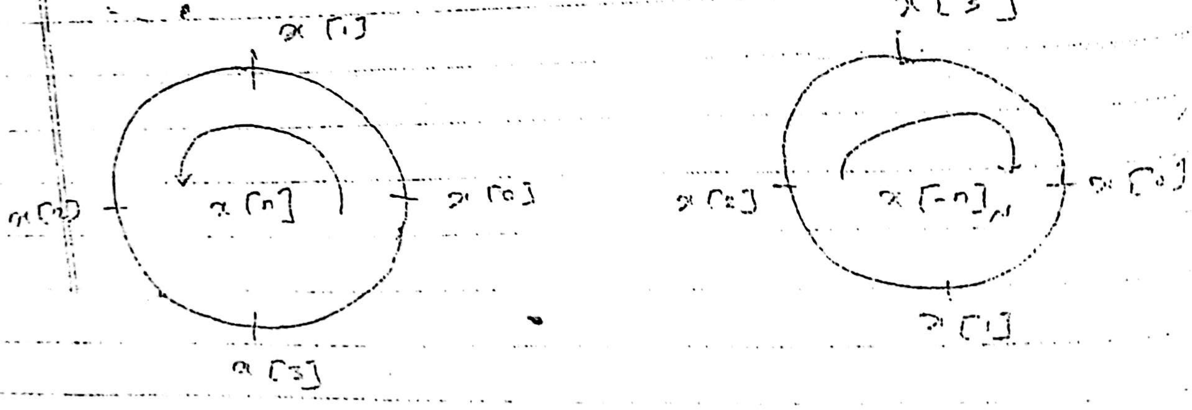
If $x[n] \xleftrightarrow[N]{DFT} X(k)$

then:

$x[-n]_N = x[N-n] \xleftrightarrow[N]{DFT} X(-k)_N = X(N-k)$

This property state that reverseing ~~the~~ a sequence in time domain is equivalent to reversing its DFT values.

i.e. magnitude of DFT values remains unchanged But the phase will be reversed.



3. Circular time shift:

If $x[n] \xleftrightarrow[N]{DFT} X(k)$

then $x[n-l]_N \xleftrightarrow[N]{DFT} e^{-jk\frac{2\pi}{N}l} X(k)$

This property states that shifting a sequence in time domain by l - units is equivalent to no change in magnitude of its DFT values but its phase will be changed by amount $-k \frac{2\pi}{N} l$.

4. Circular frequency shift:

If $x[n] \xleftrightarrow[N]{DFT} X(k)$

then $e^{jl \frac{2\pi}{N} n} x[n] \xleftrightarrow[N]{DFT} X[k-l]_N$.

This property states that multiplication of sequence $x[n]$ with complex exponential $e^{jl \frac{2\pi}{N} n}$ is equivalent to circular shift of its DFT values by l units in frequency domain.

5. Complex Conjugate:

If $x[n] \xleftrightarrow[N]{DFT} X(k)$.

then: $x^*[n] \xleftrightarrow[N]{DFT} X^*(-k)$

Proof: IDFT of $X(k)$:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{+jk \frac{2\pi}{N} n}$$

$$x^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X^*(k) e^{-jk \frac{2\pi}{N} n}$$

Also;

$$x^*[n] = \frac{1}{N} \sum_{k=\langle N \rangle} X^*(-k) e^{jk \frac{2\pi}{N} n}$$

for $n = 0, 1, \dots, N-1$

$\text{DFT}\{x^*[n]\} = X^*(-k)$

6. Multiplication of Two DFT's (Circular Convolution)

If $x_1[n] \xleftrightarrow[N]{DFT} X_1(k)$

$x_2[n] \xleftrightarrow[N]{DFT} X_2(k)$

then;

$x_1[n] \, \textcircled{N} \, x_2[n] \xleftrightarrow[N]{DFT} X_1(k) \cdot X_2(k)$

Proof:

let, $X_3(k) = X_1(k) \cdot X_2(k)$

IDFT of $X_3(k)$ is given by;

$$x_3[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) \, e^{jk\frac{2\pi}{N}n}$$

for $n = 0, 1, \ldots, N-1$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2(k) \cdot e^{jk\frac{2\pi}{N}n}$$

Also; DFT of $x_1[n]$

$$X_1(k) = \sum_{m=0}^{N-1} x_1[m] \cdot e^{-jk\frac{2\pi}{N}m}$$

for $k = 0, 1, \ldots, N-1$

Now;

$$x_3[n] = \frac{1}{N} \sum_{k=0}^{N-1} \cdot \sum_{m=0}^{N-1} x_1[m] \cdot e^{-jk\frac{2\pi}{N}m} \cdot X_2(k) e^{jk\frac{2\pi}{N}n}$$

$$= \sum_{m=0}^{N-1} x_1[m] \, \frac{1}{N} \sum_{k=0}^{N-1} X_2(k) \, e^{jk\frac{2\pi}{N}(n-m)}$$

Using circular time shift property of DFT, we get;

$$x_3[n] = \sum_{m=0}^{N-1} x_1[m] \cdot x_2[n-m]_N$$

$$= x_1[n] \; \textcircled{N} \; x_2[n]$$

sign of circular convolution

7. Multiplication of two sequences:

If $x_1[n] \xleftrightarrow[N]{DFT} X_1(k)$

$x_2[n] \xleftrightarrow[N]{DFT} X_2[k]$

then:
$$x_1[n] \cdot x_2[n] \xleftrightarrow[N]{DFT} \frac{1}{N}\left[X_1(k) \; \textcircled{N} \; X_2[k]\right]$$

Proof:

let $x_3[n] = x_1[n] \cdot x_2[n]$

DFT of $x_3[n]$ is given by:

$$X_3[k] = \sum_{n=0}^{N-1} x_3[n] e^{-jk\frac{2\pi}{N}n} \quad \text{for, } k = 0, 1, \ldots N$$

$$= \sum_{n=0}^{N-1} x_1[n] \cdot x_2[n] \cdot e^{-jk\frac{2\pi}{N}n}$$

Also, IDFT of $X_1(k)$:

$$x_1[n] = \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) e^{jl\frac{2\pi}{N}n}$$

for $n = 0, 1, \ldots N-1$

then;

$$X_3[k] = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot e^{jl\frac{2\pi}{N}n} \cdot x_2[n] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \sum_{n=0}^{N-1} x_2[n] e^{-j\frac{2\pi}{N}(k-l)n}$$

Using the property of circular frequency shift property of DFT, we get;

$$X_3(k) = \frac{1}{N} \sum_{l=0}^{N-1} X_1(l) \cdot X_2(k-l)_N$$

$$= \frac{1}{N} \left[X_1(k) \circledN X_2(k) \right]$$

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8. Co-relation:

If $x[n] \xleftrightarrow[N]{DFT} X(k)$

$y[n] \xleftrightarrow[N]{DFT} Y(k)$

where, $x[n]$ and $y[n]$ are complex valued sequence.

then;

a) Cross-correlation:

$$r_{xy}(l) \xleftrightarrow[N]{DFT} R_{xy}(k) = X(k)\, Y^{*}(k)$$

where,

$$r_{xy}(l) = \sum_{n=0}^{N-1} x[n] \cdot y^{*}[n-l]_N$$

b) Auto Correlation:

$$r_{xx}(l) \xleftrightarrow[N]{DFT} R_{xx}(k) = X(k)\, X^{*}(k)$$

$$= |X(k)|^2$$

where;

$$r_{xx}(l) = \sum_{n=0}^{N-1} x[n]\, x^{*}[n-l]_N$$

9. Parseval's Relation:

If $x[n] \xrightarrow[N]{DFT} X(k)$

$y[n] \xrightarrow[N]{DFT} Y(k)$

Where $x[n]$ & $y[n]$ are complex valued sequences, then;

$$\sum_{n=0}^{N-1} x[n]\,y^*[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot Y^*(k)$$

<u>Proof:</u>

L.H.S.

$$= \sum_{n=0}^{N-1} x[n]\cdot y^*[n]$$

We have;

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,e^{jk\frac{2\pi}{N}n}$$

for $n = 0, 1, 2, \ldots$

L.H.S.

$$= \sum_{n=0}^{N-1} \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,e^{jk\frac{2\pi}{N}n}\cdot y^*[n]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot \underline{\sum_{n=0}^{N-1} y^*[n]\,e^{jk\frac{2\pi}{N}n}}$$

Also,

$$Y(k) = \sum_{n=0}^{N-1} y[n]\,e^{-jk\frac{2\pi}{N}n}, \quad \text{for } k = 0, 1, \ldots$$

$$Y^*(k) = \sum_{n=0}^{N-1} y^*[n]\,e^{jk\frac{2\pi}{N}n}, \quad \text{for } k = 0, 1, \ldots$$

Eqn (i) becomes:

L.H.S.

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot Y^*(k).$$

= R.H.S.

proved

If $x[n] = y[n]$

then;

Energy of signal $x[n]$:

$$E = \sum_{n=0}^{N-1} |x[n]|^2 = \sum_{n=0}^{N-1} x[n] \cdot x^*[n]$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X[K] X^*(K)$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} |X(K)|^2$$

10. Periodicity:

If $x[n] \xleftrightarrow[N]{DFT} X(K)$

then,

$x[n] = x[n+N]$ for all n

$X(K) = X(K+N)$ for all K.

Example:

1. Compute the circular convolution between the two sequence: $x_1[n] = \{1, 2, 3, 4\}$ & $x_2[n] = \{2, 1, 2, 0.5\}$.

Same question In other way

1. Find IDFT of $X_3(K)$ where $X_3(K) = X_1(K)$ where $X_1(K)$ & $X_2(K)$ are four point DFT of the sequence $x_1[n] = \{1, 2, 3, 4\}$ & $x_2[n] = \{2, 1, 2, 0.5\}$ respectively.

Sol: DFT of $x_1(n)$:

$$(X_1)_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+2+3+4 \\ 1-2j-3+4j \\ 1-2+3-4 \\ 1+2j-3-4j \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

Similarly:

$$(X_2)_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2+1+3+0.5 \\ 2-j-3+0.5j \\ 2-1+3-0.5 \\ 2+j-3-0.5j \end{bmatrix}$$

$$= \begin{bmatrix} 6.5 \\ -1-0.5j \\ 3.5 \\ -1+0.5j \end{bmatrix}$$

$$(X_3)_4 = (X_1)_4 \cdot I_{4\times 4} \; (X_2)_4$$

$$= \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & -2+2j & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2-2j \end{bmatrix} \begin{bmatrix} 6.5 \\ -1-j0.5 \\ 3.5 \\ -1+0.5j \end{bmatrix}$$

$$X_3(k) = \begin{bmatrix} 65 \\ 3-j \\ -7 \\ 3+j \end{bmatrix}$$

IDFT of $X_3(k)$:

$$(x_3)_4 = \frac{1}{4} (W_4)^{*} (X_3)_4$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 65 \\ 3-j \\ -7 \\ 3+j \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 65 + 3 - \cancel{j} - 7 + 3 + \cancel{j} \\ 65 + \cancel{3j} + 1 + 7 - \cancel{3j} + 1 \\ 65 - 3 + \cancel{j} - 7 - 3 - \cancel{j} \\ 65 - \cancel{3j} - 1 + 7 + \cancel{3j} - 1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 64 \\ 74 \\ 52 \\ 70 \end{bmatrix}$$

$$= \begin{bmatrix} 16 \\ 18.5 \\ 13 \\ 17.5 \end{bmatrix}$$

$\therefore x_3[n] = \{ 16, 18.5, 13, 17.5 \}$

Another method;

We have;

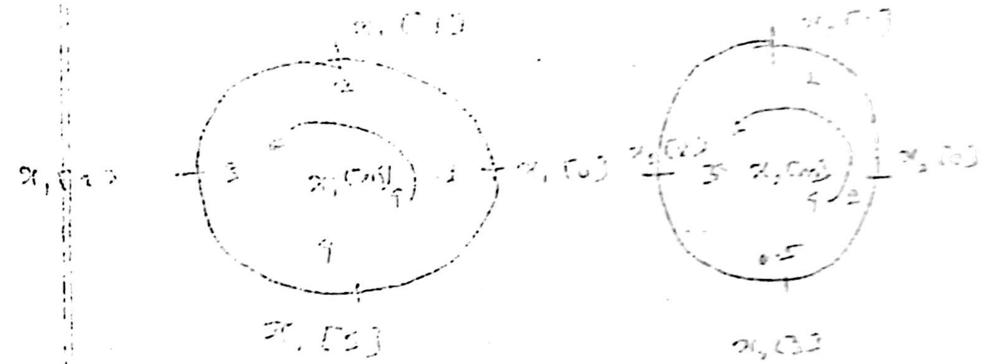
$$x_3[n] = x_1[n] \circledast x_2[n]$$

$$= \sum_{m=0}^{N-1} x_1[m]\, x_2[n-m]_N$$

for $n = 0, 1, \dots$

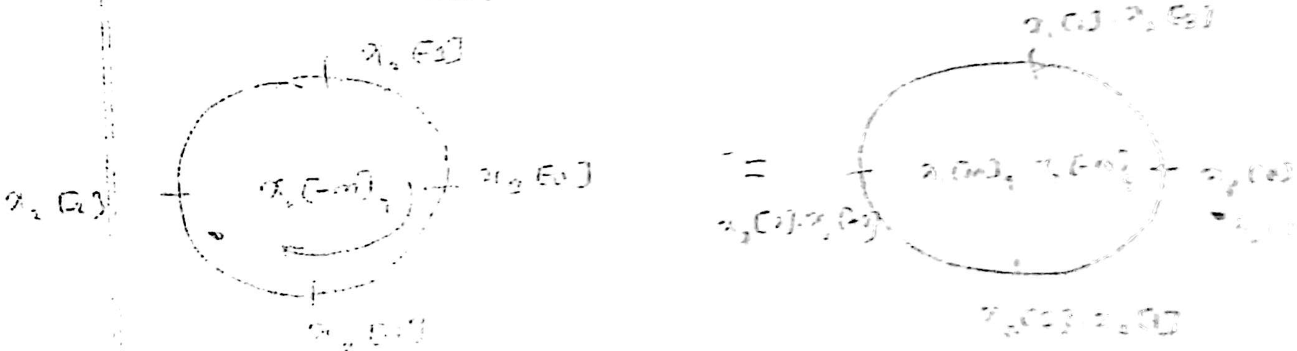
$$= \sum_{m=0}^{3} x_1[m]\, x_2[n-m]_4$$

for $n = 0, 1, \dots$



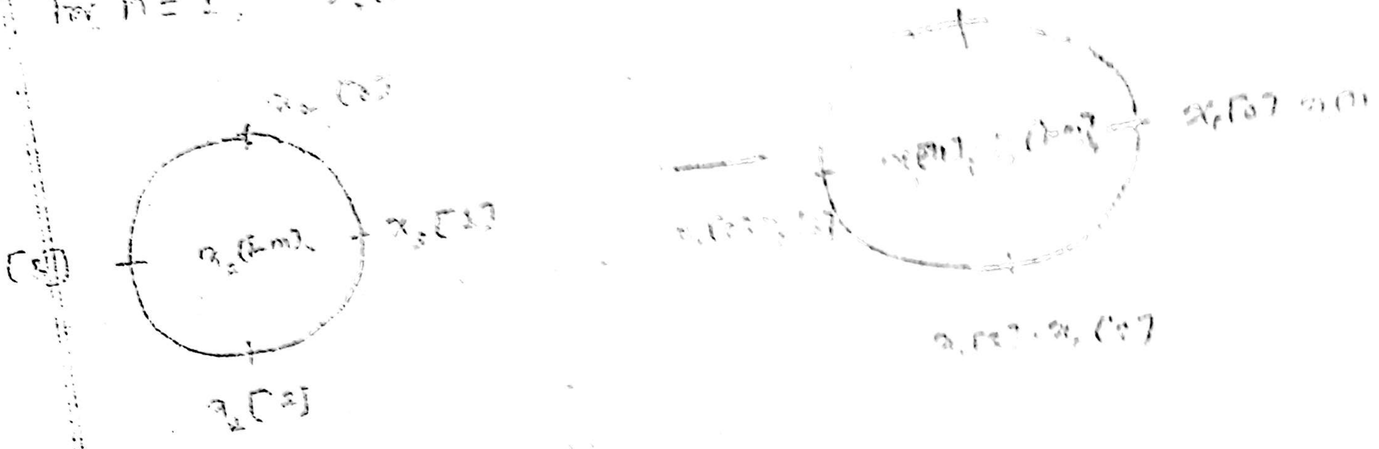
For $n = 0$;

$$x_3[0] = \sum_{m=0}^{3} x_1[m]\, x_2[-m]_4$$



$$x_3[0] = 2 + 1 + \dots + 9$$
$$= 16.$$

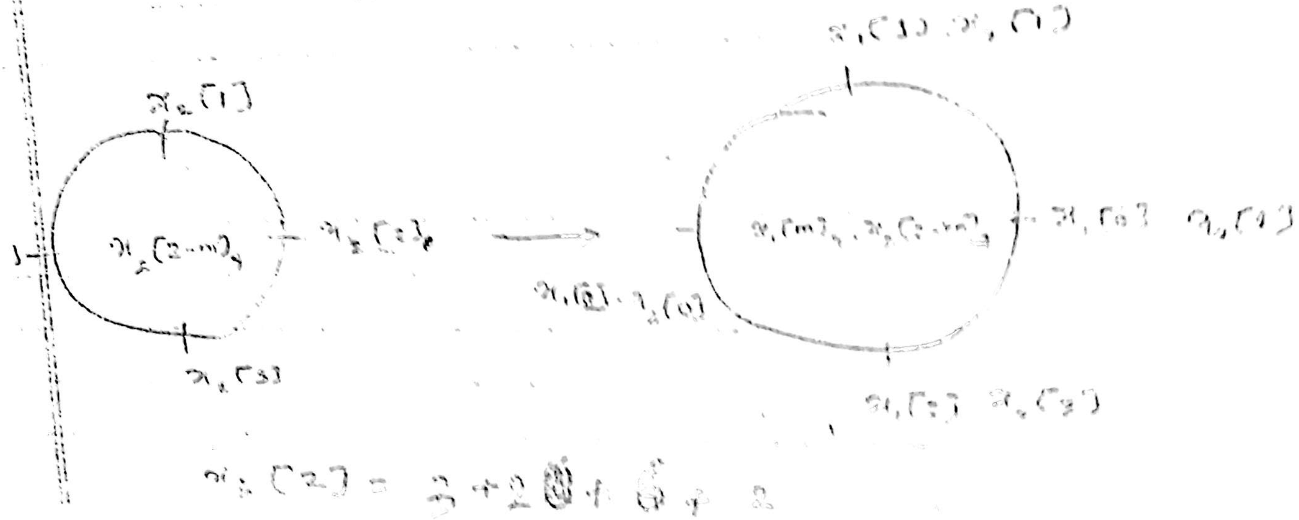
For $n = 1$, $x_3[1] = \sum_{m=0}^{3} x_1[m] \cdot x_2[1-m]_4$



$\therefore x_3[1] = 1 + 4 + 1.5 + 12 = 18.5$

For $n = 2$;

$x_3[2] = \sum_{m=0}^{3} x_1[m] \cdot x_2[2-m]_4$

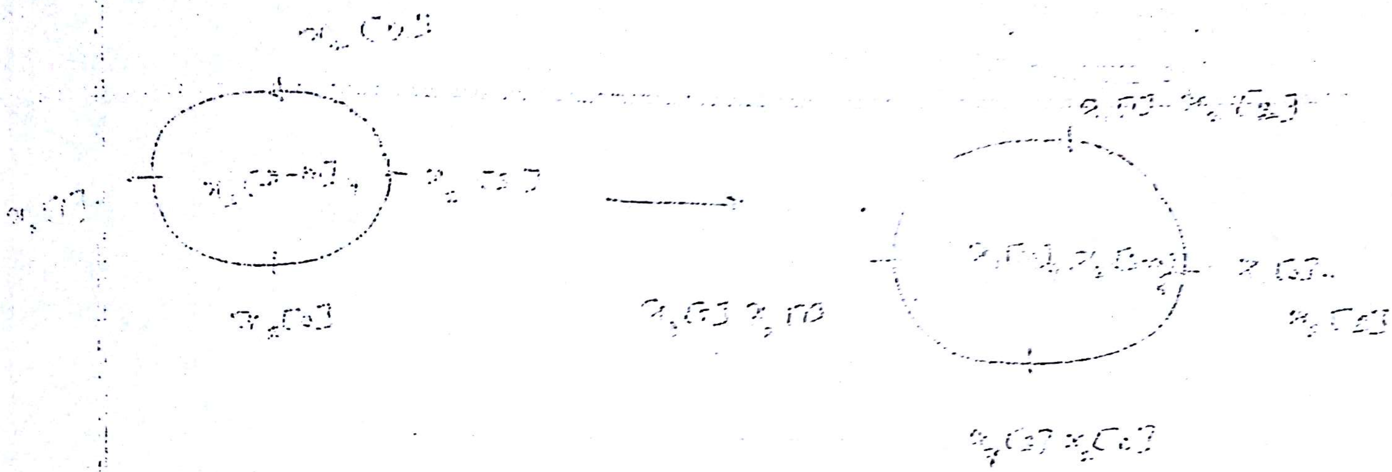


$x_3[2] = 3 + 2 + 6 + 2$

$= 13$

For $n = 3$;

$x_3[3] = \sum_{m=0}^{3} x_1[m] \cdot x_2[3-m]_4$



$$x_3[3] = 0.5 + 6 + 3 + 8$$

$$= 17.5$$

$$\therefore x_3[n] = \{16, 8.5, 13, 17.5\}$$ Ans

H.W.

Find $x_1[n]$ Ⓝ $x_2[n]$.

$x_1[n] = \{1, 4, 3, 9, 5\}$

$x_2[n] = \{-2, 3, 0.5, 2\}$

Fast fourier Transform (FFT):

Why FFT?
→ Computational Complexity of direct DFT computation:

DFT of $x[n]$.

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \quad \text{for } k = 0, 1, 2, ---, N-1$$

$$X(k) = x[0]e^{-j0} + x[1]e^{-jk\frac{2\pi}{N}} + ------ x[N-1]e^{-jk\frac{2\pi}{N}(N-1)}$$

i> No. of complex Multiplication:
For N-point DFT, it requires N^2 complex multiplication.

ii> No. of complex addition:
For N-point DFT, it requires $N(N-1)$ complex addition.

iii> $\delta\omega = \frac{2\pi}{N}$
$\delta\omega$ should be very small

iv> For smaller $\delta\omega$, N becomes larger and larger N makes high number of complex multiplication addition.

⇒ Fast fourier transform computes N-point DFT in such a efficient manner that there will be less number of complex multiplication & addition.

i.e. Less computational time

→ in the middle of 1960's

Divide and Conquer approach

Radix - 2 FFT Algorithm

↳ most widely used FFT algorithm

↳ can be used in two ways:

a) Decimation in time FFT (DITFFT)

b) Decimation in frequency FFT (DIFFFT)

a) Decimation in time FFT (DITFFT):

Let $x[n]$ be the finite length sequence having N-samples and it can be represented by odd and even values of 'n' as below:

even → $f_1[n] = x[2n]$, for $n = 0, 1, \ldots, \frac{N}{2} - 1$

odd → $f_2[n] = x[2n+1]$, for $n = 0, 1, 2, \ldots, \frac{N}{2} - 1$

$f_1[n]$ & $f_2[n]$ are obtained by decimating the sequence $x[n]$ by a factor of 2. So, this algorithm is called Decimation in time Radix FFT algorithm.

DFT of $x[n]$:

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

where,

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_N^2 = e^{-j\frac{2\pi}{N} \cdot 2} = e^{-j\frac{2\pi}{N/2}} = W_{N/2}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x[2m] W_N^{2mk} + \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_N^{(2m+1)k}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} x[2m] W_N^{2mk} + W_N^{k} \sum_{m=0}^{\frac{N}{2}-1} x(2m+1) W_N^{2mk}$$

$$= \sum_{m=0}^{\frac{N}{2}-1} f_1[m] W_{\frac{N}{2}}^{mk} + W_N^{k} \sum_{m=0}^{\frac{N}{2}-1} f_2(m) W_{\frac{N}{2}}^{mk}$$

$$X(k) = F_1(k) + W_N^{k} F_2(k) \quad \text{for } k = 0, 1, \ldots, \frac{N}{2}-1$$

where, $F_1(k)$ & $F_2(k)$ are $\frac{N}{2}$ point DFT of the sequence $f_1[m]$ & $f_2[m]$ respectively.

Since, DFT values are periodic in nature;

$$F_1(k + \tfrac{N}{2}) = F_1(k)$$

$$F_2(k + \tfrac{N}{2}) = F_2(k)$$

Now;

$$X(k + \tfrac{N}{2}) = F_1(k + \tfrac{N}{2}) + W_N^{k+\frac{N}{2}} F_2(k + \tfrac{N}{2})$$

$$= F_1(k) + W_N^{k} \cdot W_N^{N/2} F_2(k)$$

$$\because W_N^{N/2} = \left(e^{-j\frac{2\pi}{N}}\right)^{N/2}$$

$$= e^{-j\pi}$$

$$= -1$$

$$X(k + \tfrac{N}{2}) = F_1(k) - W_N^{k} F_2(k)$$

No. of complex multiplication at this stage:

$$= \left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + \frac{N}{2}$$

$$= \frac{N^2}{2} + \frac{N}{2}$$

Decimating the sequence and computing the N-point DFT, reduces the no. of multiplication from N^2 to $\left(\frac{N^2}{2} + \frac{N}{2}\right)$.

Similarly again processing for $f_1[n]$ & $f_2[n]$ we get; four (4) $N/4$-point DFT.

$f_1[n] \rightarrow v_{11}[n]$ - even, $v_{12}[n]$ - odd

$f_2[n] \rightarrow v_{21}[n]$ - even, $v_{22}[n]$ - odd

$$\left.\begin{aligned}
F_1(k) &= V_{11}(k) + W_{N/2}^{k} V_{12}(k) \\
F_2(k) &= V_{21}(k) + W_{N/2}^{k} V_{22}(k) \\
F_1\left(k + \frac{N}{4}\right) &= V_{11}(k) - W_{N/2}^{k} V_{12}(k) \\
F_2\left(k + \frac{N}{4}\right) &= V_{21}(k) - W_{N/2}^{k} V_{22}(k)
\end{aligned}\right\} \text{ for } k = 0, 1, \ldots \frac{N}{4} - 1.$$

No. of multiplication for this stage $= 4\left(\frac{N}{4}\right)^2 + 4 \cdot \frac{N}{4}$

$$= \frac{N^2}{4} + N$$

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At this stage, no. of multiplication reduces for

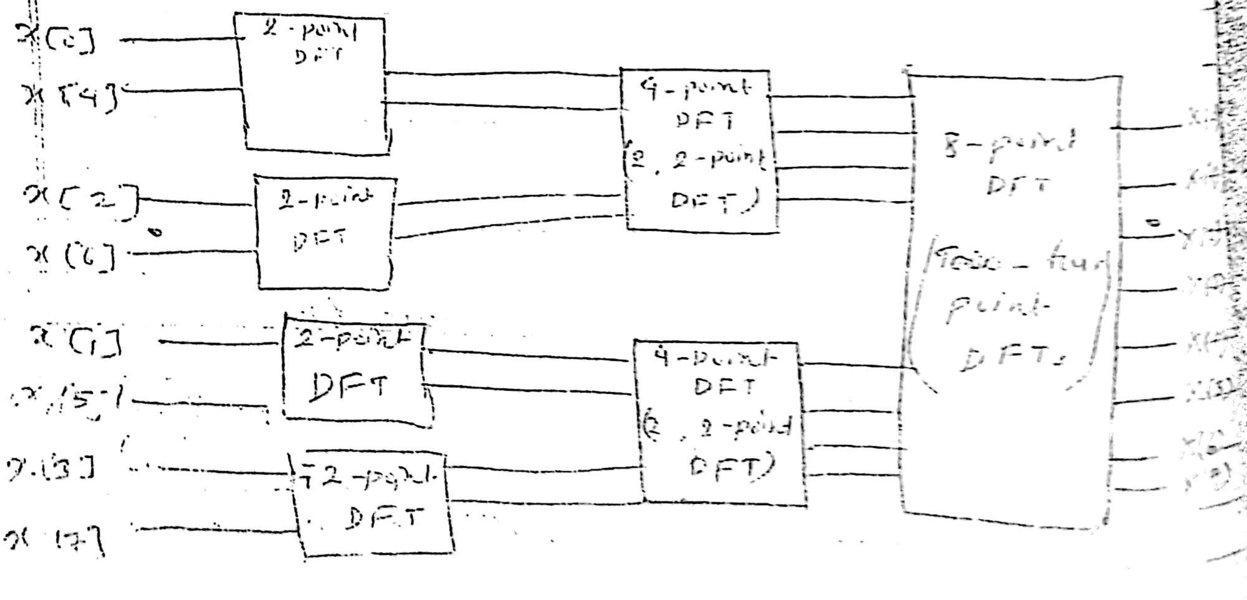
Similarly, this process continues till two-point DFTs.
And, overall no. of multiplication required in computation of N-point DFT is:

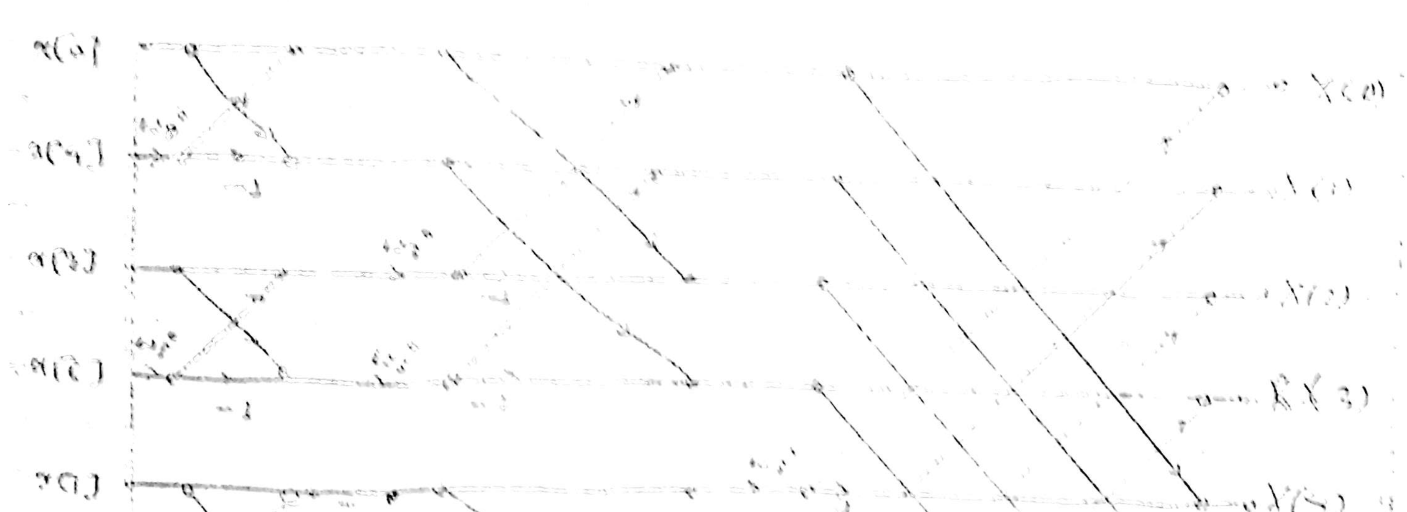
$$\Rightarrow \boxed{\frac{N}{2}\log_2 N}$$

Efficiency of FFT over direct DFT computation

N	No. of multiplication: For Direct DFT computation N^2	No. of multiplication for FFT method $\frac{N}{2}\log_2 N$	Efficient
4	16	4	4 : 1
8	64	12	5.33
16	256	32	8
32	1024	80	12.8
64	4096	192	21.33

For N = 8 :





$$\omega_8^2 = \left(e^{-j\frac{\pi}{4}}\right)^2 = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$

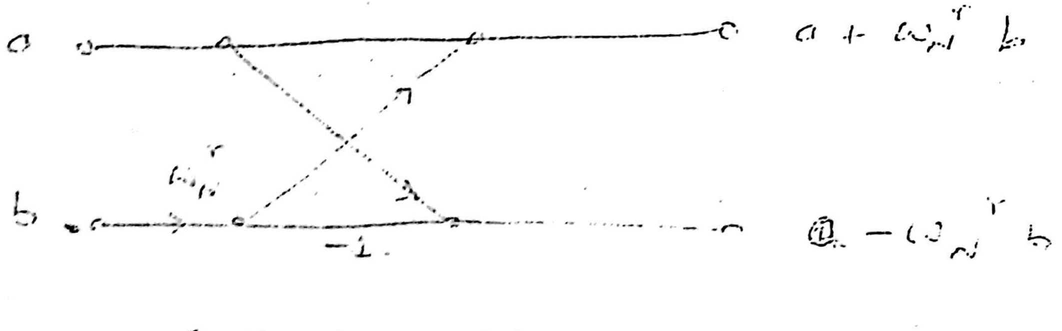
$$= -j$$

$$\omega_8^3 = \left(e^{-j\pi/4}\right)^3 = e^{-j\frac{3\pi}{4}} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$X(0) = x[0] + x[4]\omega_8^0 + \omega_8^0 x[2] + x[6]\omega_8^0\omega_8^0 + x[1]\omega_8^0 + x[5]\omega_8^0\omega_8^0 + x[3]\omega_8^0\omega_8^0 + x[7]\omega_8^0$$

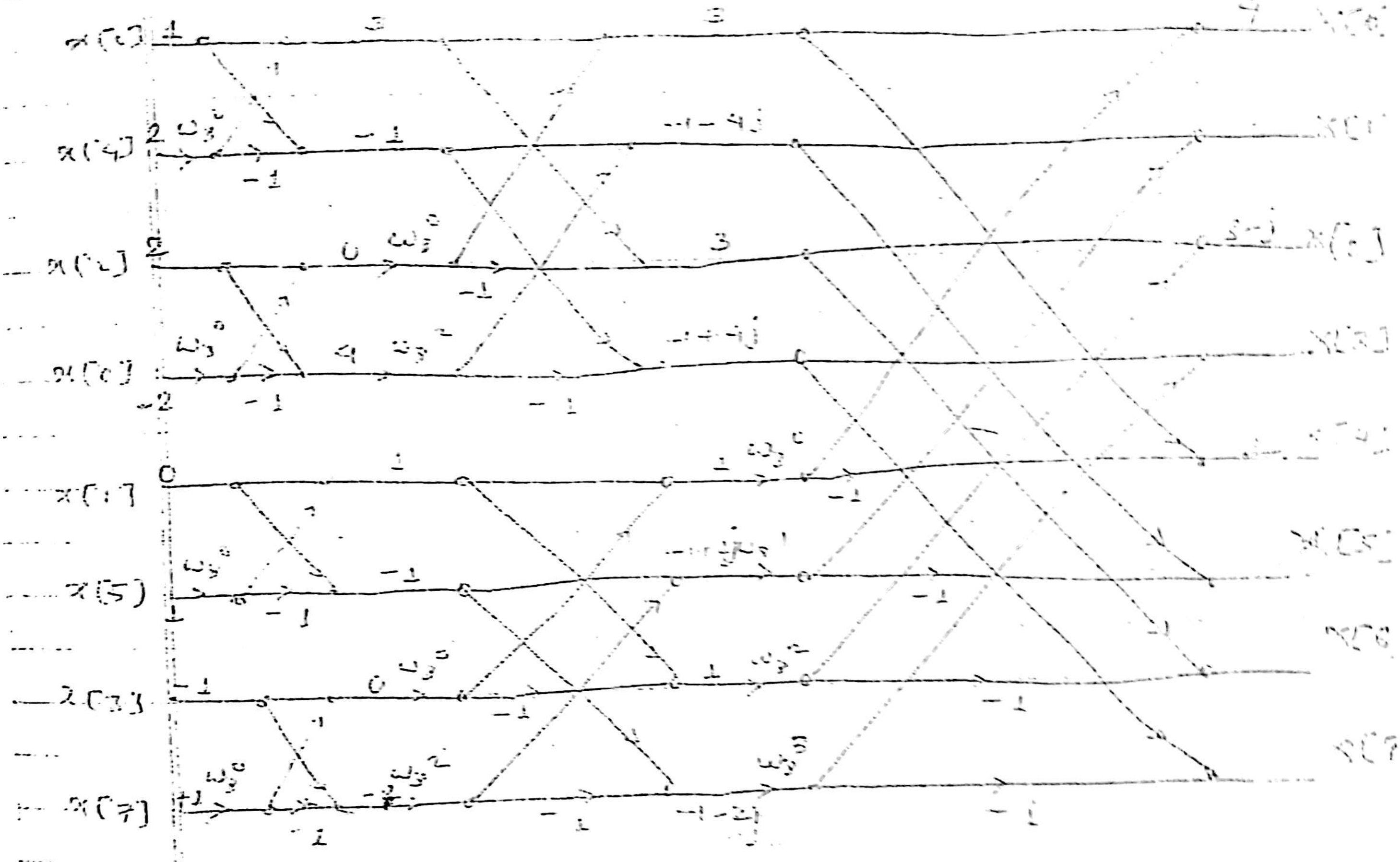
$$X(5) = x[0] + x[4]\omega_8^0(-1) + x[2]\omega_8^2 + x[6]\omega_8^0(-1) + x[1]\omega_8^1(-1) + x[5]\omega_8^1(-1)(-1)\omega_8^0 + x[3]\omega_8^2\omega_8^1(-1) + x[7]\omega_8^0(-1)\omega_8^2(-1)\omega_8^1$$



Butterfly Algorithm.

Eg:- Find the 8-point DFT of sequence [illegible] {1, 0, 2, -1, [illegible], [illegible], -2, 1} using [illegible] algorithm

Sol: $N = 8$, DIT FFT



$$\omega_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} j$$

$$\omega_8^0 = 1$$

$$\omega_8^2 = e^{-j\frac{\pi}{2}} = -j$$

$$\omega_8^3 = e^{-j\frac{3\pi}{4}} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

Now, $\cancel{x}(4) = 4$

$X(1) = (-1 - j4) + (-1 + 2j)\,W_8$

$= -1 - j4 + (-1+2j)\left(\frac{1}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right)$

$= -1 - j4 - \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + \sqrt{2}\,j + \sqrt{2}$

$= -1 - \frac{1}{\sqrt{2}} + \sqrt{2} - j\left(4 - \frac{1}{\sqrt{2}} - \sqrt{2}\right)$

$= \frac{-\sqrt{2} - 1 + 2}{\sqrt{2}} - j\left(\frac{4\sqrt{2} - 1 - 2}{\sqrt{2}}\right)$

$= \left(-1 + \frac{1}{\sqrt{2}}\right) - j\left(4 - 3\frac{1}{\sqrt{2}}\right)$

$X(2) = 3 + 1\,(W_8^{2}) = 3 - j$

$X(3) = -1 + 4j + (-1 - j2)\,W_8^{3}$

$= -1 + 4j + (-1 - j2)\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$= -1 + 4j + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + 2j\frac{1}{\sqrt{2}} + 2$

$= \left(-1 + \frac{1}{\sqrt{2}}\right) + j\left(4 + 3\frac{1}{\sqrt{2}}\right)$

$X(4) = 1 \times W_8^{0} * (-1) + 3$

$= -1 + 3$

$= 2$

$X(5) = (-1 + 2j)\,W_8^{1} * (-1) - 1 - j4$

$= (-1 + 2j)\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)(-1) - 1 - j4$

$= -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + j2\frac{1}{\sqrt{2}} - j^2 2\frac{1}{\sqrt{2}}) - 1 - j4$

$X(1)$ & $X(7)$, $X(3)$ & $X(5)$ } complex conjugate.

$$= \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} - j2\frac{1}{\sqrt{2}} - j2\frac{1}{\sqrt{2}} - 1 - j4$$

$$= \left(-1 - \frac{1}{\sqrt{2}}\right) - j\left(4 + 3\frac{1}{\sqrt{2}}\right)$$

$$X(6) = 3 + j$$

$$X(7) = -1 + 4j + (-1 - j2) * W_8^3 * (-1)$$

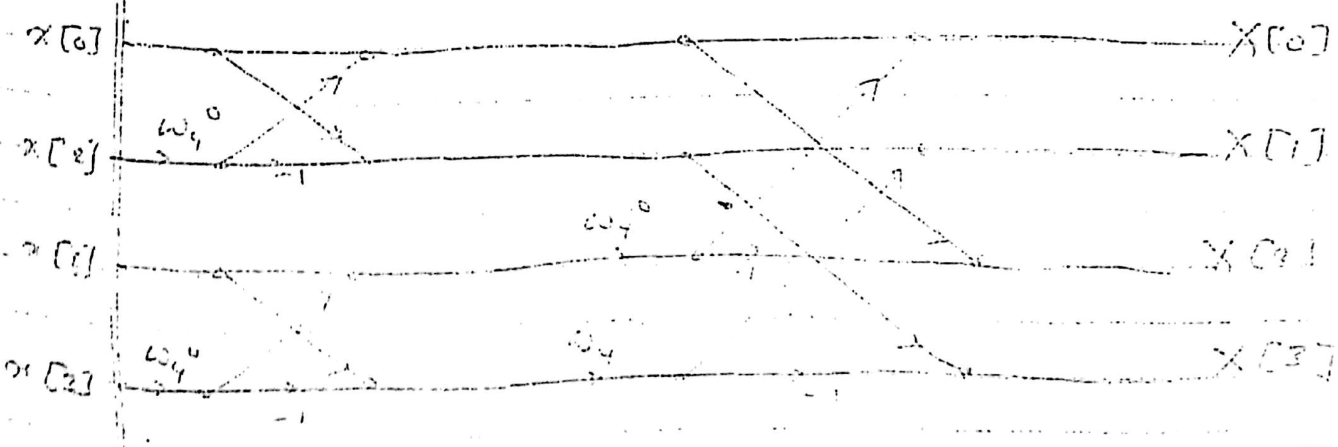
$$= -1 + j4 + (1 + j2) * \left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$$

$$= -1 + j4 - \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} - j2\frac{1}{\sqrt{2}} + j2\frac{1}{\sqrt{2}}$$

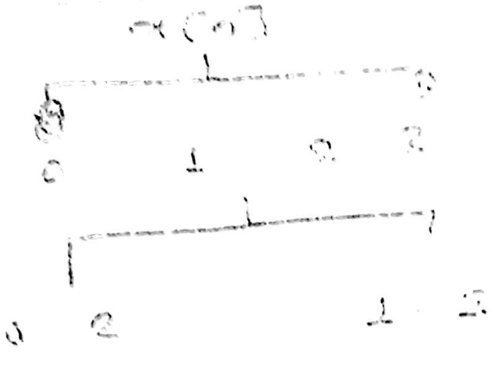
$$= \left(-1 + \frac{1}{\sqrt{2}}\right) + j\left(4 - 3\frac{1}{\sqrt{2}}\right)$$

2.10. Calculate 8-point DFT of
$x[n] = \{1, 2, 3, 4, -1, 2, 1\}$

N = 4 - DIT FFT:

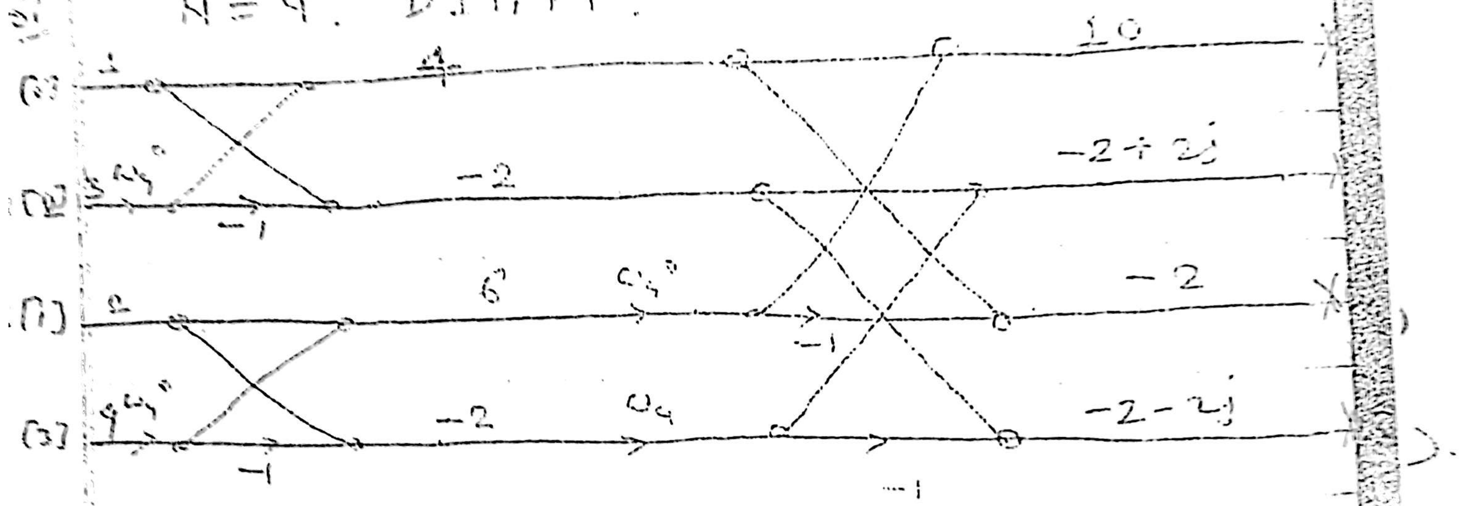


$$W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$
$$= -j$$



7. Compute 4-point DFT of $x[n] = \{1, 2, 3,$
using DIT FFT.

$N = 4$. DIT FFT :



$\therefore X(0) = 10$

$\therefore X(1) = -2 + 2j$

$\therefore X(2) = -2$

$\therefore X(3) = -2 - 2j$

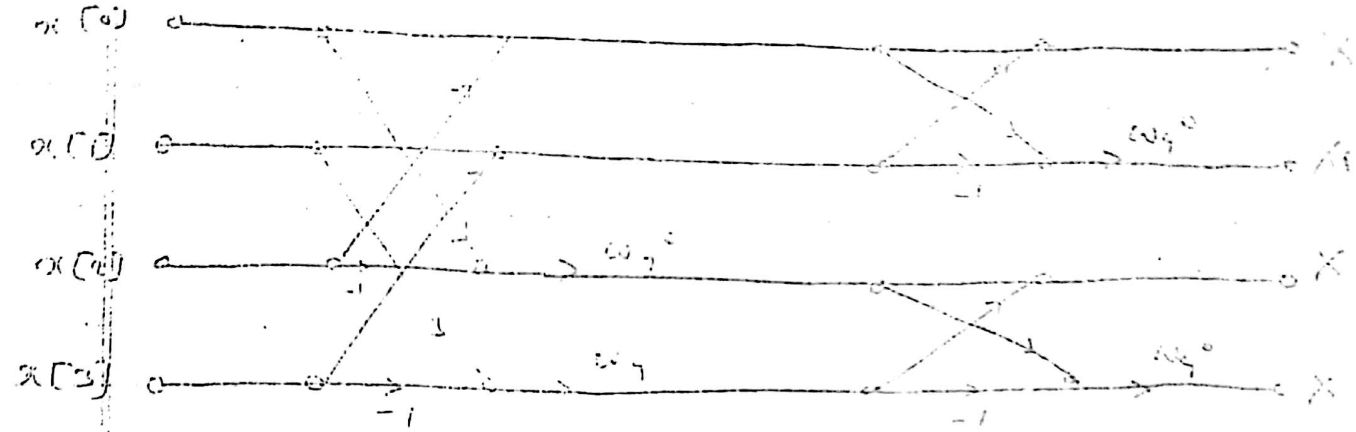
Answer

$\therefore X(K) = \{10, -2 + 2j, -2, -2 - 2j\}$

Answer

Decimation in frequency FFT (DIFFFT)

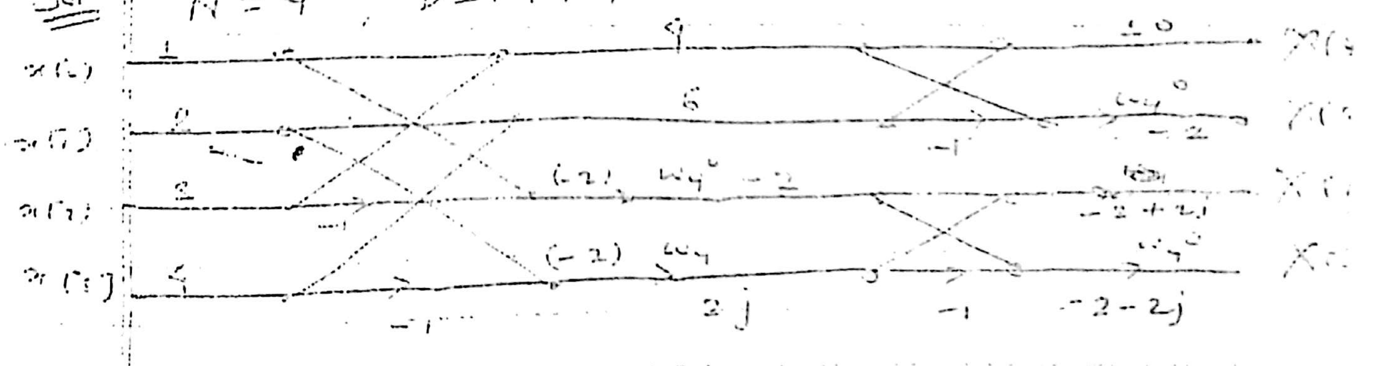
$N = 4$, DIF-FFT



$$\omega_4 = e^{-j\frac{2\pi}{4}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

Q. Compute 4-point DFT of $x[n] = \{1, 2, 3, 4\}$ using DIFFFT.

Sol. $N = 4$, DIFFFT



$$X(k) = \{10, -2+2j, -2, -2-2j\}$$

Ans

Q. Compute 8-point DFT of sequence $x[n] = \{1, 0, 2, -1, 2, 1, -2, 1\}$ using DIF FFT algorithm.

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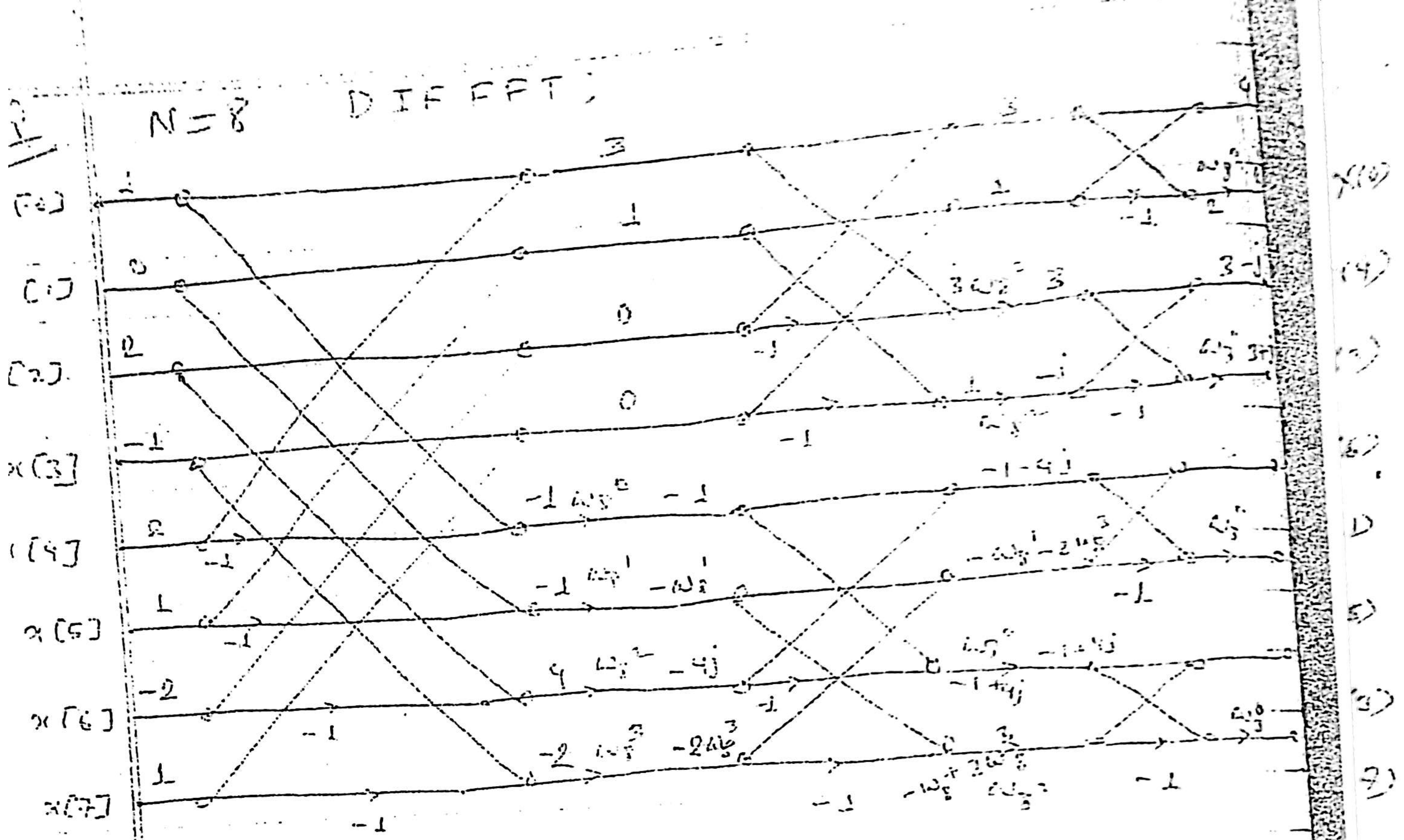
N = 8   DIF FFT ;



$$W_8 = e^{-j\frac{2\pi}{8}} = e^{-j\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - j\,\frac{1}{\sqrt{2}}$$

$$W_8^0 = 1$$

$$W_8^2 = e^{-j\frac{2\pi}{4}} = -j$$

$$W_8^3 = -\frac{1}{\sqrt{2}} - j\,\frac{1}{\sqrt{2}}$$

$X(0) = 4$, $X(4) = 2$

$$X(1) = -1 - 4j - W_8^1 - 2W_8^3$$

$$= -1 - 4j - \frac{1}{\sqrt{2}} + j\,\frac{1}{\sqrt{2}} + 2\,\frac{1}{\sqrt{2}} + 2j\,\frac{1}{\sqrt{2}}$$

$$= \left(-1 + \frac{1}{\sqrt{2}}\right) - j\left(4 - 3\,\frac{1}{\sqrt{2}}\right)$$

$X(2) = 3 - j$

$X(6) = 3 + j$

$$X(3) = -1 + 4j - \left(-W_8 + 2W_8^3\right)(-j)$$

$$= -1 + 4j - jW_8 - 2jW_8^3$$

$$= -1 + 4j + \frac{1}{\sqrt{2}}j + \frac{1}{\sqrt{2}} + 2\frac{1}{\sqrt{2}}j - 2\frac{1}{\sqrt{2}}$$

$$= \left(-1 - \frac{1}{\sqrt{2}}\right) + j\left(4 + 3\frac{1}{\sqrt{2}}\right)$$

$$X(5) = -1 - 4j + W_8 + 2W_8^3$$

$$= -1 - 4j + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j - 2\frac{1}{\sqrt{2}} + 2\frac{1}{\sqrt{2}}j$$

$$= \left(-1 - \frac{1}{\sqrt{2}}\right) - j\left(4 + 3\frac{1}{\sqrt{2}}\right)$$

$$X(7) = -1 + 4j + \left(W_8 - W_8^3\right)(-j)$$

$$= -1 + 4j - \frac{1}{\sqrt{2}}j - \frac{1}{\sqrt{2}} - \frac{2\cdot 1}{\sqrt{2}}j + 2\frac{1}{\sqrt{2}}$$

$$= \left(-1 + \frac{1}{\sqrt{2}}\right) + j\left(4 - 3\frac{1}{\sqrt{2}}\right)$$

Ans

H.W. (7) Find the 8-point DFT of sequence $x(n) = \{1, 1, 1, 0, 0, -1, -1\}$ using DIF FFT algorithm.

5.

[illegible] of Discrete time System

Structure of Discrete time System.

Most of the discrete time systems are better represented in terms of structure which may be block diagram representation or signal flow graph representation. The structure consists of inter-connection of adder, multiplier and memory element (delay elements).

The general form of constant coefficient difference equation for discrete time LTI system can be represented as:

$$\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

Taking z-transform:

$$\sum_{k=0}^{N} a_k\, z^{-k}\, Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

or, $$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Transfer function : $H(z) = \frac{Y(z)}{X(z)}$

$$= \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

If $N=0$, $H(z)$ only contains [illegible] zeros, [illegible] FIR

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \qquad [\because a_0 = 1]$$

$$= H_1(z)\, H_2(z)$$

where, $H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$\frac{Y(z)}{X(z)} = H_1(z) \cdot H_2(z)$$

or, $Y(z) = H_1(z) H_2(z) \cdot X(z)$

Direct form I

If $V(z) = H_1(z) X(z)$

then, $Y(z) = H_2(z) V(z)$

$$V(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

Taking inverse z transform

$$v[n] = \sum_{k=0}^{M} b_k x[n-k]$$

Also, $Y(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-1}} V(z)$

or, $Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = V(z)$

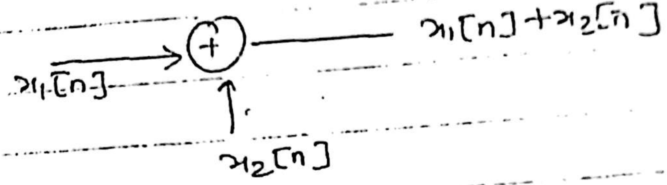
Taking inverse z-transform, we get

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = v[n]$$

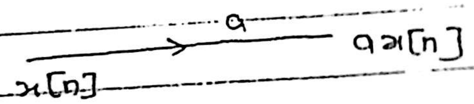
$$\therefore y[n] = v[n] - \sum_{k=1}^{N} (a_k) y[n-k]$$

For block diagram representation.

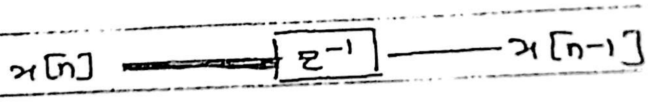
Adder



Multiplier



delay element (memory)



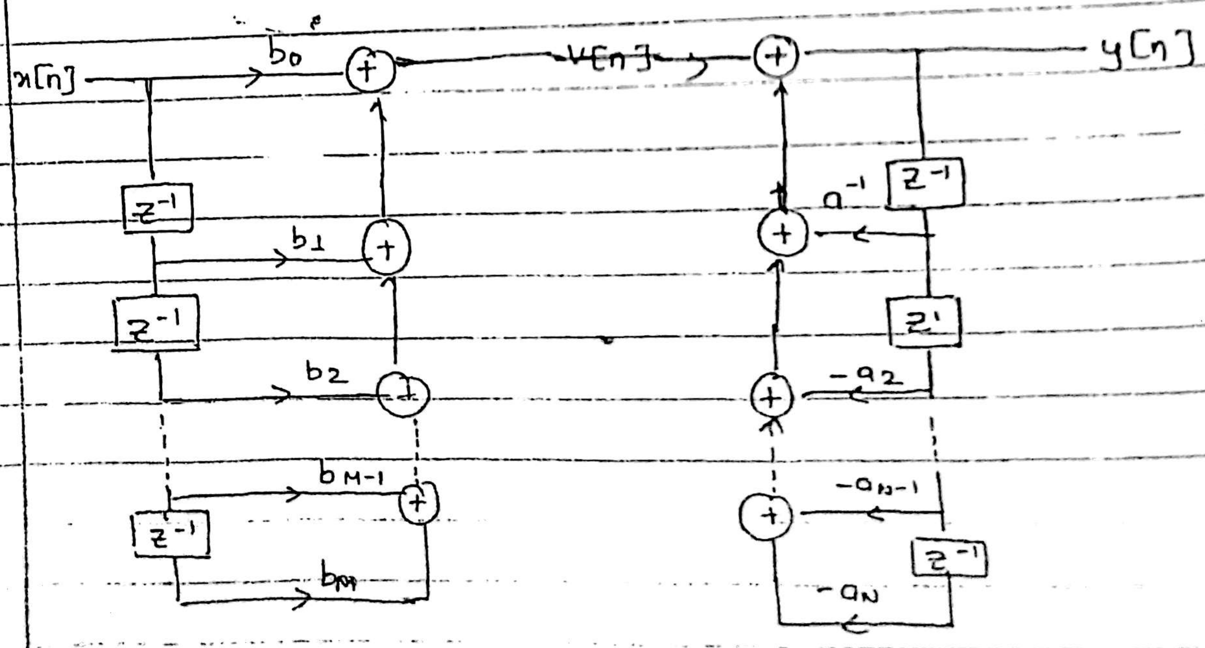


Fig:- Direct from I.

Direct ~~form~~ form II.

If $W(z) = H_2(z)X(z)$

then

$$Y(z) = H_1(z)W(z)$$

$$W(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} X(z)$$

$$W(z) + \sum_{k=1}^{N} a_k z^{-k} W(z) = X(z)$$

Taking inverse z-transform,

$$\therefore \quad w[n] + \sum_{k=1}^{N} a_k w[n-k] = x[n]$$

Also $Y(z) = \sum_{k=0}^{M} b_k z^{-k} W(z)$

Taking inverse z-transform we get.

$$y[n] = \sum_{k=0}^{M} b_k w[n-k]$$

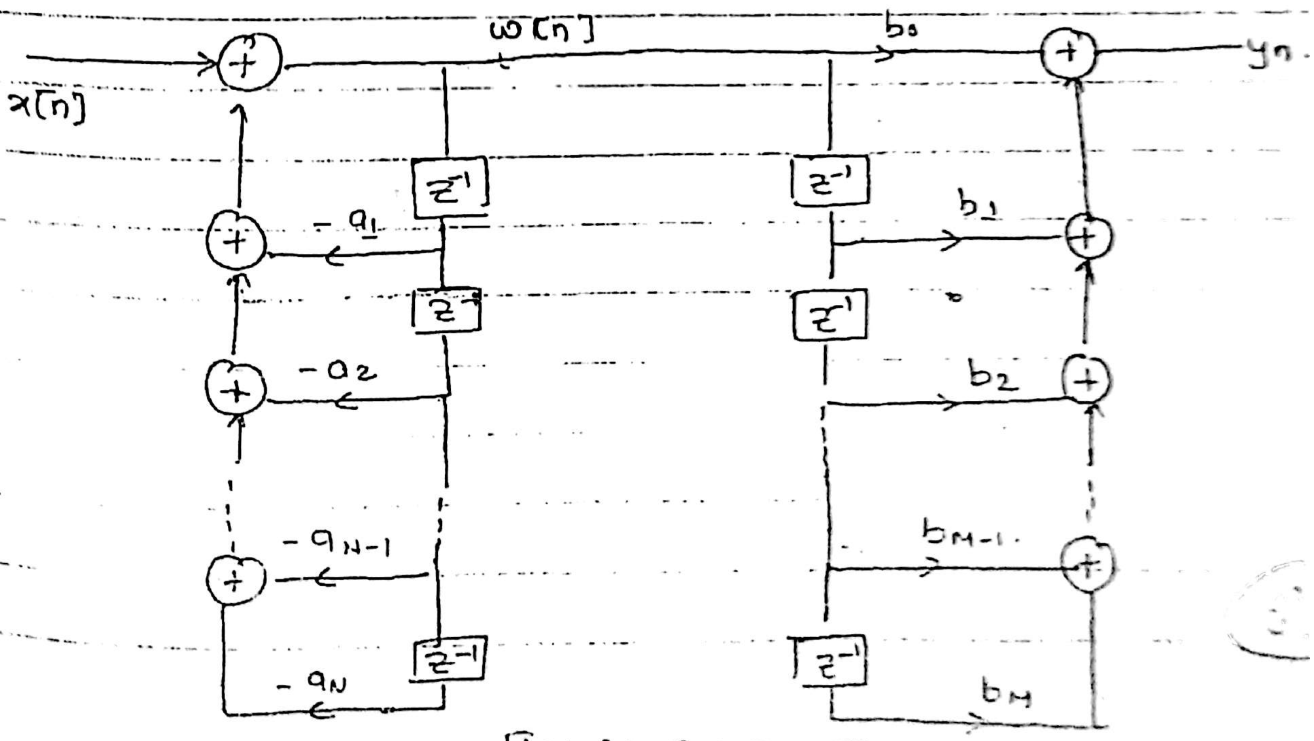


Fig:- Direct Form II.

Direct Form II (Cannonic Form)

A structure is said to be cannonic form if the number of delay elements in block diagram representation or signal flow graph representation is equal to the order of difference equation.

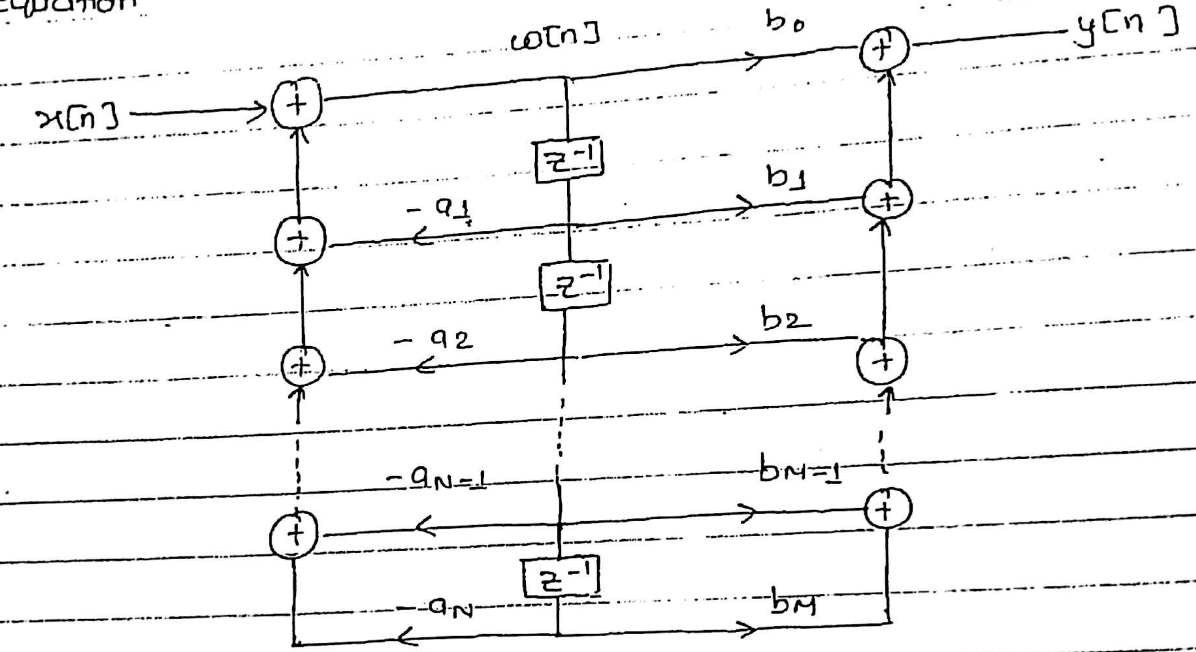


Fig:- Direct Form II.

Example:- $$H(z) = \frac{0.3 + 0.5z^{-1} - 0.9z^{-2}}{1 - 0.75z^{-1} + 0.25z^{-2}}$$

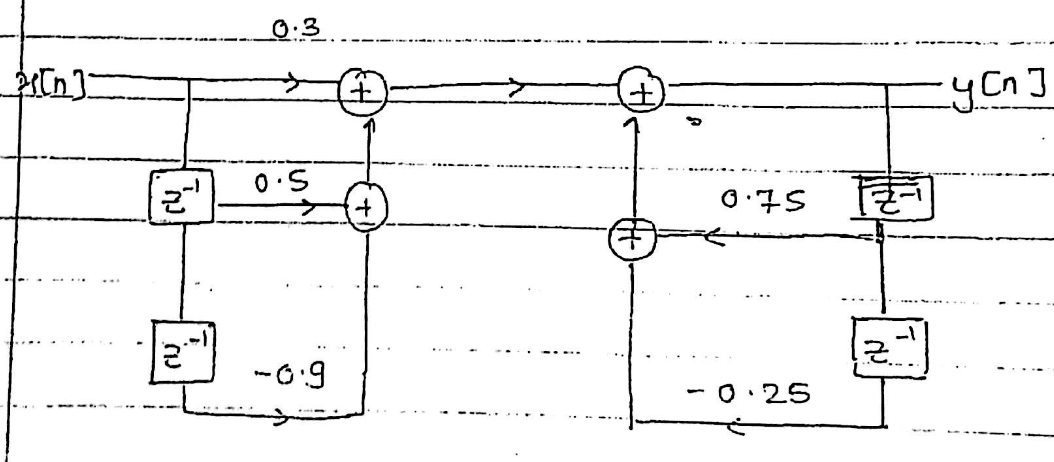
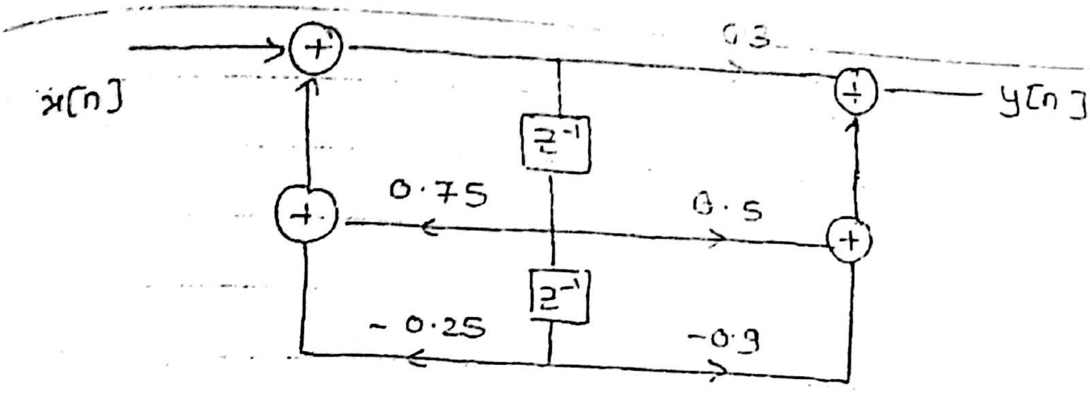


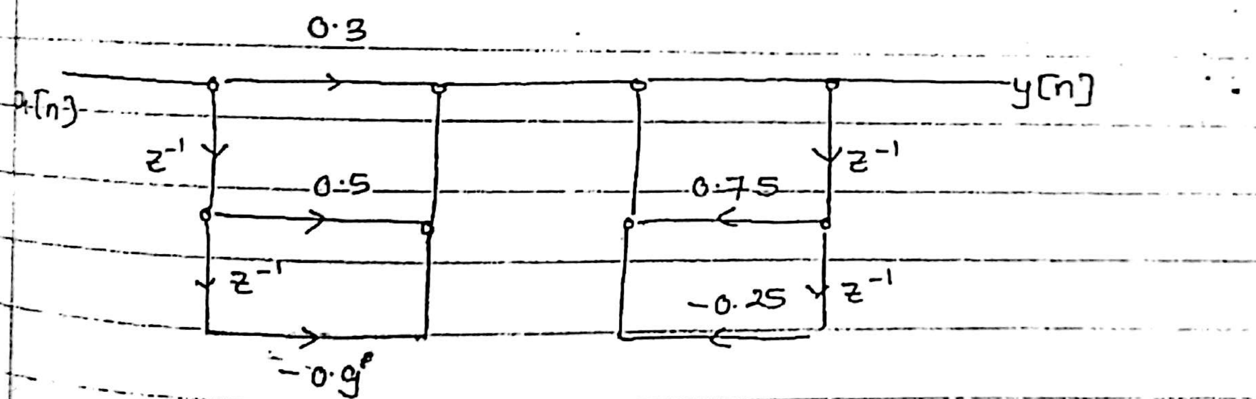
Fig:- Direct Form II



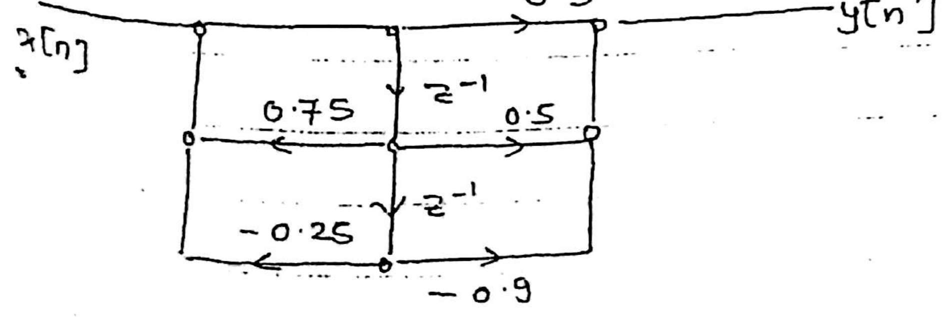
Direct Form II.

signal flow graph representation =>

↳ same as block diagram representation only few notational difference.



Direct Form (I) → signal flow graph.



Direct Form (II) → signal flow graph.

$H(z) = \dfrac{2+0.5z^{-1}-0.3z^{-2}}{1+0.55z^{-1}-0.64z^{-2}+0.8z^{-3}}$

$H(z) = \dfrac{0.8+0.4z^{-1}+0.3z^{-2}}{2-0.9z^{-1}+0.66z^{-2}} = \dfrac{0.4+0.2z^{-1}+0.15z^{-2}}{1-0.45z^{-1}+0.33z^{-2}}$

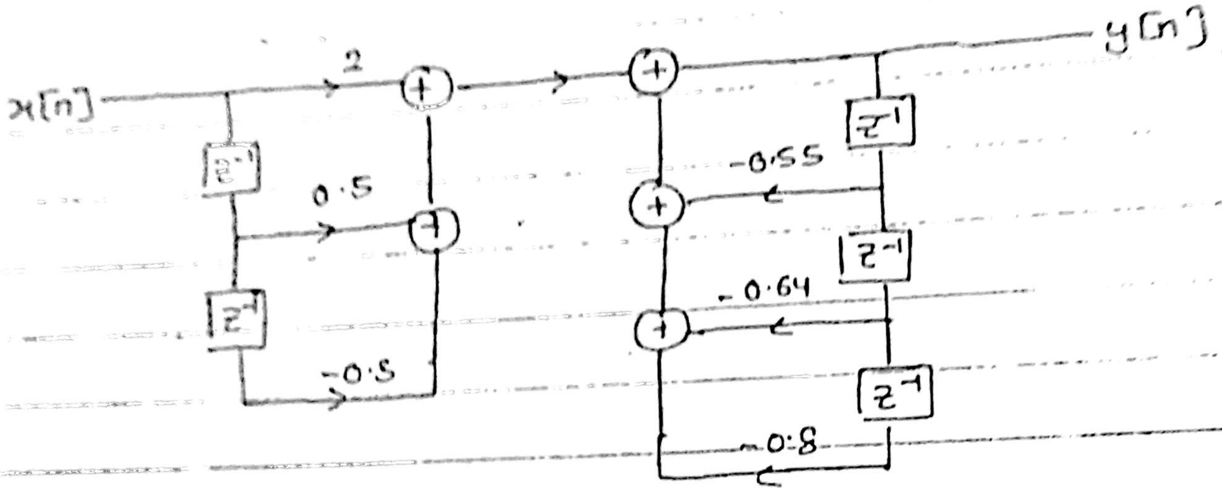


Fig:- Direct Form I

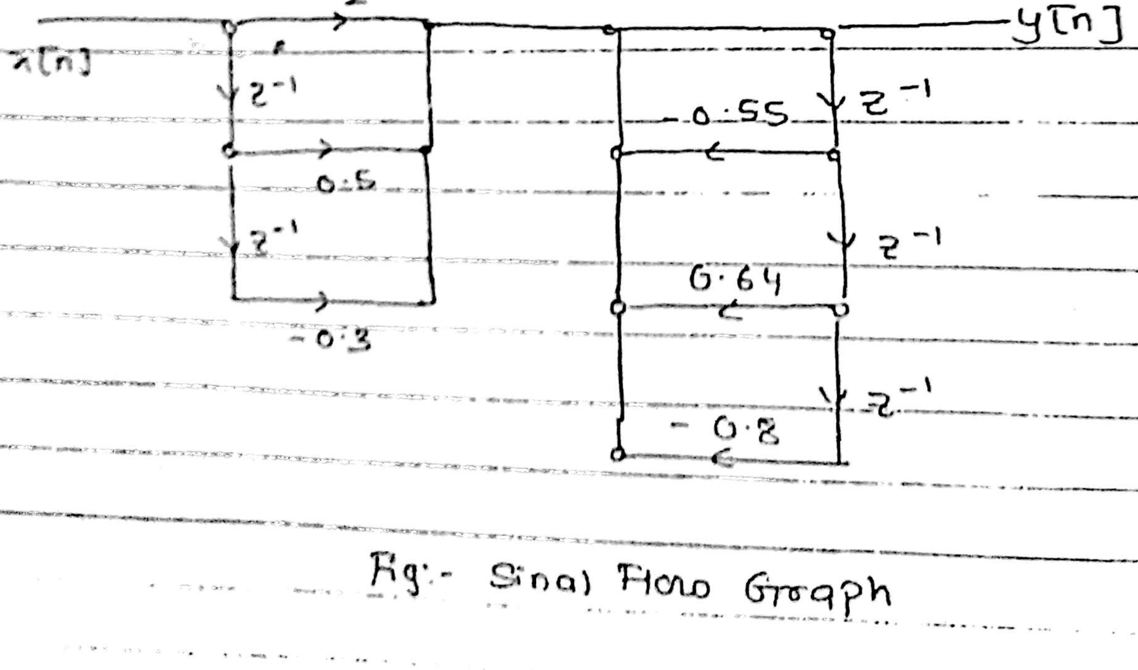


Fig:- Sinal Flow Graph

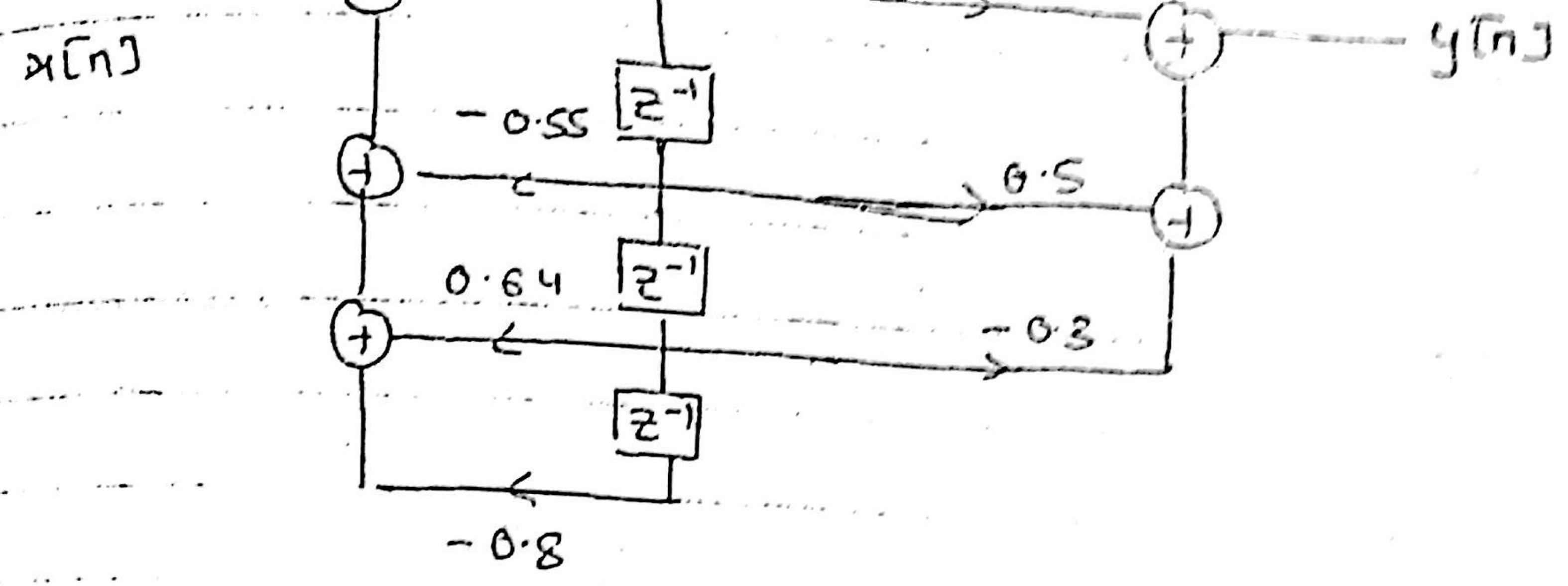


Fig:- Direct Form (II).

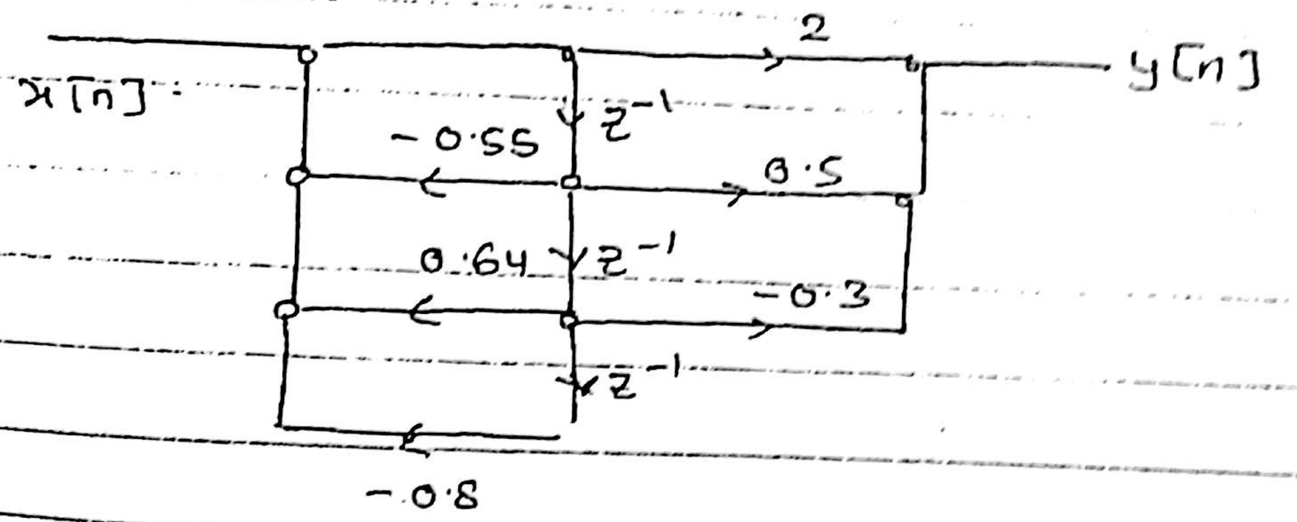


Fig:- Signal Flow Graph.

#

$$\frac{b_0}{a_0} \frac{\prod_{k=1}^{M_1}(1-c_k z^{-1}) \prod_{k=1}^{M_2}(1-d_k z^{-1})(1-d_k^* z^{-1})}{\prod_{k=1}^{N_1}(1-g_k z^{-1}) \prod_{k=1}^{N_2}(1-h_k z^{-1})(1-h_k^* z^{-1})}$$

where

$c_k \Rightarrow$ real zero of first order section

$g_k \Rightarrow$ real pole of first order section

d_k and d_k^* $\Rightarrow$ complex conjugate zeroes of 2nd order section

h_k and h_k^* $\Rightarrow$ " " poles " " "

$$N = N_1 + 2N_2$$

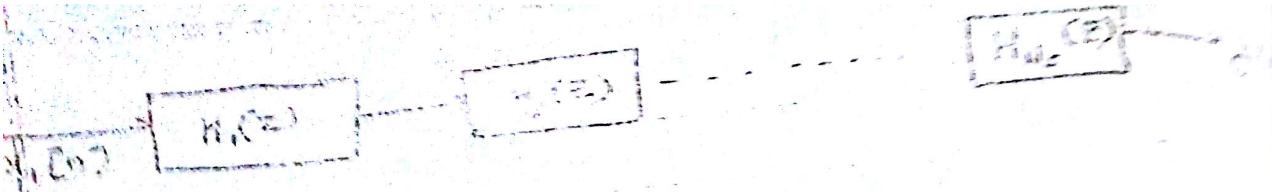
$$M = M_1 + 2M_2$$

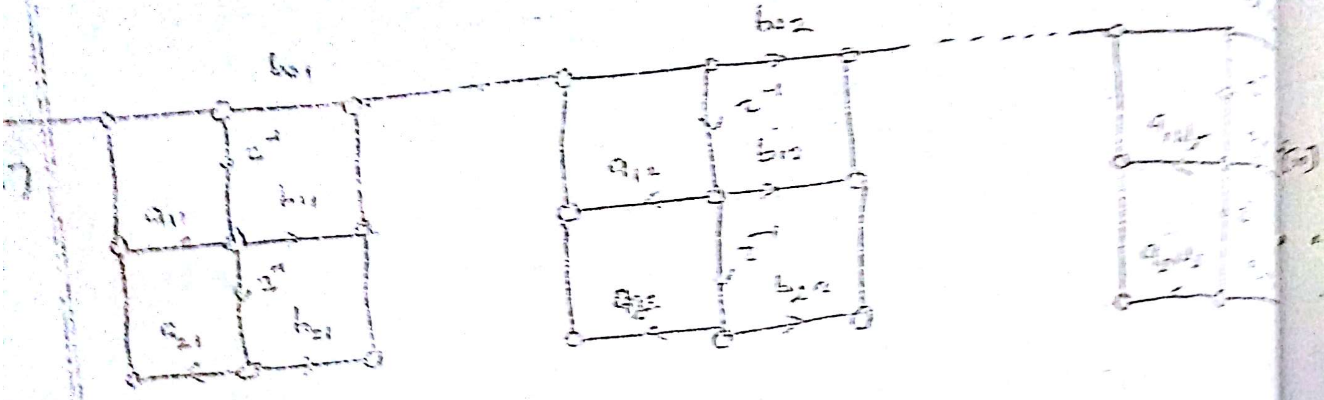
Combining real poles and zeroes with complex conjugate poles and zeroes (a pair of) in order to make 2nd order section provides better structure of a system.

$$H(z) = \prod_{k=1}^{N_s} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

$$= \prod_{k=1}^{N_s} H_k(z)$$

where $N_s \Rightarrow$ greatest integer in $\frac{N+1}{2}$





If there are N_s second order section there is $(N_s!)$ different pairings of pole[s] [and] zeroes in order to make 2^{nd} order section and $(N_s!)$ different ordering of second [order] sections or ~~the~~ a total of $(N_s!)^2$ [of] pairing & ordering of 2^{nd} order section

Overall transfer function of these pairing[s] [and] orderings is same but they are differ[ent in] interconnection.

Eg: Implement the given system in cascade [form] of 2^{nd} order section.

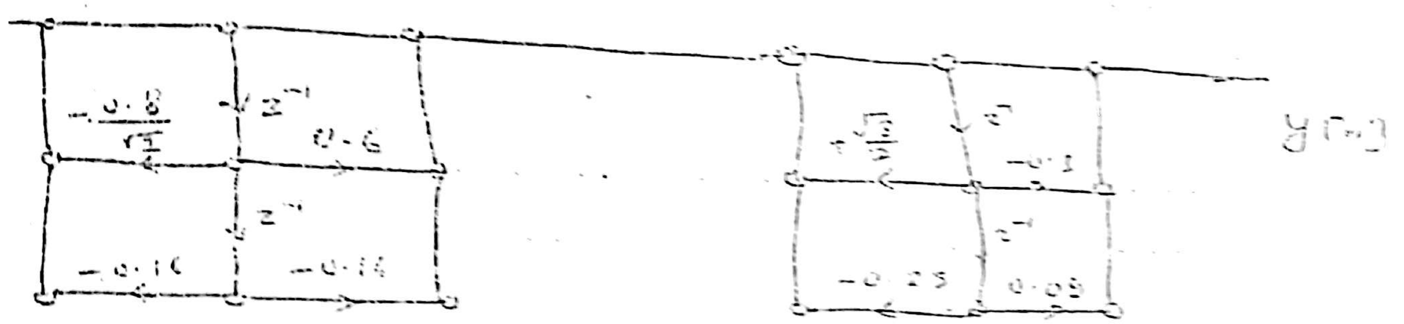
$$H(z) = \frac{(1-0.2z^{-1})(1+0.3z^{-1})(1-0.3e^{j\frac{\pi}{8}}\ldots)(1-0.3e^{\ldots})}{(1+0.4e^{j\pi/4}z^{-1})(1+0.4e^{-j\pi/4}z^{-1})(1-0.5e^{j\ldots})(1-0.5e^{\ldots})}$$

Part-2

$$H(z) = \frac{(1+0.6z^{-1}-0.16z^{-2})\cdots}{(1+2\times 0.4\cos\frac{\pi}{4}z^{-1}+(0.4)^2z^{-2})(1-2\times 0.5\cos\frac{\pi}{6}z^{-1}+0.5^2z^{-2})}$$

$$= \left(\frac{1+0.6z^{-1}-0.16z^{-2}}{1+\frac{0.8}{\sqrt{2}}z^{-1}+0.16z^{-2}}\right)\left(\frac{1-0.3z^{-1}+0.09z^{-2}}{1-\frac{\sqrt{3}}{2}z^{-1}+0.25z^{-2}}\right)$$

$$= H_1(z)\cdot H_2(z)$$



Parallel Form :-

A parallel form for general discrete time LTI system is given by :

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

as;

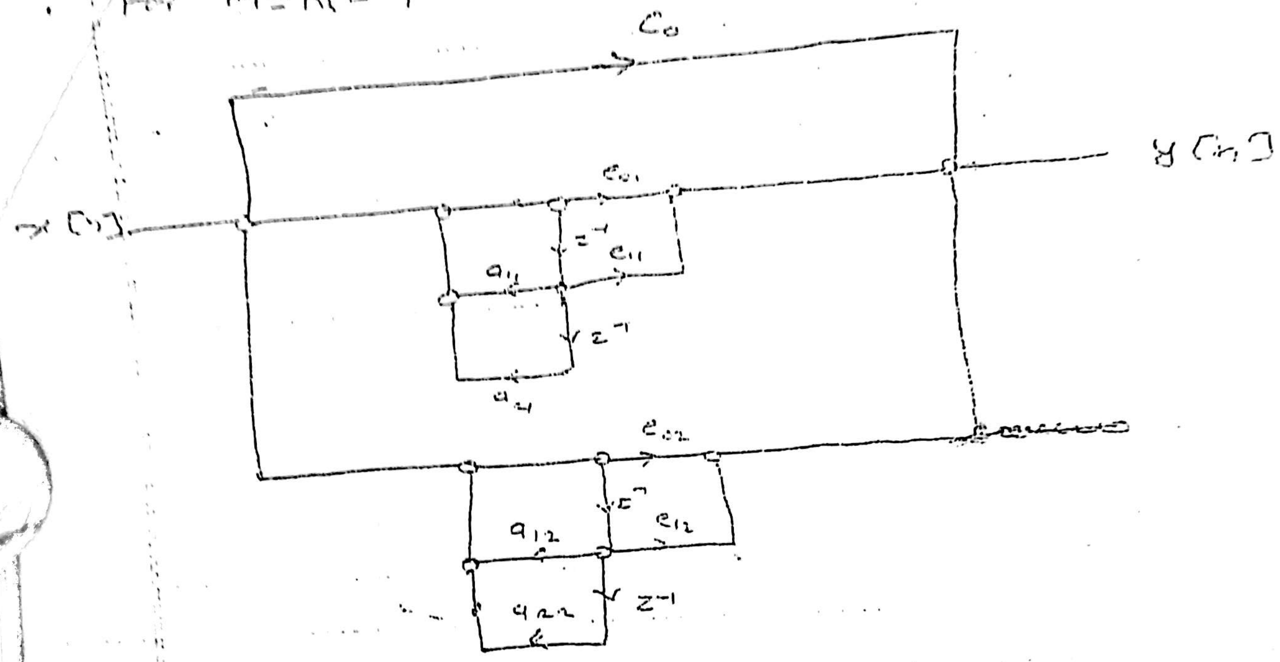
$$H(z) = \sum_{q=0}^{M-N} c_q z^{-q} + \sum_{k=1}^{N_1} \frac{A_k}{1-a_k z^{-1}} + \sum_{k=1}^{N_2} \frac{B_k \cdots}{(1-b_k z^{-1})\cdots}$$

For $M < N$, First term will not be ...

2nd and 3rd terms are combined to make 2nd order section having the form:

$$\frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

For M = N = 4



Eg: Realize the given system in parallel form:

$$H(z) = \frac{2 + 2z^{-1} + 3z^{-2}}{1 - 0.25z^{-1} + 0.75z^{-2}}$$

Sol:

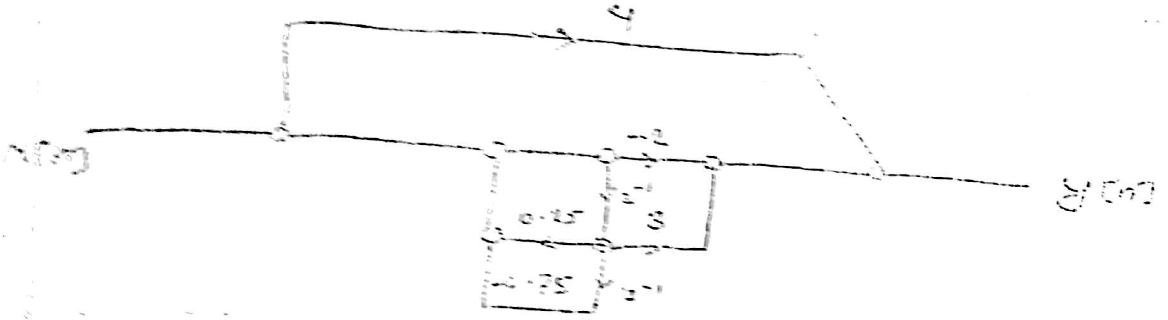
~~$1 - 0.25z^{-1} + 0.75z^{-2}) \; 1 + 2z^{-1} + 3z^{-2} (4$~~

$0.75z^{-2} + 0.25z^{-1} + 1 \;) \; 3z^{-2} + 2z^{-1} + 2 \; (4$

$3z^{-2} - z^{-1} + 4$

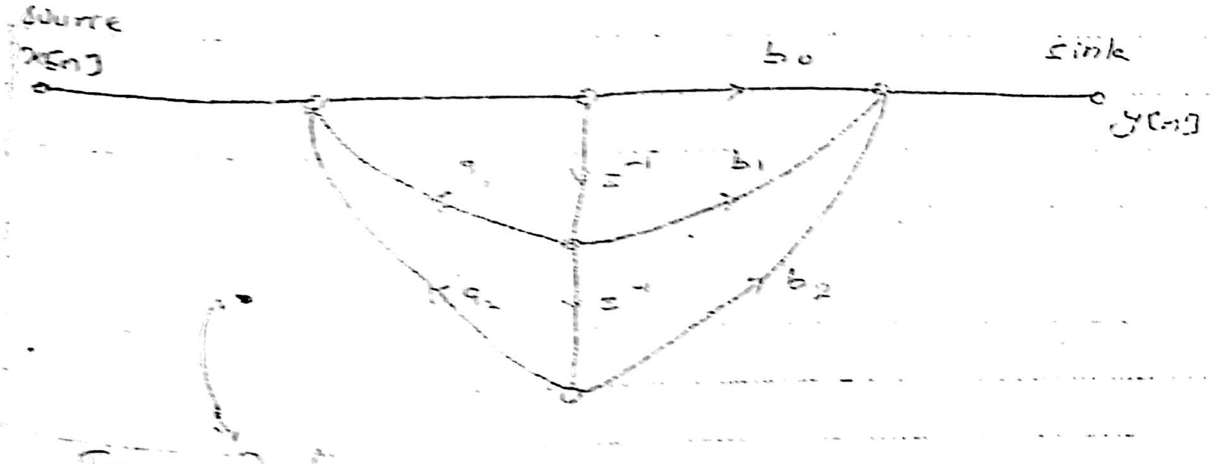
$3z^{-1} - 2$

$$\therefore H(z) = 4 + \frac{-2 \ldots}{1 - 0.25z^{-1} + 0.75z^{-2}}$$

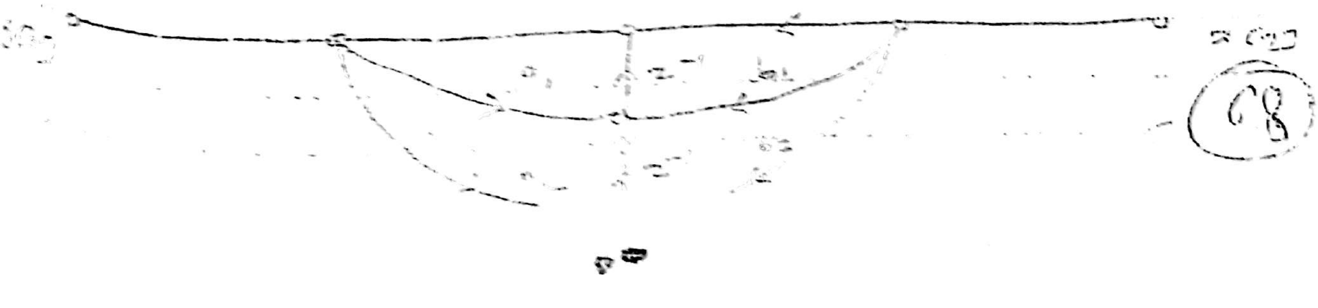


iii) Transposed form :-

If we reverse the direction of all branch ~~transmittance~~ and interchange the role the input & output in signal flow graph representation then the system remain unchanged. The resulting structure is called transposed form.



Transposed form



~~Alternate~~

Structures for FIR System:

O/p of FIR System:

$$y[n] = \sum_{k=0}^{M} h[k]\, x[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$$

Taking z-transform:

$$Y(z) = \sum_{k=0}^{M} h[k]\, z^{-k} X(z)$$

$$= H(z) \cdot X(z)$$

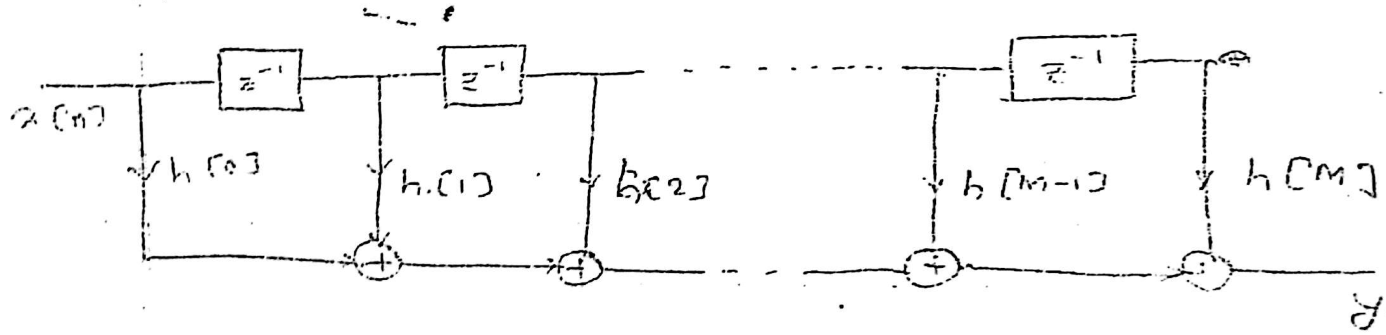
$$\therefore H(z) = \sum_{k=0}^{M} h[k] \cdot z^{-k}$$

$$= \sum_{k=0}^{M} b_k z^{-k}$$

where,

$$h[k] = \begin{cases} b_k, & 0 \le k \le M \\ 0, & \text{otherwise} \end{cases}$$

i) Direct Form:



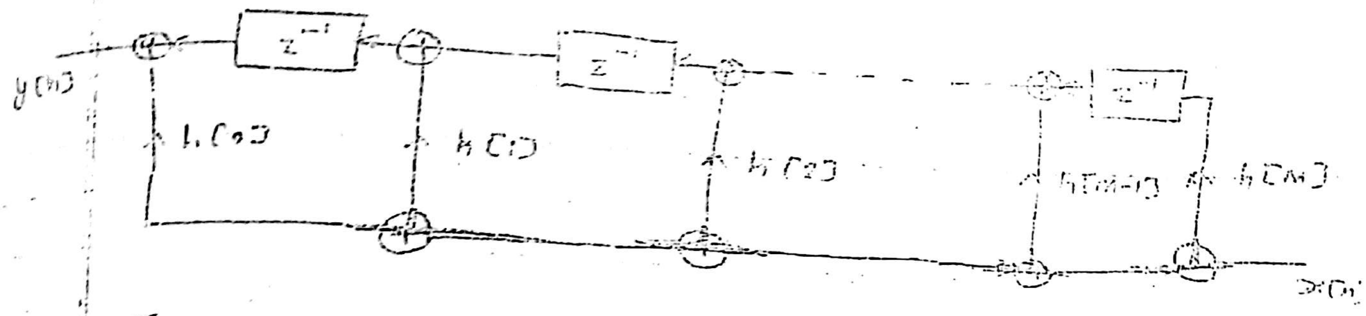
System required;

$M+1$ multiplier

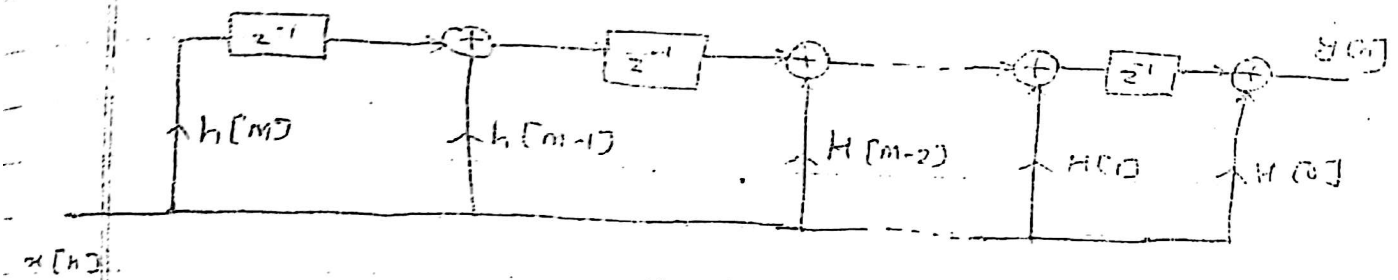
M Memory element

M Adder

Transposed form



$\equiv$



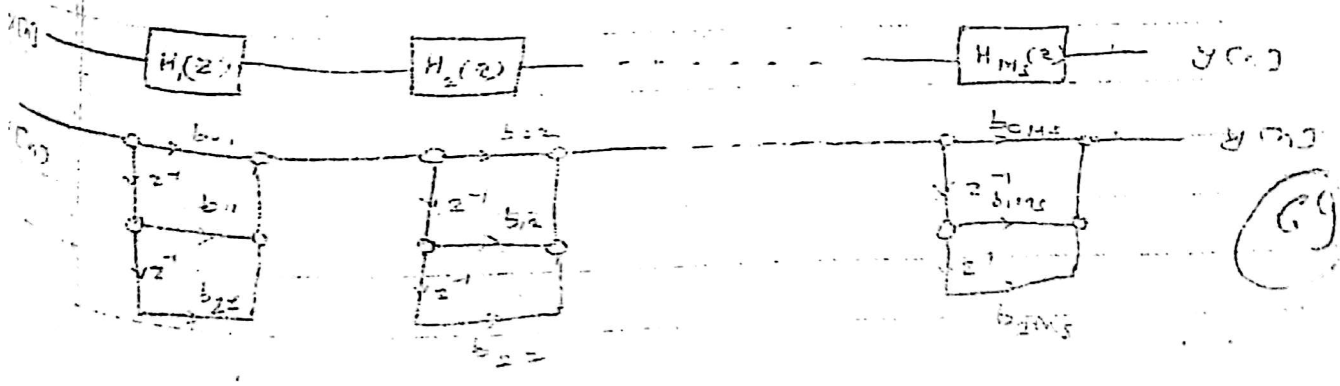
Cascaded form :-

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$= \prod_{k=1}^{M_s} \left(b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2} \right)$$

$$= \prod_{k=1}^{M_s} H_k(z)$$

where, $M_s \Rightarrow$ greatest integer in $\frac{M+1}{2}$.



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* Lattice structure for FIR system:
Lattice structure are widely used in speech processing, adaptive filtering, spectrum estimation etc.

A general form:

$$H(z) = A_m(z)$$

where,

$$A_m(z) = 1 + \sum_{K=1}^{m} \alpha_m(K) z^{-K} = \sum_{K=0}^{m} \alpha_m(K) z^{-K}$$

$m \Rightarrow$ denoted degree of polynomial

$\alpha_m(0) = 1$

For $m = 1$ (1st order polynomial)

$$H(z) = A_1(z)$$

$$\frac{Y(z)}{X(z)} = 1 + \alpha_1(1) z^{-1}$$

or, $Y(z) = X(z) + \alpha_1(1) z^{-1} X(z)$

Taking inverse z-transform; we get;

$$y[n] = x[n] + \alpha_1(1)\, x[n-1] \quad \text{---} (1)$$

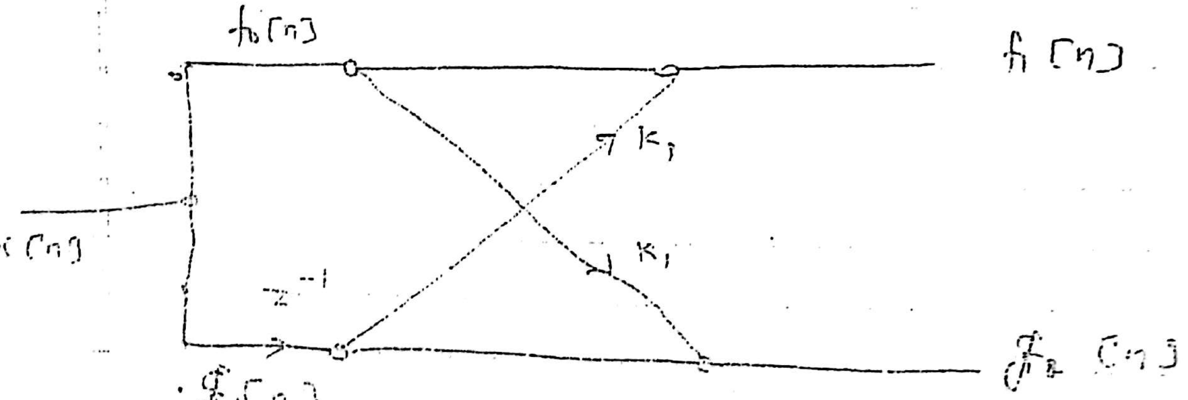

Fig: 1st stage lattice

A.O.

$$f_1[n] = f_0[n] + k_1 g_0[n-1]$$
$$g_1[n] = k_1 f_0[n] + g_0[n-1]$$

Comparing with Direct form (i.e eqn ①)

$$y[n] = f_1[n]$$
$$x[n] = f_0[n] = g_0[n]$$
$$k_1 = \alpha_1(1)$$

For $m=2$

$$H(z) = A_2(z)$$
$$\frac{Y(z)}{X(z)} = 1 + \alpha_2(1) z^{-1} + \alpha_2(2) z^{-2}$$

or $Y(z) = X(z) + \alpha_2(1) z^{-1} X(z) + \alpha_2(2) z^{-2} X(z)$

Taking inverse z-transform;

$$y[n] = x[n] + \alpha_2(1)\, x[n-1] + \alpha_2(2)\, x[n-2]$$

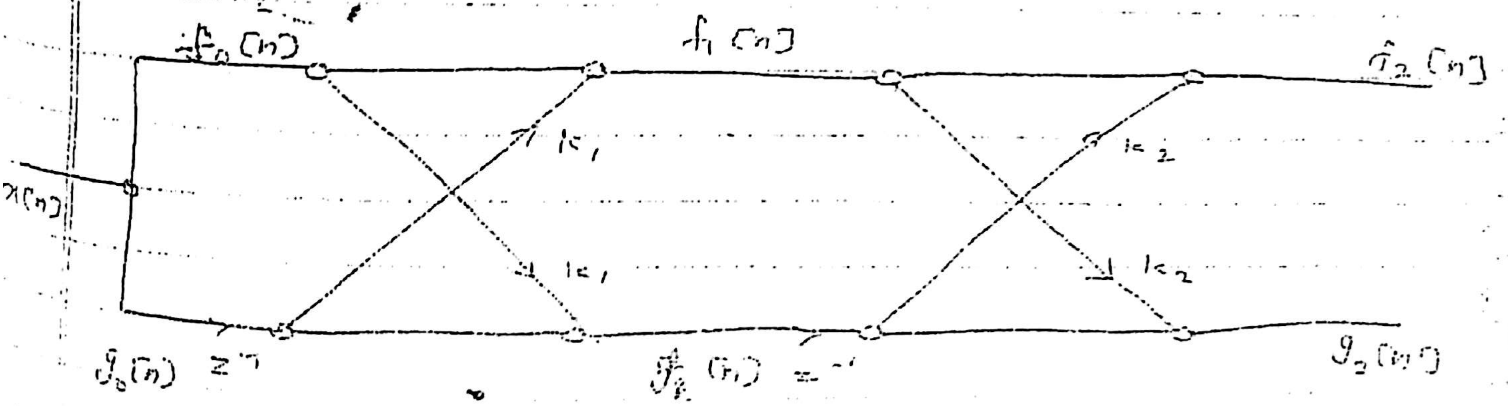

Fig: 2nd stage lattice

$$f_1[n] = f_0[n] + k_1 g_0[n-1]$$
$$g_1[n] = k_1 f_0[n] + g_0[n-1]$$

$$f_2[n] = f_1[n] + k_2 g_1[n-1]$$
$$g_2[n] = k_2 f_1[n] + g_1[n-1]$$

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Now,

$$f_2[n] = f_0[n] + K_1 g_0[n-1] + K_2 \cdot K_1 f_0[n-1] + K_2 g_0[n-2]$$

$$= x[n] + K_1 x[n-1] + K_1 K_2 x[n-1] + K_2 x[n-2]$$

$$= x[n] + K_1 (1 + K_2) x[n-1] + K_2 x[n-2]$$

Comparing with Direct form, we get:

$$y[n] = f_2[n]$$

$$\alpha_2(2) = K_2$$

$$\alpha_2(1) = K_1(1 + K_2)$$

$$\therefore K_1 = \frac{\alpha_2(1)}{1 + \alpha_2(2)}$$

$K_m \Rightarrow$ lattice coefficients.

For m^{th} stage lattice :

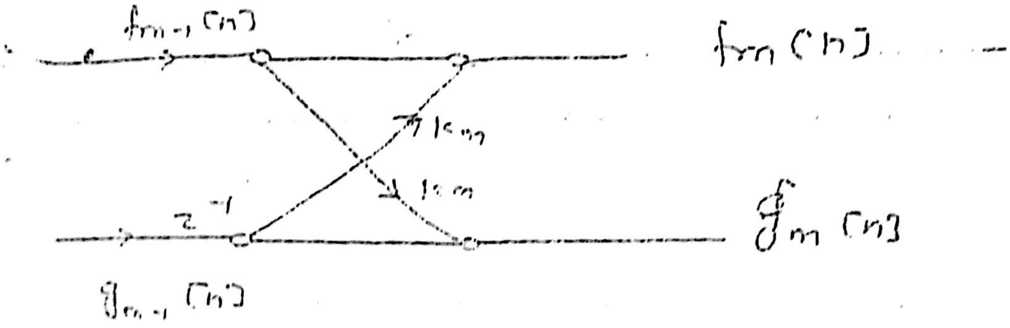


Fig: m^{th} stage Lattice

$$y[n] = f_m[n]$$

$$x[n] = f_0[n] = g_0[n]$$

$$f_m[n] = f_{m-1}[n] + K_m g_{m-1}[n-1]$$

$$g_m[n] = K_m f_{m-1}[n] + g_{m-1}[n-1]$$

Output of mth stage lattice structure:

$$y[n] = \sum_{k=0}^{m} \alpha_m(k)\, x[n-k] = f_m[n]$$

Taking z-transform, we get:

$$Y(z) = \sum_{k=0}^{m} \alpha_m(k)\, z^{-k} X(z) = F_m(z)$$

$$F_m(z) = A_m(z) \cdot X(z)$$

$$\therefore A_m(z) = \frac{F_m(z)}{X(z)}$$

Similarly:

$$g_2[n] = k_2 f_1[n] + g_1[n-1]$$

$$= k_2 \{ f_0[n] + k_1 g_0[n-1] \} + k_1 f_0[n-1] + g_0[n-2]$$

$$= k_2 x[n] + k_1 k_2 x[n-1] + k_1 x[n-1] + x[n-2]$$

$$= \underset{\beta_2(0)}{\underset{\uparrow}{k_2}}\, x[n] + \underset{\beta_2(1)}{\underset{\uparrow}{k_1(1+k_2)}}\, x[n-1] + \underset{\beta_2(2)}{\underset{\uparrow}{1}}\, x[n-2]$$

So,

$$g_m[n] = \sum_{k=0}^{m} \beta_m(k) \cdot x[n-k]$$

where, $\beta_m(k) = \alpha_m(m-k)$

Taking z-transform:

$$G_m(z) = \sum_{k=0}^{m} \beta_m(k)\, z^{-k} X(z)$$

$$= B_m(z)\, X(z)$$

$$\therefore \; B_m(z) = \frac{G_m(z)}{X(z)}$$

where,
$$B_m(z) = \sum_{k=0}^{m} \beta_m(k)\, z^{-k}$$

* Conversion from lattice coefficient to Direct

we have;

$$\left.\begin{aligned} y[n] &= f_m[n] \\ x[n] &= f_0[n] = g_0[n] \\ f_m[n] &= f_{m-1}[n] + K_m\, g_{m-1}[n-1] \\ g_m[n] &= K_m\, f_{m-1}[n] + g_{m-1}[n-1] \end{aligned}\right\}$$

Taking z- transform of eq^n (I);

$$\left.\begin{aligned} X(z) &= F_0(z) = G_0(z) \\ F_m(z) &= F_{m-1}(z) + K_m\, z^{-1} G_{m-1}(z) \\ G_m(z) &= k_m\, F_{m-1}(z) + z^{-1} G_{m-1}(z) \end{aligned}\right\} \text{(II)}$$

Dividing eq^n (II) by $X(z)$, we get;

$$1 = \frac{F_0(z)}{X(z)} = \frac{G_0(z)}{X(z)}$$

$$\boxed{1 = A_0(z) = B_0(z)}$$

$$\frac{F_m(z)}{X(z)} = \frac{F_{m-1}(z)}{X(z)} + K_m z^{-1} \frac{G_{m-1}(z)}{X(z)}$$

$$\boxed{A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)}$$

$$\frac{G_m(z)}{X(z)} = \frac{K_m F_{m-1}(z)}{X(z)} + \frac{z^{-1} G_{m-1}(z)}{X(z)}$$

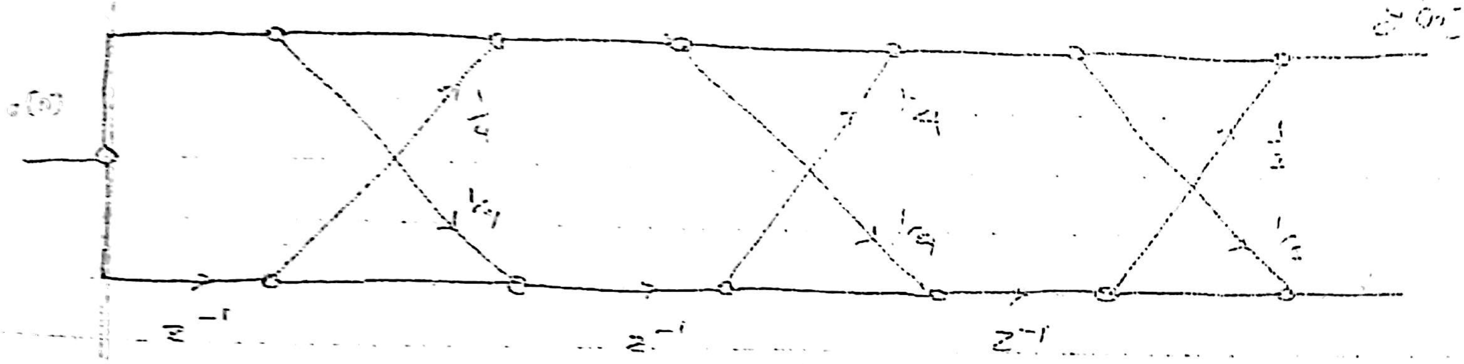
$$B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z)$$

for:

$$m = 1, 2, \ldots, M$$

Solution is obtained starting with $m = 1, 2, \ldots, M$.

example:



Draw Direct form structure for given lattice structure, with $k_1 = 1/4$, $k_2 = 1/4$, $k_3 = 1/3$

Sol

For given 3rd stage FIR system lattice structure

$$k_1 = \frac{1}{4}, \quad k_2 = \frac{1}{4}, \quad k_3 = \frac{1}{3}$$

we have,

$$A_0(z) = B_0(z) = 1$$

Also; $A_m(z) = A_{m-1}(z) + k_m z^{-1} B_{m-1}(z)$

$$A_1(z) = A_0(z) + k_1 z^{-1} B_0(z)$$

$$= 1 + \frac{1}{4} z^{-1}$$

Comparing with Direct form,

$\alpha_1(0) = 1 \quad , \quad \alpha_1(1) = \frac{1}{4}$

Also,

$$B_m(z) = \sum_{K=0}^{m} \beta_m(K) z^{-K}$$

$$= \sum_{K=0}^{m} \alpha_m(m-K) z^{-K}$$

$$B_1(z) = \frac{1}{4} + z^{-1}$$

Now,

$$A_2(z) = A_1(z) + K_2 z^{-1} B_1(z)$$

$$= 1 + \frac{1}{4} z^{-1} + \frac{1}{4} \cdot z^{-1}\left(\frac{1}{4} + z^{-1}\right)$$

$$= 1 + \frac{1}{4} z^{-1} + \frac{1}{16} z^{-1} + \frac{1}{4} z^{-2}$$

$$= 1 + \frac{5}{16} z^{-1} + \frac{1}{4} z^{-2}$$

Comparing with Direct form,

$\alpha_2(0) = 1, \quad \alpha_2(1) = \frac{5}{16}, \quad \alpha_2(2) = \frac{1}{4}$

$$B_2(z) = \frac{1}{4} + \frac{5}{16} z^{-1} + z^{-2}$$

Again,

$$A_3(z) = A_2(z) + K_3 z^{-1} B_2(z)$$

$$= 1 + \frac{5}{16} z^{-1} + \frac{1}{4} z^{-2} + \frac{1}{3} z^{-1}\left(\frac{1}{4} + \frac{5}{16} \ldots\right.$$

$$= 1 + \frac{5}{16} z^{-1} + \frac{1}{4} z^{-2} + \frac{1}{12} z^{-1} + \frac{5}{48} z^{-2} + \frac{1}{3} \ldots$$

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$$= 1 + \frac{13}{48} z^{-1} + \frac{17}{48} z^{-2} + \frac{1}{3} z^{-3}$$

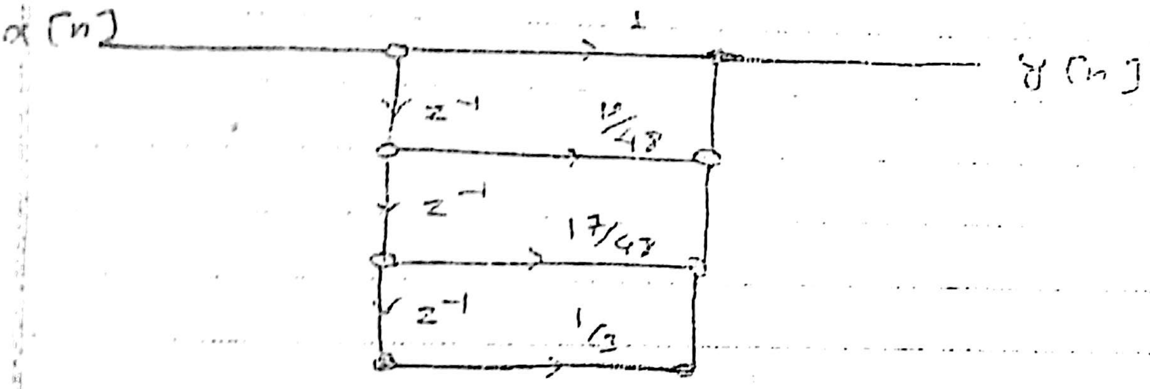
Comparing with Direct form;

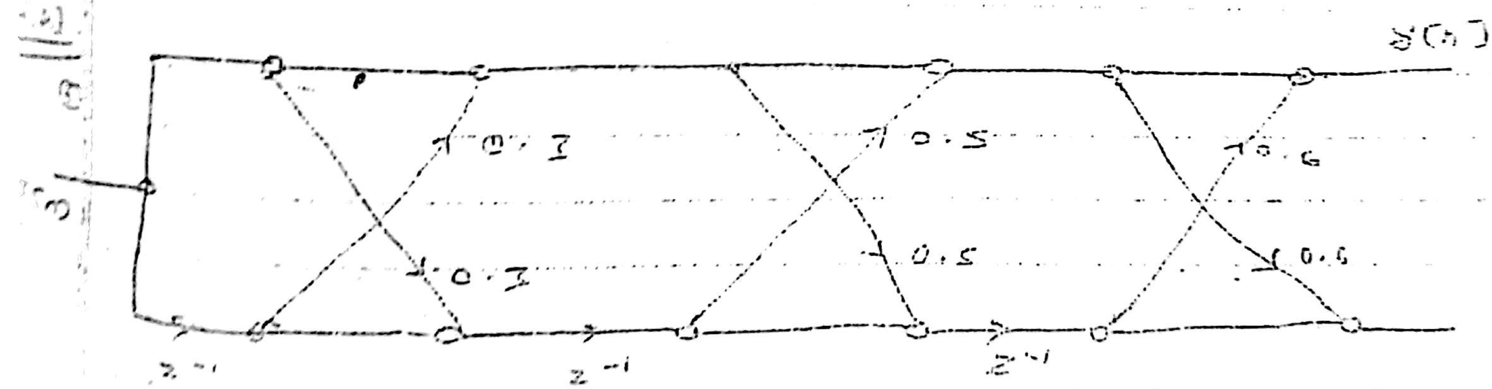
$\alpha_3(0) = 1$, $\alpha_3(1) = \frac{13}{48}$, $\alpha_3(2) = \frac{17}{48}$, $\alpha_3(3) = \frac{1}{3}$

For given system;

$H(z) = A_3(z)$

$$\therefore H(z) = 1 + \frac{13}{48} z^{-1} + \frac{17}{48} z^{-2} + \frac{1}{3} z^{-3}$$





Draw Direct form structure for given lattice s.

② Draw Direct form for FIR system having lattice coefficients:

$k_1 = 0.35$, $k_2 = 0.25$

$k_3 = 0.4$, $k_4 = 0.8$.

* Relation betn $B_m(z)$ & $A_m(z)$.

we know:

$$B_m(z) = \sum_{k=0}^{m} \beta_m(k) z^{-k}$$

where, $\beta_m(k) = \alpha_m(m-k)$

$$B_m(z) = \sum_{k=0}^{m} \alpha_m(m-k) z^{-k}$$

put $m-k = l$.

$$= \sum_{l=0}^{m} \alpha_m(l) z^{l-m}$$

$$= z^{-m} \sum_{l=0}^{m} \alpha_m(l) z^{l}$$

$$= z^{-m} A_m(z^{-1})$$

* Conversion from Direct form to lattice Coefficients

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)$$

$$B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z)$$

$$A_m(z) = A_{m-1}(z) + K_m \{ B_m(z) - K_m A_{m-1}(z) \}$$

$$\boxed{\therefore A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}}$$

$K_m = \alpha_m(m)$. for $m = M, M-1, \ldots$

example:

$$H(z) = 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}$$

Draw lattice structure for given system.

sol^n

For given 3rd order FIR system:

$$H(z) = A_3(z) = 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}$$

comparing with Direct form;

$\alpha_3(0) = 1$, $\alpha_3(1) = \frac{13}{24}$, $\alpha_3(2) = \frac{5}{8}$, $\alpha_3(3) = \frac{1}{3}$

$k_m = \alpha_m(m)$, $k_3 = \alpha_3(3) = \frac{1}{3}$

Also,
$$B_m(z) = \sum_{k=0}^{m} \alpha_m(m-k) z^{-k}$$

$$B_3(z) = \frac{1}{3} + \frac{5}{8} z^{-1} + \frac{13}{24} z^{-2} + z^{-3}$$

we have;

$$A_{m-1}(z) = \frac{A_m(z) - k_m B_m(z)}{1 - k_m^2}$$

$$A_2(z) = \frac{A_3(z) - k_3 B_3(z)}{1 - k_3^2}$$

$$= \frac{1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3} - \frac{1}{3}\left(\frac{1}{3} + \frac{5}{8} z^{-1} + \frac{13}{24} z^{-2} + z^{-3}\right)}{1 - \left(\frac{1}{3}\right)^2}$$

$$= \frac{\frac{8}{9} + \frac{8}{24} z^{-1} + \frac{32}{72} z^{-2} +}{8/9}$$

$$= 1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2}$$

Comparing with Direct form:

$\alpha_2(0) = 1$, $\alpha_2(1) = 3/8$, $\alpha_2(2) = k_2 = 1/2$

$$B_2(z) = \frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2}$$

Also,

$$A_1(z) = \frac{A_2(z) - k_2 B_2(z)}{1 - k_2^2}$$

$$= \frac{1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2} - \frac{1}{2}\left(\frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2}\right)}{1 - \left(\frac{1}{2}\right)^2}$$

$$= 1 + \frac{3}{16} \times \frac{4}{3} z^{-1} + 0$$

$$= 1 + \frac{1}{4} z^{-1}$$

Comparing with Direct form:

$\alpha_1(0) = 1$, $\alpha_1(1) = \frac{1}{4} = k_1 = \frac{1}{4}$

Lattice coefficients are:

$k_1 = \frac{1}{4}$, $k_2 = \frac{1}{2}$, $k_3 = \frac{1}{3}$

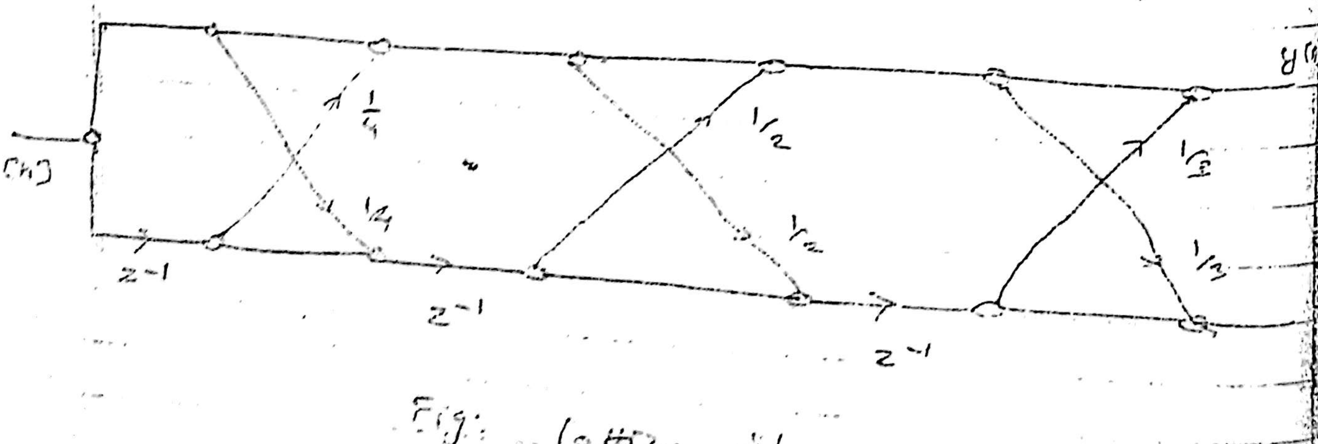
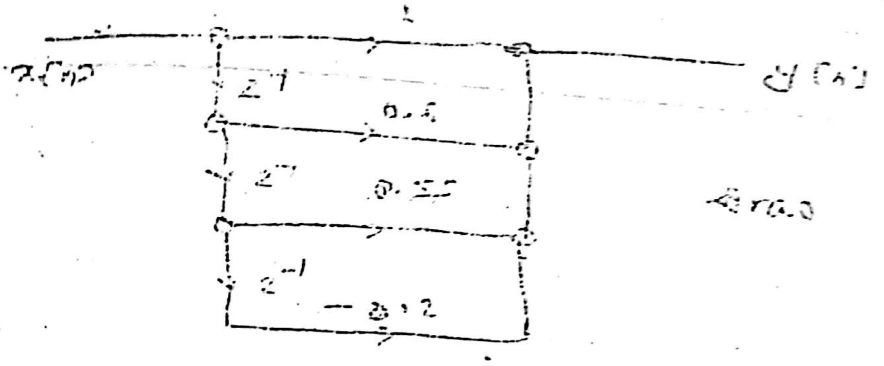


Fig: Lattice structure

i) $H(z) = 1 + 3.8z^{-1} - 0.25z^{-2} + 0.5z^{-3} + 0.2z^{-4}$

Draw lattice structure for given system.

ii) 

Draw its lattice structure

Lattice structure for all pole IIR System:

T.F. of all pole IIR system:

$$H(z) = \frac{1}{A_N(z)} = \frac{1}{1 + \sum_{K=1}^{N} \alpha_N(K) \cdot z^{-K}}$$

$$\frac{Y(z)}{X(z)} = \frac{1}{1 + \sum_{K=1}^{N} \alpha_N(K) \cdot z^{-K}}$$

or, $Y(z) + \sum_{K=1}^{N} \alpha_N(K) z^{-k} Y(z) = X(z)$

Taking inverse z-transform we get;

$$y[n] + \sum_{K=1}^{N} \alpha_N(K) y[n-K] = x[n]$$

$$y[n] = x[n] - \sum_{K=1}^{N} \alpha_N(K) y[n-K] \Rightarrow IIR$$

If we interchange role of input and output, th

$$x[n] = y[n] - \sum_{K=1}^{N} \alpha_N(K) \cdot x[n-K]$$

$$\therefore y[n] = x[n] + \sum_{K=1}^{N} \alpha_N(K) \cdot x[n-K] \quad \text{FIR}$$

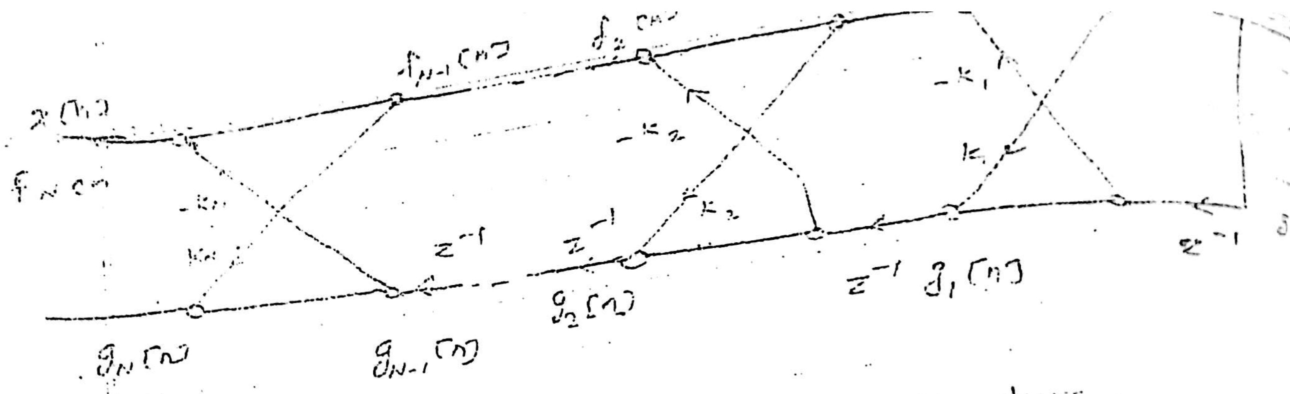


Fig: All pole lattice structure.

All pole IIR	FIR
$y[n] = f_0[n] = g_0[n]$	$y[n] = f_N[n]$
$x[n] = f_N[n]$	$x[n] = f_0[n] = g_0[n]$

Due to these changes, output of 1st stage all pole lattice;

For $N=1$;

$$x[n] = f_1[n]$$
$$f_0[n] = f_1[n] - k_1 g_0[n-1]$$
$$g_1[n] = k_1 f_0[n] + g_0[n-1]$$

Comparing with direct form;

$y[n] = f_0[n] = g_0[n]$ $\quad$ $x[n] = f_1[n]$

$k_1 = \alpha_1(1)$.

For $N=2$;

$$x[n] = f_2[n]$$
$$f_0[n] = f_1[n] - k_1 g_0[n-1]$$
$$g_1[n] = k_1 f_0[n] + g_0[n-1]$$
$$f_1[n] = f_2[n] - k_2 g_1[n-1]$$
$$g_2[n] = k_2 f_1[n] + g_1[n-1]$$

$$f_2[n] = f_2[n] - k_2 g_1[n-1] - k_1 g_0[n-1]$$
$$= f_2[n] - k_2\{k_1 f_0[n-1] + g_0[n-2]\} - k_1$$
$$= f_2[n] - k_1 k_2 y[n-1] - k_2 y[n-2] - k_1 y[n-1]$$
$$= \underbrace{x[n]}_{\alpha_2(0)} - \underbrace{k_1(1+k_2)}_{\alpha_2(1)} y[n-1] - \underbrace{k_1}_{\alpha_2(2)} y[n-2]$$

So, in general:

$$y[n] = f_0[n] = x[n] - \sum_{k=1}^{m} \alpha_m(k) \cdot y[n-k]$$

Similarly:

$$\cancel{g_2[n] = k_2\{f_2[n] - k_1 g_0[n-1]\} + g_1[n-1]}$$
$$\cancel{= k_2\{f_2[n] - k_2(k_1 f_0[n-1] + g_0[n-2])\}}$$
$$\cancel{+ k_1 f_0[n-1] + g_0[n-2]}$$

$$g_2[n] = k_2\{f_0[n] + k_1 g_0[n-1]\} + k_1 f_0[n] + g_0[n]$$
$$= k_2 y[n] + k_1 k_2 y[n-1] + k_1 y[n-1] + y[n-2]$$
$$= \underbrace{k_2}_{\alpha_2(2)} y[n] + \underbrace{k_1(k_2+1)}_{\alpha_2(1)} y[n-1] + \underbrace{y[n-2]}_{\alpha_2(0)}$$

In general:

$$g_m[n] = \sum_{k=0}^{m} \alpha_m(m-k) \cdot y[n-k]$$

arks:

i) All pole lattice structure has all zero path al goes from $g_0[n] \Longrightarrow g_m[n]$.

ii) All pole lattice structure and all zero lattice structure are characterized by same set of lattice parameter $k_1, k_2, \ldots, k_m$. But they are differ in interconnection.

iii) The roots of polynomial $A_m(z)$ lies inside the unit circle if and only if $|k_m| < 1$. So the lattice structure will be stable if and only if $|k_m| < 1$. This is known as Schur Cohn stability.

2022-4-10

xample:

$$H(z) = \frac{1}{1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z^{-3}}$$

Draw lattice structure for given system.

Sol. For given 3rd order all-pole IIR system

$$H(z) = \frac{1}{A_3(z)}$$

$$A_3(z) = 1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z^{-3}$$

Comparing with Direct form, we get:

$\alpha_3(0) = 1, \quad \alpha_3(1) = -0.9, \quad \alpha_3(2) = 0.64, \quad \alpha_3(3) = -0.576$

$K_m = \alpha_m(m)$

$K_3 = \alpha_3(3) = -0.576$

Also

$$B_m(z) = \sum_{k=0}^{m} \alpha_m(m-k)\, z^{-k}$$

$$B_3(z) = -0.576 + 0.64z^{-1} - 0.9z^{-2} + z^{-3}$$

Also,

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$$

$$A_2(z) = \frac{A_3(z) - K_3 B_3(z)}{1 - K_3^2}$$

$$A_2(z) = \frac{1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z^{-3} - (-0.576)(-0.576 + 0.64z^{-1} - 0.9z^{-2} + z^{-3})}{1 - (-0.576)^2}$$

$= \cancel{1 - 0.9z^{-1} + 0.64z^{-2} - 0.576z}$

$= 1 - 0.795z^{-1} + 0.1819z^{-2}$

Comparing with direct form;

$\alpha_2(0) = 1, \quad \alpha_2(1) = -0.795, \quad \alpha_2(2) = K_2 = 0.1819$

$$B_2(z) = 0.1819 - 0.795z^{-1} + z^{-2}$$

$$A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - K_2^2}$$

$$= \frac{1 - 0.795z^{-1} + 0.1819z^{-2} - 0.1819(0.1819 - 0.795z^{-1} + z^{-2})}{[illegible]}$$

(77)

$1 - 0.6726 z^{-1}$

Comparing with Direct form, we get:

$\alpha_1(0) = 1$, $\alpha_1(1) = K_1 = -0.6726$

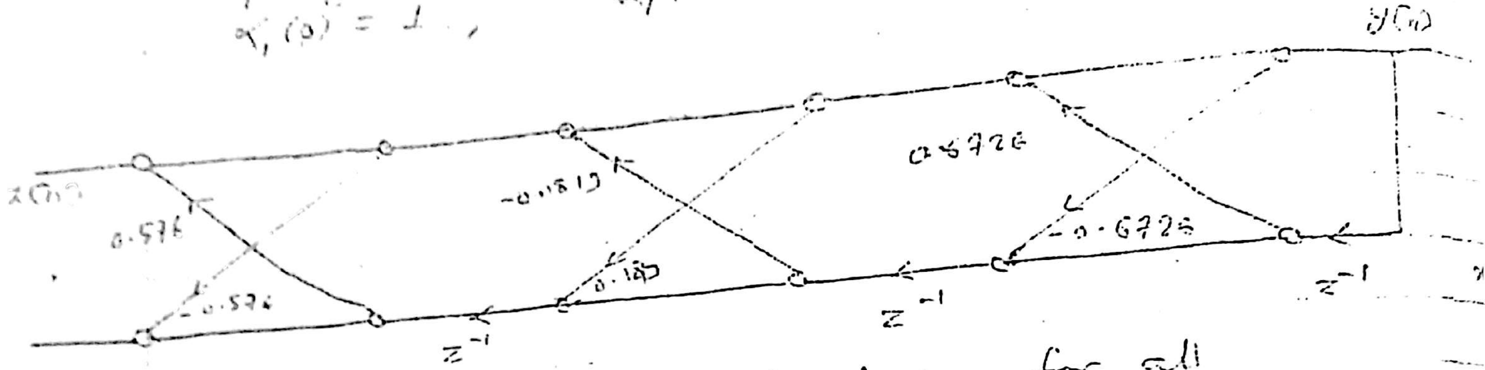


Fig:- lattice structure for all pole - system

H.W

1. $H(z) = \dfrac{1}{1 + 0.54 z^{-1} + 0.35 z^{-2} + 0.85 z^{-3}}$

Draw lattice structure for given system:

2. If $K_1 = 0.36$, $K_2 = 0.56$, $K_3 = 0.45$ then, draw Direct form of all-pole IIR system.

⇒ Lattice - ladder structure for pole - zero system:

T.F. of pole - zero system:

$$H(z) = \frac{\sum_{k=0}^{M} C_M(k) z^{-k}}{1 + \sum_{k=1}^{N} \alpha_N(k) z^{-k}} = \frac{C_M(z)}{A_N(z)} \quad \text{and } M \le N$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} C_M(k) z^{-k}}{1 + \sum_{k=1}^{N} \alpha_N(k) \cdot z^{-k}}$$

$$Y(z) + \sum_{k=1}^{N} \alpha_N(k) z^{-k} Y(z) = \sum_{k=0}^{M} C_M(k) z^{-k}$$

Taking Inverse z-transform ; we get;

$$y[n] + \sum_{k=1}^{N} \alpha_N(k) y[n-k] = \sum_{k=0}^{M} C_M(k) x[n-k]$$

$$\therefore y[n] = \sum_{k=0}^{M} C_M(k) x[n-k] - \sum_{k=1}^{N} \alpha_N(k) y[n-k]$$

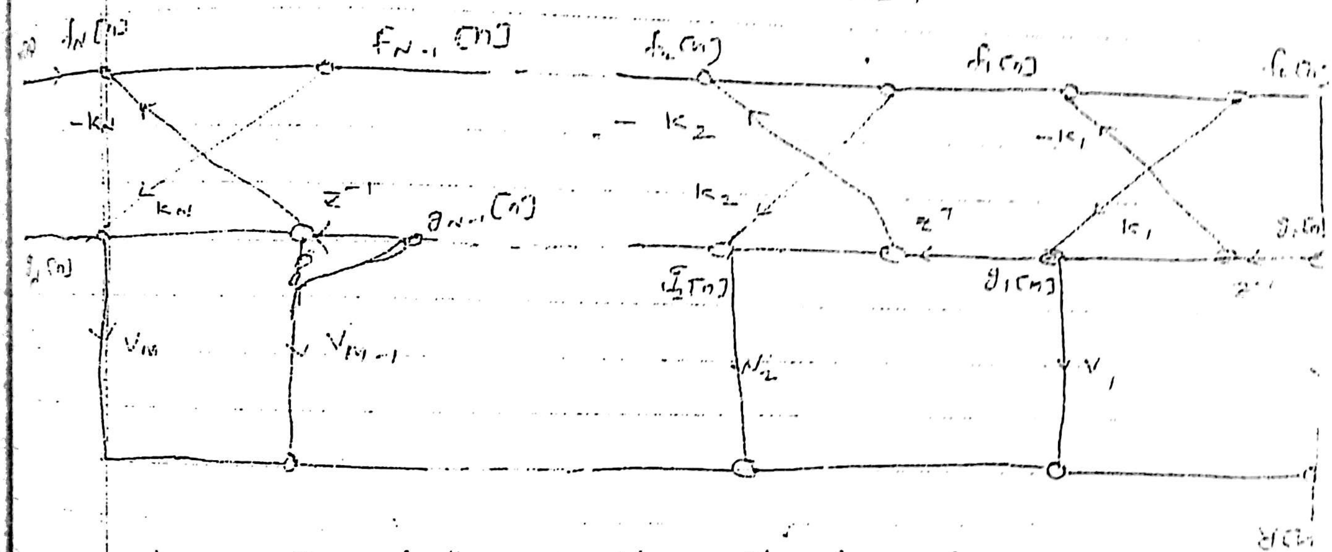


Fig: Lattice-ladder structure for pole-zero system.

O/p of this structure will be:

$$y[n] = \sum_{m=0}^{M} V_m g_m[n]$$

, V_m = ladder coefficient

Taking z-transform:

$$Y(z) = \sum_{m=0}^{M} V_m G_m(z)$$

Transfer function of structure:

$$H(z) = \frac{Y(z)}{X(z)}$$

78

for ... ;

$x[n] = f_N[n]$ $\quad$ $f_0[n] = g_0[n]$

$X(z) = F_N(z)$ $\quad$ $F_0(z) = G_0(z)$.

Now, $H(z)$ becomes:

$$H(z) = \frac{\sum_{m=0}^{M} V_m G_m(z)}{F_N(z)}$$

$$H(z) = \sum_{m=0}^{M} V_m \cdot \frac{G_m(z)}{G_0(z)} \cdot \frac{F_0(z)}{F_N(z)}$$

$$H(z) = \sum_{m=0}^{M} V_m \cdot B_m(z) \cdot \frac{1}{A_N(z)}$$

$$\therefore \boxed{H(z) = \sum_{m=0}^{M} \frac{V_m B_m(z)}{A_N(z)}}$$

But required T.F. $H(z) = \dfrac{C_M(z)}{A_N(z)}$.

So;

$$\boxed{C_M(z) = \sum_{m=0}^{M} V_m B_m(z)} \quad \text{---- (1)}$$

To designs.

1. Draw out lattice parameter K_m and B_m using polynomial

2. Use equation (1) to find out ladder parameters V_m.

To find out ladder coefficients:

$$C_m(z) = \sum_{k=0}^{m} V_k B_k(z), \quad \text{for } m = 0, 1, 2, \ldots, M$$

$$C_m(z) = \sum_{k=0}^{m-1} V_k B_k(z) + V_m B_m(z)$$

$$= C_{m-1}(z) + V_m B_m(z)$$

So,

$$\boxed{C_{m-1}(z) = C_m(z) - V_m B_m(z).}$$

Again;

$$\sum_{k=0}^{M} C_M(k) \cdot z^{-k} = \sum_{m=0}^{M} V_m B_m(z)$$

or, $C_M(0) + \cdots + C_M(m) z^{-M} = V_0 + \cdots + V_m z^{-M}$

i.e. $V_m = C_m(m)$

Example:

$$H(z) = \frac{1 + 2z^{-1} + 2z^{-2} + z^{-3}}{1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}}$$

Draw lattice ladder structure for given system.

Sol:

- For given 3rd order pole-zero system;

$$H(z) = \frac{C_3(z)}{A_3(z)}$$

$$A_3(z) = 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}, \quad k_3 = \frac{1}{3}$$

$$B_3(z) = \frac{1}{3} + \frac{5}{8} z^{-1} + \frac{13}{24} z^{-2} + z^{-3}$$

Already solved :

$$A_2(z) = 1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2}$$

$$B_2(z) = \frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2} \quad , \quad K_2 = \frac{1}{2}$$

$$A_1(z) = 1 + \frac{1}{4} z^{-1} \quad , \quad K_1 = \frac{1}{4}$$

$$B_1(z) = \frac{1}{4} + z^{-1}$$

Now,

$$C_3(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

Comparing with $C_m(z) = \sum_{k=0}^{m} C_m(k) \cdot z^{-k}$ ---- (1)

we get

$C_3(0) = 1$, $C_3(1) = 2$, $C_3(2) = 2$, $C_3(3) = 1$

$$V_m = C_m(m) \Rightarrow V_3 = C_3(3) = 1$$

Also;

$$C_{m-1}(z) = C_m(z) - V_m B_m(z)$$

$$C_2(z) = C_3(z) - V_3 B_3(z)$$

$$= 1 + 2z^{-1} + 2z^{-2} + z^{-1} - 1 \cdot \left(\frac{1}{3} + \frac{5}{8} z^{-1} + \frac{13}{24} z^{-2} \ldots\right.$$

$$= \frac{2}{3} + \frac{11}{8} z^{-1} + \frac{35}{24} z^{-2}$$

Comparing with eqn (1);

$C_2(0) = \frac{2}{3}$, $C_2(1) = \frac{11}{8}$, $C_2(2) = \frac{35}{24}$

$$\therefore V_2 = C_2(2) = \frac{35}{24}$$

Again:

$$C_1(z) = C_2(z) - V_2 B_2(z)$$

$$= \frac{?}{?} + \frac{?}{8} z^{-1} + \frac{35}{64} z^{-2} - \frac{35}{64}\left(\frac{1}{2} + \frac{?}{?} z^{-1} + z^{-2}\right)$$

$$= -\frac{1}{16} + \frac{159}{192} z^{-1}$$

$$= -\frac{1}{16} + \frac{53}{64} z^{-1}$$

Comparing with eqn ①;

$$C_1(0) = -\frac{1}{16}, \quad C_1(1) = +\frac{53}{64}$$

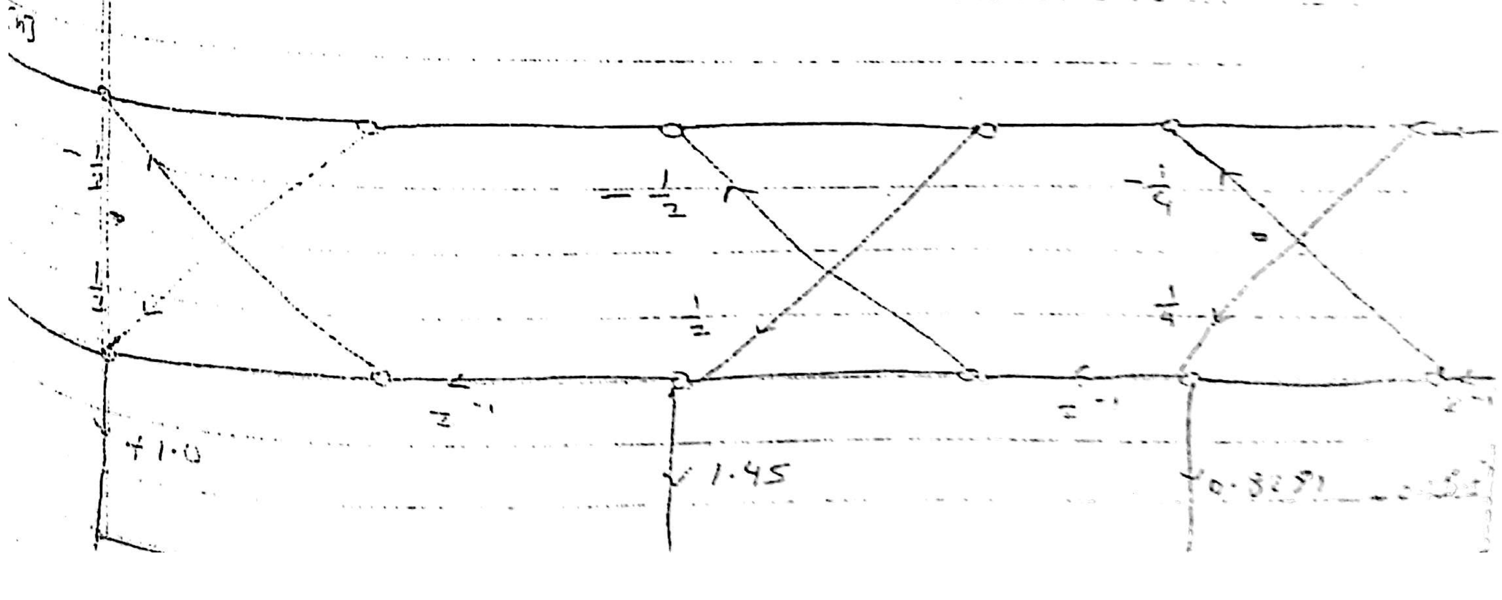
$$\therefore V_1 = C_1(1) = +\frac{53}{64} = 0.8281$$

Again:

$$C_0(z) = C_1(z) - V_1 B_1(z)$$

$$= -\frac{1}{16} - \frac{53}{64} z^{-1} + \frac{53}{64}\left(\frac{1}{4} + z^{-1}\right)$$

$$= \cancel{\frac{37}{256}} \quad -\frac{69}{256} = -0.2695$$



Draw lattice ladder structure for given system:

a) $H(z) = \dfrac{0.5 + 0.32 z^{-1} + 0.66 z^{-2} - 0.85 z^{-3}}{1 + 0.55 z^{-1} + 0.45 z^{-2} - 0.75 z^{-3}}$

b) $H(z) = \dfrac{2 + 3 z^{-1} + 0.7 z^{-2}}{1 - 0.3 z^{-1} + 0.42 z^{-2} + 0.27 z^{-3}}$ → (v_2, v_1, v_0 ...)

2) If $v_0 = 0.35$, $v_1 = 1.2$, $v_2 = 0.84$, $v_3 = 0.32$.
$k_1 = \frac{1}{2}$, $k_2 = \frac{1}{4}$, $k_3 = \frac{3}{5}$

Draw Direct form of pole-zero system.

* Characteristic of lattice structure:

1. Stability is embodied in lattice parameter ~~...~~ k_m.

2. Structure having robustness to finite word length effect.

5.3 Frequency Response :

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Taking Z - transform ;

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Transfer function :

$$H(z) = \frac{Y(z)}{X(z)}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$$= \frac{b_0 \left(1 + \frac{b_1}{b_0} z^{-1} + \frac{b_2}{b_0} z^{-2} + \cdots + \frac{b_M}{b_0} z^{-M}\right)}{a_0 \left(1 + \frac{a_1}{a_0} z^{-1} + \frac{a_2}{a_0} z^{-2} + \cdots + \frac{a_N}{a_0} z^{-N}\right)}$$

$$= \frac{b_0 \prod_{k=1}^{M} (1 - c_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - d_k z^{-1})}$$

Contains M zeros & N poles.

For $z = e^{j\omega}$

$$H(e^{j\omega}) = \frac{b_0}{a_0} \frac{\prod_{k=1}^{M} (1 - c_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - d_k e^{-j\omega})}$$

Magnitude Response of system in dB :

$$20 \log_{10} |H(e^{j\omega})| = 20 \log_{10} \left| \frac{b_0}{a_0} \right| + \sum_{k=1}^{M} 20 \log_{10} |1 - c_k e^{-j\omega}| - \sum_{k=1}^{N} 20 \log_{10} |1 - d_k e^{-j\omega}|$$

Phase Response of system :

$$\measuredangle H(e^{j\omega}) = \measuredangle \frac{b_0}{a_0} + \sum_{k=1}^{M} \measuredangle (1 - c_k e^{-j\omega}) - \sum_{k=1}^{N} \measuredangle (1 - d_k e^{-j\omega})$$

Group delay of system :

$$\operatorname{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \measuredangle H(e^{j\omega})$$

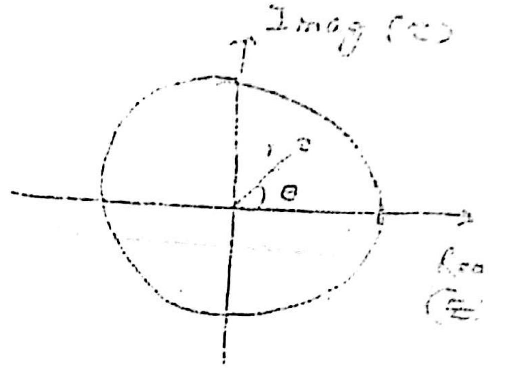
$$= \sum_{k=1}^{N} \frac{d}{d\omega} \measuredangle (1 - d_k e^{-j\omega}) - \sum_{k=1}^{M} \frac{d}{d\omega} \measuredangle (1 - c_k e^{-j\omega})$$

For a single zero:

$$H(z) = 1 - c_k z^{-1}$$

zero at $z = c_k$

where, $c_k = re^{j\theta}$



$$H(e^{j\omega}) = 1 - c_k e^{-j\omega}$$

$$= 1 - re^{j\theta} e^{-j\omega}$$

$$= 1 - re^{-j(-\theta+\omega)}$$

$$= 1 - r\cos(\omega-\theta) + jr\sin(\omega-\theta)$$

Magnitude Response in dB:

$$20\log_{10}|H(e^{j\omega})| = 20\log_{10}\sqrt{(1-r\cos(\omega-\theta))^2 + (r\sin(\omega$$

$$20\log_{10}|H(e^{j\omega})| = 10\log_{10}\left[(1-r\cos(\omega-\theta))^2 + (r\sin(\omega-\theta))^2\right.$$

$$= 10\log_{10}\left[1 - 2r\cos(\omega-\theta) + r^2\cos^2(\omega-\theta) + r^2\sin^2(\omega-\theta)\right.$$

$$= 10\log_{10}\left[1 + r^2 - 2r\cos(\omega-\theta)\right]$$

Phase Response:

$$\measuredangle H(e^{j\omega}) = \tan^{-1}\left[\frac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}\right]$$

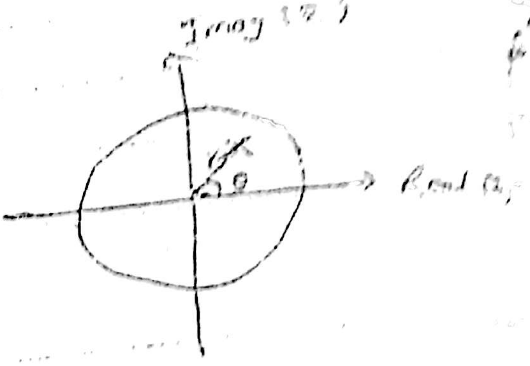
group delay:

$$grd[H(e^{j\omega})] = -\frac{d}{d\omega}\measuredangle H(e^{j\omega})$$

$$=$$

For a single Pole

$$H(z) = \frac{1}{1 - d_k z^{-1}}$$



pole at $z = d_k$

where, $d_k = re^{j\theta}$

$$H(e^{j\omega}) = \frac{1}{1 - d_k e^{-j\omega}}$$

$$= \frac{1}{1 - re^{j\theta} \cdot e^{-j\omega}}$$

$$= \frac{1}{1 - re^{-j(\omega-\theta)}}$$

$$= \frac{1}{1 - r\cos(\omega-\theta) + jr\sin(\omega-\theta)}$$

Magnitude response in dB:

$$20\log_{10}|H(e^{j\omega})| = 20\log_{10}\left(\frac{1}{\sqrt{(1 - r\cos(\omega-\theta))^2 + (r\sin(\omega-\theta))^2}}\right)$$

$$= -10\log_{10}\left[(1 - r\cos(\omega-\theta))^2 + (r\sin(\omega-\theta))^2\right]$$

$$= -10\log_{10}\left[1 + r^2 - 2r\cos(\omega-\theta)\right]$$

Phase Response:

$$\measuredangle H(e^{j\omega}) = -\tan^{-1}\left(\frac{r\sin(\omega-\theta)}{1 - r\cos(\omega-\theta)}\right)$$

group delay:

$$grd[H(e^{j\omega})] = -\frac{d}{d\omega}\measuredangle H(e^{j\omega})$$

mple: Draw the frequency response of a given system:-

(i) 1^{st} - described by difference equation:-

$$y[n] - 0.2\,y[n-1] + 0.5\,y[n-2] = x[n] - 0.1\,x[n-1] - 0.12\,x[n-2]$$

(ii) 2^{nd} - having poles are at $\Rightarrow 0.1 \pm j0.7$

zeroes are at $\Rightarrow -0.3$ and 0.4

(iii) 3^{rd} - having

poles are at	
r	θ
0.707	1.428 r
0.707	-1.428 r

zeroes are at	
r	θ
0.4	0
0.3	π

Sol: Taking z-transform of given system;

$$Y(z) - 0.2\,z^{-1}Y(z) + 0.5\,z^{-2}Y(z) = X(z) - 0.1\,z^{-1}X(z) - 0.12\,z^{-2}X(z)$$

or, $$\frac{Y(z)}{X(z)} = \frac{1 - 0.1z^{-1} - 0.12z^{-2}}{1 - 0.2z^{-1} + 0.5z^{-2}}$$

or, $$H(z) = \frac{z^2 - 0.1z - 0.12}{z^2 - 0.2z + 0.5}$$

$$= \frac{(z-0.4)(z+0.3)}{(z^2 - 0.2z + 0.5)}$$

System have zeros at $z = 0.4$ & $z = -0.3$

$\cancel{r_1 = 0.4}$, $r_1 = 0.4$, $r_2 = 0.3$

$\cancel{\theta_1 = 0}$, $\theta_1 = 0$, $\theta_2 = \pi$

System has two poles at :

$$z = \frac{-(-0.2) \pm \sqrt{0.2^2 - 4\times1\times0.5}}{2\times1}$$

(S3)

$= 0.1 \pm j0.7$

one pole at $0.1 + j0.7$

i.e. $r_3 = 0.707$

$\theta_3 = 1.428$ rad

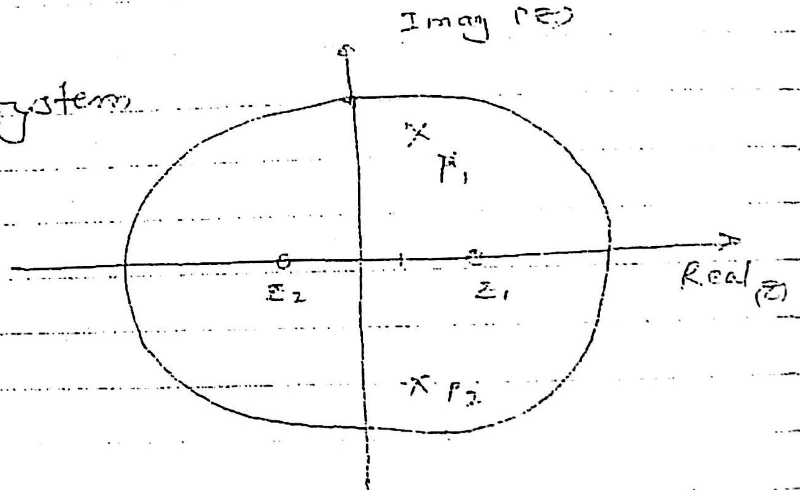
another pole at $0.1 - j0.7$

i.e. $r_4 = 0.707$

$\theta_4 = -1.428$ rad

Now
Magnitude response of system in dB;

$20 \log |H(e^{j\omega})| =$



$$20\log|H(e^{j\omega})| = 10\log_{10}[1+r_1^2-2r_1\cos(\omega-\theta_1)] + 10\log_{10}[1+r_2^2-2r_2\cos(\omega-\theta_2)]$$

$$-10\log_{10}[1+r_3^2-2r_3\cos(\omega-\theta_3)] - 10\log_{10}[1+r_4^2-2r_4\cos(\omega-\theta_4)]$$

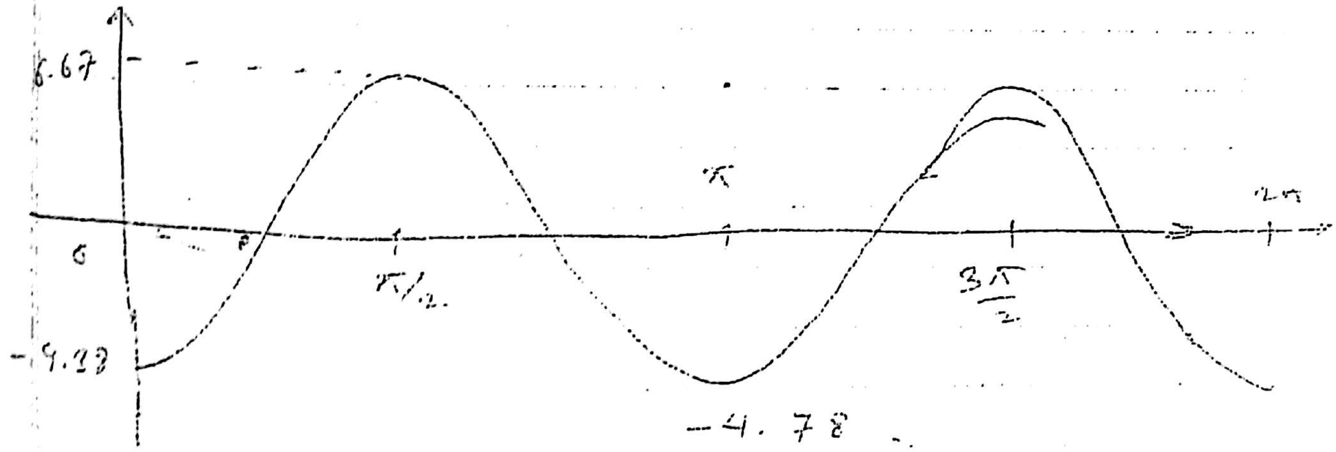
$$= 10\log_{10}[1+0.4^2-2\times0.4\cos(\omega)] + 10\log_{10}[1+0.3^2-2\times0.3\cos(\omega-\pi)]$$

$$-10\log[1+0.707^2-2\times0.707\cos(\omega-1.428)] - 10\log_{10}[1+0.707^2-2\times(+0.707)\cos(\omega+1.428)]$$

$$= 10\log[1.16-0.8\cos(\omega)] + 10\log_{10}[1.09-0.6\cos(\omega-\pi)]$$

$$-10\log[1.49984-1.414\cos(\omega-1.428)] - 10\log_{10}[1.49984-1.414\cos(\omega+1.428)]$$

ω	magnitude in dB
0	-4.427
$\pi/8$	-2.207
$\pi/4$	0.1796
$\pi/8$	3.34825
$\pi/2$	6.3856
~~[illegible]~~ $2\pi/3$	0.524
.	
.	
π	-4.7899
.	
.	
2π	-4.42



Phase Response:

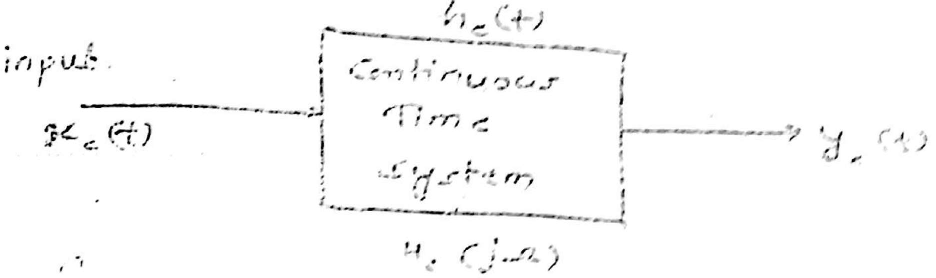
$$\angle H(e^{j\omega}) = \tan^{-1}\frac{r_1 \sin(\omega-\theta_1)}{1-r_1\cos(\omega-\theta_1)} + \tan^{-1}\left(\frac{r_2\sin(\omega-\theta_2)}{1-r_2\cos(\omega-\theta_2)}\right)$$

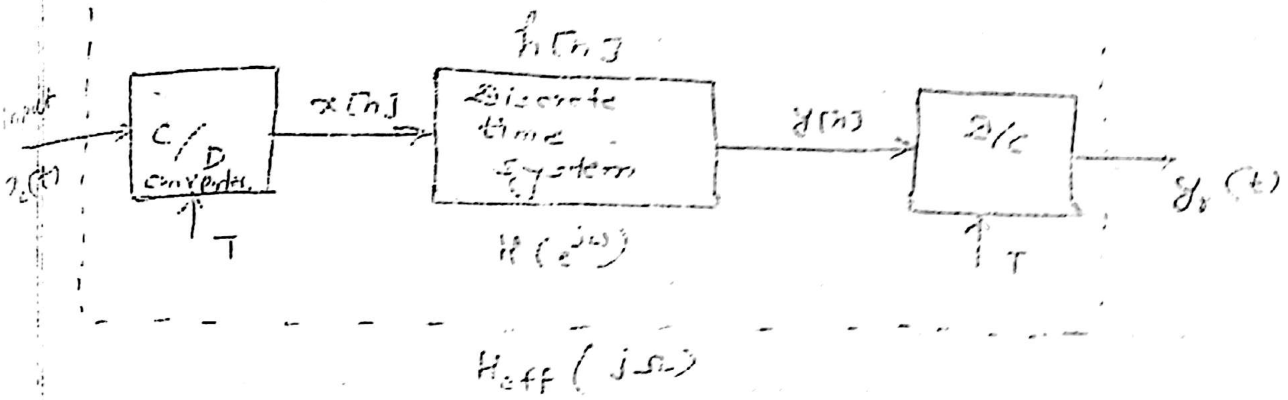
$$-\tan^{-1}\frac{r_3\sin(\omega-\theta_3)}{1-r_3\cos(\omega-\theta_3)} - \tan^{-1}\left(\frac{r_4\sin(\omega-\theta_4)}{1-r_4\cos(\omega-\theta_4)}\right)$$

(84)

0	$0.6612 - 0.6612 = 0$
$\pi/6$	$0.957388 - 0.4793 = 0.478$
$\pi/4$	$0.97757 - 0.3783 = 0.5974$
$\pi/3$	$0.8392 - 0.2738 = 0.5654$
$\pi/2$	$-0.2343 - 0.0591 = -0.2934$
π	$-0.5666 + 0.56634 = 0$

Discrete time Processing of Continuous time System





Sampling of Continuous time Signal:-

The ~~sampling~~ conversion of continuous time signal into discrete time signal is done by periodic sampling.

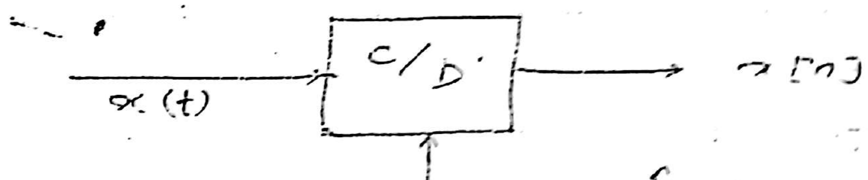


fig 1.

where, T ⇒ sampling period
f_s ⇒ sampling frequency.

$$\boxed{f_s = \frac{1}{T}}$$

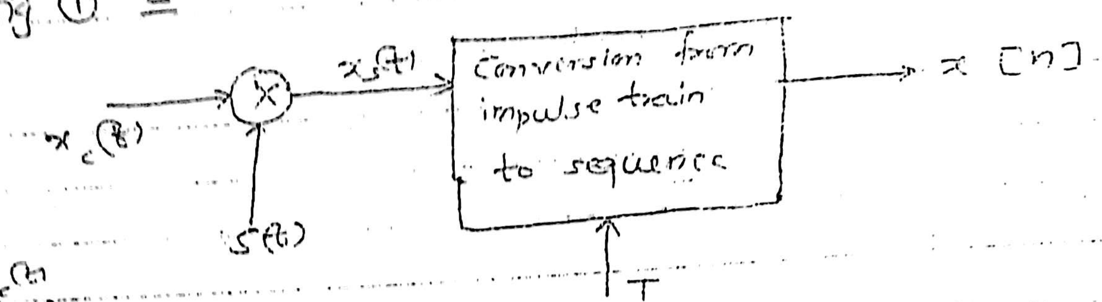
Since, sampling is non-reversable process and many continuous time signal can produce same set of discrete time samples. It is difficult to get back original continuous

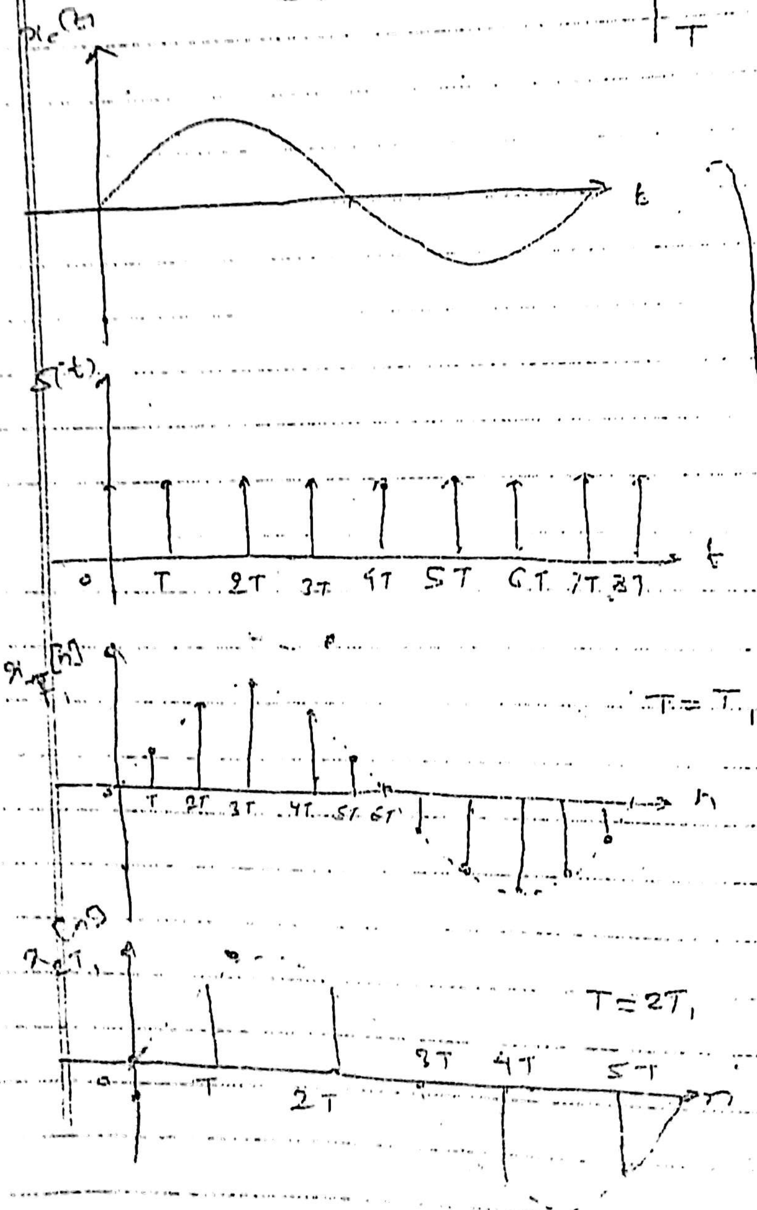
$\omega \to$

$\Omega = \frac{\omega}{T}$

time signal. So, there should be restriction of class of input signal to the sample.

fig ① ≡





$x_c(t)$ = Continuous time signal

$s(t)$ = Sampling signal

$x_s(t)$ = Sampled signal having ~~discrete~~ impulse at $t = nT$

$x[n]$ = Discrete time signal.

let $s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

then $x_s(t) = x_c(t) \cdot s(t)$

$= x_c(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t-nT)$

$= \sum_{n=-\infty}^{\infty} x_c(nT) \cdot \delta(t-nT)$

In frequency domain:

$$X_s(j\Omega) = \frac{1}{2\pi}\left[X_c(j\Omega) * S(j\Omega)\right]$$

Now:

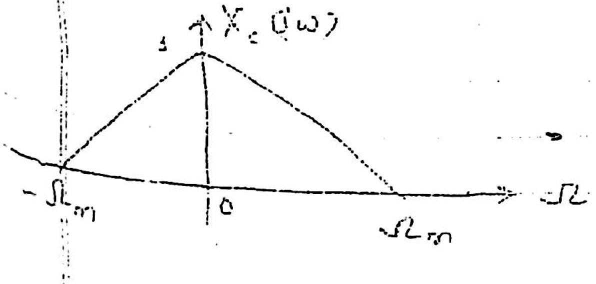
FT of $s(t)$ is:

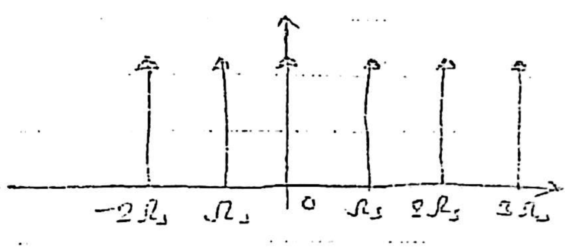
$$S(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(j\Omega - jk\Omega_s)$$

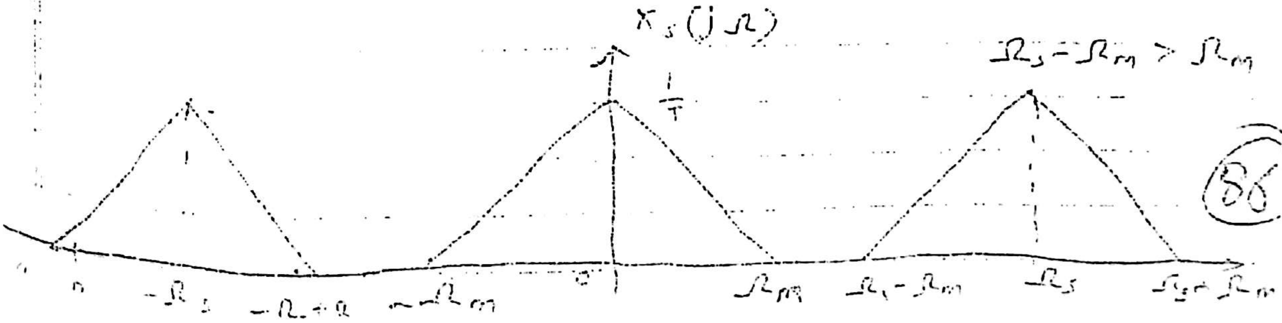
where $\Omega_s = \frac{2\pi}{T} = 2\pi f_s$

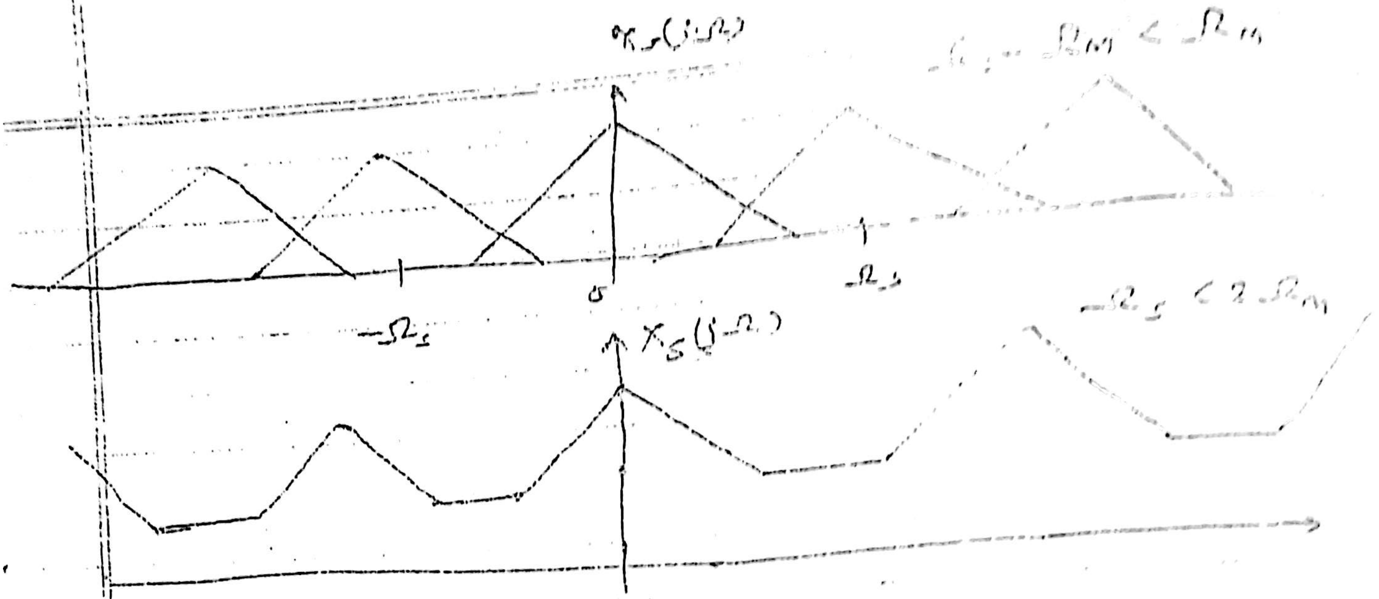
$$X_s(j\Omega) = \frac{1}{2\pi}\left[X_c(j\Omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(j\Omega - jk\Omega_s)\right]$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} \left(X_c(j\Omega - jk\Omega_s)\right)$$









For: $\Omega_s - \Omega_M < \Omega_M$

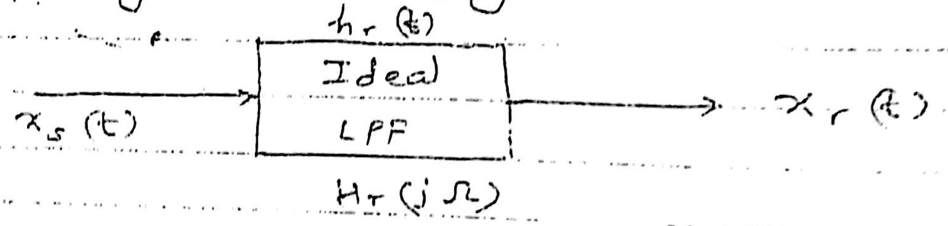
$\Omega_s < 2\Omega_M$

Original signal $x_c(t)$ can not reconstructed from its sampled signal.

For: $\Omega_s - \Omega_M > \Omega_M$

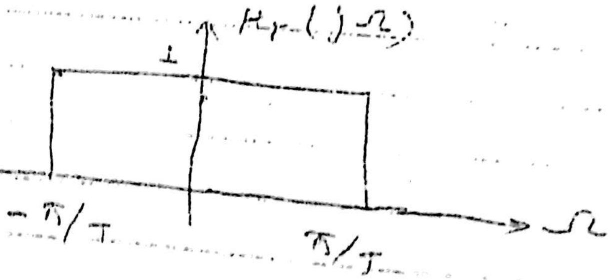
$\Omega_s > 2\Omega_M$

Then original signal $x_c(t)$ can be reconstructed via passing $x_s(t)$ through ideal LPF.



where:

$$H_r(j\Omega) = \begin{cases} 1, & \text{for } |\Omega| \le \pi/T \\ 0, & \text{for } |\Omega| > \pi/T \end{cases}$$



$$X_r(j\Omega) = H_r(j\Omega) \cdot X_s(j\Omega)$$

we have;

$$x_r(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\, \delta(t-nT)$$

FT of $x_s(t)$;

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT) \cdot e^{-j\omega nT}$$

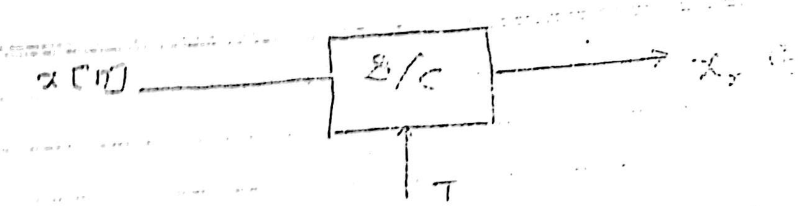
$$(x[n] = x_c(nT))$$

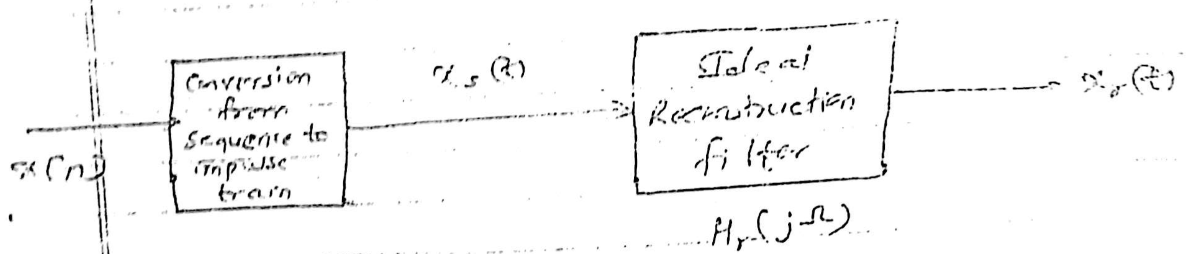
$$= \sum_{n=-\infty}^{\infty} x_c(nT)\, e^{-j\omega n}$$

$$X(e^{j\omega}) = X_s(j\Omega)\Big|_{\Omega = \frac{\omega}{T}}$$

$$= \frac{1}{T} \sum_{K=-\infty}^{\infty} X_c(j\Omega - jK\Omega_s)\Big|_{\Omega = \frac{\omega}{T}}$$

$$\boxed{X(e^{j\omega}) = \frac{1}{T} \sum_{K=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - jK\frac{2\pi}{T}\right)}$$



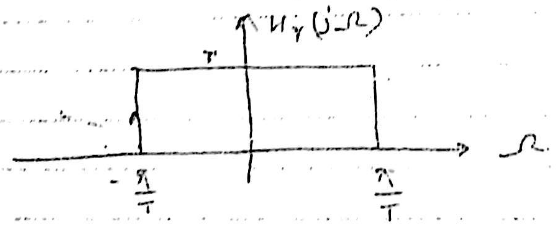


$$X_r(j\Omega) = X_s(j\Omega)\, H_r(j\Omega)$$

$$= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\Omega - jk\Omega_s) \quad H_r(j\Omega)$$

where, T.F. of ideal Reconstruction Filter:

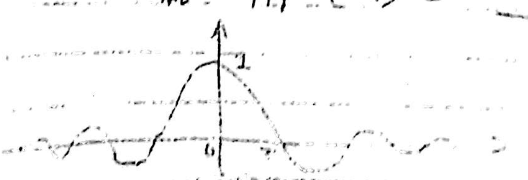
$$H_r(j\Omega) = \begin{cases} T, & \text{for } |\Omega| \le \pi/T \\ 0, & \text{for } |\Omega| > \pi/T \end{cases}$$



$$X_r(j\Omega) = \begin{cases} X_c(j\Omega), & |\Omega| \le \pi/T \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}$$

Impulse response of Ideal Reconstruction filter:

$$h_r(t) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\left(\frac{\pi t}{T}\right)}$$



$$x_r(t) = x_s(t) * h_r(t)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, \delta(t-nT) * h_r(t)$$

$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \frac{\sin \frac{\pi(t-nT)}{T}}{\frac{\pi(t-nT)}{T}}}$$

5. For a given signal;

$$x_a(t) = 5 \sin 100\pi t$$

Find i> Nyquist sampling frequency

ii> If $f_s = 200$ Hz, get $x[n]$

iii> If $f_s = 75$ Hz, get $x[n]$

iv> What will be $x_r(t)$ if signal is reconstruction from (iii).

→

$$x_a(t) = 5 \sin 2\pi \times 50\, t$$

comparing with: $x_a(t) = A \sin(2\pi f t + \phi)$

$f = 50$ Hz.

(i) $f_s \geq 2 f_{max}$

$\therefore f_N = f_s \geq 2 \times 50 = 100$ Hz

(ii) $f_s = 200$ Hz

$$x[n] = x_a(t) \Big|_{t = nT = \frac{n}{f_s}}$$

$$= 5 \sin\left(100\pi \frac{n}{200}\right)$$

$$= 5 \sin\left(\frac{\pi}{2} n\right)$$

(iii) $f_s = 75$ Hz;

$$x[n] = x_a(t)\Big|_{t=\frac{n}{f_s}}$$

$$= 5\sin\frac{100\pi n}{75}$$

$$= 5\sin\left(\frac{4\pi}{3}\right)n$$

$$= 5\sin\left(\left(2\pi - \frac{2\pi}{3}\right)n\right)$$

$$= -5\sin\left(\frac{2\pi}{3}n\right)$$

$$x[n] = -5\sin\left(\frac{2\pi}{3}n\right)$$

$$f' = \frac{1}{3}$$

$$f_r = f_s f' = 75 \times \frac{1}{3} = 25\ Hz$$

$$x_r(t) = -5\sin(2\pi \times 25t)$$

$$= -5\sin(50\pi t)$$

2072-09-18

Discrete time Property of Continuous Time signal:
(photo Copy)

C/D & D/C is not available
↳ only approximation.

→ Impulse Invariance.
↳ making analog T.F equal to Digital T.F

$$H_c(j\Omega) = H_{eff}(j\Omega)$$

$$h[n] \propto h_c(t)$$

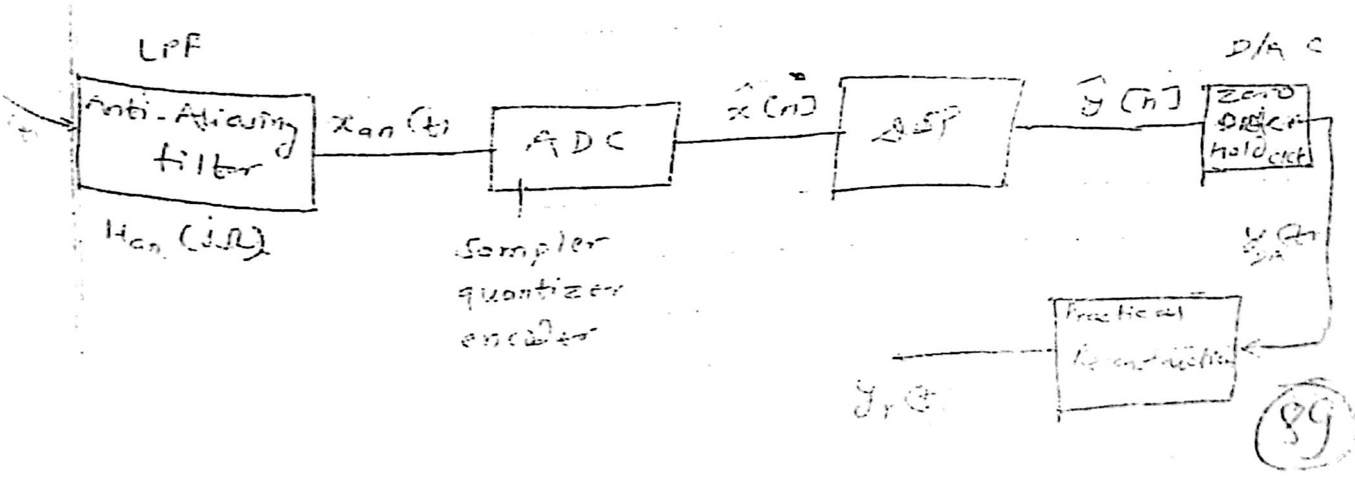
$$h[n] = T\, h_c(nT)$$

Practical Considerations:

1. C/D & D/C is not available.
approximation only.

2. Ideal LPF can not be physically realized.

3. Input signal are not available.



89

ADC:.

4. Digital Filter Design Consideration:

→ Structure :
→ Coefficient :
→ Arithmatic operation :
→ Representation:

* Number representation:.

1. Fixed Point

$$X = 10.25$$
$$= 1 \times 10 + 0 \times 10^{0} + 2 \times 10^{-1} + 5 \times 10^{-2}$$

In general:
$$X = \sum_{i=A}^{B} b_i r^{-i} \quad , 0 \le b_i \le r-1$$

$r \to$ radix or base.

$$X = \sum_{i=-A}^{B} b_i 2^{-i}$$

Integer bits :
+ve $\quad 0 \le X \le 2^{B-1}$

fraction: $\quad 0 \le X \le (1 - 2^{-B})$ eg: .0000 → min

max ⇒ 0.1111

$\frac{15}{16}$

+ve fraction bit:

$$X = \sum_{i=1}^{8} b_i \, 2^{-i}$$

$$= 0 \cdot b_1 b_2 b_3 \cdots b_8$$

-ve fraction bit:

$$X = -0 \cdot b_1 b_2 b_3 \cdots b_8$$

$$\downarrow$$

$$X_{sm} = 1 \cdot b_1 b_2 b_3 \cdots b_8$$

$$X_{1c} = 1 \cdot \bar{b}_1 \bar{b}_2 \bar{b}_3 \cdots \bar{b}_8$$

$$X_{2c} = 1 \cdot \bar{b}_1 \bar{b}_2 \bar{b}_3 \cdots \bar{b}_8 + 0.0000 \cdots 1$$

eg:

ⓐ $X = \frac{7}{8}$

0.875×2	1.75	1
0.75×2	1.5	1
0.5×2	1.0	1

$(0.111)_2$

ⓑ $X = \frac{52}{69} = 0.75362318$

$\cancel{\frac{52}{69} \times 2}$	$\cancel{\frac{104}{69}}$	$\cancel{1}$
0.75362318×2	1.507246	1
0.50724×2	1.016	1
0.016×2	0.32	0
		0

d) $X = \frac{-69}{253} = -0.272727$ (8 bit)

$X_{sm} = (1.111)_2$

$X_{1c} = (1.000)_2$

$X_{2c} = (1.001)_2$

0.272727×2	0.545454	0
0.545454×2	1.090909	1
0.090909×2	0.181818	0
0.181818×2	0.363636	0
0.363636×2	0.727272	0
0.727272×2	1.454545	1
0.454545×2	0.9090	0
0.909090×2	1.8080	1

$= (0.01000101)_2$

$X_{sm} = (1.01000101)_2$

$X_{1c} = (1.10111010)_2$

$X_{2c} = (1.10111011)_2$

Addition

~~X_{sm}~~

X_{1c} & X_{2c}

Multiplication

~~X_{sm}, X_{1c}, X_{2c}~~

eg: $\frac{4}{8} - \frac{3}{8}$

2's $\rightarrow$ $(0.100)_2 \oplus (1.101)_{2s} = \overset{\text{carry}}{1}0.001 = \frac{1}{8}$

1's $\rightarrow$ $(0.100)_2 \oplus 1.100 = 1\,0.000 + 0.001 = 0.001$

carry on LSB

(1) Floating Point Representation.

$X = 10.2 \approx$

$= 0.102 \times 10^{2}$

$X = M\,2^{E}$ → exponent

↑ mantissa

a. $X = (101)_2$

$X = 0.101\ 2^{011}$

$X = \frac{3}{8} = 0.011$

$X = 0.110\ 2^{101}$

$= 0.110\ 2^{110}$

$= 0.110\ 2^{111}$

Floating Point Representation.
(Mantissa - 6 bit)

$X = (1.001001)_2$

$= 0.100100\ 2^{001}$ → loses 2^{-6}

$X = 0.01001011$

$= 0.100101\ 2^{111}$ → loses 2^{-8}

(2)

$$H(z) = \frac{\sum_{K=0}^{M} b_K z^{-K}}{1 + \sum_{K=1}^{N} a_K z^{-K}}$$

$$\hat{H}(z) = \frac{\sum_{K=0}^{M} \hat{b}_K z^{-K}}{1 + \sum_{K=1}^{N} \hat{a}_K z^{-K}}$$

$$\hat{H}(z) = \frac{\prod_{K=1}^{M} (1 - \hat{c}_K z^{-1})}{\prod_{K=1}^{N} (1 - \hat{p}_K z^{-1})}$$

(2) $H(z) = \dfrac{1 - 0.9 z^{-1}}{1 + 0.6 z^{-1}}$

Realize with 3-fractional bit.
What will be $\hat{H}(z)$?

Sol^n

0.9×2	1.8	1
0.8×2	1.6	1
0.6×2	1.2	1

$x = (0.111)_2 \longrightarrow 0.875$

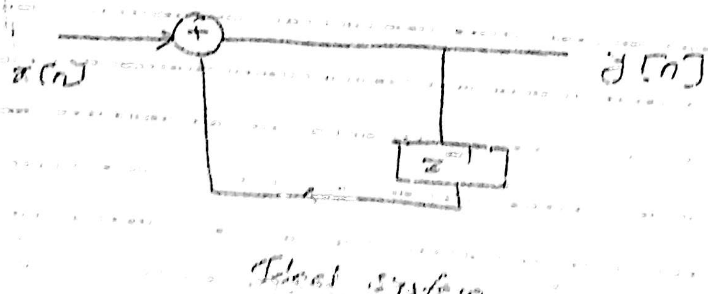
0.6×2	1.2	1
0.2×2	0.4	0
0.4×2	0.8	0

$x = (0.100)_2 \longrightarrow 0.5$

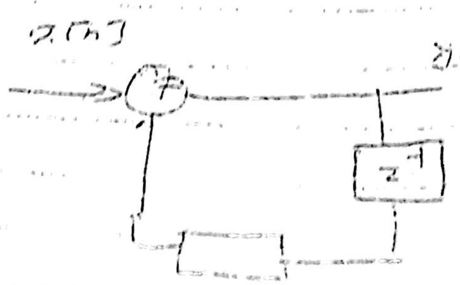
$$H(z) = \frac{1 - 0.875 z^{-1}}{1 + 0.5 z^{-1}}$$

<u>Limit Cycle effect:</u> (Round off) 2072-4-19

$y[n] = a\,y[n-1] + x[n]$



Ideal system



$a = \frac{1}{2}$

$= 0.1000$

$x[n] = \frac{15}{15}\delta[n]$

$= 0.1111\ \delta[n]$

System is relaxed, $y[n] = 0$, for $n < 0$

For, $n = 0$;

$y[0] = a\,y[-1] + x[0]$

$=$

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pter 6

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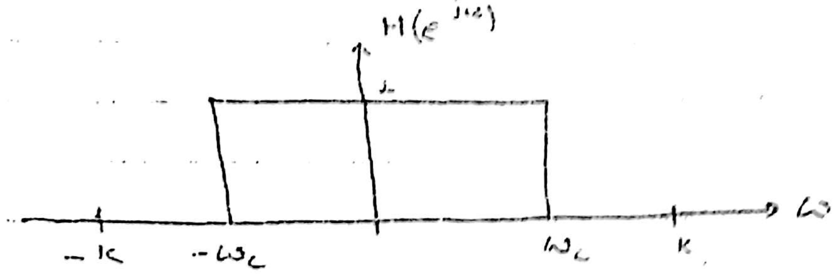
Analog Filter :

Digital Filter Design:

Ideal low pass filter :



$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

and it's impulse response :

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \cdot e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c}$$

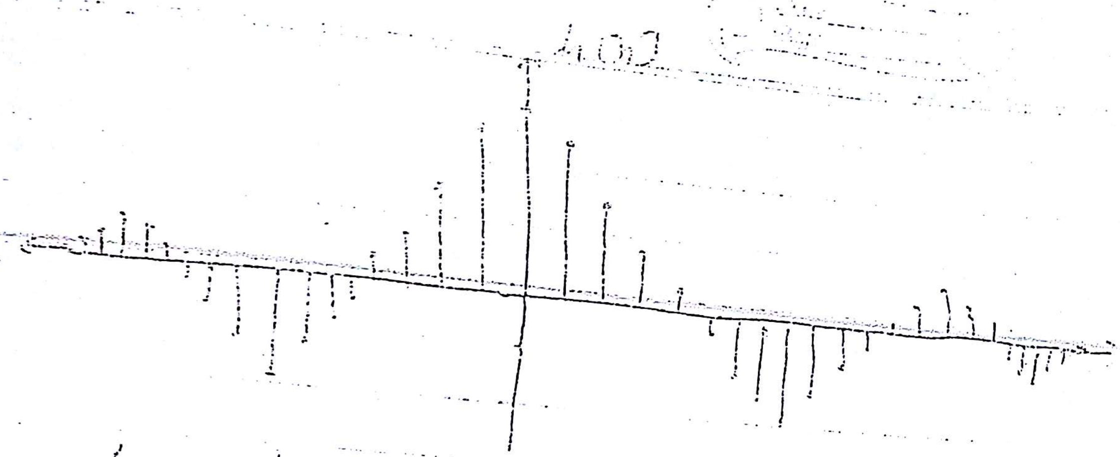
$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn}\right]$$

$$= \frac{1}{\pi n} \cdot \sin \omega_c n$$

$$= \frac{\omega_c}{\pi} \frac{\sin \pi \frac{\omega_c n}{\pi}}{\pi \cdot \frac{\omega_c n}{\pi}}$$

$$= \begin{cases} \frac{\omega_c}{\pi}, & \text{for } n = 0 \\ \frac{\sin \omega_c n}{\pi n}, & \text{for } n \neq 0 \end{cases}$$

Ch-6



Since, $h(n) \neq 0$ for $n<0$.
The given filter is non-causal and non-causal filter can not be physically realized. Hence ideal LPF can not be physically realized.

For a system to be causal:

1. The magnitude response $|H(e^{j\omega})|$ can be zero at some frequencies but can not be zero for a band of frequency.

2. The magnitude Response $|H(e^{j\omega})|$ should not be constant over a band of frequencies.

3. Transition from pass-band to stop band should not be infinitely sharp.

4. The real and imaginary part of $H(e^{j\omega})$ are interdependent and related by discrete Hilbert Transform.

Characteristics of practical filter:

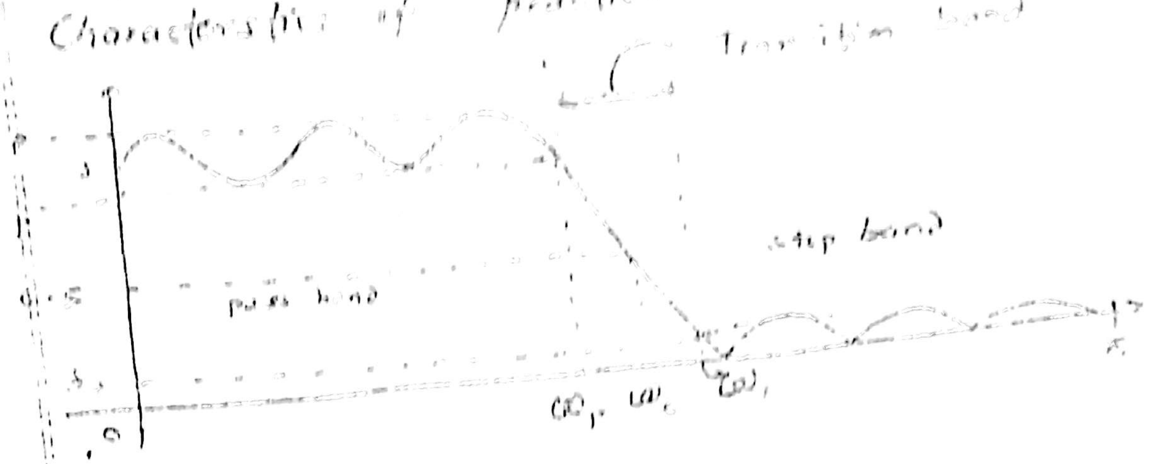


Fig: General filter response.

where,

$\delta_p \Rightarrow$ passband ripple

$\delta_s \Rightarrow$ stop band ripple

$\omega_p \Rightarrow$ pass band edge frequency

$\omega_c \Rightarrow$ cut-off or Half-power frequency

$\omega_s \Rightarrow$ stop band edge frequency.

For pass band:

$$1 - \delta_p \leq |H(e^{j\omega})| \leq 1 + \delta_p \quad \text{for} \quad |\omega| \leq \omega_p$$

Bandwidth of filter = passband = ω_p

For stopband:

$$|H(e^{j\omega})| \leq \delta_s \quad \text{for} \quad \omega_s \leq |\omega| \leq \pi$$

Maximum passband Ripple:

$$\alpha_s = \alpha_{max} = -20 \log_{10}(1 - \delta_p) \text{ dB}.$$

minimum stop band attenuation:

$$\alpha_s = \alpha_{min} = -20 \log_{10}(\delta_s)$$

Frequency Normalization:

f_p = pass band frequency
f_t = Sampling frequency

$$\boxed{\omega_p = \frac{2\pi f_p}{f_t}}$$

eg:

$$f_p = 3 \text{ kHz}$$
$$f_t = 25 \text{ kHz}.$$

$$\omega_p = \frac{3 \times 2\pi}{25}$$

$$= \frac{6\pi}{25}$$ (always < π/[illegible])

[illegible] IIR [illegible]

IIR filters are usually designed by transferring of prototype analog transfer function.

Method I ⇒ Impulse Invariance Method

→ Basic idea is to get the T.F. of digital filter which impulse response is sampled version of impulse response of prototype analog transfer function.
Let a causal & stable analog T.F. $H_a(s)$ and its impulse response is given by inverse Laplace transform.

$$h_a(t) = \mathcal{L}^{-1}\{H_a(s)\}$$

Now, taking the discrete time samples of $h_a(t)$
we get,

$$g[n] = h_a(nT), \quad \text{for } n = 0, 1, 2, \dots$$

T = Sampling period

Now, taking z-transform;

$$G(z) = Z\{g[n]\}$$

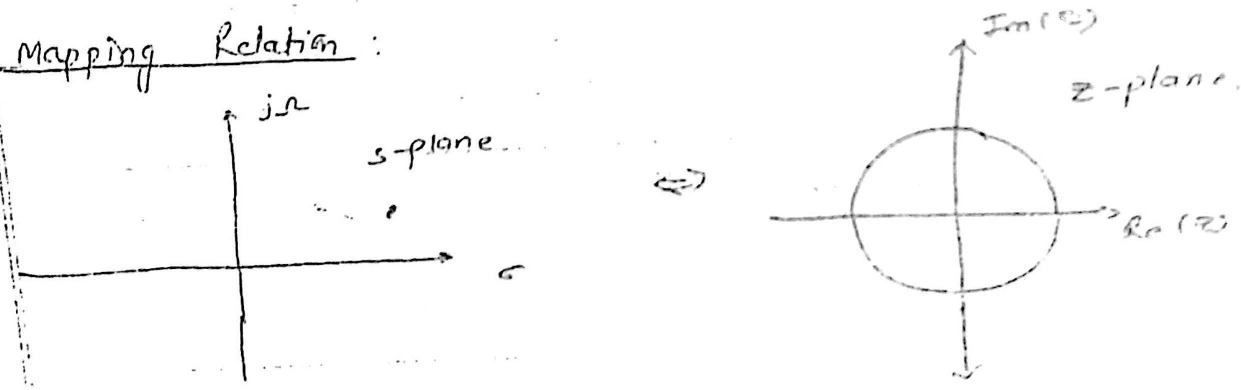
From the spectral properties of sampling theory, we get;

$$G(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(j\frac{\Omega}{T} - jk\frac{2\pi}{T}\right)$$

For, $z = e^{j\omega}$;

$$G(z) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(s - jk\frac{2\pi}{T}\right)\Bigg|_{s = \frac{1}{T}\ln(z)}$$

<u>Mapping Relation</u>:



$$s = \frac{1}{T}\ln z$$

$$z = e^{sT}$$

For $z = re^{j\omega}$ and $s = \sigma + j\Omega$

$$re^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$= e^{\sigma T} \cdot e^{j\Omega T}$$

$j\Omega = j\omega$

$\boxed{\Omega = \frac{\omega}{T}}$

$\rightarrow |z| = e^{\sigma T}$

when, $\sigma = 0$, $|z| = 1$
when, $\sigma < 0$, $|z| < 1$
when, $\sigma > 0$, $|z| > 1$

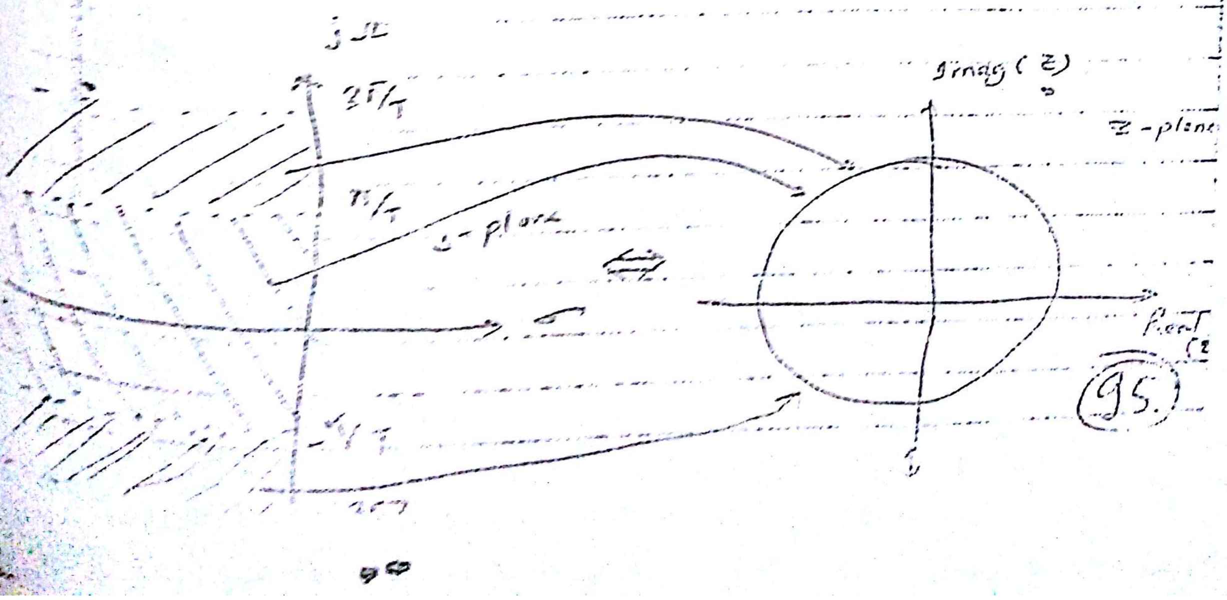
⇒ $j\Omega$-axis or imaginary axis in ~~root~~ s-plane match to unit circle in z-plane.

⇒ Left half s-plane match to inside the unit circle ~~#~~ in z-plane.

⇒ Right half s-plane match to outside the unit circle in z-plane.

$$z = e^{sT}$$
$$= e^{sT} \cdot e^{\pm jk2\pi} \quad \text{for } k = 0, \pm 1, \pm 2, \ldots$$
$$= e^{\left(s \pm jk\frac{2\pi}{T}\right)T}$$

So, every $\left(s \pm jk\frac{2\pi}{T}\right)$ maps to a single point in z-plane & this method is many to one mapping i.e. Analog frequency 'Ω' to digital frequency 'ω' is many to one.



(95.)

$-\frac{\pi}{T} < \Omega \le \frac{\pi}{T}$ maps to entire z-plane

$-\frac{3\pi}{T} \le \Omega < -\frac{\pi}{T}$ " " "

$\frac{\pi}{T} < \Omega \le \frac{3\pi}{T}$ " " "

For example :

If $H_a(j\Omega) = 0$, for $|\Omega| > \frac{\pi}{T}$

i.e. for low pass filter.

then, $G(e^{j\omega}) = \frac{1}{T} H_a\left(j \frac{\omega}{T}\right)$, $|\omega| \le \pi$.

In this case no aliasing occurs.

But for High Pass, Band Pass & Band Stop filter → aliasing will occur. Hence this method is not suitable to design HP, BP & BS filter.

How to Design :

First order analog Transfer function :

$$H_a(s) = \frac{A}{s+\alpha}, \quad \alpha > 0$$

$$h_a(t) = A e^{-\alpha t} u(t)$$

$$g[n] = h_a[nT] = A e^{-\alpha n T} u[n]$$

Taking z-transform ;

$$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

$$G(z) = \sum_{n=-\infty}^{\infty} A e^{-\alpha n T} u[n] \cdot z^{-n}$$

$$= A \sum_{n=0}^{\infty} \left(e^{-\alpha T} z^{-1}\right)^n$$

$$= \frac{A}{1 - e^{-\alpha T} z^{-1}}$$

//

2) 2nd order Analog T. F.

a) $$H_a(s) = \frac{\alpha}{(s+\beta)^2 + \lambda^2}$$

then:

$$G(z) = \frac{z e^{-\beta T} \cdot \sin \lambda T}{(s+\beta)^2 + \lambda^2}$$

b) $$H_a(s) = \frac{s+\beta}{\cancel{(s+\beta)^2 + \lambda^2}\ z^2 - 2z e^{-\beta T} \cos \lambda T + e^{-2\beta T}}$$

then,

$$G(z) = \frac{z^2 - z e^{-\beta T} \cos \lambda T}{\cancel{(s+\beta)^2 + \lambda^2}\ z^2 - 2z e^{-\beta T} \cos \lambda T + e^{-2\beta T}}$$

c) $$H_a(s) = \frac{cs + D}{(s+\beta)^2 + \lambda^2}$$

$$= \frac{c(s + D/c)}{(s+\beta)^2 + \lambda^2}$$

$$= \frac{c(s+\beta) + A\lambda}{(s+\beta)^2 + \lambda^2} \quad \text{where } A = \frac{D - \beta c}{\lambda}$$

then;

$$G(z) = \frac{cz^2 + z e^{-\beta T}(A \sin \lambda T - c \cos \lambda T)}{z^2 - 2z e^{-\beta T} \cos \lambda T + e^{-2\beta T}}$$

96

97

$\omega_p = 0.25\pi$ $\omega_s = 0.55\pi$

$\alpha_{max} = \alpha_p = 0.5\ dB$ $\alpha_s = 15\ dB = \alpha_{min}$

Sampling freq $= 1\ Hz$

$$T = \frac{1}{f_s} = 1$$

Impulse - Invariance method;

$$\Omega = \frac{\omega}{T}$$

$$\Omega_p = \omega_p = 0.25\pi$$

$$\Omega_s = \omega_s = 0.55\pi$$

- order of butterworth filter;

$$n = \frac{\log_{10}\left\{\left(10^{\alpha_{min}/10} - 1\right) / \left(10^{\alpha_{max}/10} - 1\right)\right\}}{2\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{\log_{10}\left\{\left(10^{15/10} - 1\right) / \left(10^{0.5/10} - 1\right)\right\}}{2\log_{10}\left(\frac{0.55\pi}{0.25\pi}\right)}$$

$= 3.5058$
$\simeq 4$

From table;

$$H_4(S) = \frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)}$$

with $\Omega_c = 1$.

Now taking partial fraction;

$$H_4(S) = \frac{A_1 s + A_2}{s^2+0.7654s+1} + \frac{B_1 s + B_2}{s^2+1.8478s+1}$$

Solve 4 above eq^{ns} after comparing to get A_1, A_2, B_1 and B_2.

$$H_4(S) = \frac{-0.9238s - 0.7071}{s^2+0.7654s+1} + \frac{0.9238s + 1.7071}{s^2+1.8478s+1}$$

For given specification; $1/2N$

$$\Omega_c = \Omega_p \left(\frac{1}{10^{\alpha_{max}/10} - 1}\right)^{1/2N}$$

$$= 0.25\pi \left(\frac{1}{10^{3.5/10} - 1}\right)^{\frac{1}{2\times 4}}$$

$= 1.08158$

Change to new $\Omega_c = 1.08158$

Replace $s \rightarrow \frac{s}{\Omega_c}$

$$H_{4_{new}}(s) = H_4(s)\Big|_{s = \frac{s}{1.021568}}$$

$$= \frac{-0.9238\frac{s}{1.021568} - 0.7071}{\left(\left(\frac{s}{1.021568}\right)^2 + 0.7654\frac{s}{1.021568} + 1\right)} + \frac{0.9238\frac{s}{1.021568} + 1.7071}{\left\{\left(\frac{s}{1.021568}\right)^2 + 1.84776\frac{s}{1.021568} + 1\right\}}$$

$$\frac{-0.94381s - 0.73798}{(s+0.3909)^2 + (0.94383)^2} + \frac{0.94381s + 1.7816}{(s+0.94383)^2 + (0.39069)^2}$$

Now, comparing with general form, $\frac{Cs + D}{(s+\beta)^2 + \lambda^2}$, we get,

1) $C_1 = -0.94381$ $D_1 = -0.73798$

$\beta_1 = 0.3909$ $\lambda_1 = 0.94383$

$$A_1 = \frac{D_1 - \beta_1 C_1}{\lambda_1} = 0.39093$$

2) $C_2 = 0.94381$ $D_2 = 1.7816$

$\beta_2 = 0.94323$ $\lambda_2 = 0.39069$

$$A_2 = \frac{D_2 - \beta_2 C_2}{\lambda_2} = 2.2789$$

$$G_1(z) = \frac{C_1 z^2 + z e^{-\beta_1 T}(A_1 \sin\lambda_1 T - C_1 \cos\lambda_1 T)}{z^2 - 2z e^{-\beta T}\cos\lambda_1 T + e^{-2\beta_1 T}}$$

$$= \frac{-0.94381z^2 + z e^{-0.3909T}(0.39093\sin(0.94383) - (-0.94381)\cos(0.94383))}{z^2 - 2z e^{-0.3909}\cos(0.94383) + e^{-2\times 0.3909}}$$

$$= \frac{-0.94381\, z^2 + 0.1604\, z}{z^2 - 0.7937\, z + 0.4575}$$

$$= \frac{-0.94381 + 0.1604\, z^{-1}}{1 - 0.7937\, z^{-1} + 0.4575\, z^{-2}}$$

Similarly,

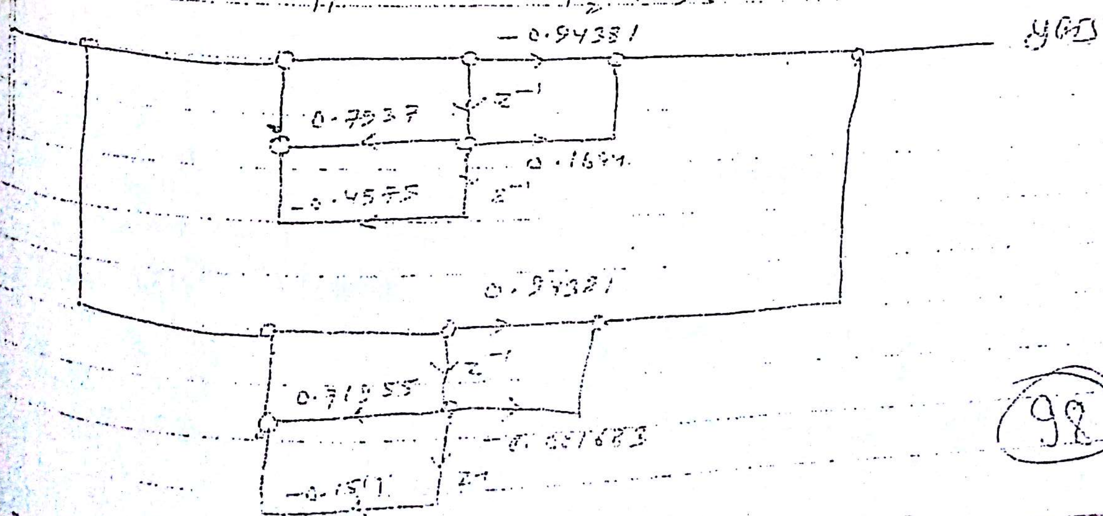
$$G_2(z) = \frac{C_2 z^2 + z e^{-\beta_2 T}\left(A_2 \sin \lambda_2 T - C_2 \cos \lambda_2 T\right)}{z^2 - 2 z e^{-\beta_2 T} \cos \lambda_2 T + e^{-2\beta_2 T}}$$

$$= \frac{0.94381\, z^2 - 0.001633\, z}{z^2 - 0.71955\, z + 0.1514}$$

$$= \frac{0.94381 - 0.001633\, z^{-1}}{1 - 0.71955\, z^{-1} + 0.1514\, z^{-2}}$$

Required Transfer function:

$$G(z) = G_1(z) + G_2(z)$$



$\omega_p = 0.25\pi$, $\omega_s = 0.55\pi$

$\alpha_p = 0.5\ dB$ $\alpha_s = 15\ dB$

$T = 1$

Chebyshev Filter:
Impulse Invariance method.

2020 - 4 - 25

Method 2

Bilinear transformation method::

⇒ Impulse invariance method to have aliasing problem that is due to many to one mapping from s-plane to z-plane. To avoid aliasing, there should one-to-one mapping, and Bilinear Transformation attemps to do this.

Mapping from s-plane to z-plane is given by:

$$S = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$G(z) = H_a(s)\Big|_{s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}}$$

Steps to perform Bilinear transformation::

1. ~~Frequency~~ Inverse Bilinear transformation is applied to the digital ~~frequency~~ filter specification of prototype analog filter.
2. Analog Transfer function is determined.

5. Bilinear Transformation is applied on analog transfer function to get digital transfer function.

Assume $T = 2$

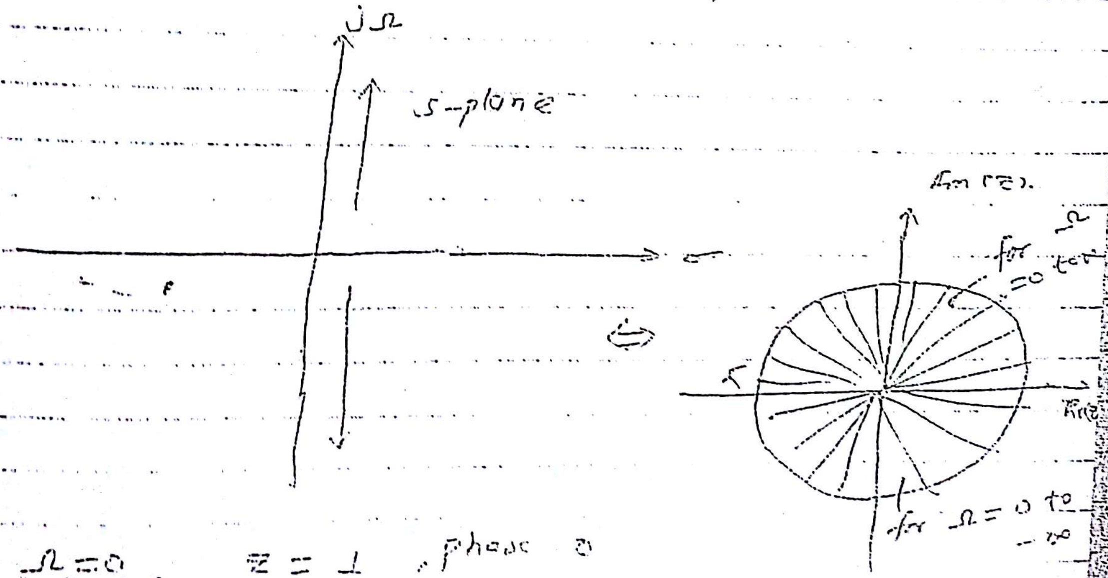
$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

or, $$z = \frac{1+s}{1-s}$$

For, $s = j\Omega$, $\quad z = \frac{1+j\Omega}{1-j\Omega} = \frac{1+j\Omega}{1-j\Omega} \times \frac{1+j\Omega}{1+j\Omega}$

$|z| = 1 \qquad = \frac{1-\Omega^2}{1+\Omega^2} + j\,\frac{2\Omega}{1+\Omega^2}$

Imaginary - axis or $j\Omega$ axis in s-plane maps to unit circle in z-plane.



For, $\Omega = 0$, $\quad z = 1$, phase 0

For, $\Omega = \infty$, $\quad z = -1 + j0$, phase π

When, $\Omega = 0$ to ∞, it maps to upper-half circle in z-plane

For, $\Omega = -\infty$, $\quad z = -1 - j0$, phase $-\pi$

When, $\Omega = 0$ to $-\infty$, it maps to lower half-circle in z-plane

Since, $s = j\Omega$ for $-\infty < \Omega < \infty$ maps to $|z| = 1$ for $-\pi < \omega < \pi$, there is no multiple mapping.

Again;
$$z = \frac{1+s}{1-s}$$

$$s = \sigma + j\Omega$$

$$z = \frac{1+\sigma+j\Omega}{1-\sigma-j\Omega}$$

$$|z|^2 = z \cdot z^* = \frac{1+\sigma+j\Omega}{1-\sigma-j\Omega} \times \frac{1+\sigma-j\Omega}{1-\sigma+j\Omega}$$

$$= \frac{(1+\sigma)^2+\Omega^2}{(1-\sigma)^2+\Omega^2}$$

when, $\sigma = 0$,

$$|z| = 1$$

Imaginary axis or $j\Omega$ axis in s-plane maps to unit circle in z-plane.

When, $\sigma < 0$; $|z| < 1$

→ Left half s-plane maps to inside the unit circle in z-plane.

when, $\sigma > 0$, $|z| > 1$.

→ Right half s-plane maps to outside the unit circle in z-plane.

Relationship between Ω & ω

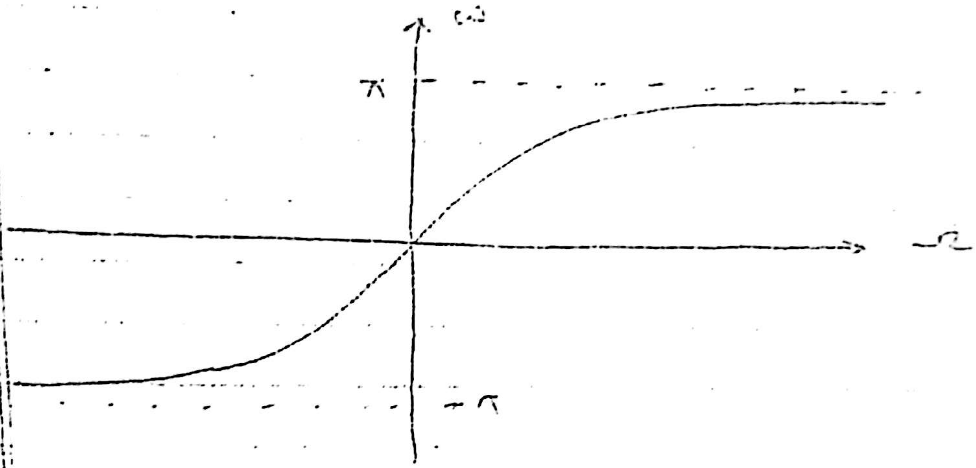
For $s = j\Omega$, $|z| = 1$

$z = e^{j\omega}$

$$s = \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$j\Omega = \frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = j \tan \frac{\omega}{2}$$

$$\Omega = \tan \frac{\omega}{2}$$



Since : $s = j\Omega$ for $-\infty < \Omega < \infty$ maps to $|z| = 1$ for $-\pi < \omega < \pi$; there is some distortion called <u>frequency warping</u>

Example: Design the digital IIR low pass filter using Bilinear transformation method, the designed filter must be Butterworth LP, the maximum gain of desired filter is 0.5 dB below 0 dB at a frequency of 0.25 π rad. The gain of filter is -15 dB at a stop band edge frequency 0.55 π rad. Assume sampling frequency 0.5 Hz.

(100)

Given: $\omega_p = 0.25\pi$ $\omega_s = 0.55\pi$

$\alpha_{min} = \alpha_p = 0.5\,dB$ $\alpha_{max} = \alpha_s = 15\,dB$

Sampling frequency $= 0.5\,Hz$

$$T = \frac{1}{0.5} = 2$$

For Bilinear transformation:

$$\boxed{\Omega = \frac{2}{T} \tan\frac{\omega}{2}}$$

$$\Omega_p = \tan\frac{\omega_p}{2} = \tan\frac{0.25\pi}{2} = 0.41421$$

$$\Omega_s = \tan\frac{\omega_s}{2} = \tan\frac{0.55\pi}{2} = 1.17085$$

Order of Butterworth filter:

$$N = \frac{\log_{10}\left[\left(10^{\alpha_{min}/10}-1\right)/\left(10^{\alpha_{max}/10}-1\right)\right]}{2\log_{10}\left(\frac{\Omega_s}{\Omega_p}\right)}$$

$$= \frac{\log_{10}\left[\left(10^{15/10}-1\right)/\left(10^{0.5/10}-1\right)\right]}{2\log\left(\frac{1.17085}{0.41421}\right)}$$

$$= 2.6586$$

$$\simeq 3$$

From table;

$$H_3(s) = \frac{1}{(s+1)(s^2+s+1)}$$

with $\Omega_c = 1$

$$H_N(s) = \prod_{k=0}^{N-1} \frac{1}{(s-s_k)}$$

where $s_k = \Omega_c\, e^{j\pi/2} \cdot e^{\frac{j(2k+1)\pi}{2N}}$

for $k = 0, 1, 2, \ldots N-1$

For given specification;

$$\Omega_c = \Omega_p \left(\frac{1}{10^{\alpha_{max}/10} - 1} \right)^{1/2N}$$

$$= 0.41421 \left[\frac{1}{10^{0.5/10} - 1} \right]^{\frac{1}{2\times 3}}$$

$$= 0.588143$$

To set, $\Omega_c = 0.588143$

Replace $s \rightarrow \frac{s}{\Omega_c}$

$$H_{3_{new}}(s) = H_3(s) \Big|_{s = \frac{s}{0.588143}}$$

$$= \frac{1}{\left(\frac{s}{0.588143} + 1\right)\left(\left(\frac{s}{0.588143}\right)^2 + \frac{s}{0.588143} + 1\right)}$$

$$= \frac{0.2034}{(s + 0.588143)(s^2 + 0.588143s + 0.345912)}$$

Applying Bilinear transformation;

$$s \Rightarrow \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}$$

$$s \Rightarrow \frac{1 - z^{-1}}{1 + z^{-1}}$$

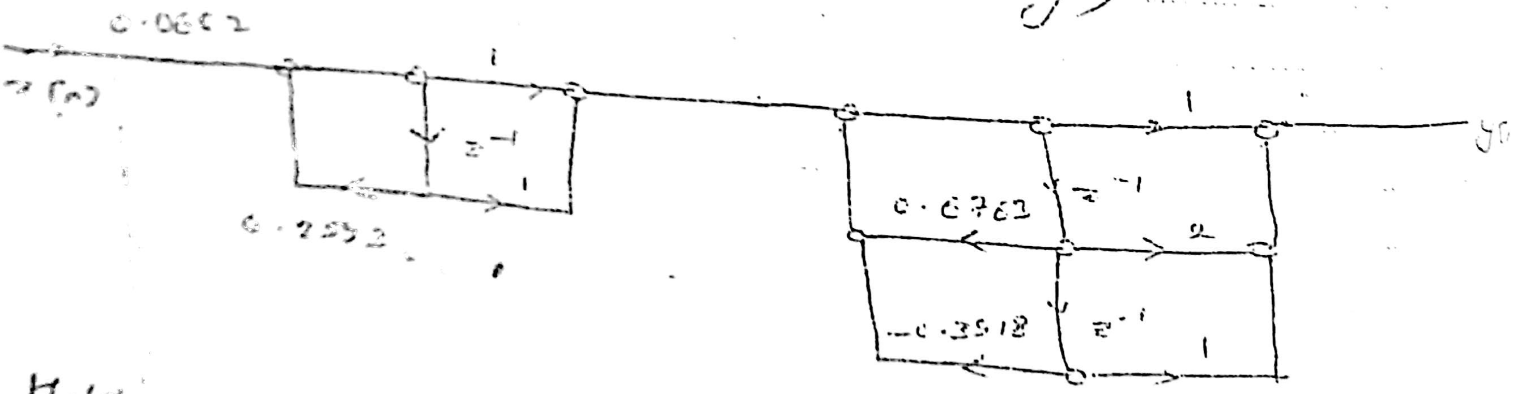
$$G(z) = H_{3,new}(s) \Big|_{s = \frac{1 - z^{-1}}{1 + z^{-1}}}$$

$$\left(\frac{1-z^{-1}}{1+z^{-1}}+0.588143\right)\left\{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+0.588143\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.395912\right\}$$

$$= \frac{0.2074\,(1+z^{-1})^3}{(1.588143-0.4185z^{-1})(1.534-1.3081z^{-1}+0.7577z^{-2})}$$

$$= 0.0662\left(\frac{1+z^{-1}}{1-0.2593z^{-1}}\right)\left(\frac{1+2z^{-1}+z^{-2}}{1-0.6763z^{-1}+0.3518z^{-2}}\right)$$

(Cascading)



H.W

$\omega_p = 0.25\pi$, $\omega_s = 0.55\pi$

$\alpha_p = 0.5$ dB , $\alpha_s = 15$ dB

$T=2$

Chebyshev filter.

⑧ Given:

$$H_a(s) = \frac{s+0.1}{s^2+0.2s+9.01}$$

Convert it into digital filter by using impulse invariance method with $T=0.1$.

Sol:

$$H_a(s) = \frac{s+0.1}{s^2+0.2s+9.01}$$

$$= \frac{s+0.1}{s^2+2 \times 0.1 + (0.1)^2 + 3^2}$$

$$= \frac{s+0.1}{(s+0.1)^2+3^2}$$

$$\left\{ \frac{s+\beta}{(s+\beta)^2+\alpha^2} \right\}$$

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2-(+j3)^2}$$

$$= \frac{s+0.1}{(s+0.1-j3)(s+0.1+j3)}$$

$$= \frac{1/2}{s+0.1+j3} + \frac{1/2}{s+0.1-j3}$$

* $$\frac{A}{s+\alpha} \xrightarrow[\text{Invariance}]{\text{Impulse}} \frac{A}{1-e^{-\alpha T}z^{-1}}$$

$$G(z) = \frac{1/2}{1-e^{-(0.1+j3)T}z^{-1}} + \frac{1/2}{1-e^{-(0.1-j3)T}z^{-1}}$$

$T=0.1 \Rightarrow$

$$G(z) = \frac{1/2}{1-e^{-0.01}e^{-j0.3}z^{-1}} + \frac{1/2}{1-e^{-0.01}e^{j0.3}z^{-1}}$$

$$\left(1 - e^{-0.01} e^{-j0.3\pi} z^{-1}\right)\left(1 - e^{-0.01} e^{j0.3\pi} z^{-1}\right)$$

$$= \frac{1 - 2 * e^{-0.01} \cos(0.3) z^{-1}}{\left(1 - 2 * e^{-0.01} \cos 0.3 Z^{-1} + e^{-0.02} Z^{-2}\right)}$$

Given $H_a(s) = \dfrac{s + 0.1}{s^2 + 0.2s + 16.01}$

Convert it into Digital filter with $\omega_r = \pi/2$ using Bilinear transformation

Given. $H_a(s) = \dfrac{\Omega_c}{s + \Omega_c}$

$$H_a(s) = \frac{s + \beta}{(s+\beta)^2 + \Omega_r^2}$$

Ω_c = Cut off frequency

Convert into digital filter with $\omega_c = 0.2\pi$ using Bilinear transformation.

Matched Z-transform Method :)

In this method, poles and zeroes of analog filter in S-domain is directly mapped into poles and zeroes of Z-plane.

In analog filter

$$H_a(s) = \frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}$$

Then in digital domain:

$$H(z) = \frac{\prod_{k=1}^{M}(1 - e^{z_k T} z^{-1})}{\prod_{k=1}^{N}(1 - e^{p_k T} z^{-1})}$$

where p_k and z_k are poles and zeroes of analog filter and T is sampling period. In this method each $(s-a)$ factor in analog domain is mapped to $(1 - e^{aT} z^{-1})$

i.e. $(s-a) \longrightarrow (1 - e^{aT} z^{-1})$

Example :-

$$H_a(s) = \frac{s + 0.1}{s^2 + 0.2s + 9.01}$$

Convert it into digital filter by using impulse invariance and matched z-transform method. Compare both the result ~~time~~ selecting $T = 0.1$ sec.

By using impulse Invariance method:

$$G(z) = \frac{1 - 2e^{-0.01}\cos(0.3)z^{-1}}{1 - 2e^{-0.01}\cos(0.3)z^{-1} + e^{-0.02}z^{-2}}$$

Now using matched z-transform method:

$$H_a(s) = \frac{s+0.1}{(s+0.1+j3)(s+0.1-j3)}$$

$$(s-a) \xrightarrow[\text{z-transform}]{\text{matched}} (1 - e^{aT}z^{-1})$$

$$G(z) = \frac{1 - e^{-0.1T}}{(1 - e^{(-0.1-j3)T}z^{-1})(1 - e^{(-0.1+j3)T}z^{-1})}$$

$$= \frac{1 - e^{-0.01}z^{-1}}{(1 - e^{-0.01}e^{-j0.3}z^{-1})(1 - e^{-0.01}e^{j0.3}z^{-1})}$$

$$= \frac{1 - e^{-0.01}z^{-1}}{1 - 2e^{-0.01}\cos(0.3)z^{-1} + e^{-0.02}z^{-2}}$$

The digital filter obtained from the both method has the poles at the same place but location of zero at different.

* Spectral Transformation (Frequency Transformation)

To design highpass, bandpass and band-stop digital IIR filter, we have two options

⟨i⟩ Perform frequency transformation of analog prototype filter and then convert it into a digital filter by mapping from s-plane to z-plane.

⟨ii⟩ Convert LP analog filter into LP digital filter and perform frequency transformation in z-domain.

Spectral transformation (Frequency Transformation) in analog domain:

1) Low pass to low pass:

Ω_p = old pass band edge frequency of LPF

Ω_p' = new " " " " "

$$s \rightarrow \frac{\Omega_p}{\Omega_p'} s$$

$$\therefore H_{LP,new}(s) = H_{LP}\left(\frac{\Omega_p}{\Omega_p'} s\right)$$

2) Low pass to High Pass:

Ω_p = pass band edge frequency of LPF

Ω_p' = pass band endge " " HPF

$$s \rightarrow \frac{\Omega_p \Omega_p'}{s}$$

$$\therefore H_{HP}(s) = H_{LP}\left(\frac{\Omega_p \Omega_p'}{s}\right)$$

3) Lowpass to band pass:

$\Omega_p \Rightarrow$ pass band edge frequency of BPF

$\Omega_l \Rightarrow$ lower band edge freq. of BPF

$\Omega_u \Rightarrow$ upper " " " " "

$$s \rightarrow \frac{\Omega_p(s^2 + \Omega_u \Omega_l)}{s(\Omega_u - \Omega_l)}$$

$$\therefore H_{BP}(s) = H_{LP}\left[\frac{\Omega_p(s^2 + \Omega_u \Omega_l)}{s(\Omega_u - \Omega_l)}\right]$$

4) LP to Band stop:

$\Rightarrow$ Inverse of band pass

(10<1)

$$s \rightarrow \Omega_p \cdot \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$$

$$\therefore H_{BS}(s) = H_{LP}\left(\Omega_p \cdot \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}\right)$$

Spectral transformation (frequency transformation) in digital domain:-

⇒ In digital domain frequency transformation is similar to as in analog domain. Replace variable z^{-1} by a rational function $g(z^{-1})$ which must satisfy following properties:

1) Mapping z^{-1} to $g(z^{-1})$ must mapped points inside the unit circle of the z-plane into itself.

2) The unit circle must also be mapped into unit circle itself.

$$z^{-1} \longrightarrow g(z^{-1}) \quad , |z| = r = 1$$

i.e. $e^{-j\omega} \longrightarrow g(e^{-j\omega})$

If we have, low-pass digital filter with pass band edge frequency ω_p then; it is converted into Highpass, Band pass & Band-stop digital filter by selecting z^{-1} to $g(z^{-1})$ in following way:

Type of Transformation	Transformation	Parameters
Low Pass to Low pass	$z^{-1} = \dfrac{z^{-1} - a}{1 - a z^{-1}}$	$\omega_p' \Rightarrow$ new pass band edge freq. of LPF $a = \dfrac{\sin\left(\frac{\omega_p - \omega_p'}{2}\right)}{\sin\left(\frac{\omega_p + \omega_p'}{2}\right)}$
Low pass to High pass	$z^{-1} \Rightarrow -\dfrac{z^{-1} + a}{1 + a z^{-1}}$	$\omega_p' \Rightarrow$ passband edge freq. of HPF. $a = -\dfrac{\cos\left(\frac{\omega_p + \omega_p'}{2}\right)}{\cos\left(\frac{\omega_p - \omega_p'}{2}\right)}$
Low pas to Bandpass	$z^{-1} \Rightarrow -\left(\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}\right)$	$\omega_u \Rightarrow$ upper bandedge freq. of BPF $\omega_l =$ lower bandedge freq. of BPF $a_1 = -\dfrac{2k\alpha}{k+1}$ $a_2 = \dfrac{k-1}{k+1}$

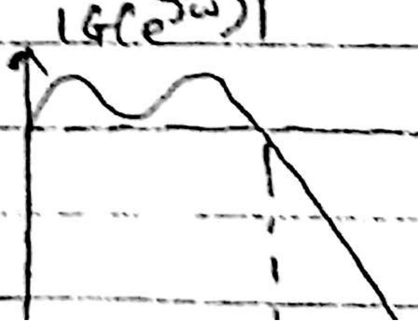
$$\alpha = \frac{\cos\left(\frac{\omega_u + \omega_l}{2}\right)}{\cos\left(\frac{\omega_u - \omega_l}{2}\right)}$$

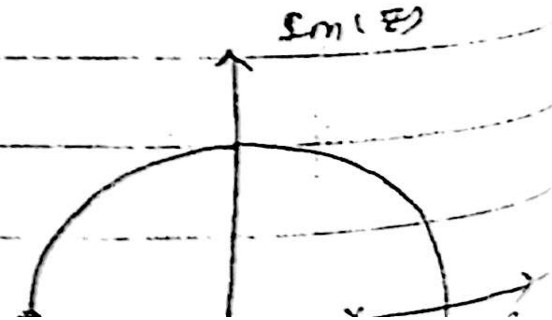
$$G_{LP}(z) = 0.245\left(\frac{1+z^{-1}}{1-0.509z^{-1}}\right)$$

with $\omega_p = 0.2\pi$

Convert it into Band pass filter having lower band edge frequency $(\omega_L) = 2\pi/5$ & upper band edge frequency $(\omega_u) = 3\pi/5$.

Sol





$k = \cos(0 \cdot \pi) \tan(0 \cdot \pi)$

$= 1$

$$\alpha = \frac{\cos\left(\frac{\omega_u + \omega_l}{2}\right)}{\cos\left(\frac{\omega_u - \omega_l}{2}\right)} = \frac{\cos\left(\frac{\frac{2\pi}{3} + \frac{2\pi}{5}}{2}\right)}{\cos\left(\frac{\frac{2\pi}{3} - \frac{2\pi}{5}}{2}\right)}$$

$$= \frac{\cos \pi/2}{\cos \pi/10}$$

$$= 0$$

$$a_1 = \frac{2k\alpha}{k+1} = 0$$

$$a_2 = \frac{k-1}{k+1} = \frac{1-1}{1+1} = 0$$

$$z^{-1} \Rightarrow - \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

$$z^{-1} \Rightarrow - z^{-2}$$

Date ______
Page ______

Q. $G_{LP}(z) = 0.245\left(\frac{1+z^{-1}}{1-0.509z^{-1}}\right)$ with $\omega_p = 0.2\pi$

Convert it into High pass filter with $\omega_p' \Rightarrow 0.3\pi$

Q. Design a digital IIR filter using bilinear transformation method, the designed filter must be B.W. LP with following digital specification: $\omega_p = 0.25\pi$ $\omega_s = 0.55\pi$

$\alpha_p = 0.5\ dB$ $\alpha_s = 15\ dB$

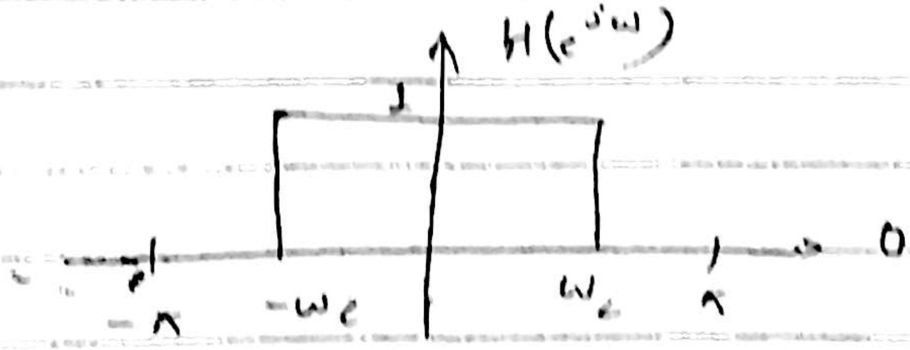
$T = 2\ sec.$

Convert this LPF into HPF having passband edge frequency $\omega_p = 0.35\pi$.

where, $h[n]$ has finite length.

Ideal low pass filter:

$$H(e^{j\omega}) = \begin{cases} 1 & , \text{ for } |\omega| \leq \omega_c \\ 0 & , \text{ for } \omega_c < |\omega| \leq \pi \end{cases}$$



and its impulse response:

$$h[n] = \begin{cases} \omega_c/\pi & , \text{ for } n = 0 \\ \dfrac{\sin \omega_c n}{n\pi} & , \text{ for } n \neq 0 \end{cases}$$

Since, impulse response of ideal response filter is non-causal, this can not be physically realized. To make this impulse response for finite duration, first truncate the impulse response & then shift the truncated response to make it causal. This introduces the concept of windowing.

2072-5-01
Tuesday:

Least Integral squared error design of FIR filter:

Let $H_d(e^{j\omega})$ be the desired frequency response and given by;

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d[n]\, e^{-j\omega n}$$

where;

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

↳ desired impulse response.

Desired frequency response is piecewise constant with infinitely sharp transition from pass band to stop band. Desired impulse response is of infinite duration and non-causal. Try to find finite duration impulse response $h_t[n]$ of length $(2M+1)$ which frequency response is given by;

$$H_t(e^{j\omega}) = \sum_{n=-M}^{M} h_t[n]\, e^{-j\omega n}$$

where;

$$h_t[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_t(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Due to truncation there will be error and Integral

From Parseval's relation;

$$\phi = \sum_{n=-\infty}^{\infty} \left| h_t[n] - h_d[n] \right|^2$$

This error is minimum when $h_t[n] = h_d[n]$ for: $-M \le n \le M$.

Thus obtained impulse response is of finite duration and obtained by truncating the desired impulse response $h_d[n]$ from $n = (-M \text{ to } M)$ $(-M \le n \le M)$.

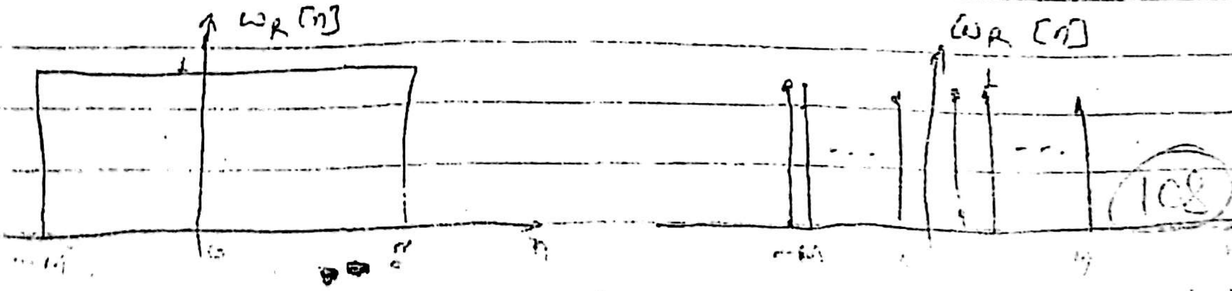
Thus obtained impulse response is delayed by M - samples to make it causal.

$$h[n] = h_t[n-M]$$
↳ Causal FIR

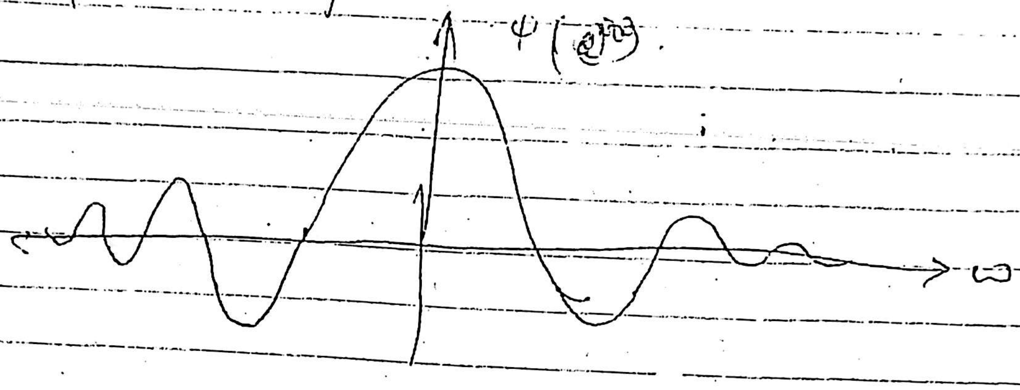
$$h_t[n] = h_d[n] \cdot w[n]$$
↳ window function

Let us take Rectangular window function;

$$w_R[n] = 1 \quad , \text{ for } -M \le n \le M$$



And its frequency response $\Psi(e^{j\omega})$;

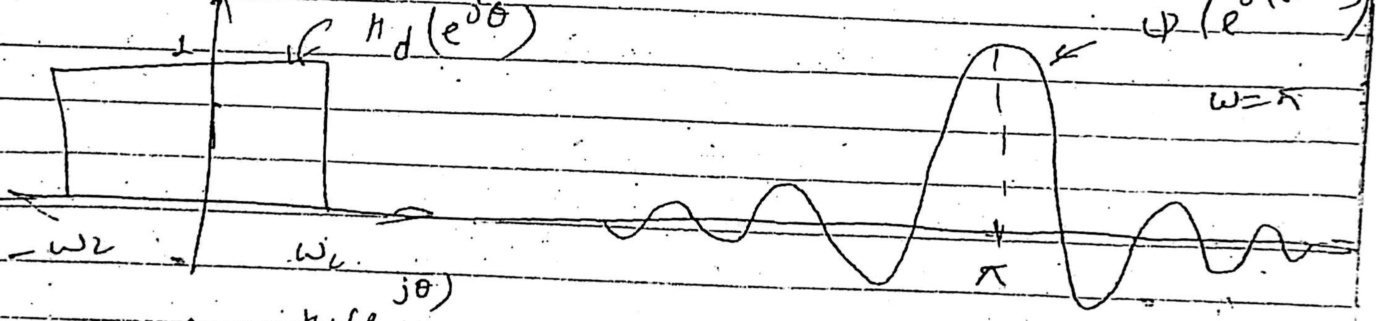


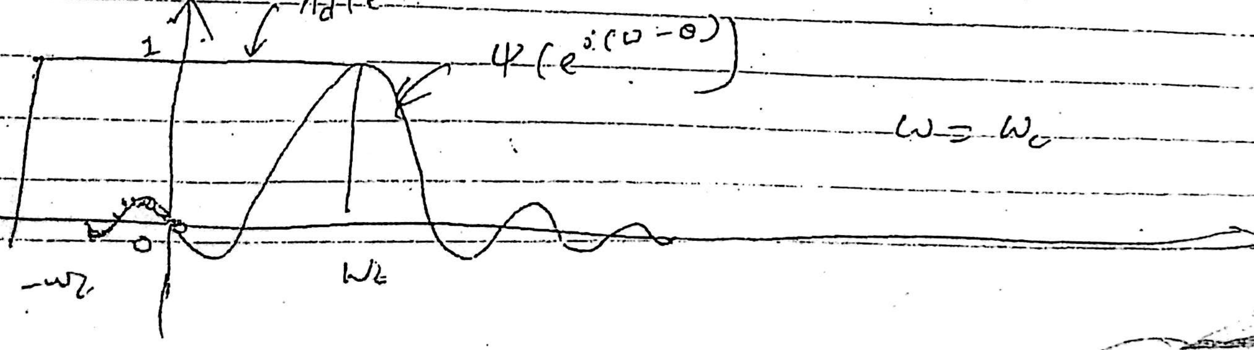
$$h_t[n] = h_d[n] \cdot \omega_R[n].$$

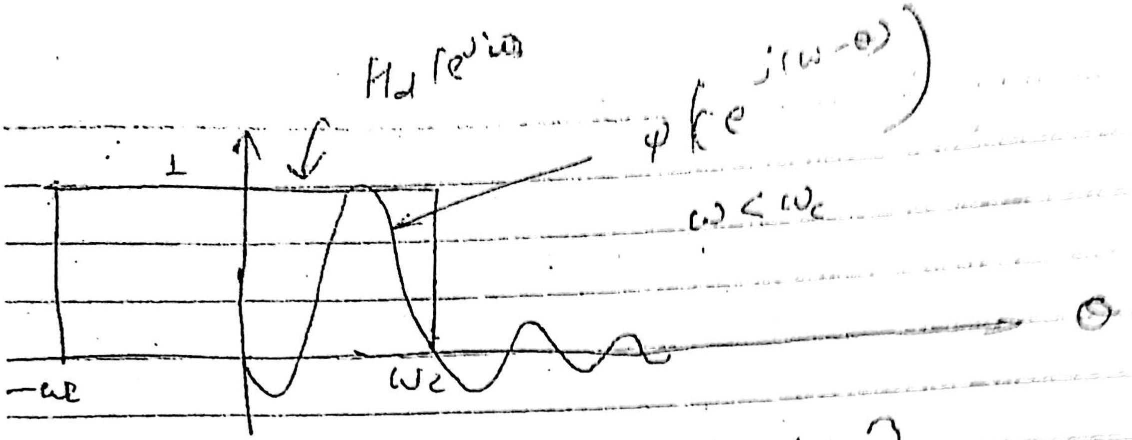
For frequency domain:

$$H_t(e^{j\omega}) = \frac{1}{2\pi}\left[H_d(e^{j\omega}) \circledast \Psi(e^{j\omega})\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega}) \cdot \Psi(e^{j(\omega-\theta)})\, d\theta$$



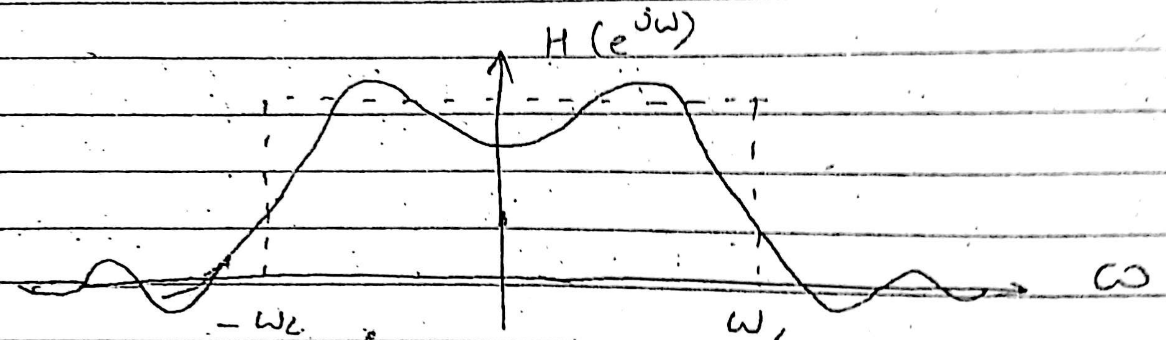




$$h_t[n] = FT^{-1}\{H_t(e^{j\omega})\}$$

$$h[n] = h_t[n-M]$$

$$H(e^{j\omega}) = \sum_{n=0}^{2M} h[n]\, e^{-j\omega n}$$

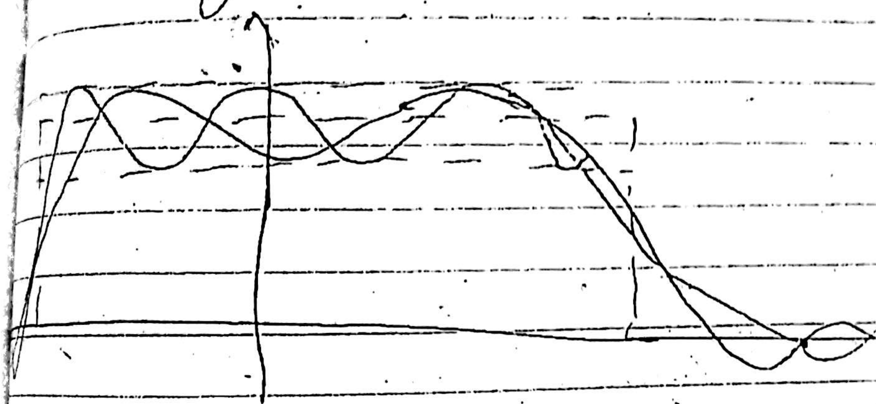


This modulation causes ripple effects known as Gibb's phenomena.

Gibb's Phenomena: The approximation of frequency response $X_N(e^{j\omega})$ to the function $X(e^{j\omega})$ at the point of discontinuity is called Gibb's phenomena.

Thus obtained frequency response exhibits oscillatory behaviour in pass band and stop band.

the Length of filter increases the number of ripple stopband or pass band also increases with corresponding decrease in width of ripples. But the height of largest ripple remains same.



For Rectangular window:

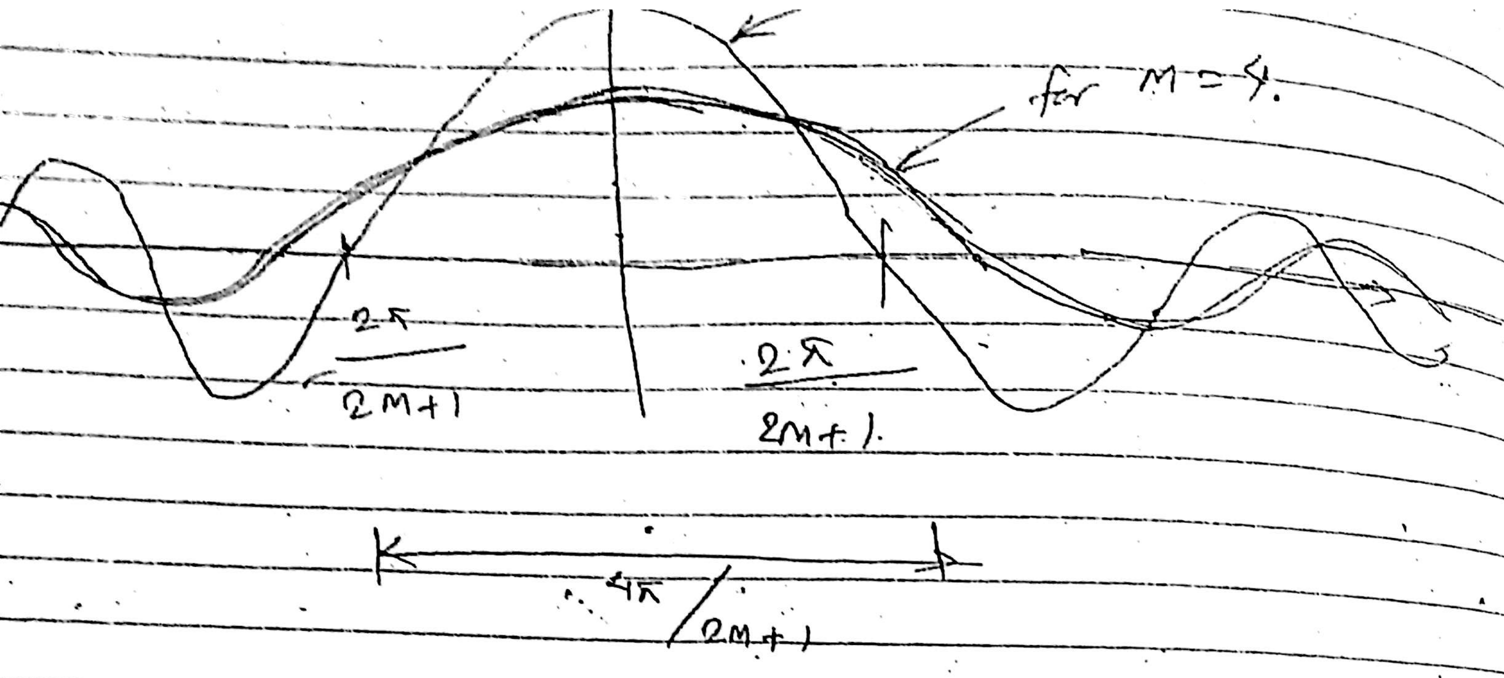
$$w_R[n] = \begin{cases} 1, & \text{for } -M \le n \le M \\ 0, & \text{otherwise.} \end{cases}$$

&

$$\Psi(e^{j\omega}) = \sum_{n=-\infty}^{\infty} w_R[n]\, e^{-j\omega n}$$

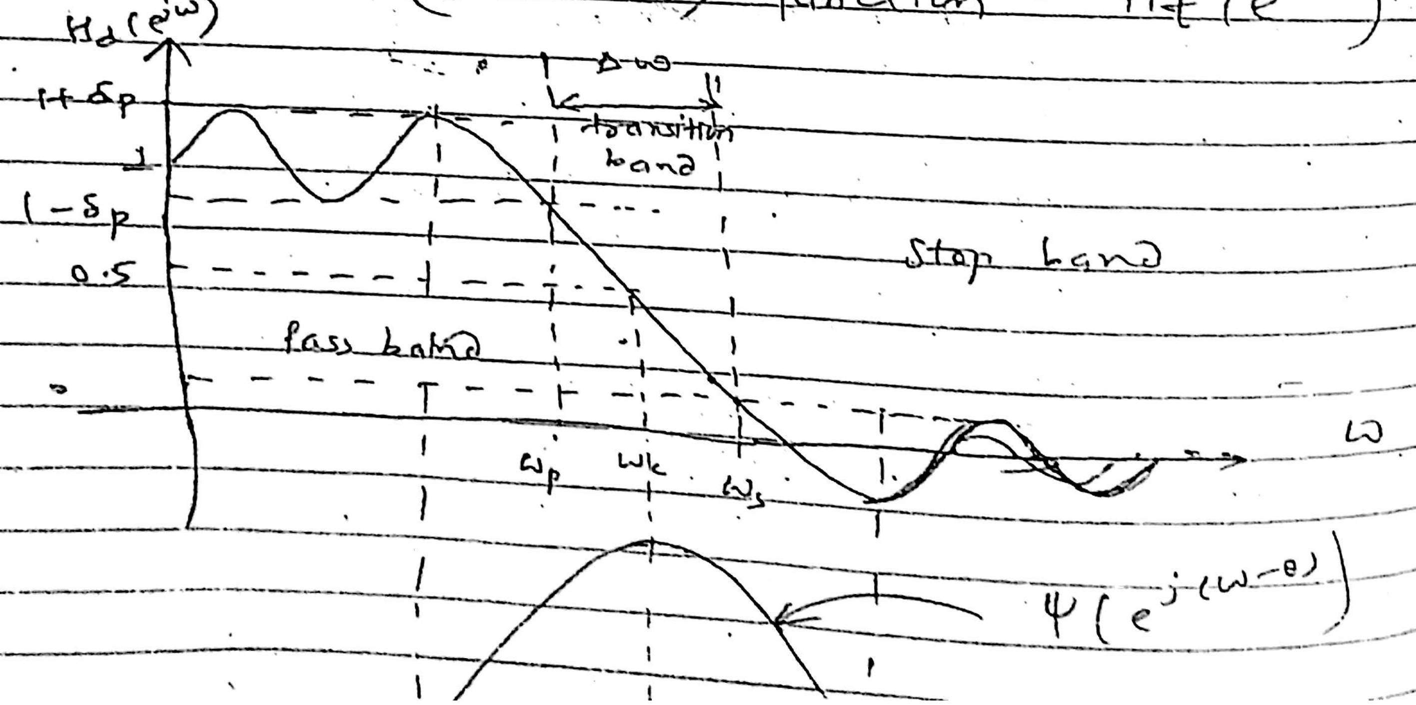
$$= \sum_{n=-M}^{M} e^{-j\omega n}$$

$$= \frac{\sin\left(\omega\,\frac{2M+1}{2}\right)}{\sin\left(\omega/2\right)}$$



Relationship between :

- Ideal LPF $H_d(e^{j\omega})$
- window function $\Psi(e^{j\omega})$
- windowed (truncated) function $H_t(e^{j\omega})$



...ed window function:

... window function
...riable (Adjustable) window function.

...ed window function:

Type:	$w[n]$ for $-M \le n \le M$
1) Rectangular	1
2) Hanning	$\frac{1}{2}\left[1+\cos\frac{2\pi n}{2M+1}\right]$
3) Hamming	$0.54+0.46\cos\left(\frac{2n\pi}{2M+1}\right)$
4) Blackman	$0.42+0.5\cos\frac{2\pi n}{2M+1}+0.08\cos\frac{4n\pi}{2M+1}$

...e:

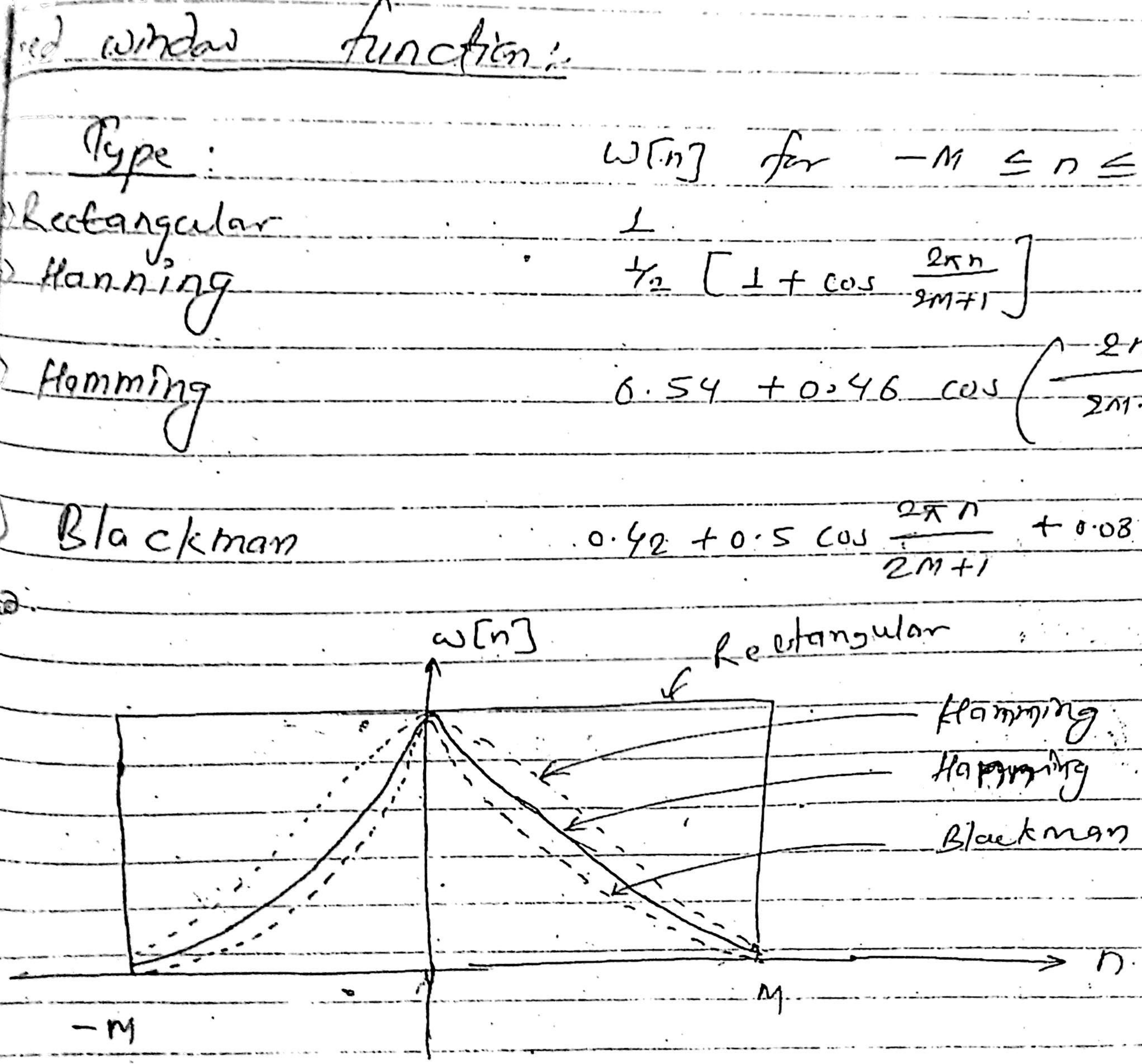


Fig: window function.

re:

ΔM_L = Main lobe width (Distance between nearest zero crossing on both sides of main lobe).

ΔW = Transition band

ΔSL = Relative sidelobe level. (difference in dB between largest side lobe and main lobe)

Drawback of Rectangular Window:

- Abrupt Transition at $n = \pm M$
- Causes Gibb's phenomena
- Gibbs phenomena can be reduced by using tappered window, so that there will be smooth transition from pass band to stop band. Tappered window causes height of side lobe to deminise with corresponding ~~increase~~ increase in main lobe width resulting in a wider transition.

(Characteristics) of fixed window function

Main lobe width (Δ_{ML})	Relative sidelobe level (Δ_{SL})	Minimum stop band attenuation α_s	Transition bandwidth $\Delta\omega$
$\frac{4\pi}{2M+1}$	13.3 dB	20.9 dB	0.92 π/M
$\frac{8\pi}{2M+1}$	31.5 dB	43.9 dB	3.11 π/M
$8\pi/2M+1$	42.7 dB	54.5 dB	3.32 π/M
$\frac{12\pi}{2M+1}$	58.1 dB	73.5 dB	5.58 $\frac{\pi}{M}$

...iable (Adjustable) window function:

In fixed window function Ripple's in pass band ...r in stop band can not be controlled but ... adjustable window function ripples can be ...ntrolled.

Kaiser window:

→ widely used adjustable window function:

$$\omega[n] = \frac{I_0\left\{\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right\}}{I_0\{\beta\}}, \quad -M \le n \le M$$

→ $\beta \Rightarrow$ adjustable parameter

$I_0(u) \Rightarrow$ modified zeroth order Bessel function expressed as;

$$I_0(u) = 1 + \sum_{r=1}^{\infty} \left[\frac{(u/2)^r}{r!} \right]^2$$

Which is +ve for all real value of u ($r=20$ sufficient)

β - parameter controls attenuation in stop band of windowed filter response ($H_t(e^{j\omega})$) and sidelobe ripple of window function. But Kaiser window doesn't provide independent control over pass band ripple.

$$\beta = \begin{cases} 0.1102\,(\alpha_s - 8.7) & , \alpha_s > 50\,dB \\ 0.5842\,(\alpha_s - 21)^{0.4} + 0.07886\,(\alpha_s - 21), & 21 \le \alpha_s \le 50\,dB \\ 0; & \text{for } \alpha_s < 21\,dB \end{cases}$$

Filter length N is given by;

$$N = 2M + 1$$

$$N = \begin{cases} \dfrac{\alpha_s - 7.95}{14.36\,\Delta f} + 1, & \text{for } \alpha_s > 21\,dB \\ \dfrac{0.9222}{\Delta f} + 1, & \alpha_s \le 21\,dB \end{cases}$$

Where, $\Delta f = \dfrac{\omega_s - \omega_p}{2\pi}$

Q. Design a low pass FIR digital filter having passband edge frequency $\omega_p = 0.3\pi$ rad and stop band edge frequency $\omega_s = 0.5\pi$ and stop band attenuation $\alpha_s = 40$ dB. Using appropriate window function.

Sol: $\omega_p = 0.3\pi$, $\omega_s = 0.5\pi$, $\alpha_s = 40$ dB

1) Using Kaiser window:

for $\alpha_s = 40$ dB, we have formula for β as:

$$\beta = 0.5842(\alpha_s - 21)^{0.4} + 0.07886(\alpha_s - 21) \quad \text{for } 21 \le \alpha_s \le 50 \text{ dB}$$

$$= 0.5842(40-21)^{0.4} + 0.7886(40-21)$$

$$= 3.3953$$

$$N = \frac{\alpha_s - 7.95}{14.36\Delta f} + 1 \quad \text{for } \alpha_s > 21 \text{ dB}$$

$$\Delta f = \frac{\omega_s - \omega_p}{2\pi} = \frac{0.5\pi - 0.3\pi}{2\pi}$$

$$= 0.1$$

Now,

$$N = \frac{40 - 7.95}{14.36 \times 0.1}$$

$$= 23.31854$$

$$\simeq 24$$

To make compatible with the relation : $N = 2M+1$

we have to select, $M = 12$

so, $N = 25$

Window function:

$$\omega[n] = \frac{I_0\left\{\beta\sqrt{1-\left(\frac{n}{M}\right)^2}\right\}}{I_0\{\beta\}} \quad \text{for } -M \le n \le M$$

$$I_0(u) = 1 + \sum_{r=1}^{\infty} \left[\frac{\left(\frac{u}{2}\right)^r}{r!} \right]^2$$

n	ω[n]
−12	
−11	
⋮	
0	
⋮	
11	
12	

Impulse response of Ideal LPF is given by ;

$$h_d[n] = \begin{cases} \frac{\omega_c}{\pi} & , \text{for } n = 0 \\ \frac{\sin(\omega_c n)}{n\pi} & , \text{for } n \neq 0 \end{cases}$$

$$\omega_c = \frac{\omega_s + \omega_p}{2} = 0.4\pi$$

n	$h_d[n]$
−12	
−1°	
⋮	
0	
⋮	
12	

Now, we have;

$$h_2[n] = h_d[n] \cdot \omega[n]$$

-12		
⋮		
0	6.4	
⋮		
+12		

Now,

Causal FIR filter is;

$$h[n] = h_t[n-M]$$

$$= \begin{cases} \frac{\omega_c}{\pi}, & \text{for } n = M \\ \frac{\sin \omega_c (n-M)}{(n-M)\pi} \cdot \frac{I_0\left\{\beta\sqrt{1-\left(\frac{n-M}{M}\right)^2}\right\}}{I_0\{\beta\}} & \text{for } 0 \le n \le 2M, \; n \ne M \end{cases}$$

n	h[n]
0	
1	
⋮	
M	
⋮	
2M	

(11) Using fixed window function:

Since, stop band attenuation is 50 dB, so we can choose, Hanning, Hamming and Blackman window.

Now, selecting Hamming window

For Hamming window;

$$\Delta\omega = \frac{3.32\pi}{M}$$

$$\omega_s - \omega_p = \frac{3.32\pi}{M}$$

$$\text{or, } M = \frac{3.32\pi}{(0.5-0.3)\pi} = 16.6 \simeq 17$$

Now;

$$w[n] = 0.54 + 0.46\cos\left(\frac{2n\pi}{2M+1}\right), \quad -M \le n \le M$$

n	$w[n]$
-17	
⋮	
0	
⋮	
17	

Impulse response of Ideal low pass filter is given by;

$$h_d[n] = \begin{cases} \frac{\omega_c}{\pi}, & \text{for } n=0 \\ \sin\left(\frac{\omega_c n}{n\pi}\right), & \text{for } n \neq 0 \end{cases}$$

$$\omega_c = \frac{\omega_s + \omega_p}{2} = 0.4\pi$$

n	$h_d[n]$
-17	
⋮	
0	
⋮	
17	

Now, we have;

$$h_t[n] = h_d[n] \cdot \omega[n]$$

$$= \begin{cases} \frac{\omega_c}{\pi} & \text{for } n = 0 \\ \frac{\sin \omega_c n}{n\pi} \left\{ 0.54 + 0.46 \cos\left(\frac{2n\pi}{2M+1}\right) \right\} & \text{for } -M \le n \le M,\ n \ne 0 \end{cases}$$

n	$h_t[n] = h_d[n] \cdot \omega[n]$	
-17		
⋮		
0		
⋮		
17		

Causal FIR filter:

$$h[n] = h_t[n-M]$$

$$h[n] = \begin{cases} \dfrac{w_c}{\pi}, & \text{for } n = M \\ \dfrac{\sin\{w_c(n-M)\}}{(n-M)\pi} \left\{ 0.54 + 0.46 \cos\left(\dfrac{2(n-M)\pi}{2M+1}\right) \right\} \end{cases}$$

for $0 \le n \le 2M$
$n \ne M$

n	$h[n]$
0	
1	
⋮	
M	0.4
⋮	
2M	

Transfer function of causal FIR filter:

$$H(z) = \sum_{n=0}^{2M} h[n]\, z^{-n}$$

$= \ldots$

H.W

$\omega_s = 0.45\pi$

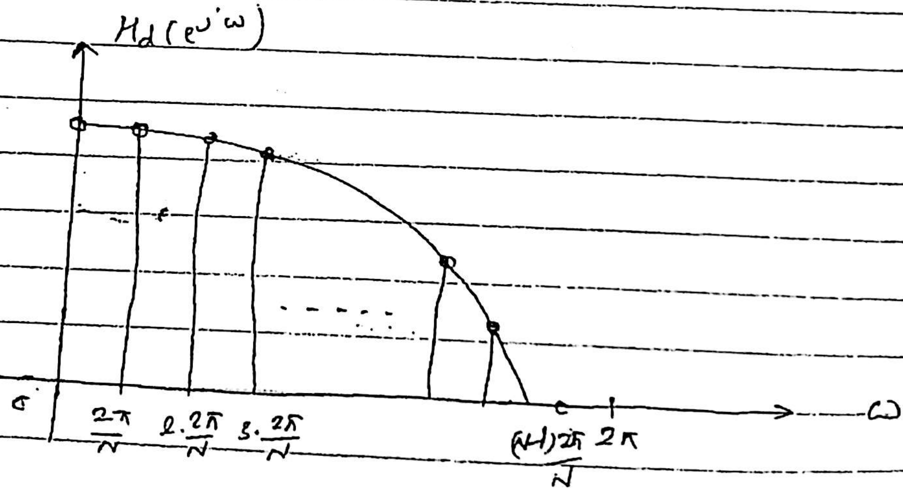
$\omega_p = 0.2\pi$

$\alpha_s = 50\,dB$

Design FIR lowpass filter:

FIR filter Design by using Frequency Sampling Approach

In this method, desired frequency response $H_d(e^{j\omega})$ is uniformly sampled at N equally spaced frequency samples.



Since, desired frequency response is periodic of 2π, it is only necessary to sample from $\omega = 0$ to 2π.

$$H_d(e^{j\omega}) = \sum_{l=-\infty}^{\infty} h_d[l]\, e^{-j\omega l}$$

where, $h_d[l] \Rightarrow$ desired impulse response.

This frequency response is sampled at

$$\omega = k \cdot \frac{2\pi}{N}, \quad \text{for } K = 0, 1, \dots, N-1$$

$$H_d\left(e^{jk\frac{2\pi}{N}}\right) = \sum_{l=-\infty}^{\infty} h_d[l]\, e^{-jk\frac{2\pi}{N}l}$$

for $K = 0, 1, \dots, N-1$

$$H(K) = \sum_{l=-\infty}^{\infty} h_d[l]\, e^{-jk\frac{2\pi}{N}l}$$

$$= \sum_{l=-\infty}^{\infty} h_d[l] \cdot \omega_N^{kl}$$

where, $\omega_N = e^{-j\frac{2\pi}{N}}$

Now, taking Inverse Discrete Fourier transform (IDFT) of $H(K)$, we get,

$$h[n] = \frac{1}{N} \sum_{K=0}^{N-1} H(K)\, \omega_N^{-kn}$$

for $n = 0, 1, \dots, N-1$

$$= \frac{1}{N} \sum_{K=0}^{N-1} \sum_{l=-\infty}^{\infty} h_d[l]\, \omega_N^{kl} \cdot \omega_N^{-kn}$$

$$= \sum_{l=-\infty}^{\infty} h_d[l] \cdot \left\{ \frac{1}{N} \sum_{K=0}^{N-1} \omega_N^{-k(n-l)} \right\}$$

Since;

$$\frac{1}{N} \sum_{K=0}^{N-1} \omega_N^{-k(n-l)} = \begin{cases} 1, & \text{for } l = n + mN \quad m; \text{integer} \\ 0, & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} \sum_{m=-\infty}^{\infty} h_d[n+mN], & \text{for } l = n+mN \\ 0, & \text{otherwise.} \end{cases}$$

So, the causal FIR filter is;

$$\boxed{\underbrace{h[n]}_{\text{designed}} = \sum_{m=-\infty}^{\infty} \underbrace{h_d[n+mN]}_{\text{desired}}, \quad \text{for } 0 \le n \le N-1}$$

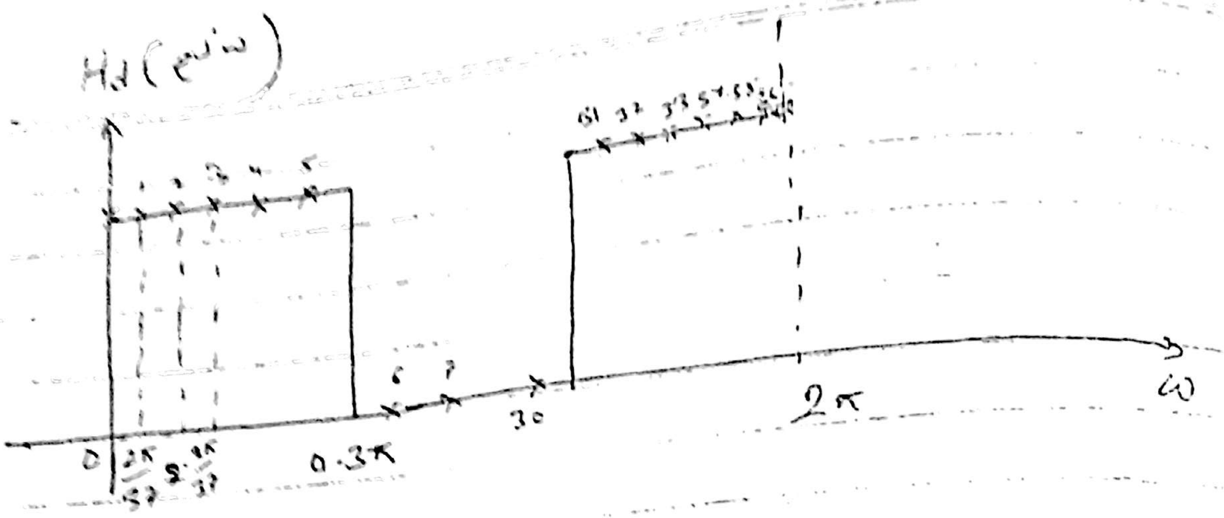
Designed impulse response $h[n]$ is obtained from infinite number of shifted replicas of desired impulse response $h_d[n]$ shifted by N samples.

If $h_d[n]$ is finite ~~sample~~ length sequence $(\le N)$, then $h[n]$ will be $h_d[n]$.

$$h[n] = h_d[n], \quad \text{for } 0 \le n \le N-1$$

Otherwise, there will be time domain aliasing.

example: Consider a design of linear phase FIR filter with passband edge frequency $\omega_p = 0.3\pi$ using frequency



Now,
select $N = 37$ (← any)

By using frequency sampling approach, we get frequency samples $H(K)$ as;

$$H(K) = \begin{cases} e^{-j\frac{2\pi k}{37}\cdot 18} & \text{for } k = 0,1,2,3,4,5, \\ & 31,32,33,34,35,36 \\ 0, & \text{for } k = 6,7,8,9,----,30 \end{cases}$$

$$H(K) = \sum_{n=0}^{N-1} e^{-j\frac{2\pi k}{N}n}$$

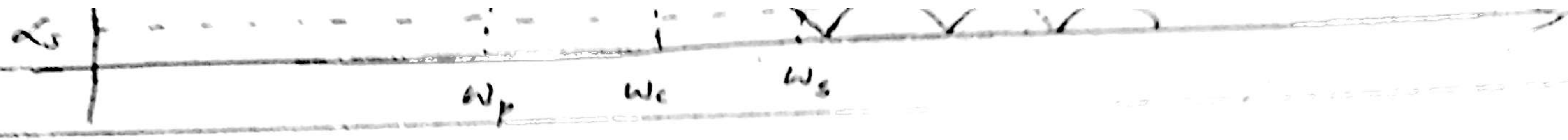
$$\left(\frac{N-1}{2} = 18\right)$$

Now take 37-point IDFT to get impulse response;

$$h[n] = \frac{1}{37}\sum_{K=0}^{36} H(K)\cdot W_N^{-kn}, \quad \text{for } n=0,1,\ldots,36$$

Taking fourier transform,

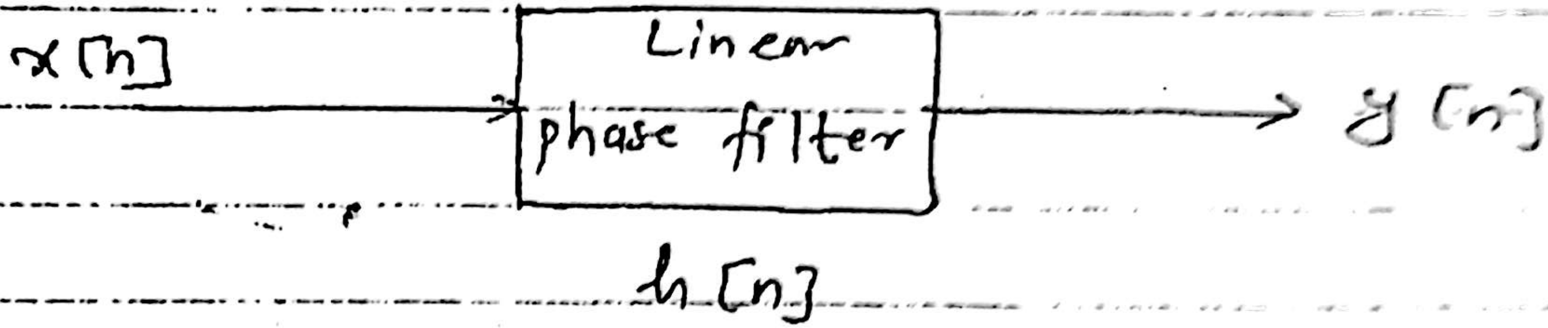
$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$$



Stop band ripple $\alpha_s = -17.1886$ dB
pass-band ripple at $\alpha_p = -1.15$ dB.

Linear Phase Filter :

$$H_{LP}(e^{j\omega}) = \begin{cases} e^{-j\omega\theta}, & \text{for } |\omega| \le \omega_c \\ 0, & \text{for } \omega_c < |\omega| \le \pi \end{cases}$$



then O/p ;

$$y[n] = x[n] \cdot H_{Lr}(e^{j\omega})$$

$$= A e^{j\omega} \cdot e^{-j\omega\theta}$$

$$= A e^{j\omega(n-\theta)}$$

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$$= h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3} + h[4]z^{-4} + h[5]z^{-5}$$

$$= h[0]\{1+z^{-5}\} + h[1]\{z^{-1}+z^{-4}\} + h[2]\{z^{-2}+z^{-3}\}$$

$$= z^{-5/2}\left\{h[0]\{z^{5/2}+z^{-5/2}\} + h[1]\{z^{3/2}+z^{-3/2}\} + h[2]\{z^{\frac{1}{2}}+z^{-\frac{1}{2}}\}\right\}$$

for, $z = e^{j\omega}$;

$$H(e^{j\omega}) = e^{-j\frac{5}{2}\omega}\left[\left\{h[0]\{e^{j\frac{5}{2}\omega} + e^{-j\frac{5}{2}\omega}\} + h[1]\{e^{j\frac{3}{2}\omega} + e^{-j\frac{3}{2}\omega}\} + h[2]\{e^{\frac{1}{2}j\omega} + e^{-j\frac{1}{2}\omega}\}\right]\right.$$

$$= e^{-j\frac{5}{2}\omega}\left\{2h[0]\cos\frac{5}{2}\omega + 2h[1]\cos\frac{3}{2}\omega + 2h[2]\cos\frac{1}{2}\omega\right\}$$

$$= e^{-j\frac{5}{2}\omega}\,\breve{H}(\omega)$$

↳ real function of ω.

$$|H(e^{j\omega})| = |\breve{H}(\omega)|$$

$$\angle H(e^{j\omega}) = -\frac{5}{2}\omega + \beta.$$

Linear phase.

where, $\beta = \begin{cases} 0, & \text{if } \breve{H}(\omega) \text{ is +ve} \\ \pi, & \text{if } \breve{H}(\omega) \text{ is -ve} \end{cases}$

$$h[n] = -h[N-n] \quad , \quad 0 \leq n \leq N$$

Total length = odd

for $N=6$

Linear phase FIR filter :

Type I Linear phase FIR filter :
Symmetric impulse response with odd length and impulse response satisfies

$$h[n] = h[N-n] \quad , \quad 0 \le n \le N$$

Total length = odd

For ;

$N = 6$:

$h[0] = h[6]$; $h[1] = h[5]$, $h[2] = h[4]$

T.F.

$$H(z) = \sum_{n=0}^{6} h[n] z^{-n}$$

$$= h[0] + h[1] z^{-1} + h[2] z^{-2} + h[3] z^{-3} + h[4] z^{-4} + h[5] z^{-5} + h[6] z^{-6}$$

$$= h[0]\{1 + z^{-6}\} + h[1]\{z^{-1} + z^{-5}\} + h[2]\{z^{-2} + z^{-4}\} + h[3]\{z^{-3}\}$$

$$= z^{-3}\left[h[0]\{z^{3} + z^{-3}\} + h[1]\{z^{2} + z^{-2}\} + h[2]\{z + z^{-1}\} + h[3]\right]$$

For, $z = e^{j\omega}$;

$$H(e^{j\omega}) = e^{-j3\omega}\left[h[0]\{e^{j3\omega} + e^{-j3\omega}\} + h[1]\{e^{j2\omega} + e^{-j2\omega}\} + h[2]\{e^{j\omega} + e^{-j\omega}\} + h[3]\right]$$

$$= e^{-j3\omega}\left[2h[0]\cos 3\omega + 2h[1]\cos 2\omega + 2h[2]\cos\omega\right.$$

$= e^{-j3\omega} \breve{H}(\omega)$

real function of ω

$|H(e^{j\omega})| = |\breve{H}(\omega)|$

$\angle H(e^{j\omega}) = -3\omega + \beta$, where $\beta = \begin{cases} 0 & \text{if } \breve{H}(\omega) \text{ is +ve} \\ \pi & \text{if } \breve{H}(\omega) \text{ is -ve} \end{cases}$

linear phase.

General frequency response of Type I linear phase FIR filter is given by:

$$H(e^{j\omega}) = e^{-j\omega\frac{N}{2}} \left[\sum_{n=0}^{N/2} a[n] \cos n\omega \right]$$

where: $a[0] = h[N/2]$

$$a[n] = 2h\left[\frac{N}{2} - n\right], \quad 1 \le n \le N/2$$

Type II linear phase FIR filter:

Symmetric impulse response with even length & impulse response satisfies:

$$h[n] = h[N-n] \; ; \quad 0 \le n \le N$$

Total length = even

for $N=5$;

$h[0] = h[5]$, $h[1] = h[4]$, $h[2] = h[3]$

$\therefore \; H(z) = \sum^{5} h[n] z^{-n}$

Type IV linear phase filter:
Anti-symmetric impulse response with even length and impulse response satisfies:

$$h[n] = -h[N-n], \quad 0 \le n \le N$$

Total length = even

for $N = 5$;

(121)

$h[0] = -h[5]$, $h[1] = -h[4]$, $h[2] = -h[3]$.

T.F.:

$$H(z) = \sum_{n=0}^{5} h[n] z^{-n}$$

$$= h[0] z^{0} + h[1] z^{-1} + h[2] z^{-2} + h[3] z^{-3} + h[4] z^{-4}$$
$$+ h[5] z^{-5}$$

$$= h[0] \{1 - z^{-5}\} + h[1] \{z^{-1} - z^{-4}\} +$$
$$h[2] \{z^{-2} - z^{-3}\}$$

$$= z^{-5/2} \{ h[0] \{ z^{5/2} - z^{-5/2} \} + h[1] \{ z^{\frac{3}{2}} - z^{-\frac{3}{2}} \}$$
$$+ h[2] \{ z^{\frac{1}{2}} - z^{-\frac{1}{2}} \} \}$$

eg:

(i) $h[n] = \{0, 1, -1, 0\}$ $\rightarrow$ Type IV

(ii) $h[n] = \{1, 0, -1, 0, 1\}$ $\rightarrow$ Type I

(iii) $h[n] = \{1, 2, 2, 1\}$ $\rightarrow$ Type II

(iv) $h[n] = \{2, 0, -1, 0, -2\}$ $\rightarrow$ Type III

(v) $h[n] = \{1, 2, 3, 1, 2, 3\}$ $\rightarrow$ $\times$

Computer Aided Design of Digital Filters:

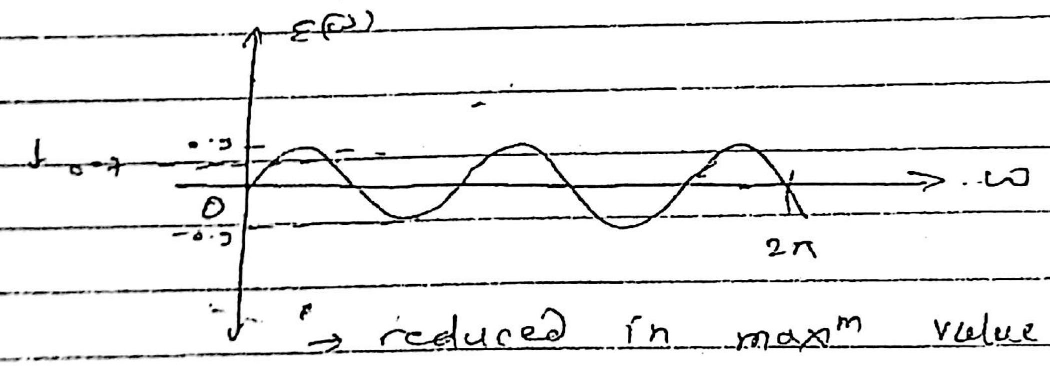
$D(\omega) \Rightarrow$ designed Magnitude response

$H(e^{j\omega}) \Rightarrow$ designed frequency response

$E(\omega) = P(\omega)\left[D(\omega) - |H(e^{j\omega})|\right]$

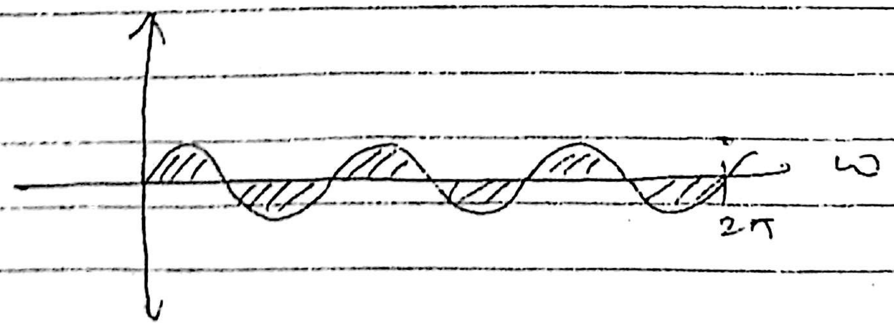
→ user defined function

Minimax or Chebyshev method:



→ reduced in max^m value of error

Least P - Criteria:

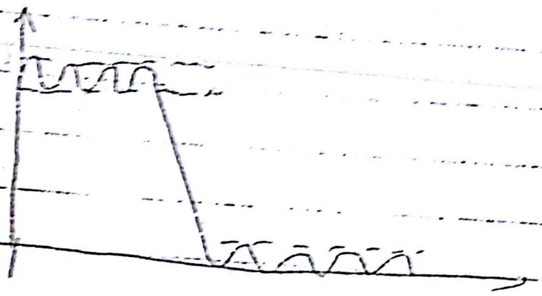


→ over all error is added & reduced

Optimal filter:

→ after reduction of error

→ provides equi ripple in pass band & stop band.



$$\mathcal{E}(\omega) = P(\omega)\left[D(\omega) - |H(e^{j\omega})|\right]$$

$$= P(\omega)\left[D(\omega) - \sum_{n=0}^{N/2} a[n]\cos n\omega\right]$$

?

* Park Mcclelan's algorithm:

↳ to find $a[n]$.

↳ uses Alternation theorem

↓

M+2 external frequency:

$\omega_1 < \omega_2 < \omega_3 < \cdots < \omega_{M+2}$

$$\mathcal{E}(\omega_1) = -\mathcal{E}(\omega_2)$$

$$\mathcal{E}(\omega_i) = -\mathcal{E}(\omega_{i+1}), \quad 1 \leq i \leq M+2$$

$$P(\omega_i)\left[D(\omega_i) - \sum_{n=0}^{M} a[n]\cos n\omega_i\right] = (-1)^i \mathcal{E}, \quad 1 \leq i \leq M+2$$

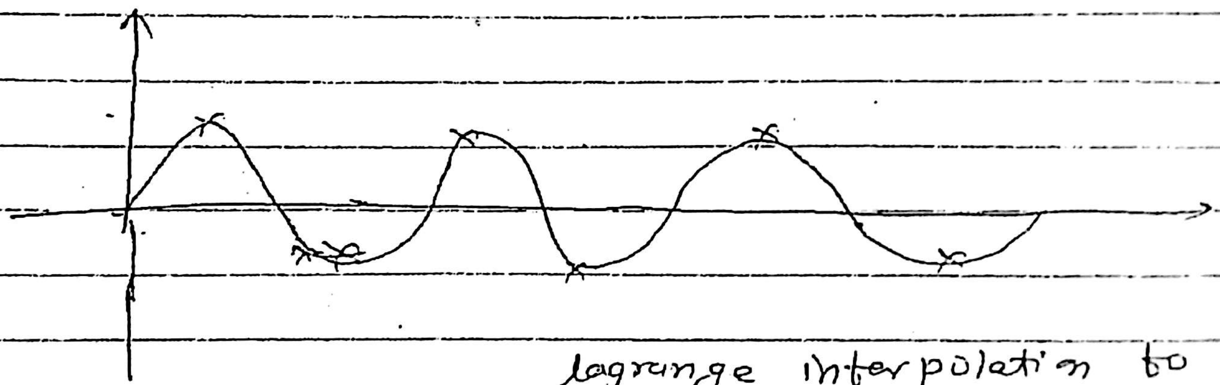
$$\sum_{n=0}^{M} a[n]\cos n\omega_i + (-1)^{i-1}\frac{\mathcal{E}}{P(\omega_i)} = D(\omega_i)$$

new M+2 external frequency:

Remez exchange algorithm:

Algorithm:

1. Choose M+2 external frequencies.
2. To find the value of ϵ.
3. Calculate error.



lagrange interpolation to draw error curve

4. Select new M+2 external frequency from above curve

5. change → go to step 2

no chage → continue.

6. a[n] is best.

(Ch-8)

DSP Implementation

Agree System

ADSP – 3000 series

Texas Instruments

Motorola

Features :-

↳ weight and dimension

→ as low as possible

↳ power consumption

⇒ as low as possible

↳ Sampling rate

→ Audio ⇒ Above 8 KHZ

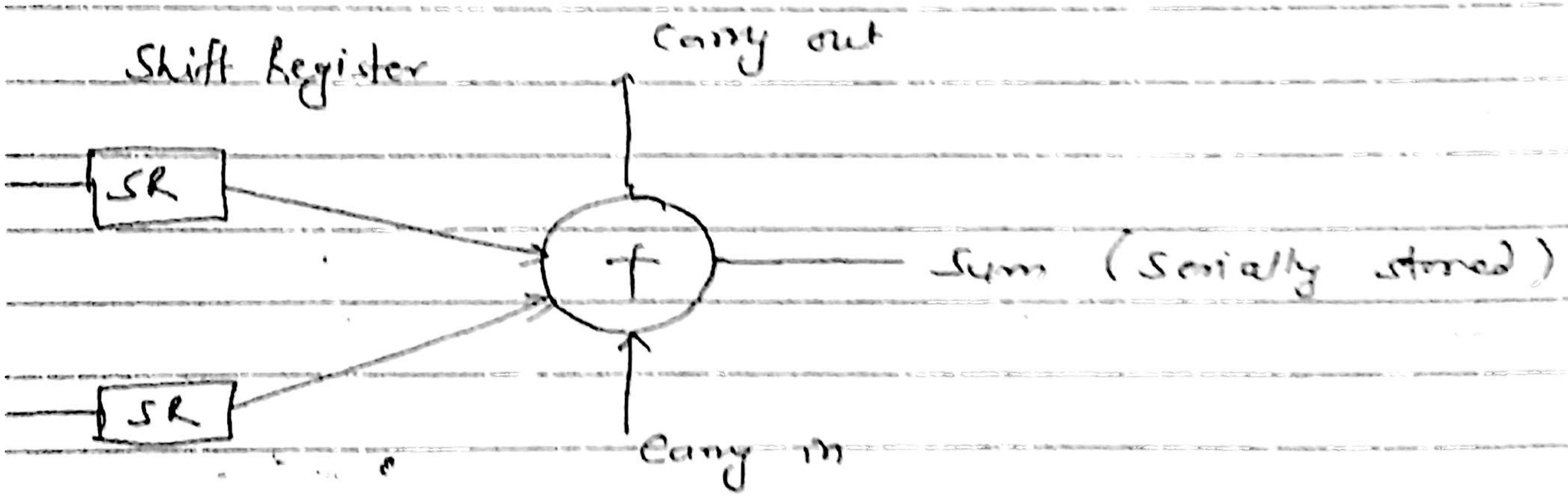
→ Video ⇒ Above few MHz

↳ Quality and speed.

↳ as better as possible

(adder)

Full Adder:



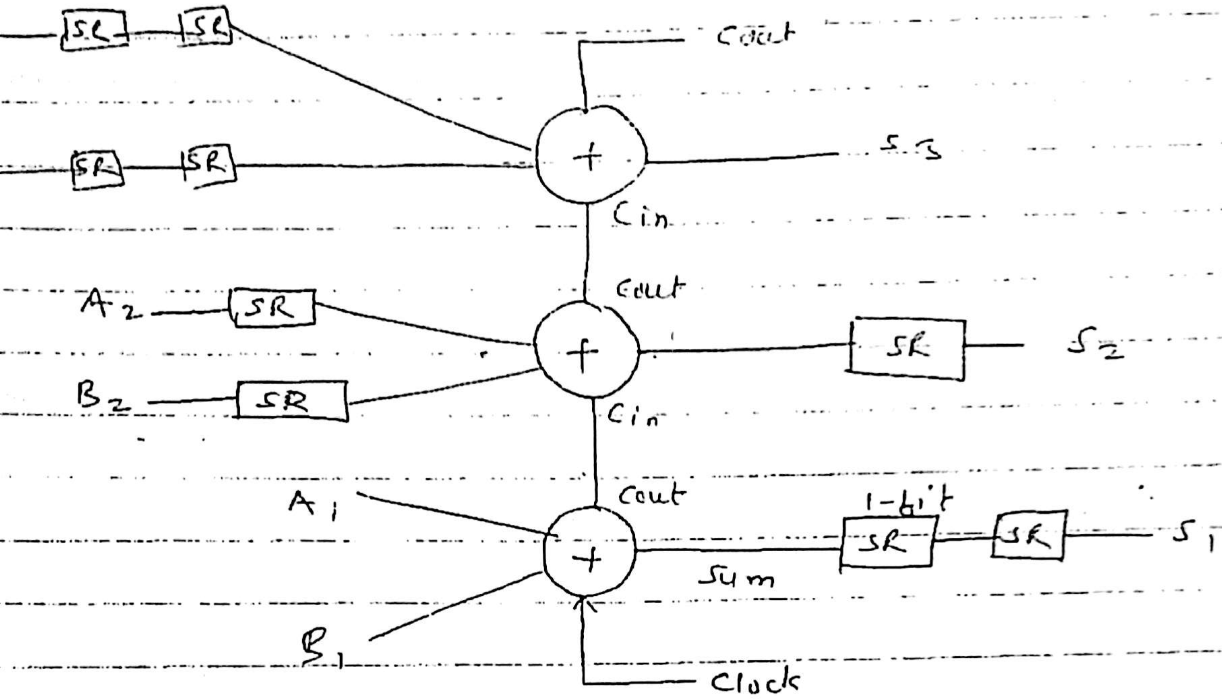
Disadvantage

no. of clocks high
i.e. high time consuption

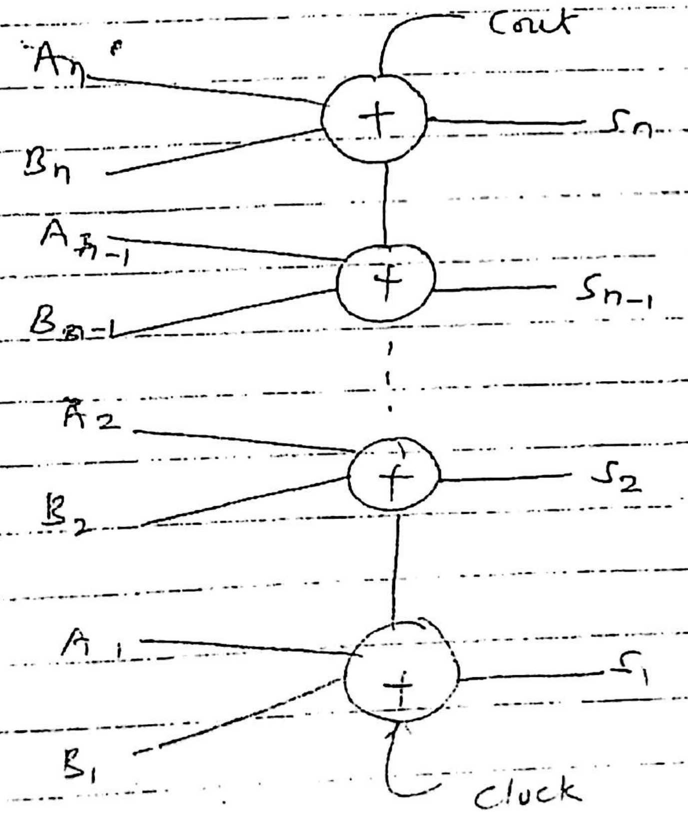
Advantage

- require less
component.

To boost up the speed of bit serial adder, pipeline implementation:



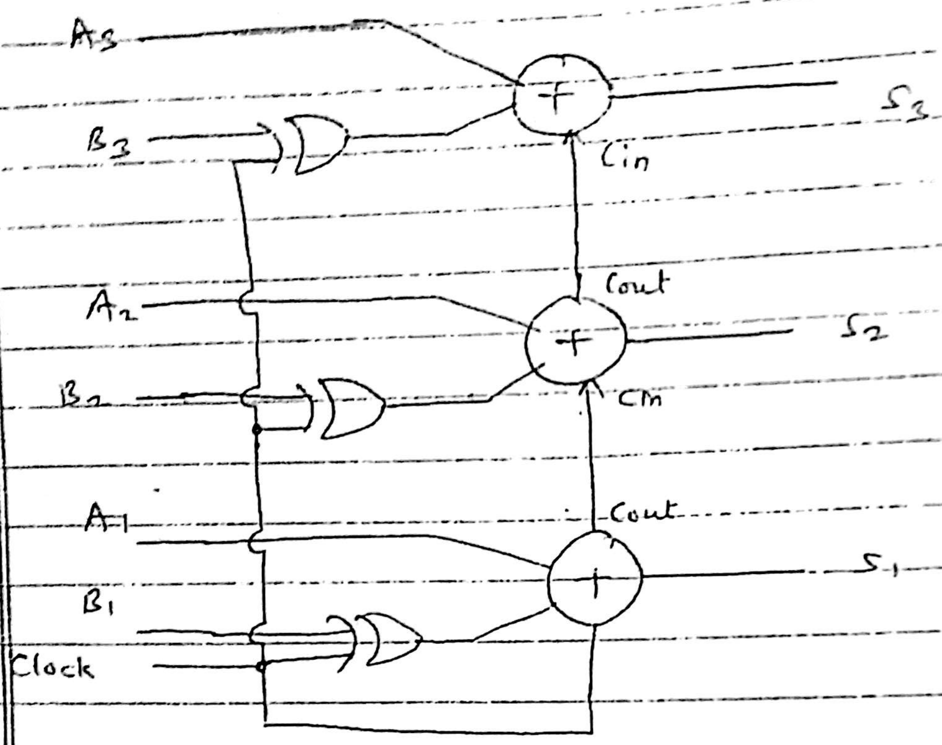
Bit parallel Adder:



Total delay $= nT_c$

T_c = Clock time.

Adder / Subtraction circuit

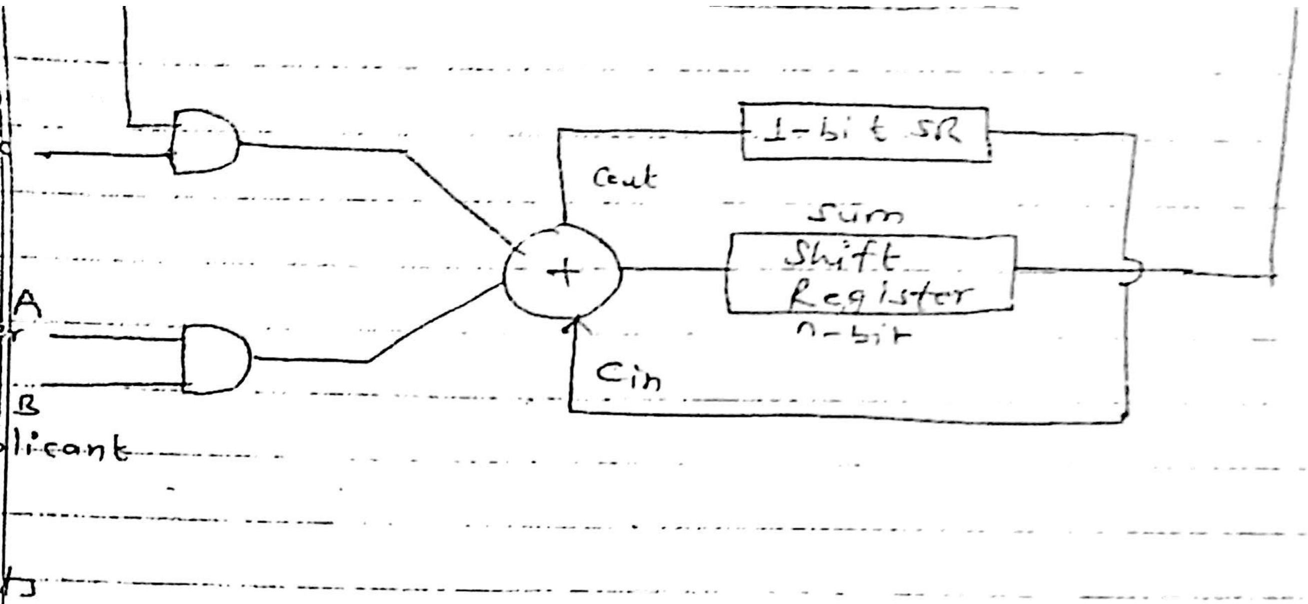


If Clock = 0, $Sum = A_i + B_i$

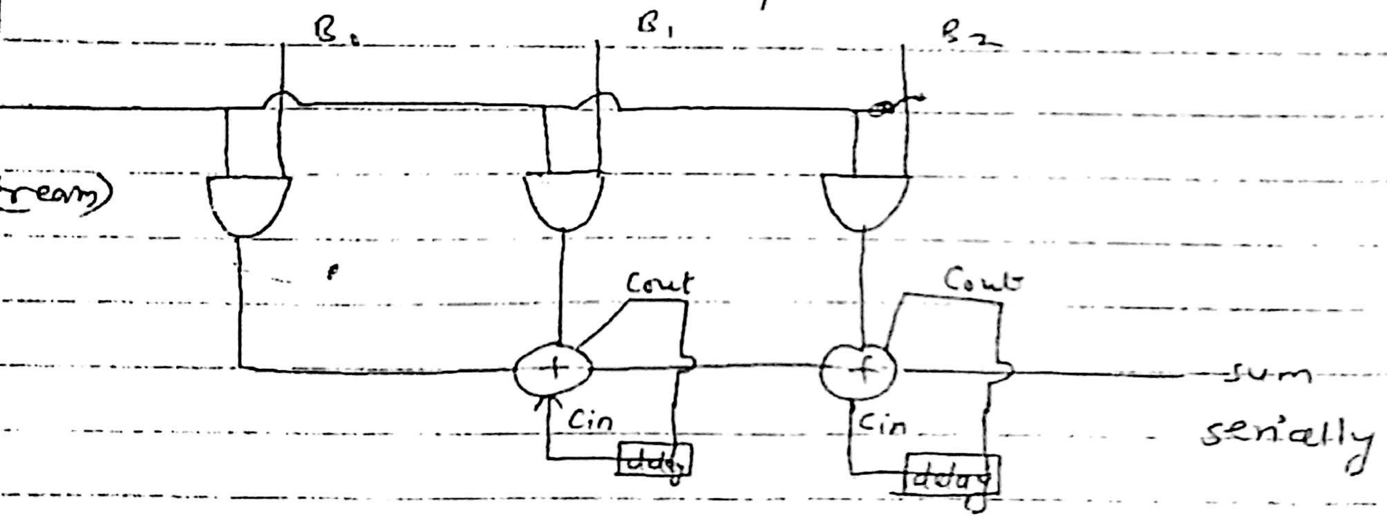
else, clock = 1, $Sum = A_i - B_i$

Multiplication :

```
      1 0 1 0 1
    × 1 0 1 1 1
      1 0 1 0 1
    0 0 0 0 0 _
  1 0 1 0 1 _ _
0 0 0 0 0 _ _ _
+ 1 0 1 0 1 _ _ _ _
```



Bit Serial / parallel multiplier:



$A_2 \; A_1 \; A_0$

$B_2 \; B_1 \; B_0$

$B_0 \quad A_1B_0 \quad A_0B_0$

$B_1 \quad A_0B_1 \quad -$

FIR	IIR
of impulse response	1. length of impulse response infinite
	eg:
= {1, 2, -1, 2}	$h[n] = (\frac{1}{2})^n u[n]$
recursive	2. Recursive
feedback	3. Feedback
contains non-zero	4. Atleast one pole
f H(z) exists, will be entire possibly except z = ∞	5. If T.F H(z) exists then ROC will be interior or exterior of circle with radius i.e. $\lvert z \rvert > r_{max}$ or $\lvert z \rvert < r_{min}$