Hilbert Spaces

수학과 오서영

Basic Definition

Metric

A metric is a way of measuring distance between two points.

Definition 2.1. *A metric space* $(\mathcal{M}, d)$ *is a set* $\mathcal{M}$ *together with a function* $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ *called a metric satisfying four conditions:*

1. $d(x, y) \geq 0$ *for all* $x, y \in \mathcal{M}$.
2. $d(x, y) = 0$ *if and only if* $x = y$.
3. $d(x, y) = d(y, x)$ *for all* $x, y \in \mathcal{M}$.
4. $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in \mathcal{M}$.

Basic Definition

ball

Definition 2.2. *Let* $(\mathcal{M}, d)$ *be a metric space. The open* r*-ball centered at* x *is the set* $B_r(x) = \{y \in \mathcal{M} : d(x,y) < r\}$ *for any choice of* $x \in \mathcal{M}$ *and* $r > 0$*. A closed* r*-ball centered at* x *is the set* $\overline{B}_r(x) = \{y \in \mathcal{M} : d(x,y) \leq r\}$*.*

norm

Definition 2.3. *A (complex) normed linear space* $(\mathcal{V}, \|\cdot\|)$ *is a (complex) linear space* $\mathcal{V}$ *together with a function* $\|\cdot\| : \mathcal{V} \to \mathbb{C}$ *called a norm satisfying the following conditions:*

1. $\|v\| \geq 0$ *for all* $v \in \mathcal{V}$*.*
2. $\|v\| = 0$ *if and only if* $v = 0$*.*
3. $\|\lambda v\| = |\lambda| \, \|v\|$ *for all* $v \in \mathcal{V}$ *and* $\lambda \in \mathbb{C}$*.*
4. $\|v + w\| \leq \|v\| + \|w\|$ *for all* $v, w \in \mathcal{V}$*.*

Basic Definition

Inner product

Definition 2.4. *A (complex) inner product space* $(\mathcal{V}, < \cdot, \cdot >)$ *is a (complex) linear space together with a function* $< \cdot, \cdot >: \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ *called an inner product satisfying the following conditions:*

1. $< v, v > \geq 0$ *for all* $v \in \mathcal{V}$.
2. $< v, v >= 0$ *if and only if* $v = 0$.
3. $< \lambda v, w >= \lambda < v, w >$ *for all* $v, w \in \mathcal{V}$ *and* $\lambda \in \mathbb{C}$.
4. $< v, \lambda w >= \overline{\lambda} < v, w >$ *for all* $v, w \in \mathcal{V}$ *and* $\lambda \in \mathbb{C}$.
5. $< v, w >=< \overline{w, v} >$ *for all* $v, w \in \mathcal{V}$.
6. $< v, w + u >=< v, w > + < v, u >$ *for all* $u, v, w \in \mathcal{V}$.
7. $< v + u, w >=< v, w > + < u, w >$ *for all* $u, v, w \in \mathcal{V}$.

Basic Definition

Hilbert space

Definition 2.5. *A Hilbert space is a vector space H with an inner product $< f, g >$ such that the norm defined by $\|f\| = \sqrt{< f, f >}$ turns H into a complete metric space. Complete means that the Cauchy sequences converge.*

Fourier Series

Definition 2.7. *The Fourier Series for a function f on the interval $[-\pi, \pi]$ is*

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mt + b_m \sin mt)$$

Where

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) cos(mt) dt$$

and

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) sin(mt) dt.$$

Basic Definition

Inner Product

Definition 2.8. *[Kat76] $L^2(D)$, is the set of complex valued functions $f(t)$ on the real number line with $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$. $L^2(D)$ is known as the space of square integrable functions. Its inner product is defined as: $< f, g >= \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt$.*

Convergence of sequence

Definition 2.9. *Let f_n, $n = 1, 2, ...$ and f be complex valued functions on a set D. The sequence (f_n) converges pointwise (on D) to the function f if for every $x \in D$, the sequence $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$ i.e.*

$$f_n(x) \longrightarrow f(x) \text{ as } n \to \infty.$$

Basic Definition

Uniformly convergence

Definition 2.10. *Let* f_n, $n = 1, 2, \ldots$ *and* f *be complex valued functions on a set* D. *The sequence* (f_n) *converges uniformly (on* D*) to the function* f *if for every* $\varepsilon > 0$, *the closed ball* $\overline{B}_\varepsilon(f)$ *absorbs the sequence* (f_n). *i.e. For all* $\varepsilon > 0$, *there exists an* $N \in \mathbb{N}$ *such that for all* $n \geq N$ *and all* $x \in D$, *we have:*

$$|f_n(x) - f(x)| \leq \varepsilon.$$

Basic Definition

In mean convergence

Definition 2.11. *Convergence in the norm ("in mean" convergence)*

Let f_n, $n = 1, 2, \ldots,$ *and* f *be functions in* $L^2(D)$. *We say that the sequence* (f_n) *converges in norm if:*

$$\lim_{n\to\infty} \|f_n - f\|_2 = 0.$$

In the case of the Fourier series of f, *we have that the Fourier series of* f *converges to* f *in mean if:*

$$\lim_{n\to\infty} \left[\int_{-\pi}^{\pi} \left[f(x) - \left(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + \sum_{k=1}^{n} b_k \sin(kx) \right) \right]^2 dx \right] = 0.$$

Basic Definition

Parallelogram identity
Generalization of the Pythagorean theorem
-> way to determine whether or not the norm is induced by an inner product

Definition 2.12. *Let $\mathcal{V}$ be a normed linear space and x, y be elements of $\mathcal{V}$. We say that x and y satisfy the parallelogram identity if*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

Examples of Hilbert Spaces

Euclidean space = real Hilbert space

Example 3.1. *The first example of a Hilbert space is* $\mathbb{R}^n$ *with the inner product,* $\langle (x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \rangle = x_1 y_1 + x_2 y_2 + \cdots x_n y_n$.
This inner product induces the norm $\|(x_1, x_2, ...x_n)\|$ *and the metric* $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \ldots + (x_n - y_n)^2}$.

Complex linear spaces

Example 3.4. *Another example of a Hilbert space is* $L^2(\mathbb{R})$*, as defined in Definition* 2.8*, is the set of complex valued functions* $f(t)$ *on the real number line with* $\int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$ *(that is,* f *is square integrable).* $L^2(\mathbb{R})$ *is known as the space of square integrable functions. Its inner product is defined as* $< f, g >= \int_{-\infty}^{\infty} f(t)\overline{g(t)} dt$.

Types of Convergence

When does a Fourier series converge to its function?
If it does converge, what type of convergence?

Uniform and Pointwise convergence

Proposition 4.1. *Let D be a subset of $\mathbb{R}$, and $f_n, n = 1, 2, \ldots$ and f be complex valued functions on D. If $f_n \to f$ uniformly, then $f_n \to f$ pointwise.*

Mean Convergence

Proposition 4.2. *Let $f_n, f \in L^2(D)$. If $f_n \to f$ uniformly, then $f_n \to f$ in mean.*

Convergence of the Fourier Series

Definition 4.1. *[PZ97] The space E is the space of piecewise continuous functions on the interval* $[-\pi, \pi]$.

Definition 4.2. *[PZ97] The space E' is the space of of all functions $f(x) \in E$ such that the right-hand derivative, $D_+ f(x)$, exists for all $-\pi \leq x < \pi$ and the left-hand derivative, $D_- f(x)$, exists for all $-\pi \leq x < \pi$.*

Every continuous function is a piecewise continuous function i.e belonging to E.
Every function which is differentiable on [-π, π] belongs to E'.
All functions in E belong to L^2([-π, π])

Convergence of the Fourier Series

Theorem 4.3. *[PZ97] If* $f \in E'$ *and*

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k cos(kx) + b_k sin(kx))$$

then the series converges pointwise to

$$\frac{f(x_+) + f(x_-)}{2}.$$

That is

$$S_N(x) \to \frac{f(x_+) + f(x_-)}{2} \quad as\ N \to \infty.$$

In particular, we have

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k cos(kx) + b_k sin(kx))$$

at every point $x \in [-\pi, \pi]$ *where f(x) is continuous.*

Convergence of the Fourier Series

Example 4.4. *Given the function:*

$$f(x) = \begin{cases} 1 & : -\pi \leq x < 0 \\ 0 & : 0 \leq x < \pi \end{cases}$$

Show that its Fourier series is:

$$\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{((-1)^n - 1)}{n} sin(nx)$$

Recall the Fourier series is given by:

$$f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m cos(mt) + b_m sin(mt))$$

Proof

The Fourier coefficients of f for $k \geq 1$ are given by:

$$a_k = \frac{\langle f, cos(kt)\rangle}{\langle cos(kt), cos(kt)\rangle} \text{ and } b_k = \frac{\langle f, sin(kt)\rangle}{\langle sin(kt), sin(kt)\rangle}$$

$$where\ \langle cos(kt), cos(kt)\rangle = \pi \text{ and } \langle sin(kt), sin(kt)\rangle = \pi$$

To see why this is true, let $k = 1$ and therefore

$$\langle cost, cost\rangle = \int_{-\pi}^{\pi} cos^2 t dt$$

Since, $cos^2 t = \frac{1}{2}cos(2t) + \frac{1}{2}$, we have,

$$\int_{-\pi}^{\pi} \frac{1}{2}cos(2t) + \frac{1}{2} = \left[\frac{1}{4}sin(2t) + \frac{1}{2}t\right]_{-\pi}^{\pi} = \pi$$

Similarly, this is also true for $\langle sin(kt), sin(kt)\rangle$.

Convergence of the Fourier Series

When $k = 0$,

$$\frac{a_0}{2} = \frac{\langle f, cos0 \rangle}{\langle cos0, cos0 \rangle} = \frac{\langle f, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\pi}{2\pi} = \frac{1}{2}$$

Therefore the first term in the Fourier series is $\frac{a_0}{2} = \frac{1}{2}$.

When $k \geq 1$, then,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)cos(kt)dt \text{ and } b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)sin(kt)dt$$

For $k \geq 1$, we have

$$a_k = \frac{1}{\pi}\int_{-\pi}^{0} f(t)cos(kt)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)cos(kt)dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{0} cos(kt)dt + \frac{1}{\pi}\int_{0}^{\pi} 0dt$$
$$= \left[\frac{1}{kt}sin(kt)\right]_{-\pi}^{0} + 0 = 0$$

Thus, $a_k = 0$ for all $k \geq 1$ since $sin(k\pi) = 0$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{0} f(t)sin(kt)dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)sin(kt)$$
$$= \frac{1}{-\pi}\int_{-\pi}^{0} sin(kt)dt + \frac{1}{\pi}\int_{0}^{\pi} 0dt$$
$$= \left[\frac{-1}{k\pi}cos(kt)\right]_{-\pi}^{0}$$

$= \frac{-1}{k\pi}[cos0 + cos(k\pi)] = \frac{-1}{k\pi}[1 \pm cos(k\pi)]$ depending on the value of k.

$$cos\,(k\pi) = \begin{cases} -1 & : \text{k is even} \\ 1 & : \text{k is odd} \end{cases}$$

When k is odd, $b_k = 0$. When k is even, then b_k is the series:

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}sin(nx).$$

Hence, the Fourier Series for $f(x)$ is:

Convergence of the Fourier Series

$$\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)$$

If we assume this series converges in mean to f, we should have:

$$\lim_{n\to\infty}\left[\int_{-\pi}^{\pi}\left[f(x) - \left(\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)\right)\right]^2 dx\right] = 0$$

We will first evaluate:

$$\left\|f(x) - \left(\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)\right)\right\|^2 =$$

$$\left\langle f(x) - \left(\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)\right), f(x) - \left(\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)\right)\right\rangle$$

$$= \langle f, f\rangle - 2\left\langle f, \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)\right\rangle$$

$$+ \left\langle \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx), \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)\right\rangle$$

Convergence of the Fourier Series

$$= \langle f, f\rangle - \left\langle f, 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)\right\rangle$$

$$+ \left\langle \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx), \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)\right\rangle =$$

$$\langle f, f\rangle - \langle f, 1\rangle - \left\langle f, \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)\right\rangle + \int_{-\pi}^{\pi}\left|\frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx)\right|^2 dx$$

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\int_{-\pi}^{0} sin(nx)dx$$

$$+ \int_{-\pi}^{\pi}\left(\frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{((-1)^n - 1)}{n} sin(nx) + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n}\right)^2 sin^2(nx)\right) dx$$

(Since $\langle f, f\rangle = \langle f, 1\rangle = \pi$.)

$$= -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\left[\frac{-1}{n} cos(nx)\right]_{-\pi}^{0} + \frac{1}{4}(2\pi) + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1)}{n}\right)^2\int_{-\pi}^{\pi} sin^2(nx)$$

Convergence of the Fourier Series

$(Since \int_{-\pi}^{\pi} sin(nx)dx = 0)$.

$$= \frac{-2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\left(\frac{-1}{n} + \frac{1}{n}(-1)^n\right) + \frac{\pi}{2} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{-1)^n - 1}{n}\right)^2 \cdot \pi$$

$$= \frac{-2}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{2n+1}\right)^2 + \frac{\pi}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{2n+1}\right)^2$$

Thus,

$$\frac{\pi}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{2}{2n+1}\right)^2 = 0$$

And,

$$\frac{\pi}{2} = \frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}\right)^2$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\left(\frac{1}{2n+1}\right)^2$$

Hence, the series converges in mean.

The Projection Theorem

Theorem 5.2. *(The Projection Theorem) [RS80] Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ a closed subspace. Then every $x \in \mathcal{M}$ can be uniquely written $x = z + w$ where $z \in \mathcal{M}$ and $w \in \mathcal{M}^{\perp}$*

Proof. Let x be in $\mathcal{H}$. Then by the lemma, there is a unique element $z \in \mathcal{M}$ closest to x. Define $w = x - z$, then we have $x = z + w$. Let $y \in \mathcal{M}$ and $t \in \mathbb{R}$. If $d = \|x - z\|$, then

$$d^2 \leq \|x - (z + ty)\|^2 = \|w - ty\|^2 = d^2 - 2tRe(w, y) + t^2 \|y\|^2$$

Thus, $-2tRe(w, y) + t^2 \|y\|^2 \geq 0$ for all t, and $Re(w, y) = 0$. Similarly, substituting ti instead of t produces $Im(w, y) = 0$. Hence, $w \in \mathcal{M}^{\perp}$.

To show uniqueness, we need to show that we have a unique z and w. Choose $z_1 \in \mathcal{M}$ and $w_1 \in \mathcal{M}^{\perp}$. We have $x = z + w = z_1 + w_1$. Thus, $x = z - z_1 = w_1 - w$. Since $z - z_1 \in \mathcal{M}$ and $w_1 - w \in \mathcal{M}^{\perp}$, the only element in both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ is 0. Hence, $z - z_1 = 0$ and $w_1 - w = 0$, so $z = z_1$ and $w = w_1$.

The projection theorem contends that the closest function o f in the span of the orthogonal set $\{f_k\}$ is the orthogonal projection onto the space spanned by this set

Convergence of the Fourier Series in the $L^2 - norm$

Theorem 7.1. *[Sax01] Assume that*

1. $\{d_k\}_{k=1}^{\infty}$ *is a sequence of real numbers such that* $\sum_{k=1}^{\infty} d_k^2$ *converges, and*
2. $\mathcal{V}$ *is a Hilbert space with complete orthonormal sequence* $\{f_k\}_{k=1}^{\infty}$.

Then there is an element $f \in \mathcal{V}$ *whose Fourier coefficients with respect to* $\{f_k\}_{k=1}^{\infty}$ *are the numbers* d_k *and*

$$\|f\|^2 = \sum_{k=1}^{\infty} d_k^2$$

Proof. Define $s_n = \sum_{k=1}^{n} d_k f_k$. For $m > n$, the square of the distance between s_n and s_m is as follows:

$$\|s_n - s_m\|^2 = \sum_{j=n+1}^{m} \sum_{k=n+1}^{m} d_j d_k \langle f_j, f_k \rangle = \sum_{k=n+1}^{m} d_k^2.$$

This is true since when $j = k$, $\langle f_j, f_k \rangle = 1$. Hence, $s_n = \sum_{k=1}^{n} d_k f_k$ is Cauchy. By assumption, if $f \in \mathcal{V}$, which is a Hilbert space, then there is an $f \in \mathcal{V}$ such that

$$\lim_{n \to \infty} \|s_n - f\| = 0.$$

Therefore, $f = \sum_{k=1}^{\infty} d_k f_k$ and $d_k = \langle f, f_k \rangle$. Since Parseval's theorem states $\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 = \|f\|^2$, it follows that $\|f\|^2 = \sum_{k=1}^{\infty} d_k^2$. □

The sum of squares of the Fourier coefficients (d_k) is finite.

Convergence of the Fourier Series in the $L^2 - norm$

Theorem 7.2. *[Sax01] For an orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ in $L^2([-\pi,\pi],m)$, the following are equivalent:*

1. *$\{f_k\}_{k=1}^{\infty}$ is a complete orthonormal sequence.*
2. *For every $f \in L^2$ and $\epsilon > 0$ there is a finite linear combination*

$$g = \sum_{k=1}^{n} d_k f_k$$

   such that $\|f - g\| \leq \epsilon$

3. *If the Fourier coefficients with respect to $\{f_k\}_{k=1}^{\infty}$ of a function in L^2 are all 0, then the function is equal to 0 almost everywhere.*

Relationship between Fourier series and Hilbert spaces

The space of periodic L^2 functions (say with period 2π) forms a Hilbert space. (Here L^2 means that $\int_0^{2\pi} f(x)^2\,dx$ exists.)

The inner product of two functions is given by $\int_0^{2\pi} f(x)g(x)dx$. (Here and above I am thinking of real-valued functions; for complex valued functions the formulas are similar.)

Now we consider two facts, one about L^2-functions, and one about Hilbert space

- Every L^2-function can be expanded as a Fourier series.
- Every Hilbert space admits an orthonormal basis, and each vector in the Hilbert space can be expanded as a series in terms of this orthonormal basis.

It turns out that the first of these facts is a special case of the second: we can interpret the trigonometric functions as an orthonormal basis of the space of L^2-functions, and then the Fourier expansion of an arbitrary L^2-function *is* the same thing as its Hilbert space-theoretic expansion in terms of the orthonormal basis.

References

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https://math.stackexchange.com/questions/184390/relationship-of-fourier-series-and-hilbert-spaces