Pattern Formation

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Introduction

Pattern Formation
: topic in mathematical biology that studies how structures and patterns in nature evolve over time

- Alan Turing : we could obtain patterns with relatively simple PDE called **reaction-diffusion equations**.

-> we can get animal coat patterning to form due to interactions between chemicals that are sensitive to spatial and temporal factors.

$$“\ \frac{\partial u}{\partial t} = \underbrace{D_u \frac{\partial^2 u}{\partial x^2}}_{\text{diffusion term}} + \underbrace{f(u)}_{\text{reaction term}}\ ”$$

diffusion constant

Reactions

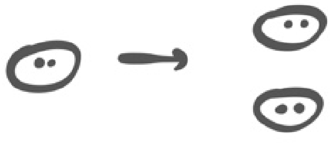
If s are reacting with zero -> #  will increase or decrease

ex) Exponential growth (1D)

Let u = # bacteria, then

$$\frac{du}{dt} = \alpha u \quad \text{with} \quad \alpha > 0 \quad \Rightarrow \quad u(t) = u_0 e^{\alpha t}, \quad u_0 = u(0)$$

In real life, bacteria will reproduce via cell division



let $u_0 = 1$, $\alpha = \ln 2$

$\Rightarrow u(t) = 2^t$

Increase exponentially

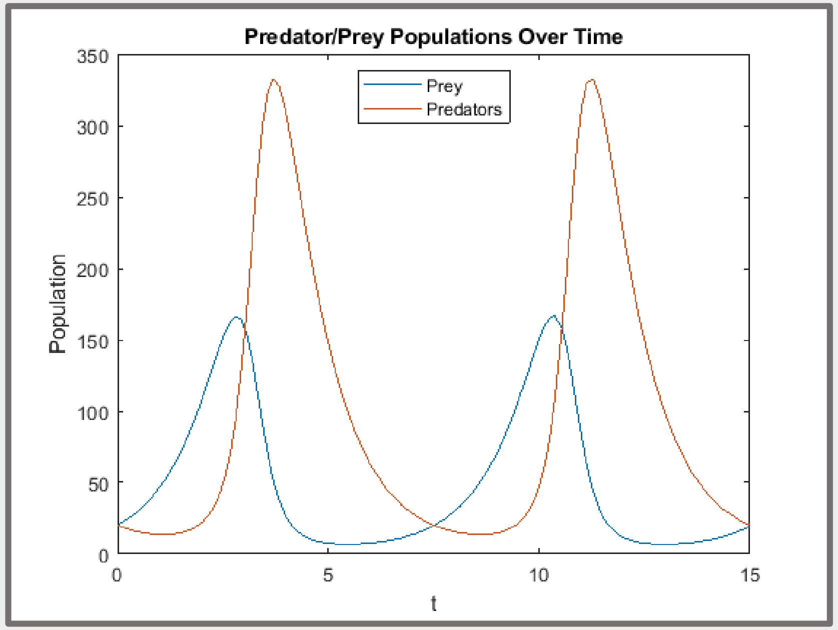
But in real life,

there are external factors

Reactions

ex) Predator – Prey model (2D)
Let u = # prey, v = # predator

" $\frac{du}{dt} = u - euv$, $\frac{dv}{dt} = vu - v$ "



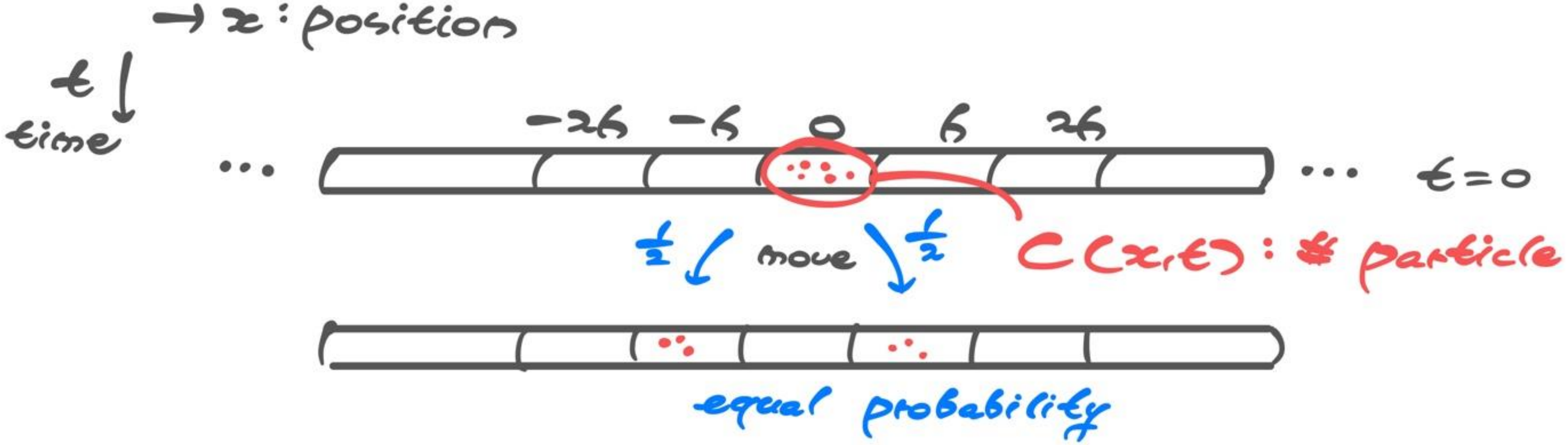
Populations of both species fluctuate but are cyclic

Differential Equations describing reactions
can produce interesting behavior
and phenomena in nature

Diffusion

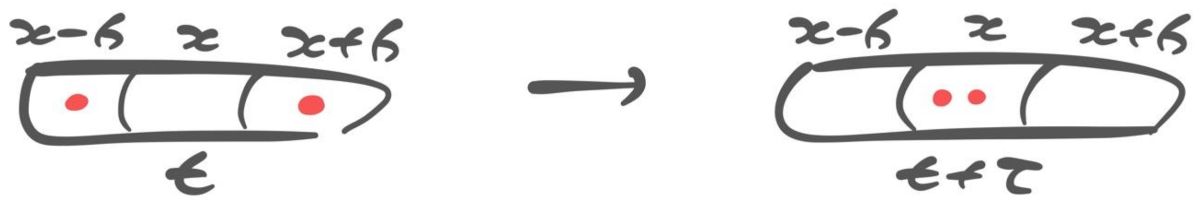
Diffusion describes how particles intermingle and move from areas of high concentration to low concentration

- Random walk on 1D (Brownian motion)



Diffusion

- Random walk on 1D (Brownian motion)



let $c(x,t) = \mathbb{E}(\#\text{ of particles})$,

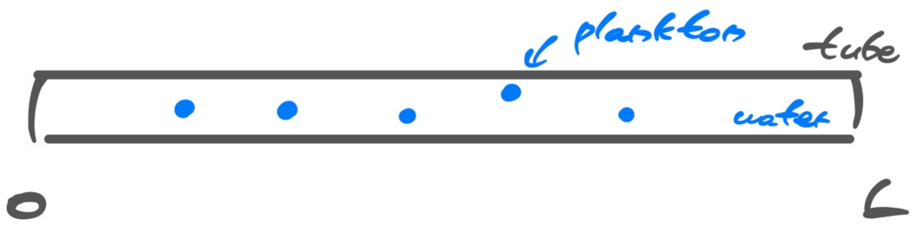
$$c(x, t+\tau) = \frac{1}{2}c(x+h, t) + \frac{1}{2}c(x-h, t)$$

$\xrightarrow[\text{Taylor Expansion}]{}$ as $\tau, h \to 0$, $\dfrac{\partial c}{\partial t} = D\dfrac{\partial^2 c}{\partial x^2}$, $D = \dfrac{h^2}{2\tau}$

Diffusion equation can be derived from Brownian motion

Spatial Domain

- 1D case



Plankton cannot survive outside of this tube
-> zero boundary conditions at both ends

K = reproduction rate, D = Diffusion constant
Reaction-Diffusion equations

: $$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + kc$$

Spatial Domain

Solution (Separation of variables and Fourier series)

$$c(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{(k - n^2\pi^2 D / L^2)t},$$

infinite sum

ft of space → product ← ft of time

$$B_n = \frac{2}{L}\int_0^L c_0 \sin\left(\frac{n\pi x}{L}\right) dx$$

Fourier coefficient

n = Fourier mode

Spatial Domain

Check the dominant behavior of c

For $n=1$, $C(x,t) = B_1 \sin\left(\frac{\pi x}{L}\right) e^{(k-\pi^2 D/L^2)t}$

- Sign of the argument

1. > 0, the population of plankton : "Growth"
2. < 0, "stable"
3. = 0, "decay"

as t -> inf

Spatial Domain

$$= k - \pi^2 \frac{D}{L^2} = 0 \Rightarrow L^2 = \pi^2 \frac{D}{k} \Rightarrow L = \pi \sqrt{\frac{D}{k}} = L_c$$

Critical length

-> how the population depends on the length L

1. L > Lc, the population of plankton will “increase”
2. L < Lc, “die out”
3. L = Lc, “converge to a nontrivial steady state”
   (zero at the end, maximun in the middle)

Spatial Domain

stabilizing destabilizing

$$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} + kc$$

$$L_c = \pi\sqrt{\frac{D}{k}}$$

1. L > Lc, reactive term dominates
2. L < Lc, diffusive term dominates
3. L = Lc, diffusive and reactive forces balance

Importance of spatial domains on the behavior of reaction-diffusion equations

Turing instability and Pattern formation

$$\text{“} \frac{\partial u}{\partial t} = \underbrace{D_u \frac{\partial^2 u}{\partial x^2}}_{\text{diffusion}} + \underbrace{f(u)}_{\text{reaction}} \text{”}$$

Alan Turing

: Reaction-diffusion systems could lead to patterns via diffusion driven instabilities

-> mimic a lot of patterns we observe in nature

Turing instability and Pattern formation

Activator-inhibitor systems
ex) Lengyel-Epstein model

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + k_1 \left(v - \frac{uv}{1+v^2} \right)$$

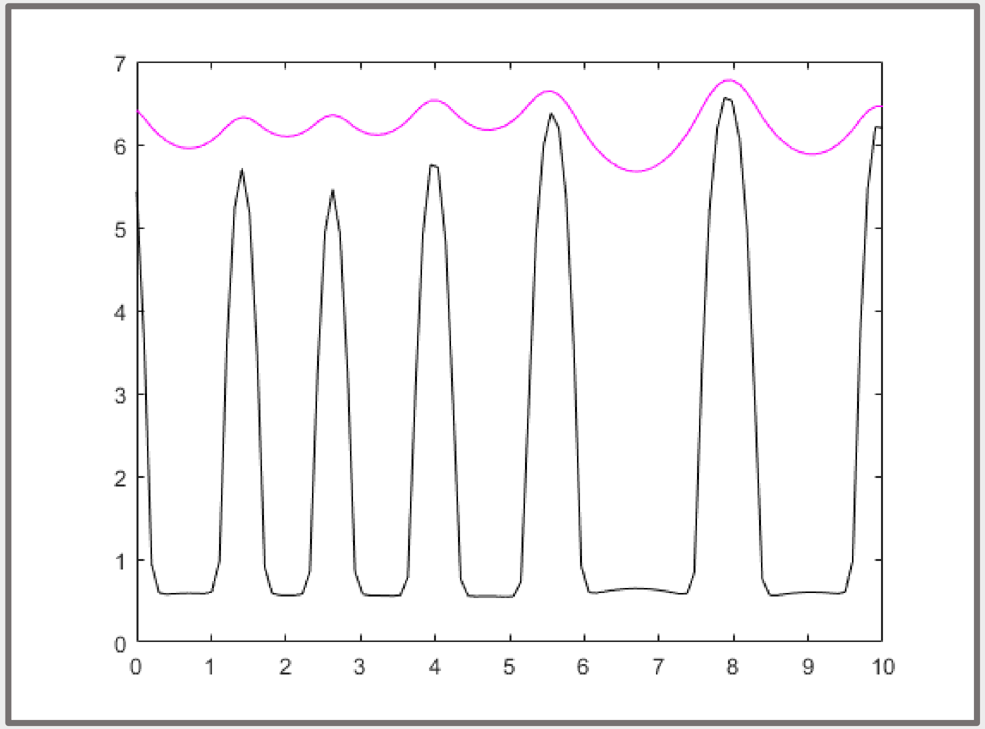
$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + k_2 - v - \frac{4uv}{1+v^2}$$

c = (u, v) representing hypothetical chemical morphogens in the integument
- Two chemicals that react with each other

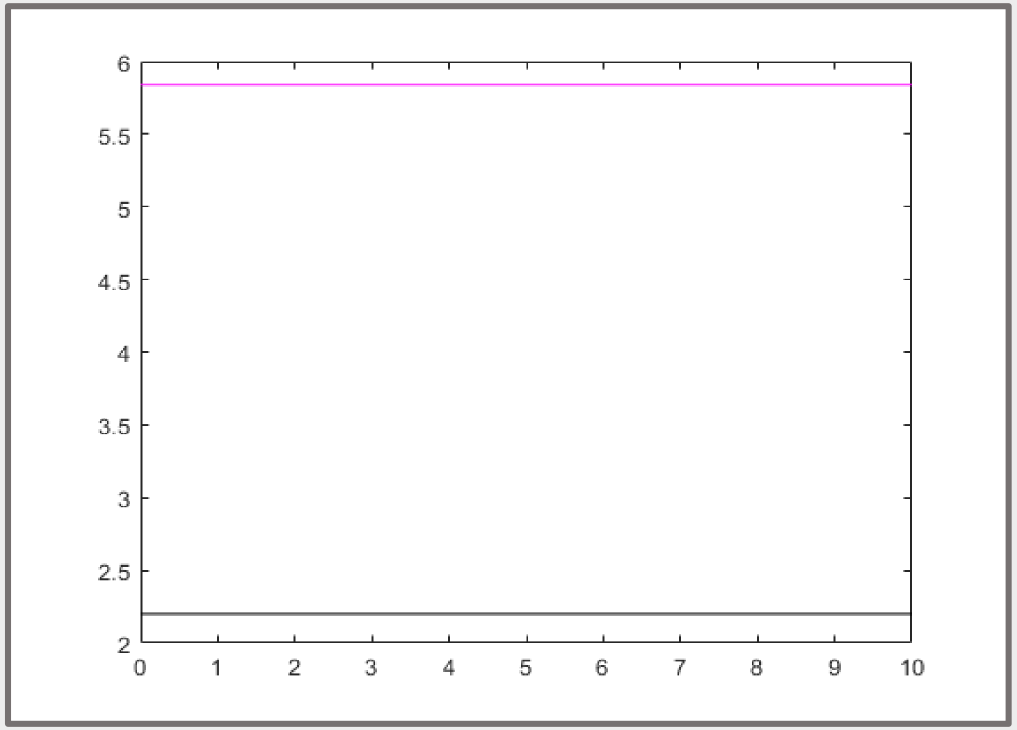
u : inhibitor -> increase the rate of degradation of the activator
v : activator -> promote the production of both

Turing instability and Pattern formation

1D



Dv = 0.01, K1 = 1

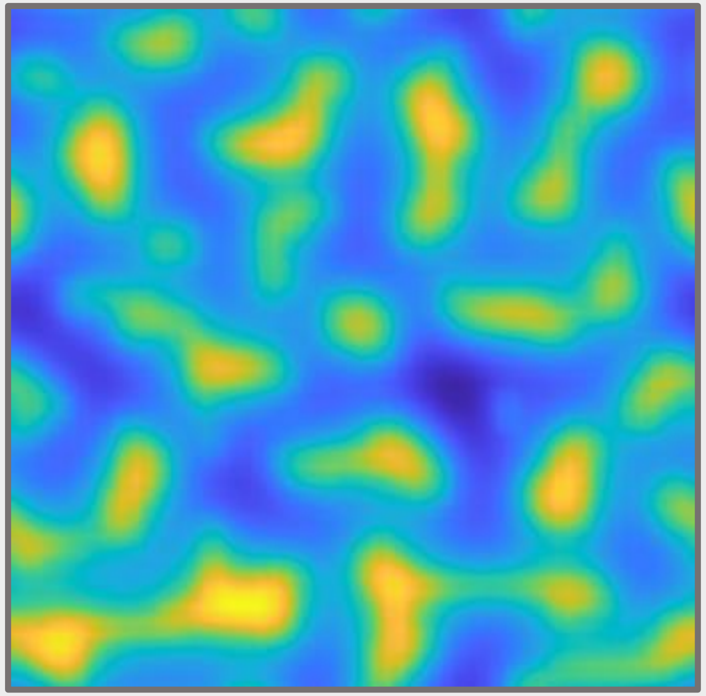


Dv = 0.06, K1 = 11

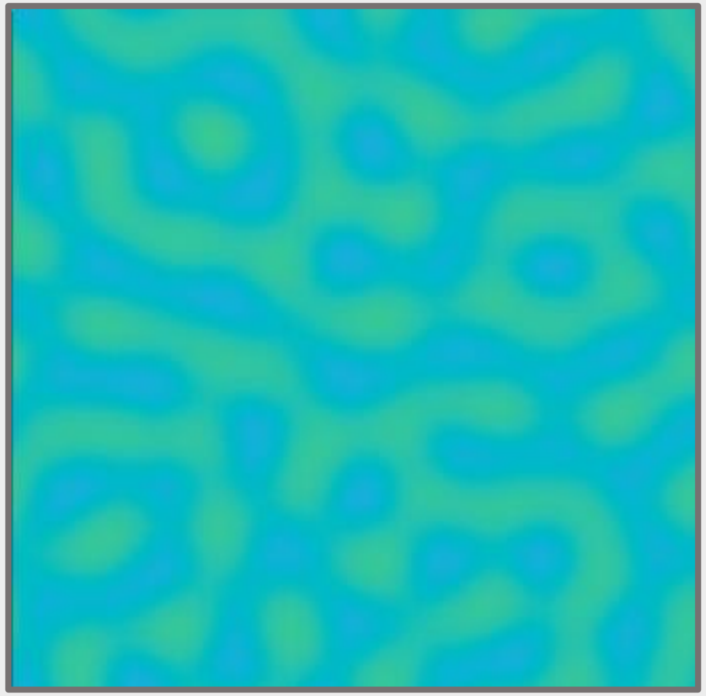
Turing instability and Pattern formation

2D

We can vary the size of domain on two directions



Dv = 0.01, K1 = 1



Dv = 0.06, K1 = 11

Turing instability and Pattern formation

Lengyel-Epstein model
-> Produce many of the commonly occurring pattern-types observed in animal skins

Why do tigers have stripes
- This model suggest that patterns come out of balancing out the reaction and diffusion chemicals with a spatial domain that they diffuse in
-> Why aren't human spotted and striped?
→ More variables.. Experiments…

References

[1] Solve Predator-Prey Equations, https://kr.mathworks.com/help/matlab/math/numerical-integration-of-differential-equations.html?lang=en

[2] Othmer, Hans G., et al. "The intersection of theory and application in elucidating pattern formation in developmental biology." *Mathematical modelling of natural phenomena* 4.4 (2009): 3-82.

[3] TheShapeofMath – The mathematics of patterns, https://www.youtube.com/playlist?list=PLILvrDLrWErxwurHeO0WnnV1ucCF1_6bl