Ten lectures on wavelets

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2.7. Parallels with the continuous windowed Fourier transform
2.8. The continuous transforms as tools to build useful operators
2.9. The continuous wavelet transform as a mathematical zoom :
The characterization of local regularity

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Parallels with the continuous windowed FT

- Windowed FT of function f

  : $(T^{win} f)(\omega, t) = \langle f, g^{\omega,t} \rangle$ where $g^{\omega,t}(x) = e^{i\omega x} g(x-t)$

- For $\forall f_1, f_2 \in L^2(\mathbb{R})$, $\iint d\omega\, dt\, (T^{win} f_1)(\omega, t) \overline{(T^{win} f_2)(\omega, t)} = 2\pi \|g\|^2 \langle f_1, f_2 \rangle$

  $\Rightarrow f = (2\pi \|g\|^2)^{-1} \iint d\omega\, dt\, (T^{win} f)(\omega, t) g^{\omega,t}$ for $\forall g \in L^2$

— convenient normalization for g is $\|g\|_{L^2} = 1$

Parallels with the continuous windowed FT

- Continuous windowed FT can be viewed as a map from $L^2(\mathbb{R})$ to reproducing kernel Hilbert space r.k.H.s

↪ the function $F \in T^{win}L^2(\mathbb{R})$ are all in $L^2(\mathbb{R}^2)$ and

$$F(\omega, t) = \frac{1}{2\pi} \iint d\omega' dt' \, K(\omega, t ; \omega', t') \, F(\omega', t')$$

where $K(\omega, t ; \omega', t') = \langle g^{\omega', t'}, g^{\omega, t} \rangle$

- $\exists$ special choice for g which reduce this r.k.H.s. to a Hilbert space of analytic functions

↪ For $g(x) = \pi^{-1/4} \exp(-x^2/2)$.

entire function

$$(T^{win} f)(\omega, t) = \exp\left(-\frac{1}{4}(\omega^2 + t^2) - \frac{i}{2}\omega t\right) \phi(\omega + it).$$

Parallels with the continuous windowed FT

- canonical coherent states

: $g^{w.t}$ obtained from $g(x) = g_0(x) = \pi^{-1/4} \exp(-x^2/2)$

↳ the associate continuous windowed FT

↳ applying the differential operator $H = -\frac{d^2}{dx^2} + x^2 - 1$ to $g_0(x)$

leads to $\left(-\frac{d^2}{dx^2} + x^2 - 1\right)\pi^{-1/4} \exp\left(\frac{-x^2}{2}\right) = 0$

i.e. g_0 is an eigenfunction of H with eigenvalue 0

↳ the other eigenfunctions of H are given by higher order Hermite functions.

$\phi_n(x) = \pi^{-1/4} 2^{-n/2} (n!)^{-1/2} \left(x - \frac{d}{dx}\right)^n \exp(-x^2/2)$ which satisfy $H\phi_n = 2n\phi_n$

The continuous transform as tools to build useful operators

- The resolution of identity

① $C_\psi^{-1} \iint \frac{da\,db}{a^2} \langle \cdot , \psi^{a,b} \rangle \psi^{a,b} = Id$, ② $\frac{1}{2\pi} \iint d\omega\, dt \langle \cdot , g^{\omega,t} \rangle g^{\omega,t} = Id$

where $\langle \cdot , \phi \rangle \phi$ stands for the operator on $L^2(\mathbb{R})$ that $f \to \langle f, \phi \rangle \phi$

↪ a rank one projection operator

i.e. ① its square and its adjoint are both identical to the operator itself

② its range is 1D

↪ ①, ② : "superposition" with equal weights. of the rank one projection operators

↪ If the weight function is (un)bounded. corresponding operator as well

The continuous transform as tools to build useful operators

ex insert a weight function $w(p,q)$ to windowed FT

: $W = \frac{1}{2\pi} \iint dp\, dq\, w(p,q) \langle \cdot, g^{p,q} \rangle g^{p,q}$

$\hookrightarrow$ $w \notin L^\infty(\mathbb{R}^2)$ then W : unbounded ($\because$ not everywhere defined)

∘ used to build time-frequency localization operators

Let S be any measurable subset of $\mathbb{R}^2$.

operator L_S corresponding to the indicator function of S.

$$a(\omega,t) = \begin{cases} 1 & \text{if } (\omega,t) \in S \\ 0 & \text{if } (\omega,t) \notin S \end{cases}, \quad L_S = \frac{1}{2\pi} \iint_{(\omega,t)\in S} d\omega\, dt \langle \cdot, g^{\omega,t} \rangle g^{\omega,t}$$

$$\Rightarrow \langle L_S f, f \rangle = \frac{1}{2\pi} \iint_{(\omega,t)\in S} d\omega\, dt\, |\langle f, g^{\omega,t} \rangle|^2 \le \frac{1}{2\pi} \iint d\omega\, dt\, |\langle f, g^{\omega,t} \rangle|^2 = \|f\|^2$$

by resolution of identity

On the other hand, $\langle L_S f, f \rangle \ge 0$ i.e $0 \le L_S \le Id$

The continuous transform as tools to build useful operators

special operator s.t $\sum_n |\langle Ae_n, e_n \rangle|$ is finite for $\forall$ orthogonal basis in $\mathcal{H}$.

$(\operatorname{tr} A = \sum_n \langle Ae_n, e_n \rangle)$

If S is bounded set, operator L_S is trace-class

$(\because)$ For any orthogonal basis $(u_n)_{n \in \mathbb{N}}$ in $L^2(\mathbb{R})$,

$$\sum_n \langle L_S u_n, u_n \rangle = \frac{1}{2\pi} \iint_{(\omega,t)\in S} d\omega\, dt \sum_n |\langle u_n, g^{\omega,t} \rangle|^2$$

by Lebesgue's dominated convergence thm

$$= \frac{1}{2\pi} \iint_{(\omega,t)\in S} d\omega\, dt\, \|g^{\omega,t}\|^2 = |S| \text{ measure of } S$$

It follows that $\exists$ a complete set of eigenvectors for L_S, with eigenvalues decreasing to zero.

$L_S \phi_n = \lambda_n \phi_n$, $\lambda_n \geq \lambda_{n+1} \geq 0$, $\lim_{n\to\infty} \lambda_n = 0$.

$\{\phi_n \mid n \in \mathbb{N}\}$ orthogonal basis for $L^2(\mathbb{R})$

The continuous transform as tools to build useful operators

- If window function g is well localized & centered around 0 in both time & freq then $\langle f, g^{\omega,t}\rangle g^{\omega,t}$: elementary component

↪ $L_S f$ is the sum of only those components for which $(\omega, t) \in S$

↪ For most choices of S, g, the eigenfunctions & eigenvalues of L_S are hard to characterize, and this construction is of limited usefulness.

↪ there is one choice of g & one family of sets S for which everything is transparent

Take $g(x) = g_0(x) = \pi^{-1/4} \exp(-x^2/2)$ & $S_R = \{(\omega, t) \mid \omega^2 + t^2 \le R^2\}$

corresponding localization operator $L_R = \frac{1}{2\pi} \iint_{\omega^2+t^2 \le R^2} d\omega\, dt \langle \cdot, g_0^{\omega,t}\rangle g_0^{\omega,t}$

↪ L_R commutes with the harmonic oscillator Hamiltonian $H = -\frac{d^2}{dx^2} + x^2 - 1$

The continuous transform as tools to build useful operators

ex $C_\psi^{-1} \iint \frac{da\,db}{a^2} \langle \cdot, \psi^{a,b} \rangle \psi^{a,b} = Id$

↪ restrict the integral to a subset S of (a,b) - space.

↪ one special ψ is $\hat{\psi}(\xi) = 2\xi e^{-\xi}$ for $\xi \geq 0$, 0 for $\xi \leq 0$

associated resolution of identity

: $C_\psi^{-1} \int_0^\infty \frac{da}{a^2} \int_{-\infty}^{\infty} db \left[\langle \cdot, \psi_+^{a,b} \rangle \psi_+^{a,b} + \langle \cdot, \psi_-^{a,b} \rangle \psi_-^{a,b} \right] = 1$

where $\psi_+ = \psi$, $\hat{\psi}_-(\xi) = \hat{\psi}(-\xi)$

$L_c = C_\psi^{-1} \iint_{(a,b) \in S_c} \frac{da\,db}{a^2} \left[\langle \cdot, \psi_+^{a,b} \rangle \psi_+^{a,b} + \langle \cdot, \psi_-^{a,b} \rangle \psi_-^{a,b} \right]$

with $S_c = \{ (a,b) \in \mathbb{R}_+ \times \mathbb{R} \mid a^2 + b^2 + 1 \leq 2ac \}$ & $c \geq 1$

↪ In the upper half complex plane, $z = b + ia$

S_c correspond to disks $|z - ic|^2 \leq c^2 - 1$

The continuous transform as tools to build useful operators

↪ The role of harmonic oscillator Hamiltonian

$$: \ H(f)^{\sim}(\xi) = \left(-\xi \frac{d^2}{d\xi^2} - \frac{d}{d\xi} + \xi + \frac{1}{\xi}\right) \tilde{f}(\xi)$$

↪ For this H, $\exp(-iHt)\psi_+^{a,b} = e^{i\alpha_t(a,b)} \psi_+^{a(t),b(t)}$

where $b(t) + ia(t) = z(t) = \dfrac{z\cos t + \sin t}{\cos t - z\sin t}$ with $z = b + ia$

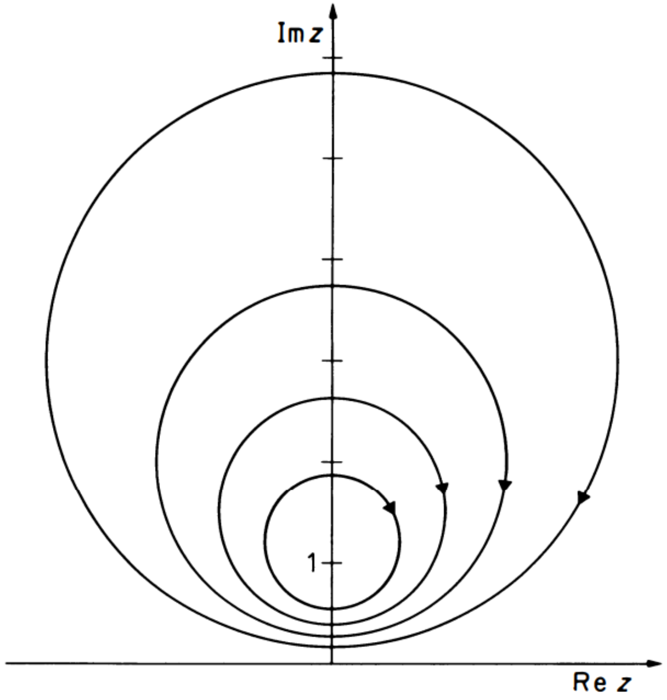

FIG. 2.5. *Flow lines for* $z(t) = \frac{z\cos t + \sin t}{\cos t - z\sin t}$.

The continuous transform as tools to build useful operators

↪ The eigenvalues of H have degeneracy 2 for $\forall$ eigenvalue $E_n = 3+2n$

⇒ 2 eigenfunctions

$$(\psi_n^+)^\sim(\xi) = \begin{cases} 2\sqrt{2}\,[(n+2)(n+1)]^{-1/2}\,\xi\, L_n^2(2\xi)e^{-\xi} & \text{for } \xi \geq 0 \\ 0 & \text{for } \xi \leq 0 \end{cases}$$

$$(\psi_n^-)^\sim(\xi) = (\psi_n^+)^\sim(-\xi)$$

Laguerre polynomial

$$\oplus\ L_n^\alpha(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n}(e^{-x} x^{n+\alpha}) = \sum_{m=0}^{n} (-1)^m \frac{\Gamma(n+\alpha+1)}{\Gamma(n-m+1)\Gamma(\alpha+m+1)} \frac{1}{m!} x^m$$

The continuous wavelet transform as a mathematical zoom

THEOREM 2.9.1. *Suppose that* $\int dx(1+|x|)\ |\psi(x)| < \infty$, *and* $\hat{\psi}(0) = 0$. *If a bounded function* f *is Hölder continuous with exponent* α, $0 < \alpha \leq 1$, *i.e.,*

$$|f(x) - f(y)| \leq C|x-y|^{\alpha} \ ,$$

then its wavelet transform satisfies

$$|T^{\mathrm{wav}}(a,b)| = |\langle f, \psi^{a,b}\rangle| \leq C'\ |a|^{\alpha+1/2} \ .$$

Proof. Since $\int dx\ \psi(x) = 0$ we have

$$\langle \psi^{a,b}, f\rangle = \int dx\ |a|^{-1/2}\ \psi\left(\frac{x-b}{a}\right) [f(x) - f(b)] \ ;$$

hence

$$\begin{aligned} |\langle \psi^{a,b}, f\rangle| &\leq \int dx\ |a|^{-1/2} \left|\psi\left(\frac{x-b}{a}\right)\right| C|x-b|^{\alpha} \\ &\leq C\ |a|^{\alpha+1/2} \int dy\ |\psi(y)|\ |y|^{\alpha} \\ &\leq C'\ |a|^{\alpha+1/2} \ . \qquad \blacksquare \end{aligned}$$

The continuous wavelet transform as a mathematical zoom

converse theorem

THEOREM 2.9.2. *Suppose that ψ is compactly supported. Suppose also that $f \in L^2(\mathbb{R})$ is bounded and continuous. If, for some $\alpha \in]0,1[$, the wavelet transform of f satisfies*

$$|\langle f, \psi^{a,b} \rangle| \leq C|a|^{\alpha+1/2} , \tag{2.9.1}$$

then f is Hölder continuous with exponent α.

Proof.

1. Choose ψ_2 compactly supported and continuously differentiable, with $\int dx\ \psi_2(x) = 0$. Normalize ψ_2 so that $C_{\psi,\psi_2} = 1$. Then, by Proposition 2.4.2,

$$f(x) = \int_{-\infty}^{\infty} \frac{da}{a^2} \int_{-\infty}^{\infty} db\ \langle f, \psi^{a,b} \rangle\ \psi_2^{a,b}(x) .$$

We will split the integral over a into two parts, $|a| \leq 1$ and $|a| \geq 1$, and call the two terms $f_{SS}(x)$ (small scales) and $f_{LS}(x)$ (large scales).

The continuous wavelet transform as a mathematical zoom

2. First of all, note that f_{LS} is bounded uniformly in x:

$$\begin{aligned} |f_{LS}(x)| &\leq \int_{|a|\geq 1} \frac{da}{a^2} \int_{-\infty}^{\infty} db\ |\psi_2^{a,b}(x)|\ \|f\|_{L^2}\ \|\psi\|_{L^2} \\ &\leq C \int_{|a|\geq 1} \frac{da}{a^2} \int_{-\infty}^{\infty} db\ |a|^{-1/2} \left|\psi_2\left(\frac{x-b}{a}\right)\right| \\ &\leq C\ \|\psi_2\|_{L^1} \int_{|a|\geq 1} da\ |a|^{-3/2} = C' < \infty\ . \end{aligned} \tag{2.9.2}$$

Next, we look at $|f_{LS}(x+h) - f_{LS}(x)|$ for $|h| \leq 1$:

$$|f_{LS}(x+h) - f_{LS}(x)| \leq \int_{|a|\geq 1} \frac{da}{|a|^3} \int_{-\infty}^{\infty} db \int_{-\infty}^{\infty} dy\ |f(y)| \\ \left|\psi\left(\frac{y-b}{a}\right)\right| \left|\psi_2\left(\frac{x+h-b}{a}\right) - \psi_2\left(\frac{x-b}{a}\right)\right| \tag{2.9.3}$$

The continuous wavelet transform as a mathematical zoom

Since $|\psi_2(z+t)-\psi_2(z)| \le C|t|$, and since support ψ, support $\psi_2 \subset [-R,R]$ for some $R<\infty$, we can bound this by

$$
\begin{aligned}
(2.9.3)\quad &\le C'\,|h| \int_{|a|\ge 1} da\; a^{-4} \int_{\substack{|x-b|\le |a|R+1 \\ |y-b|\le |a|R}} db \int dy\; |f(y)| \\
&\le C''\,|h| \int_{|a|\ge 1} da\; |a|^{-3} \int_{|y-x|\le 2|a|R+1} dy\; |f(y)| \\
&\le C''\,|h|\; \|f\|_{L^2} \int_{|a|\ge 1} da\; |a|^{-3}(4|a|R+2)^{1/2} \le C'''\,|h|\,.
\end{aligned}
$$

This holds for all $|h|\le 1$; together with the bound (2.9.2), we conclude that $|f_{LS}(x+h)-f_{LS}(x)| \le C|h|$ for all h, uniformly in x. Note that we did not even use (2.9.1) in this estimate: f_{LS} is always regular.

The continuous wavelet transform as a mathematical zoom

3. The small scale part f_{SS} is also uniformly bounded:

$$
\begin{aligned}
|f_{SS}(x)| &\leq C \int_{|a|\leq 1} \frac{da}{a^2} \int_{-\infty}^{\infty} db\ |a|^{\alpha+1/2}\ |a|^{-1/2} \left|\psi_2\left(\frac{x-b}{a}\right)\right| \\
&\leq C\|\psi_2\|_{L^1} \int_{|a|\leq 1} da\ |a|^{-1+\alpha} = C' < \infty\ .
\end{aligned}
$$

4. We therefore again only have to check $|f_{SS}(x+h) - f_{SS}(x)|$ for small h, such as $|h| \leq 1$. Using again $|\psi_2(z+t) - \psi_2(z)| \leq C|t|$, we have

$$
\begin{aligned}
&|f_{SS}(x+h) - f_{SS}(x)| \\
&\quad\leq \int_{|a|\leq|h|} \frac{da}{a^2} \int_{-\infty}^{\infty} db\ |a|^{\alpha} \left(\left|\psi_2\left(\frac{x-b}{a}\right)\right| + \left|\psi_2\left(\frac{x+h-b}{a}\right)\right|\right) \\
&\qquad\quad + \int_{|h|\leq|a|\leq 1} \frac{da}{a^2} \int_{|x-b|\leq|a|R+|h|} db\ |a|^{\alpha}\ C\left|\frac{h}{a}\right| \\
&\quad\leq C'\left[\|\psi_2\|_{L^2} \int_{|a|\leq|h|} da\ |a|^{-1+\alpha} + |h| \int_{|h|\leq|a|\leq 1} da\ |a|^{-3+\alpha}(|a|R+|h|)\right] \\
&\quad= C''\ |h|^{\alpha}\ .
\end{aligned}
$$

The continuous wavelet transform as a mathematical zoom

THEOREM 2.9.3. *Suppose that* $\int dx\ (1+|x|)\ |\psi(x)| < \infty$ *and* $\int dx\ \psi(x) = 0$. *If a bounded function* f *is Hölder continuous in* x_0*, with exponent* $\alpha \in]0,1]$*, i.e.,*

$$|f(x_0+h) - f(x_0)| \leq C|h|^\alpha \ ,$$

then

$$|\langle f, \psi^{a,x_0+b}\rangle| \leq C|a|^{1/2}\ (|a|^\alpha + |b|^\alpha)\ .$$

Proof. By translating everything we can assume that $x_0 = 0$. Because $\int dx\ \psi(x) = 0$, we again have

$$\begin{aligned} |\langle f, \psi^{a,b}\rangle| &\leq \int dx\ |f(x) - f(0)|\ |a|^{-1/2}\ \left|\psi\left(\frac{x-b}{a}\right)\right| \\ &\leq C\int dx\ |x|^\alpha\ |a|^{-1/2}\ \left|\psi\left(\frac{x-b}{a}\right)\right| \\ &\leq C|a|^{\alpha+1/2}\int dy\ \left|y+\frac{b}{a}\right|^\alpha\ |\psi(y)| \\ &\leq C'\ |a|^{1/2}\ (|a|^\alpha + |b|^\alpha)\ . \qquad \blacksquare \end{aligned}$$