Quantum Limits (Noise, Measurement, Metrology, and Information)

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Conventions and scope. Unless otherwise stated, $S(\omega)$ denotes a (classical) one-sided power spectral density when used in RMS-in-bandwidth formulas (e.g., $\Delta I_{\mathrm{rms}} = \sqrt{S_I\,\Delta f}$), while $\bar{S}(\omega)$ denotes a *symmetrized* (quantum) noise spectral density. When comparing to one-sided engineering conventions for stationary noise, a common conversion is $S^{(1)}(\omega > 0) = 2\,\bar{S}(\omega)$ (and similarly for cross-spectra). Noise temperature conventions can differ by constant factors if one uses the full quantum Nyquist expression for the source (this is definitional, not new physics). Several entries are *standard quantum limits* (SQLs): they are rigorous under stated assumptions (e.g., coherent probes, phase-preserving detection, linear response), and they can be surpassed by changing the assumptions (e.g., squeezed states, entanglement, quantum nondemolition measurement, or back-action evasion).

#	**Limit/ bound name**	**Equation (LaTeX)**	**Category/ context**	**Key symbols, assumptions, and "how to beat"**	**Primary ref.**
1	Poisson shot-noise current (RMS in bandwidth Δf)	$\Delta I_{\mathrm{rms}} = \sqrt{2qI\,\Delta f}$	Electronic transport; photon/electron counting; Poisson statistics	I average current, q carrier charge, Δf measurement bandwidth. Equivalent one-sided current-noise PSD: $S_I^{(1)} = 2qI$. *Not an ultimate limit:* sub-Poisson sources (Fano factor < 1) reduce it.	(standard)
2	Coherent-beam number–phase noise constraint (optical shot noise/ SQL for phase readout)	$S_{\dot{N}\dot{N}}(\omega)\,S_{\phi\phi}(\omega) \geq \frac{1}{4},\ S_{\phi\phi}^{\mathrm{coh}}(\omega) = \frac{1}{4\dot{N}} = \frac{\hbar\omega}{4\bar{P}}$	Quantum optics; homodyne/heterodyne phase sensing; interferometry	$\dot{N}$ photon flux, $\bar{P} = \hbar\omega\dot{N}$ optical power, ϕ phase. *Beatable:* phase squeezing (reduce $S_{\phi\phi}$) at cost of increased conjugate-quadrature noise; see squeezed-input interferometry.	[7, 6]
3	Phase-preserving amplifier quantum limit (Haus–Caves)	$n_{\mathrm{add}} \geq \frac{1}{2}\ (G \gg 1),\ T_{\mathrm{N}} \geq \frac{hf}{2k_{\mathrm{B}}}$	Linear amplification with equal treatment of both quadratures ("phase preserving")	n_{add} added quanta referred to the input; T_{N} input-referred noise temperature; f signal frequency ($\omega = 2\pi f$). The second line matches Eq. (143) in [19]. *Definition caveat:* alternative noise-temperature conventions (e.g., using the full quantum equilibrium source noise) can introduce constants like $\ln 3$ while leaving the half-quantum added-noise physics unchanged. *Evasion:* phase-sensitive amplification can add ≈ 0 quanta to *one* quadrature.	[5, 7, 19]
4	Free-mass standard quantum limit (SQL) for displacement monitoring (time-domain form)	$(\Delta x)_{\mathrm{SQL}} = \sqrt{\frac{2\hbar\,\tau}{m}}$	Continuous position readout of a free mass; gravitational-wave interferometers	m test-mass, τ averaging/integration time. *Interpretation:* optimal balance of measurement imprecision and radiation-pressure back-action in a standard (phase-preserving) readout. *Beatable:* squeezing, variational readout, speed-meters, QND strategies.	[6, 3]

#	Limit/ bound name	Equation (LaTeX)	Category/ context	Key symbols, assumptions, and "how to beat"	Primary ref.
5	Imprecision–back-action (Heisenberg) product for a linear continuous measurement (simplest form)	$\bar{S}_{xx}^{\mathrm{imp}}(\omega)\,\bar{S}_{FF}^{\mathrm{ba}}(\omega) \geq \frac{\hbar^2}{4}$	General linear measurement; quantum noise tradeoff	$\bar{S}_{xx}^{\mathrm{imp}}$ imprecision noise, $\bar{S}_{FF}^{\mathrm{ba}}$ back-action force noise. *Assumption:* no useful imprecision–back-action correlations are exploited (cf. row 9). *Beatable in a narrow sense:* correlations and/or QND observables reshape total noise.	[3, 7, 16]
6	Heisenberg scaling for single-parameter phase estimation (entanglement-enabled)	$\Delta\phi \geq \frac{1}{\sqrt{\nu}\,N}$	Quantum metrology (unitary parameter encoding)	N quanta per probe state; ν independent repetitions. Often contrasted with shot-noise scaling $\Delta\phi \sim 1/\sqrt{\nu N}$. *Caveat:* decoherence/noise can destroy N-scaling, restoring shot-noise scaling in many realistic settings.	[9]
7	Shot-noise limit (SNL) for phase estimation (separable probes)	$\Delta\phi \geq \frac{1}{\sqrt{\nu\,N}}$	Classical-like metrology with independent probes (no entanglement)	N uncorrelated particles/photons per repetition; ν repetitions. *Beatable:* entanglement (Heisenberg scaling) or squeezing (improved constant factor).	[9]
8	Quantum Cramér–Rao bound (QCRB)	$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{\nu\,F_Q(\theta)},\ \Delta\theta \geq \frac{1}{\sqrt{\nu\,F_Q(\theta)}}$	Ultimate single-parameter estimation bound; quantum Fisher information (QFI)	$F_Q(\theta)$ quantum Fisher information of ρ_θ; ν repetitions. Shot-noise and Heisenberg scalings are corollaries of upper bounds on F_Q under different resource assumptions.	[4, 9]
9	General quantum noise inequality with imprecision–back-action correlations	$\bar{S}_{xx}^{\mathrm{imp}}(\omega)\,\bar{S}_{FF}^{\mathrm{ba}}(\omega) - \left\lvert \bar{S}_{xF}(\omega) \right\rvert^2 \geq \frac{\hbar^2}{4}$	Linear-response detector theory; correlated quantum noise	$\bar{S}_{xF}$ is the (symmetrized) cross-spectrum. *Meaning:* correlations can reduce total added noise at selected frequencies (variational measurement), though the inequality still constrains what is possible.	[7]
10	Quantum nondemolition (QND) measurement efficiency bound (measurement vs dephasing)	$\eta \equiv \frac{\Gamma_{\mathrm{meas}}}{\Gamma_\varphi} \leq 1$	Continuous QND qubit readout; measurement chains	Γ_{meas} measurement rate; Γ_φ measurement-induced dephasing rate. $\eta = 1$ is the quantum-limited (ideal) case; technical loss/noise yields $\eta < 1$.	[7]
11	Landauer bound (bit erasure cost)	$Q_{\mathrm{diss}} \geq k_{\mathrm{B}} T \ln 2$	Information thermodynamics; quantum/classical computing hardware	Q_{diss} heat dissipated to a bath at temperature T by logically irreversible erasure. *Not "purely quantum":* thermodynamic, but fundamental; relevant to quantum information engines and control costs.	[13]

#	Limit/ bound name	Equation (LaTeX)	Category/ context	Key symbols, assumptions, and “how to beat”	Primary ref.
12	Holevo bound (accessible classical information from a quantum ensemble)	$I_{\mathrm{acc}}(X{:}Y) \le \chi,$ $\chi \equiv S(\rho) - \sum_x p_x S(\rho_x),\ \rho = \sum_x p_x \rho_x$	Quantum Shannon theory; communication; measurement limits	$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$ is von Neumann entropy. *Beatable?* Not in general without changing the task: χ is the task-dependent upper bound on accessible information for the ensemble.	[11, 20]
13	Helstrom bound (minimum error for binary quantum state discrimination)	$P_{e,\min} = \frac{1}{2}\Big(1 - \lVert p_0\rho_0 - p_1\rho_1 \rVert_1\Big)$	Quantum hypothesis testing; optimal measurements	$\lVert \cdot \rVert_1$ trace norm; priors p_0, p_1; states ρ_0, ρ_1. *Meaning:* ultimate performance of any measurement for binary discrimination.	[10, 20]
14	Frequency-domain SQL for linear displacement sensing (uncorrelated imprecision and back-action)	$\bar{S}_{xx}^{\mathrm{SQL}}(\omega) = \hbar\,\lvert\chi_{xx}(\omega)\rvert\,,$ $S_{xx}^{(1),\mathrm{SQL}}(\omega > 0) = 2\hbar\,\lvert\chi_{xx}(\omega)\rvert$	Continuous displacement measurement; optomechanics; interferometry; noise budgeting	$\chi_{xx}(\omega)$ is the mechanical susceptibility (response of x to force). *Assumptions:* linear response; phase-preserving measurement; no use of imprecision–back-action correlations. *Useful special case:* for a free mass, $\lvert\chi_{xx}(\omega)\rvert = 1/(m\omega^2)$ so $S_{xx}^{(1),\mathrm{SQL}}(\omega > 0) = 2\hbar/(m\omega^2)$. *Beatable:* variational readout (correlations), back-action evasion, and QND choices of measured observable.	[7, 3]
15	Quantum speed limit (QSL) for state evolution (Mandelstam–Tamm and Margolus–Levitin)	$\tau \ge \max\!\Big(\frac{\pi\hbar}{2\,\Delta E},\ \frac{\pi\hbar}{2\,(\langle E\rangle - E_0)}\Big)$	Quantum control; gate-time lower bounds; bandwidth–time tradeoffs	ΔE is the energy standard deviation in the initial state; E_0 the ground-state energy. *Use:* lower-bounds how fast one can drive an isolated system to an orthogonal state (or, more generally, a target fidelity). *Caveat:* open-system dynamics and constrained controls modify practical limits; see modern QSL reviews.	[14, 15, 8]
16	Quantum Chernoff bound (QCB) for asymptotic binary state discrimination	$P_{e,\min}^{(n)} \le \frac{1}{2}\exp\!\Big[-n\,\xi_{\mathrm{QCB}}\Big]\,,$ $\xi_{\mathrm{QCB}} = -\ln\!\Big(\min_{0\le s\le 1} \mathrm{Tr}\Big[\rho_0^s \rho_1^{1-s}\Big]\Big)$	Quantum hypothesis testing; readout classification; sensing error exponents	n is the number of independent copies; ρ_0, ρ_1 are the hypotheses. *Meaning:* sets the optimal exponential rate at which error decays with repeated trials; complements the single-shot Helstrom bound (row 13).	[2, 20]

#	Limit/ bound name	Equation (LaTeX)	Category/ context	Key symbols, assumptions, and "how to beat"	Primary ref.
17	Quantum Ziv–Zakai bound (QZZB) for mean-square error (Bayesian/global)	$\Sigma \geq \frac{1}{2}\int_0^\infty d\tau\,\tau\,\mathcal{V}\int_{-\infty}^\infty dx$ $2\min\Big[P_X(x), P_X(x+\tau)\Big]$ $\times P_{e,\mathrm{rel}}(x, x+\tau),$ $P_{e,\mathrm{rel}}(x, x+\tau) \geq \frac{1}{2}\left[1-\sqrt{1-F(\rho_x, \rho_{x+\tau})}\right]$	Bayesian metrology; global parameter estimation; low-SNR/non-asymptotic regimes	Σ mean-square error (MSE); P_X prior density; $\mathcal{V}$ is a "valley-filling" tightening operation used in ZZB/QZZB constructions. $P_{e,\mathrm{rel}}$ is the minimum binary discrimination error probability between hypotheses x and $x+\tau$ with the *relative* priors induced by P_X. $F(\rho,\sigma) = (\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}})^2$ is the Uhlmann fidelity. *Why useful:* complements the (local) QCRB (row 8) when the likelihood is multimodal, the prior is non-negligible, or performance is far from the asymptotic regime. *"Beatable":* only by changing the estimation task (prior), resources, or physical model; otherwise it is a lower bound on achievable mean-square error (MSE).	[18, 21]
18	Multi-parameter quantum Cramér–Rao bound (QCRB; QFIM/SLD form)	$\mathrm{Cov}(\hat{\boldsymbol{\theta}}) \succeq \frac{1}{\nu}\,\mathbf{F}_Q^{-1},$ $[\mathbf{F}_Q]_{ij} = \mathrm{Tr}\Big[\rho_\theta\,\frac{L_iL_j+L_jL_i}{2}\Big],$ $\partial_i\rho_\theta = \frac{L_i\rho_\theta+\rho_\theta L_i}{2}$	Multi-parameter metrology; imaging; simultaneous estimation of noncommuting generators	$\boldsymbol{\theta}$ parameter vector; $\mathbf{F}_Q$ quantum Fisher information matrix (QFIM); L_i symmetric logarithmic derivatives (SLDs). *Attainability caveat:* unlike the single-parameter case, the SLD-QCRB can be *not simultaneously saturable* when optimal measurements for different parameters are incompatible. *What to use when incompatible:* the Holevo Cramér–Rao bound (row 19) is the standard tight benchmark for general multi-parameter problems.	[10, 17]
19	Holevo Cramér–Rao bound (HCRB; tight multi-parameter quantum precision bound)	$\mathrm{Tr}\Big[W\,\mathrm{Cov}(\hat{\boldsymbol{\theta}})\Big] \geq \frac{1}{\nu}\,C_H(W),$ $C_H(W) = \min_{\{X_i\}}\Big\{\mathrm{Tr}[W\,\mathrm{Re}\,Z]$ $+\Big\|\sqrt{W}(\mathrm{Im}\,Z)\sqrt{W}\Big\|_1\Big\},$ $Z_{ij} = \mathrm{Tr}(\rho_\theta\,X_iX_j)$	Multi-parameter metrology; simultaneous estimation; benchmark when measurements are incompatible	$W \succeq 0$ is a user-chosen weight matrix for the scalar figure of merit $\mathrm{Tr}[W\,\mathrm{Cov}]$. The minimization is over Hermitian operator sets $\{X_i\}$ satisfying unbiasedness constraints (e.g., $\mathrm{Tr}[\rho_\theta X_i] = 0$ and $\mathrm{Tr}[X_i\,\partial_j\rho_\theta] = \delta_{ij}$ in common formulations). $\|\cdot\|_1$ is the trace norm. *Use:* when the SLD-QCRB (row 18) is not jointly attainable, the HCRB is the standard "right" quantum benchmark for what is fundamentally possible.	[12, 1, 17]

Acronym Glossary

Acronym	Expansion
RMS	Root-mean-square
MSE	Mean-square error
PSD	Power spectral density
SQL	Standard quantum limit
SNL	Shot-noise limit
QCRB	Quantum Cramér–Rao bound
QFI	Quantum Fisher information
QFIM	Quantum Fisher information matrix
QND	Quantum nondemolition
TLS	Two-level system
ZPF	Zero-point fluctuations
QSL	Quantum speed limit
QCB	Quantum Chernoff bound
MT	Mandelstam–Tamm
ML	Margolus–Levitin
ZZB	Ziv–Zakai bound
QZZB	Quantum Ziv–Zakai bound
SLD	Symmetric logarithmic derivative
HCRB	Holevo Cramér–Rao bound

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