

Quantum Limits (Noise, Measurement, Metrology, and Information)

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Conventions and scope. Unless otherwise stated, $S(\omega)$ denotes a (classical) one-sided power spectral density when used in RMS-in-bandwidth formulas (e.g., $\Delta I_{\text{rms}} = \sqrt{\bar{S}_I \Delta f}$), while $\bar{S}(\omega)$ denotes a *symmetrized* (quantum) noise spectral density. When comparing to one-sided engineering conventions for stationary noise, a common conversion is $S^{(1)}(\omega > 0) = 2\bar{S}(\omega)$ (and similarly for cross-spectra). Noise temperature conventions can differ by constant factors if one uses the full quantum Nyquist expression for the source (this is definitional, not new physics). Several entries are *standard quantum limits* (SQLs): they are rigorous under stated assumptions (e.g., coherent probes, phase-preserving detection, linear response), and they can be surpassed by changing the assumptions (e.g., squeezed states, entanglement, quantum nondemolition measurement, or back-action evasion).

#	Limit/ bound name	Equation (LaTeX)	Category/ context	Key symbols, assumptions, and “how to beat”	Primary ref.
1	Poisson shot-noise current (RMS in bandwidth Δf)	$\Delta I_{\text{rms}} = \sqrt{2qI \Delta f}$	Electronic transport; photon/electron counting; Poisson statistics	I average current, q carrier charge, Δf measurement bandwidth. Equivalent one-sided current-noise PSD: $S_I^{(1)} = 2qI$. <i>Not an ultimate limit</i> : sub-Poisson sources (Fano factor < 1) reduce it.	(standard)
2	Coherent-beam number–phase noise constraint (optical shot noise/ SQL for phase readout)	$S_{\dot{N}\dot{N}}(\omega) S_{\phi\phi}(\omega) \geq \frac{1}{4}, S_{\phi\phi}^{\text{coh}}(\omega) = \frac{1}{4\dot{N}} = \frac{\hbar\omega}{4\bar{P}}$	Quantum optics; homodyne/heterodyne phase sensing; interferometry	\dot{N} photon flux, $\bar{P} = \hbar\omega\dot{N}$ optical power, ϕ phase. <i>Beatable</i> : phase squeezing (reduce $S_{\phi\phi}$) at cost of increased conjugate-quadrature noise; see squeezed-input interferometry.	[7, 6]
3	Phase-preserving amplifier quantum limit (Haus–Caves)	$n_{\text{add}} \geq \frac{1}{2} \ (G \gg 1), T_{\text{N}} \geq \frac{\hbar f}{2k_{\text{B}}}$	Linear amplification with equal treatment of both quadratures (“phase preserving”)	n_{add} added quanta referred to the input; T_{N} input-referred noise temperature; f signal frequency ($\omega = 2\pi f$). The second line matches Eq. (143) in [19]. <i>Definition caveat</i> : alternative noise-temperature conventions (e.g., using the full quantum equilibrium source noise) can introduce constants like $\ln 3$ while leaving the half-quantum added-noise physics unchanged. <i>Evasion</i> : phase-sensitive amplification can add ≈ 0 quanta to <i>one</i> quadrature.	[5, 7, 19]
4	Free-mass standard quantum limit (SQL) for displacement monitoring (time-domain form)	$(\Delta x)_{\text{SQL}} = \sqrt{\frac{2\hbar\tau}{m}}$	Continuous position readout of a free mass; gravitational-wave interferometers	m test-mass, τ averaging/integration time. <i>Interpretation</i> : optimal balance of measurement imprecision and radiation-pressure back-action in a standard (phase-preserving) readout. <i>Beatable</i> : squeezing, variational readout, speed-meters, QND strategies.	[6, 3]

#	Limit/ bound name	Equation (LaTeX)	Category/ context	Key symbols, assumptions, and “how to beat”	Primary ref.
5	Imprecision–back-action (Heisenberg) product for a linear continuous measurement (simplest form)	$\bar{S}_{xx}^{\text{imp}}(\omega) \bar{S}_{FF}^{\text{ba}}(\omega) \geq \frac{\hbar^2}{4}$	General linear measurement; quantum noise tradeoff	$\bar{S}_{xx}^{\text{imp}}$ imprecision noise, \bar{S}_{FF}^{ba} back-action force noise. <i>Assumption:</i> no useful imprecision–back-action correlations are exploited (cf. row 9). <i>Beatable in a narrow sense:</i> correlations and/or QND observables reshape total noise.	[3, 7, 16]
6	Heisenberg scaling for single-parameter phase estimation (entanglement-enabled)	$\Delta\phi \geq \frac{1}{\sqrt{\nu} N}$	Quantum metrology (unitary parameter encoding)	N quanta per probe state; ν independent repetitions. Often contrasted with shot-noise scaling $\Delta\phi \sim 1/\sqrt{\nu N}$. <i>Caveat:</i> decoherence/noise can destroy N -scaling, restoring shot-noise scaling in many realistic settings.	[9]
7	Shot-noise limit (SNL) for phase estimation (separable probes)	$\Delta\phi \geq \frac{1}{\sqrt{\nu N}}$	Classical-like metrology with independent probes (no entanglement)	N uncorrelated particles/photons per repetition; ν repetitions. <i>Beatable:</i> entanglement (Heisenberg scaling) or squeezing (improved constant factor).	[9]
8	Quantum Cramér–Rao bound (QCRB)	$\text{Var}(\hat{\theta}) \geq \frac{1}{\nu F_Q(\theta)}, \Delta\theta \geq \frac{1}{\sqrt{\nu F_Q(\theta)}}$	Ultimate single-parameter estimation bound; quantum Fisher information (QFI)	$F_Q(\theta)$ quantum Fisher information of ρ_θ ; ν repetitions. Shot-noise and Heisenberg scalings are corollaries of upper bounds on F_Q under different resource assumptions.	[4, 9]
9	General quantum noise inequality with imprecision–back-action correlations	$\bar{S}_{xx}^{\text{imp}}(\omega) \bar{S}_{FF}^{\text{ba}}(\omega) - \left \bar{S}_{xF}(\omega) \right ^2 \geq \frac{\hbar^2}{4}$	Linear-response detector theory; correlated quantum noise	\bar{S}_{xF} is the (symmetrized) cross-spectrum. <i>Meaning:</i> correlations can reduce total added noise at selected frequencies (variational measurement), though the inequality still constrains what is possible.	[7]
10	Quantum nondemolition (QND) measurement efficiency bound (measurement vs dephasing)	$\eta \equiv \frac{\Gamma_{\text{meas}}}{\Gamma_\varphi} \leq 1$	Continuous QND qubit readout; measurement chains	Γ_{meas} measurement rate; Γ_φ measurement-induced dephasing rate. $\eta = 1$ is the quantum-limited (ideal) case; technical loss/noise yields $\eta < 1$.	[7]
11	Landauer bound (bit erasure cost)	$Q_{\text{diss}} \geq k_B T \ln 2$	Information thermodynamics; quantum/classical computing hardware	Q_{diss} heat dissipated to a bath at temperature T by logically irreversible erasure. <i>Not “purely quantum”:</i> thermodynamic, but fundamental; relevant to quantum information engines and control costs.	[13]

#	Limit/ bound name	Equation (LaTeX)	Category/ context	Key symbols, assumptions, and “how to beat”	Primary ref.
12	Holevo bound (accessible classical information from a quantum ensemble)	$I_{\text{acc}}(X:Y) \leq \chi,$ $\chi \equiv S(\rho) - \sum_x p_x S(\rho_x), \quad \rho = \sum_x p_x \rho_x$	Quantum Shannon theory; communication; measurement limits	$S(\rho) = -\text{Tr}(\rho \log \rho)$ is von Neumann entropy. <i>Beatable?</i> Not in general without changing the task: χ is the task-dependent upper bound on accessible information for the ensemble.	[11, 20]
13	Helstrom bound (minimum error for binary quantum state discrimination)	$P_{e,\min} = \frac{1}{2} \left(1 - \ p_0 \rho_0 - p_1 \rho_1\ _1 \right)$	Quantum hypothesis testing; optimal measurements	$\ \cdot\ _1$ trace norm; priors p_0, p_1 ; states ρ_0, ρ_1 . <i>Meaning:</i> ultimate performance of any measurement for binary discrimination.	[10, 20]
14	Frequency-domain SQL for linear displacement sensing (uncorrelated imprecision and back-action)	$\bar{S}_{xx}^{\text{SQL}}(\omega) = \hbar \chi_{xx}(\omega) ,$ $S_{xx}^{(1),\text{SQL}}(\omega > 0) = 2\hbar \chi_{xx}(\omega) $	Continuous displacement measurement; optomechanics; interferometry; noise budgeting	$\chi_{xx}(\omega)$ is the mechanical susceptibility (response of x to force). <i>Assumptions:</i> linear response; phase-preserving measurement; no use of imprecision–back-action correlations. <i>Useful special case:</i> for a free mass, $ \chi_{xx}(\omega) = 1/(m\omega^2)$ so $S_{xx}^{(1),\text{SQL}}(\omega > 0) = 2\hbar/(m\omega^2)$. <i>Beatable:</i> variational readout (correlations), back-action evasion, and QND choices of measured observable.	[7, 3]
15	Quantum speed limit (QSL) for state evolution (Mandelstam–Tamm and Margolus–Levitin)	$\tau \geq \max\left(\frac{\pi\hbar}{2\Delta E}, \frac{\pi\hbar}{2\langle E \rangle - E_0}\right)$	Quantum control; gate-time lower bounds; bandwidth–time tradeoffs	ΔE is the energy standard deviation in the initial state; E_0 the ground-state energy. <i>Use:</i> lower-bounds how fast one can drive an isolated system to an orthogonal state (or, more generally, a target fidelity). <i>Caveat:</i> open-system dynamics and constrained controls modify practical limits; see modern QSL reviews.	[14, 15, 8]
16	Quantum Chernoff bound (QCB) for asymptotic binary state discrimination	$P_{e,\min}^{(n)} \leq \frac{1}{2} \exp\left[-n \xi_{\text{QCB}}\right],$ $\xi_{\text{QCB}} = -\ln\left(\min_{0 \leq s \leq 1} \text{Tr}[\rho_0^s \rho_1^{1-s}]\right)$	Quantum hypothesis testing; readout classification; sensing error exponents	n is the number of independent copies; ρ_0, ρ_1 are the hypotheses. <i>Meaning:</i> sets the optimal exponential rate at which error decays with repeated trials; complements the single-shot Helstrom bound (row 13).	[2, 20]

#	Limit/ bound name	Equation (LaTeX)	Category/ context	Key symbols, assumptions, and “how to beat”	Primary ref.
17	Quantum Ziv–Zakai bound (QZZB) for mean-square error (Bayesian/global)	$\Sigma \geq \frac{1}{2} \int_0^\infty d\tau \tau \mathcal{V} \int_{-\infty}^\infty dx$ $2 \min[P_X(x), P_X(x + \tau)]$ $\times P_{e,\text{rel}}(x, x + \tau),$ $P_{e,\text{rel}}(x, x + \tau) \geq \frac{1}{2} \left[1 - \sqrt{1 - F(\rho_x, \rho_{x+\tau})} \right]$	Bayesian metrology; global parameter estimation; low-SNR/non-asymptotic regimes	Σ mean-square error (MSE); P_X prior density; \mathcal{V} is a “valley-filling” tightening operation used in ZZB/QZZB constructions. $P_{e,\text{rel}}$ is the minimum binary discrimination error probability between hypotheses x and $x + \tau$ with the <i>relative</i> priors induced by P_X . $F(\rho, \sigma) = (\text{Tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}})^2$ is the Uhlmann fidelity. <i>Why useful</i> : complements the (local) QCRB (row 8) when the likelihood is multimodal, the prior is non-negligible, or performance is far from the asymptotic regime. <i>“Beatable”</i> : only by changing the estimation task (prior), resources, or physical model; otherwise it is a lower bound on achievable mean-square error (MSE).	[18, 21]
18	Multi-parameter quantum Cramér–Rao bound (QCRB; QFIM/SLD form)	$\text{Cov}(\hat{\boldsymbol{\theta}}) \succeq \frac{1}{\nu} \mathbf{F}_Q^{-1},$ $[\mathbf{F}_Q]_{ij} = \text{Tr} \left[\rho_\theta \frac{L_i L_j + L_j L_i}{2} \right],$ $\partial_i \rho_\theta = \frac{L_i \rho_\theta + \rho_\theta L_i}{2}$	Multi-parameter metrology; imaging; simultaneous estimation of noncommuting generators	$\boldsymbol{\theta}$ parameter vector; \mathbf{F}_Q quantum Fisher information matrix (QFIM); L_i symmetric logarithmic derivatives (SLDs). <i>Attainability caveat</i> : unlike the single-parameter case, the SLD-QCRB can be <i>not simultaneously saturable</i> when optimal measurements for different parameters are incompatible. <i>What to use when incompatible</i> : the Holevo Cramér–Rao bound (row 19) is the standard tight benchmark for general multi-parameter problems.	[10, 17]
19	Holevo Cramér–Rao bound (HCRB; tight multi-parameter quantum precision bound)	$\text{Tr} [W \text{Cov}(\hat{\boldsymbol{\theta}})] \geq \frac{1}{\nu} C_H(W),$ $C_H(W) = \min_{\{X_i\}} \left\{ \text{Tr} [W \text{Re } Z] \right.$ $\left. + \left\ \sqrt{W} (\text{Im } Z) \sqrt{W} \right\ _1 \right\},$ $Z_{ij} = \text{Tr}(\rho_\theta X_i X_j)$	Multi-parameter metrology; simultaneous estimation; benchmark when measurements are incompatible	$W \succeq 0$ is a user-chosen weight matrix for the scalar figure of merit $\text{Tr}[W \text{Cov}]$. The minimization is over Hermitian operator sets $\{X_i\}$ satisfying unbiasedness constraints (e.g., $\text{Tr}[\rho_\theta X_i] = 0$ and $\text{Tr}[X_i \partial_j \rho_\theta] = \delta_{ij}$ in common formulations). $\ \cdot\ _1$ is the trace norm. <i>Use</i> : when the SLD-QCRB (row 18) is not jointly attainable, the HCRB is the standard “right” quantum benchmark for what is fundamentally possible.	[12, 1, 17]

Acronym Glossary

Acronym	Expansion
RMS	Root-mean-square
MSE	Mean-square error
PSD	Power spectral density
SQL	Standard quantum limit
SNL	Shot-noise limit
QCRB	Quantum Cramér–Rao bound
QFI	Quantum Fisher information
QFIM	Quantum Fisher information matrix
QND	Quantum nondemolition
TLS	Two-level system
ZPF	Zero-point fluctuations
QSL	Quantum speed limit
QCB	Quantum Chernoff bound
MT	Mandelstam–Tamm
ML	Margolus–Levitin
ZZB	Ziv–Zakai bound
QZZB	Quantum Ziv–Zakai bound
SLD	Symmetric logarithmic derivative
HCRB	Holevo Cramér–Rao bound

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