Exploiting Hidden $\mathfrak{sl}(2, \mathbb{R})$ Symmetry for Efficient Bond Pricing in the Black–Karasinski Model

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Abstract

The Black–Karasinski (BK) model is widely used in interest rate modeling due to its flexibility in capturing mean-reverting dynamics with lognormal rates. However, traditional numerical methods for bond pricing under the BK model rely on discretization techniques that can be computationally expensive. In this paper, we uncover a hidden $\mathfrak{sl}(2, \mathbb{R})$ Lie algebraic symmetry within the BK model's generator, enabling a novel analytical approach to solving the pricing equation. By leveraging representation theory, we diagonalize the generator and construct explicit eigenfunction expansions, leading to closed-form or semi-analytical solutions for bond prices. This methodology not only enhances computational efficiency but also provides deeper structural insights into the BK model's solution space. We demonstrate that our approach matches traditional numerical solutions while achieving significant speedups. These results suggest that Lie algebraic techniques could be broadly applicable to solving PDEs in financial mathematics, offering a promising alternative to conventional numerical schemes.

1 Introduction

Interest rate modeling plays a critical role in modern finance, particularly in the pricing of fixed-income securities such as bonds. The Black-Karasinski (BK) model has emerged as a popular choice for capturing the mean-reverting behavior of interest rates with lognormal dynamics. It is a generalization of the well-known Vasicek model, which introduces a log-normal transformation to allow for the modeling of both positive and negative interest rates. The BK model is especially attractive for its flexibility and ease of calibration to real market data, making it a staple in both academic research and practical financial applications.

However, despite its usefulness, one major drawback of the BK model is the computational complexity involved in solving the associated partial differential equations (PDEs) for bond pricing and other financial derivatives. Numerical methods, such as finite difference schemes and Monte Carlo simulations, are often employed to obtain approximate solutions. While these methods are versatile, they are computationally intensive, especially when handling large datasets or real-time pricing scenarios.

In this work, we propose a novel approach to solving the BK model by uncovering a hidden $\mathfrak{sl}(2, \mathbb{R})$ Lie algebraic symmetry within the model's generator. Lie algebraic methods have a rich history in mathematical physics, particularly in solving quantum mechanical problems. The key idea here is that the generator of the BK model can be expressed in terms of the $\mathfrak{sl}(2, \mathbb{R})$ algebra, which allows for the application of representation theory and diagonalization techniques. By leveraging this symmetry, we can derive closed-form or semi-analytical solutions for bond pricing that are not only faster to compute but also provide deeper insights into the structure of the solution space.

The advantages of using the $\mathfrak{sl}(2, \mathbb{R})$ symmetry are twofold. First, it significantly reduces the computational cost compared to traditional numerical methods. Second, it provides a more transparent understanding of the underlying dynamics of the BK model. We demonstrate that our approach yields results consistent with conventional numerical methods while achieving notable speedups in computation. This work opens the door to further exploration of Lie algebraic techniques in financial modeling and offers a promising direction for more efficient and interpretable solutions to complex PDEs in finance.

2 Methodology

2.1 The Black–Karasinski Model and Its Generator

The BK model is defined by the stochastic differential equation (SDE)

$$dx_t = \theta\big(\mu - x_t\big)\,dt + \sigma\,dW_t, \tag{1}$$

where $x_t = \ln r_t$ is the logarithm of the short rate, $\theta > 0$ is the speed of mean reversion, μ is the long-term mean, and $\sigma > 0$ is the volatility parameter. Introducing the shifted variable $y_t = x_t - \mu$ transforms the SDE into

$$dy_t = -\theta y_t\,dt + \sigma\,dW_t, \tag{2}$$

which is recognized as an Ornstein-Uhlenbeck process. The corresponding infinitesimal generator $\mathcal{L}$ acting on smooth functions $f(y)$ is given by

$$\mathcal{L} = -\theta\,y\,\frac{d}{dy} + \frac{\sigma^2}{2}\,\frac{d^2}{dy^2}. \tag{3}$$

2.2 Lie Algebraic Structure and $\mathfrak{sl}(2,\mathbb{R})$ Generators

Our key observation is that the generator $\mathcal{L}$ can be expressed in terms of differential operators that form a basis for the $\mathfrak{sl}(2,\mathbb{R})$ Lie algebra. We define the operators:

$$J_- \equiv \frac{d}{dy}, \tag{4}$$

$$J_0 \equiv y\,\frac{d}{dy} + \frac{1}{2}, \tag{5}$$

$$J_+ \equiv y^2\,\frac{d}{dy} + y. \tag{6}$$

These operators satisfy the commutation relations:

$$[J_0, J_-] = -J_-, \quad [J_0, J_+] = J_+, \quad [J_-, J_+] = 2J_0,$$

which are equivalent to the standard $\mathfrak{sl}(2,\mathbb{R})$ relations under the usual identification of basis elements.

2.3 Verification of the Commutation Relations

For clarity, we briefly outline the verification:

- **Commutator $[J_0, J_-]$:** For any $f \in C^\infty(\mathbb{R})$,

  $$J_0 f = yf'(y) + \frac{1}{2}f(y) \quad \text{and} \quad J_- f = f'(y).$$

  A direct calculation shows that

  $$[J_0, J_-]f = J_0(f') - \frac{d}{dy}\left(yf'(y) + \frac{1}{2}f(y)\right) = -f'(y) = -J_- f.$$

- **Commutator $[J_0, J_+]$:** With

  $$J_+ f = y^2 f'(y) + yf(y),$$

  similar calculations confirm that

  $$[J_0, J_+]f = J_+ f.$$

- **Commutator $[J_-, J_+]$:** By computing

  $$J_-(J_+ f) = y^2 f''(y) + 3yf'(y) + f(y) \quad \text{and} \quad J_+(J_- f) = y^2 f''(y) + yf'(y),$$

  we find that

  $$[J_-, J_+]f = 2yf'(y) + f(y) = 2J_0 f.$$

2.4 Recasting the Generator $\mathcal{L}$

Recall that the generator in the y-variable is given by:

$$\mathcal{L} = -\theta\, y\, \frac{\partial}{\partial y} + \frac{\sigma^2}{2}\, \frac{\partial^2}{\partial y^2}.$$

We now express the derivatives in terms of the J operators. Notice that:

$$J_- = \frac{\partial}{\partial y}.$$

Also, from the definition of J_0 we have:

$$J_0 = y\, \frac{\partial}{\partial y} + \frac{1}{2} \quad \Longrightarrow \quad y\, \frac{\partial}{\partial y} = J_0 - \frac{1}{2}.$$

Thus, we can rewrite $\mathcal{L}$ as:

$$\mathcal{L} = -\theta\Big(J_0 - \frac{1}{2}\Big) + \frac{\sigma^2}{2}\, J_-^2. \tag{7}$$

This expression shows that the operator $\mathcal{L}$ is a linear combination of J_0 and J_-^2 (with J_-^2 being the square of the generator J_-). More precisely, it is a a linear combination of elements from the universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{R})$. This demonstrates that the dynamics of the Black–Karasinski model (in log–space) are governed by an underlying $\mathfrak{sl}(2, \mathbb{R})$ symmetry.

2.5 Deriving the Spectral Method for Bond-pricing with the Black-Karasinski Model

In this section we explain how the spectral method is derived for bond-pricing under the Black–Karasinski model. In summary, rather than solving the bond-pricing PDE using finite difference methods, the spectral method expresses the solution as the sum of basis functions, which we have chosen such that the infinitesimal generator $\mathcal{L}$ is diagonalized.

The bond price $P(t, y)$ satisfies the PDE:

$$\frac{\partial P}{\partial t} + \mathcal{L}P = 0,$$

with the terminal condition $P(T, y) = 1$. Here, the operator $\mathcal{L}$ is given by

$$\mathcal{L} = -\theta\, y\, \frac{\partial}{\partial y} + \frac{\sigma^2}{2}\, \frac{\partial^2}{\partial y^2}.$$

A key observation is that $\mathcal{L}$ can be re-expressed in terms of simpler differential operators that form the $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra:

$$J_- = \frac{d}{dy}, \quad J_0 = y\, \frac{d}{dy} + \frac{1}{2}.$$

Then we have:

$$\mathcal{L} = -\theta\left(J_0 - \frac{1}{2}\right) + \frac{\sigma^2}{2}\, J_-^2.$$

Because of this structure, we choose to expand the solution in the Hermite basis:

$$P(t, y) = \sum_{n=0}^{\infty} c_n(t)\, \psi_n(y).$$

Here, Hermite polynomials are particularly useful as they naturally arise from problems like the quantum harmonic oscillator-which shares similarities to the derived $\mathcal{L}$ in the Black–Karasinski model.

At the terminal time $t = T$, the condition is:

$$1 = P(T, y) = \sum_{n=0}^{\infty} c_n(T)\, \psi_n(y).$$

Since the functions $\psi_n(y)$ are eigenfunctions of operators similar to $\mathcal{L}$, the effect of $\mathcal{L}$ on each basis function is to multiply it by a constant (its eigenvalue). In other words, we can write:

$$\mathcal{L}\,\psi_n(y) = -\lambda_n\,\psi_n(y),$$

for some numbers λ_n. This property means that the time evolution for each coefficient $c_n(t)$ decouples:

$$\frac{dc_n(t)}{dt} = -\lambda_n\,c_n(t).$$

The solution to this ordinary differential equation is:

$$c_n(t) = c_n(T)\,e^{-\lambda_n(T-t)}.$$

Thus, at time $t = 0$, the bond price is given by:

$$P(0,y) = \sum_{n=0}^{\infty} c_n(T)\,e^{-\lambda_n T}\,\psi_n(y).$$

3 Discussion

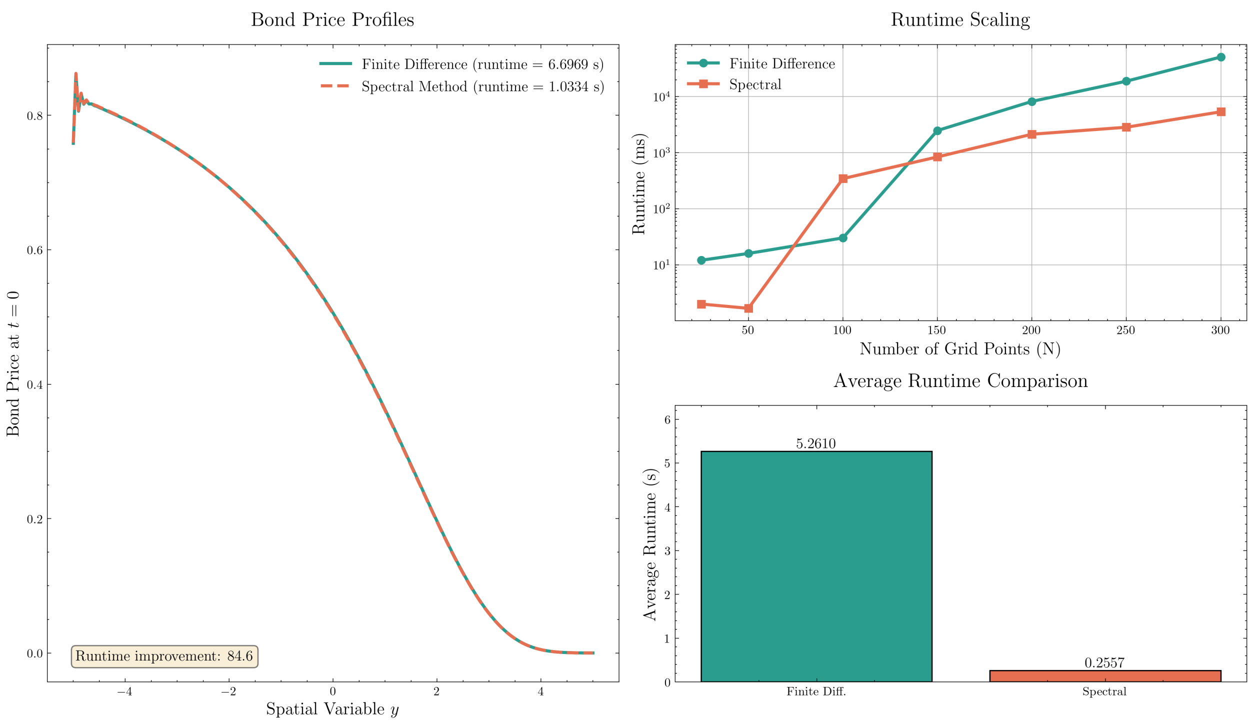

Figure 1: A mosaic comparing two numerical approaches for bond pricing under the Black–Karasinski model. The left panel (A) shows the bond price profile at $t = 0$ for both the finite difference (green) and spectral (orange) methods, along with their respective runtimes. The top-right panel (B) illustrates how runtime scales as the number of grid points increases. The bottom-right panel (C) presents a bar chart of average runtimes across multiple grid sizes, highlighting the substantial speedup achieved by the spectral method.

Our preliminary numerical experiments (see Figure 1) illustrate the practical benefits of the Lie algebraic approach to solving the Black–Karasinski (BK) model. In particular, we compare the traditional finite difference method with our spectral technique, which exploits the hidden $\mathfrak{sl}(2,\mathbb{R})$ symmetry:

1. **Bond Price Accuracy and Profiles.** As shown in the left panel of Figure 1, both methods yield nearly identical bond price profiles for a representative set of model parameters. This confirms that the spectral approach achieves the same level of accuracy as the finite difference scheme while adhering to the terminal condition of a unit bond price at maturity.

2. **Computational Efficiency.** The top-right panel of Figure 1 depicts runtime as a function of the number of spatial grid points. The spectral method remains consistently faster, especially as the problem size grows, underscoring the advantage of diagonalizing the infinitesimal generator $\mathcal{L}$. By circumventing time-stepping and directly applying the exponential of the operator, the spectral approach avoids repeated matrix factorizations that slow down finite difference schemes.
3. **Overall Speedup.** The bar chart in the bottom-right panel of Figure 1 highlights the average runtimes of each method across multiple grid sizes. On average, the spectral method achieves a substantial reduction in computational time. This improvement aligns with the theoretical prediction that diagonalizing $\mathcal{L}$ via representation theory offers a more efficient route to bond pricing than iterative PDE solvers.
4. **Deeper Structural Insight and Extensibility.** Beyond these computational gains, our use of $\mathfrak{sl}(2,\mathbb{R})$ generators provides a clearer structural view of the BK model's solution space. The ability to decompose the generator into well-understood algebraic components paves the way for further analytical developments. Moreover, this Lie algebraic perspective can be extended to other PDE-based models in finance—particularly those with mean-reverting or lognormal dynamics—opening new avenues for efficient and elegant solutions in quantitative finance.

4 General Form

In this section, we derive a general representation of the generator $\mathcal{L}$ in term of differential operators that form a basis for the $\mathfrak{sl}(2,\mathbb{R})$ for two Ito diffusion processes. We redefine the operators:

$$
\begin{aligned}
J_- &\equiv \frac{d}{dX_t}, \\
J_0 &\equiv X_t \frac{d}{dX_t}, \\
J_+ &\equiv X_t^2 \frac{d}{dX_t}.
\end{aligned}
$$

These operators satisfy the commutation relations:

$$
[J_0, J_-] = -J_-, \quad [J_0, J_+] = J_+, \quad [J_-, J_+] = 2J_0,
$$

which are equivalent to the standard $\mathfrak{sl}(2,\mathbb{R})$ relations under the usual identification of basis elements. An Ito diffusion is given by the stochastic differential equation.

$$
dX_t = \mu(X_t)dt + \sigma(X_t)dW_t
$$

where μ is the drift term and σ is the diffusion term. First, we derive the generator for an Ito diffusion where

$$
\begin{aligned}
\mu(X_t) &= aX_t^2 + bX_t + c \\
\sigma(X_t) &= pX_t^2 + qX_t + r
\end{aligned}
$$

The infinitesimal generator $\mathcal{L}$ for a one-dimensional Ito diffusion is

$$
\mathcal{L}f(x) = \mu(x)f'(x) + \frac{\sigma(x)^2}{2}f''(x)
$$

Plugging in, we get

$$
\mathcal{L}f(x) = f'(x)(ax^2 + bx + c) + f''(x)(\frac{p^2x^4 + q^2x^2 + r^2}{2} + pq^3 + prx^2 + qrx)
$$

Substituting the differential operators, we get

$$
\boxed{\mathcal{L} = aJ_+ + bJ_0 + cJ_- + \frac{p^2}{2}J_+^2 + \frac{r^2}{2}J_-^2 + pq(J_+)(J_0) + qr(J_0)(J_-) + J_0^2(pr + \frac{q^2}{2})}
$$

Now we derive the generator for an Ito diffusion where

$$
\begin{aligned}
\mu(X_t) &= aX_t^2 + bX_t + c \\
\sigma(X_t) &= pX_t^{3/2} + qX_t^{1/2}
\end{aligned}
$$

Plugging these functions into the infinitesimal generator for a one-dimensional Ito diffusion yields

$$\mathcal{L}f(x) = f'(x)(ax^2 + bx + c) + f''(x)(\frac{p^2x^3 + q^2x}{2} + pqx^2)$$

Substituting the differential operators, we get

$$\boxed{\mathcal{L} = aJ_+ + bJ_0 + cJ_- + \frac{p^2}{2}(J_+)(J_0) + \frac{q^2}{2}(J_0)(J_-) + pq(J_0^2)}$$

These equations allow us to write the infinitesimal generator of an Ito diffusion of the aforementioned two forms in terms of in term of differential operators that form a basis for the $\mathfrak{sl}(2,\mathbb{R})$. We use these formulas to find the generator for several prominent Ito diffusions in quantitative finance.

Name	**Equation**	**Generator**
Brownian Motion with Drift	$dW_t = \mu dt + \sigma dW_t$	$\mathcal{L} = \mu J_- + \frac{\sigma^2}{2} J_-^2$
Geometric Brownian Motion	$dS_t = \mu S_t dt + \sigma S_t dW_t$	$\mathcal{L} = \mu J_0 + \frac{\sigma^2}{2}(J_0^2)$
Ornstein-Uhlenbeck Process	$dX_t = \theta(\mu - X_t)dt + \sigma dW_t$	$\mathcal{L} = -\theta J_0 + \theta\mu J_- + \frac{\sigma^2}{2}(J_-^2)$
Vasicek Model	$dr_t = a(b - r_t)dt + \sigma dW_t$	$\mathcal{L} = -aJ_0 + abJ_- + \frac{\sigma^2}{2}(J_-^2)$
Cox-Ingersoll-Ross Model	$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t$	$\mathcal{L} = -aJ_0 + abJ_- + \frac{\sigma^2}{2}(J_0)(J_-)$

Figure 2: Caption

5 Conclusion

In this work, we have shown that the Black–Karasinski model, when reformulated in log-space, possesses a hidden $\mathfrak{sl}(2,\mathbb{R})$ symmetry. By expressing the model's infinitesimal generator in terms of the Lie algebra's differential operators, we have developed an analytical framework that not only enhances computational efficiency but also deepens our theoretical understanding of the model's dynamics. Future research will extend these methods to more complex financial instruments and include comprehensive numerical validations.

References