- Δ - method

  가정

$$\sqrt{n}(X_n - \theta) \xrightarrow{D} N(0, \sigma^2)$$

  $g(x)$ is differentiable at θ, $g'(\theta) \neq 0$.

$$\sqrt{n}(g(X_n) - g(\theta)) \xrightarrow{D} N(0, \sigma^2(g'(\theta))^2)$$

  proof) Taylor expansion

  $g(X_n)$을 θ에서 테일러 전개

$$g(X_n) = g(\theta) + g'(\theta)(X_n - \theta) + \underbrace{\frac{1}{2}g''(\theta)(X_n - \theta) + \cdots}_{remainder\ O_p(|X_n - \theta|) \Rightarrow 0}$$

$$\boxed{\begin{aligned} g(X_n) &= g(\theta) + g'(\theta)(X_n - \theta) \\ g(X_n) - g(\theta) &= g'(\theta)(X_n - \theta) \\ \sqrt{n}(g(X_n) - g(\theta)) &= \sqrt{n}(g'(\theta)(X_n - \theta)) \end{aligned}}$$

$$\therefore\ g'(\theta) \cdot \sqrt{n}(X_n - \theta) \xrightarrow{D} N(0, \sigma^2(g'(\theta))^2)$$

- MGF Technique

  목표 : $X_n \xrightarrow{D} X$

  If

$$M_{X_n}(t)\ :\ mgf\ of\ X_n, \quad M(t)\ :\ mgf\ of\ X$$

  ,

$$\lim_{n\to\infty} M_{X_n}(t) = M(t), \quad |t| \leq h.$$

  $\Rightarrow$ 즉, mgf가 같으면 해당 분포로 분포 수렴.

- 중심극한정리Central Limit Theorem

  : 모든 분포를 "정규분포"로 수렴시킬수 있다. (n개의 "합"의 분포, $\sum X_i$)

Definition

$X_1, \dots, X_n \sim (\mu, \sigma^2)$

$$Y_n = \frac{\sum X_i - n\mu}{\sqrt{n}\sigma} \xrightarrow{D} N(0,1)$$

기초통계 : $\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$

proof)

1. mgf technique, Taylor expansion

   $X_1, \dots, X_n \sim (\mu, \sigma^2),\ E(\sum X_i) = n\mu,\ Var(\sum X_i) = n\sigma^2$

$$Y_n = \frac{\sum X_i - n\mu}{\sqrt{n}\sigma} = \frac{\sqrt{n}(\bar{X}_n - mu)}{\sigma}$$

   $mgf\ of\ (X - \mu),\ M(t) = E(e^{tx})$

$$m(t) = E(e^{t(x-\mu)}) = e^{-\mu t}M(t), \quad |t| \leq h$$

$$\begin{aligned} m(0) &= 1 \\ m'(0) &= E((x-\mu)e^{t(x-\mu)}|_{t=0} = E(x-\mu) = 0 \\ m''(0) &= E((x-\mu)^2 e^{t(x-\mu)}|_{t=0} = E((x-\mu)^2) = \sigma^2 \end{aligned}$$

   Taylor expansion

$$\begin{aligned} m(t) &= m(0) + m'(0)t + \frac{m''(\xi)t^2}{2}, \quad 0 < \xi < t \\ &= 1 + \frac{m''(\xi)t^2}{2} \\ &= 1 + \frac{\sigma^2 t^2}{2} + \frac{(m''(\xi) - \sigma^2)t^2}{2} \end{aligned}$$

Consider $M(t;n)$

$$
\begin{aligned}
M(t;n) &= E\left[exp\left(t\frac{\sum X_i - n\mu}{\sigma\sqrt{n}}\right)\right] \\
&= E\left[exp\left(t\frac{X_1 - n\mu}{\sigma\sqrt{n}}\right)\cdot exp\left(t\frac{X_2 - n\mu}{\sigma\sqrt{n}}\right)\cdots exp\left(t\frac{X_n - n\mu}{\sigma\sqrt{n}}\right)\right] \\
&= E\left[exp\left(t\frac{X_1 - n\mu}{\sigma\sqrt{n}}\right)\right]\cdots E\left[exp\left(t\frac{X_n - n\mu}{\sigma\sqrt{n}}\right)\right] \\
&= \left\{E\left[exp\left(t\frac{X - n\mu}{\sigma\sqrt{n}}\right)\right]\right\}^n \\
&= \left[m\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n, \qquad -h < \frac{t}{\sigma\sqrt{n}} < h \\
&= \left\{1 + \frac{t^2}{2n} + \frac{[m''(\xi) - \sigma^2]t^2}{2n\sigma^2}\right\}^n, \qquad 0 < \xi < \frac{t}{\sigma\sqrt{n}},\ -h\sigma\sqrt{n} < t < h\sigma\sqrt{n} \\
\lim_{n\to\infty} M(t;n) &= \lim_{n\to\infty}\left\{1 + \frac{t^2}{2n} + \frac{[m''(\xi) - \sigma^2]t^2}{2n\sigma^2}\right\}^n \\
&= \lim_{n\to\infty}\left[1 + \frac{t^2/2}{n}\right]^n \\
&= e^{\frac{t^2}{2}}, \quad mgf\ of\ N(0,1)
\end{aligned}
$$

5.2.16

$\lim_{n\to\infty}\psi(n) = 0$,

$$\lim_{n\to\infty}\left[1 + \frac{b}{n} + \frac{\psi(n)}{n}\right]^{cn} = \lim_{n\to\infty}\left(1 + \frac{b}{n}\right)^{cn} = e^{bc}$$

2. 로피탈의 정리를 이용한 증명

L'Hopital's Theorem

극한값이 부정형 (0/0, ∞/∞) 일 경우 미분을 이용하여 계산할 수 있도록 하는 방법
함수 $f(x), g(x)$가 미분가능하고 $f(a) = 0, g(a) = 0$이고, $\lim_{x\to a} g(x) \neq 0$ 일때,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Law of Large Numbers

(Strong)

$$\bar{X}_n \to \mu \text{ as } n \to \infty \text{ with prob 1.}$$

(Weak)

$$\text{For any } C > 0,\ P(|\bar{X}_n - \mu| > c) \to 0 \text{ as } n \to \infty$$

⋆ $\bar{X}_n - \mu \to 0$ with prob 1, but what dose the distribution of $\bar{X}_n$ look like?

CLT

$$\sqrt{n}\,\frac{\bar{X}_n-\mu}{\sigma} \to N(0,1) \text{ in distribution.}$$

Equvalently

$$\frac{\sum_{j=1}^{n} X_j - n\mu}{\sqrt{n}\sigma} \to N(0,1) \text{ in distribution.}$$

proof) (assume MGF $M(t)$ of X_j exists)

Can assume $\mu = 0, \sigma = 1$, since consider $\dfrac{1}{\sqrt{n}}\displaystyle\sum_{j=1}^{n}\frac{X_j-\mu}{\sigma}$. Let $S_n = \sum_{j=1}^{n} X_j$, show MGF of $\frac{S_n}{\sqrt{n}}$ goes to $N(0,1)$ MGF.

$$\begin{aligned}
E\left(e^{tS_n/\sqrt{n}}\right) &= E\left(e^{tX_1/\sqrt{n}}\right)\cdots E\left(e^{tX_n/\sqrt{n}}\right)\\
&= \left(M\left(\frac{t}{\sqrt{n}}\right)\right)^n
\end{aligned}$$

Take logs

$$\begin{aligned}
\lim_{n\to\infty} n\log M\left(\frac{t}{\sqrt{n}}\right) &= \lim_{n\to\infty}\frac{\log M\left(\frac{t}{\sqrt{n}}\right)}{1/n}, \quad y=\frac{1}{\sqrt{n}}\\
&= \lim_{y\to 0}\frac{\log M(yt)}{y^2}\\
&= \lim_{y\to 0}\frac{tM'(yt)}{2yM(yt)}, \quad \lim_{y\to 0} M(yt)=1\\
&= \frac{t}{2}\lim_{y\to 0}\frac{M'(yt)}{y}\\
&= \frac{t^2}{2}\lim_{y\to 0}\frac{M''(yt)}{1}\\
&= \frac{t^2}{2}
\end{aligned}$$

Which is the log of $e^{\frac{t^2}{2}}$ ($N(0,1)$ MGF)

$$\begin{aligned}
M(t) &= E(e^{tx})\\
M(0) &= 1\\
M'(0) &= E(te^{tx})|_{t=0} = 0\\
M''(0) &= E(e^{tx}+t^2e^{tx})|_{t=0} = 1
\end{aligned}$$