고급수리통계학 중간고사

2020221005
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2021. 4. 22.

1. Let X have a geometric distribution. Show that

$$P(X \geq k+j|X \geq) = P(X \geq j),$$

where k and j are nonnegative integers. Note that we sometimes say in this situation that X is memoryless.

(Solution)

$$\begin{aligned} P(X \geq k+j|X \geq) &= \frac{P(X \geq k+j, X \geq k)}{P(X \geq k)} = \frac{P(X \geq k+j)}{P(X \geq k)} \\ &= \frac{1-F(k+j)}{1-F_X(k)} = \frac{1-(1-(1-p)^{k+j})}{1-(1-(1-p)^k)} \\ &= \frac{(1-p)^{k+j}}{(1-p)^k} = (1-p)^j = 1-(1-(1-p)^j) \\ &= 1-F_X(j) = P(X \geq j) \end{aligned}$$

cdf of geometric distribution

$$\begin{aligned} F_X(x) = P(X \leq x) &= \sum_{k=1}^{x} P(X=k) = \sum_{k=1}^{x} (1-p)^{k-1}p \\ &= p(1+(1-p)+(1-p)^2+\cdots+(1-p)^{x-1}) \\ &= p\frac{1-(1-p)^x}{1-(1-p)} = 1-(1-p)^x \end{aligned}$$

2. Let X equal the number of independent tosses of a fair coin that are required to observe heads on consecutive tosses. Let u_n equal the nth Fibonacci number, where $u_1 = u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$, $n = 3, 4, 5, \ldots$. Show that the pmf of X is

$$P_X(x) = P(X = x) = \frac{u_{x-1}}{2^x}, \quad x = 2, 3, 4, \ldots.$$

(Solution)

x	Set	경우의 수
2	$\{HH\}$	$u_1 = 1$
3	$\{THH\}$	$u_2 = 1$
4	$\{TTHH, HTHH\}$	$u_3 = 2$
5	$\{TTTHH, HTTHH, THTHH\}$	$u_4 = 3$
6	$\{TTTTHH, HTTTHH, THTTHH, TTHTHH, HTHTHH\}$	$u_5 = 5$
7	$\{TTTTTHH, HTTTTHH, THTTTHH, TTHTTHH, TTTHTHH,$ $HTHTTHH, THTHTHH, HTTHTHH\}$	$u_6 = 8$
$\vdots$	$\vdots$	$\vdots$

Head 가 연속적으로 관찰 되었을 때의 경우의 수는 위와 같다. 즉 마지막에 두번 head 가 관찰된 경우이다. $x \geq 3$일 경우 마지막 3번 관찰값은 $\{THH\}$ 로 고정이 되고, 앞의 관찰에서는 head 가 연속적으로 나오지 않는 경우의 수를 구한다. x가 증가하면서 경우의 수는 $\{1, 1, 2, 3, 5, 8, \ldots\}$ 으로 Fibonacci 수열 형태를 따르게 된다.

3. Let the independent random variable X_1 and X_2 have binomial distributions

$$X_1 \sim Bin(n_1, 0.5), \quad X_2 \sim Bin(n_2, 0.5).$$

Show that

$$Y = X_1 - X_2 + n_2 \sim Bin(n_1 + n_2, 0.5).$$

(Solution)

$$M_{X_1}(t) = (\frac{1}{2}e^t + \frac{1}{2})^{n_1}, \qquad M_{X_2}(t) = (\frac{1}{2}e^t + \frac{1}{2})^{n_2}$$

$$\begin{aligned} M_Y(t) &= E(e^{X_1 - X_2 + n_2}) = E(e^{tx_1 - tx_2 - tn_2}) \\ &= (\frac{1}{2}e^t + \frac{1}{2})^{n_1}(\frac{1}{2}e^{-t} + \frac{1}{2})^{n_2}e^{tn_2} \\ &= (\frac{1}{2}e^t + \frac{1}{2})^{n_1}(\frac{1}{2} + \frac{1}{2}e^t)^{n_2} \\ &= (\frac{1}{2}e^t + \frac{1}{2})^{n_1 + n_2} \\ \therefore \quad & Y \sim Bin(n_1 + n_2, 0.5) \end{aligned}$$

4. Let X have the uniform distribution with pdf $f(x) = 1,\ 0 < x < 1$, zero elsewhere. Find the cdf of $Y = -2\log X$. What is the pdf of Y?

(Solution)

$$X = e^{-\frac{Y}{2}}, \quad dx = -\frac{1}{2}e^{-\frac{Y}{2}}dy$$

$$f_Y(y) = f_X(e^{-\frac{y}{2}})\frac{1}{2}e^{-\frac{y}{2}} = \frac{1}{2}e^{-\frac{y}{2}}, \quad 0 < y < \infty$$

5. Find the mean and variance of the β distribution $X \sim Beta(\alpha, \beta)$.

(Solution)

$$
\begin{aligned}
f_X(x) &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad 0 < x < 1 \\
E(X) &= \int_0^1 x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+1)-1}(1-x)^{\beta-1} dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+\beta+1)} \\
&= \frac{\alpha}{\alpha+\beta} \\
E(X^2) &= \int_0^1 x^2 \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{(\alpha+2)-1}(1-x)^{\beta-1} dx \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+\beta+2)} \\
&= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} \\
Var(X) &= E(X^2) - \{E(X)\}^2 \\
&= \frac{\alpha(\alpha+1)}{(\alpha+\beta)(\alpha+\beta+1)} + \frac{\alpha^2}{(\alpha+\beta)^2} \\
&= \frac{(\alpha^2+\alpha)(\alpha+\beta) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}
\end{aligned}
$$

6. Show, for $k = 1, 2, \ldots, n$, that

$$
\int_p^1 \frac{n!}{(k-1)!(n-k)!} z^{k-1}(1-z)^{n-k} dz = \sum_{x=0}^{k-1} \binom{n}{x} p^x (1-p)^{n-x}.
$$

This demonstrates the relationship between the cdfs of the β and binomial distributions.

(Solution)

$$
\int_p^1 \frac{n!}{(k-1)!(n-k)!} z^{k-1}(1-z)^{n-k} dz = A
$$

$$
\begin{aligned}
A &= \frac{n!}{(k-1)!(n-k)!}\left[(-1)\frac{1}{n-k+1}z^{k-1}(1-z)^{n-k+1}\Big|_p^1 + \frac{k-1}{n-k+1}\int_p^1 z^{k-2}(1-z)^{n-k+1}dz\right] \\
&= \binom{n}{k-1}p^{k-1}(1-p)^{n-(k-1)} + \frac{n!}{(k-2)!(n-k+1)!}\int_p^1 z^{k-2}(1-z)^{n-k+1}dz \\
&= \binom{n}{k-1}p^{k-1}(1-p)^{n-(k-1)} + \binom{n}{k-2}p^{k-2}(1-p)^{n-(k-2)} \\
&\qquad\qquad\qquad + \frac{n!}{(k-3)!(n-k+2)!}\int_p^1 z^{k-3}(1-z)^{n-k+2}dz \\
&\qquad\qquad\qquad \vdots \\
&= \binom{n}{k-1}p^{k-1}(1-p)^{n-(k-1)} + \binom{n}{k-2}p^{k-2}(1-p)^{n-(k-2)} + \cdots \\
&\qquad\qquad\qquad + \frac{n!}{(k-k)!(n-k+k-1)!}\int_p^1 z^{k-k}(1-z)^{n-k+(k-1)}dz \\
&= \binom{n}{k-1}p^{k-1}(1-p)^{n-(k-1)} + \binom{n}{k-2}p^{k-2}(1-p)^{n-(k-2)} + \cdots \\
&\qquad\qquad\qquad + \frac{n!}{(n-1)!}\int_p^1 (1-z)^{n-1}dz \\
&= \binom{n}{k-1}p^{k-1}(1-p)^{n-(k-1)} + \binom{n}{k-2}p^{k-2}(1-p)^{n-(k-2)} + \cdots + (1-p)^n \\
&= \binom{n}{k-1}p^{k-1}(1-p)^{n-(k-1)} + \binom{n}{k-2}p^{k-2}(1-p)^{n-(k-2)} + \cdots + \binom{n}{0}p^0(1-p)^n \\
&= \sum_{x=0}^{k-1}\binom{n}{x}p^x(1-p)^{n-x}
\end{aligned}
$$

7. Let the random variable X is $N(\mu, \sigma^2)$, $\sigma^2 > 0$. Show that the random variable

$$V = \frac{(X-\mu)^2}{\sigma^2}$$

is $\chi^2(1)$.

(Solution)

$$Z = \frac{(X-\mu)}{\sigma} \sim N(0,1) \qquad \Rightarrow \qquad V = Z^2$$

$$
\begin{aligned}
F_V(v) &= P(V \le v) = P(Z^2 \le v) = P(-\sqrt{v} \le z \le \sqrt{v}) \\
&= \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = 2\int_0^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \qquad z^2 = y \\
&= 2\int_0^{v} \frac{1}{\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{2\sqrt{y}} dy, \qquad dz = \frac{1}{2\sqrt{y}} dy \\
&= \int_0^{v} \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} dy \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{v} y^{-\frac{1}{2}} e^{-\frac{y}{2}} dy \\
f_V(v) &= F'(v) = \frac{1}{\sqrt{2\pi}} v^{-\frac{1}{2}} e^{-\frac{v}{2}} \\
&= \frac{1}{2^{\frac{1}{2}}\Gamma(\frac{1}{2})} v^{\frac{1}{2}-1} e^{-\frac{v}{2}} \quad \sim \quad Gamma(\frac{1}{2}, 2) \equiv \chi^2_{(1)}
\end{aligned}
$$

8. 연속형 확률변수 X가 평균이 0, 표준편차가 $\sigma > 0$ 인 정규분포 $X \sim N(0, \sigma^2)$을 따른다고 하자. (단, $\sigma \neq 1$)

(a) X의 적률생성함수 $M_X(t)$를 구하시오.

$$
\begin{aligned}
M_X(t) = E(e^{tx}) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2} - tx} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2 - 2\sigma^2 tx)} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2 - \sigma^2 t)^2} e^{-\frac{1}{2\sigma^2}(-\sigma^4 t^2)} dx \\
&= e^{-\frac{1}{2\sigma^2}(-\sigma^4 t^2)} \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x^2 - \sigma^2 t)^2} dx \\
&= e^{\frac{\sigma^2 t^2}{2}}
\end{aligned}
$$

(b) 모든 자연수 n에 대하여 X의 n-차 적률 $E[X^n]$를 구하는 식을 일반화하시오.

i) 홀수

$$
E(X^n) = \int_{-\infty}^{\infty} x^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = 0
$$

ii) 짝수

$$
\begin{aligned}
M_X(t) = e^{\frac{\sigma^2 t^2}{2}} &= \sum_{n=0}^{\infty} \frac{(\frac{\sigma^2 t^2}{2})^n}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{\sigma^2 t^2}{2})^n (2n)!}{n!(2n)!} \\
&= \frac{\frac{\sigma^{2n}(2n)!}{2^n n!} t^{2n}}{(2n)!} \\
E(X^{2n}) &= \frac{\sigma^{2n}(2n)!}{2^n n!}
\end{aligned}
$$

9. Let X_1 and X_2 be independent r.v, each with pdf

$$f(x) = e^{-x}, \quad 0 < x < \infty.$$

Let $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$. Find (a) f_{Y_1,Y_2}, (b) f_{Y_1}, (c) f_{Y_2}

(Solution)

X_1, X_2 *pdf*

$$f(x) = e^{-x}, \quad 0 < x < \infty$$
$$Y_1 = X_1 - X_2, \quad Y_2 = X_1 + X_2 \rightarrow X_1 = \frac{Y_1 + Y_2}{2}, \quad X_2 = \frac{Y_2 - Y_1}{2}$$

Joint *pdf*

$$f_{X_1,X_2}(x_1, x_2) = e^{-(x_1+x_2)}$$

Jacobian

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}$$

By CoV (a)

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(\frac{Y_1 + Y_2}{2}, \frac{Y_2 - Y_1}{2})\frac{1}{2} \\ &= exp(-(\frac{Y_1 + Y_2}{2} + \frac{Y_2 - Y_1}{2}))\frac{1}{2} \\ &= \frac{1}{2}e^{-y_2} \end{aligned}$$

(1) $x_1 = 0, x_2 > 0 \Rightarrow \frac{Y_1+Y_2}{2} = 0, \frac{Y_2-Y_1}{2} > 0 \Rightarrow y_2 = -y_1, y_2 > y_1$

(2) $x_1 > 0, x_2 = 0 \Rightarrow \frac{Y_1+Y_2}{2} > 0, \frac{Y_2-Y_1}{2} = 0 \Rightarrow y_2 = y_1, y_2 > -y_1$

$\Rightarrow 0 < |y_1| \leq y_2 < \infty, \ -\infty < y_1 < \infty$

Marginal *pdf* (b), (c)

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{|y_1|}^{\infty} \frac{1}{2}e^{-y_2} dy_2 = \frac{1}{2}(-e^{-\infty} + e^{-|y_1|}) \\ &= \frac{1}{2}e^{-|y_1|}, \quad -\infty < y_1 < \infty \\ f_{Y_2}(y_2) &= \int_{-y_2}^{y_2} \frac{1}{2}e^{-y_2} dy_1 = \frac{1}{2}e^{-y_2}(y_2 + y_2) \\ &= y_2 e^{-y_2} \\ &= \frac{1}{\Gamma(2)1^2} y_2^{2-1} e^{-\frac{y_2}{1}}, \quad 0 < y_2 < \infty \end{aligned}$$

10. 변수변환 테크닉을 사용하여 F 분포의 pdf를 유도하시오.

(Solution)

$U \sim \chi^2_{r_1},\ V \sim \chi^2_{r_2}$

$$W = \frac{U/r_1}{V/r_2}, \quad Z = V \quad \Rightarrow \quad U = \frac{r_1}{r_2}WZ, \quad V = Z$$

Joint *pdf*

$$f_{U,V}(u,v) = \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}} u^{\frac{r_1}{2}-1} v^{\frac{r_2}{2}-1} e^{-\frac{(u+v)}{2}}$$

Jacobian

$$J = \begin{bmatrix} \frac{r_1}{r_2}v & 0 \\ 0 & 1 \end{bmatrix} = \frac{r_1}{r_2}v = \frac{r_1}{r_2}z$$

By CoV

$$\begin{aligned} f_{W,Z}(w,z) &= f_{U,V}(\frac{r_1}{r_2}wz, z)\frac{r_1}{r_2}z \\ &= \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}}(\frac{r_1}{r_2}wz)^{\frac{r_1}{2}-1} z^{\frac{r_2}{2}-1} e^{-\frac{z}{2}(\frac{r_1}{r_2}w+1)} \frac{r_1}{r_2}z \end{aligned}$$

Marginal *pdf*

$$\begin{aligned} f_W(w) &= \frac{(r_1/r_2)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}} \int_0^\infty z^{\frac{r_1+r_2}{2}-1} exp\left(-\frac{z}{2}(\frac{r_1}{r_2}w+1)\right) dz \\ & y = \frac{z}{2}(\frac{r_1}{r_2}w+1), \quad z = \frac{2y}{\frac{r_1}{r_2}w+1}, \quad dz = \frac{2}{\frac{r_1}{r_2}w+1}dy \\ &= \frac{(r_1/r_2)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}} \int_0^\infty \left(\frac{2y}{\frac{r_1}{r_2}w+1}\right)^{\frac{r_1+r_2}{2}-1} e^{-y}\frac{2}{\frac{r_1}{r_2}w+1}dy \\ &= \frac{(r_1/r_2)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}} \frac{2^{\frac{r_1+r_2}{2}}}{(\frac{r_1}{r_2}w+1)^{\frac{r_1+r_2}{2}}} \int_0^\infty y^{\frac{r_1+r_2}{2}-1} e^{-y} dy \\ &= \frac{\Gamma(\frac{r_1+r_2}{2})(r_1/r_2)^{\frac{r_1}{2}}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} \frac{w^{\frac{r_1}{2}-1}}{(\frac{r_1}{r_2}w+1)^{\frac{r_1+r_2}{2}}}, \quad 0 < w < \infty \end{aligned}$$