1 감마 분포 Gamma distribution

- X : α번 사건이 발생할때까지 걸리는 시간
- X의 cdf

$$
\begin{aligned}
F_X(x) &= P(X \le x) = 1 - P(X > x) \\
&= 1 - P([0,x]\text{에 } \alpha\text{보다 적게 사건이 발생} \\
&= 1 - \sum_{k=0}^{\alpha-1} P(Y = k), \quad Y \sim Poisson(\lambda x) \\
&= 1 - \sum_{k=0}^{\alpha-1} \frac{e^{-\lambda x}(\lambda x)^k}{k!} \\
&= 1 - e^{-\lambda x} - \sum_{k=1}^{\alpha-1} \frac{e^{-\lambda x}(\lambda x)^k}{k!}
\end{aligned}
$$

- X의 pdf

$$
\begin{aligned}
f_X(x) &= F'_X(x) \\
&= \lambda e^{-\lambda x} - e^{-\lambda x} \sum_{k=1}^{\alpha-1} \left[\frac{k(\lambda x)^{k-1}\lambda}{k!} - \frac{(\lambda x)^k \lambda}{k!} \right] \\
&= \lambda e^{-\lambda x} - e^{-\lambda x} \left[\lambda - \frac{\lambda(\lambda x)^{\alpha-1}}{(\alpha-1)!} \right] \\
&= \frac{\lambda(\lambda x)^{\alpha-1}}{(\alpha-1)!} e^{-\lambda x} \\
&= \frac{\lambda^\alpha}{(\alpha-1)!} x^{\alpha-1} e^{-\lambda x}, \quad x > 0 \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}}, \quad x > 0
\end{aligned}
$$

- pdf check!

$$
\begin{aligned}
f_X(x) &= \frac{1}{(\alpha-1)!} \lambda(\lambda x)^{\alpha-1} e^{-\lambda x}, \quad x > 0,\ \alpha : \text{자연수} \\
(\alpha-1)! &= \int_0^\infty \lambda(\lambda x)^{\alpha-1} e^{-\lambda x} dx, \quad \lambda x = y \\
(\alpha-1)! &= \int_0^\infty y^{\alpha-1} e^{-y} dy = \Gamma(\alpha)
\end{aligned}
$$

- Gamma function

$$\begin{aligned}
\Gamma(\alpha) &= \int_0^\infty y^{\alpha-1}e^{-y}dy \\
&= -e^{-y}y^{\alpha-1}|_0^\infty + \int_0^\infty e^{-y}(\alpha-1)y^{\alpha-2}dy \\
&= (\alpha-1)\int_0^\infty e^{-y}y^{\alpha-2}dy = (\alpha-1)\Gamma(\alpha-1) \\
&= (\alpha-1)(\alpha-2)\Gamma(\alpha-2) \\
&\qquad\vdots \\
&= (\alpha-1)(\alpha-2)\cdots 1
\end{aligned}$$

- $\Gamma(\frac{1}{2})$

$$\boxed{\begin{aligned}
\int_{-\infty}^\infty e^{-\frac{y^2}{2}}dy &= \sqrt{2\pi} \\
\int_0^\infty e^{-\frac{y^2}{2}}dy &= \frac{\sqrt{2\pi}}{2}, \qquad \frac{y^2}{2} = t \\
\int_0^\infty e^{-t}\frac{1}{y}dt &= \int_0^\infty e^{-t}\frac{1}{\sqrt{2}t^{\frac{1}{2}}} \\
&= \int_0^\infty \frac{1}{\sqrt{2}}t^{-\frac{1}{2}}e^{-t}dt \\
&= \frac{1}{\sqrt{2}}\int_0^\infty t^{-\frac{1}{2}}e^{-t}dt \\
&= \frac{\sqrt{2\pi}}{2}
\end{aligned}}$$

$$\begin{aligned}
\frac{1}{\sqrt{2}}\Gamma(\frac{1}{2}) &= \frac{\sqrt{2\pi}}{2} \\
\Gamma(\frac{1}{2}) &= \frac{\sqrt{2}\sqrt{2\pi}}{2} = \sqrt{\pi}
\end{aligned}$$

- mgf of X

$$
\begin{aligned}
M_X(t) &= E(e^{tx}) \\
&= \int_0^\infty e^{tx}\frac{1}{\Gamma(\alpha)\theta^\alpha}x^{\alpha-1}e^{-\frac{x}{\theta}}dx \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty x^{\alpha-1}e^{-x(\frac{1}{\theta}-t)}dx \quad x(\frac{1}{\theta}-t)=y,\ y: \text{양수}\ \frac{1}{\theta}>t \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha}\int_0^\infty (\frac{y}{\frac{1}{\theta}-t})^{\alpha-1}e^{-y}\frac{1}{\frac{1}{\theta}-t}dy \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha}\frac{1}{(\frac{1}{\theta}-t)^\alpha}\int_0^\infty y^{\alpha-1}e^{-y}dy \\
&= \frac{1}{\theta^\alpha(\frac{1}{\theta}-t)^\alpha} \\
&= \frac{1}{(1-\theta t)^\alpha} \\
&= (1-\theta t)^{-\alpha}, \quad t<\frac{1}{\theta} \\
M_X'(t) &= \alpha\theta(1-\theta t)^{-\alpha-1}, \quad E(X)=M_X'(0)=\alpha\theta \\
M_X''(t) &= \alpha(\alpha+1)\theta^2(1-\theta t)^{-\alpha-2}, \quad M_X''(0)=\alpha(\alpha+1)\theta^2 \\
Var(X) &= E(X^2)-\{E(X)\}^2=\alpha(\alpha+1)\theta^2-\alpha^2\theta^2=\alpha\theta^2
\end{aligned}
$$

2 포아송과정Poisson process

- λ : "단위시간" 당 평균 사건 발생횟수
- $\theta = \frac{1}{\lambda}$: "한번" 사건이 발생할 평균 시간
- 확률변수 T_j : j번 사건이 발생할 때까지 걸린 시간, $j=1,\ldots,n$
- 확률변수 N_t : 시점 t까지 발생한 사건의 수
- $N_t \sim Poisson(\lambda t)$

3 Chi-square distribution

$$f_X(x)=\frac{1}{\Gamma(\frac{r}{2})2^{\frac{r}{2}}}x^{\frac{r}{2}-1}e^{-\frac{x}{2}}$$

- $X\sim\Gamma(\frac{r}{2},2)\triangleq\chi^2_{(r)}$, $\frac{r}{2}$번 발생까지 걸리는 시간, $\lambda=\frac{1}{2}$
- $E(X)=\alpha\theta=\frac{r}{2}2=r$
- $Var(X)=\alpha\theta^2=\frac{r}{2}4=2r$
- $M_X(t)=(1-\theta t)^{-\alpha}=(1-2t)^{-\frac{r}{2}}, \quad t<\frac{1}{2}$

4 Distributions of functions of a random variable

- Change-of-variable technique
  - 목적 : Y의 pdf를 구하는 것 ($f_Y(y)$)
  - 방법 : Y의 cdf를 X의 cdf로 표현한후 미분
  - 알고있는것 : $f_X(x)$, $Y = g(x)$, $g(\cdot)$: 연속, 일대일 함수(증가함수 or 감소함수) $\rightarrow g^{-1}$ 존재

- $g(\cdot)$: 증가함수

$$\begin{aligned}
F_Y(y) &= P(Y \le y) = P(g(x) \le y) \\
&= P(X \le g^{-1}(y)), \quad \text{증가함수이기 때문에 부등호가 바뀌지 않음} \\
&= F_X(g^{-1}(y)) \\
f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) \\
&= \frac{d}{dx} F_X(g^{-1}(y)) \frac{dx}{dy}, \quad \text{연쇄법칙(chain rule)} \\
&= f_X(g^{-1}(y)) \frac{dx}{dy}, \quad \text{g 가 증가함수이기 때문에 } \frac{dx}{dy} > 0 \\
&= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|
\end{aligned}$$

- $g(\cdot)$: 감소함수

$$\begin{aligned}
F_Y(y) &= P(Y \le y) = P(g(x) \le y) \\
&= P(X \ge g^{-1}(y)), \quad \text{감소함수이기 때문에 부등호가 바뀜} \\
&= 1 - F_X(g^{-1}(y)) \\
f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{d}{dy} (1 - F_X(g^{-1}(y))) \\
&= \frac{d}{dx} (1 - F_X(g^{-1}(y))) \frac{dx}{dy}, \quad \text{연쇄법칙(chain rule)} \\
&= -f_X(g^{-1}(y)) \frac{dx}{dy}, \quad \text{g 가 감소함수이기 때문에 } \frac{dx}{dy} < 0 \\
&= f_X(g^{-1}(y)) \left| \frac{dx}{dy} \right|, \quad \frac{dx}{dy} = -\left| \frac{dx}{dy} \right|
\end{aligned}$$

- Example. $X \sim \Gamma(\alpha, \theta),\ \alpha > 0, \theta > 0,\ Y = e^X$

$$f_X(x) = \frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}}$$

By COV,

$$
\begin{aligned}
f_Y(y) &= f_X(\ln y)\left|\frac{dx}{dy}\right| \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha}(\ln y)^{\alpha-1}e^{-\frac{\ln y}{\theta}}e^{\ln y}, \quad 1 < y < \infty \\
&= \frac{1}{\Gamma(\alpha)\theta^\alpha y}(\ln y)^{\alpha-1}e^{-\frac{\ln y}{\theta}}, \quad 1 < y < \infty
\end{aligned}
$$

- CoV on multivariate random variables
  - $X = (X_1, \ldots, X_n)$, $(X_1, \ldots, X_n)$의 joint pdf : $f_X(x_1, \ldots, x_n)$
  - $Y = (Y_1, \ldots, Y_n) = g(X), \quad g(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$
  - $f_Y(y_1, \ldots, y_n) = f_X(g^{-1}(y))\left|\frac{d\vec{x}}{d\vec{y}}\right|$
  - Jacobian matrix, $\quad J$: determinant of Jacobian matrix

$$
|J| = \left|\frac{d\vec{x}}{d\vec{y}}\right| = \left|\left|\frac{d(x_1, \ldots, x_n)}{d(y_1, \ldots, y_n)}\right|\right| = \left|\left|\begin{bmatrix} \frac{dx_1}{dy_1} & \cdots & \frac{dx_n}{dy_1} \\ \vdots & \ddots & \vdots \\ \frac{dx_1}{dy_n} & \cdots & \frac{dx_n}{dy_n} \end{bmatrix}\right|\right|
$$

- β **distribution**

  X_1, X_2 : independent

$$
X_1 \sim \Gamma(\alpha, \theta), \quad X_2 \sim \Gamma(\beta, \theta), \quad Y_1 = \frac{X_1}{X_1 + X_2} \quad Y_2 = X_1 + X_2
$$

$$
f(x_1, x_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}}x_1^{\alpha-1}x_2^{\beta-1}exp\left(-\frac{x_1 + x_2}{\theta}\right), \quad 0 < x_1, x_2 < \infty
$$

$$
\begin{aligned}
f_Y(y_1, y_2) &= f_X(y_1y_2, y_2(1-y_1))|J| \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)\theta^{\alpha+\beta}}(y_1y_2)^{\alpha-1}y_2^{\beta-1}(1-y_1)^{\beta-1}exp\left(-\frac{y_2}{\theta}\right)y_2
\end{aligned}
$$

Y_1의 marginal pdf

$$
\begin{aligned}
f_{Y_1}(y_1) &= \int_0^\infty f_Y(y_1, y_2)dy_2 \\
&= \frac{y_1^{\alpha-1}(1-y_1)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\frac{1}{\theta^{\alpha+\beta}}\int_0^\infty y_2^{\alpha-1}y_2^{\beta-1}e^{-\frac{y_2}{\theta}}dy_2 \\
&= \frac{y_1^{\alpha-1}(1-y_1)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)}\frac{1}{\theta^{\alpha+\beta}}\int_0^\infty y_2^{(\alpha+\beta)-1}e^{-\frac{y_2}{\theta}}dy_2, \quad y = -\frac{y_2}{\theta} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y_1^{\alpha-1}(1-y_1)^{\beta-1}, \quad 0 < y_1 < 1
\end{aligned}
$$

$$
Y \sim Beta(\alpha, \beta), \quad 0 < Y < 1
$$

– "확률"의 분포로 사용 → 모수 α, β를 자유롭게 조절하면서 다양한 모양을 만들수 있음.

– 베이지안 통계학 → 확률 모수의 사전 분포로 활용

- t-**distribution**

$Z \sim N(0,1),\ V \sim \chi^2_{(r)}$

$$T = \frac{Z}{\sqrt{\frac{V}{r}}}, \quad W = V$$

$$f_{Z,V}(z,v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{1}{\Gamma(\frac{r}{2}) 2^{\frac{r}{2}}} v^{\frac{r}{2}-1} e^{-\frac{v}{2}} = \frac{1}{\sqrt{2\pi}\Gamma(\frac{r}{2})2^{\frac{r}{2}}} v^{\frac{r}{2}-1} e^{-\frac{z^2+v}{2}}$$

Jacobian

$$\frac{d(z,v)}{d(t,w)} = \begin{bmatrix} \frac{dz}{dt} & \frac{dv}{dt} \\ \frac{dz}{dw} & \frac{dv}{dw} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{v}{r}} & 0 \\ 0 & 1 \end{bmatrix}, \qquad |J| = \sqrt{\frac{v}{r}}$$

By CoV,

$$\begin{aligned} f_{T,W}(t,w) &= f_{Z,V}(\sqrt{\frac{w}{r}}t, w)\sqrt{\frac{w}{r}} \\ &= \frac{1}{\sqrt{2\pi}\Gamma(\frac{r}{2})2^{\frac{r}{2}}} w^{\frac{r}{2}-1} e^{-\frac{t^2\frac{w}{r}+v}{2}} \sqrt{\frac{w}{r}}, \qquad -\infty < t < \infty,\ w > 0 \end{aligned}$$

T 의 marginal *pdf*

$$\begin{aligned} f_T(t) &= \int_0^\infty \frac{1}{\sqrt{2\pi}\Gamma(\frac{r}{2})2^{\frac{r}{2}}} w^{\frac{r}{2}-1} e^{-\frac{t^2\frac{w}{r}+v}{2}} \sqrt{\frac{w}{r}} dw \\ &= \frac{1}{\sqrt{2\pi}\Gamma(\frac{r}{2})2^{\frac{r}{2}}\sqrt{r}} \int_0^\infty w^{\frac{r+1}{2}-1} e^{\frac{w}{2}(1+\frac{t^2}{r})} dw, \quad y = \frac{w}{2}(1+\frac{t^2}{r}) \\ &= \frac{1}{\sqrt{2\pi r}\Gamma(\frac{r}{2})2^{\frac{r}{2}}} \int_0^\infty \left(\frac{2y}{1+\frac{t^2}{r}}\right)^{\frac{r+1}{2}-1} e^{-y} \frac{1}{1+\frac{t^2}{r}} dy \\ &= \frac{2^{\frac{r+1}{2}}}{\sqrt{2\pi r}\Gamma(\frac{r}{2})2^{\frac{r}{2}}(1+\frac{t^2}{r})^{\frac{r+1}{2}}} \int_0^\infty y^{\frac{r+1}{2}-1} e^{-y} dy \\ &= \frac{\Gamma(\frac{r+1}{2})}{\sqrt{\pi r}\Gamma(\frac{r}{2})} \frac{1}{(1+\frac{t^2}{r})^{\frac{r+1}{2}}}, \quad -\infty < t < \infty \end{aligned}$$

- **Double-exponential distribution**

X_1, X_2 *pdf*

$$f(x) = e^{-x}, \quad 0 < x < \infty$$

$$Y_1 = X_1 - X_2, \quad Y_2 = X_1 + X_2 \to X_1 = \frac{Y_1 + Y_2}{2}, \quad X_2 = \frac{Y_2 - Y_1}{2}$$

Joint *pdf*

$$f_{X_1,X_2}(x_1,x_2) = e^{-(x_1+x_2)}$$

Jacobian

$$J = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}$$

By CoV

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= f_{X_1,X_2}(\frac{Y_1+Y_2}{2}, \frac{Y_2-Y_1}{2})\frac{1}{2} \\ &= exp(-(\frac{Y_1+Y_2}{2} + \frac{Y_2-Y_1}{2}))\frac{1}{2} \\ &= \frac{1}{2}e^{-y_2} \end{aligned}$$

(1) $x_1 = 0, x_2 > 0 \Rightarrow \frac{Y_1+Y_2}{2} = 0, \frac{Y_2-Y_1}{2} > 0 \Rightarrow y_2 = -y_1, y_2 > y_1$

(2) $x_1 > 0, x_2 = 0 \Rightarrow \frac{Y_1+Y_2}{2} > 0, \frac{Y_2-Y_1}{2} = 0 \Rightarrow y_2 = y_1, y_2 > -y_1$

$\Rightarrow 0 < |y_1| \leq y_2 < \infty,\ -\infty < y_1 < \infty$

Marginal *pdf*

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{|y_1|}^{\infty} \frac{1}{2}e^{-y_2}dy_2 = \frac{1}{2}(-e^{-\infty} + e^{-|y_1|}) \\ &= \frac{1}{2}e^{-|y_1|}, \quad -\infty < y_1 < \infty \\ f_{Y_2}(y_2) &= \int_{-y_2}^{y_2} \frac{1}{2}e^{-y_2}dy_1 = \frac{1}{2}e^{-y_2}(y_2 + y_2) \\ &= y_2e^{-y_2} \\ &= \frac{1}{\Gamma(2)1^2}y_2^{2-1}e^{-\frac{y_2}{1}}, \quad 0 < y_2 < \infty \end{aligned}$$

- F **distribution**

$U \sim \chi^2_{r_1},\ V \sim \chi^2_{r_2}$

$$W = \frac{U/r_1}{V/r_2}, \quad Z = V \quad \Rightarrow \quad U = \frac{r_1}{r_2}WZ, \quad V = Z$$

Joint *pdf*

$$f_{U,V}(u,v) = \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}}u^{\frac{r_1}{2}-1}v^{\frac{r_2}{2}-1}e^{-\frac{(u+v)}{2}}$$

Jacobian

$$J = \begin{bmatrix} \frac{r_1}{r_2}v & 0 \\ 0 & 1 \end{bmatrix} = \frac{r_1}{r_2}v = \frac{r_1}{r_2}z$$

By CoV

$$\begin{aligned} f_{W,Z}(w,z) &= f_{U,V}(\frac{r_1}{r_2}wz, z)\frac{r_1}{r_2}z \\ &= \frac{1}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}}(\frac{r_1}{r_2}wz)^{\frac{r_1}{2}-1}z^{\frac{r_2}{2}-1}e^{-\frac{z}{2}(\frac{r_1}{r_2}w+1)}\frac{r_1}{r_2}z \end{aligned}$$

Marginal *pdf*

$$f_W(w) = \frac{(r_1/r_2)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}} \int_0^\infty z^{\frac{r_1+r_2}{2}-1} exp\left(-\frac{z}{2}(\frac{r_1}{r_2}w+1)\right) dz$$

$$y = \frac{z}{2}(\frac{r_1}{r_2}w+1), \quad z = \frac{2y}{\frac{r_1}{r_2}w+1}, \quad dz = \frac{2}{\frac{r_1}{r_2}w+1}dy$$

$$= \frac{(r_1/r_2)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}} \int_0^\infty \left(\frac{2y}{\frac{r_1}{r_2}w+1}\right)^{\frac{r_1+r_2}{2}-1} e^{-y}\frac{2}{\frac{r_1}{r_2}w+1}dy$$

$$= \frac{(r_1/r_2)^{\frac{r_1}{2}} w^{\frac{r_1}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}} \frac{2^{\frac{r_1+r_2}{2}}}{(\frac{r_1}{r_2}w+1)^{\frac{r_1+r_2}{2}}} \int_0^\infty y^{\frac{r_1+r_2}{2}-1}e^{-y}dy$$

$$= \frac{\Gamma(\frac{r_1+r_2}{2})(r_1/r_2)^{\frac{r_1}{2}}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} \frac{w^{\frac{r_1}{2}-1}}{(\frac{r_1}{r_2}w+1)^{\frac{r_1+r_2}{2}}}, \quad 0 < w < \infty$$