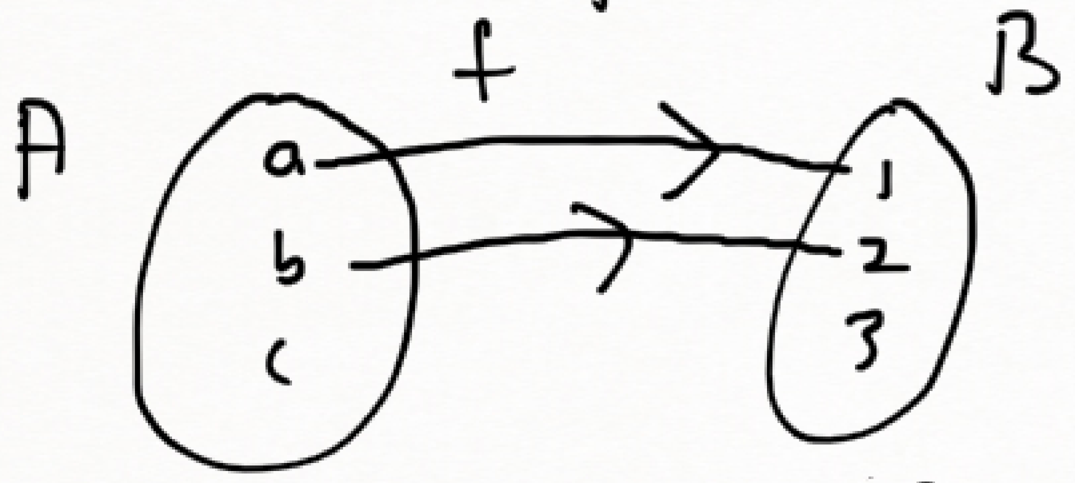
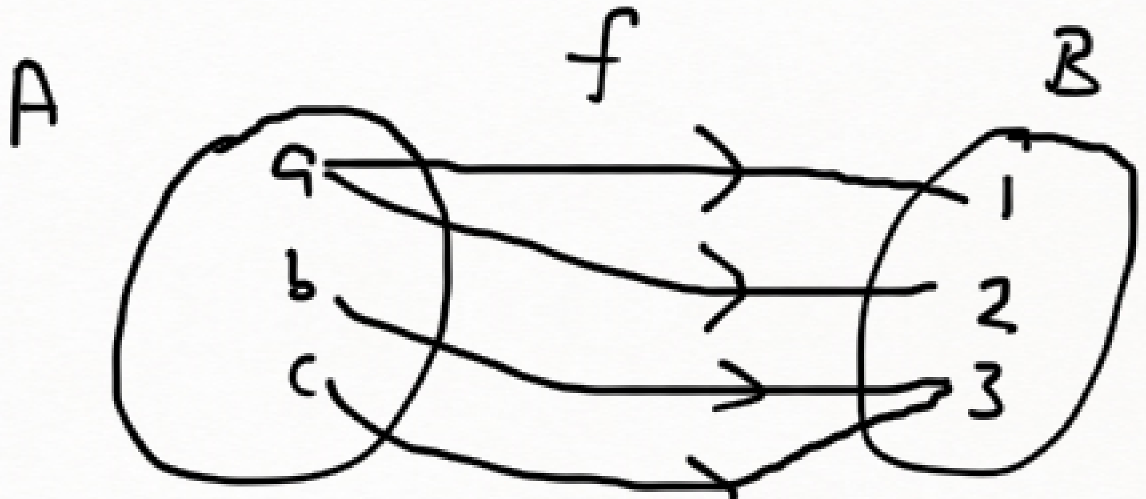
Function

Let A, B be two nonempty sets. A function $f: A \to B$ is a rule of correspondance that assigns every value of A to a unique value of B.



$f: A \to B$ is not a function as f is not assigned any value at c.



Here also $f: A \to B$ is not a function as f takes two different values at a.

If $f: A \to B$ be a function then A is said to be the domain of f and B is said to be the codomain.

Range of f is defined to be $f(A) = \{f(x) \mid x \in A\}$.

Clearly $f(A) \subseteq B$.

<u>Definition</u> — Let $f: A \to B$ be a function. Then

i) f is called injective (or one one) if $\forall\, a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. ie distinct elements of the domain are mapped to distinct images.

ii) f is called surjective (or onto) if $f(A) = B$. ie for every element $b \in B$, $\exists$ atleast one $a \in A$ such that $f(a) = b$.

iii) f is bijective if f is both injective and surjective.

Remark — Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is injective iff every horizontal line intersects the graph in atmost one point, f is surjective iff every horizontal line intersects the graph in atleast one point, f is bijective iff every horizontal line intersects the graph in exactly one point.

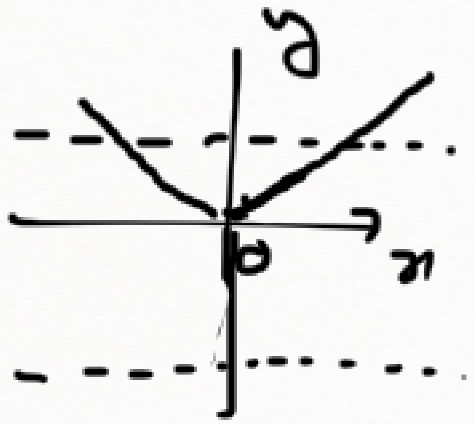
Examples

(1) Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = |x|$.

$f(1) = f(-1) \Rightarrow f$ is not injective

$-1 \in \mathbb{R}$. There does not exist any $x \in \mathbb{R}$ such that $f(x) = -1$.

f is not surjective.

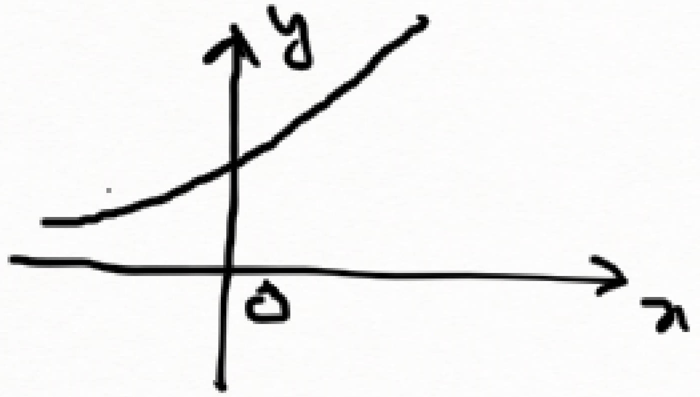


$\to$ horizontal line intersects graph in two points (not injective)

$\to$ horizontal line does not intersect graph in any point (not surjective)

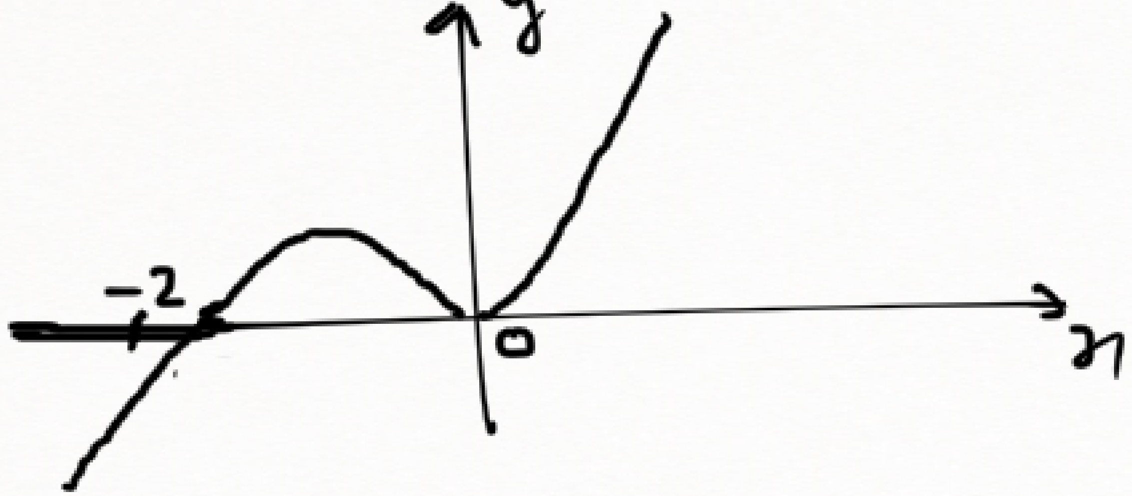
$y = |x|$

(2) Let $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2^x$.



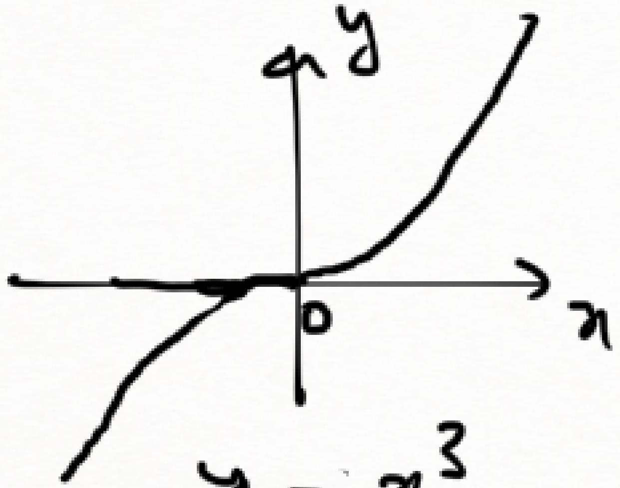
Clearly g is injective but not surjective.

(3) $h: \mathbb{R} \to \mathbb{R}$ by $h(x) = x^3 + 3|x|$



h is not injective but surjective

(4) $K: \mathbb{R} \to \mathbb{R}$ by $K(x) = x^3$.



$y = x^3$.

Every horizontal line intersects the graph in exactly one point so K is bijective.

Inverse trigonometry functions

Let $A, B, C \subseteq \mathbb{R}$.

Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions.

Then the composition of f and g, denoted by $g \circ f$ is defined as

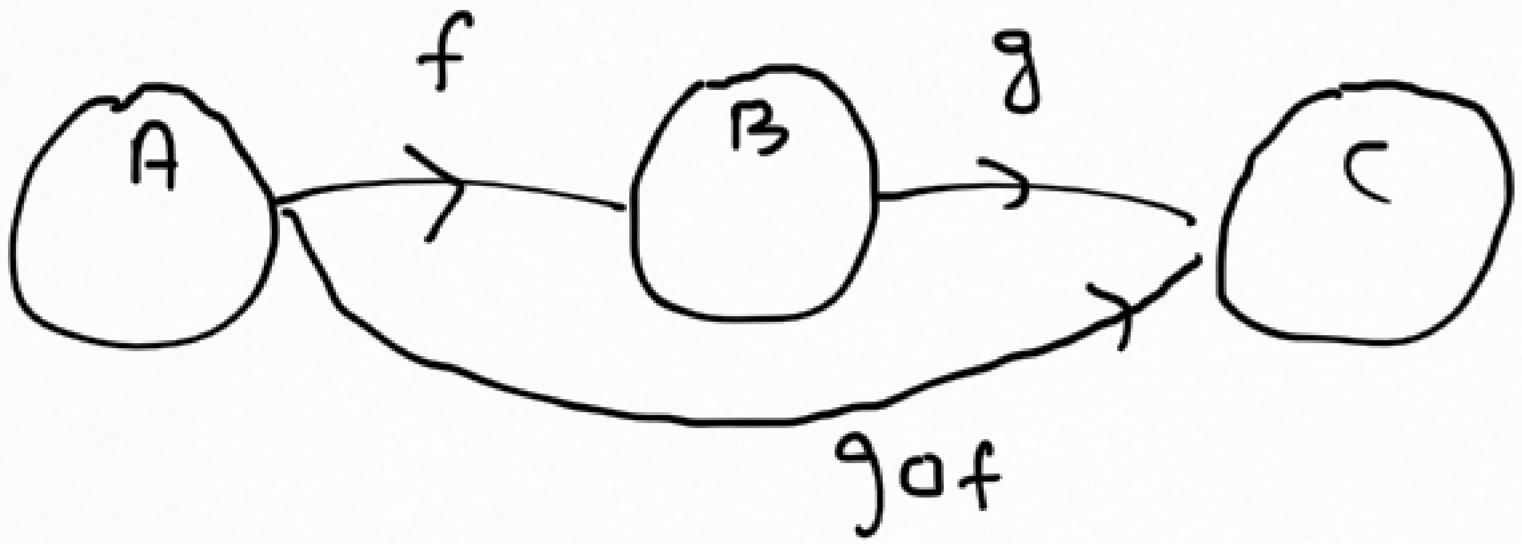
$g \circ f : A \longrightarrow C$ by

$(g \circ f)(x) = g(f(x))$, $x \in A$.

$x \in A$

$\Rightarrow f(x) \in B$

$\Rightarrow g(f(x)) \in C$.



Example — Let $f(x) = x^2$, $g(x) = \sin x$, $x \in \mathbb{R}$.

Then $(g o f)(x) = g(x^2) = \sin x^2$, $x \in \mathbb{R}$.

Definition of Invertible function — Let $f : A \rightarrow B$ be any function.

f is said to be invertible if $\exists\ g : B \rightarrow A$ Such that

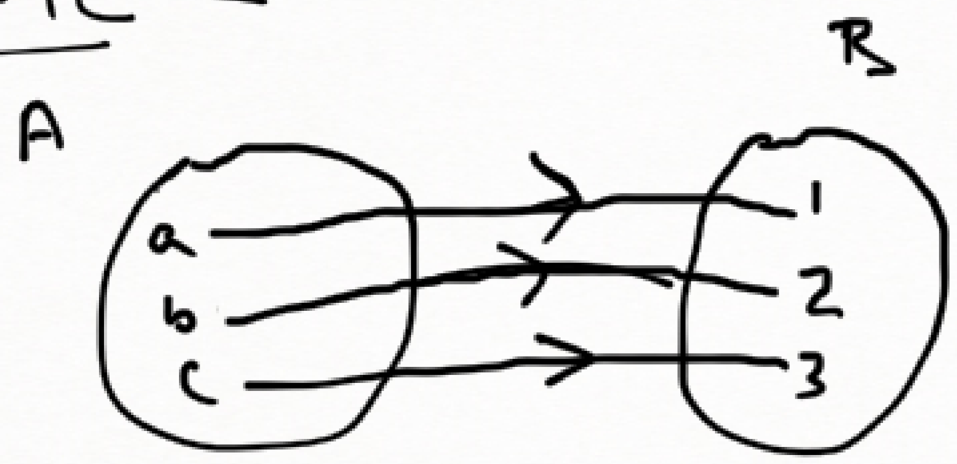
$f o g = I_B$ & $g o f = I_A$ — Identity function on A.

g is called the inverse of A.

If f is bijective then f is invertible.

Example —

1



Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$.

Let $f : A \to B$ be a function defined by

$$f(a) = 1,\ f(b) = 2,\ f(c) = 3.$$

Clearly f is bijective.

f^{-1} exists and $f^{-1} : B \to A$ is given by

$$f^{-1}(1) = a,\ f^{-1}(2) = b,\ f^{-1}(3) = c.$$

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x+1$.

Clearly f is bijective so f^{-1} exists.

Let $y \in \mathbb{R}$. $f(x) = y \Rightarrow x+1 = y \Rightarrow x = y-1$.

$f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by $f^{-1}(y) = y-1$.

(3) Let $f : [0, \infty) \to [0, \infty)$ be given by $f(x) = \sqrt{x}$.

Then f is bijective

If $\sqrt{x} = y \Rightarrow x = y^2$.

$f^{-1} : [0, \infty) \to [0, \infty)$ be given by $f^{-1}(y) = y^2$.

If a function is not bijective then we can't find its inverse. But we may find restricted domain on which f is bijective and so on that restricted domain f^{-1} exists.

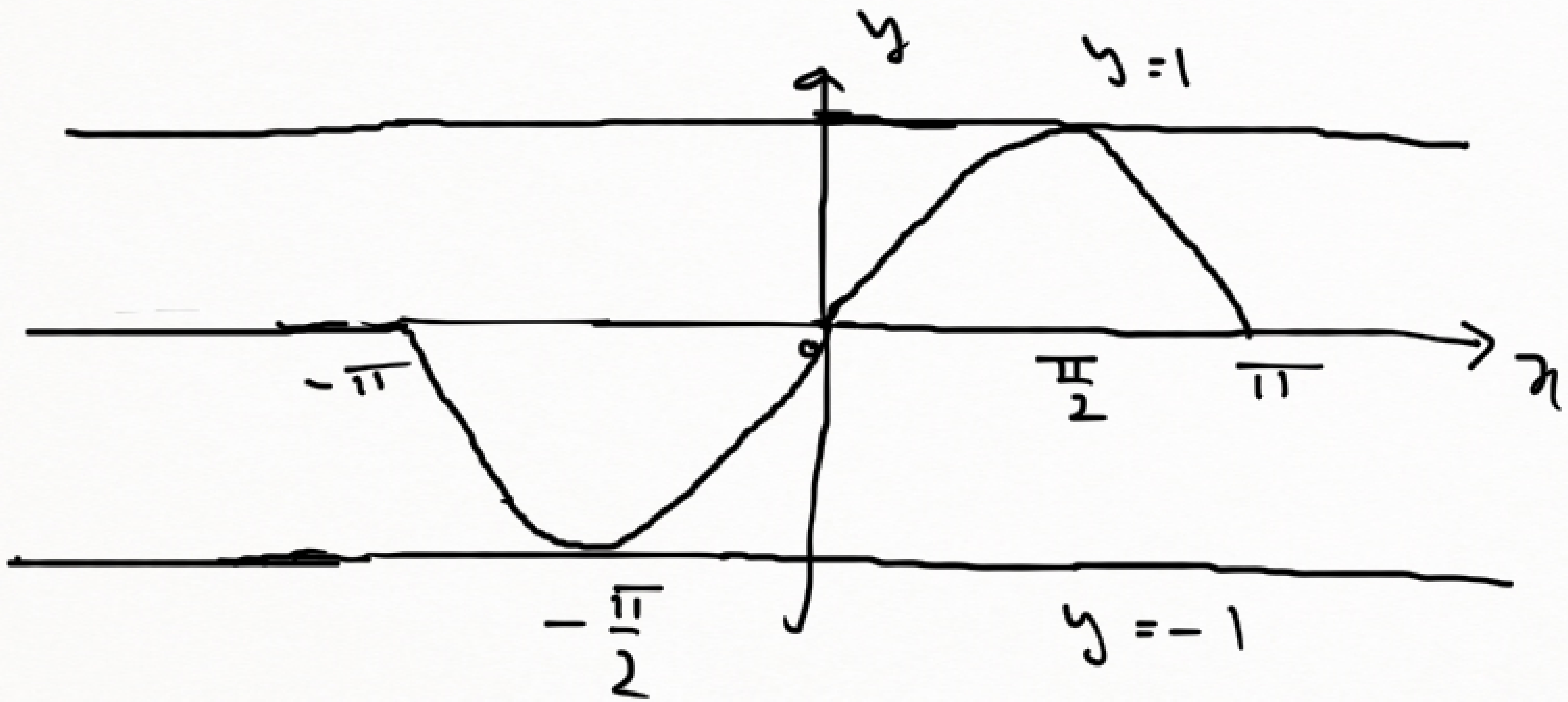
Let $f(x) = \sin x$, $x \in \mathbb{R}$.

Clearly f is not one one as $f(0) = f(2n\pi)$ $\forall n \in \mathbb{Z}$.

But f is one one on $[-\pi, \pi]$.

So $\sin : [-\pi, \pi] \longrightarrow [-1, 1]$ is a bijective map

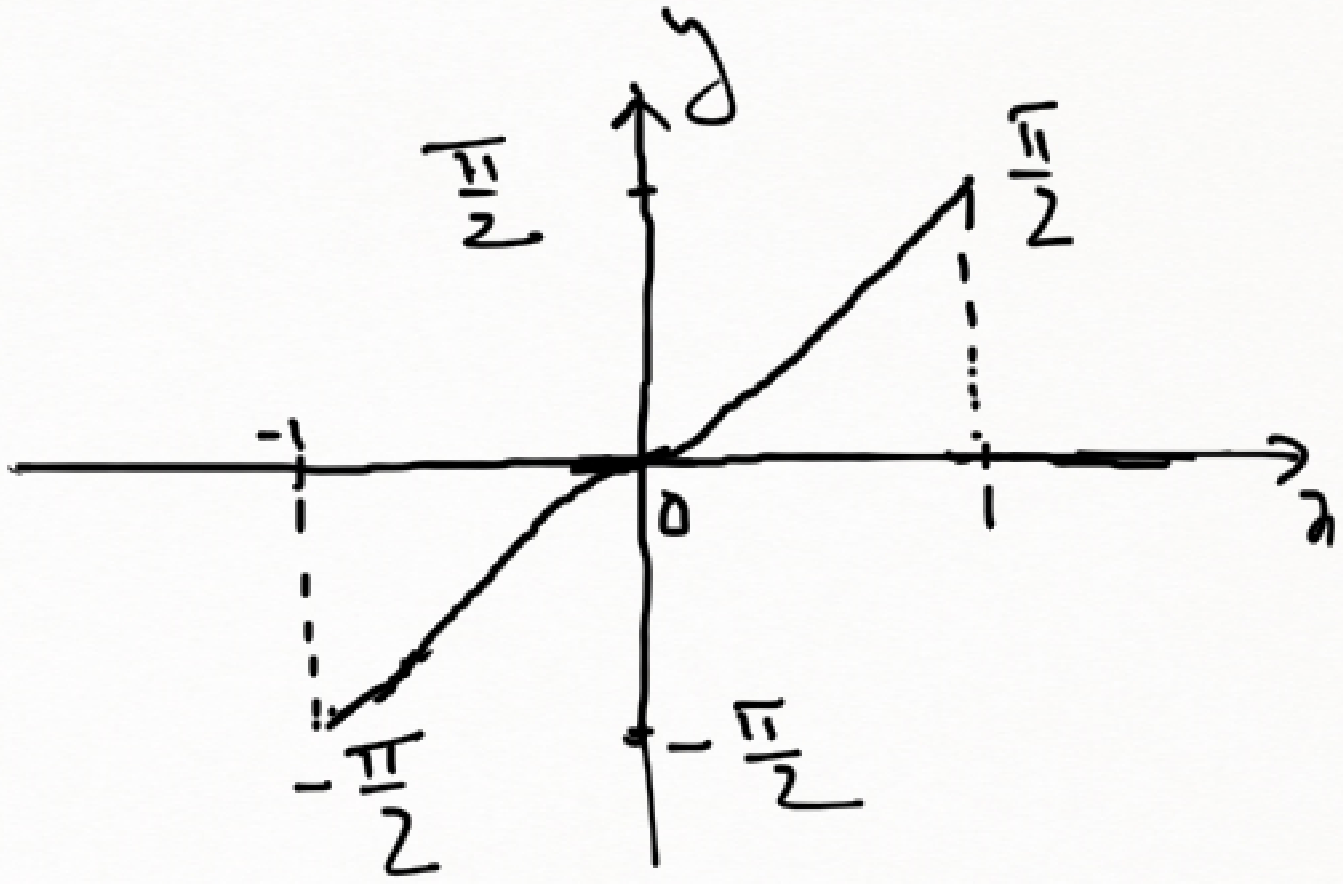
Inverse of Sin exists on $[-1, 1]$, which we denote by $\sin^{-1} x$, $|x| \le 1$.



$y = \sin x$ graph

$y = \sin^{-1} x$ graph

$|x| \le 1$



For $|x| \le 1$, $\exists$ unique $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin\theta = x$.

θ is the principal value of $\sin^{-1}x$

Function	Domain	Range
$y = \sin^{-1}x \Leftrightarrow \sin y = x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}x \Leftrightarrow \cos y = x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1}x \Leftrightarrow \tan y = x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1}x \Leftrightarrow \cot y = x$	$\mathbb{R}$	$(0, \pi)$
$y = \sec^{-1}x \Leftrightarrow \sec y = x$	$\lvert x\rvert \ge 1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$y = \operatorname{cosec}^{-1}x \Leftrightarrow \operatorname{cosec} y = x$	$\lvert x\rvert \ge 1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ .. are also denoted by arc $\sin x$, arc $\cos x$, arc $\tan x$ respectively.

Example - 1 Find $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$.

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \theta \Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\frac{3\pi}{4}$

$\Rightarrow \theta = \frac{3\pi}{4}$ (here $0 < \frac{3\pi}{4} < \pi$).

2 Find $\tan^{-1}(-1)$.

Let $\tan^{-1}(-1) = \theta \Rightarrow \tan\theta = -1 = \tan\left(-\frac{\pi}{4}\right)$

$\Rightarrow \theta = -\frac{\pi}{4}$ (here $-\frac{\pi}{2} < -\frac{\pi}{4} < \frac{\pi}{2}$).