UNIVERSITY OF RWANDA
COLLEGE OF SCIENCE & TECHNOLOGY
SCHOOL OF SCIENCE

Theory of Generalized Functions to Analyse the Cauchy Problem of Wave Equation.

A Research Project Submitted for the Partial Fulfillment of the Requirements for the Degree of Bachelor's of Science

in

MATHEMATICS (Applied Mathematics)

by

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DECLARATION

I **KANAMUGIRE Oliver,** hereby declare that the work presented in this research project is entirely my own and that I did not use any other sources and references than the listed ones. Neither this work nor significant parts of it were part of another examination procedure. I have not published this work in whole or in part before.

KANAMUGIRE Olivier
Kigali, Rwanda.
On *April 2023*

APPROVAL

This research project entitled "Theory of Generalized Functions to Analyse the Cauchy Problem of Wave Equation" has been submitted to the Department of Mathematics for Evaluation with the approval of Assoc.Prof Banzi Wellars.

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DEDICATION

This Research project is dedicated to:

- My lovely family
- My professors and mentors
- My collegues.

ACKNOWLEDGEMENTS

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ABSTRACT

Partial Differential Equations (PDEs) provide the ability to understand how natural phenomena evolve with respect to changes in the various factors on which such phenomena depend. Famous equations such as the heat equation, which describes how heat propagates, and the wave equation, as well as the Schrödinger equation, which describes how the quantum state of a physical system changes over time, have been studied; however, there is still a shortage of complexity in analyzing such systems. The goal of this research project is to solve the initial boundary value problem of the wave equation in its linear form using generalized functions. A specific emphasis is placed on the treatment and generalization of initial and boundary data because this is an essential aspect that boosts the applicability and versatility of solutions to the wave equation. By solving and analyzing initial boundary value problems in one dimension (d'Alembert formula) and two dimensions (Poisson formula) through the use of generalized functions, the unique characteristics and properties inherent in the solutions show that they can be generalized to n dimensions and used to solve specific problems governed by the wave equation.

Keywords: PDEs, Generalized Functions, Wave Equation, Initial boundary problem, Fourier Transform.

Contents

1 INTRODUCTION

1.1 General introduction

The theory of generalized functions, often referred to as distribution theory, is a crucial mathematical framework with a wide range of applications. In this research, we explore the application of distribution theory to analyze the solution of a Cauchy problem for partial differential equations (PDEs), specifically focusing on the wave equation.

In the classical sense, the solution to a PDE is a function that satisfies the provided differential equation. However, complications may arise when the solution is expressed as an infinite series. In such cases, concerns arise regarding the convergence of the infinite series within the specified domain.

Assuming convergence within the domain, another crucial question emerges: does the series converge to the actual solution of the PDE under consideration? It is noteworthy that Fourier series, for instance, may converge to non-differentiable functions or even non-continuous ones, as discussed in [1]. In light of this, it becomes imperative to investigate whether our series converges to solutions that adhere to the properties of the original PDE.

Sobolev and Schwartz [2] introduced the theory of distributions, which has proven to be remarkably effective in solving mathematical problems. Generalized functions are crafted to meet what seems to be a mutually contradictory set of requirements: they exhibit sufficient regularity in certain aspects, being infinitely differentiable and thus having the potential to serve as solutions to partial differential equations (PDEs). An intriguing aspect is that when considered as ordinary functions, they can simultaneously be non-differentiable, non-continuous, and even non-finite [1].

Distribution functions have played a significant role in a wide range of partial differential equations as well as in a substantial portion of analysis. Their ability to reconcile seemingly conflicting characteristics makes them a powerful tool in addressing complex mathematical problems.

The main motivations for introducing generalized functions are summarized as follows:

- **Extension of Classical Operations:** Many fundamental operations, such as differentiation and the Fourier transform, are not defined for all classical functions. Generalized functions extend these operations to a broader class of objects.

- **Incorporation of Singular Objects:** Generalized functions provide a rigorous framework for handling singular entities such as the Dirac delta distribution, which frequently arise in applications.

- **Weak Solutions of Differential Equations:** Solutions of differential equations, as well as derivatives of functions, may fail to exist in the classical sense. The theory of generalized functions enables a precise formulation and analysis of such weak solutions.

- **Unification of Series and Integral Methods:** Distributions allow discrete sequences to be represented using delta functions, thereby interpreting series as integrals. This leads to a unified treatment of summation and integration, including Fourier series as a special case of the Fourier transform [3].

1.2 Problem statement

1.3 Problem Statement

Let $\varphi, f \in C^\infty(\mathbb{R}^{1+n})$. Consider the inhomogeneous wave equation

$$\Box\varphi := \partial_t^2\varphi - \Delta\varphi = f \tag{1.1}$$

with initial conditions

$$\varphi|_{t=0} = \varphi_0, \qquad \partial_t\varphi|_{t=0} = \varphi_1.$$

Here, $\Box$ denotes the wave (d'Alembert) operator and Δ denotes the Laplacian in the spatial variables. The objective is to determine the solution φ and analyze its properties.

1.4 Objectives

1.4.1 General objective

The general objective of the research is to find the general solution to n-dimensional wave equation using the theory of distribution and do analyses of the this generalized solution in contrast to the classical one.

1.4.2 Specific objectives

- To express the solution(s) of wave equation(s) in terms of generalized functions.

- To analyse solution of IVP/BVP for wave equation(s) using generalized functions.

1.5 Limitation of the research

- This research focuses only on the wave equation in n dimensional space but we only find the explicit formulae for the wave equation in one and two dimensions, namely the D'Alembertian formula and the Poisson formula for the in-homogeneous and homogeneous wave equation respectively.

- We mainly use the Fourier transform only on the distributional solution to the wave equation.

- The wave equation considered is linear and, moreover, in 2 dimensions, is homogeneous.

2

LITERATURE REVIEW

2.1 Historical background

2.1.1 Distribution in the development of functional analysis

In what Frechet called functional calculus [2], he initiated the study of abstract function space. The aim of Frechet and his predecessors was to find suitable definition of the differential of a functional so that it would have extremum at the points where the differential vanishes. However, it was not successful because they were too general and had no linear structure.

Later Linear function emerged from work on integral equations. In the work of David Hilbert's(1862-1943)*"Grundzuge einer Allgemeinen Theorie der Linearen Integralgleichungen [1912]"* (Foundations of a General Theory of Linear Integral Equations), David Hilbert developed a unified theory for the solution of integral equations. This work was an important contribution to functional analysis and provided a new approach to solve important problems in physics and engineering. Hilbert's theory introduced a rigorous mathematical framework for the analysis of linear integral equations, including Fredholm integral equations, and developed new techniques for the solution of these equations. Hilbert's work provided a foundation for modern functional analysis, and his methods are still widely used today in the study of integral equations [2]. Eventhough he used a great tool, he didn't consider L^2 or l^2 spaces.

By 1913 the technical tool required for generalization of the function concept in terms of measures had been discovered. However, neither functionals nor measures were used for this purpose until 1936 when Sobolev began to use functionals as generalized functions [2] in his work on partial differential equation.

2.1.2 Generalized differentiation and generalized solution to the PDEs"

Here Generalized differentiation means "differentiation " of the functions which are not smooth [2]. similarly, generalized solution to an nth order PDE means a "solution" in some sense which is not n times differentiable.

In the recent years vibrating string has been studied thoroughly in order to shed the light on the concept of functions used in the eighteenth century. In seventeenth century, mathematicians had one class of functions,"the analytic expressions". therefore, the classical solutions were analytical expressions by Euler called continuous functions, whereas generalized solutions were the other types of functions called "Discontinuous expressions."

The development of the theory of PDEs continued in the 19^{th} century with great tools and fruitful problems. During the time a clear understanding of classical concepts and theories was necessary condition for development of the more procedures which culminated in the theory of distribution [2].

2.1.3 Brief history of mathematicians on the development of theory of the distribution

Laplace was one of the first mathematicians to study partial differential equations, which describe the behavior of physical systems in space and time. He introduced the concept of the Laplace operator, which is a differential operator that can be used to solve certain types of partial differential equations. Gauss also made significant contributions to the theory of partial differential equations, particularly in the area of potential theory. He introduced the concept of the Green's function, which is a fundamental solution to the Laplace equation, a type of partial differential equation that arises in many physical problems. The concept of fundamental solution was further developed in the early 20th century by mathematicians such as Hadamard and Schwartz. They introduced the theory of distributions, which provides a rigorous framework for dealing with functions that are not well-defined in the usual sense.

In this framework, a fundamental solution is a distribution that solves a particular type of partial differential equation, known as a homogeneous linear equation. The fundamental solution is often used as a building block for constructing solutions to more complicated partial differential equations.

2.2 General concept for Wave equation

Most of the time, we find a solution to the PDEs contain arbitrary functions(constants) which are determined by Initial and boundary conditions. such problems are called initial value/boundary value problem. Wave equation arises naturally in different states, dimensions. The following are variety of models of wave equation and their prototype [4].

- **Vibrating string**

  This phenomena occurs like when someone is playing grand-piano, guitar,... It was first derived by Jean Baptiste le Rond d'Alembert. it is the case where in equation (??), n=1 and $f = 0$ (homogeneous equation of dimension $\mathbb{R}^{1+1}$)

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2}$$

  where $y(x,t)$ is the displacement of the string at position x and time t, T is the tension in the string, and μ is the linear density of the string.
  This equation relates the acceleration of a small element of the string to the tension and curvature of the string. It describes the propagation of a wave along the string, and its solutions represent the various modes of vibration of the string.

- **Light in vacuum**

It is obvious that Maxwell equations from electromagnetism , each component of electric and magnetic fields satisfy (??) with $f = 0$ and $n = 3$.

$$\nabla.E = \rho/\epsilon_0$$
$$\nabla.B = 0$$
$$\nabla \times E = \frac{-\partial B}{\partial t}$$
$$\nabla \times B = \mu_0(J + \epsilon_0 \frac{\partial E}{\partial t})$$

where E is the electric field, B is the magnetic field, ρ is the charge density, J is the current density, ϵ_0 is the electric constant (permittivity of free space), and μ_0 is the magnetic constant (permeability of free space).

- **Propagation of sound.**

  The wave equation (1) arises as the linear approximation of the compressible Euler equations, which describe the behavior of compressible fluids (e.g., air).
  One important equation in this set is the continuity equation, which can be written as:
  $$\frac{\partial \rho}{\partial t} + \nabla.(\rho v) = 0,$$
  where ρ is the density of the fluid, v is the velocity vector of the fluid, and ∇ is the gradient operator

- **Gravitational wave.**

  A suitable geometric generalization of the wave equation (??) turns out to be the linear approximation of the Einstein equations, which is the basic equation of the theory of general relativity for gravity.

  one of important equation is linearized gravitational equation around a flat space-time background.
  $$\Box h_{\mu\nu} = -16\pi G T_{\mu v},$$
  where $\Box$ is the d'Alembertian operator, G is the gravitational constant, $h_{\mu\nu}$ is the perturbation to the metric tensor that describes the gravitational wave, and $T_{\mu\nu}$ is the stress-energy tensor that represents the sources of the gravitational field.

Since in the research we restrict on the wave equation, for a great warm up let's give an example of wave equation in one dimension:
The initial-value problem of a vibrating string is the problem of finding the solution of the wave equation

$$u_{tt} = c^2 u_{xx}, \tag{2.1}$$

satisfying the initial conditions

$$u(x, t_0) = u_0(x) \text{ and } u_t(x, t_0) = v_0$$

In initial-value problems, the initial values usually refer to the data assigned at $t = t_0$(initial time).Here we say initial position and initial velocity. It is not essential that these values be given along the line $t = t_0$; they may very well be prescribed along some curve L_0 in the x-t plane. In such a context, the problem is called **the Cauchy problem** instead of the initial-value problem [1], although the two names are actually synonymous.

Example: equation (2.1) may be assigned the following data

$$\begin{aligned} u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \\ u(0,t) &= 0 \end{aligned}$$

where $0 \leq x < \infty$ and $t > 0$

In the theory of partial differential equation, it is not common to seek solutions to the PDEs using theory of distributions rather classical solution. Consider the wave equation (2.1) together with initial conditions and this boundary condition $u(0,t) = 0$. Its solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau \qquad x > ct \\ u(x,t) &= \frac{1}{2}[f(x+ct) - f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau \qquad x < ct \end{aligned}$$

Looking at the above solutions, We see that it is essential for f to be twice continuously differentiable, and the same goes for g but not necessarily twice. The greatest challenge we can face is that, "what if f, g are not differentiable at all in the usual sense?"

Well, the theory of generalized solutions allows us to handle this challenge. Using the distribution theory, we can transfer the strong form of the partial differential equation into the weak form of the partial differential equation. If the equation satisfies a strong solution, then it also satisfies a weak equation, although the converse is not necessarily true.

Before we define generalized function or distribution, it is essential to define it's domain. it is called "Space of test functions".

Definition 1. *Test/Fundamental function [1]*

The function $\phi : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{C}$ is said to be a test function or fundamental function if it is infinitely differentiable with compact support(that is $\phi \in C_c^\infty$).

Compact support means that there exist a compact set $K \subset \Omega$ such that $\phi(x) = 0$ whenever $x \notin K$.

We denote the space of all such functions as $\mathcal{D}(\Omega)$ which is called fundamental space.

Example:

$$\Psi(x) = \begin{cases} e^{\frac{-1}{1-x^2}}, & \text{whenever } |x| < 1 \\ 0, & \text{elsewhere.} \end{cases} \tag{2.2}$$

We see that $supp(\psi) = \overline{]-1,1[}$. Python codes used to plot the Figure 2.1 can be found in Appendix section.

Definition 2. *Distribution*

A map $\mathcal{T} : \mathcal{D}(\Omega) \to \mathbb{R}$ given by $\phi \rightsquigarrow \mathcal{T}(\phi)$ is called distribution if the following properties hold:

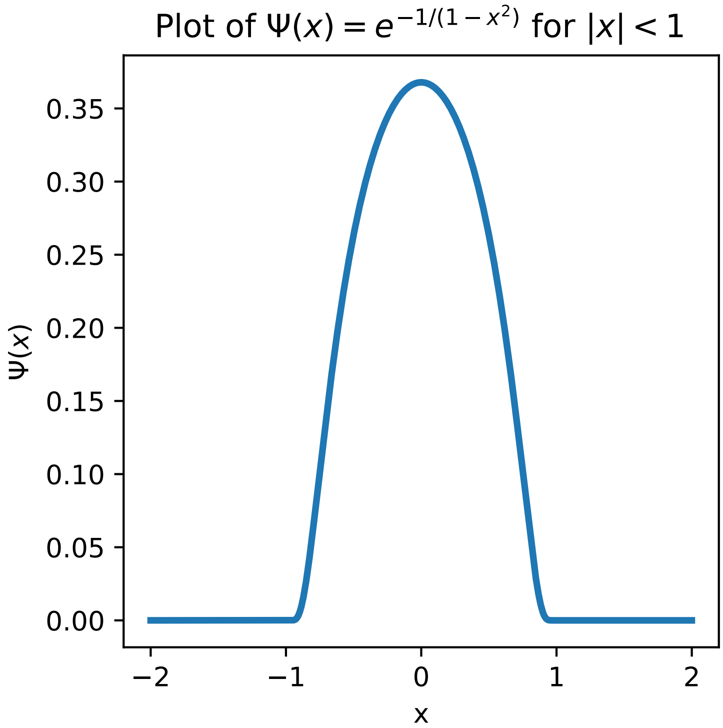


Figure 2.1: This graph is for the function in the example 2.2

1. *It is linear; For* $\phi,\psi \in \mathcal{D}(\Omega)$ *and* a,b *belong in* $\mathbb{C}$
   $\mathcal{T}(a\phi + b\psi) = a\mathcal{T}(\phi) + b\mathcal{T}(\psi)$

2. *It is continuous, that is*
   $\psi_m \to \psi$ *in* $\mathcal{D}(\Omega) \implies \mathcal{T}(\psi_m) \to \mathcal{T}(\psi)$ *in* $\mathbb{C}$

Moreover, the space of distribution can be denoted by $\mathcal{D}'$.

2.3 Dirac delta distribution

Dirac delta distribution is also known as unit impulse. It is a mathematical construct that was first introduced by the English physicist Paul Dirac in 1927.

Dirac introduced the delta function as a way of describing the behavior of quantum mechanical particles in a way that was consistent with the principles of classical mechanics. One can read this paper through this link [All about the Dirac Delta Function]. It became the powerful tool in mathematical representation of many physical phenomena.

Definition 3. *Delta function*

$$\delta(x) = \begin{cases} +\infty \quad \text{when} \quad x = 0 \\ 0 \quad \text{when} \quad x \neq 0 \end{cases}$$

Although using the integral of smooth function with test function is one way to define distribution. That is regular distribution $\langle T_f, \varphi \rangle = \int f\varphi dx$ We can find other way to define distribution.

$$\delta(\varphi) = \varphi(0)$$

It is direct to see that the above equation defines a distribution.

Proof. Linearity: for $\phi, \varphi \in \mathcal{D}$ and scalars α, β

$$\begin{aligned} \delta(\alpha\phi + \beta\varphi) &= \alpha\phi(0) + \beta\varphi(0) \\ &= \alpha\delta(\phi) + \beta\delta(\varphi) \end{aligned}$$

Continuity: Suppose that $\varphi_m \longrightarrow \varphi$. $\delta(\varphi_m) = \varphi_m(0) \longrightarrow \varphi(0) = \delta(\varphi)$

□

2.3.1 Basic Properties of Dirac delta distribution

1. **Integration property** : This property is often referred to as the normalization property of the delta distribution. The integration of δ over the real line is one.

   $\int \delta(x)dx = 1$ and $\int \delta(x-y)f(y)dy = f(x)$

2. **Scaling property** $\delta(ax) = \dfrac{1}{|a|}\delta(x)$ where a is a scalar.

3. **Shifting property**: This property is sometimes called the the sampling property of the delta distribution, because it "samples" the value of the function $f(x)$ at the point $x = 0$. $\int f(x)\delta(x)dx = f(0)$

4. **Translation property:** $\int f(x)\delta(x-a)dx = f(a)$

5. **Convolution property**: Intuitively, we can think of the convolution as a kind of "blending" or "mixing" of the two functions, where the shape of the resulting function depends on both functions.

   $\delta(x) * f(x) = f(x)$ this property can be derived using properties of delta distribution.

2.3.2 Graph of δ distribution

A variety of functions are commonly used as approximations to the Dirac delta distribution. Figure 2.2 illustrates four such delta sequences: the Laplace-type exponential, Gaussian, Cauchy (Lorentzian), and sinc-based approximations. Each family depends on a positive parameter c, which controls the width of the peak around the origin. $c \to 0$, the functions become increasingly concentrated near $x = 0$ while preserving unit integral, thereby converging to the Dirac delta in the sense of distributions. Python codes can be found in the appendix section of listing 2.

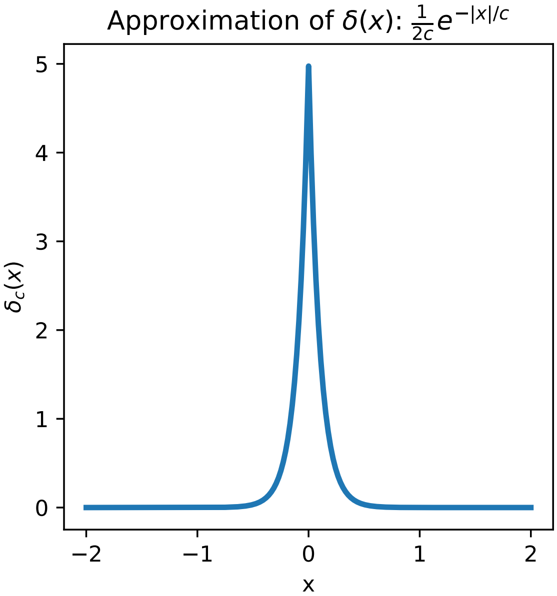
If we have to plot $\delta(x-a)$ we shift the graph to the value of $a \in \mathbb{R}_*$ (if we are in one dimension). the spike spouts from the value of a

2.4 Some operations on the distribution theory

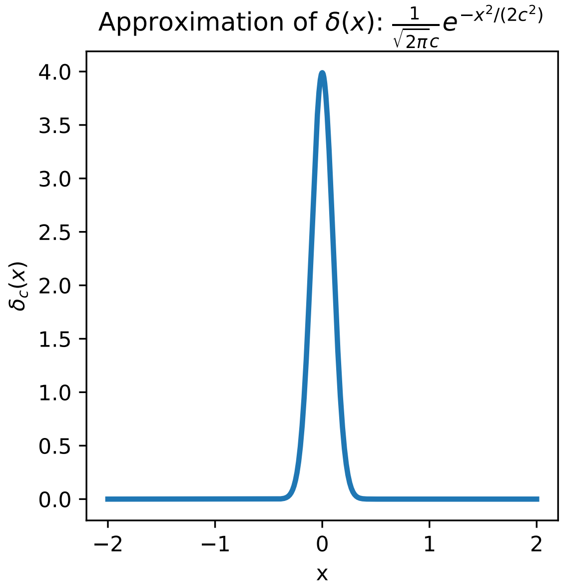
2.4.1 Derivatives

In order to use distributions in differential equations, we need to know what it means to differentiate them.

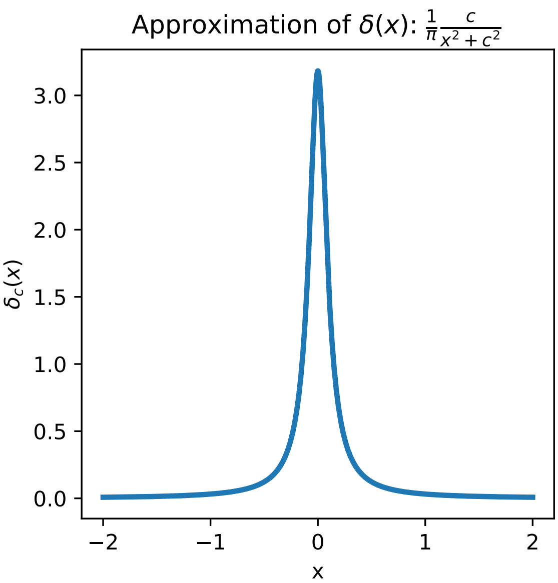
Definition 4. *Let T be a distribution on $\mathbb{R}^n$. Then the i^{th} partial derivative of T is defined by*



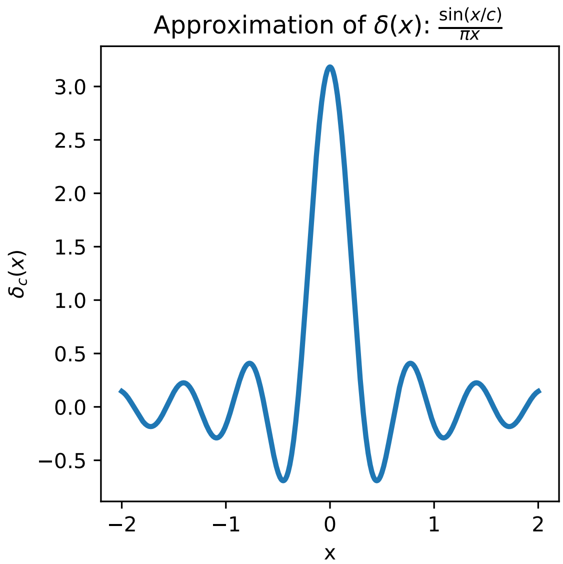
(a) Laplace



(b) Gaussian



(c) Cauchy



(d) sinc-based

Figure 2.2: Four common approximations of the Dirac delta function.

$$\left\langle \frac{\partial T}{\partial x_i}, \varphi \right\rangle = -\left\langle T, \frac{\partial \varphi}{\partial x_i} \right\rangle$$

Where φ is a test function on $\mathbb{R}^n$
More general we can write differentiation like this,

$$\left\langle \frac{\partial^\alpha T}{\partial x^\alpha}, \varphi \right\rangle = (-1)^{|\alpha|} \left\langle T, \frac{\partial^\alpha \varphi}{\partial x^\alpha} \right\rangle$$

Example: consider the function

$$\Theta(x) = \begin{cases} 0 & when \quad x < 0 \\ 1 & when \quad x > 0 \end{cases} \tag{2.3}$$

We find the derivative of this function. It is called Heaviside function. named after Oliver Heaviside (1850–1925), a self-taught British electrical engineer, mathematician, and physicist. One has $\Theta' = \delta$ [3].

$$\langle \Theta', \varphi \rangle = -\langle \Theta, \varphi' \rangle = -\int_{-\infty}^{+\infty} \Theta(x)\varphi'(x)dx = \int_{0}^{+\infty} \varphi'(x)dx = \varphi(o) = \langle \delta, \varphi \rangle$$

2.4.2 Properties of distributions

- **Addition of two distributions**

  $\langle T_1 + T_2, \varphi \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle$ given T_1 and T_2 in $\mathcal{D}'(\mathbb{R}^n)$

- **Multiplication by a scalar**
  $\langle \alpha T, \varphi \rangle = \alpha \langle T, \varphi \rangle$

- **The shifting of a distribution**
  $\langle T(t-\tau), \varphi \rangle = \langle T, \varphi(t+\tau) \rangle$

[1]

[1] There are many properties of distributions, for more, read [3]. it includes convolutions, product with continuously differentiable functions, etc.

3
METHODS AND TECHNIQUES

In the theory of Partial Differential Equations (PDEs), various methods exist for finding solutions. Analytical methods include the method of characteristics, the Fourier method, and the Laplace transform method, among others. Numerical methods encompass the Finite Element Method, Finite Volume Method, and Finite Difference Method, which are widely employed techniques.

In our research, we will demonstrate how to utilize distribution theory to obtain the fundamental solution to partial differential equations, specifically applying it to the Cauchy problem of the wave equation. We will commence with the one-dimensional space and subsequently extend our approach to the two-dimensional space.

3.1 How does distribution work to obtain solution to the PDEs?

We denote linear differential operator as [5]

$$P(D) := L(x, D) = \sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha \tag{3.1}$$

We will define linear differential equation of order m in terms of above differential operator L.

$$\sum_{|\alpha|=0}^{m} a_\alpha(x) D^\alpha u = f \tag{3.2}$$

Note that this equation is homogeneous if and only if $f = 0$.
The equation (3.2) becomes

$$L(x, D)u = f \tag{3.3}$$

In general, we are looking for a distribution $u \in \mathcal{D}'(\Omega)$which satisfies (3.3). Notice that this solution may not be a usual solution we are used to, for it need not be a function, and even if it is a function, its derivatives need not be functions. This is where the concept of generalised solution comes from.

3.2 Basic definitions and theorems

Definition 5. *A distribution $u \in \mathcal{D}'(\Omega)$ is a generalized solution of the equation$L(x, D)u = f$ in the region $\mathcal{A} \subseteq \mathbb{R}^n$ if*

$$\langle L(x, D)u, \psi \rangle = \langle f, \psi \rangle \tag{3.4}$$

for every $\psi \in \mathcal{D}(\Omega)$ and supp$(\psi) \subseteq \mathcal{A}$

Definition 6. *Let $L(D)$ be a differential operator with constant coefficients. We call distribution $E \in \mathcal{D}'(\Omega)$ a **fundamental solution** of differential operator $L(D)$ if E satisfies the equation*

$$L(D)E = \delta \qquad\qquad \textit{in } \mathcal{D}'(\Omega)$$

where δ is Dirac delta function.

Theorem 3.2.1. *Let $L(D)$ be a differential operator with constant coefficients and E a tempered fundamental solution of it. Let also f be a Schwartz function on $\mathbb{R}^n$. Then, a solution to the equation $L(D)u = f$ is given by*

$$u = E * f$$

*where * represents the convolution operation.*
The proof of this theorem can be found in [3].

3.3 Derivation of d'Alembertian using fundamental solution aproach

We begin with simple case for n=1. that is we find the solution to the d'Alembertian of the form

$$\Box = \partial_t^2 - \partial_x^2.$$

step 1: we factorize and we choose simple coordinate system in which factors are coordinate derivatives.

$$\partial_t^2 - \partial_x^2 = (\partial_t - \partial_x)(\partial_t + \partial_x)$$

we choose $u = \dfrac{1}{2}(t - x)$ and $v = \dfrac{1}{2}(t + x)$

hence

$$\partial_u = \frac{1}{2}(\partial_t - \partial_x) \text{ and } \partial_v = \frac{1}{2}(\partial_t + \partial_x)$$

thus d'alembertian becomes

$$\Box = 4\partial_u \partial_v$$

3.4 The change of variables for distributions

Theorem 3.4.1. *Change of variables [4].*

let $\Phi : X_1 \to X_2$ be a diffeomorphism where X_1 and X_2 are open subsets of $\mathbb{R}^k$. $\forall h \in \mathcal{D}'(X_2)$ there is a way to associate a unique distribution $h \circ \Phi \in \mathcal{D}'(X_1)$ so that $u \circ \Phi$ agrees with usual composition for $h \in C_c^\infty \subseteq \mathcal{D}'(X_2)$ and the following holds:

The mapping $\mathcal{D}'(X_2) \to \mathcal{D}'(X_1)$

. $h \rightsquigarrow h \circ \Phi$ *is linear and continuous in h.*

In fact, for $\phi \in C_c^\infty$ then

$$\langle u \circ \Phi | \phi \rangle = \langle u | \frac{1}{|det(\Phi)|} \phi \circ \Phi^{-1} \rangle$$

Proof. Uniqueness is obvious since C_c^∞ is dense in $\mathcal{D}'(X_2)$.

Let $h_j \to h$ be the sequence in C_c^∞ converges to h in sense of distribution
Write $\Phi(x) = (y^1(x), y^2(x), y^3(x), ..., y^k(x))$ and
$\dfrac{\partial(y^1, y^2, ..., y^k)}{\partial(x^1, x^2, ..., x^k)} = det(\Phi)$ and then $\dfrac{\partial(x^1, x^2, ..., x^k)}{\partial(y^1, y^2, ..., y^k)} = det(\Phi^{-1})$

For any $\phi \in C_c^\infty$ we have

$$\begin{aligned} \langle h_j \circ \Phi | \phi \rangle &= \int u_n \Phi(x) \phi(x) dx \\ &= \int u_n(y) \phi(\Phi^{-1}(y)) \frac{\partial(x^1, x^2, ..., x^k)}{\partial(y^1, y^2, ..., y^k)} dy \end{aligned}$$

By changing the coordinate $x = \Phi^{-1}(y)$
It is clear that $\phi(\Phi^{-1}(y)) \dfrac{\partial(x^1, x^2, ..., x^k)}{\partial(y^1, y^2, ..., y^k)}$ is a test function on X_2

$$\begin{aligned} \int u_n(y) \phi(\Phi^{-1}(y)) \frac{\partial(x^1, x^2, ..., x^k)}{\partial(y^1, y^2, ..., y^k)} dy &= \langle u_n(y) | \phi(\Phi^{-1}(y)) \frac{\partial(x^1, x^2, ..., x^k)}{\partial(y^1, y^2, ..., y^k)} \rangle \\ &= \langle u_n(y) | \phi(\Phi^{-1}(y)) \frac{1}{|det(\Phi)|} \rangle \end{aligned}$$

$$\begin{aligned} \langle u_n(y) | \phi(\Phi^{-1}(y)) \frac{1}{|det(\Phi)|} \rangle &\longrightarrow \langle u(y) | \phi(\Phi^{-1}(y)) \frac{1}{|det(\Phi)|} \rangle \\ &= \langle u | \frac{1}{|det(\Phi)|} \phi \circ \Phi^{-1} \rangle \end{aligned}$$

□

Now we apply the transformation to delta distribution to see previous result.

Corollary 3.4.1. *let $\Phi : \mathbb{R}^k \longrightarrow \mathbb{R}^k$ be an invertible linear map. then we have*

$$\delta_0 \circ \Phi = \frac{1}{|det(\Phi)|} \delta_0 \text{ the initial conditions.}$$

Applying this corollary

$$\delta|_{(t,x)=(0,0)} = \frac{1}{|det\Phi|}\delta|_{(u,v)=(0,0)}$$

here

$$det\Phi = \begin{vmatrix} \frac{\partial u}{\partial t} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial t} & \frac{\partial v}{\partial x} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Substituting the results gives

$$\delta|_{(t,x)=(0,0)} = \frac{1}{|1/2|}\delta|_{(u,v)=(0,0)} = 2\delta|_{(u,v)=(0,0)}$$

We continue to find the fundamental solution call it E and by using the definition (5) we have

$$4\partial_u\partial_v E = 2\delta_0 \Longleftrightarrow \partial_u\partial_v E = \frac{1}{2}\delta_0 \tag{3.5}$$

We impose the ansatz that $E(u,v)$ is the tensor product $\frac{1}{2}E_1(u)E_2(v)$

$$\partial_u E_1 = \delta_{u=0} \text{ and } \partial_v E_2 = \delta_{v=0}$$

we have found out the function whose derivative is a delta function ie Heaviside function $E_1(u) = H(u) + c_1$ and $E_2(v) = H(v) + c_2$

$$E(u,v) = \frac{1}{2}(H(u) + c_1)(H(v) + c_2)$$

How do we choose the arbitrary constants c_1 and $c_2 \in \mathbb{R}$? We look for the forward fundamental solution, i.e. a solution E_+ to equation (5) which is supported in the half space $\{t \geq 0\}$.
Then we see that we are forced to choose $c_1 = c_2 = 0$ [4] hence

$$E_+(u,v) = \frac{1}{2}H(u)H(v)$$

we come back to our original coordinates and we get the final result

$$E_+(u,v) = \frac{1}{2}H(x-t)H(x+t) \tag{3.6}$$

3.4.1 Properties of the fundamental solution

The fundamental solution (3.6) have the following properties

1. $\Box E_+ = \delta_0$
2. $suppE_+ \subseteq \{(t,x) \in \mathbb{R}^{1+1} : 0 \leq |x| \leq t\}$

Theorem 3.4.2. *Suppose that a forward fundamental solution E+ with the properties above exists.Then it is the unique forward fundamental solution, i.e., any fundamental solution E with $suppE \subseteq \{t \leq 0\}$ equals E_+.*

Proof. suppose E to be a fundamental solution that is $\Box E = \delta_0$ and $suppE \subseteq \{t \leq 0\}$ then

$$E = \delta_0 * E = \Box E_+ * E = E_+ * \Box E = E_+ * \delta_0 = E_+$$

$\Box$

Theorem 3.4.3. *Representation formula [4]*
Suppose that a forward fundamental solution E_+ with the properties defined before exists.Then given any solution ϕ to the equation $\Box\phi = f$ with $\phi, f \in C_c^\infty(\mathbb{R}^{1+k})$ we have the so called representation formula

$$\phi = E_+ * \phi_1 \delta_{t=0} + \partial_t E_+ * \phi_0 \delta_{t=0} + E_+ * f 1_{\{t \geq 0\}}.$$

where $(\phi_0, \phi_1) = (\phi, \partial_t \phi)|_{t=0}$

4

SOLUTION TO THE CAUCHY PROBLEM OF THE WAVE EQUATION

4.1 Solution in $\mathbb{R}^{1+1}$

Using the representation formula, we compute each term individually. Before proceeding, we state the following corollary.

Corollary 4.1.1. *Suppose a forward fundamental solution E_+ with the properties defined previously exists. Let $\phi \in C_c^\infty(\mathbb{R}^{1+k})$ solve the inhomogeneous wave equation*

$$\Box\phi = f$$

with initial data

$$(\phi, \partial_t\phi)|_{t=0} = (\phi_0, \phi_1),$$

for $t > 0$ and $(t, x) \in \mathbb{R}^{1+k}$.

If

$$f(s, y) = 0 \quad \text{for } 0 < s < t, \ |y - x| \leq t - s$$

and

$$\phi_0(y) = \phi_1(y) = 0 \quad \text{for } |y - x| \leq t,$$

then

$$\phi(t, x) = 0.$$

Now, we compute each term in the representation formula:

$$\begin{aligned}
E_+ * \phi_1\delta_{t=0} &= \frac{1}{2}\Big\langle H(t - s - (x - y))H(t - s + x - y) \,|\, \phi_1(y)\delta_0(s)\Big\rangle_{(y,s)} \\
&= \frac{1}{2}\Big\langle H(t - (x - y))H(t + x - y) \,|\, \phi_1(y)\Big\rangle_y \\
&= \frac{1}{2}\int_{x-t}^{x+t} \phi_1(y)\, dy,
\end{aligned}$$

$$
\begin{aligned}
E_+ * f1_{\{t\geq 0\}} &= \frac{1}{2}\Big\langle H(t-s-(x-y))H(t-s+x-y) \,|\, f(s,y)H(s)\Big\rangle_{(y,s)} \\
&= \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s,y)\,dy\,ds.
\end{aligned}
$$

Moreover, the term $\partial_t E_+ * \phi_0\delta_{t=0}$ reproduces the d'Alembert solution in one space and one time dimension. Indeed, we have

$$
\begin{aligned}
\partial_t E_+ * \phi_0\delta_{t=0} &= \partial_t\big(E_+ * \phi_0\delta_{t=0}\big) \\
&= \partial_t\left(\frac{1}{2}\int_{x-t}^{x+t}\phi_0(y)\,dy\right) \\
&= \frac{1}{2}\big(\phi_0(x+t)+\phi_0(x-t)\big),
\end{aligned}
$$

which is precisely the d'Alembert formula in $\mathbb{R}^{1+1}$.

Combining all terms, we arrive at the final solution:

$$
\phi(t,x) = \frac{1}{2}\left[\phi_0(x+t)+\phi_0(x-t)+\int_{x-t}^{x+t}\phi_1(y)\,dy + \int_0^t\int_{x-t+s}^{x+t-s} f(s,y)\,dy\,ds\right]. \tag{4.1}
$$

4.2 Initial value solution in R^{1+n}

4.2.1 Review of Fourier transform on generalized functions

Recall that the Fourier transform pair is defined as

$$
\mathcal{F}_{x\to\omega}\{f(x)\} = \hat{f}(\omega) = \int f(x)e^{-ix\cdot\omega}dx \tag{4.2}
$$

and

$$
\mathcal{F}^{-1}_{\omega\to x}\{\hat{f}(\omega))\} = f(x) = \frac{1}{(2\pi)^{n/2}}\int \hat{f}(\omega)e^{ix\cdot\omega}d\omega \tag{4.3}
$$

where $f \in L^1(\mathbb{R}^n)$. Note that $\hat{f}(0)$ is the integral of f. The operator $\mathcal{F} : L^1 \longrightarrow L^\infty$ is very far from invertible.

It is considered very advantageous to have spaces where the Fourier transform acts as an isomorphism [6], meaning that it preserves the structure and properties of the original space. One such space that possesses this property is the Schwartz function space.

Definition 7. *Schwartz functions let $\chi : \mathbb{R}^n \longrightarrow \mathbb{C}$ be a function. We say that χ is a Schwartz function if*

1. $\chi \in \mathbb{C}^\infty$
2. $\forall\ m \in \mathbb{Z}^+,\ k \in (\mathbb{Z}^+)^n$, *then* $\exists C_{m,k}$ *such that* $|t|^m|D^k\chi(t)| \leq C_{m,k}$

The space of all Schwartz functions is called Schwartz space, and it is denoted by $\mathcal{S}(\mathbb{R})$. They are usually called the functions of rapid descent.

Definition 8. *A linear functional u from Schwartz space to complex numbers is called Tempered distributions if $\phi_k \to 0$ in $\mathcal{S}(\mathbb{R}^n)$ we have $\langle u, \phi_k\rangle \longrightarrow 0$. That is, u must be linear and continuous. We denote the space of all tempered distributions as $\mathcal{S}'(\mathbb{R}^n)$ and Some authors also call tempered distributions, "the distributions of slow growth".*

4.2.2 Fourier transform on tempered distribution

Definition 9. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. We define the Fourier transform as $\mathcal{F}u = \hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ by the formula:*

$$\langle \hat{u}, \varphi \rangle := \langle u, \hat{\varphi} \rangle \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$$

Theorem 4.2.1. *On $\mathbb{R}^n$, we have:*

$$\hat{\delta_0} = 1$$
$$\hat{1} = (2\pi)^n \delta_0$$

Note that if S and T are tempered distributions then the convolution between them is also a tempered distribution and moreover, $\widehat{S * T} = \hat{S}.\hat{T}$.

4.2.3 Practical use of the Fourier transform

The Fourier transform is beneficial in differential equations because it can reformulate them as problems which are easier to solve. In addition, many transformations can be made simply by applying predefined formulas to the problems of interest. A small table of transforms and some properties is given below

Description	Function	Transform
Delta function in x	$\delta(x)$	1
Delta function in k	1	$2\pi\delta(k)$
Exponential in x	$e^{-a\|x\|}$	$\frac{2a}{a^2+k^2}$
Exponential in k	$\frac{2a}{a^2+x^2}e^{-a\|k\|}$	$2\pi e^{-\frac{x^2}{2}}$
Gaussian	$e^{-\frac{x^2}{2}}$	$\sqrt{2\pi}e^{-\frac{k^2}{2}}$
Derivative in x	$f_0'(x)$	$ikF(k)$
Derivative in k	$xf(x)$	$iF_0'(k)$
Integral in x	$\int_{-\infty}^{x} f(x_0)dx_0$	$\frac{F(k)}{ik}$
Translation in x	$f(x-a)$	$e^{-iak}F(k)$
Translation in k	$e^{iax}f(x)$	$F(k-a)$
Dilation in x	$f(ax)$	$\frac{1}{a}F\left(\frac{k}{a}\right)$
Convolution	$f(x) * g(x)$	$F(k)G(k)$

Table 4.1: Fourier transforms of basic functions

4.3 Distributional solution to the wave equation

Consider the initial value problem(IVP):

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \sum_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}; \qquad \alpha \in \mathbb{R}, x \in \mathbb{R}^n \tag{4.4}$$

with

$$u(t,x)|_{t=0} = \varphi(x) \quad \& \quad \partial_t u(t,x)|_{t=0} = \phi(x)$$

We apply Fourier transform with respect to spatial coordinates x to (4.4) and we get:

$$\frac{\partial^2 \hat{u}}{\partial t^2} - \alpha^2 \sum_{k=1}^{n} (i\xi_k)^2 \hat{u} = 0 \tag{4.5}$$

with

$$\hat{u}(t,\xi)|_{t=0} = \hat{\varphi}(\xi) \quad \& \quad \partial_t \hat{u}(t,\xi)|_{t=0} = \hat{\phi}(\xi)$$

Under algebraic manipulation and using $i^2 = -1$ and $\sum_{k=1}^{n} \xi_k^2 = |\xi|^2$; where $|.|$ defines an euclidean norm. The equation (4.5) becomes

$$\frac{\partial^2 \hat{u}}{\partial t^2} + \alpha^2 |\xi|^2 \hat{u} = 0 \tag{4.6}$$

And since there is only time derive, we can rewrite the equation (4.6) as ordinary differential equation

$$\frac{d^2 \hat{u}}{dt^2} + \alpha^2 |\xi|^2 \hat{u} = 0 \tag{4.7}$$

whose solution is given as

$$\hat{u}(t,\xi) = A(\xi) cos(\alpha|\xi|t) + B(\xi) sin(\alpha|\xi|t) \tag{4.8}$$

After applying initial condition, we find that $A(\xi) = \hat{\varphi}(\xi)$ and $B(\xi) = \frac{\hat{\phi}(\xi)}{\alpha|\xi|}$. Hence substitute them in the equation (4.8), it becomes

$$\hat{u}(t,\xi) = \hat{\varphi}(\xi) cos(\alpha|\xi|t) + \frac{\hat{\phi}(\xi)}{\alpha|\xi|} sin(\alpha|\xi|t) \tag{4.9}$$

Note that the above formula shows that if φ, ϕ are Schwartz functions then there is a unique solution u(t,x) which is a Schwartz function for all t and it's Fourier transform is given by (4.9). In the same sense, if φ, ϕ are tempered distributions then u(x,t) is solution which is a tempered distribution [?].

Applying inverse Fourier transform we directly see that

$$u(t,x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix\cdot\xi} \left(\hat{\varphi}(\xi) cos(\alpha|\xi|t) + \frac{\hat{\phi}(\xi)}{\alpha|\xi|} sin(\alpha|\xi|t) \right) d\xi \tag{4.10}$$

If $\varphi = 0$ and $\phi = \delta$ the solution given by equation (4.9) is called the **fundamental solution** or the **Riemann function** and it given by [?]

$$\hat{\mathcal{R}}(t,\xi) = (2\pi)^{(n/2)} \frac{1}{|\xi|} sint|\xi| \tag{4.11}$$

For simplicity we have taken $\alpha = 1$.
We need to find the exact expression for $\mathcal{R}(t,x)$ and to deduce this formula, let us consider this formula

$$\hat{Q}(y,\xi) = (2\pi)^{-n/2} e^{-y|\xi|}$$

For n=1, it is obvious that

$$Q(y,x) = \frac{1}{\pi} \frac{y}{y^2 + x^2}$$

For $n \geq 2$,it's not so elementary like n=1 instead we use this subordination identity [7]

$$e^{-yB} = \frac{y}{2\pi^{1/2}} \int_0^\infty e^{-y^2/4t} e^{tB^2} t^{-3/2} dt, \qquad B > 0 y > 0$$

Let $B = |\xi|$, for any $n \geq 1$

$$\begin{aligned} Q(y,x) &= 1/(2\pi)^n \int e^{-y|\xi| + ix\cdot\xi} d\xi \\ &= \frac{1}{(2\pi)^n} \frac{y}{(2\pi)^{1/2}} \int_0^\infty e^{-y^2/4t} \left[\int e^{-t|\xi|^2 + ix\cdot\xi} d\xi \right] t^{-3/2} dt \end{aligned}$$

[1]

$$Q(y,x) = \left(\sqrt{\frac{1}{4\pi}}\right)^{n+1} y \int_0^\infty e^{-y^2/4t} e^{-|x|^2/4t} t^{-(n+3)/2} dt$$

$\Updownarrow$ Use Fourier transformation

$$Q(y,x) = \theta(n) \frac{y}{(y^2 + |x|^2)^{(n+1)/2}} \tag{4.12}$$

where $\theta(n) = \left(\sqrt{\frac{1}{\pi}}\right)^{n+1} \Gamma(\frac{n+1}{2})$ and Γ is the Euler gamma function.

From (4.12), if

$$\hat{M}(y,\xi) = (2\pi)^{-n/2} \frac{1}{\xi} e^{y\xi} \tag{4.13}$$

Then

$$M(y,x) = \theta'(n)(y^2 + |x|^2)^{-(n-1)/2} \tag{4.14}$$

where $\theta'(n) = \frac{\theta(n)}{n-1}$

In view of continuity of Fourier transform on $S'(\mathbb{R}^n)$, [7]we deduce that if

$$\hat{P}(t,\xi) = (2\pi)^{-n/2} \frac{1}{\xi} e^{it|\xi|} \tag{4.15}$$

then it's inverse Fourier transform is given by

$$P(t,x) = lim_{t \searrow \epsilon} \left[\theta'(n)(|x|^2 - (t - i\epsilon)^2)^{-(n-1)/2}\right] \tag{4.16}$$

Consequently, for the Riemann function, we have

$$\mathcal{R}(t,x) = lim_{\epsilon \searrow 0} \left[\theta'(n) Im(|x|^2 - (t - i\epsilon)^2)^{-(n-1)/2}\right] \tag{4.17}$$

4.3.1 Description on the Riemann function

Case 1: $|x| > |t|$

$\mathcal{R}(t,x) = 0$

This is because $lim_{\epsilon \searrow 0} \left[\theta'(n)(|x|^2 - (t - i\epsilon)^2)^{-(n-1)/2}\right] = \theta'(n)(|x|^2 - t^2)^{-(n-1)/2}$ this means that this part is a real number [7]. Therefore by (4.17), we take imaginary part which is zero. Notice that this is finite speed propagation

Case 2: For n odd

It is clear that $supp((\mathcal{R})(t,\cdot)) \subset \{x \in \mathbb{R}^n : |x| = t\}$

So the equation (4.17) vanishes for $|x| < t$ This is the Huygens principle [7] and it does not hold for an even n.

When $n = 2$, the computation of the limit in (4.17) is elementary. We have

$$\mathcal{R}(t,x) = \begin{cases} \theta'(2) \frac{1}{\sqrt{(t^2 - |x|^2)}} \cdot sgn(t), & for \quad |x| < |t|, \\ 0, & for \quad |x| > |t|. \end{cases} \tag{4.18}$$

[1]Recall the Fourier transform of Gaussian function. Given $g(x) = e^{-k|x|^2}$ then it's Fourier transform is given by $\hat{g}(\xi) = \left(\sqrt{\frac{\pi}{k}}\right)^n e^{-k|\xi|^2}$.

Theorem 4.3.1. *Given $\varphi, \phi \in \mathcal{S}'(\mathbb{R}^n)$, there is unique solution $u \in C^\infty(\mathbb{R}, \mathcal{S}'(\mathbb{R}^n))$ to the initial value problem (4.4) and it is given by (4.10) which can be expressed as*

$$u(t,x) = \mathcal{R}(t,x) * \phi(x) + \partial_t(\mathcal{R}(t,x) * \varphi(x)) \tag{4.19}$$

where * is the convolution operation.

This theorem above explains the the general solution of wave equation in n dimensional space.

Now it's time to derive the Poisson equation. That is, the solution of wave equation in 2 dimensions.

We first find the constant $\theta'(2)$

$$\begin{aligned} \theta'(2) &= \theta(2) \\ &= \left(\sqrt{\frac{1}{\pi}}\right)^3 \Gamma(\frac{3}{2}) \\ &= (\sqrt{\pi})^{-3} \cdot \frac{1}{2}\Gamma(\frac{1}{2}) = (\sqrt{\pi})^{-3} \cdot \frac{1}{2} \cdot \sqrt{\pi} \\ &= \frac{1}{2\pi} \end{aligned}$$

$\therefore$ The Riemann function becomes

$$\mathcal{R}(t,x) = \begin{cases} \frac{1}{2\pi} \cdot \frac{1}{\sqrt{(t^2 - |x|^2)}} \cdot sgn(t), & for \quad |x| < |t|, \\ 0, & for \quad |x| > |t|. \end{cases}$$

Now we can write the solution

$$\begin{aligned} u(t,x) &= \mathcal{R}(t,x) * \phi(x) + \partial_t(\mathcal{R}(t,x) * \varphi(x)) && (4.20) \\ &= \frac{1}{2\pi} \cdot \frac{1}{\sqrt{(t^2 - |x|^2)}} * \phi(x) + \partial_t \left(\frac{1}{2\pi} \cdot \frac{1}{\sqrt{(t^2 - |x|^2)}} * \varphi(x) \right) && (4.21) \end{aligned}$$

Theorem 4.3.2. *Poisson's formula suppose u(t,x) is a solution to the wave equation $\Box u(t,x) = 0$ with u(t,x) $\in C^\infty(\mathbb{R}^{(1+2)})$. Then we have the formula*

$$u(t,x) = \frac{1}{2\pi} \int_{\{|x| \le t\}} \frac{\phi(y)}{\sqrt{(t^2 - |x-y|^2)}} dy + \partial_t \left(\int_{\{|x| \le t\}} \frac{\varphi(y)}{\sqrt{(t^2 - |x-y|^2)}} dy \right) \tag{4.22}$$

5
ANALYSIS AND DISCUSSION

5.1 Remarks and analysis: Solution in $\mathbb{R}^{1+1}$

- We found out the fundamental solution to be $E_+(u,v) = \dfrac{1}{2}H(x-t)H(x+t)$. The wave equation solution can be written as the product of two Heaviside functions, and thus its derivative can be calculated as the product of the derivatives of the individual Heaviside functions, which are delta distributions. This gives us a distributional solution to the wave equation that is not differentiable in the usual sense, but is well defined over test functions.
- The last term in the equation (6.1) is called **" Duhamel's Principle "** and the first two terms are called **" d'Alembert's Formula "** eventhough most of the time the formula is all called **" d'Alembert's Formula "**
- The wave equation has finite propagation speed and domains of influence as we have defined then in corollary 5.1. This is because the equation describes the behavior of a wave that travels through a medium and spreads out over time. The speed of the wave, which is the speed of energy transfer, is limited and finite. This means that the wave can only travel so far in a given amount of time. The domain of influence of a wave refers to the region in which the wave has a noticeable effect. This is determined by the speed of the wave and the time it takes for the wave to reach a given location. The finite propagation speed and domains of influence are inherent properties of waves and are not dependent on the specific form of the wave equation used to describe them.
- solutions of the wave equation do not become smoother in time. this is due to the nature of the wave equation. Over time, the wave equation generates new wave crests and troughs that interact with one another, leading to a continuously changing wave pattern. The wave equation does not impose any restrictions on the smoothness of its solutions, so it is possible for solutions to become rougher or more irregular as time progresses.

5.1.1 Remarks on Poisson formula given by theorem 4.3.2

- The denominator of the solution is zero at $x = \pm t$, which corresponds to the boundary of the light cone originating from the point (0,0) in the space-time plane. This means that the wavefront travels along this boundary at the speed of light.
- The solution is symmetric with respect to the x-axis, which means that the wave propagates equally in all directions along the x-axis.

In other words, The forward fundamental solution of the wave equation in two dimensions, $\mathcal{R}(t,x) = \dfrac{1}{\sqrt{(t^2 - |x|^2)}}$, is symmetric with respect to the x-axis because it only depends on the magnitude of x, $|x|$, and not on its sign.

To see this, consider the value of $\mathcal{R}$ at two points on the x-axis, say (t, 0) and (-t, 0). We have: $\mathcal{R}(t,0) = \dfrac{1}{2\pi} \cdot \dfrac{1}{(t^2 - 0)^{1/2}} = \mathcal{R}(-t,0)$. This means that the amplitude of the wave at these two points is the same, even though one is to the right of the origin and the other is to the left. Therefore, the wave propagates equally in all directions along the x-axis, without any preference for either the positive or negative x-axis direction. This symmetry is a consequence of the isotropy of space in the two-dimensional case, which implies that there is no preferred direction in the plane.

As a result,the wave propagates equally in all directions along the x-axis.

- The amplitude of the wave decreases as the distance from the source increases. This is because the denominator of the solution increases as the distance from the source increases, leading to a decrease in amplitude.

- The solution satisfies the wave equation in two dimensions, which means that it is a valid solution to the wave equation as we have seen from the previous calculations and since the solution is a fundamental solution, it can be used to construct other solutions to the wave equation by convolution with a suitable initial data.

- The solution is singular at t = 0 and $x = \pm t$, which means that it cannot be interpreted as a physical wave at these points. However, the singularity can be removed by taking a suitable limit as $t \to 0$ or $x \to \pm t$.

- The solution can be used to study the behavior of waves in two-dimensional media, such as waves on the surface of a lake or waves in a thin membrane.

6
CONCLUSION AND FUTURE WORK

6.1 Summary

Our research project in mathematics focused on using generalized functions to extend classical solution methods to more general situations where the initial data may not be smooth or well-behaved.

- Specifically, we derived the solution to the one dimensional linear non-homogeneous wave equation with initial conditions, which led us to the d'Alembert formula. This formula is expressed as a convolution with the Dirac delta function and the Heaviside function, which are essential functions in the theory of distributions from chapter 4. The formula is given by:

$$\phi(t,x) = \frac{1}{2}\left[(\phi_0(x+t) + (\phi_0)(x-t)) + \int_{x-t}^{x+t} \phi_1(y)dy + \int_0^t \int_{x-t+s}^{x+t-s} f(s,y)dyds\right]$$

- We also derived the Riemann function which was used as fundamental solution to n dimensional wave equation where we specifically computed explicit formula for n=2 and found

$$\mathcal{R}(t,x) = \begin{cases} \theta'(2)\dfrac{1}{\sqrt{(t^2-|x|^2)}} \cdot sgn(t), & for \quad |x| < |t|, \\ 0, & for \quad |x| > |t|. \end{cases} \tag{6.1}$$

  where $\theta'(2)$ is a constant to be calculated using the it's formula from equation (4.14).

- By using the Riemann function we derived Poisson formula,

$$u(t,x) = \frac{1}{2\pi}\int_{\{|x|\le t\}} \frac{\phi(y)}{\sqrt{(t^2-|x-y|^2)}}dy + \partial_t\left(\int_{\{|x|\le t\}} \frac{\varphi(y)}{\sqrt{(t^2-|x-y|^2)}}dy\right)$$

  We provided some remarks on this equation like symmetry of the waves and how wave amplitude is related with distance from the source.

6.2 Conclusion

In this project, we investigated the use of the theory of generalized functions to analyze the Cauchy problem of wave equation where we found that the theory of generalized functions provides a powerful framework for analyzing the Cauchy problem of wave equation, allowing us to extend the classical solution methods to more general situations. Specifically, we showed that the use of distributions and their derivatives enables us to define weak solutions to the wave equation that satisfy the initial conditions in the sense of distributions. Our results have important implications for the study of partial differential equations and their applications in physics and engineering, where the Cauchy problem of wave equation is a fundamental model for wave propagation phenomena.

6.3 Future work

While our method provides a powerful tool for analyzing the Cauchy problem of wave equation, it has some limitations and assumptions that need to be carefully considered in practice. For instance, the method may not be applicable to some types of nonlinear or higher-order partial differential equations, and the existence and uniqueness of solutions may require additional assumptions or conditions. Moreover, there are several potential extensions and generalizations of our method that could be explored in future research, such as the use of other types of generalized functions or the study of more general initial boundary value problems. Another possible direction for future research is to investigate the use of other types of generalized functions, such as hyper-functions or ultra-distributions, to analyze the Cauchy problem of wave equation. Well, not only that but also for future research, one may consider to study the stability and convergence of the weak solutions obtained by our method, and to explore their connections with other solution concepts in the theory of partial differential equations.

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Appendix

Listing 1: Python code for plotting the smooth compactly supported function $\Psi(x)$

```python
# Libraries
import numpy as np
import matplotlib.pyplot as plt

# Define the psi function
def psi(x):
    return np.where(np.abs(x) < 1, np.exp(-1 / (1 - x**2)), 0)

# Create part of the domain for plotting
x = np.linspace(-2, 2, 1000)
y = psi(x)

# plotting
plt.figure(figsize=(4, 4))
plt.plot(x, y, linewidth=2.5)
plt.xlabel("x")
plt.ylabel(r"$\Psi(x)$")
plt.title(r"Plot of $\Psi(x) = e^{-1/(1-x^2)}$ for $|x|<1$")
plt.grid(False)

# saving the figure
plt.savefig("psi_function.pdf", bbox_inches="tight")
plt.show()
```

Listing 2: Python code for plotting four approximations of the Dirac delta function

```python
import numpy as np
import matplotlib.pyplot as plt

# Laplace-type delta sequence
def delta_laplace(x, c):
    return (1 / (2 * c)) * np.exp(-np.abs(x) / c)

# Gaussian delta sequence
def delta_gaussian(x, c):
    return (1 / (np.sqrt(2 * np.pi) * c)) * np.exp(-x**2 / (2 * c
        **2))

# Cauchy (Lorentzian) delta sequence
def delta_cauchy(x, c):
    return (1 / np.pi) * (c / (x**2 + c**2))

# Sinc-based delta sequence with removable singularity at x = 0
def delta_sinc(x, c):
    y = np.zeros_like(x)
    y[x != 0] = np.sin(x[x != 0] / c) / (np.pi * x[x != 0])
    y[x == 0] = 1 / (np.pi * c)  # limiting value as x -> 0
    return y

# Spatial domain and width parameter
```

```python
x = np.linspace(-2, 2, 4000)
c = 0.1

# List of delta approximations and output filenames
functions = [
    (delta_laplace, r"$\frac{1}{2c}e^{-|x|/c}$", "delta_laplace.pdf"),
    (delta_gaussian, r"$\frac{1}{\sqrt{2\pi}c}e^{-x^2/(2c^2)}$", "delta_gaussian.pdf"),
    (delta_cauchy, r"$\frac{1}{\pi}\frac{c}{x^2+c^2}$", "delta_cauchy.pdf"),
    (delta_sinc, r"$\frac{\sin(x/c)}{\pi x}$", "delta_sinc.pdf")
]

# Plot and save each approximation independently
for func, label, filename in functions:
    y = func(x, c)

    plt.figure(figsize=(4, 4))
    plt.plot(x, y, linewidth=2.5)
    plt.xlabel("x")
    plt.ylabel(r"$\delta_c(x)$")
    plt.title(rf"Approximation of $\delta(x)$: {label}")
    plt.grid(False)

    plt.savefig(filename, bbox_inches="tight")
    plt.show()
```