Chapter1: Introduction to Optimization

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Introduction of Optimization

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Outline

1. Plan for 3 weeks.
2. References
3. What is Optimization?
4. Brief Historical Reference
5. Optimization Today
6. Introduction of Optimization Problem

Outline

Plan for 3 week.

1. Part1:
   - Review some Background of Linear Algebra
   - Introduction of Linear Programming
   - Simplex Method for Linear Programming
   - Introduction to Mathematica Programming or Matlab.
2. Part 2:
   - Introduction to Dual Problem
   - Introduction to Constraint and Unconstraint for Non- Linear Programming
3. Part 3:
   - The Convex Optimization
   - The Lagrange's method for Optimization.
   - The Karush Kunh Tucker condition

Outline

References

1. Elementary Linear Programming with Applications by Bernard Kolman, Robert E. Beck
2. Jan A. Snyman · Daniel N. Wilke,Practical Mathematical Optimization,2005
3. Edwin K. P. Chong ,Stanislaw H.Zak, AN INTRODUCTION TO OPTIMIZATION ,2013.
4. INTRODUCTION TO NONLINEAR OPTIMIZATION Theory, Algorithms, and cations with MATLAB, Amir Beck.

Outline

What is Optimization?

What is Optimization?

Searching for the " Best" possible decision.

- Minimize , traveled distance, waste, cost
- Maximize ,benefit, revenue, profit.

Outline

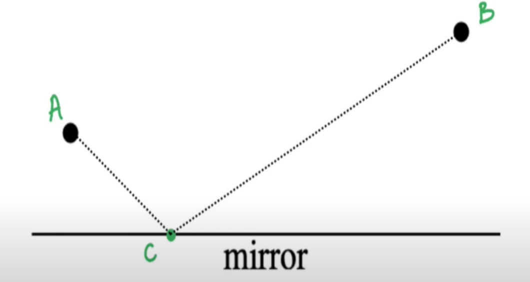
Brief Historical Reference

⋆ 1st century A.D: the Alexandrian mathematician Heron solved the problem of finding the shortest path between two point by way of the mirror.

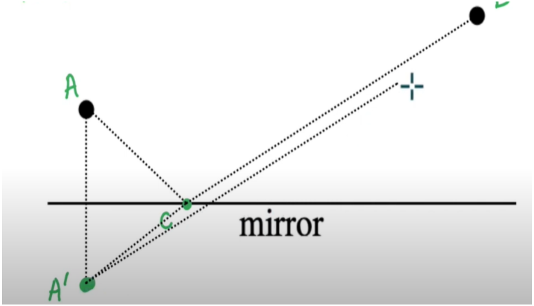
" Heron's Theorem of the light ray " - the origin of the theory of geometrical optics.

Brief Historical Reference

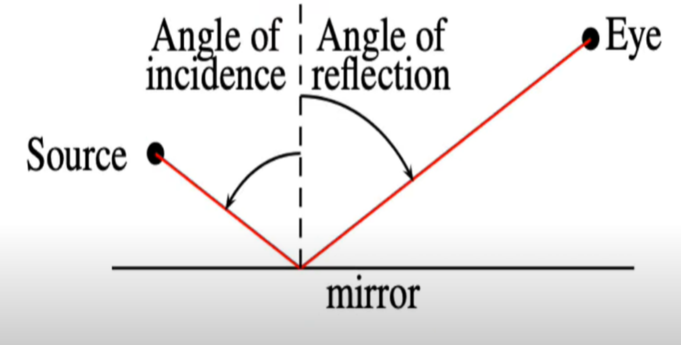
" Heron's Theorem of the light ray "
find C: $\min |AC| + |CB|$



Heron's Theorem of the light ray

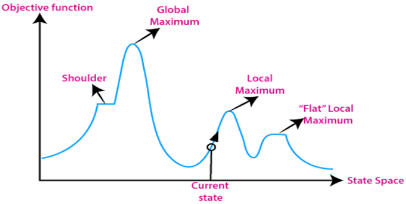


Heron's Theorem of the light ray



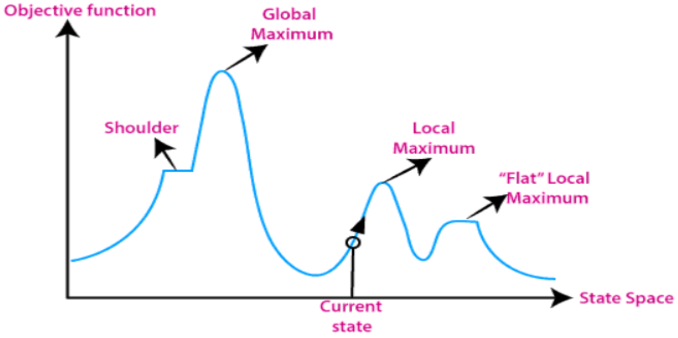
Brief Historical Reference

⋆ 17th century: Leonhard Euler (1707–1783) ,the problem of finding extreme value serve as one of motivation in the invention of differential calculus.



Brief Historical Reference

What is the global maximmum , Local maximum?



Brief Historical Reference

" Nothing happens in the universe that does not have a sense of either certain maximum or minimum" -Leonhard Euler (1707–1783)

Brief Historical Reference

- 20$^{\text{th}}$ century: Invention of the digital computer $\rightarrow$ development of numerical optimization.
- 1939: Leonid Kantorovich uses linear optimization techniques for optimizing production in a plywood industry. (1975 Nobel Prize in Economics)
- World War II: Optimization algorithms are used to solve military logistics and operations problems: linear programming.
- 1947: George Dantzig published the simplex method;
  John von Neumann developed the duality theory.

Outline

Optimization Today

Decision making tools based on optimization procedures are successfully applied to a wide range of practical problems arising in virtually any sphere of human activities, including

- healthcare
- biomedicine
- energy management
- aerospace research
- telecommunications
- finance
- ...

Optimization Today

- Linear Programming
- (Mixed) Integer Programming
- Network Optimization
- Nonlinear Programming
- Combinatorial Optimization
- Global Optimization
- Stochastic Optimization
- ...

Outline

Introduction of Optimization Problem

An optimization problem consists of three main ingredients:

- a decision space X,
- a constraint set $C \subset X$,and
- an objective function $f : U \to R$, where $U \subset X$ is nonempty,

Aim at finding a point $x^\star \in C$ such that $f(x^\star) \leq f(x)$ for all $x \in C$.This can be conventionally written as

$$Opt\,(f, C) \quad \begin{cases} \min & f(x) \\ s.t & x \in C \end{cases}$$

For the problem to make sense, we may require $C \subset U \neq 0$.

Assume that $C \subset U \neq 0$ .We shall develop a shorthand notation $x^\star \in Opt(f, C)$ denote that $x^\star$ is a solution to the optimization problem $Opt(f, C)$,Concerning with the problem $Opt(f, C)$, the point $x \in X$ is said to be feasible if it belongs to the constraint set C. If $C = X$ is feasible and the optimization problem is said to be unconstrained.

Notice that the constraint set C above can be just anything, hence the general form of $Opt(f, C)$ is said to be equipped with an abstract constraint. Of course, the constraint C can be described in a more explicit form, i.e. by inequalities and equalities:

$$C = \{x \in X | \, g_i(x) \leq 0, \quad h_j(x) = 0, \forall i =, 1, 2, ..., r, \forall i =, 1, 2, ..., l\}$$

where the (vector) functions $g = (g_1, g_2, ..., g_r) : X \to \mathbb{R}^r$ and $h = (h_1, h_2, ..., h_l) : X \to \mathbb{R}^l$ are given. In this case.$Opt(f, C)$ will be represented by $Opt(f, g, h)$ with

$$Opt\,(f, g, h) \quad \begin{cases} \min \quad f(x) \\ s.t \quad g_i\,(x) \leq 0 \quad, \forall i = 1, 2, ..., r \\ \qquad h_i(x) = 0 \quad, \forall i = 1, 2, ..., l. \end{cases}$$

Some Application?

Economics:The consumer's demand functions

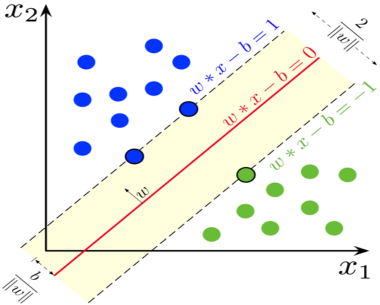
A consumer's preferences are represented by the utility function

$$u(x_1, x_2) = 2lnx_1 + lnx_2 \tag{1}$$

If the budget constraint is $p_1x_1 + p_2x_2 = M$,
Determine the demand functions, that is, the optimal values $x_1^\star$ and $x_2^\star$; in terms ofp_1, p_2 and M.

How to solve this problem?

Statistics :Support Vector Machines



Minimize

$$\frac{1}{2}||w||^2$$

subject to

$$y_i(w^T x_i - b) \geq 1, \forall i = 1, 2, ..., n$$

How to solve this problem?

A portfolio constructed from n different stocks can be described in terms of their weights

$$w_i = \frac{x_i S_i(0)}{V(0)}$$

We can arrange it into a one-row matrix

$$w_i = [w_1, w_2, ..., w_n]$$

so, the weights add up to one, which is

$$uw^T = 1$$

where

$$u = [1, 1...1]$$

Portfolio Selection

The expected return $\mu_v = E(K_v)$ and variance $\sigma_V^2 = V(K_V)$ of a portfolio with weights w are given by

$$\mu_v = mw^T$$

$$\sigma_V^2 = wCw^T$$

where, $m = [\mu_1, \mu_2, ...\mu_n]$ is the expected return matrix and C is the covariance matrix between return of n stocks.

Portfolio Selection

Minimum Variance Problem:
minimize

$$wCw^T \tag{2}$$

subject to

$$uw^T = 1 \tag{3}$$

where
w vector of portfolio weightings,
C covariance matrix,
u vector of 1s

How to solve this problem?

Markowitz Model

Minimum Variance Line Problem:
minimize

$$wCw^T \tag{4}$$

subject to

$$mw^T = \mu_b, uw^T = 1 \tag{5}$$

where,
μ_b target portfolio expected return , w vector of portfolio weightings, C covariance matrix, m expected return matrix, u vector of 1s

How to solve this problem?

Thank you for attention.