Mathematical Concepts

Exercise 1: Optimality in 1d

Let $f : [-1, 2] \to \mathbb{R}, x \mapsto \exp(x^3 - 2x^2)$

(a) Compute f'

(b) Plot f and f' with `Python or Matlab programming` .

(c) Find all possible candidates x^* for maxima and minima.
Hint: exp is a strictly monotone function.

(d) Compute f''

(e) Determine if the candidates are local maxima, minima or neither.

(f) Find the global maximum and global minimum of f

Exercise 2: Optimality in 2 dimensions

Let $f : \mathbb{R}^2 \to \mathbb{R}, (x_1, x_2) \mapsto -\cos(x_1^2 + x_2^2 + x_1x_2)$

(a) Create a contour plot of f in the range $[-2, 2] \times [-2, 2]$ with `Python or Matlab programming` .

(b) Compute ∇f

(c) Compute $\nabla^2 f$

Now, we define the restriction of f to $S_r = \{(x_1, x_2) \in \mathbb{R}^2 |\ x_1^2 + x_2^2 + x_1x_2 < r\}$ with $r \in \mathbb{R}, r > 0$, i.e., $f_{|S_r} : S_r \to \mathbb{R}, (x_1, x_2) \mapsto f(x_1, x_2)$.

(d) Show that $f_{|S_{\overline{r}}}$ with $\overline{r} = \pi/4$ is convex.

(e) Find the local minimum $\mathbf{x}^*$ of $f_{|S_{\overline{r}}}$

(f) Is $\mathbf{x}^*$ a global minimum of f?

Exercise 3: Optimality in d dimensions

Let $\mathbf{X}$ be a d-dimensional random vector and let $\mathbf{Y}$ be a one-dimensional random vector with $\mathsf{Var}(\mathbf{X}) = \mathbf{\Sigma_X}$ and $\mathsf{Cov}(\mathbf{X}, \mathbf{Y}) = \mathbf{\Sigma_{X,Y}} \in \mathbb{R}^{d\times 1}$.
Further, let $f : \mathbb{R}^d \to \mathbb{R}, \mathbf{w} \mapsto \mathsf{Var}(\mathbf{w}^\top \mathbf{X} - \mathbf{Y})$.

(a) Show that f is convex.

(b) Compute ∇f and $\nabla^2 f$

(c) Under which condition exists a unique minimizer $\mathbf{w}^*$ of f. Is this a global minimum? (if it exists)

(d) Given the samples $(\mathbf{x}_i, y_i) \sim \mathbb{P}_{\mathbf{X},\mathbf{Y}}$, under which condition is the least squares estimator a consistent estimator of $\mathbf{w}^*$ in general?

Exercise 4:
Find the minimum value of

$$f(x, y, z) = 2x^2 + y^2 + z^2$$

subject to the constraint

$$2x - 3y - 4z = 49$$

.

Exercise 5:
Find the maximum value of

$$f(x, y) = xy$$

,where $x > 0, y > 0$ subject to the constraint

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Exercise 6:
Consider the function

$$f(x_1, x_2) = (x_1^2 + x_2^2 - 1)^2 + (x_2^2 - 1)^2$$

over $\mathbb{R}^2$. Find all the stationary points of f over $\mathbb{R}^2$ and classify them.

Exercise 7:
Find maximize

$$-(x-2)^2 - 3(y-1)^2$$

subject to

$$x + 4y \leq 3$$
$$x \geq y$$

Exercise 8:
Find the extremizing the objective function

$$f(x) = x_1^2 + x_2^2$$

on the ellipse

$$\{[x_1, x_2]^T : h(x) = x_1^2 + 2x_2^2 - 1 = 0\}$$

Exercise 9:
Find the maximum and minimum value of

$$f(x, y, z) = x + y + z^2$$

subject to constraint $x + y + z = 1$ and $x^2 + z^2 = 1$.

Exercise 10:
Consider the problem
Minimize

$$x_1^2 + (x_2 + 1)^2, \quad x_1, x_2 \in \mathbb{R}$$

subject to

$$x_2 \leq e^{x_1}$$

Let $x^\star = [x_1^\star, x_2^\star]^T$ be the solution to the problem.

(a) Write down the KKT condition that must be satisfied by $x^\star$.

(b) Prove that $x_2^\star = e^{x_1^\star}$

(c) Prove that $-2 < x_1^\star < 0$