Chapter2 : Introduction to Linear Programming Problem.

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Outline

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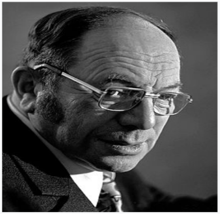
Leonid Kantorovich in 1975

THE BEST USE OF
ECONOMIC RESOURCES

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Leonid Vitalyevich Kantorovich (Russian); 19 January 1912 – 7 April 1986) was a Soviet mathematician and economist, known for his theory and development of techniques for the optimal allocation of resources. He is regarded as the founder of linear programming. He was the winner of the Stalin Prize in 1949 and the Nobel Memorial Prize in Economic Sciences in 1975.

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Introduction

The objective of a linear programming problem is to obtain an optimal solution. Linear programming problems deal with the problem of minimizing or maximizing a linear objective function in the presence of a system of linear inequalities. The linear objective function represents cost or profit. A large and complex problem can be formulated in the form of a linear programming problem, and users can solve such a large problem in a definite amount of time using the simplex method and computer.

In this chapter we study a graphical method for solving linear programming problems. This method is helpful to choose the best feasible point among the many possible feasible points. A point minimizing the objective function and satisfying the set of linear constraints is called a "feasible point"

A linear programming problem is a mathematical problem that consists of the following components:

- An objective function, f, of n decision variables, $x_1, x_2, \ldots, x_n$, that is to be maximized or minimized. This function is linear; that is it can be written in the form

$$z = f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \tag{1}$$

  where each c_i belongs to $\mathbb{R}$

- A set of m constraints or inequalities. Each constraint is linear in that it takes the form

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i \tag{2}$$

  where $1 \leq i \leq m$ and where each a_{ij} and b_i be longs to $\mathbb{R}$.

- Possible sign restrictions placed on any of the decision variables.

We will define the vector $x = (x_1, x_2, ..., x_n)^T$ in $\mathbb{R}^n$. to be a feasible solution of the LPP if it satisfies all constraints and sign restrictions of the LP. The feasible region is the set of all feasible solutions, and an optimal solution is a feasible solution whose corresponding objective function value is greater than or equal to that of any other feasible solution for a maximization problem and less than or equal to that of any other feasible solution for a minimization problem

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Introduction

Given a description of the problem, you should be able to formulate it as an LP problem following the steps below:

1. Identify the decision variables,
2. Identify the objective function,
3. Identify the constraints, and
4. Write down the entire problem adding nonnegativity constraints if necessary.

Let's follow these steps to formulate the associated LP problem in the following three examples.

The Linear Programming Problem

Example 1

(A financial problem). Suppose that the financial advisor of a university's endowment fund must invest exact/y $100,000$ in two types of securities: bond AAA, paying a dividend of 7%, and stock BB, paying a dividend of 9%. The advisor has been told that no more than $30,000$ can be invested in stock BB, whereas the amount invested in bond AAA must be at least twice the amount invested in stock BB. How much should be invested in each security to maximize the university's return?

MATHEMATICAL MODEL.

Let x and y denote the amounts invested in bond AAA and stock BB, respectively. We must then have

$$\begin{aligned} x + y &= 100\ 000 \\ x &\geq 2y \\ y &\leq 30000 \end{aligned}$$

Of course, we also require that $x \geq 0$ and $y \geq 0$

The return to the university, which we seek to maximize, is

$$z = 0.07x + 0.09y \tag{3}$$

Thus, our mathematical model is:
Find values of x and y that will,
maximize

$$z = 0.07x + 0.09y$$

subject to

$$\begin{aligned} x + y &= 100\ 000 \\ x - 2y &\geq 0 \\ y &\leq 30000 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Example 2

A company produces two products, A and B. One unit of product A is sold for 50, while one unit of product B is sold for 35. Each unit of product A requires 3 kg of raw material and 5 labor hours for processing, while each unit of product B requires 4 kg of raw material and 4 labor hours for processing. The company can buy 200 kg of raw material every week. Moreover, the company has currently 4 employees that work 8-hour shifts per day (Monday - Friday). The company wants to find the number of units of each product that should produce in order to maximize its revenue.

- Let $x_1$and x_2 be the number of units of product A and B per week,respectively.
- Next, we define the objective function. The company wants to maximize its revenue and we already know that the price for product A is 50 per unit and the price for product B is 35 per unit. Hence, the objective function is:
  max
  $$z = 50x_1 + 35x_2 \tag{4}$$
- First of all, there is a constraint about the raw material that should be used to produce the two products. Each unit of product A requires 3 kg of raw material and each unit of product B requires 4 kg of raw material, while the raw material used every week cannot exceed 200 kg. Hence, the first constraint is given by
  $$3x_1 + 4x_2 \leq 200 \tag{5}$$

- There is another technological constraint about the labor hours that should be used to produce the two products. Each unit of product A requires 5 labor hours for processing and each unit of product B requires 4 labor hours for processing, while the available labor hours every week cannot exceed 160 h (4 employees $\times$5days per week $\times$8-h shifts per day). Hence, the second constraint is given by

$$5x_1 + 4x_2 \leq 160 \tag{6}$$

Moreover, we also add the nonnegativity constraints for variables x_1 and x_2. Hence, the LP problem is the following:
max

$$z = 50x_1 + 35x_2 \tag{7}$$

subject to

$$\begin{aligned} 3x_1 + 4x_2 &\leq 200 \\ 5x_1 + 4x_2 &\leq 160 \\ x_1 \geq 0, x_2 &\geq 0 \end{aligned}$$

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General form of LPP

Following the form of the previous examples, the general linear programming problem can be stated as follow.
Find values of $x_1, x_2, ..., x_n$ that will maximize or minimize

$$z = c_1x_1 + c_2x_2 + ... + c_nx_n \tag{8}$$

subject to

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le (\ge)\,(=) \quad b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le (\ge)\,(=) \quad b_2 \\ \vdots \qquad\qquad\qquad\quad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{2n}x_n \le (\ge)\,(=) \quad b_m \end{array} \qquad (\star)$$

where in each inequality in $(\star)$ one and only one of the symbols $, \leq, \geq, =$occurs. The linear function in (1) is called the objective function. The equalities or inequalities in $(\star)$are called constraints. Note that the left-hand sides of all the inequalities or equalities in (2)are linear functions of the variables $x_1, x_2, ..., x_n$, just as the objective function is.

A problem in which not all the constraints or the objective function are linear functions of the variables is a nonlinear programming problem.

Standard Form of LPP

We shall say that a linear programming problem is in standard form if it is in the following form:
Maximize

$$z = c_1x_1 + c_2x_2 + ... + c_nx_n \tag{9}$$

subject to

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\leq \ b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\leq \ b_2 \\
\vdots \qquad\quad \vdots \qquad \vdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{2n}x_n &\leq \ b_m
\end{aligned}$$

$$x_j \geq 0, j = 1, 2, .., n$$

Canonical Form LPP

Maximize

$$z = c_1x_1 + c_2x_2 + ... + c_nx_n \tag{10}$$

subject to

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{2n}x_n &= b_m
\end{aligned}$$

$$x_j \geq 0, j = 1, 2, .., n$$

Some Example

Minimization Problem as a Maximization Problem

Every minimization problem can be viewed as a maximization problem and conversely. This can be seen from the observation that

$$\min \sum_{i=1}^{n} c_i x_i = -\max \left(\sum_{i=1}^{n} -c_i x_i \right). \tag{11}$$

That is, to minimize the objective function we could maximize its negative instead and then change the sign of the answer.

Reversing an Inequality

If we multiply the inequality

$$k_1x_1 + k_2x_2 + \cdots + k_nx_n \geq b$$

by -1, we obtain the inequality

$$-k_1x_1 - k_2x_2 - \cdots - k_nx_n \leq -b$$

.

Example

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Changing an Equality to an Inequality

In the general case the equation ,

$$\sum_{j=1}^{n} a_{ij}x_i = b_i$$

can be written as the pair of inequalities,

$$\begin{cases} \sum_{j=1}^{n} a_{ij}x_i \leq b_i \\ \sum_{j=1}^{n} -a_{ij}x_i \leq -b_i. \end{cases}$$

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Matrix Notation

It is convenient to write linear programming problems in matrix notation. Consider the standard linear programming problem: we can write our given linear programming problem as:
Find a vector $x \in \mathbb{R}^n$ that will
maximize

$$z = c^T x \tag{12}$$

subject to

$$Ax \leq b \tag{13}$$

$$x \geq 0$$

where,

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Example

Find a vector $x \in \mathbb{R}^2$ that will maximize

$$z = (120 \quad 100) \begin{pmatrix} x \\ y \end{pmatrix}$$

subject to

$$\begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 8 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \geq 0$$

Definition 3

A vector $x \in \mathbb{R}^n$ satisfying the constraints of a linear programming problem is called a feasible solution to the problem. A feasible solution that maximizes or minimizes the objective function of a linear programming problem is called an optimal solution.

Find a vector $x \in \mathbb{R}^2$ that will maximize

$$z = (120 \quad 100) \begin{pmatrix} x \\ y \end{pmatrix}$$

subject to

$$\begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 8 \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \geq 0$$

The vector

$$x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ x_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ x_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

are feasible solutions. For example,

$$\begin{pmatrix} 2 & 2 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 13 \end{pmatrix} \leq \begin{pmatrix} 8 \\ 15 \end{pmatrix}$$

and

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \geq 0$$

Therefore, x_2 is a feasible solution. The vectors x_1 and x_3 can be checked.

Changing an Inequality to an Equality

Consider the constraint

$$a_{i1} + a_{i2} + \ldots + a_{in}x_n \leq b_i \tag{14}$$

We may convert (14) into an equation by introducing a new variable, u_i, and writing

$$a_{i1} + a_{i2} + \ldots + a_{in}x_n + u_i = b_i. \tag{15}$$

The variable u_i is nonnegative and is called a slack variable because it "takes up the slack" between the left side of constraint (8) and its right side.

Example

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Geometry of a Constraint

A single constraint of a linear programming problem in standard form, say the ith one,

$$a_{i1} + a_{i2} + \ldots + a_{in}x_n \leq b_i,$$

can be written as

$$a^T x \leq b_i, \tag{16}$$

where, $a^T = [a_{i1}, a_{i2}, ..., a_{in}]$

The set of points $x = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ that satisfy this constraint is called a closed half-space.
If the inequality is reversed, the set of points $x = (x_1, x_2, ..., x_n)$ in $\mathbb{R}^n$ satisfying

$$a^T x \geq b_i \tag{17}$$

is also called a closed half-space.

Example

Consider the constraint $2x + 3y \leq 6$ and the closed half-space.

$$H = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid (2\ 3) \begin{pmatrix} x \\ y \end{pmatrix} \leq 6 \right\}$$

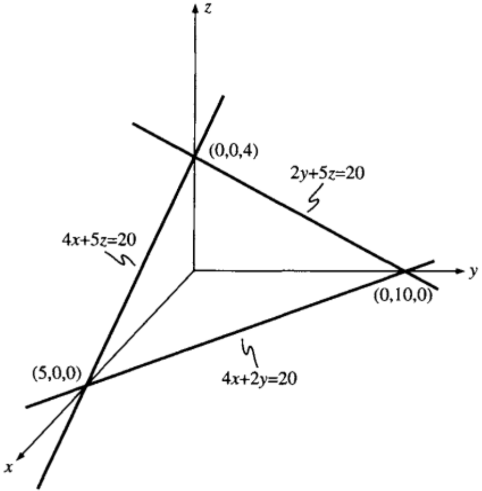
which consists of the points satisfying the constraint. Note that the points $(3, 0)$ and $(1, 1)$ satisfy the inequality and therefore are in H. Also, the points $(3, 4)$ and $(-1, 3)$ do not satisfy the inequality and therefore are not in H.
Every point on the line $2x + 3y = 6$ satisfies the constraint and thus lies in H.

Example

The constraint in three variables, $4x + 2y + 5z \leq 20$, defines the closed half-space H in $\mathbb{R}^3$, where

$$H = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| (4\ 2\ 5) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \leq 20 \right\}$$

We can graph H in $\mathbb{R}^3$ by graphing the plane $4x + 2y + 5z = 20$


z
(0,0,4)
2y+5z=20
4x+5z=20
y
(0,10,0)
(5,0,0)
4x+2y=20
x

A typical constraint of a linear programming problem in canonical form has the equation

$$a^T x = b \tag{18}$$

Its graph in $\mathbb{R}^n$ is a hyperplane. If this equation were an inequality,namely,

$$a^T x \leq b \tag{19}$$

then the set of points satisfying the inequality would be a closed half-space. Thus, a hyperplane is the boundary of a closed half-space. Intuitively, it consists of the points that are in the half-space, but on its edge.

The hyperplane H defined by $a^T x = b$ divides $\mathbb{R}^n$ into the two closed half-spaces

$$H_1 = \{x \in \mathbb{R}^{nn} | a^T x \leq b\}$$

and

$$H_2 = \{x \in \mathbb{R}^{nn} | a^T x \geq b\}$$

We also see that

$$H_1 \cap H_2 = \hat{H}$$

the original hyperplane. In other words, a hyperplane is the intersection of two closed half-spaces.

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Inequality and Equality Constrained Problems

Theorem: KKT condition

Let $x^\star$ be a local minimum of the problem

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & g_i(x) \leq 0, \quad i = 1, 2, ..., m, \\ & b_j(x) = 0, \quad j = 1, 2, ..., p, \end{aligned}$$

where $f, g_1, ..., g_m, b_1, ...b_p$,are continuously differentiable functions over $\mathbb{R}^n$. Suppose that the gradients of the active constraints and the equality constraints,

$$\{\nabla g_i(x^\star) : i \in I(x^\star)\} \cup \{\nabla b_j(x^\star) : j = 1, 2, ..., p\}$$

, are linearly independent(where as before $I\,(x^\star) = \{i : g_i(x^\star) = 0\}$), then there exist multipliers $\lambda_1, \lambda_2, ..., \lambda_m \geq 0$ and $\mu_1, \mu_2, ..., \mu_p \in \mathbb{R}$ such that

$$\nabla f(x^\star) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^\star) + \sum_{i=1}^{m} \mu_i \nabla b_i(x^\star) = 0,$$

$$\lambda_i g_i(x^\star) = 0, \quad i = 1, 2, ..., m.$$

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Thank you for attention.