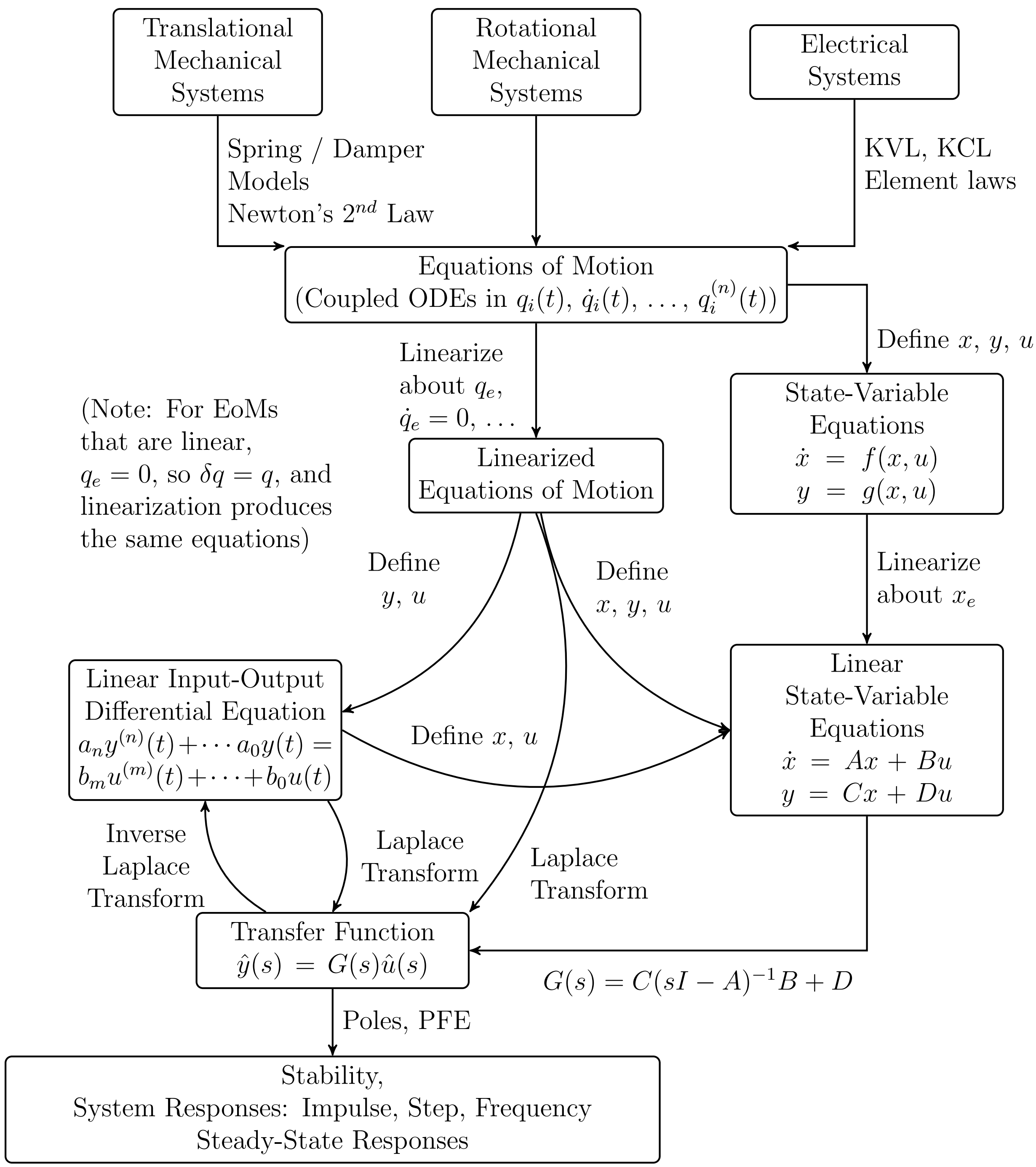
Linear State-Variable Equations

ME 340 Fall 2019. Instructor: Hasan Poonawala

We have looked at the following physical systems and ways to represent the EoMs:



1 Linear State-Variable Equations

Suppose that we are given the state-variable equations

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$
$$y(t) = Cx(t) + Du(t) \tag{2}$$

where $x(t)$ is the state, $y(t)$ is the output, $u(t)$ is the input, and t represents time.

Suppose we know the initial condition (IC) $x(t_0)$, and input $u(t)$ for $t \in [t_0, t_f]$. We want to understand how the output $y(t)$ will behave over the time interval $[t_0, t_f]$. To do so, we may either

- Explicitly solve for $x(t)$, because $y(t) = Cx(t) + Du(t)$
- Use A, B, C, and D to predict the behavior of solutions $x(t)$ given ICs and input.

We saw that transfer functions allow us to do something similar for Input-Output Differential Equations.

1.1 Explicit Solution

In earlier calculus classes, you may have seen methods to solve linear ODEs by computing homogenous and particular solutions. In the notes on Laplace transforms, we solve first-order input-output differential equations in $y(t)$ using this method. This section shows the relationship between that method and the linear state-variable equations given by matrices A, B, C, and D.

Matrix Exponential. Given a matrix A, we define the *matrix* e^{At} as the infinite sequence

$$e^{At} = I + \frac{1}{1}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots \tag{3}$$

This definition implies that $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$.

Let's calculate the derivative of e^{At}:

$$\begin{aligned}
\frac{d}{dt}e^{At} &= \frac{d}{dt}\left(I + \frac{1}{1}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots\right) \\
&= \frac{d}{dt}(I) + \frac{d}{dt}\left(\frac{1}{1}At\right) + \frac{d}{dt}\left(\frac{1}{2!}A^2t^2\right) + \frac{d}{dt}\left(\frac{1}{3!}A^3t^3\right) + \frac{d}{dt}\left(\frac{1}{4!}A^4t^4\right) + \cdots \\
&= 0 + A + \frac{1}{2!}A^2(2t) + \frac{1}{3!}A^3(3t^2) + \frac{1}{4!}A^4(4t^3) + \cdots \\
&= 0 + A + \frac{1}{1}A^2t + \frac{1}{2!}A^3t^2 + \frac{1}{3!}A^4t^3 + \cdots \\
&= A\left(I + \frac{1}{1}At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\right) \\
&= Ae^{At}
\end{aligned}$$

Suppose we define $z(t) = e^{At}v$, where $v = e^{-At_0}z(t_0)$, Then,

$$\begin{aligned}
\dot{z}(t) &= \frac{d}{dt}\left(e^{At}v\right) = \frac{d}{dt}\left(e^{At}\right)v \\
&= \left(Ae^{At}\right)v = A\left(e^{At}v\right) \\
&= Az(t)
\end{aligned}$$

In other words, $x(t) = e^{A(t-t_0)}x(t_0)$ is the solution to the differential equation

$$\dot{x}(t) = Ax(t)$$

with initial condition $x(t_0)$.

Suppose we have the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t).$$

Multiply on the left by e^{-At} and rearrange to get

$$e^{-At}\dot{x}(t) - Ae^{-At}x(t) = e^{-At}Bu(t)$$

But,

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}\dot{x}(t) - Ae^{-At}x(t),$$

so that we may write

$$\frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}Bu(t).$$

Integrate this equation on both sides:

$$\int_{t_0}^{t} \frac{d}{dt}\left(e^{-A\tau}x(\tau)\right)d\tau = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$\implies e^{-At}x(t) - e^{-At_0}x(t_0) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

Now, multiply both sides by e^{At}:

$$e^{At}e^{-At}x(t) - e^{At}e^{-At_0}x(t_0) = e^{At}\int_{t_0}^{t} e^{-A\tau}Bu(\tau)d\tau$$

$$\implies x(t) - e^{A(t-t_0)}x(t_0) = \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$\implies x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

$$\implies y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

$$= \text{free response} + \text{forced response}$$

The forced response is $\int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$, which may be difficult to calculate. The first term in the forced response is exactly a convolution operation between the function Ce^{At} and $Bu(t)$.

1.2 Laplace Transform

Again, if the goal is to explicitly calculate $y(t)$, we may prefer to work in the s-domain, which implies we work with Laplace transforms.

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{4}$$

$$y(t) = Cx(t) + Du(t) \tag{5}$$

$$s\hat{x}(s) - x(t_0) = A\hat{x} + B\hat{u}(s) \tag{6}$$

$$\implies (sI - A)\hat{x}(s) = x(t_0) + B\hat{u}(s) \tag{7}$$

$$\hat{x}(s) = (sI - A)^{-1}x(t_0) + (sI - A)^{-1}B\hat{u}(s) \tag{8}$$

$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s) \tag{9}$$

$$\implies \hat{y}(s) = C(sI - A)^{-1}x(t_0) + C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s) \tag{10}$$

To find the Laplace transform, set $x(t_0) = 0$ to obtain

$$
\begin{aligned}
&\hat{y}(s) = C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s) && (11)\\
\implies &\hat{y}(s) = \big(C(sI - A)^{-1}B + D\big)\,\hat{u}(s) = G(s)\hat{u}(s) && (12)\\
\implies &G(s) = C(sI - A)^{-1}B + D && (13)
\end{aligned}
$$

The remainder of this subsection shows that the explicit time-domain solution would be the same as if we used the s-domain and the inverse Laplace transform to calculate $y(t)$.

Fact. Let A be an $n \times n$ matrix. Then,

$$
L\left\{e^{At}\right\} = (sI - A)^{-1}e^{At_0} \implies L^{-1}\left\{(sI - A)^{-1}\right\} = e^{A(t-t_0)}
$$

To derive this fact, apply the Differentiation rule:

$$
\begin{aligned}
&L\left\{e^{At}\right\} = L\left\{e^{At}\right\} && \text{(Always true)}\\
\implies &sL\left\{e^{At}\right\} - e^{At_0} = L\left\{\frac{d}{dt}\left(e^{At}\right)\right\} && \text{(Differentiation rule)}\\
\implies &sL\left\{e^{At}\right\} - e^{At_0} = L\left\{\left(Ae^{At}\right)\right\} && \text{(derived earlier)}\\
\implies &sL\left\{e^{At}\right\} - e^{At_0} = AL\left\{e^{At}\right\} && \text{(Linearity of LT)}\\
\implies &(sI - A)L\left\{e^{At}\right\} = e^{At_0} && \text{(Rearrange terms)}\\
\implies &L\left\{e^{At}\right\} = (sI - A)^{-1}e^{At_0} && \text{(Matrix inversion)}
\end{aligned}
$$

So, we can take the inverse Laplace transform of (10) to obtain

$$
\begin{aligned}
L^{-1}\left\{\hat{y}(s)\right\} &= L^{-1}\left\{C(sI - A)^{-1}x(t_0) + C(sI - A)^{-1}B\hat{u}(s) + D\hat{u}(s)\right\}\\
y(t) &= L^{-1}\left\{C(sI - A)^{-1}x(t_0)\right\} + L^{-1}\left\{C(sI - A)^{-1}B\hat{u}(s)\right\} + L^{-1}\left\{D\hat{u}(s)\right\}\\
y(t) &= Ce^{A(t-t_0)}x(t_0) + \left(L^{-1}\left\{C(sI - A)^{-1}\right\} * L^{-1}\left\{B\hat{u}(s)\right\}\right)(t) + DL^{-1}\left\{\hat{u}(s)\right\}\\
&\quad \text{(Convolution of functions of } t \text{ is product of Laplace transforms of those functions)}\\
y(t) &= Ce^{A(t-t_0)}x(t_0) + \left(Ce^{At} * Bu(t)\right)(t) + Du(t)\\
y(t) &= Ce^{At}x(0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)
\end{aligned}
$$

2 Matrix Computations

To calculate the transfer function, we must compute $(sI - A)^{-1}$, where M^{-1} is the inverse of a square matrix M.

To calculate the inverse of M, we need to calculate the determinant of M.

2.1 Determinant

The determinant of M, and $n \times n$ matrix, is always a scalar number.

Let $M_{i,j}$ be the $(i,j)^{\text{th}}$ element of M.

Let S_n be the set of $n!$ possible permutations of an ordered set of n numbers.

The determinant of a $n \times n$ square matrix M is given by

$$\det M = \sum_{\sigma \in S_n} (-1)^{N_\sigma} \Pi_i^n M_{i,\sigma(i)}$$

where N_σ is the number of pairwise exchanges of elements of σ required to reach the order $(1, 2, \ldots, n)$.

Example 1. Let

$$M = \begin{bmatrix} a \end{bmatrix}$$

.

Calculate $\det M$.

Solution:

$S_1 = \{(1)\} = \{\sigma_1\}$. $N_{\sigma_1} = 0$, because we don't need to switch any elements to get to the permutation (1). $\det M$ has only one term:

$$\begin{aligned}
\det M &= (-1)^{N_{\sigma_1}} \Pi_i^1 M_{i,\sigma_1(i)} && (1)\\
&= (-1)^0 M_{1,\sigma_1(1)} && (2)\\
&= 1 \cdot M_{1,1} && (3)\\
&= a && (4)
\end{aligned}$$

Example 2. Let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

.

Calculate $\det M$.

Solution: Since M is a 2×2 matrix, we will need to work with S_2, which has $2! = 2$ elements: $S_2 = \{(1,2), \quad (2,1)\} = \{\sigma_1, \sigma_2\}$.

$\det M$ is the sum of two terms:

$$\det M = (-1)^{N_{\sigma_1}} \Pi_i^2 M_{i,\sigma_1(i)} + (-1)^{N_{\sigma_2}} \Pi_i^2 M_{i,\sigma_2(i)} \tag{5}$$

Consider the first term corresponding to $\sigma_1 = (1,2)$. $N_{\sigma_1} = 0$, because we don't need to permute any entries to reach the permutation $(1,2)$. Since $\sigma_1 = (1,2)$, we have

$$\sigma_1(1) = 1, \sigma_1(2) = 2.$$

We need to evaluate $(-1)^{N_{\sigma_1}} \Pi_i^2 M_{i,\sigma_1(i)}$:

$$\begin{aligned}
(-1)^{N_{\sigma_1}} \Pi_i^2 M_{i,\sigma_1(i)} &= (-1)^{N_{\sigma_1}} M_{1,\sigma_1(1)} M_{2,\sigma_1(2)} && (6)\\
&= (-1)^0 M_{1,1} M_{2,2} && (7)\\
&= ab && (8)
\end{aligned}$$

For $\sigma_2 = \{(2,1)\}$: $N_{\sigma_2} = 1$, because we must switch the 1 and 2 to obtain the permutation $(1,2)$. We have

$$\sigma_2(1) = 2, \sigma_2(2) = 1.$$

$$\begin{aligned}
(-1)^{N_{\sigma_2}} \Pi_i^2 M_{i,\sigma_2(i)} &= (-1)^{N_{\sigma_2}} M_{1,\sigma_2(1)} M_{2,\sigma_2(2)} && (9)\\
&= (-1)^1 M_{1,2} M_{2,1} && (10)\\
&= -bc && (11)
\end{aligned}$$

Therefore, $\det M = ad - bc$

Example 3. Let

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

.

Calculate $\det M$.

Solution: $M \in \mathbb{R}^{3\times 3}$. Therefore, the permutations we consider belong to S_3. There are $3! = 6$ such permutations. They are:

$S_3 = \{(1,2,3), \quad (1,3,2), \quad (2,1,3), \quad (2,3,1), \quad (3,1,2), \quad (3,2,1)\} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6\}$

$\det M$ is the sum of six terms:

$$\begin{aligned}
\det M = (-1)^{N_{\sigma_1}} \Pi_i^3 M_{i,\sigma_1(i)} + (-1)^{N_{\sigma_2}} \Pi_i^3 M_{i,\sigma_2(i)} + (-1)^{N_{\sigma_3}} \Pi_i^3 M_{i,\sigma_3(i)} \qquad (12)\\
+ (-1)^{N_{\sigma_4}} \Pi_i^3 M_{i,\sigma_4(i)} + (-1)^{N_{\sigma_5}} \Pi_i^3 M_{i,\sigma_5(i)} + (-1)^{N_{\sigma_6}} \Pi_i^3 M_{i,\sigma_6(i)}
\end{aligned}$$

Consider the first term corresponding to $\sigma_1 = (1,2,3)$. $N_{\sigma_1} = 0$, because we don't need to permute any entries to reach the permutation $(1,2,3)$. Since $\sigma_1 = (1,2,3)$, we have

$$\sigma_1(1) = 1, \sigma_1(2) = 2, \sigma_1(3) = 3.$$

We need to evaluate $(-1)^{N_{\sigma_1}}\Pi_i^3 M_{i,\sigma_1(i)}$:

$$
\begin{aligned}
(-1)^{N_{\sigma_1}}\Pi_i^3 M_{i,\sigma_1(i)} &= (-1)^0 M_{1,\sigma_1(1)} M_{2,\sigma_1(2)} M_{3,\sigma_1(3)} && (13)\\
&= M_{1,1}M_{2,2}M_{3,3} && (14)\\
&= aei && (15)
\end{aligned}
$$

So, the first term in $\det M$ is $+aei$.

We repeat this process for the second term in $\det M$ corresponding to $\sigma_2 = (1,3,2)$. $N_{\sigma_2} = 1$, because we need to exchange the last two elements of σ_2 to get the permutation $(1,2,3)$. Since $\sigma_2 = (1,3,2)$, we have

$$\sigma_2(1) = 1, \sigma_2(2) = 3, \sigma_2(3) = 2.$$

$$
\begin{aligned}
(-1)^{N_{\sigma_2}}\Pi_i^3 M_{i,\sigma_2(i)} &= (-1)^1 M_{1,\sigma_2(1)} M_{2,\sigma_2(2)} M_{3,\sigma_2(3)} && (16)\\
&= -M_{1,1}M_{2,3}M_{3,2} && (17)\\
&= -afh && (18)
\end{aligned}
$$

Continuing this process, we get

$$\det M = aei - afh + dhc - dib + gbf - gce.$$

2.2 Matrix Inverse

sad Let M be an $n \times n$ matrix.

The inverse of M, denoted M^{-1}, is a matrix whose $(i,j)^{\text{th}}$ element $M^{-1}_{i,j}$ is given by

$$M^{-1}_{i,j} = (-1)^{(i+j)}\frac{\det M_{[i,j]}}{\det M},$$

where $M_{[i,j]}$ is an $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} **column** and j^{th} **row** of M.

Example 4. Let

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

.

Calculate M^{-1}.

Solution:

$$\det M = ad - bc.$$

By deleting the i^{th} **column** and j^{th} **row** of M, we get

$$M_{[1,1]} = d \tag{19}$$
$$M_{[1,2]} = b \tag{20}$$
$$M_{[2,1]} = c \tag{21}$$
$$M_{[2,2]} = a \tag{22}$$

The $(i,j)^{\text{th}}$ entry of M^{-1} is then

$$M^{-1}_{1,1} = (-1)^{(1+1)}\frac{\det M_{[1,1]}}{\det M} = \frac{d}{ad-bc} \tag{23}$$
$$M^{-1}_{1,2} = (-1)^{(1+2)}\frac{\det M_{[1,2]}}{\det M} = \frac{-b}{ad-bc} \tag{24}$$
$$M^{-1}_{2,1} = (-1)^{(2+1)}\frac{\det M_{[2,1]}}{\det M} = \frac{-c}{ad-bc} \tag{25}$$
$$M^{-1}_{2,2} = (-1)^{(2+2)}\frac{\det M_{[2,2]}}{\det M} = \frac{a}{ad-bc} \tag{26}$$

Therefore,

$$M^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 5.

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, B = 0, D = 0 \tag{27}$$

Let

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find the free response using Laplace transforms.
Note: $L\left\{e^{At}\right\} = (sI-A)^{-1}e^{At_0}$

Solution:

The free response $\hat{y}_{free}(s)$ is

$$\hat{y}_{free}(s) = C(sI-A)^{-1}x(t_0)$$

Let's first construct $M = (sI-A)$:

$$\begin{aligned} M = (sI-A) &= s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} (s+1) & -1 \\ 0 & (s+2) \end{bmatrix} \end{aligned}$$

Calculate the determinant:

$$\begin{aligned}
\det M &= (s+1)(s+2) - (-1)\cdot(0) && (28)\\
&= (s+1)(s+2) && (29)
\end{aligned}$$

We've derived the expression for the inverse of a 2×2 matrix, so that

$$\begin{aligned}
M^{-1} = (sI - A)^{-1} &= \frac{1}{(s+1)(s+2)} \begin{bmatrix} (s+2) & 1 \\ 0 & (s+1) \end{bmatrix} && (30)\\
&= \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} && (31)
\end{aligned}$$

$$\begin{aligned}
\hat{y}_{free}(s) &= C(sI - A)^{-1} x(t_0) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{(s+1)(s+2)} \\
&= \frac{1}{s+1} - \frac{1}{s+2} \\
\implies y_{free}(t) &= L^{-1}\left\{ \frac{1}{s+1} \right\} - L^{-1}\left\{ \frac{1}{s+2} \right\} \\
&= e^{-t} - e^{-2t}
\end{aligned}$$