ME 599/699 Robot Modeling & Control
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Homogenous Coordinate Transformations

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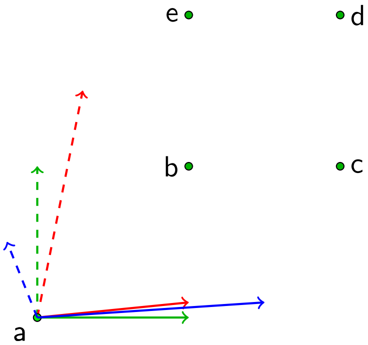
Same Vector Space, Different Bases



A basis and an origin together form a **coordinate frame** or **reference frame**.

Change Of Basis

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Let A, B, C, ... be different coordinate frames.

A point p then has coordinates p^A, p^B, p^C... corresponding to each basis.

Change Of Vector Space Basis

Given p^A, what is p^B, or p^C?

Answer:

$$p^B = \left(T_B^A\right)^{-1} p^A,$$

where

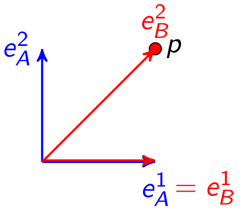
$$T_B^A = \left[\left(e_B^1\right)^A \quad \left(e_B^2\right)^A \quad \cdots \quad \left(e_B^n\right)^A\right],$$

and $\left(e_B^i\right)^A$ is the coordinates in frame A of the i^{th} basis vector of frame B.

Example

Problem: Find p^B if $p^A = (1, 1)$.
Solution: From the diagram,

$$e_B^1 = e_A^1$$
$$e_B^2 = e_A^1 + e_A^2$$
$$\implies T_B^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



Apply the formula:

$$p^B = \left(T_B^A\right)^{-1} p^A$$
$$= T_A^B p^A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The columns of T_B^A tell us how to draw the basis of B in A

Change Of Vector Space Basis

Full derivation:

The vector e_B^i has coordinates $\left(e_B^i\right)^A = (T_{1i}, T_{2i}, \ldots, T_{ni})$ in frame A.

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$$p \iff \sum_j^n \alpha_j^A e_A^j$$

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Note that p is an abstract point equivalent to the coordinate-given combination of the basis $\{e_A^1, e_A^2, \ldots, e_A^n\}$.

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Note that p is an abstract point equivalent to the coordinate-given combination of the basis $\{e_A^1, e_A^2, \ldots, e_A^n\}$.

Similarly, if $p^B = (\beta_1^B, \beta_2^B, \ldots, \beta_n^B)$, then

$$p \iff \sum_i^n \beta_i^B e_B^i$$

Change Of Vector Space Basis

So, we can write

$$e_B^i = \sum_i^n T_{ji} e_A^i; \quad p \iff \sum_i^n \beta_i^B e_B^i; \quad p \iff \sum_j^n \alpha_j^A e_A^j \qquad (1)$$

Change Of Vector Space Basis

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$$e_B^i = \sum_i^n T_{ji} e_A^i; \quad p \iff \sum_i^n \beta_i^B e_B^i; \quad p \iff \sum_j^n \alpha_j^A e_A^j \tag{1}$$

Combining the first and second equation in (1), we get

$$\begin{aligned} p &\iff \sum_i^n \beta_i^B e_B^i = \sum_i^n \beta_i^B \left(\sum_j^n T_{ji} e_A^j \right) \\ &\iff \sum_j^n \left(\sum_i^n \left(\beta_i^B T_{ji} \right) \right) e_A^j \end{aligned} \tag{2}$$

Comparing (2) to the third equation in (1), we get

$$\alpha_j^A = \sum_i^n \left(\beta_i^B T_{ji} \right).$$

Change Of Vector Space Basis

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$$p \iff \begin{bmatrix} e_A^1 & e_A^2 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} \alpha_1^A \\ \alpha_2^A \\ \vdots \\ \alpha_n^A \end{bmatrix} = \begin{bmatrix} e_B^1 & e_B^2 & \cdots & e_B^n \end{bmatrix} \begin{bmatrix} \beta_1^B \\ \beta_2^B \\ \vdots \\ \beta_n^B \end{bmatrix}$$

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The coordinates of e_B^i in frame A give:

$$e_B^1 = \begin{bmatrix} e_A^1 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} T_{11} \\ \vdots \\ T_{n1} \end{bmatrix}, e_B^2 = \begin{bmatrix} e_A^1 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} T_{12} \\ \vdots \\ T_{n2} \end{bmatrix}, \ldots$$

Change Of Vector Space Basis

We can collect these expressions for point e_B^i as

$$\begin{bmatrix} e_B^1 & e_B^2 & \cdots & e_B^n \end{bmatrix} = \begin{bmatrix} e_A^1 & e_A^2 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix},$$

So that

$$\begin{bmatrix} e_B^1 & e_B^2 & \cdots & e_B^n \end{bmatrix} \begin{bmatrix} \beta_1^B \\ \beta_2^B \\ \vdots \\ \beta_n^B \end{bmatrix} = \begin{bmatrix} e_A^1 & e_A^2 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} \begin{bmatrix} \beta_1^B \\ \beta_2^B \\ \vdots \\ \beta_n^B \end{bmatrix}$$

Change Of Vector Space Basis

Since

$$p = \begin{bmatrix} e_A^1 & e_A^2 & \cdots & e_A^n \end{bmatrix} \begin{bmatrix} \alpha_1^A \\ \alpha_2^A \\ \vdots \\ \alpha_n^A \end{bmatrix},$$

we find that transforming coordinates is a linear operation represented by matrix operations:

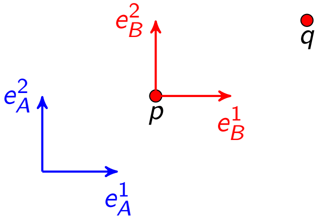
$$\begin{bmatrix} \alpha_1^A \\ \alpha_2^A \\ \vdots \\ \alpha_n^A \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{bmatrix} \begin{bmatrix} \beta_1^B \\ \beta_2^B \\ \vdots \\ \beta_n^B \end{bmatrix}$$

More compactly: $p^B = \left(T_B^A\right)^{-1} p^A$, where ▸ to example

$$T_B^A = \left[\left(e_B^1\right)^A \quad \left(e_B^2\right)^A \quad \cdots \quad \left(e_B^n\right)^A \right].$$

Change Of Origin

Suppose points p, q have coordinates p^A, q^A in a frame A. Consider frame B whose origin is at p, with the same basis elements for its vector space as the frame A. What is q^B?



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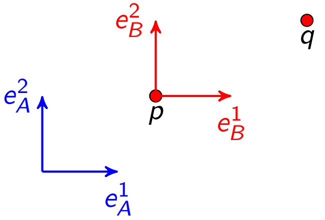

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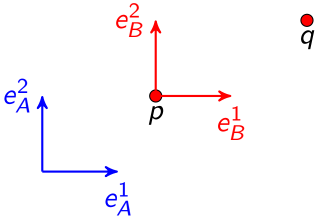

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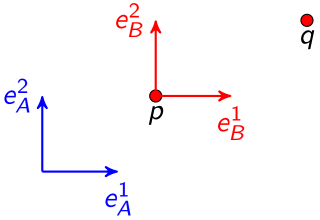

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Precisely because vectors are free, the coordinates of v in frame B will be the same as that in frame A. So, $q^B = q^A - p^A$.

In general, $q^B = q^A -$ (coordinates of origin of B in A)

Change Of Frames

Combining previous discussions, we get that to map coordinates from one frame to another we :

1. express the coordinates of the basis vectors of one frame in the other (through, say, matrix T_B^A),
2. express the coordinates of the origin of one frame in another (through, say coordinate vector o_B^A),
3. use the map

$$p^B = \left(T_B^A\right)^{-1} (p^A - o_B^A)$$

Checkpoint

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If all bases for the plane give us two numbers, what's special about a basis where the two elements are at 90 degrees , and have the same 'length'?

Norms and Distances

Let's reconsider our earlier example:

$$T_B^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \quad q^A = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \implies q^B = \left(T_B^A \right)^{-1} q^A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

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Note that $\|q^B\|_B = \|\left(T_B^A\right)^{-1} q^A\|_A$.

Q: What kinds of matrices preserve the norms of the vectors they act upon?

Special Orthogonal Group in Three Dimensions

if $T_B^A \in SO(3)$, then we'd have norm-preservation.

Definition ($SO(3)$)

The Special Orthogonal Group $SO(3)$ is the set of matrices $R \in \mathbb{R}^{3\times 3}$ such that

$$R^T R = Id, \text{and } \det R = 1$$

.

$SO(3)$ is known as the orientation group **and** the rotation group.

Example

Problem: Find p^B if $p^A = (1, 1)$.
Solution: From the diagram,

$$e_B^1 = e_A^1$$
$$e_B^2 = e_A^1 + e_A^2$$
$$\implies T_B^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

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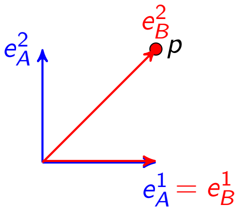

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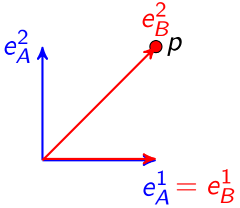

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$$\left(T_A^B\right)^T T_A^B = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

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$$e_B^1 = \frac{1}{\sqrt{2}} e_A^1 - \frac{1}{\sqrt{2}} e_A^2$$
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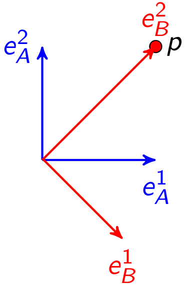

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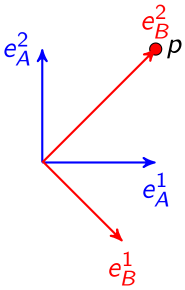

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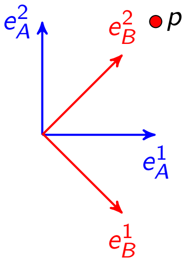

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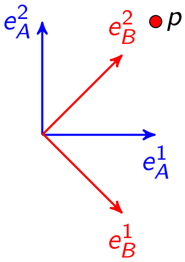

norm-preserving!

$$\left(T_A^B\right)^T T_A^B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Orthonormal Vectors

We have seen that

$$T_B^A = \left[\left(e_B^1\right)^A \quad \left(e_B^2\right)^A \quad \cdots \quad \left(e_B^n\right)^A \right].$$

Therefore,

$$\left(T_B^A\right)^T T_B^A = \begin{bmatrix} \left(\left(e_B^1\right)^A\right)^T \\ \left(\left(e_B^2\right)^A\right)^T \\ \vdots \\ \left(\left(e_B^1\right)^A\right)^T \end{bmatrix} \left[\left(e_B^1\right)^A \quad \left(e_B^2\right)^A \quad \cdots \quad \left(e_B^n\right)^A \right]$$

Orthonormal Vectors

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$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Effectively, the coordinates of basis vectors of B in frame A are unit length and perpendicular to each other.

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- To preserve distance, the linear part of the affine map must be in SO(3)
- $T_B^A \in \mathrm{SO}(3)$ when basis vectors are all unit length, mutually perpendicular.

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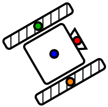
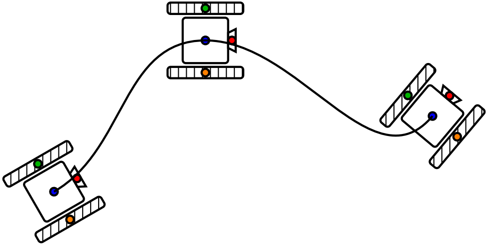
- Can transform vector coordinates of a point in different frames using an affine map
- To preserve distance, the linear part of the affine map must be in SO(3)
- $T_B^A \in SO(3)$ when basis vectors are all unit length, mutually perpendicular.
- The coordinate transformation is then $p^B = \left(R_B^A\right)^{-1}\left(p^A - o_B^A\right)$ mobile robot

Coordinate Transformation Vs Rigid Motion



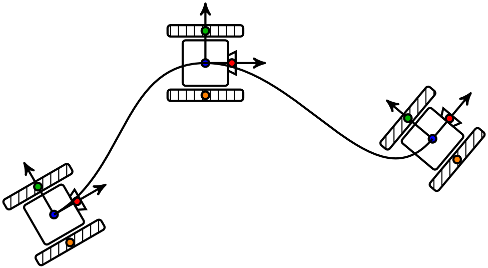
Consider a robot with a center, a camera in ‘front’, and two wheels to the side.

Coordinate Transformation Vs Rigid Motion



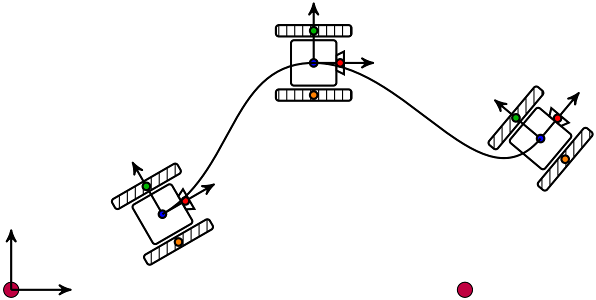
Whenever we move the robot, the distances between these points don't change.

Coordinate Transformation Vs Rigid Motion



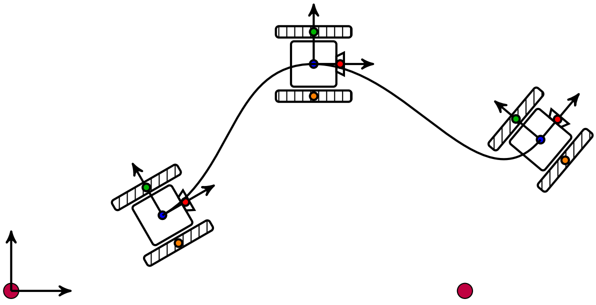
As the robot moves, we can take a snapshot of these points, and they each define a coordinate frame for Euclidean space.

Coordinate Transformation Vs Rigid Motion



Q1: How would the robot compare observations of either purple point over time? A1: Coordinate transformations

Coordinate Transformation Vs Rigid Motion



Q1: How would the robot compare observations of either purple point over time? A1: Coordinate transformations

Q2: How do we keep track of all the points on the robots?

A2: Coordinate transformations, but reinterpreted. ▸ rigid motion

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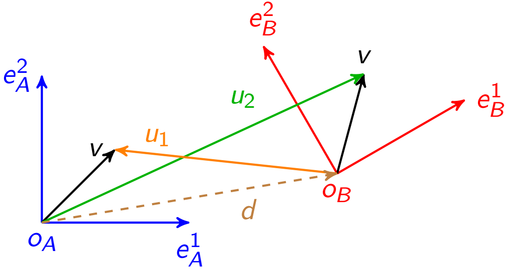

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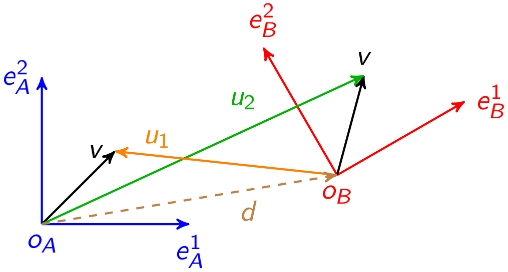
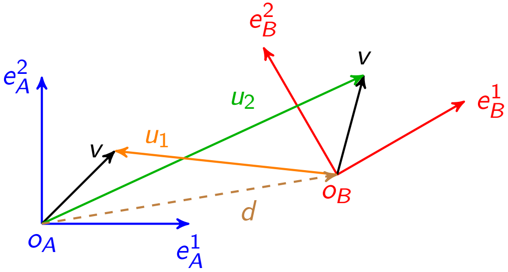


- If we view u_1 as coordinates in frame B, we've changed coordinates of v from world to body frame.
- If we view u_2 as coordinates in frame A, we've moved the point $o_A \oplus v$ relative to frame A.

Rigid Body Pose



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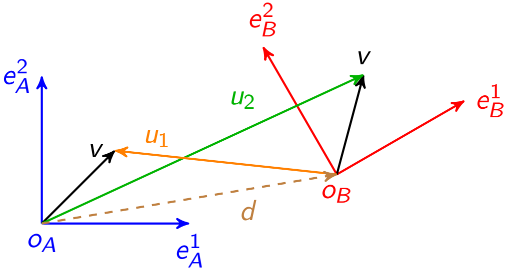

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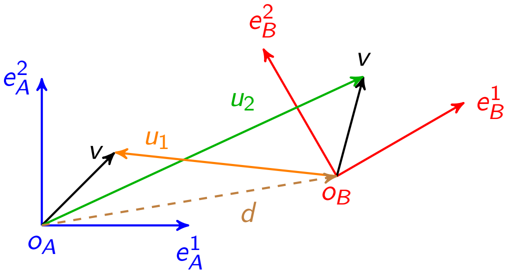

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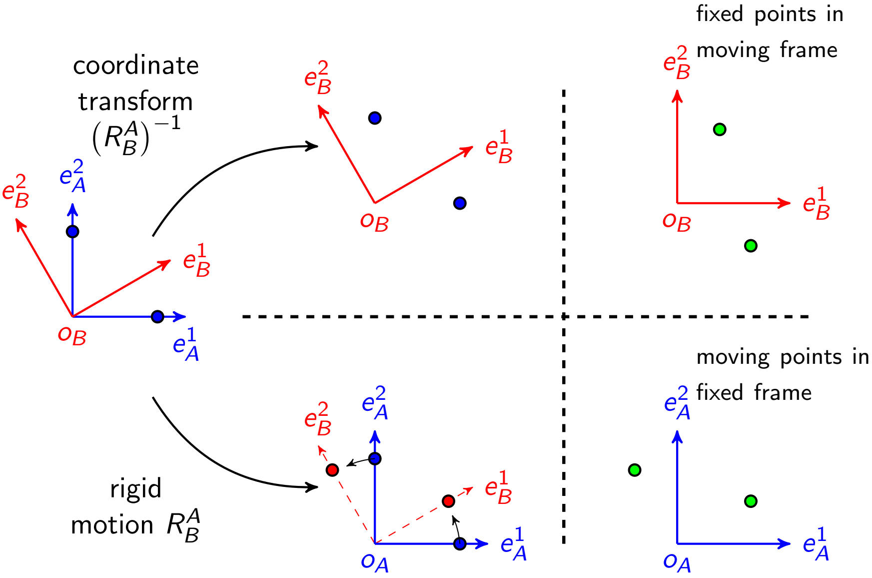
Move in frame $A =$ reorient by R and then move by d : $Rv + d$

Example



coordinate transform $\left(R_B^A\right)^{-1}$
rigid motion R_B^A
fixed points in moving frame
moving points in fixed frame
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o_B

Special Euclidean Group $\mathrm{SE}(3)$

Coordinates of points in 3D Euclidean space $= p^A \in \mathbb{R}^3$
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For a G-Torsor, we don't define origin+basis (not a vector space).

Instead, we define an identity element (it's a group): the reference coordinate frame.

Homogenous Transformations

We can convert the affine map between two Euclidean spaces of dimension 3 into a linear map between two subsets of $\mathbb{R}^4$.

Define a homogenization $h\colon \mathbb{R}^3 \mapsto \mathbb{R}^4$ as $h\left(p^A\right) = \begin{bmatrix} p^A \\ 1 \end{bmatrix}$.

If $p^A = Rp^B + d$, then

$$h\left(p^A\right) = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} h\left(p^B\right). \tag{6}$$

The matrix $\begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$ represents a homogenous transformation, and forms a group.

Checkpoint

- The coordinate transformation is $p^B = \left(R_B^A\right)^{-1}\left(p^A - o_B^A\right)$
- Norm-preserving coordinate transformation $=$ rigid motion of points within the same coordinate frame.
- Set of rigid body poses/rigid motions forms a group: $\mathrm{SE}(3)$
- After choosing a reference frame, we assign coordinates – aka rigid body pose – (d, R) to frame (Torsor structure)

Back to $\mathbf{SO}(3)$

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- The G-Torsor nature is why SO(3) is called both the rotation group and the orientation group.
- Assigning coordinates to an orientation is the same as defining the rotation that generates that frame.

Basic Rotations

Consider three frames rotated about each one of the world frame axes by an angle θ.

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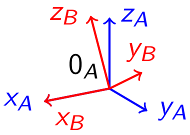

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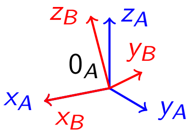

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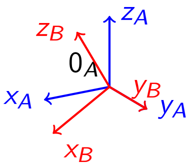

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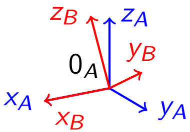

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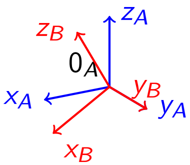
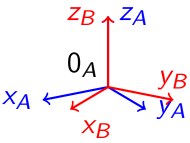


$$R_{z,\theta} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



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For Euclidean vector spaces, the order of a sequence of (vector space) operations didn't matter: $v + w = w + v$.

For rotations, they do. In general, $R_1 R_2 \neq R_2 R_1$.

Change-of-Basis For Orientations

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Which rotation R_B^C below correctly defines orientation of B relative to orientation C?

1. $R_B^C = R_B^A R_A^C$
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How would you pick the right transformation? Why did we not consider R_A^B?

Change-of-Basis For Orientations

- For example, imagine you, a driver, and a passenger in a car. Your orientation frames are aligned: Forward: x, upwards: z.
- When the car stops, the passenger opens the door spins to their right ($R_C^A = R_{z,-90^\circ}$)
- You lean back in your driver's seat ($R_B^A = R_{y,-20^\circ}$)
- What is your orientation according to the passenger?
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To change the coordinates of vectors from A to C, we must pre-multiply by $\left(R_C^A\right)^{-1} = R_A^C$. So,

$$R_B^C = R_A^C R_B^A$$

.

Change-of-Basis For Orientations

Alternatively, The rigid motion in A corresponding to moving to frame B is R_B^A; the rigid motion in frame C corresponding to moving to frame A is R_A^C.

The combined rigid motion in C is achieved by first moving by R_B^A **in C**, then moving the result by R_A^C.
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Instead of orientation R_B^A in frame A, what if we define rotation R^A in frame A.
How do we represent this rotation in frame C?

Change-of-Basis For Rotations

- The rotation R^A is relative to frame A.
- A general orientation P has coordinates R_P^A in frame A
- Rotating this orientation results in a new orientation $R^A R_P^A$ in frame A:

$$R_P^A \mapsto R^A R_P^A$$

- But, note that $R_P^A = R_C^A R_P^C$
- Therefore :

$$R_C^A R_P^C \mapsto R^A R_C^A R_P^C, \text{or}$$

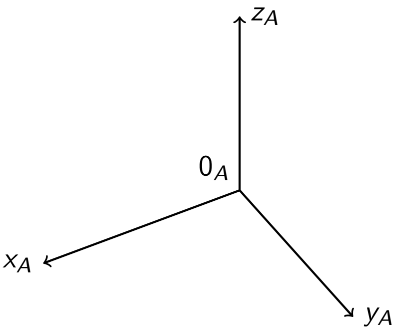
$$R_P^C \mapsto \left(R_C^A\right)^{-1} R^A R_C^A R_P^C, \text{or}$$

- Therefore, a rotation R^A in frame A becomes a rotation

$$R^C = \left(R_C^A\right)^{-1} R^A R_C^A$$

in frame C.

Extrinsic vs Intrinsic Rotations



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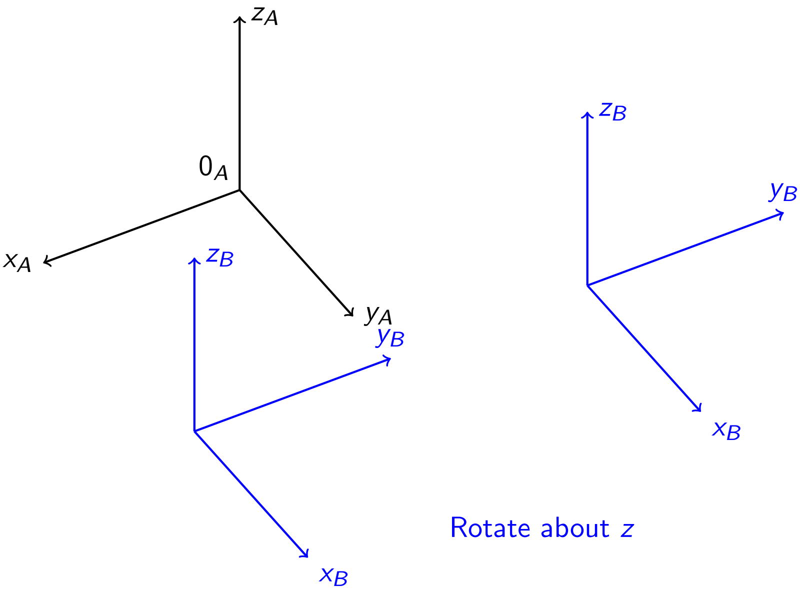
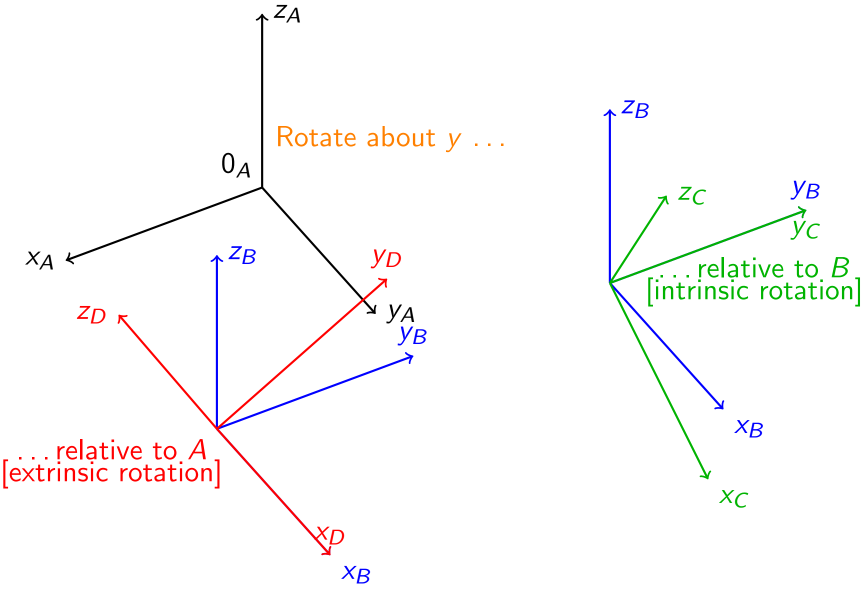


Rotate about z

Extrinsic vs Intrinsic Rotations



Extrinsic vs Intrinsic Rotations

- A first rigid motion corresponding to rotation R_1 relative to a frame A produces frame B
- A second rigid motion rotation R_2 can be applied relative to either A or B.
- When applied relative to B, the second rotation is an intrinsic rotation. $R = R_1 R_2$.
- When applied relative to A, the second rotation is an extrinsic rotation. $R = R_2 R_1$.