Complex Numbers and PFEs

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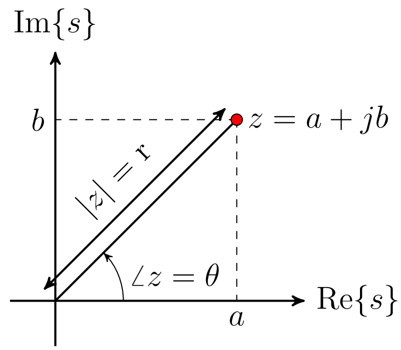
Contents

1 Review of Complex Numbers

Let $j^2 = -1$, or equivalently, $j = \sqrt{-1}$.

We represent a complex number $z \in \mathbb{C}$ in two ways.



The first is $z = a + jb$, where a and b are real numbers. We refer to a and b as the real and imaginary part of z respectively. We denote these parts of z as $\mathrm{Re}\{z\}$ $(= a)$ and $\mathrm{Im}\{z\}$ $(= b)$.

The second is $z = re^{j\theta}$, where r and θ are real numbers. The numbers r and θ are the magnitude and argument of z respectively.

Note that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$, so that

$$\mathrm{Re}\{z\} = a = r\cos\theta, \quad \mathrm{Im}\{z\} = b = r\sin\theta.$$

To any complex number $z = a + jb = re^{j\theta}$, we can associate the following quantities:

- a magnitude $|z| = \sqrt{a^2 + b^2} = r$,
- an argument

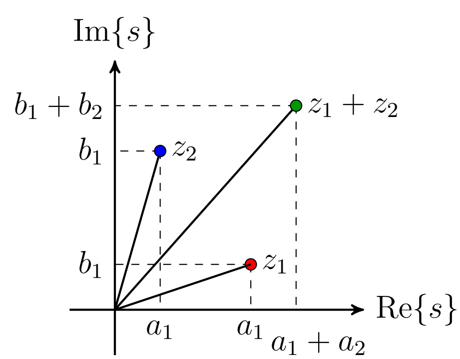
$$\angle z = \theta = \begin{cases} \tan^{-1}\frac{b}{a} & \text{if } a > 0 \\ \pi + \tan^{-1}\frac{b}{a} & \text{if } a < 0 \\ \pi/2 & \text{if } a = 0 \text{ and } b > 0 \\ -\pi/2 & \text{if } a = 0 \text{ and } b < 0 \end{cases}.$$

- a complex conjugate $\bar{z} = a - jb$, and

Just as for real numbers, we can define the operations of addition and multiplication, which depend on the same operations that are defined for real numbers.

Addition. For two numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, we define the sum

$$z_1 + z_2 = (a_1 + a_2) + j(b_1 + b_2).$$

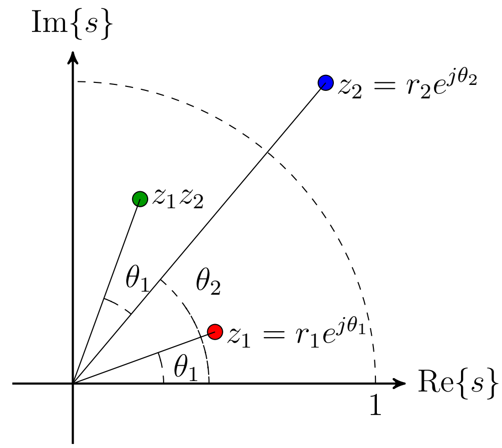


Multiplication. For two numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, we define the product

$$z_1 z_2 = (a_1 + jb_1)(a_2 + jb_2) \tag{1}$$
$$= (a_1a_2 - b_1b_2) + j(a_1b_2 + a_2b_1). \tag{2}$$

Alternatively, if $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, then

$$z_1 z_2 = r_1 r_2 e^{j(\theta_1+\theta_2)}$$

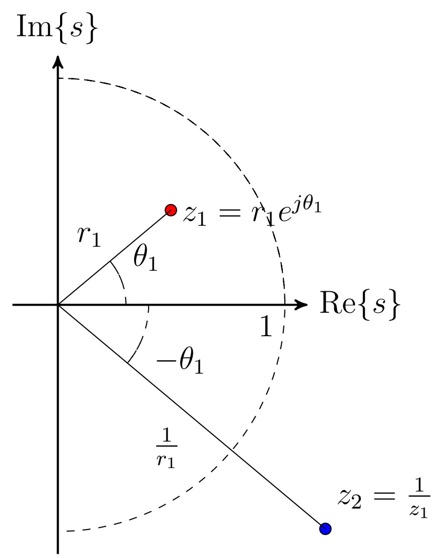


Inversion. If $z = a + jb = re^{j\theta}$, then

$$z^{-1} = \frac{1}{z} = \frac{1}{a+jb} \tag{3}$$
$$= \frac{1}{(a+jb)}\frac{a-jb}{a-jb} \tag{4}$$
$$= \frac{a-jb}{a^2+b^2} \tag{5}$$
$$= \frac{a}{a^2+b^2} - j\frac{b}{a^2+b^2}. \tag{6}$$

Alternatively,

$$z^{-1} = \frac{1}{r}e^{-j\theta}.$$



Division. For two numbers $z_1 = a_1 + jb_1$ and $z_2 = a_2 + jb_2$, we define division as multiplication by z_2^{-1}

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{(a_1a_2 + b_1b_2) - j(a_1b_2 + a_2b_1)}{a_2^2 + b_2^2}.$$

Alternatively, if $z_1 = r_1e^{j\theta_1}$ and $z_2 = r_2e^{j\theta_2}$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{j(\theta_1-\theta_2)}$$

Definition 1 (Roots Of Complex Polynomials)**.** Let $\alpha(s)$ be a polynomial in the complex variable s, with complex coefficients. If $\alpha(p) = 0$ for $p \in \mathbb{C}$, then p is a *root* of $\alpha(s)$

Definition 2 (Multiplicity)**.** Let p be a root of $\alpha(s)$,

$$\lim_{s\to p} \frac{\alpha(s)}{(s-p)^n} \neq 0, \text{ and}$$

$$\lim_{s\to p} \frac{\alpha(s)}{(s-p)^{n-1}} = 0,$$

then p is a root of $\alpha(s)$ with multiplicity n.

Example 1. Let $\alpha(s) = (s-2)(s-1)^2 s^4$. By our definition above, $p_1 = 2$ is a root of $\alpha(s)$ with multiplicity 1, $p_2 = 1$ is a root with multiplicity 2, and $p_3 = 0$ is a root with multiplicity 4. $\square$

2 Complex Functions

- A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is differentiable with continuous derivative.
- A meromorphic function $f : \mathbb{C} \to \mathbb{C}$ is locally holomorphic except for a discrete set of points.
  - In a neighborhood $D$of S, a singular point of f, $f = g/h$, where $g,\ h$ are holomorphic functions on D
- Cauchy Theorem:
  - For any holomorphic function f
  - And simply connected open set Ω
  - containing a closed contour γ
  - $\int_\gamma f(z)dz = 0$
- Cauchy Integral Theorem (holomorphic)
  - For any holomorphic function f
  - γ is a simple closed curve which encloses a bounded region U in the anticlockwise direction.
  - $2\pi j f(z_0) = \int_\gamma \frac{f(z)}{z-z_0} dz$
- Cauchy Residue Theorem
  - Let f be a meromorphic function on a simply connected domain Ω, and let γ be a closed contour in Ω enclosing a bounded region U anticlockwise, and avoiding all the singularities of f. Show that
    $\int_\gamma f(z)\ dz = 2\pi i \sum_\rho \mathrm{Res}(f;\rho)$
    where ρ is summed over all the poles of f that lie in U.

3 Partial Fraction Expansion

The expression $\hat{y}(s)$ for the solution of linear time-invariant (LTI) ODEs, in the s-domain, is the ratio of polynomials in s.

In other words,

$$\hat{y}(s) = \frac{N(s)}{D(s)}.$$

When we want to compute $y(t)$, we need to compute

$$y(t) = L^{-1}\left\{\hat{y}(s)\right\} = L^{-1}\left\{\frac{N(s)}{D(s)}\right\}. \tag{7}$$

We use some related ideas to simplify this computation:

- $L^{-1}\{1/(s-a)\}$ equals e^{at}, when a is real
- any polynomial $a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0$ can be rewritten as $a_n(s-p_1)(s-p_2)\cdots(s-p_n)$, where $p_1, \ldots, p_n$ are complex numbers
- For polynomials with real coefficients, if one complex number is a root, its conjugate is always a root.

<u>Loosely</u> speaking, a partial fraction expansion (PFE) of $\hat{y}(s)$ will be of the form

$$\hat{y}(s) = k_0 + \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \ldots$$

for complex numbers k_0, k_1, etc. and where $D(s) = (s-p_1)(s-p_2)\cdots(s-p_n)$. Then, $y(t) = L^{-1}\{\hat{y}(s)\}$ is simply

$$y(t) = k_0\delta(t) + k_1 e^{-p_1 t} + k_2 e^{-p_2 t} + \ldots.$$

The expression above doesn't always apply, and we go over the different cases below. In general,

$$\hat{y}(s) = \frac{N_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}, \tag{8}$$

where m is the order of polynomial $N(s)$, n is the order of polynomial $D(s)$, z_i for $i = 1, \ldots, m$ are complex numbers, and p_i for $i = 1, \ldots, n$ are complex numbers.

Some terminology:

1. If $n = m$, then $N(s)/D(s)$ is **exactly proper**.
2. If $n > m$, then $N(s)/D(s)$ is **strictly proper**.
3. The complex numbers z_i are the roots of $N(s)$ and are called **zeros**.
4. The complex numbers p_i are the roots of $D(s)$ and are called **poles**.

The partial fraction expansion of $\hat{y}(s)$ depends on the values of n, m, N_m, z_i, and p_i.

3.1 Case 1: All roots of $D(s)$ are distinct

The PFE of $N(s)/D(s)$ is exactly

$$\hat{y}(s) = k_0 + \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n}, \tag{9}$$

where k_0, k_1, …, $k_n \in \mathbb{C}$. Furthermore,

$$k_0 = \begin{cases} N_m & \text{if } n = m \\ 0 & \text{if } n > m. \end{cases}$$

Example 2. Find the PFE and inverse Laplace transform of

$$\hat{y}(s) = \frac{4(s+2)}{(s+1)(s+5)}.$$

Solution:

$$\begin{aligned} \hat{y}(s) &= k_0 + \frac{k_1}{s+1} + \frac{k_2}{s+5} && (10) \\ &= \frac{k_1}{s+1} + \frac{k_2}{s+5} \qquad (n=2, m=1, n>m) && (11) \\ \frac{4(s+2)}{(s+1)(s+5)} &= \frac{k_1}{s+1} + \frac{k_2}{s+5} && (12) \end{aligned}$$

Consider multiplying (12) by $s+1$:

$$\frac{4(s+2)}{(s+5)} = k_1 + \frac{k_2(s+1)}{s+5}$$

When we set $s = -1$, we get

$$\left.\frac{4(s+2)}{(s+5)}\right|_{s=-1} = k_1 + 0.$$

□

We get a **general method for distinct roots**. If

$$\hat{y}(s) = k_0 + \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_n}{s-p_n}, \tag{13}$$

then

$$k_i = \left[\hat{y}(s)(s-p_i)\right]\big|_{s=p_i}.$$

Example 3. Find the PFE and inverse Laplace transform of

$$\hat{y}(s) = \frac{3(s+1)(s+2)(s+3)}{(s+4)(s+2-j)(s+j+2)}.$$

Solution:
We get

$$\begin{aligned} \hat{y}(s) &= k_0 + \frac{k_1}{s+4} + \frac{k_2}{s+2-j} + \frac{k_2}{s+j+2} && (14) \\ &= 3 + \frac{k_1}{s+4} + \frac{k_2}{s+2-j} + \frac{k_2}{s+j+2} && (15) \end{aligned}$$

To calculate k_1:

$$\begin{aligned} k_1 &= \hat{y}(s)(s+4)\big|_{s=-4} && (16) \\ &= \frac{3(s+1)(s+2)(s+3)}{(s+4)(s+2-j)(s+j+2)}(s+4)\bigg|_{s=-4} && (17) \\ &= \frac{3(s+1)(s+2)(s+3)}{(s+2-j)(s+j+2)}\bigg|_{s=-4} && (18) \\ &= \frac{3(-4+1)(-4+2)(-4+3)}{(-4+2-j)(-4+j+2)} && (19) \\ &= \frac{3(-3)(-2)(-1)}{(-2-j)(j-2)} && (20) \\ &= \frac{-18}{5} && (21) \end{aligned}$$

To calculate k_2:

$$\begin{aligned} k_2 &= \hat{y}(s)(s+2-j)\big|_{s=-2+j} && (22) \\ &= \frac{3(s+1)(s+2)(s+3)}{(s+4)(s+2-j)(s+j+2)}(s+2-j)\bigg|_{s=-2+j} && (23) \\ &= \frac{3(s+1)(s+2)(s+3)}{(s+4)(s+j+2)}\bigg|_{s=-2+j} && (24) \\ &= \frac{3(-2+j+1)(-2+j+2)(-2+j+3)}{(-2+j+4)(-2+j+j+2)} && (25) \\ &= \frac{3(-1+j)(j)(1+j)}{(2+j)(2j)} && (26) \\ &= \frac{-3}{2+j} && (27) \\ &= \frac{-3(2-j)}{5} \qquad \text{(by inversion, Section 1)} && (28) \end{aligned}$$

To calculate k_3:

$$
\begin{aligned}
k_2 &= \hat{y}(s)(s+2+j)\big|_{s=-2-j} && (29)\\
&= \frac{3(s+1)(s+2)(s+3)}{(s+4)(s+2-j)(s+j+2)}(s+2+j)\bigg|_{s=-2-j} && (30)\\
&= \frac{3(s+1)(s+2)(s+3)}{(s+4)(s+2-j)}\bigg|_{s=-2-j} && (31)\\
&= \frac{3(-2-j+1)(-2-j+2)(-2-j+3)}{(-2-j+4)(-2-j+2-j)} && (32)\\
&= \frac{3(-1-j)(-j)(1-j)}{(2-j)(-2j)} && (33)\\
&= \frac{-3}{2-j} && (34)\\
&= \frac{-3(2+j)}{5} \qquad \text{(by inversion, Section 1)} && (35)
\end{aligned}
$$

□

$$
\begin{aligned}
\hat{y}(s) &= k_0 + \frac{k_1}{s+4} + \frac{k_2}{s+2-j} + \frac{k_2}{s+j+2} && (36)\\
&= 3 + \frac{-18}{5(s+4)} + \frac{-3(2-j)}{5(s+2-j)} + \frac{-3(2+j)}{5(s+j+2)} && (37)\\
&= 3 + \frac{-18}{5(s+4)} + \frac{-3}{5}\left(\frac{4(s+2)-8}{(s+2)^2+1^2}\right) && (38)
\end{aligned}
$$

We combine the last two terms because we will be able to take the inverse Laplace transform of the result. Instead of slogging through the algebra, we can use complex number algebra to handle this step. Notice that if $z = 2 - j$, the last two terms are

$$
\begin{aligned}
\text{last two terms} &= \frac{-3}{5}\left(\frac{z}{s+z} + \frac{\bar{z}}{s+\bar{z}}\right) && (39)\\
&= \frac{-3}{5}\left(\frac{z(s+\bar{z}) + \bar{z}(s+z)}{(s+z)(s+\bar{z})}\right) && (40)\\
&= \frac{-3}{5}\frac{(z+\bar{z})s + 2z\bar{z}}{(s^2 + (\bar{z}+z)s + z\bar{z})} && (41)
\end{aligned}
$$

Now, $z + \bar{z} = 2\text{Re}\{z\} = 2 \cdot 2$, and $z\bar{z} = |z|^2 = 2^2 + 1^2 = 5$. Therefore, we get

$$\text{last two terms} = \frac{-3}{5}\left(\frac{4s+10}{s^2+4s+5}\right).$$

This looks a little nicer, in part because

$$L^{-1}\left\{\frac{(s+a)}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt, \text{ and}$$

$$L^{-1}\left\{\frac{c}{(s+a)^2+b^2}\right\} = \frac{c}{b}e^{-at}\sin bt.$$

and we will be able to apply this rule. The first step is to simplify the denominator, by completing squares:

$$s^2+4s+5 \to s^2+4s+4+1 \to (s+2)^2+1^2.$$

This step also tells us how to modify the numerator:

$$4s+10 \to 4(s+2-2)+10 \to 4(s+2)+10-8 \to 4(s+2)+2.$$

We now get the last two terms into the form

$$\text{last two terms} = \frac{-3}{5}\left(\frac{4(s+2)+2}{(s+2)^2+1^2}\right).$$

We are now ready to take the inverse of

$$\hat{y}(s) = 3 + \frac{-18}{5(s+4)} + \frac{-3}{5}\left(\frac{4(s+2)+2}{(s+2)^2+1^2}\right). \tag{42}$$

$$\begin{aligned}
L^{-1}\{\hat{y}(s)\} &= L^{-1}\left\{3 + \frac{-18}{5(s+4)} + \frac{-3}{5}\left(\frac{4(s+2)+2}{(s+2)^2+1^2}\right)\right\} && (43)\\
&= \mathcal{L}^{-1}\{3\} + \mathcal{L}^{-1}\left\{\frac{-3.6}{(s+4)}\right\} + \mathcal{L}^{-1}\left\{\frac{-3}{5}\left(\frac{4(s+2)+2}{(s+2)^2+1^2}\right)\right\} && (44)\\
&= 3\delta(t) - 3.6e^{-4t} + \mathcal{L}^{-1}\left\{\frac{-3}{5}\left(\frac{4(s+2)}{(s+2)^2+1^2}\right)\right\} + \mathcal{L}^{-1}\left\{\frac{-3}{5}\left(\frac{2}{(s+2)^2+1^2}\right)\right\} && (45)\\
&= 3\delta(t) - 3.6e^{-4t} + \mathcal{L}^{-1}\left\{\left(\frac{-2.4(s+2)}{(s+2)^2+1^2}\right)\right\} + \mathcal{L}^{-1}\left\{-1.2\left(\frac{1}{(s+2)^2+1^2}\right)\right\} && (46)\\
&= 3\delta(t) - 3.6e^{-4t} - 2.4e^{-2t}\cos t - 1.2e^{-2t}\sin t, && (47)
\end{aligned}$$

Which is the solution to Example 3.

3.2 Case 2: Roots of $D(s)$ are repeated

$$\hat{y}(s) = \frac{N_m(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)^{l_1}(s-p_2)^{l_2}\cdots(s-p_q)^{l_q}}, \tag{48}$$

where $n = l_1 + l_2 + \cdots l_q$. The PFE in this case is

$$\begin{aligned}\hat{y}(s) = k_0 &+ \frac{k_1}{(s-p_1)^{l_1}} + \frac{k_2}{(s-p_1)^{l_1-1}} + \cdots + \frac{k_{l_1}}{(s-p_1)} + \frac{k_{l_1+1}}{(s-p_2)^{l_2}} \\ &+ \cdots + \frac{k_{n-l_q}}{(s-p_q)^{l_q}} + \frac{k_{n-l_q+1}}{(s-p_q)^{l_q-1}} + \cdots + \frac{k_n}{s-p_q}.\end{aligned} \tag{49}$$

where k_0, k_1, $\ldots$, $k_n \in \mathbb{C}$. Again,

$$k_0 = \begin{cases} N_m & \text{if } n = m \\ 0 & \text{if } n > m. \end{cases}$$

Example 4. Find the PFE and inverse Laplace transform of

$$\hat{y}(s) = \frac{1}{(s+2)(s+1)^2}.$$

Solution: The roots are: $p_1 = -2$ with multiplicity 1, and $p_2 = -1$ with multiplicity 2. Therefore.

$$\hat{y}(s) = k_0 + \frac{k_1}{s+2} + \frac{k_2}{s+1} + \frac{k_3}{(s+1)^2} \tag{50}$$

$$= \frac{k_1}{s+2} + \frac{k_2}{s+1} + \frac{k_2}{(s+1)^2} \qquad (n = 3, m = 0, n > m) \tag{51}$$

Since p_1 has multiplicity 1, we can obtain k_1 using the same rule as for distinct roots:

$$k_1 = \hat{y}(s)(s+2)\big|_{s=-2} \tag{52}$$

$$= \frac{1}{(s+2)(s+1)^2}(s+2)\bigg|_{s=-2} \tag{53}$$

$$= \frac{1}{(s+1)^2}\bigg|_{s=-2} \tag{54}$$

$$= \frac{1}{(-2+1)^2} \tag{55}$$

$$= 1 \tag{56}$$

$$\tag{57}$$

This rule works for distinct roots p_i because we know all other terms have to go to zero when evaluating at $s = p_i$. When we have a root p_j with multiplicity greater than 1, multiplying by $(s - p_j)$ won't work. Let's see why:

$$\frac{1}{(s+2)(s+1)^2} = \frac{k_1}{s+2} + \frac{k_2}{s+1} + \frac{k_3}{(s+1)^2} \tag{58}$$

$$\implies \frac{1}{(s+2)(s+1)^2}(s+1) = \frac{k_1}{s+2}(s+1) + \frac{k_2}{s+1}(s+1) + \frac{k_3}{(s+1)^2}(s+1) \tag{59}$$

$$\implies \frac{1}{(s+2)(s+1)} = \frac{k_1(s+1)}{s+2} + k_2 + \frac{k_3}{(s+1)} \tag{60}$$

We can't plug in $s = -1$, so that the following equation suggested by Equation (51) is incorrect:

$$\text{Incorrect:} \quad k_2 = \hat{y}(s)(s+1)\big|_{s=-1}.$$

As you might guess, the only thing that makes sense is multiplying by $(s - p_j)^l$, where l is the multiplicity of root p_j. In our example:

$$\frac{1}{(s+2)(s+1)^2} = \frac{k_1}{s+2} + \frac{k_2}{s+1} + \frac{k_3}{(s+1)^2} \tag{61}$$

$$\implies \frac{1}{(s+2)(s+1)^2}(s+1)^2 = \frac{k_1}{s+2}(s+1)^2 + \frac{k_2}{s+1}(s+1)^2 + \frac{k_3}{(s+1)^2}(s+1)^2 \tag{62}$$

$$\implies \frac{1}{(s+2)} = \frac{(s+1)^2}{s+2} + k_2(s+1) + k_3 \tag{63}$$

If $s = -1$, the only terms remaining are k_3 and the left hand side which is $\hat{y}(s)(s+1)^2$. It turns out that we could have used the same pattern as in the case of distinct roots only for the term containing the $(s - p_j)^l$, which here is k_3:

$$\text{Correct:} \quad k_3 = \hat{y}(s)(s+1)^2\big|_{s=-1}. = \left.\frac{1}{s+2}\right|_{s=-1} = \frac{1}{-1+2} = 1$$

In other words, we can use the following more general rule: If the PFE of $\hat{y}(s)$ contains the term $k_i/(s - p_j)^l$, then

$$k_i = \hat{y}(s)(s - p_j)^l\big|_{s=p_j}, \text{ only when } l \text{ is the multiplicity of pole } p_j.$$

This rule includes the case of poles with multiplicity 1.

What about terms of the form $k_i/(s - p_j)^{l'}$, where l' is **less** than the multiplicity l of p_j? First, note that we would expect $l-1$ such terms, as defined in the PFE (49) for the repeated root case. We use the following approach:

1. Multiply the expression involving the PFE by $(s - p_j)^l$, where l is the multiplicity of pole p_j.

2. Differentiate the expression with respect to s, a total of $l-1$ times, using the expression after each time you differentiate to calculate one of the $l-1$ coefficients by plugging in $s = p_j$.

So, in our still running example:

$$\frac{1}{(s+2)(s+1)^2} = \frac{k_1}{s+2} + \frac{k_2}{s+1} + \frac{k_3}{(s+1)^2} \tag{64}$$

$$\implies \frac{1}{(s+2)} = \frac{(s+1)^2}{s+2} + k_2(s+1) + k_3 \qquad \text{(Multiplying by } (s+1)^2 \text{)} \tag{65}$$

Notice that when we substitute in $s = -1$, on the right hand side only the coefficient in front of the term without $(s+1)$ remains. How do we make that coefficient be k_2? The easy way is to differentiate. This does two things: 1) k_3 disappears 2) the terms with higher powers of $(s+1)$ will still contain $(s+1)$, and so we don't have to explicitly evaluate the derivative:

$$\frac{d}{ds}\left(\frac{1}{(s+2)} = \frac{(s+1)^2}{s+2} + k_2(s+1) + k_3\right) \tag{66}$$

$$\implies \frac{-1}{(s+2)^2} = \frac{d}{ds}\left(\frac{(s+1)^2}{s+2}\right) + k_2 + 0 \tag{67}$$

Again, we don't worry about the first term on the RHS for now because it evaluates to 0 when we plug in $s = -1$. So, let's plug in $s = -1$

$$\frac{-1}{(-1+2)^2} = 0 + k_2 \implies k_2 = -1.$$

So, we have now completed the PFE.

$$\hat{y}(s) = \frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2} \tag{68}$$

Let's take the inverse Laplace transform:

$$y(t) = \mathcal{L}^{-1}\left\{\hat{y}(s)\right\} \tag{69}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+1} + \frac{1}{(s+1)^2}\right\} \tag{70}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} \tag{71}$$

$$= e^{-2t} - e^{-t} + te^{-t} \qquad \text{(by the } s\text{-shift and multiplication-by-time rules),} \tag{72}$$

which is the solution to Example 4 □

3.3 Summary of Partial Fraction Expansions

If there are m roots, no matter what their multiplicities are, we will be able to obtain m coefficients in the PFE by direct calculation. The PFE contains the term $k_i/(s-p_j)^l$ where l is the multiplicity of p_j, and

$$k_i = \hat{y}(s)(s-p_j)^l\Big|_{s=p_j}.$$

For the terms of the form $k_i/(s-p_j)^{l'}$, where l' is less than the multiplicity of p_j, we do the following:

1. Multiply the expression involving the PFE of $\hat{y}(s)$ by $(s-p_j)^l$, where l is the multiplicity of pole p_j.
2. Differentiate the expression with respect to s, a total of $l-1$ times, using the expression after each time you differentiate to calculate one of the $l-1$ coefficients by plugging in $s=p_j$.

Example 5. Find the PFE and inverse Laplace transform of

$$\hat{y}(s) = \frac{3(s+2)(s+1)}{(s+5)^3}.$$

Solution steps:

1. Calculate the poles
2. Write down the form of the PFE, containing unknown coefficients
3. Use n and m to calculate k_0
4. Calculate coefficients for term corresponding to highest multiplicity of pole directly
5. Calculate the remaining coefficient corresponding to repeated roots using differentiation
6. Express $\hat{y}(s)$ using the calculated coefficients
7. Calculate $y(t)$ using the inverse Laplace transform

The roots are: $p_1 = -5$ with multiplicity 3. Therefore

$$\hat{y}(s) = k_0 + \frac{k_1}{(s+5)^3} + \frac{k_2}{(s+5)^2} + \frac{k_3}{(s+5)}. \tag{73}$$

Since $n = 3$ and $n = 2$, $k_0 = 0$. Therefore,

$$\hat{y}(s) = \frac{k_1}{(s+5)^3} + \frac{k_2}{(s+5)^2} + \frac{k_3}{(s+5)}. \tag{74}$$

We can calculate k_1 directly:

$$\begin{aligned} k_1 &= \hat{y}(s)(s+5)^3\big|_{s=-5} && (75)\\ &= \left.\frac{3(s+2)(s+1)}{(s+5)^3}(s+5)^3\right|_{s=-5} && (76)\\ &= 3(s+2)(s+1)|_{s=-5} && (77)\\ &= 3(s+2)(s+1)|_{s=-5} && (78)\\ &= 3(-5+2)(-5+1) && (79)\\ &= 36 && (80) \end{aligned}$$

To get k_2 and k_3, first multiply the PFE by $(s+5)^3$

$$3(s+2)(s+1) = k_1 + k_2(s+5) + k_3(s+5)^2.$$

Differentiate with respect to s

$$3(s+2) + 3(s+1) = 0 + k_2 + k_3 2(s+5) \tag{81}$$

Set $s = -5$, to get

$$3(-5+2) + 3(-5+1) = k_2 + 0k_3 \implies k_2 = -21.$$

Differentiate (81) with respect to s

$$3 + 3 = 0 + 0 + 2k_3 \tag{82}$$

'Plugging in' $s = -5$ gives $k_3 = 3$.

So,

$$\hat{y}(s) = \frac{36}{(s+5)^3} - \frac{21}{(s+5)^2} + \frac{3}{(s+5)}, \text{ and} \tag{83}$$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\hat{y}(s)\right\} && (84)\\ &= \mathcal{L}^{-1}\left\{\frac{36}{(s+5)^3} - \frac{21}{(s+5)^2} + \frac{3}{(s+5)}\right\} && (85)\\ &= \mathcal{L}^{-1}\left\{\frac{36}{(s+5)^3}\right\} - \mathcal{L}^{-1}\left\{\frac{21}{(s+5)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s+5)}\right\} && (86)\\ &= 18t^2e^{-5t} - 21te^{-5t} + 3e^{-5t}, \qquad \text{(using the multiplication-by-}t^n\text{ rule),} && (87) \end{aligned}$$

□