MTAT.03.015 Computer Graphics (Fall 2013)

Sample solutions to math exercises (Lectures II-V)

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1. Let s be a straight line in $\mathbb{R}^2$, passing through the origin. It can be described parametrically as

$$\mathbf{x} = \lambda \mathbf{s}, \qquad \lambda \in \mathbb{R},$$

or implicitly as

$$\mathbf{n}^T \mathbf{x} = 0\,.$$

Express the coordinates of the normal vector $\mathbf{n}$ via the coordinates of the direction vector $\mathbf{s}$.

Solution. Fix a vector $\mathbf{s}$ and consider a set of points $\{\mathbf{x} := \lambda\mathbf{s},\ \lambda \in \mathbb{R}\}$. The task is to find a vector $\mathbf{n}$, such that for any λ the condition

$$\mathbf{n}^T(\lambda\mathbf{s}) = 0,$$

is satisfied, and, vice-versa, if, for some $\mathbf{x}$ it holds that $\mathbf{n}^T\mathbf{x} = 0$ then it is necessarily true that $\mathbf{x} = \lambda\mathbf{s}$.

Necessity: For simplicity, assume that[1] $s_1 \neq 0$. Fix any $\lambda \neq 0$. Then,

$$\begin{aligned} \mathbf{n}^T(\lambda\mathbf{s}) &= 0, \\ \mathbf{n}^T\mathbf{s} &= 0, \\ n_1 s_1 + n_2 s_2 &= 0, \\ n_1 &= -n_2 s_2 / s_1 . \end{aligned}$$

Hence, if we pick any $t \in \mathbb{R}$ and construct $\mathbf{n}$ as

$$\mathbf{n} = \begin{pmatrix} -t s_2 / s_1 \\ t \end{pmatrix}$$

then all points of the form $\lambda\mathbf{s}$ will also satisfy $\mathbf{n}^T\mathbf{x} = 0$.

[1] If it is not the case we can assume $s_2 \neq 0$ and proceed with the proof in the same way. If both $s_1 = s_2 = 0$ we must treat this as a special case and demonstrate that no matching $\mathbf{n}$ exists then.

Multiplying both sides by s_1 produces a somewhat more conventional answer:

$$\mathbf{n} = t \begin{pmatrix} -s_2 \\ s_1 \end{pmatrix}.$$

Sufficiency: We complete the proof by showing that for $t \neq 0$ any $\mathbf{x}$ that satisfies $\mathbf{n}^T\mathbf{x} = 0$ (when $\mathbf{n}$ is chosen as shown above) is also of the form $\mathbf{x} = \lambda\mathbf{x}$:

$$\begin{aligned}
\mathbf{n}^T\mathbf{x} &= 0, \\
n_1x_1 + n_2x_2 &= 0, \\
-ts_2x_1 + ts_1x_2 &= 0, \\
x_2 &= x_1s_2/s_1, \\
\mathbf{x} &= \begin{pmatrix} x_1 \\ x_1s_2/s_1 \end{pmatrix} = \left(\frac{x_1}{s_1}\right) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \lambda\mathbf{s}.
\end{aligned}$$

Consequently, for any direction vector $\mathbf{s}$ the corresponding normal indeed exists and must be of the form

$$\mathbf{n} = t \begin{pmatrix} -s_2 \\ s_1 \end{pmatrix}, \, t \neq 0.$$

2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be points in $\mathbb{R}^2$. Find the coordinates of the intersection point of segments $[\mathbf{a}, \mathbf{b}]$ and $[\mathbf{c}, \mathbf{d}]$. Hint: Use the parametric representation.

Solution. The set of all points lying on the first segment can be parametrically described as

$$\{\mathbf{x} := t\mathbf{a} + (1-t)\mathbf{b}, \, t \in [0,1]\}.$$

Analogously, the set of point on the second segment is

$$\{\mathbf{x} := s\mathbf{c} + (1-s)\mathbf{d}, \, s \in [0,1]\}.$$

The intersection point must belong to both sets, and hence can be found by solving

$$t\mathbf{a} + (1-t)\mathbf{b} = s\mathbf{c} + (1-s)\mathbf{d}.$$

This is a system of two equations with two unknowns that can be solved using conventional means. Here a more elegant solution by Raimond-

Hendrik:

$$\begin{aligned}
\mathbf{b} + t(\mathbf{a} - \mathbf{b}) &= \mathbf{c} + s(\mathbf{c} - \mathbf{d}), \\
&\text{Box product on both sides with } (\mathbf{c} - \mathbf{d}), \\
[(\mathbf{b} + t(\mathbf{a} - \mathbf{b})) \times (\mathbf{c} - \mathbf{d})] &= [(\mathbf{c} + s(\mathbf{c} - \mathbf{d})) \times (\mathbf{c} - \mathbf{d})], \\
&\text{Linearity of the box product,} \\
[\mathbf{b} \times (\mathbf{c} - \mathbf{d})] + t[(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{d})] &= [\mathbf{c} \times (\mathbf{c} - \mathbf{d})] + s[(\mathbf{c} - \mathbf{d}) \times (\mathbf{c} - \mathbf{d})], \\
&\text{Box product of a vector with itself is } 0, \\
[\mathbf{b} \times (\mathbf{c} - \mathbf{d})] + t[(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{d})] &= [\mathbf{c} \times (\mathbf{c} - \mathbf{d})], \\
t &= \frac{[\mathbf{c} \times (\mathbf{c} - \mathbf{d})] - [\mathbf{b} \times (\mathbf{c} - \mathbf{d})]}{[(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{d})]} \\
t &= \frac{[(\mathbf{c} - \mathbf{b}) \times (\mathbf{c} - \mathbf{d})]}{[(\mathbf{a} - \mathbf{b}) \times (\mathbf{c} - \mathbf{d})]}.
\end{aligned}$$

Now if the denominator and the numerator are 0, the segments lie on the same line. In this case we check whether the endpoints of one segment are within the other. If the denominator is 0 and the numerator is not, the segments are parallel and do not intersect. If both are non-zero, it remains to check whether $t \in [0, 1]$ and if so, find the corresponding point as

$$t\mathbf{a} + (1 - t)\mathbf{b}.$$

3. Prove that the (Euclidean) norm $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$ satisfies the *triangle inequality*:

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Derive from this inequality also the inequalities

$$\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Solution. First part:

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\|, \\
\|\mathbf{x} + \mathbf{y}\|^2 &\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\|, \\
(\mathbf{x} + \mathbf{y})^T(\mathbf{x} + \mathbf{y}) &\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\|, \\
\mathbf{x}^T\mathbf{x} + \mathbf{y}^T\mathbf{x} + \mathbf{x}^T\mathbf{y} + \mathbf{y}^T\mathbf{y} &\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\|, \\
\mathbf{x}^T\mathbf{x} + 2\mathbf{x}^T\mathbf{y} + \mathbf{y}^T\mathbf{y} &\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\|, \\
\|\mathbf{x}\|^2 + 2\mathbf{x}^T\mathbf{y} + \|\mathbf{y}\|^2 &\leq \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\|, \\
\mathbf{x}^T\mathbf{y} &\leq \|\mathbf{x}\|\|\mathbf{y}\|. \\
\|\mathbf{x}\|\|\mathbf{y}\|\cos\alpha &\leq \|\mathbf{x}\|\|\mathbf{y}\|.
\end{aligned}$$

Second part:

$$\begin{aligned}\|(\mathbf{x}-\mathbf{y})+\mathbf{y}\| &\le \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}\|\\ \|\mathbf{x}\| &\le \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}\|\\ \|\mathbf{x}\|-\|\mathbf{y}\| &\le \|\mathbf{x}-\mathbf{y}\|\end{aligned}$$

and

$$\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{x}+(-\mathbf{y})\| \le \|\mathbf{x}\|+\|-\mathbf{y}\| = \|\mathbf{x}\|+\|\mathbf{y}\|$$

4. Let $\mathbf{p}$ and $\mathbf{q}$ be orthonormal vectors in $\mathbb{R}^3$. What transformation does the matrix $\mathbf{p}\mathbf{p}^T + \mathbf{q}\mathbf{q}^T$ correspond to? Prove it.

Solution. Intuitively, $\mathbf{p}\mathbf{p}^T$ is the orthogonal projector onto the axis defined by $\mathbf{p}$. Similarly $\mathbf{q}\mathbf{q}^T$ is the projector onto the axis defined by $\mathbf{q}$. Consequently, $\mathbf{p}\mathbf{p}^T + \mathbf{q}\mathbf{q}^T$ is an orthogonal projector onto the plane defined by $\mathbf{p}$ and $\mathbf{q}$.

Formally, let $\mathbf{r}$ be a third vector, that makes up an orthogonal basis together with $\mathbf{p}$ and $\mathbf{q}$. Pick any vector $\mathbf{x}$. As $(\mathbf{p},\mathbf{q},\mathbf{r})$ forms an orthonormal basis, we can represent $\mathbf{x}$ in it, so let

$$\mathbf{x} = x_1\mathbf{p} + x_2\mathbf{q} + x_3\mathbf{r}.$$

Now apply transformation $\mathbf{p}\mathbf{p}^T + \mathbf{q}\mathbf{q}^T$ to $\mathbf{x}$:

$$\begin{aligned}&(\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T)(x_1\mathbf{p}+x_2\mathbf{q}+x_3\mathbf{r}) =\\ &\quad x_1(\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T)\mathbf{p}+x_2(\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T)\mathbf{q}+x_3(\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T)\mathbf{r} =\\ &\qquad x_1\mathbf{p}+x_2\mathbf{q},\end{aligned}$$

i.e. the transformation is indeed an orthogonal projector onto the **p-q** plane.

5. Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ be an orthonormal basis in $\mathbb{R}^3$. Prove that $\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T+\mathbf{r}\mathbf{r}^T = \mathbf{I}$, where $\mathbf{I}$ denotes a unit matrix.

Solution. Same as above. Show that for any $\mathbf{x} = x_1\mathbf{p}+x_2\mathbf{q}+x_3\mathbf{r}$ the application of $\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T+\mathbf{r}\mathbf{r}^T$ leaves the vector intact. This can only be the case when the transformation is the identity matrix. Alternative solution would be to note that

$$\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T+\mathbf{r}\mathbf{r}^T$$

is equal to $\mathbf{A}^T\mathbf{A}$, where $\mathbf{A}$ is the matrix with $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ as the rows. It easily follows then that $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.

6. Orthogonalize the following set of vectors using the Gram-Schmidt algorithm:

$$\mathbf{e}_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)^T$$
$$\mathbf{e}_2 = (-1, 1, -1)^T$$
$$\mathbf{e}_3 = (0, -2, -2)^T$$

Solution. Following the algorithm, pick the first vector as-is:

$$\mathbf{e}'_1 = \mathbf{e}_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)^T,$$

Next, $\mathbf{e}'_2$ is $\mathbf{e}_2$ minus its projection onto $\mathbf{e}'_1$. As $\mathbf{e}'_1$ and $\mathbf{e}_2$ are already orthogonal, this projection is zero, hence in our case

$$\mathbf{e}'_2 = \mathbf{e}_2 = (-1, 1, -1)^T,$$

Next, $\mathbf{e}'_3$ is $\mathbf{e}_3$ minus its projections onto $\mathbf{e}'_1$ and $\mathbf{e}'_2$. The vectors $\mathbf{e}'_2$ and $\mathbf{e}_3$ are already orthogonal, so we only need to subtract the projection onto $\mathbf{e}'_1$. Here we can further simplify by noting that $\|\mathbf{e}'_1\| = 1$:

$$\mathbf{e}'_3 = \mathbf{e}_3 - \mathbf{e}'_1\mathbf{e}'^T_1\mathbf{e}_3 = (0, -2, -2)^T - (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)^T \cdot (-\sqrt{2}) =$$
$$(0, -2, -2)^T + (1, 1, 0)^T = (1, -1, -2)^T$$

7. Compute the area of a triangle given by vertices

$$\mathbf{a} = (1, 2, 3)^T,$$
$$\mathbf{b} = (-2, 2, 4)^T,$$
$$\mathbf{c} = (7, -8, 0)^T.$$

Solution. Let $\mathbf{p} = \mathbf{b} - \mathbf{a}$ and $\mathbf{q} = \mathbf{c} - \mathbf{a}$. The area of the triangle is then simply half the length of the cross-product $\mathbf{p} \times \mathbf{q}$.

$$S = \frac{1}{2}\|\mathbf{p} \times \mathbf{q}\|.$$

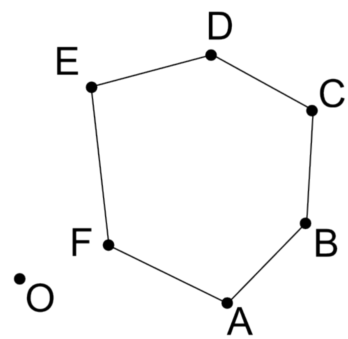
Computing it with the given numbers:

$$\mathbf{p} = (-3, 0, 1)^T$$
$$\mathbf{q} = (6, -10, -3)^T$$
$$\mathbf{p} \times \mathbf{q} = (10, -3, 30)^T$$
$$S = \frac{1}{2}\sqrt{10^2 + 3^2 + 30^2} \approx 15.88$$

8. Points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n \in \mathbb{R}^2$ are vertices of a simple polygon[2] listed in counter-clockwise order in a right-handed basis. Prove that the area of the polygon S can be computed as

$$S = \frac{1}{2}(|\mathbf{p}_1 \quad \mathbf{p}_2| + |\mathbf{p}_2 \quad \mathbf{p}_3| + \cdots + |\mathbf{p}_n \quad \mathbf{p}_1|)$$

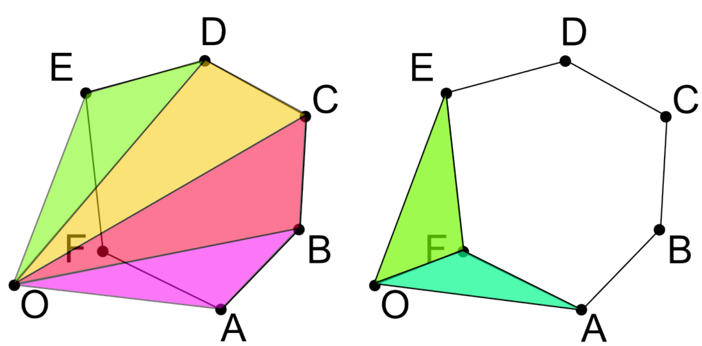
Solution. Recall that the expression $\frac{1}{2}|\mathbf{p} \quad \mathbf{q}|$ computes the signed area of the triangle $O\mathbf{pq}$, where O is the origin of the coordinate system. Now consider an example:



Note that the area ABCDEF can be computed as the difference between the area OABCDE = OAB+OBC+OCD+ODE, and the area OEFA = OEF + OFA. This is, however, exactly what the expression

$$S = \frac{1}{2}(|A \quad B| + |B \quad C| + |C \quad D| + |D \quad E| + |E \quad F| + |F \quad A|)$$

computes.



The construction applies for any simple polygon.

[2]A *simple polygon* is a polygon, whose edges do not intersect each other.

9. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function that satisfies $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ for each $\mathbf{x}, \mathbf{y}$. Show that it then necessarily follows that for each $\alpha \in \mathbb{R}$ and each $\mathbf{x}$
$$f(\alpha\mathbf{x}) = \alpha f(\mathbf{x}),$$
i.e. f must be linear.

Solution.

(a) First show that the required condition holds for any $\alpha = n \in \mathbb{N}^+$:
$$f(n\mathbf{x}) = f(\mathbf{x}+\mathbf{x}+\cdots+\mathbf{x}) = f(\mathbf{x}) + f(\mathbf{x}) + \cdots + f(\mathbf{x}) = nf(\mathbf{x}).$$

(b) Now, for $\alpha = 0$:
$$f(0\mathbf{x}) = f(0) = f(\mathbf{x}-\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x}) = 0 = 0f(\mathbf{x}).$$

(c) For $\alpha = -1$:
$$0 = f(\mathbf{x}-\mathbf{x}) = f(\mathbf{x}) + f(-\mathbf{x}), \text{ hence } f(-\mathbf{x}) = -f(\mathbf{x}).$$
Consequently, condition holds for all $\alpha \in \mathbb{N}$.

(d) Now let $\alpha = \frac{1}{m}$ for $m \in \mathbb{N}$.
$$f(\mathbf{x}) = f(m\frac{1}{m}\mathbf{x}) = mf(\frac{1}{m}\mathbf{x}),$$
hence
$$f(\frac{1}{m}\mathbf{x}) = \frac{1}{m}f(\mathbf{x}).$$

(e) Now, combining results (c) and (d), we conclude that condition holds for any rational $\alpha \in \mathbb{Q}$:
$$f(\frac{n}{m}\mathbf{x}) = nf(\frac{1}{m}\mathbf{x}) = \frac{n}{m}f(\mathbf{x}).$$

(f) Finally, let $\alpha \in \mathbb{R}$. Consider a sequence of rational numbers α_i, that converges to α:
$$\lim_{i\to\infty} \alpha_i = \alpha$$
Then, combining (e) and continuity of f:
$$\begin{aligned} f(\alpha\mathbf{x}) &= f(\lim_i \alpha_i\mathbf{x}) = \lim_i f(\alpha_i\mathbf{x}) \\ &= \lim_i \alpha_i f(\mathbf{x}) = (\lim_i \alpha_i) f(\mathbf{x}) = \alpha f(\mathbf{x}). \end{aligned}$$

10. Consider a polyhedron with vertices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$. Let $\mathbf{n}_1$, $\mathbf{n}_2, \ldots, \mathbf{n}_l$ be the normals for the faces of the polyhedron. Let us apply a linear transformation $\mathbf{F}$ to all the vertices of the polyhedron. The vertices of the new polyhedron are thus $\mathbf{Fp}_1, \mathbf{Fp}_2, \ldots, \mathbf{Fp}_k$. Express the normals of the new polyhedron in terms of the original normals.

Solution. Consider a face of the polyhedron, that includes vertices $\mathbf{p}_1$ and $\mathbf{p}_2$. The normal to this face $\mathbf{n}_i$ must satisfy: $\mathbf{n}_i^T(\mathbf{p}_1 - \mathbf{p}_2) = 0$. Let $\mathbf{n}'_i$ be the transformed normal. After transformation, the normal should stay perpendicular to the transformed face, i.e. it must hold that $\mathbf{n}_i'^T(\mathbf{F}\mathbf{p}_1 - \mathbf{F}\mathbf{p}_2) = 0$.

Let $\mathbf{n}'_i = (\mathbf{F}^{-1})^T\mathbf{n}$. Then

$$\begin{aligned} \mathbf{n}_i'^T(\mathbf{F}\mathbf{p}_1 - \mathbf{F}\mathbf{p}_2) &= ((\mathbf{F}^{-1})^T\mathbf{n})^T\mathbf{F}(\mathbf{p}_1 - \mathbf{p}_2) \\ &= \mathbf{n}^T\mathbf{F}^{-1}\mathbf{F}(\mathbf{p}_1 - \mathbf{p}_2) \\ &= \mathbf{n}^T(\mathbf{p}_1 - \mathbf{p}_2) = 0. \end{aligned}$$

In fact it is easy to see that if we transform the normals using the matrix $(\mathbf{F}^{-1})^T$ all the angles between normals and faces are preserved. The matrix $(\mathbf{F}^{-1})^T$ is thus known as the *normal transformation matrix.*

11. Let the horizontal field of view (*fov-X*) of some *view-frustum* be 75 degrees. Let the screen dimensions be 1280×1024. Find the corresponding vertical field of view (*fov-Y*).

Solution. From

$$\frac{\tan(\text{fovX}/2)}{\tan(\text{fovY}/2)} = \frac{1280}{1024}$$

we compute

$$\text{fovY} = 2\text{arctan}\left(\frac{\tan(75/2)}{1280/1024}\right) \approx 63.09$$

12. Consider a perspective projection in two-dimensional space. We shall be projecting to the line $y = 1$ with $(0, 0)$ as the center of projection.
    - Find the projection matrix in homogeneous coordinates.
    - Explain what linear transformation does this matrix correspond to in the three-dimensional homogeneous space. Illustrations are welcome.

Solution. First, observe that the necessary transformation is

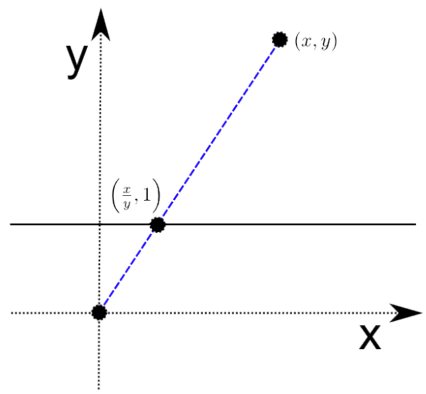
$$(x, y)^T \to (x/y, 1)^T$$

(see figure below). In homogeneous coordinates this would be

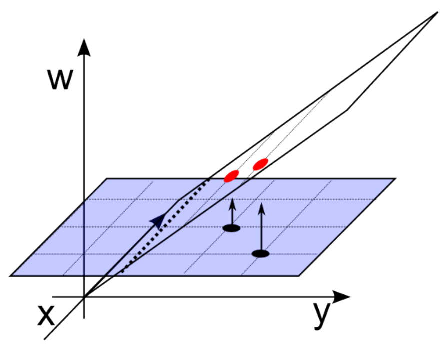
$$(x, y, 1)^T \to (x/y, 1, 1)^T = (x, y, y)^T,$$

which corresponds to the multiplication by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$



When viewed as a transformation $\mathbb{R}^3 \to \mathbb{R}^3$, this matrix acts as a projection onto the $(\mathbf{x}, \mathbf{y}+\mathbf{z})$ plane. That is, it takes points from the $y = 1$ plane and "raises" them up to $y = w$ plane. See illustration below. Think how perspective division after this transformation will lead all points back to the $w = 1$ plane and, coincidentally, also the $y = 1$ line.



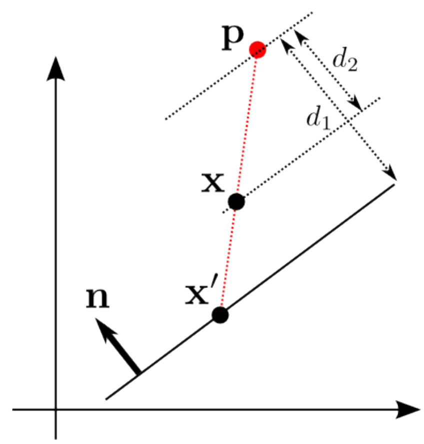
13. Let $ax+by+cz+d = 0$ be some plane in three-dimensional space and let $P = (p_x, p_y, p_z)$ be a point not located on this plane. Find a matrix, that performs a perspective projection from P onto this plane (in homogeneous coordinates).

Solution. Let

$$\begin{aligned} \mathbf{w} &:= (a, b, c, d), \\ k &:= (a^2+b^2+c^2)^{-1} \\ \mathbf{p} &:= (p_x, p_y, p_z, 1), \end{aligned}$$

Now due to the properties of implicit plane equation, for any point $\mathbf{x} = (x, y, z, 1)$ the signed distance from this point to the plane is equal to $k\mathbf{w}^T\mathbf{x}$. In particular, the distance from P to the plane is $k\mathbf{w}^T\mathbf{p}$.

Now observe that the transformation we seek is the following:



$$\mathbf{x}' = \mathbf{p} + (\mathbf{x} - \mathbf{p})\frac{d_1}{d_2} = \mathbf{p} + (\mathbf{x} - \mathbf{p})\frac{k\mathbf{w}^T\mathbf{p}}{k\mathbf{w}^T\mathbf{p} - k\mathbf{w}^T\mathbf{x}}$$

Regrouping and simplifying:

$$\begin{aligned}
(\mathbf{w}^T\mathbf{p} - \mathbf{w}^T\mathbf{x})\mathbf{x}' &= \mathbf{p}(\mathbf{w}^T\mathbf{p} - \mathbf{w}^T\mathbf{x}) + (\mathbf{x} - \mathbf{p})\mathbf{w}^T\mathbf{p} \\
&= \mathbf{p}\mathbf{w}^T\mathbf{p} - \mathbf{p}\mathbf{w}^T\mathbf{x} + \mathbf{x}\mathbf{w}^T\mathbf{p} - \mathbf{p}\mathbf{w}^T\mathbf{p} \\
&= (\mathbf{w}^T\mathbf{p})\mathbf{x} - \mathbf{p}\mathbf{w}^T\mathbf{x} \\
&= ((\mathbf{w}^T\mathbf{p})\mathbf{I} - \mathbf{p}\mathbf{w}^T)\mathbf{x}
\end{aligned}$$

Recall that in homogeneous coordinates the points $\mathbf{x}'$ and $\alpha\mathbf{x}'$ are considered equivalent, hence the left side of the above equation is already a representation of $\mathbf{x}'$.

Consequently, the matrix we are looking for is simply

$$(\mathbf{w}^T\mathbf{p})\mathbf{I} - \mathbf{p}\mathbf{w}^T.$$

It is straightforward to verify that the derivation holds for any $\mathbf{x}$ independently of what side of the plane the point is located on (note that this affects the sign of $\mathbf{w}^T\mathbf{x}$).

14. Let $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ – be points in space. Consider some attribute $\mathcal{A}$ (e.g. color) assigned to the points. Suppose that point P_1 is assigned attribute value a_1, point P_2 — value a_2 and on the line between them the attribute varies linearly.

    Let P_1^*, P_2^* — be the perspective projections of points P_1 and P_2 onto the plane $z = z_n$ with $(0, 0, 0)$ as the center of projection. Let P_t^* be a point obtained by interpolating between P_1^* and P_2^*:

$$P_t^* = tP_1^* + (1-t)P_2^*,$$

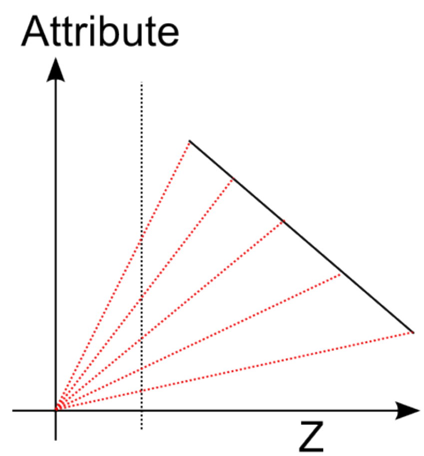
and let $P_t = (x_t, y_t, z_t)$ be the point of the segment $[P_1, P_2]$ that projects into P_t^*. Show that the value a_t of the attribute at point P_t satisfies

$$\frac{a_t}{z_t} = t\frac{a_1}{z_1} + (1-t)\frac{a_2}{z_2}\,.$$

Try to find a simple geometric proof to this fact.

It follows from this result, than when you are rasterizing a triangle, which was obtained via perspective projection, you cannot simply interpolate attribute values (e.g. colors or texture coordinates) along the screen as you did in the practice session[3].

Solution. Intuitively, the following diagram explains the idea: depict z on the horizontal axis and the attribute value on the vertical axis. The fact that screen coordinates vary linearly means that $\tan\alpha$ of the corresponding projection lines must vary linearly. The value of $\tan\alpha$ for each line is, however, exactly a_i/z_i.



A more formal proof is given in Lengyel's book, Chapter 4.4.

[3]http://en.wikipedia.org/wiki/Texture_mapping#Perspective_correctness