MTAT.03.015 Computer Graphics (Fall 2013)

Sample solutions to math exercises

Konstantin Tretyakov

1. Let s be a straight line in $\mathbb{R}^2$, passing through the origin. It can be described parametrically as

$$\mathbf{x} = \lambda \mathbf{s}, \qquad \lambda \in \mathbb{R},$$

or implicitly as

$$\mathbf{n}^T \mathbf{x} = 0\,.$$

Express the coordinates of the normal vector $\mathbf{n}$ via the coordinates of the direction vector $\mathbf{s}$.

Solution. Fix a vector $\mathbf{s}$ and consider a set of points $\{\mathbf{x} := \lambda\mathbf{s},\ \lambda \in \mathbb{R}\}$. The task is to find a vector $\mathbf{n}$, such that for any λ the condition

$$\mathbf{n}^T(\lambda\mathbf{s}) = 0,$$

is satisfied, and, vice-versa, if, for some $\mathbf{x}$ it holds that $\mathbf{n}^T\mathbf{x} = 0$ then it is necessarily true that $\mathbf{x} = \lambda\mathbf{s}$.

Necessity: For simplicity, assume that[1] $s_1 \neq 0$. Fix any $\lambda \neq 0$. Then,

$$\begin{aligned} \mathbf{n}^T(\lambda\mathbf{s}) &= 0, \\ \mathbf{n}^T\mathbf{s} &= 0, \\ n_1 s_1 + n_2 s_2 &= 0, \\ n_1 &= -n_2 s_2 / s_1. \end{aligned}$$

Hence, if we pick any $t \in \mathbb{R}$ and construct $\mathbf{n}$ as

$$\mathbf{n} = \begin{pmatrix} -t s_2/s_1 \\ t \end{pmatrix}$$

then all points of the form $\lambda\mathbf{s}$ will also satisfy $\mathbf{n}^T\mathbf{x} = 0$.

[1] If it is not the case we can assume $s_2 \neq 0$ and proceed with the proof in the same way. If both $s_1 = s_2 = 0$ we must treat this as a special case and demonstrate that no matching $\mathbf{n}$ exists then.

Multiplying both sides by s_1 produces a somewhat more conventional answer:

$$\mathbf{n} = t \begin{pmatrix} -s_2 \\ s_1 \end{pmatrix}.$$

Sufficiency: We complete the proof by showing that for $t \neq 0$ any $\mathbf{x}$ that satisfies $\mathbf{n}^T\mathbf{x} = 0$ (when $\mathbf{n}$ is chosen as shown above) is also of the form $\mathbf{x} = \lambda\mathbf{s}$:

$$\begin{aligned}
\mathbf{n}^T\mathbf{x} &= 0, \\
n_1x_1 + n_2x_2 &= 0, \\
-ts_2x_1 + ts_1x_2 &= 0, \\
x_2 &= x_1s_2/s_1, \\
\mathbf{x} &= \begin{pmatrix} x_1 \\ x_1s_2/s_1 \end{pmatrix} = \left(\frac{x_1}{s_1}\right) \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = \lambda\mathbf{s}.
\end{aligned}$$

Consequently, for any direction vector $\mathbf{s}$ the corresponding normal indeed exists and must be of the form

$$\mathbf{n} = t \begin{pmatrix} -s_2 \\ s_1 \end{pmatrix}, \, t \neq 0.$$

2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be points in $\mathbb{R}^2$. Find the coordinates of the intersection point of segments $[\mathbf{a}, \mathbf{b}]$ and $[\mathbf{c}, \mathbf{d}]$. Hint: Use the parametric representation.

Solution. The set of all points lying on the first segment can be parametrically described as

$$\{\mathbf{x} := t\mathbf{a} + (1-t)\mathbf{b}, \, t \in [0,1]\}.$$

Analogously, the set of point on the second segment is

$$\{\mathbf{x} := s\mathbf{c} + (1-s)\mathbf{d}, \, s \in [0,1]\}.$$

The intersection point must belong to both sets, and hence can be found by solving

$$t\mathbf{a} + (1-t)\mathbf{b} = s\mathbf{c} + (1-s)\mathbf{d}.$$

This is a system of two equations with two unknowns that can be solved using conventional means. Here is a more elegant solution by Raimond-

Hendrik:

$$
\begin{aligned}
\mathbf{b}+t(\mathbf{a}-\mathbf{b}) &= \mathbf{c}+s(\mathbf{c}-\mathbf{d}),\\
&\text{Box product on both sides with } (\mathbf{c}-\mathbf{d}),\\
[(\mathbf{b}+t(\mathbf{a}-\mathbf{b}))\times(\mathbf{c}-\mathbf{d})] &= [(\mathbf{c}+s(\mathbf{c}-\mathbf{d}))\times(\mathbf{c}-\mathbf{d})],\\
&\text{Linearity of the box product,}\\
[\mathbf{b}\times(\mathbf{c}-\mathbf{d})]+t[(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{d})] &= [\mathbf{c}\times(\mathbf{c}-\mathbf{d})]+s[(\mathbf{c}-\mathbf{d})\times(\mathbf{c}-\mathbf{d})],\\
&\text{Box product of a vector with itself is 0,}\\
[\mathbf{b}\times(\mathbf{c}-\mathbf{d})]+t[(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{d})] &= [\mathbf{c}\times(\mathbf{c}-\mathbf{d})],\\
t &= \frac{[\mathbf{c}\times(\mathbf{c}-\mathbf{d})]-[\mathbf{b}\times(\mathbf{c}-\mathbf{d})]}{[(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{d})]}\\
t &= \frac{[(\mathbf{c}-\mathbf{b})\times(\mathbf{c}-\mathbf{d})]}{[(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{d})]}.
\end{aligned}
$$

Now if the denominator and the numerator are 0, the segments lie on the same line. In this case we check whether the endpoints of one segment are within the other. If the denominator is 0 and the numerator is not, the segments are parallel and do not intersect. If both are non-zero, it remains to check whether $t \in [0,1]$ and if so, find the corresponding point as

$$t\mathbf{a}+(1-t)\mathbf{b}.$$

3. Prove that the (Euclidean) norm $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$ satisfies the *triangle inequality*:

$$\|\mathbf{x}+\mathbf{y}\| \le \|\mathbf{x}\|+\|\mathbf{y}\|.$$

Derive from this inequality also the inequalities

$$\|\mathbf{x}\|-\|\mathbf{y}\| \le \|\mathbf{x}-\mathbf{y}\| \le \|\mathbf{x}\|+\|\mathbf{y}\|.$$

Solution. First part:

$$
\begin{aligned}
\|\mathbf{x}+\mathbf{y}\| &\le \|\mathbf{x}\|+\|\mathbf{y}\|,\\
\|\mathbf{x}+\mathbf{y}\|^2 &\le \|\mathbf{x}\|^2+\|\mathbf{y}\|^2+2\|\mathbf{x}\|\|\mathbf{y}\|,\\
(\mathbf{x}+\mathbf{y})^T(\mathbf{x}+\mathbf{y}) &\le \|\mathbf{x}\|^2+\|\mathbf{y}\|^2+2\|\mathbf{x}\|\|\mathbf{y}\|,\\
\mathbf{x}^T\mathbf{x}+\mathbf{y}^T\mathbf{x}+\mathbf{x}^T\mathbf{y}+\mathbf{y}^T\mathbf{y} &\le \|\mathbf{x}\|^2+\|\mathbf{y}\|^2+2\|\mathbf{x}\|\|\mathbf{y}\|,\\
\mathbf{x}^T\mathbf{x}+2\mathbf{x}^T\mathbf{y}+\mathbf{y}^T\mathbf{y} &\le \|\mathbf{x}\|^2+\|\mathbf{y}\|^2+2\|\mathbf{x}\|\|\mathbf{y}\|,\\
\|\mathbf{x}\|^2+2\mathbf{x}^T\mathbf{y}+\|\mathbf{y}\|^2 &\le \|\mathbf{x}\|^2+\|\mathbf{y}\|^2+2\|\mathbf{x}\|\|\mathbf{y}\|,\\
\mathbf{x}^T\mathbf{y} &\le \|\mathbf{x}\|\|\mathbf{y}\|.\\
\|\mathbf{x}\|\|\mathbf{y}\|\cos\alpha &\le \|\mathbf{x}\|\|\mathbf{y}\|.
\end{aligned}
$$

Second part:

$$\begin{aligned}
\|(\mathbf{x}-\mathbf{y})+\mathbf{y}\| &\leq \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}\| \\
\|\mathbf{x}\| &\leq \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}\| \\
\|\mathbf{x}\|-\|\mathbf{y}\| &\leq \|\mathbf{x}-\mathbf{y}\|
\end{aligned}$$

and

$$\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{x}+(-\mathbf{y})\| \leq \|\mathbf{x}\|+\|-\mathbf{y}\| = \|\mathbf{x}\|+\|\mathbf{y}\|$$

4. Let $\mathbf{p}$ and $\mathbf{q}$ be orthonormal vectors in $\mathbb{R}^3$. What transformation does the matrix $\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T$ correspond to? Prove it.

   Solution. Intuitively, $\mathbf{p}\mathbf{p}^T$ is the orthogonal projector onto the axis defined by $\mathbf{p}$. Similarly $\mathbf{q}\mathbf{q}^T$ is the projector onto the axis defined by $\mathbf{q}$. Consequently, $\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T$ is an orthogonal projector onto the plane defined by $\mathbf{p}$ and $\mathbf{q}$.

   Formally, let $\mathbf{r}$ be a third vector, that makes up an orthogonal basis together with $\mathbf{p}$ and $\mathbf{q}$. Pick any vector $\mathbf{x}$. As $(\mathbf{p},\mathbf{q},\mathbf{r})$ forms an orthonormal basis, we can represent $\mathbf{x}$ in it, so let

   $$\mathbf{x} = x_1\mathbf{p}+x_2\mathbf{q}+x_3\mathbf{r}.$$

   Now apply transformation $\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T$ to $\mathbf{x}$:

   $$\begin{aligned}
   &(\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T)(x_1\mathbf{p}+x_2\mathbf{q}+x_3\mathbf{r}) = \\
   &\quad x_1(\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T)\mathbf{p}+x_2(\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T)\mathbf{q}+x_3(\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T)\mathbf{r} = \\
   &\qquad x_1\mathbf{p}+x_2\mathbf{q},
   \end{aligned}$$

   i.e. the transformation is indeed an orthogonal projector onto the **p-q** plane.

5. Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ be an orthonormal basis in $\mathbb{R}^3$. Prove that $\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T+\mathbf{r}\mathbf{r}^T=\mathbf{I}$, where $\mathbf{I}$ denotes a unit matrix.

   Solution. Same as above. Show that for any $\mathbf{x}=x_1\mathbf{p}+x_2\mathbf{q}+x_3\mathbf{r}$ the application of $\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T+\mathbf{r}\mathbf{r}^T$ leaves the vector intact. This can only be the case when the transformation is the identity matrix. Alternative solution would be to note that

   $$\mathbf{p}\mathbf{p}^T+\mathbf{q}\mathbf{q}^T+\mathbf{r}\mathbf{r}^T$$

   is equal to $\mathbf{A}^T\mathbf{A}$, where $\mathbf{A}$ is the matrix with $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ as the rows. It easily follows then that $\mathbf{A}^T\mathbf{A}=\mathbf{I}$.

6. Orthogonalize the following set of vectors using the Gram-Schmidt algorithm:

$$\mathbf{e}_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)^T$$
$$\mathbf{e}_2 = (-1, 1, -1)^T$$
$$\mathbf{e}_3 = (0, -2, -2)^T$$

Solution. Following the algorithm, pick the first vector as-is:

$$\mathbf{e}'_1 = \mathbf{e}_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)^T,$$

Next, $\mathbf{e}'_2$ is $\mathbf{e}_2$ minus its projection onto $\mathbf{e}'_1$. As $\mathbf{e}'_1$ and $\mathbf{e}_2$ are already orthogonal, this projection is zero, hence in our case

$$\mathbf{e}'_2 = \mathbf{e}_2 = (-1, 1, -1)^T,$$

Next, $\mathbf{e}'_3$ is $\mathbf{e}_3$ minus its projections onto $\mathbf{e}'_1$ and $\mathbf{e}'_2$. The vectors $\mathbf{e}'_2$ and $\mathbf{e}_3$ are already orthogonal, so we only need to subtract the projection onto $\mathbf{e}'_1$. Here we can further simplify by noting that $\|\mathbf{e}'_1\| = 1$:

$$\mathbf{e}'_3 = \mathbf{e}_3 - \mathbf{e}'_1\mathbf{e}'^T_1\mathbf{e}_3 = (0, -2, -2)^T - (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)^T \cdot (-\sqrt{2}) =$$
$$(0, -2, -2)^T + (1, 1, 0)^T = (1, -1, -2)^T$$

7. Compute the area of a triangle given by vertices

$$\mathbf{a} = (1, 2, 3)^T,$$
$$\mathbf{b} = (-2, 2, 4)^T,$$
$$\mathbf{c} = (7, -8, 0)^T.$$

Solution. Let $\mathbf{p} = \mathbf{b} - \mathbf{a}$ and $\mathbf{q} = \mathbf{c} - \mathbf{a}$. The area of the triangle is then simply half the length of the cross-product $\mathbf{p} \times \mathbf{q}$.

$$S = \frac{1}{2}\|\mathbf{p} \times \mathbf{q}\|.$$

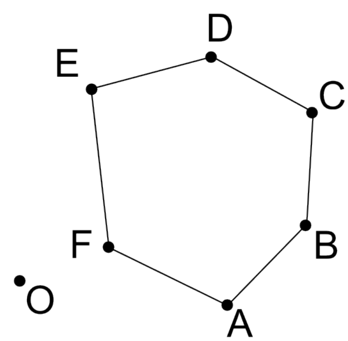
Computing it with the given numbers:

$$\mathbf{p} = (-3, 0, 1)^T$$
$$\mathbf{q} = (6, -10, -3)^T$$
$$\mathbf{p} \times \mathbf{q} = (10, -3, 30)^T$$
$$S = \frac{1}{2}\sqrt{10^2 + 3^2 + 30^2} \approx 15.88$$

8. Points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n \in \mathbb{R}^2$ are vertices of a simple polygon[2] listed in counter-clockwise order in a right-handed basis. Prove that the area of the polygon S can be computed as

$$S = \frac{1}{2}(|\mathbf{p}_1 \quad \mathbf{p}_2| + |\mathbf{p}_2 \quad \mathbf{p}_3| + \cdots + |\mathbf{p}_n \quad \mathbf{p}_1|)$$

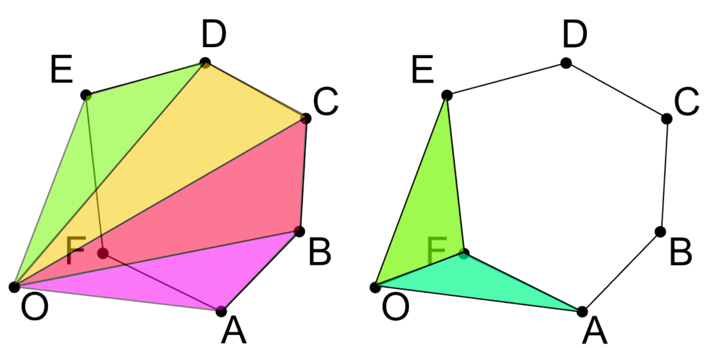
Solution. Recall that the expression $\frac{1}{2}|\mathbf{p} \quad \mathbf{q}|$ computes the signed area of the triangle $O\mathbf{pq}$, where O is the origin of the coordinate system. Now consider an example:



Note that the area ABCDEF can be computed as the difference between the area OABCDE = OAB+OBC+OCD+ODE, and the area OEFA = OEF + OFA. This is, however, exactly what the expression

$$S = \frac{1}{2}(|A \quad B| + |B \quad C| + |C \quad D| + |D \quad E| + |E \quad F| + |F \quad A|)$$

computes.



The construction applies for any simple polygon.

[2]A *simple polygon* is a polygon, whose edges do not intersect each other.

9. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function that satisfies $f(\mathbf{x}+\mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ for each $\mathbf{x}, \mathbf{y}$. Show that it then necessarily follows that for each $\alpha \in \mathbb{R}$ and each $\mathbf{x}$

$$f(\alpha\mathbf{x}) = \alpha f(\mathbf{x}),$$

i.e. f must be linear.

Solution.

(a) First show that the required condition holds for any $\alpha = n \in \mathbb{N}^+$:

$$f(n\mathbf{x}) = f(\mathbf{x}+\mathbf{x}+\cdots+\mathbf{x}) = f(\mathbf{x}) + f(\mathbf{x}) + \cdots + f(\mathbf{x}) = nf(\mathbf{x}).$$

(b) Now, for $\alpha = 0$:

$$f(0) = f(0+0) = f(0) + f(0).$$

which can only hold when $f(0) = 0$, i.e. $f(0\mathbf{x}) = 0f(\mathbf{x})$.

(c) For $\alpha = -1$:

$$0 = f(\mathbf{x}-\mathbf{x}) = f(\mathbf{x}) + f(-\mathbf{x}), \text{ hence } f(-\mathbf{x}) = -f(\mathbf{x}).$$

Consequently, condition holds for all $\alpha \in \mathbb{N}$.

(d) Now let $\alpha = \frac{1}{m}$ for $m \in \mathbb{N}$.

$$f(\mathbf{x}) = f(m\frac{1}{m}\mathbf{x}) = mf(\frac{1}{m}\mathbf{x}),$$

hence

$$f(\frac{1}{m}\mathbf{x}) = \frac{1}{m}f(\mathbf{x}).$$

(e) Now, combining results (c) and (d), we conclude that condition holds for any rational $\alpha \in \mathbb{Q}$:

$$f(\frac{n}{m}\mathbf{x}) = nf(\frac{1}{m}\mathbf{x}) = \frac{n}{m}f(\mathbf{x}).$$

(f) Finally, let $\alpha \in \mathbb{R}$. Consider a sequence of rational numbers α_i, that converges to α:

$$\lim_{i\to\infty} \alpha_i = \alpha$$

Then, combining (e) and continuity of f:

$$\begin{aligned} f(\alpha\mathbf{x}) &= f(\lim_i \alpha_i\mathbf{x}) = \lim_i f(\alpha_i\mathbf{x}) \\ &= \lim_i \alpha_i f(\mathbf{x}) = (\lim_i \alpha_i) f(\mathbf{x}) = \alpha f(\mathbf{x}). \end{aligned}$$

10. Consider a polyhedron with vertices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$. Let $\mathbf{n}_1$, $\mathbf{n}_2, \ldots, \mathbf{n}_l$ be the normals for the faces of the polyhedron. Let us apply a linear transformation $\mathbf{F}$ to all the vertices of the polyhedron. The vertices of the new polyhedron are thus $\mathbf{F}\mathbf{p}_1, \mathbf{F}\mathbf{p}_2, \ldots, \mathbf{F}\mathbf{p}_k$. Express the normals of the new polyhedron in terms of the original normals.

Solution. Consider a face of the polyhedron, that includes vertices $\mathbf{p}_1$ and $\mathbf{p}_2$. The normal to this face $\mathbf{n}_i$ must satisfy: $\mathbf{n}_i^T(\mathbf{p}_1 - \mathbf{p}_2) = 0$. Let $\mathbf{n}'_i$ be the transformed normal. After transformation, the normal should stay perpendicular to the transformed face, i.e. it must hold that $\mathbf{n}_i'^T(\mathbf{F}\mathbf{p}_1 - \mathbf{F}\mathbf{p}_2) = 0$.

Let $\mathbf{n}'_i = (\mathbf{F}^{-1})^T\mathbf{n}$. Then

$$
\begin{aligned}
\mathbf{n}_i'^T(\mathbf{F}\mathbf{p}_1 - \mathbf{F}\mathbf{p}_2) &= ((\mathbf{F}^{-1})^T\mathbf{n})^T\mathbf{F}(\mathbf{p}_1 - \mathbf{p}_2) \\
&= \mathbf{n}^T\mathbf{F}^{-1}\mathbf{F}(\mathbf{p}_1 - \mathbf{p}_2) \\
&= \mathbf{n}^T(\mathbf{p}_1 - \mathbf{p}_2) = 0.
\end{aligned}
$$

In fact it is easy to see that if we transform the normals using the matrix $(\mathbf{F}^{-1})^T$ all the angles between normals and faces are preserved. The matrix $(\mathbf{F}^{-1})^T$ is thus known as the *normal transformation matrix*.

11. Let the horizontal field of view (*fov-X*) of some *view-frustum* be 75 degrees. Let the screen dimensions be 1280×1024. Find the corresponding vertical field of view (*fov-Y*).

Solution. From

$$\frac{\tan(\text{fovX}/2)}{\tan(\text{fovY}/2)} = \frac{1280}{1024}$$

we compute

$$\text{fovY} = 2\arctan\left(\frac{\tan(75/2)}{1280/1024}\right) \approx 63.09$$

12. Consider a perspective projection in two-dimensional space. We shall be projecting to the line $y = 1$ with $(0, 0)$ as the center of projection.

    - Find the projection matrix in homogeneous coordinates.
    - Explain what linear transformation does this matrix correspond to in the three-dimensional homogeneous space. Illustrations are welcome.

Solution. First, observe that the necessary transformation is

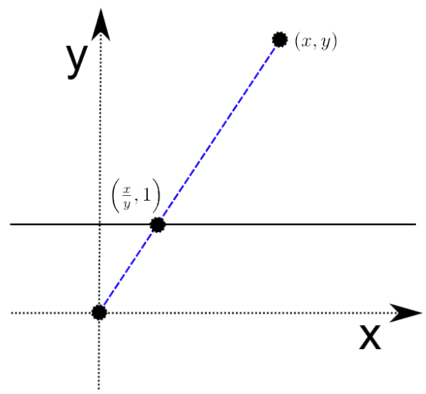
$$(x, y)^T \rightarrow (x/y, 1)^T$$

(see figure below). In homogeneous coordinates this would be

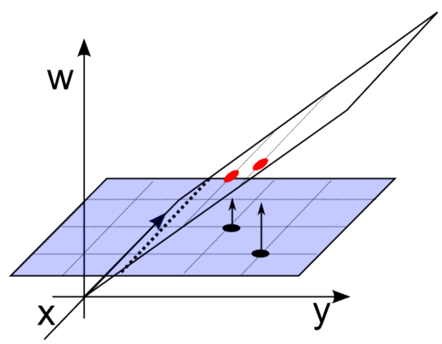
$$(x, y, 1)^T \rightarrow (x/y, 1, 1)^T = (x, y, y)^T,$$

which corresponds to the multiplication by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$



When viewed as a transformation $\mathbb{R}^3 \to \mathbb{R}^3$, this matrix acts as a projection onto the $(\mathbf{x}, \mathbf{y}+\mathbf{z})$ plane. That is, it takes points from the $y = 1$ plane and "raises" them up to $y = w$ plane. See illustration below. Think how perspective division after this transformation will lead all points back to the $w = 1$ plane and, coincidentally, also the $y = 1$ line.



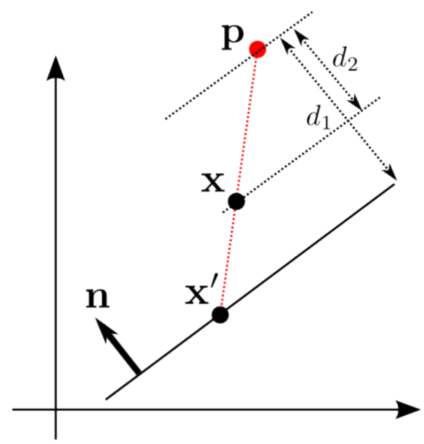
13. Let $ax + by + cz + d = 0$ be some plane in three-dimensional space and let $P = (p_x, p_y, p_z)$ be a point not located on this plane. Find a matrix, that performs a perspective projection from P onto this plane (in homogeneous coordinates).

Solution. Let

$$\begin{aligned} \mathbf{w} &:= (a, b, c, d), \\ k &:= (a^2 + b^2 + c^2)^{-1} \\ \mathbf{p} &:= (p_x, p_y, p_z, 1), \end{aligned}$$

Now due to the properties of implicit plane equation, for any point $\mathbf{x} = (x, y, z, 1)$ the signed distance from this point to the plane is equal to $k\mathbf{w}^T\mathbf{x}$. In particular, the distance from P to the plane is $k\mathbf{w}^T\mathbf{p}$.

Now observe that the transformation we seek is the following:



$$\mathbf{x}' = \mathbf{p} + (\mathbf{x} - \mathbf{p})\frac{d_1}{d_2} = \mathbf{p} + (\mathbf{x} - \mathbf{p})\frac{k\mathbf{w}^T\mathbf{p}}{k\mathbf{w}^T\mathbf{p} - k\mathbf{w}^T\mathbf{x}}$$

Regrouping and simplifying:

$$\begin{aligned}
(\mathbf{w}^T\mathbf{p} - \mathbf{w}^T\mathbf{x})\mathbf{x}' &= \mathbf{p}(\mathbf{w}^T\mathbf{p} - \mathbf{w}^T\mathbf{x}) + (\mathbf{x} - \mathbf{p})\mathbf{w}^T\mathbf{p} \\
&= \mathbf{p}\mathbf{w}^T\mathbf{p} - \mathbf{p}\mathbf{w}^T\mathbf{x} + \mathbf{x}\mathbf{w}^T\mathbf{p} - \mathbf{p}\mathbf{w}^T\mathbf{p} \\
&= (\mathbf{w}^T\mathbf{p})\mathbf{x} - \mathbf{p}\mathbf{w}^T\mathbf{x} \\
&= ((\mathbf{w}^T\mathbf{p})\mathbf{I} - \mathbf{p}\mathbf{w}^T)\mathbf{x}
\end{aligned}$$

Recall that in homogeneous coordinates the points $\mathbf{x}'$ and $\alpha\mathbf{x}'$ are considered equivalent (for $\alpha \neq 0$), hence the left side of the above equation is already a representation of $\mathbf{x}'$ (unless $(\mathbf{w}^T\mathbf{p} - \mathbf{w}^T\mathbf{x}) = 0$, i.e. the vector $\mathbf{p} - \mathbf{x}$ is perpendicular to the plane).

Consequently, the required transformation matrix in homogeneous coordinates is

$$(\mathbf{w}^T\mathbf{p})\mathbf{I} - \mathbf{p}\mathbf{w}^T.$$

It is straightforward to verify that the derivation holds for any $\mathbf{x}$ independently of what side of the plane the point is located on (note that this affects the sign of $\mathbf{w}^T\mathbf{x}$).

14. Let $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2)$ – be points in space. Consider some attribute $\mathcal{A}$ (e.g. color) assigned to the points. Suppose that point P_1 is assigned attribute value a_1, point P_2 — value a_2 and on the line between them the attribute varies linearly.

Let P_1^*, P_2^* — be the perspective projections of points P_1 and P_2 onto the plane $z = z_n$ with $(0,0,0)$ as the center of projection. Let P_t^* be a point obtained by interpolating between P_1^* and P_2^*:

$$P_t^* = tP_1^* + (1-t)P_2^*,$$

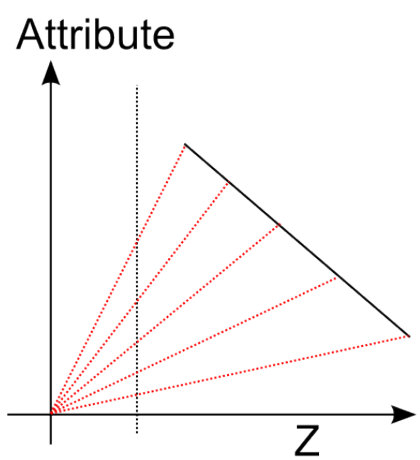
and let $P_t = (x_t, y_t, z_t)$ be the point of the segment $[P_1, P_2]$ that projects into P_t^*. Show that the value a_t of the attribute at point P_t satisfies

$$\frac{a_t}{z_t} = t\frac{a_1}{z_1} + (1-t)\frac{a_2}{z_2}\,.$$

Try to find a simple geometric proof to this fact.

It follows from this result, than when you are rasterizing a triangle, which was obtained via perspective projection, you cannot simply interpolate attribute values (e.g. colors or texture coordinates) along the screen as you did in the practice session[3].

Solution. Intuitively, the following diagram explains the idea: depict z on the horizontal axis and the attribute value on the vertical axis. The fact that screen coordinates vary linearly means that $\tan\alpha$ of the corresponding projection lines must vary linearly. The value of $\tan\alpha$ for each line is, however, exactly a_i/z_i.



A more formal proof is given in Lengyel's book, Chapter 4.4.

[3]http://en.wikipedia.org/wiki/Texture_mapping#Perspective_correctness

15. Bring examples of a two-piece linear spline curve, which happens to be:

    (a) C^1-smooth at the connection point.

    (b) G^1-smooth, but not C^1-smooth at the connection point.

    Solution.

    (a) Let the first piece be a segment between $(0,0)$ and $(1,0)$:

    $$\mathbf{p}(t) = (1-t)\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}t\\0\end{pmatrix}$$

    and the second piece – a segment between $(1,0)$ and $(2,0)$:

    $$\mathbf{q}(t) = (1-t)\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}2\\0\end{pmatrix} = \begin{pmatrix}t+1\\0\end{pmatrix}$$

    The gradient of the first piece at the endpoint $\mathbf{p}(1)$ is $(1,0)^T$, and is thus exactly equal to the gradient of the second piece at $\mathbf{q}(0)$, hence the combined curve is C^1 (in fact, C^∞)-smooth.

    (b) Like in the previous case, let the first piece be a segment between $(0,0)$ and $(1,0)$:

    $$\mathbf{p}(t) = (1-t)\begin{pmatrix}0\\0\end{pmatrix} + t\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}t\\0\end{pmatrix}.$$

    Let the second piece be a segment, continuing from $(1,0)$ in the same direction, but faster, having $(3,0)$ as the endpoint:

    $$\mathbf{q}(t) = (1-t)\begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}3\\0\end{pmatrix} = \begin{pmatrix}2t+1\\0\end{pmatrix}$$

    The gradient at $\mathbf{q}(0)$ is $\mathbf{q}'(0) = (2,0)^T = 2 \cdot \mathbf{p}'(1)$, i.e. the direction is the same, but the magnitude is different.

16. Construct the basis matrices for:

    (a) the linear Bézier' curve,

    (b) the quadratic Bézier' curve.

    Solution.

    (a) By definition, the basis functions for a linear Bézier' curve are

    $$B_0^{(1)}(t) = t^0(1-t)^{1-0} = (1-t),$$
    $$B_1^{(1)}(t) = t^1(1-t)^{1-1} = t,$$

    hence the basis matrix is

    $$\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}$$

(b) Analogously, the basis functions for a quadratic curve are

$$\begin{aligned} B_0^{(2)}(t) &= t^0(1-t)^{2-0} = 1 - 2t + t^2, \\ B_1^{(2)}(t) &= 2t^1(1-t)^{2-1} = 2t - 2t^2, \\ B_2^{(2)}(t) &= t^2(1-t)^{2-2} = t^2, \end{aligned}$$

hence the basis matrix is

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

17. Construct the basis matrices for:

(a) the linear Lagrange' curve,

(b) the quadratic Lagrange' curve (assume the parameter vector for the control points to be $t = (0, 0.5, 1)$).

Solution.

(a) Following the construction of the Lagrange' basis matrix presented on the lecture, let

$$\mathbf{A} = \begin{pmatrix} 0^0 & 1^0 \\ 0^1 & 1^1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

then the basis matrix is

$$\mathbf{M}_L = \mathbf{A}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

(b) Analogously, let

$$\mathbf{A} = \begin{pmatrix} 0^0 & 0.5^0 & 1^0 \\ 0^1 & 0.5^1 & 1^1 \\ 0^2 & 0.5^2 & 1^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0.25 & 1 \end{pmatrix}$$

hence the basis matrix is

$$\mathbf{M}_L = \mathbf{A}^{-1} = \begin{pmatrix} 1 & -3 & 2 \\ 0 & 4 & -4 \\ 0 & -1 & 2 \end{pmatrix}$$

18. Construct a one-dimensional quadratic Lagrange' curve defined by control points $(0, 1, 0)$. Provide the answer as a polynomial in t.

Solution. We seek for a parabola, that starts at 0 for $t = 0$, rises to 1 at $t = 0.5$ and then falls down to 0 by time point $t = 1$. Intuitively, it is easy to note that the polynomial must have $t = 0$ and $t = 1$ as its roots, hence it is of the form $\alpha t(t-1)$, where α can be found from the condition $\mathbf{p}(0.5) = 1$. Alternatively, you might recognize the "jumping" motion path familiar to you from several practice sessions.

A more methodical way would be to follow the standard $\mathbf{PMT}_2(t)$ formula together with the basis matrix derived in the previous exercise:

$$\mathbf{p}(t) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -3 & 2 \\ 0 & 4 & -4 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix} = 4t - 4t^2.$$

19. Consider the curve in the previous exercise. Convert it to the Bézier' representation. That is, find the Bézier' control points for exactly the same curve.

Solution. Intuitively, it is possible to note that

$$4t - 4t^2 = 2 \cdot (2(1-t)t) = 0 \cdot B_0^{(2)}(t) + 2 \cdot B_1^{(2)}(t) + 0 \cdot B_2^{(2)}(t),$$

hence the Bézier' control points for the same polynomial must be $(0, 2, 0)$.

A more methodical approach is to follow the conversion method suggested on the lecture,

$$\mathbf{P}_B = \mathbf{P}_L \mathbf{M}_L \mathbf{M}_B^{-1} = \begin{pmatrix} 0 & 2 & 0 \end{pmatrix}$$

20. Prove that degree n Bernstein polynomials $B_i^{(n)}, i \in \{0, \ldots, n\}$ sum to one, i.e.:

$$\sum_{i=0}^{n} B_i^{(n)}(t) = 1, \text{ for all } n \in \mathbb{N}, t \in [0, 1].$$

Hint: one way to show it is to note that the Bernstein polynomials are somehow related to a well-known probability distribution.

Solution. If you are familiar with probability theory, you could note that $B_i^{(n)}(t) = \Pr[x = i]$ where x is a random variable with a binomial distribution $\mathrm{Bin}(n, t)$. Consequently,

$$\sum_{i=0}^{n} B_i^{(n)}(t) = \sum_{i=0}^{n} \Pr[x = i] = \Pr[x \in \{0, \ldots, n\}] = 1.$$

More formally, we could use the binomial theorem:

$$1 = 1^n = (t + (1-t))^n = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} = \sum_{i=0}^{n} B_i^{(n)}(t).$$

21. A Hermite' curve is a reparameterization of the cubic Bézier' curve, that is specified by its start and end points $\mathbf{p}_0$, $\mathbf{p}_3$ and its direction vectors (i.e. gradients) $\mathbf{s}_0$, $\mathbf{s}_3$ at those points. In other words, the geometry matrix for the Hermite' curve is $\mathbf{G} = (\mathbf{p}_0, \mathbf{p}_3, \mathbf{s}_0, \mathbf{s}_3)$. Derive the basis matrix $\mathbf{M}_H$ of the Hermite curve.

Solution. Recall that

$$\begin{aligned} \mathbf{s}_0 &= 3(\mathbf{p}_1 - \mathbf{p}_0) \\ \mathbf{s}_3 &= 3(\mathbf{p}_3 - \mathbf{p}_2) \end{aligned}$$

Consequently, the relationship between the Bezier' and the Hermite geometry matrices is:

$$\begin{aligned} \mathbf{P}_H &= \begin{pmatrix} \mathbf{p}_0 & \mathbf{p}_3 & \mathbf{s}_0 & \mathbf{s}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{p}_0 & \mathbf{p}_3 & (3\mathbf{p}_1 - 3\mathbf{p}_0) & (3\mathbf{p}_3 - 3\mathbf{p}_2) \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{pmatrix} \\ &= \mathbf{P}_B \mathbf{M} \end{aligned}$$

Now, given a Bezier curve $\mathbf{p}(t) = \mathbf{P}_B \mathbf{M}_B \mathbf{T}_3(t)$, the equivalent Hermite representation would be

$$\mathbf{p}(t) = \mathbf{P}_B \mathbf{M}_B \mathbf{T}_3(t) = (\mathbf{P}_H \mathbf{M}^{-1}) \mathbf{M}_B \mathbf{T}_3(t) = \mathbf{P}_H (\mathbf{M}^{-1} \mathbf{M}_B) \mathbf{T}_3(t),$$

where the basis matrix is

$$\mathbf{M}^{-1} \mathbf{M}_B = \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

22. Prove that a cubic B-spline will pass a control point, if it is repeated three times.

Solution. Consider a segment of a cubic B-spline corresponding to four control points $\mathbf{p}_0, \mathbf{p}_0, \mathbf{p}_0, \mathbf{p}_3$ (i.e. the first point is repeated three times). The corresponding curve segment (when represented via blending functions) is:

$$\mathbf{p}(t) = \mathbf{p}_0(b_0(t) + b_1(t) + b_2(t)) + \mathbf{p}_3 b_3(t),$$

However, $b_3(0) = \frac{0^3}{6} = 0$, hence $(b_0(0) + b_1(0) + b_2(0))$ must be 1, and thus

$$\mathbf{p}(0) = \mathbf{p}_0 \cdot 1 + \mathbf{p}_3 \cdot 0 = \mathbf{p}_0,$$

i.e. the segment passes through $\mathbf{p}_0$ at $t = 0$.

Analogously, if the control points are $\mathbf{p}_0, \mathbf{p}_3, \mathbf{p}_3, \mathbf{p}_3$, then the curve is

$$\mathbf{p}(t) = \mathbf{p}_0 b_0(t) + \mathbf{p}_3(b_1(t) + b_2(t) + b_3(t)),$$

and as $b_0(1) = \frac{(1-1)^3}{6} = 0$,

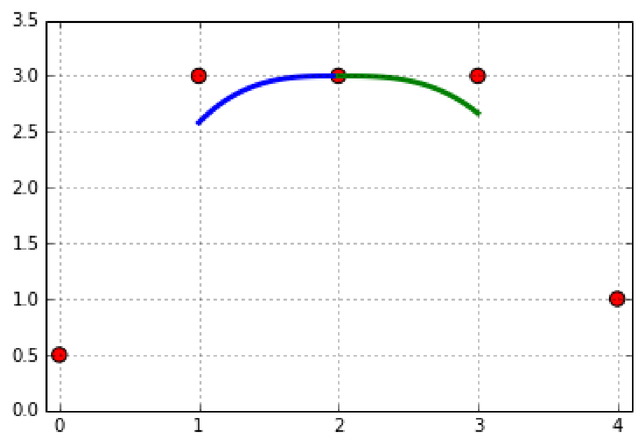
$$\mathbf{p}(1) = \mathbf{p}_3,$$

i.e. the curve passes through $\mathbf{p}_3$ at $t = 1$.

23. We derived the cubic B-spline to be a curve that is C^2-smooth at all points, no matter what the control points are. However, we also somewhy said that repeating control points "reduces smoothness" and even saw an example where repeating the control point three times results in a curve with a sharp corner. This seems like a contradiction. Explain this.

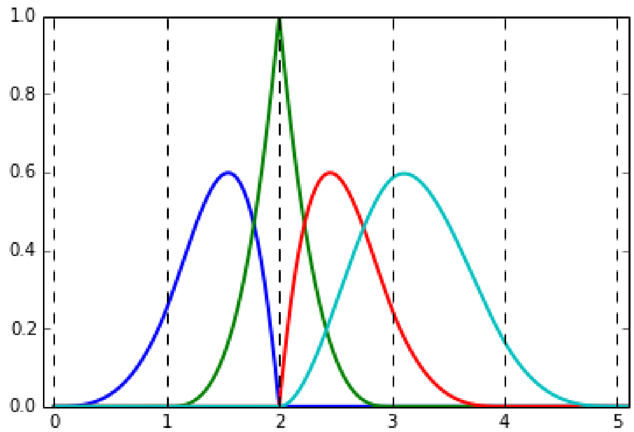
Solution. When a control point is repeated three times the curve will pass through it, which often results in visually sharp corners. Surprisingly, however, the curve function will still stay C^2-smooth. This happens because, in terms of movement, the curve "comes to a halt" at the repeated control point (its first and second derivatives become zero for a moment), and then gradually builds up speed in a different direction.

To see it better, consider a one-dimensional B-spline with control points $0.5, 3, 3, 3, 1$ as a function of t:



Note how the curve moves in a positive direction, reaches the point 3 at $t = 2$, halts there for a moment and starts moving backwards away from 3. If you visualize this movement as a one-dimensional curve viewed "from above", you might mistake this for a C^2-discontinuity.

Note that this makes the effect of repeating control points different from the effect of repeating knots in a non-uniform B-spline. In the latter case the basis functions really loose smoothness. See image below.



The basis functions of a four-control-point 3rd degree NURBS curve with knot vector $(0, 1, 2, 2, 2, 3, 4, 5)$. Note how the second basis function is not C^1-smooth at $t = 2$, and hence the whole curve will not be.

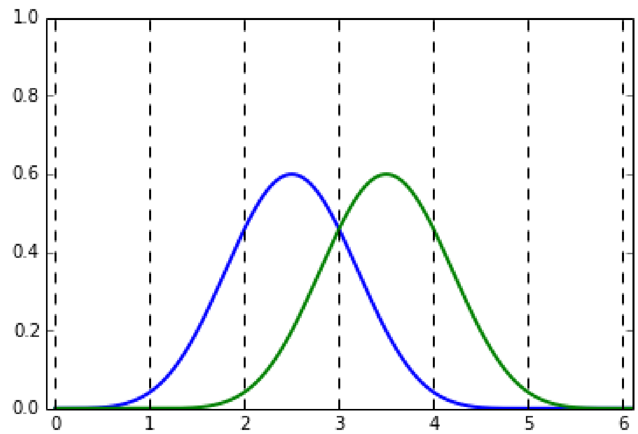
24. Prove that a (uniform) B-spline of degree k with n control points has $n + k + 1$ knots.

    Hint: count the number of curve segments and add the "virtual" segments on both sides, where the basis functions for the first and last control points are still nonzero.

    Solution. A single segment of a degree k B-spline, being a polynomial curve, is determined by $k + 1$ control points. Consequently, one control point affects $k + 1$ different segments of the spline, i.e. it is weighed by a basis function that must span $k + 2$ knots (the parameter values, corresponding to endpoints and connection points of $k + 1$ segments). This proves that for $n = 1$ the total number of knots is indeed $n + k + 1$.

    Now add a new control point. This introduces a new basis function. The domain of the added basis function includes k previous segments and 1 new (see illustration below). Consequently, one extra knot needs to be added for each new control point. For 2 points there will thus be $2+k+1$ knots, for 3 points $3 + k + 1$ knots, and the claim holds by induction.

    Alternatively, note that the parameter region where the basis function of the control point $\mathbf{p}_i$ is nonzero is $[i, i+k+2]$. Consequently, the parameter space where at least one basis function is nonzero starts at 0 (the leftmost end of the basis function for $\mathbf{p}_0$) and ends at $(n-1)+k+2$ (the rightmost end of the basis function for $\mathbf{p}_{n-1}$).



Two consecutive basis functions of a 4th degree B-spline. Each spans 5 segments and thus requires 6 knots. The two functions share a region of 4 common segments. The total number of knots is $7 = 4 + 2 + 1$.