Computer Graphics

Mathematical background: Transformations

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Sep 18, 2013

In the previous episode

- Vectors

In the previous episode

- Tools for working with vectors

In the previous episode

- Distances: _____
- Projections: _____
- Areas & Volumes: _____
- Perpendiculars: _____
- Orthogonalization: _____
- Represent straight line using:
  - _____
  - _____

In the previous episode

- Distances: **norm**
- Projections: **inner product**
- Areas & Volumes: **box product**
- Perpendiculars: **cross product**
- Orthogonalization: **inner/cross product**
- Represent straight line using:
  - **Linear combinations (parametric)**
  - **Inner product (implicit)**

Quiz

- Derive an implicit representation for a two-dimensional line using the box product.

In the previous episode

- Distances: **norm**
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- Perpendiculars: **cross product**
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- Represent straight line using:
  - **Linear combinations (parametric)**
  - **Inner product (implicit)**

In the previous episode

- Distances: **norm**
- Projections: **inner product**
- Areas & Volumes: **box product**
- Perpendiculars: **cross product**
- Orthogonalization: **inner/cross product**
- Represent straight line using:
  - **Linear combinations** **(parametric)**
  - **Inner product** **(implicit)**

(Bi)linear operations

(Bi)linearity

$$\langle \alpha \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle$$

$$|\alpha \boldsymbol{x} + \boldsymbol{y} \quad \boldsymbol{z}| = \alpha |\boldsymbol{x} \ \ \boldsymbol{z}| + |\boldsymbol{y} \ \ \boldsymbol{z}|$$

$$(\alpha \boldsymbol{x} + \boldsymbol{y}) \times \boldsymbol{z} = \alpha (\boldsymbol{x} \times \boldsymbol{z}) + (\boldsymbol{y} \times \boldsymbol{z})$$

$$\boldsymbol{A}(\alpha \boldsymbol{x} + \boldsymbol{y}) = \alpha \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{y}$$

(Bi)linearity

$$f(x+y) = f(x) + f(y)$$

$$f(\alpha x) = \alpha f(x)$$

Today

Linear transformations

http://www.thinkgeek.com/

Quiz

- Which of those are **not** linear transformations?
  - $f(x) = x$
  - $f(x) = -4x$
  - $f(x) = 4x + 4$
  - $f(x) = x^2$
  - $f(x) = 3$
  - $f(x) = 0$

Quiz

- Which of those are **not** linear transformations?
  - $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$
  - $f(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{x}$
  - $f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x}$
  - $f(\boldsymbol{x}) = |\boldsymbol{a}\ \ \boldsymbol{b}\ \ \boldsymbol{x}|$
  - $f(\boldsymbol{x}) = \boldsymbol{a} \times \boldsymbol{x}$
  - $f(x) = \boldsymbol{a}^T\boldsymbol{x} + |\boldsymbol{a}\ \ \boldsymbol{b}\ \ \boldsymbol{x}| + \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{A}\boldsymbol{x}$

Linear transformations

- Each linear transformation is uniquely defined by how it transforms the basis:

$$f\begin{pmatrix}2\\-3\\4\end{pmatrix}=$$

Linear transformations

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$$f\begin{pmatrix}2\\-3\\4\end{pmatrix} = f\left(2\begin{pmatrix}1\\0\\0\end{pmatrix} - 3\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

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$$= 2f\begin{pmatrix}1\\0\\0\end{pmatrix} - 3f\begin{pmatrix}0\\1\\0\end{pmatrix} + 4f\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Linear transformations

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$$= 2\boldsymbol{f}_1 - 3\boldsymbol{f}_2 + 4\boldsymbol{f}_3$$

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$$= |\boldsymbol{f}_1 \ \boldsymbol{f}_2 \ \boldsymbol{f}_3|\begin{pmatrix}2\\-3\\4\end{pmatrix}$$

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$$f\begin{pmatrix}2\\-3\\4\end{pmatrix} = f\left(2\begin{pmatrix}1\\0\\0\end{pmatrix} - 3\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$= \boldsymbol{F}\begin{pmatrix}2\\-3\\4\end{pmatrix}$$

Linear transformations

Each linear transformation corresponds to a matrix.

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Columns of a matrix show how it transforms the canonical basis

Quiz

- How does this matrix transform the (canonical) basis?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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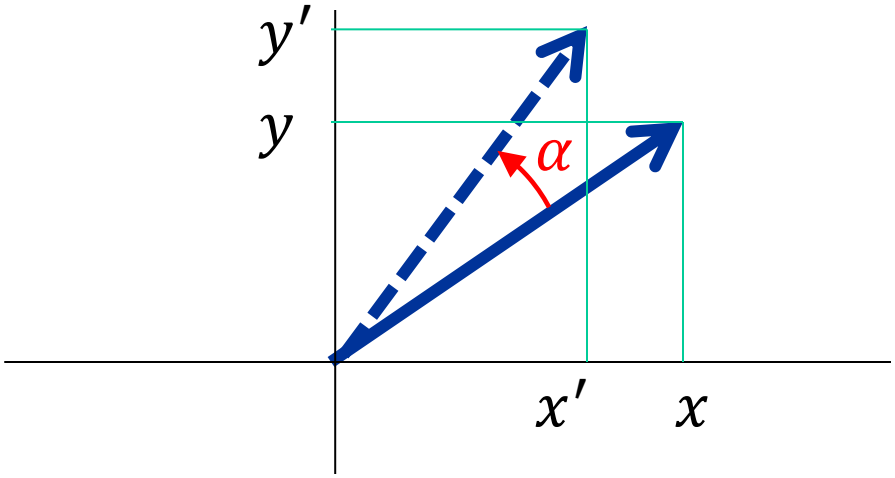
$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Quiz

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

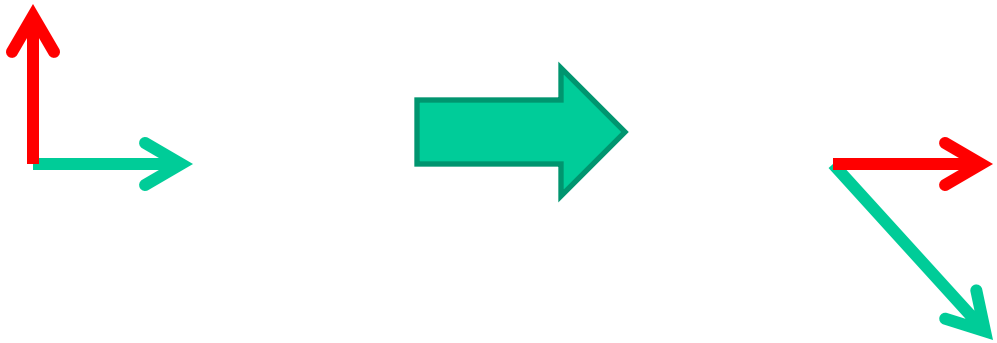
- Let (x, y) be a 2D vector

  Let (x', y') be obtained from (x, y) via rotation by angle α. Express x' and y' in terms of x and y.



Quiz

- Which matrix does the following?



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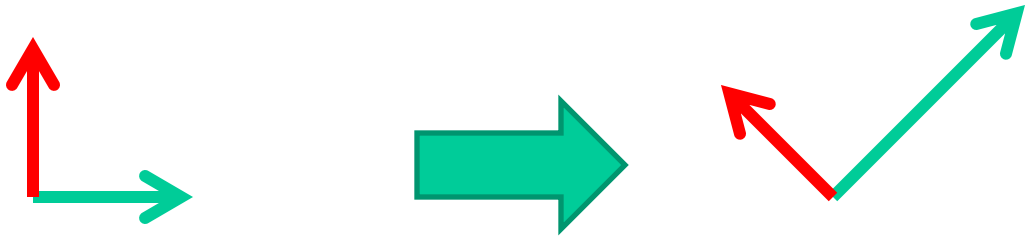

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Composition of transformations corresponds to matrix multiplication:

$$(f \circ g)(\boldsymbol{x}) = f\big(g(\boldsymbol{x})\big) = \boldsymbol{FGx}$$

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Function composition is associative, hence matrix multiplications is too:

$$(f \circ g) \circ h = f \circ (g \circ h)$$
$$(\boldsymbol{FG})\boldsymbol{H} = \boldsymbol{F}(\boldsymbol{GH})$$

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Sum of transformations corresponds to matrix sum:

$$(f + g)(\boldsymbol{x}) = f(\boldsymbol{x}) + g(\boldsymbol{x}) = (\boldsymbol{F} + \boldsymbol{G})\boldsymbol{x}$$

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Composition is distributive wrt sum:

$$(f + g) \circ h = f \circ h + g \circ h$$
$$(\boldsymbol{F} + \boldsymbol{G})\boldsymbol{H} = \boldsymbol{F}\boldsymbol{H} + \boldsymbol{G}\boldsymbol{H}$$

Rank

- Consider a linear transformation $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ it will always either:
  - Map the whole 3D space to itself somehow
  - Project the whole 3D space to a plane
  - Project the whole 3D space to a line
  - Map all points to 0.

Rank

- The dimensionality of the resulting space is the ***rank*** of f.
  - If f is full rank (i.e. $\text{rank}(f) = 3$ in our case), it is *invertible*. Otherwise it is not.
  - f is invertible $\Leftrightarrow \det(\boldsymbol{F}) \neq 0$

Orthogonal transformations

- A transformation $\boldsymbol{F}$ is called orthogonal if it maps the canonical basis into an **orthonormal basis**.

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  - $\boldsymbol{F}^T \boldsymbol{F} = ?$

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  - It must keep lengths and angles intact, i.e. it is always a **rotation** (possibly mirrored).
  - $\boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{I}$, because the columns are orthonormal
  - Hence, $\boldsymbol{F}^{-1} = \boldsymbol{F}^T$

Orthogonal transformations

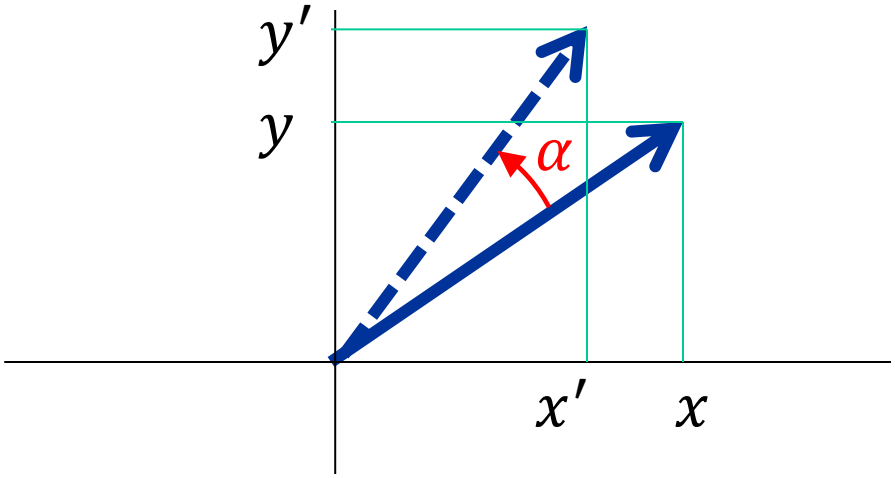
To compute the inverse of an orthogonal matrix, simply transpose it.

Quiz

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

- Let (x, y) be a 2D vector

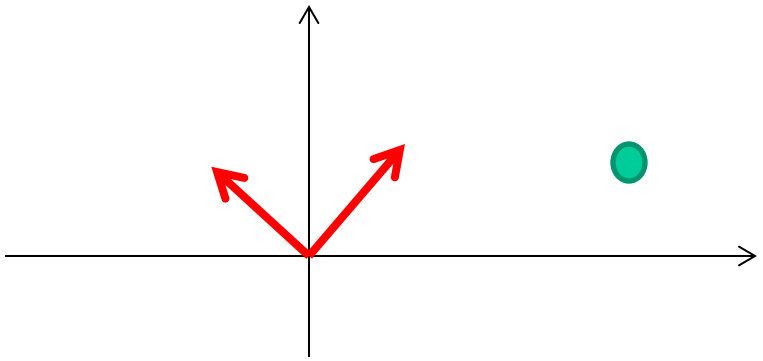
  Let (x', y') be obtained from (x, y) via rotation by angle α. Express x and y in terms of x' and y'.



Quiz

- You are standing at the origin, rotated with respect to the coordinate system, looking in the direction $(0.6, 0.8)$ (your local "x" axis).

- At position $(7,2)$ there is an object. What are the coordinates of this object with respect to you?



Examples

- Rotation: $\boldsymbol{R}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$
- Scaling: $\boldsymbol{S}(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
- Mirroring: $\mathbf{Mir}_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Shear: $\mathbf{Sh_x}(\mathrm{a}) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

Examples

- Rotation around z axis:

$$\boldsymbol{R_z}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Rotation around y axis:

$$\boldsymbol{R_z}(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

Shift

- Shift (translation) is not a linear transformation.

- To deal with shifts we must introduce the notion of an *affine space* and *affine transformations*.

Affine space

- Vector space
- Affine space

Affine space

- Vector space
  - Vectors $\boldsymbol{v} \in \mathbb{R}^3$
  - Basis: $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$
  - Linear transformations $f(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{x}$
- Affine space

Affine space

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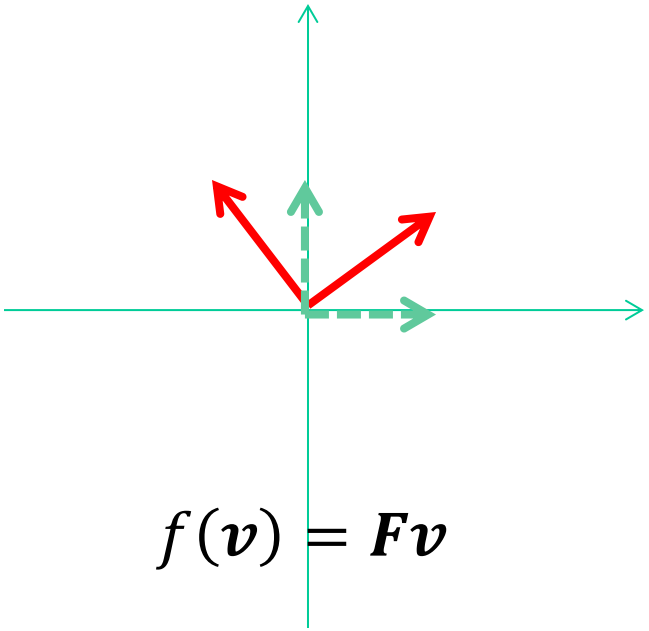
  - Vectors $\boldsymbol{v} \in \mathbb{R}^3$
  - Basis: $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$
  - Linear transformations $f(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{x}$
- Affine space
  - Vectors $\boldsymbol{v} \in \mathbb{R}^3$
  - Points $P \in \mathbb{R}^3$
    - point+vector = point
  - Frame: $(O, \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\})$
  - Affine transformations: $f(\boldsymbol{v}) = \boldsymbol{F}\boldsymbol{v}$ $f(P) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$

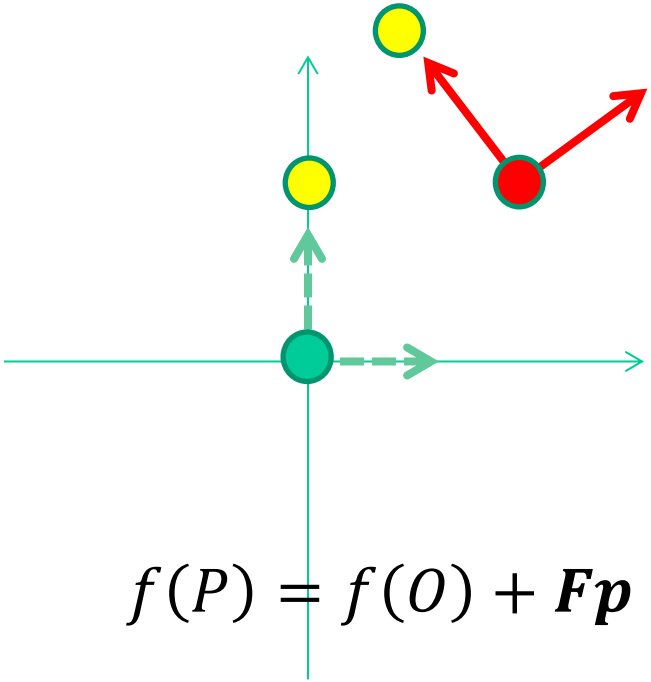
Affine space

- Vector space



$$f(\boldsymbol{v}) = \boldsymbol{F}\boldsymbol{v}$$

- Affine space



$$f(P) = f(O) + \boldsymbol{F}\boldsymbol{p}$$

Affine transformations

$$f(\boldsymbol{p}) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$$

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$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Affine transformations

$$f(\boldsymbol{p}) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$$

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$$\begin{pmatrix} q_1 \\ q_2 \\ 1 \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & t_1 \\ f_{21} & f_{22} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix}$$

Homogeneous coordinates

- We shall represent the **points** of an affine space using 3-dimensional vectors of the form $(p_1, p_2, 1)^T$
- We shall represent the **vectors** of an affine space using 3-dimensional vectors of the form $(v_1, v_2, 0)^T$
- Any affine transformation is a matrix

$$\left(\begin{array}{cc|c} f_{11} & f_{12} & t_1 \\ f_{21} & f_{22} & t_2 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Homogeneous coordinates

- Analogously, for 3D space we use 4-dimensional vectors and 4x4 matrices.
- E.g. the following transformation rotates around z axis and shifts along x axis by 0.5:

$$\left(\begin{array}{ccc|c} \cos\phi & -\sin\phi & 0 & 0.5 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Homogeneous coordinates

- Note how the representation implicitly enforces the rules:
  - vector + vector = vector
  - point + vector = point
  - point + point = undefined
  - convex combination of points = point

Homogeneous coordinates

- Rotation: $\boldsymbol{R}(\alpha) = \left(\begin{array}{cc|c} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$
- Scaling: $\boldsymbol{S}(a,b) = \left(\begin{array}{cc|c} a & 0 & 0 \\ 0 & b & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$
- Translation: $\boldsymbol{T}(x,y) = \left(\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ \hline 0 & 0 & 1 \end{array}\right)$

Quiz

- Construct a matrix, that performs a rotation by 10 degrees around the point (20, 30) in homogeneous coordinates.

Quiz

- Construct a matrix, that performs a rotation by 10 degrees around the point (20, 30) in homogeneous coordinates.

$$\boldsymbol{T}(20{,}30)\boldsymbol{R}(10)\boldsymbol{T}(-20,-30)$$

Quiz

- How to construct a matrix, that performs a rotation (in 3D) by 10 degrees around the axis given by the direction vector $(1, 2, 3)$

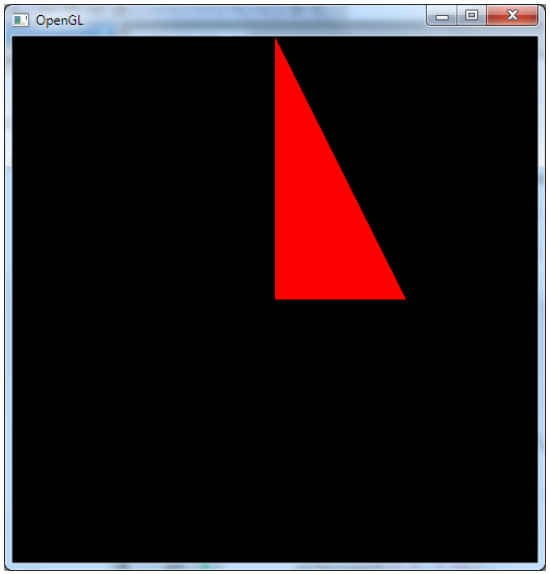
Mathematical background

- Matrices:
  - Linear transformations
  - Invertibility, rank, determinant
  - Orthogonal transformations
  - Affine transformations
  - Homogeneous coordinates

OpenGL Example (v1.0 - 3.0)

- The following code draws a triangle with vertices $(0, 0)^T$, $(0.5, 0)^T$, $(0, 1)^T$.

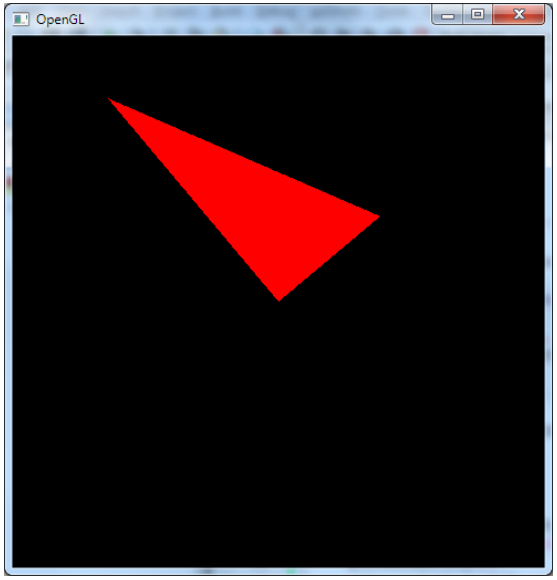
```
glBegin(GL_TRIANGLES);
     glVertex2f(0.0, 0.0);
     glVertex2f(0.5, 0.0);
     glVertex2f(0.0, 1.0);
glEnd();
```



OpenGL Example

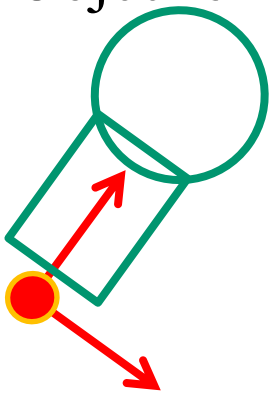
- The following code draws the same triangle, rotated by 40 degrees.

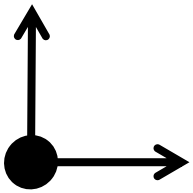
```
glRotatef(40, 0.0, 0.0, 1.0);
glBegin(GL_TRIANGLES);
      glVertex2f(0.0, 0.0);
      glVertex2f(0.5, 0.0);
      glVertex2f(0.0, 1.0);
glEnd();
```



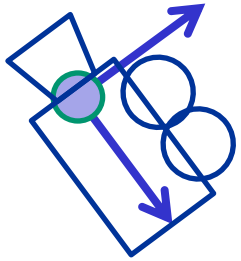
World/Object/Camera frames

Object's frame





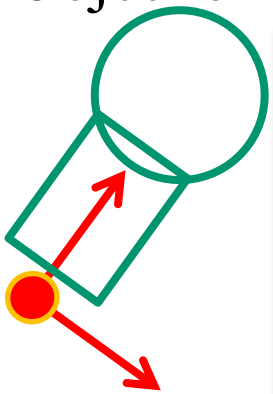
World frame



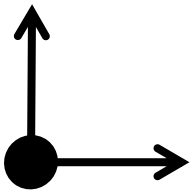
Camera frame

World/Object/Camera frames

Object's frame



The object is described in its own frame using vertices $p_1, p_2, \ldots$



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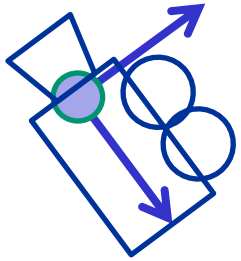
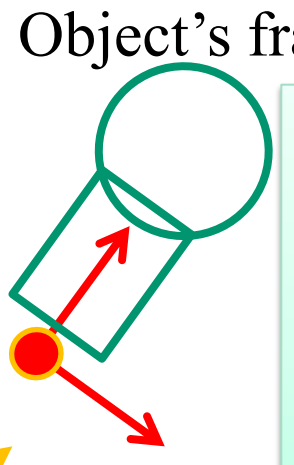


Camera frame

World/Object/Camera frames

The position of the object's frame wrt the world frame is given by the (affine) *modeling transform* matrix $\boldsymbol{M}$.



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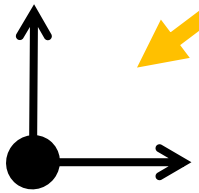

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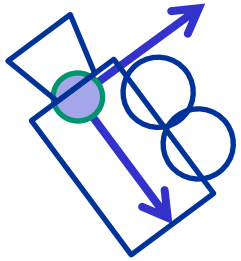

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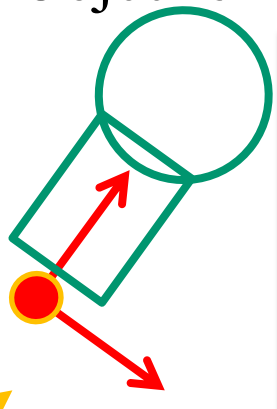
World/Object/Camera frames

Object's frame

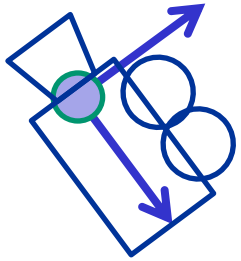


The position of the object's frame wrt the world frame is given by the (affine) *modeling transform* matrix $\boldsymbol{M}$.

The object is described in its own frame using vertices $\boldsymbol{p_1}, \boldsymbol{p_2}, \ldots$.

In world coordinates, the object's vertices are therefore $\boldsymbol{Mp_1}, \boldsymbol{Mp_2}, \ldots$

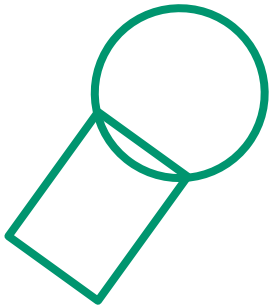
World frame

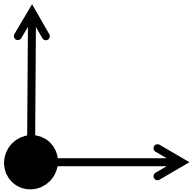


Camera frame

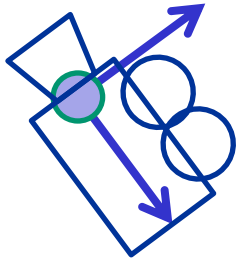
World/Object/Camera frames

In world coordinates, the object's vertices are therefore
$\boldsymbol{Mp_1}, \boldsymbol{Mp_2}, \ldots$





World frame



Camera frame

World/Object/Camera frames

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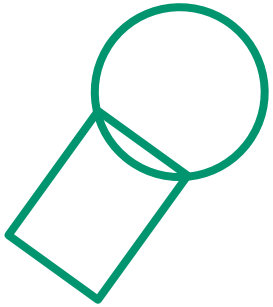
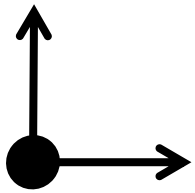
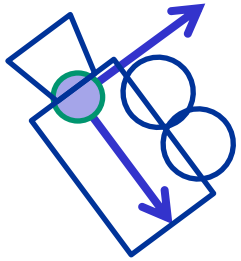


The world is observed via a camera.



World frame



Camera frame

World/Object/Camera frames

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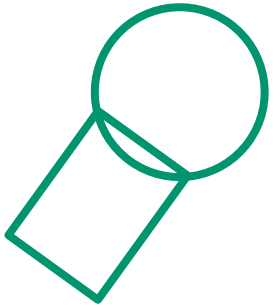


The world is observed via a camera.

Transformation from world coordinates to camera coordinates is given by the *view matrix* $\boldsymbol{V}$

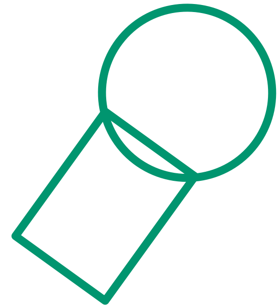
World frame

Camera frame

World/Object/Camera frames

In world coordinates, the object's vertices are therefore $\boldsymbol{Mp_1}, \boldsymbol{Mp_2}, \ldots.$

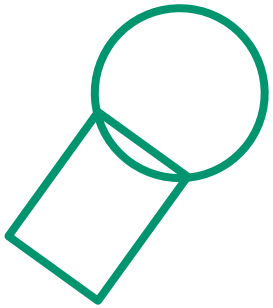
Object's vertices in camera coordinates are $\boldsymbol{VMp_1}, \boldsymbol{VMp_2}, \ldots .$



The world is observed via a camera.

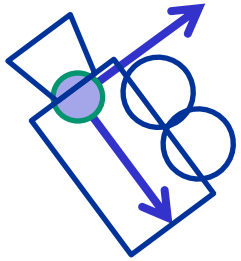
Transformation from world coordinates to camera coordinates is given by the *view matrix* $\boldsymbol{V}$

World/Object/Camera frames



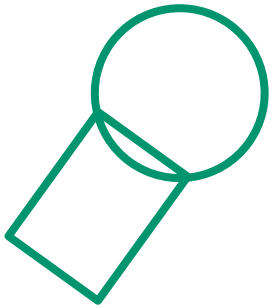
Object's vertices in camera coordinates are

$$\boldsymbol{VMp_1}, \boldsymbol{VMp_2}, \ldots.$$



Camera frame

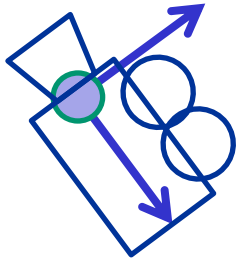
World/Object/Camera frames



Object's vertices in camera coordinates are

$VMp_1, VMp_2, \ldots.$

Model-view transform



Camera frame

Model-view matrix

- The model-view matrix $\mathcal{M}$ is part of OpenGL **state**.
- It is a 4x4 matrix (usually an affine transformation).
- Every vertex is automatically transformed using $\mathcal{M}$ before display.

Model-view matrix

- Whenever you write
  - `glVertex**(x, y, z)`
- The following conceptually takes place:

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} := \mathcal{M} \cdot \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

The point (x', y') is then used for 2D rasterization

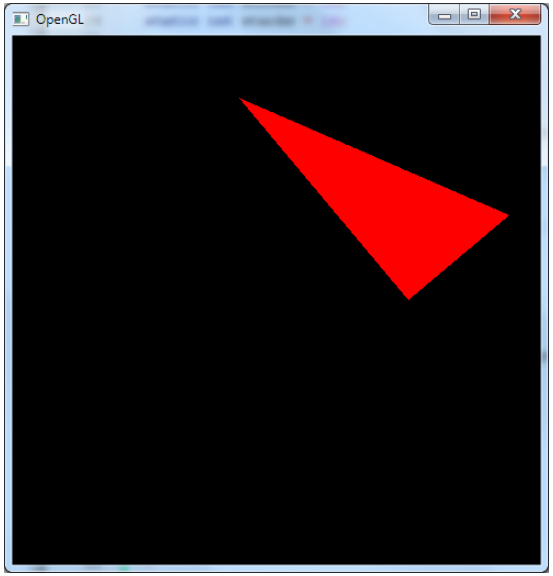
Model-view matrix

- The Model-view matrix can be provided explicitly
  - `glLoadMatrix*(. . . . . .)`
- Or, more commonly, constructed by multiplying with elementary matrices **on the right**.
  - `glLoadIdentity();` $\mathcal{M} = I$
  - `glTranslatef(…);` $\mathcal{M} = IT$
  - `glRotatef(…);` $\mathcal{M} = ITR$

OpenGL Example

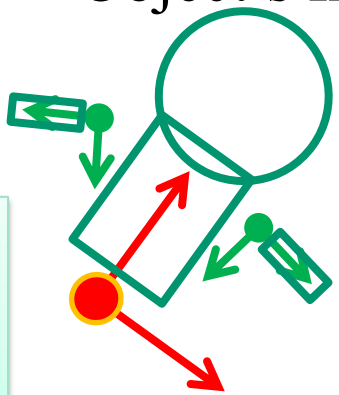
- The following code draws the same triangle, rotated by 40 degrees **and then** translated by 0.5 along x axis.

```
glTranslatef(0.5, 0.0, 0.0);
glRotatef(40, 0.0, 0.0, 1.0);
glBegin(GL_TRIANGLES);
     glVertex2f(0.0, 0.0);
     glVertex2f(0.5, 0.0);
     glVertex2f(0.0, 1.0);
glEnd();
```

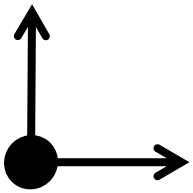


Scene graph

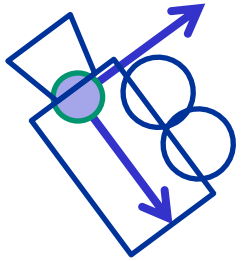
Object's frame



Objects usually consist of multiple parts, each described in their own frame



World frame



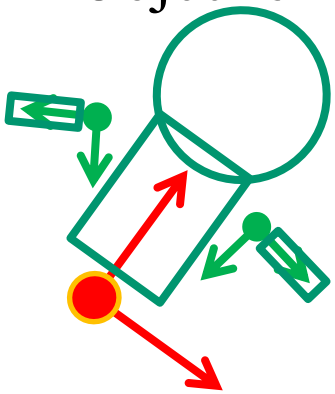
Camera frame

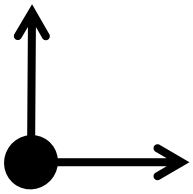
Scene graph

Each sub-object is first transformed to the frame of its parent:

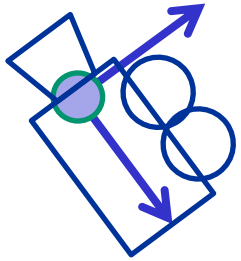
- Left arm: $\boldsymbol{M}_{left \rightarrow body}$
- Right arm: $\boldsymbol{M}_{right \rightarrow body}$

Object's frame





World frame



Camera frame

Scene graph

Each sub-object is first transformed to the frame of its parent:

- Left arm: $\boldsymbol{M}_{left \rightarrow body}$
- Right arm: $\boldsymbol{M}_{right \rightarrow body}$

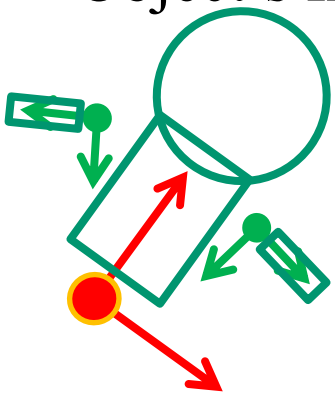
The whole object is then transformed to world

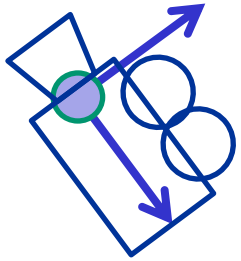
$$\boldsymbol{M}_{body \rightarrow world}$$

and finally to camera frame:

$$\boldsymbol{V}$$

Object's frame





Camera frame

Scene graph

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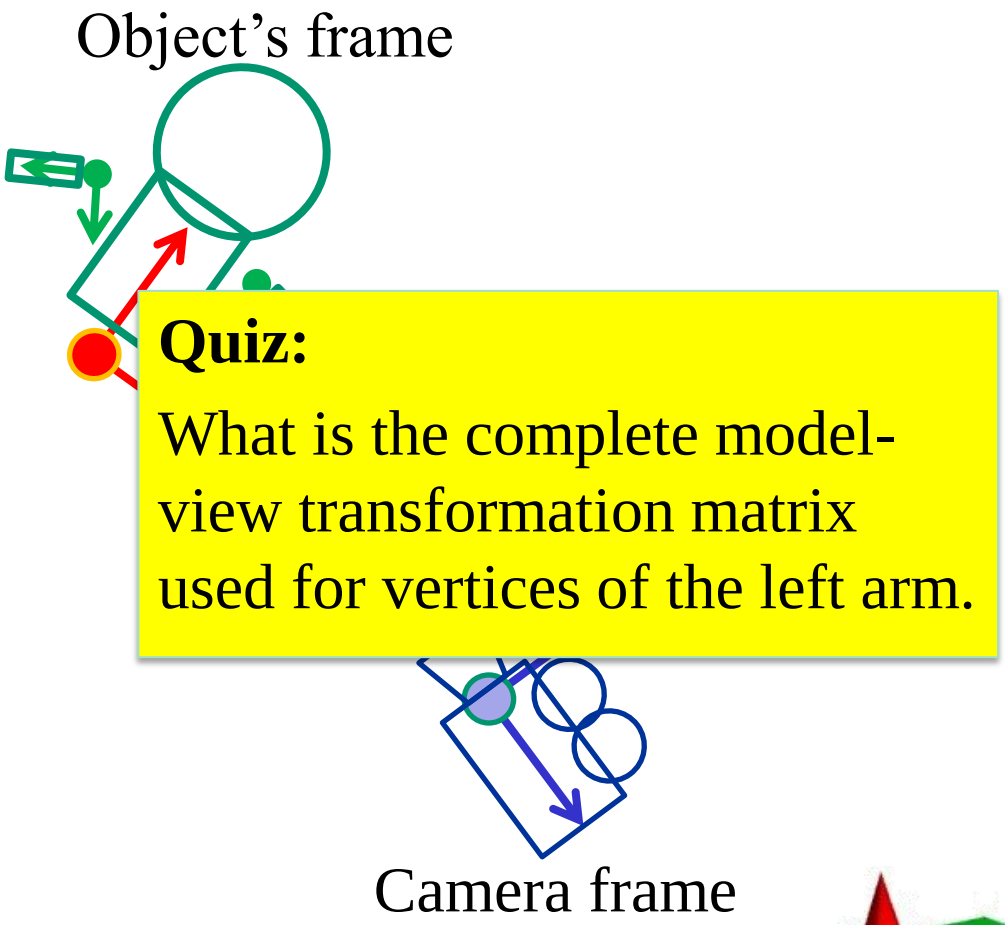

Scene graph

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- Left arm: $\boldsymbol{M}_{left \rightarrow body}$
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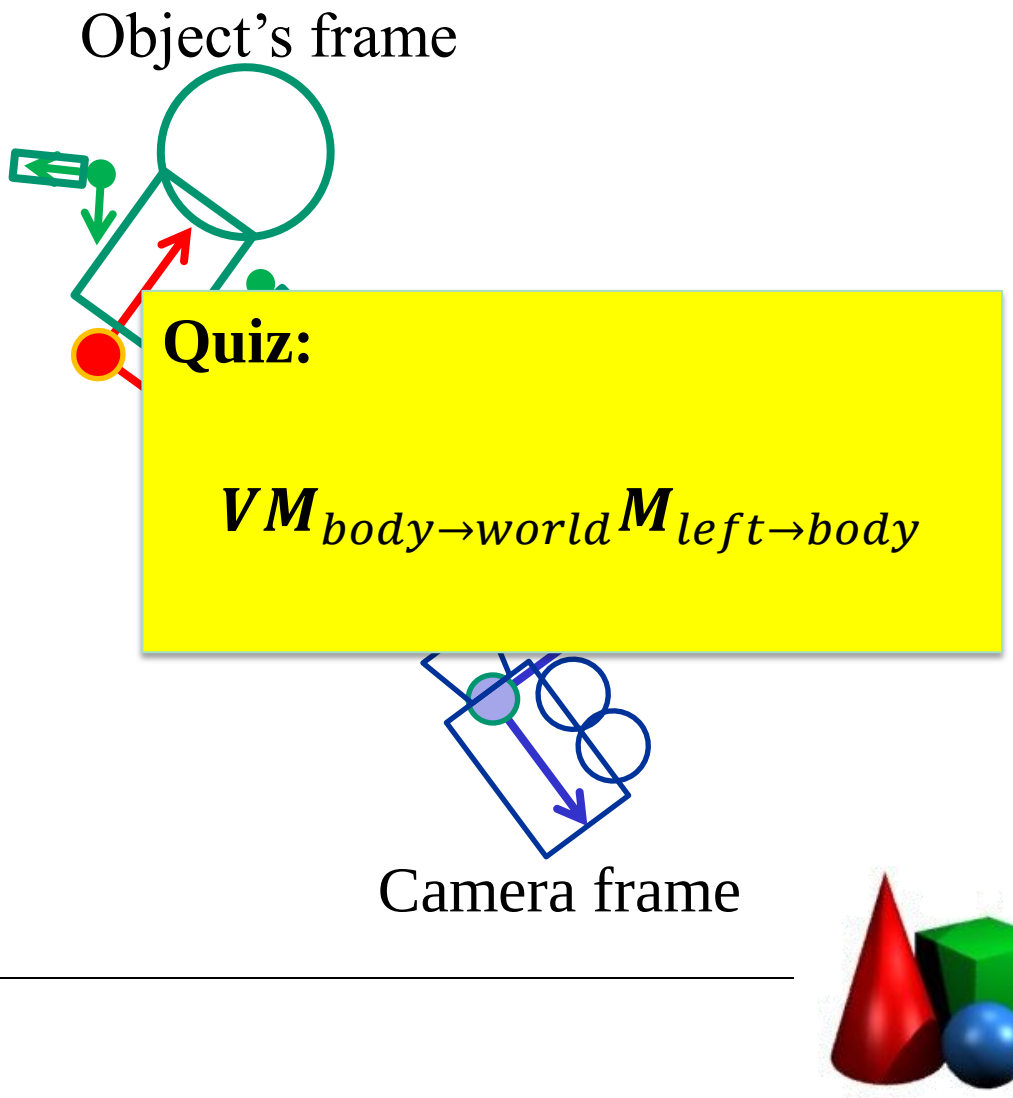
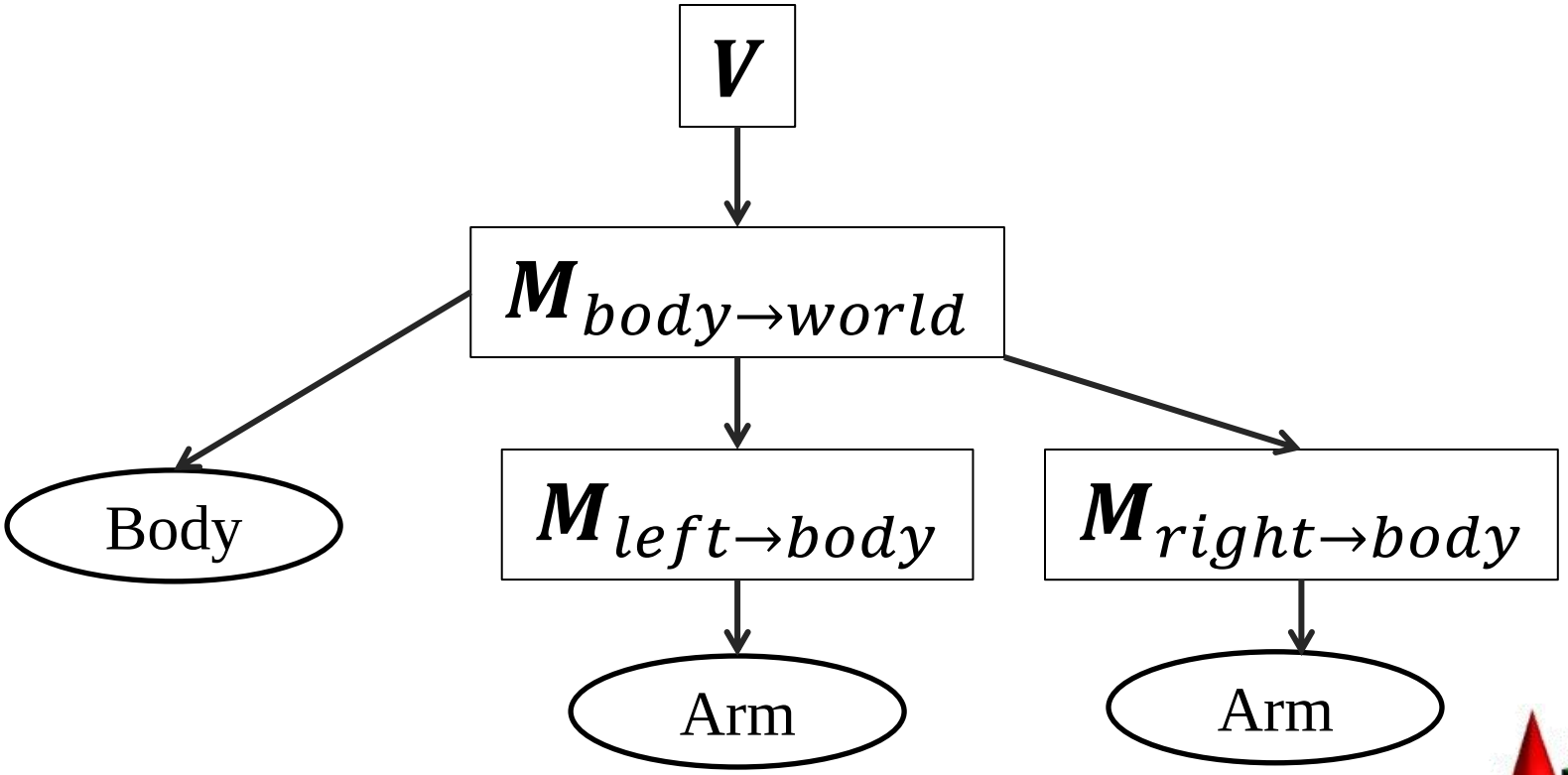
and finally to camera frame:

$$\boldsymbol{V}$$



Scene graph

- The whole scene is thus described as a tree – the **scene graph**.



Example

```
glTranslatef(0.0, 0.0, -5.0);          // View transform

    glRotatef(1.0, 0.0, 0.0, 1.0);     // Body → world
    draw_body();

        glTranslatef(0.0, 0.5, 2.0);   // Left arm → body
        glRotatef(0.5, 0.0, 1.0, 0.0);
        draw_arm();
        "unmultiply" left -> body

        glTranslate(0.0, -0.5, -2.0);  // Right arm → body
        draw_arm();
        "unmultiply" right -> body
    "unmultiply" body -> world
```

Example

```
glTranslatef(0.0, 0.0, -5.0);          // View transform
glPushMatrix();
   glRotatef(1.0, 0.0, 0.0, 1.0);      // Body → world
   draw_body();
   glPushMatrix();
      glTranslatef(0.0, 0.5, 2.0);     // Left arm → body
      glRotatef(0.5, 0.0, 1.0, 0.0);
      draw_arm();
   glPopMatrix();
   glPushMatrix();
      glTranslate(0.0, -0.5, -2.0);    // Right arm → body
      draw_arm();
   glPopMatrix();
glPopMatrix();
```

VRML example

```
Transform {
   translation 0 0 -5
   children Transform {
      rotation 0 0 1 1.0
      children [
         USE BODY
         Transform {
            translation 0 0.5 2.0
            rotation 0 1 0 0.5
            children USE ARM
         }
         Transform {
            translation 0 -0.5 -2.0
            children USE ARM
         }
      ]
   }
}
```

Projection transform

- After the vertices are transformed to the camera frame, they are *projected* to the camera plane using a *projection transform* $\boldsymbol{P}$.

- Thus, the total transformation pipeline for each object vertex $\boldsymbol{p_i}$ is:

$$\boldsymbol{PVMp_i}$$

- This is the topic for the next lecture.

Summary

- Linear transformations & matrices
- Rank, invertibility & determinant
- Orthogonal transformations
- Affine transformations & frames
- Homogeneous coordinates
- Model-view transform
- Scene graph