Computer Graphics

Mathematical background: Transformations

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In the previous episode

- Vectors

In the previous episode

- Tools for working with vectors

In the previous episode

- Distances: ____________________
- Projections: ___________________
- Areas & Volumes: _____________
- Perpendiculars: _______________
- Orthogonalization: ____________
- Represent straight line using:
  - ______________________
  - ______________________

In the previous episode

- Distances: **norm**
- Projections: **inner product**
- Areas & Volumes: **box product**
- Perpendiculars: **cross product**
- Orthogonalization: **inner/cross product**
- Represent straight line using:
  - **Linear combinations (parametric)**
  - **Inner product (implicit)**

Quiz

- Derive an implicit representation for a two-dimensional line using the box product.

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(Bi)linear operations

(Bi)linearity

$$\langle \alpha \boldsymbol{x} + \boldsymbol{y}, \boldsymbol{z} \rangle = \alpha \langle \boldsymbol{x}, \boldsymbol{y} \rangle + \langle \boldsymbol{x}, \boldsymbol{z} \rangle$$

$$|\alpha \boldsymbol{x} + \boldsymbol{y} \quad \boldsymbol{z}| = \alpha |\boldsymbol{x} \;\; \boldsymbol{z}| + |\boldsymbol{y} \;\; \boldsymbol{z}|$$

$$(\alpha \boldsymbol{x} + \boldsymbol{y}) \times \boldsymbol{z} = \alpha (\boldsymbol{x} \times \boldsymbol{z}) + (\boldsymbol{y} \times \boldsymbol{z})$$

$$\boldsymbol{A}(\alpha \boldsymbol{x} + \boldsymbol{y}) = \alpha \boldsymbol{A} \boldsymbol{x} + \boldsymbol{A} \boldsymbol{y}$$

(Bi)linearity

$$f(x + y) = f(x) + f(y)$$

$$f(\alpha x) = \alpha f(x)$$

Today

Linear transformations

http://www.thinkgeek.com/

Quiz

- Which of those are **not** linear transformations?
  - $f(x) = x$
  - $f(x) = -4x$
  - $f(x) = 4x + 4$
  - $f(x) = x^2$
  - $f(x) = 3$
  - $f(x) = 0$

Quiz

- Which of those are **not** linear transformations?
  - $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$
  - $f(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{x}$
  - $f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x}$
  - $f(\boldsymbol{x}) = |\boldsymbol{a} \;\; \boldsymbol{b} \;\; \boldsymbol{x}|$
  - $f(\boldsymbol{x}) = \boldsymbol{a} \times \boldsymbol{x}$
  - $f(x) = \boldsymbol{a}^T\boldsymbol{x} + |\boldsymbol{a} \;\; \boldsymbol{b} \;\; \boldsymbol{x}| + \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{A}\boldsymbol{x}$

Linear transformations

- Each linear transformation is uniquely defined by how it transforms the basis:

$$f\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} =$$

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$$f\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = f\left(2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

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$$= 2f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Linear transformations

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$$= 2\boldsymbol{f}_1 - 3\boldsymbol{f}_2 + 4\boldsymbol{f}_3$$

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$$= |\boldsymbol{f}_1 \;\; \boldsymbol{f}_2 \;\; \boldsymbol{f}_3| \begin{pmatrix}2\\-3\\4\end{pmatrix}$$

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$$= \boldsymbol{F}\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

Linear transformations

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Columns of a matrix show how it transforms the canonical basis

Quiz

- How does this matrix transform the (canonical) basis?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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Quiz

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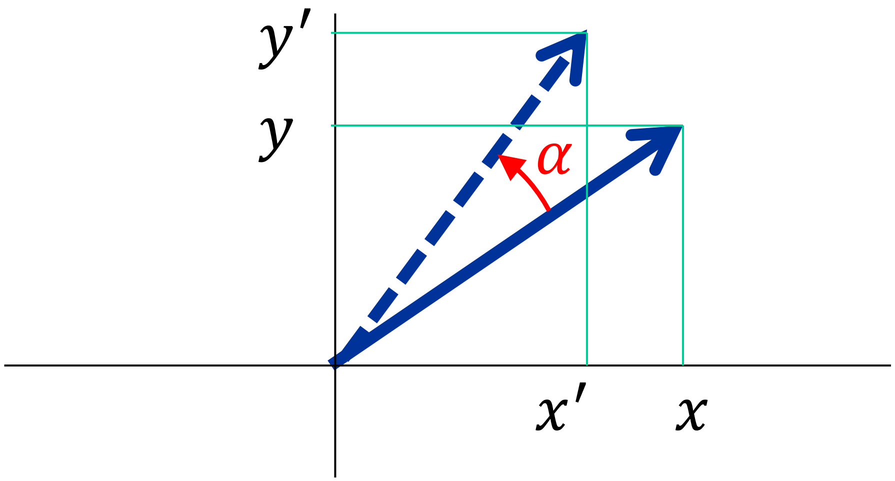
$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Quiz

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

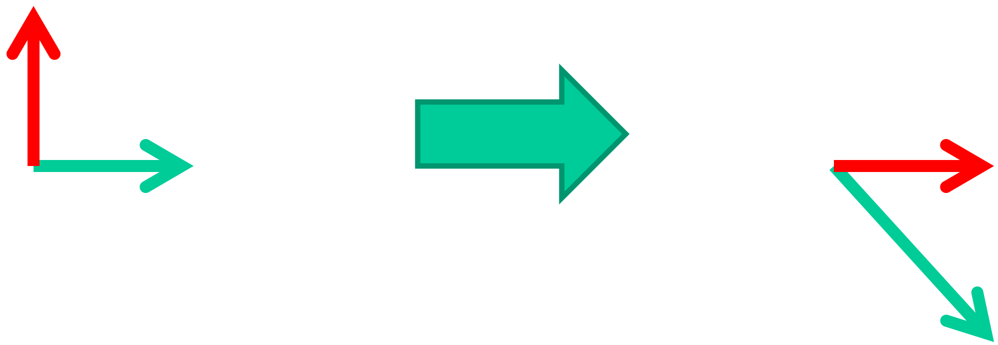
- Let (x, y) be a 2D vector

  Let (x', y') be obtained from (x, y) via rotation by angle α. Express x' and y' in terms of x and y.



Quiz

- Which matrix does the following?



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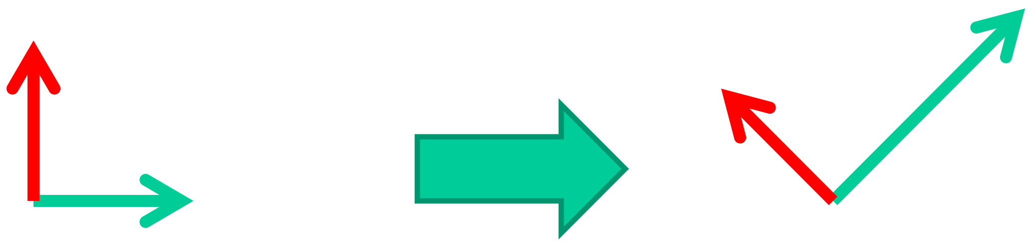

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Composition of transformations corresponds to matrix multiplication:

$$(f \circ g)(\boldsymbol{x}) = f\big(g(\boldsymbol{x})\big) = \boldsymbol{F}\boldsymbol{G}\boldsymbol{x}$$

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Function composition is associative, hence matrix multiplications is too:

$$(f \circ g) \circ h = f \circ (g \circ h)$$
$$(\boldsymbol{F}\boldsymbol{G})\boldsymbol{H} = \boldsymbol{F}(\boldsymbol{G}\boldsymbol{H})$$

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Sum of transformations corresponds to matrix sum:

$$(f+g)(\boldsymbol{x}) = f(\boldsymbol{x}) + g(\boldsymbol{x}) = (\boldsymbol{F}+\boldsymbol{G})\boldsymbol{x}$$

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Composition is distributive wrt sum:

$$(f + g) \circ h = f \circ h + g \circ h$$
$$(\boldsymbol{F} + \boldsymbol{G})\boldsymbol{H} = \boldsymbol{F}\boldsymbol{H} + \boldsymbol{G}\boldsymbol{H}$$

Rank

- Consider a linear transformation $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ it will always either:
  - Map the whole 3D space to itself somehow
  - Project the whole 3D space to a plane
  - Project the whole 3D space to a line
  - Map all points to 0.

Rank

- The dimensionality of the resulting space is the ***rank*** of f.

  - If f is full rank (i.e. $\mathrm{rank}(f) = 3$ in our case), it is *invertible*. Otherwise it is not.

  - f is invertible $\Leftrightarrow \det(\boldsymbol{F}) \neq 0$

Orthogonal transformations

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  - $\boldsymbol{F}^T\boldsymbol{F} = ?$

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  - Hence, $\boldsymbol{F}^{-1} = \boldsymbol{F}^T$

Orthogonal transformations

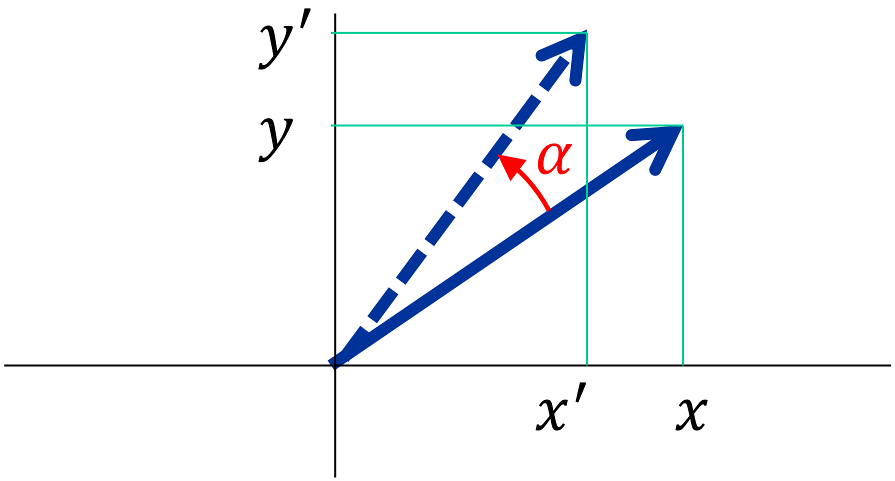
To compute the inverse of an orthogonal matrix, simply transpose it.

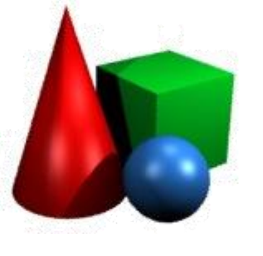
Quiz

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

- Let (x, y) be a 2D vector

  Let (x', y') be obtained from (x, y) via rotation by angle α. Express x and y in terms of x' and y'.

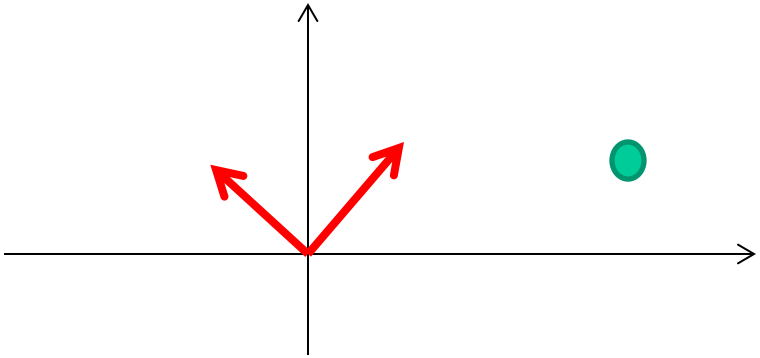




Quiz

- You are standing at the origin, rotated with respect to the coordinate system, looking in the direction $(0.6, 0.8)$ (your local "x" axis).

- At position $(7,2)$ there is an object. What are the coordinates of this object with respect to you?



Examples

- Rotation: $\boldsymbol{R}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$
- Scaling: $\boldsymbol{S}(a,b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
- Mirroring: $\mathbf{Mir}_{\boldsymbol{y}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Shear: $\mathbf{Sh}_{\mathbf{x}}(\mathrm{a}) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

Examples

- Rotation around z axis:

$$\boldsymbol{R_z}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Rotation around y axis:

$$\boldsymbol{R_z}(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

Shift

- Shift (translation) is not a linear transformation.

- To deal with shifts we must introduce the notion of an *affine space* and *affine transformations*.

Affine space

- Vector space

- Affine space

Affine space

- Vector space
  - Vectors $\boldsymbol{v} \in \mathbb{R}^3$
  - Basis: $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$
  - Linear transformations $f(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{x}$
- Affine space

Affine space

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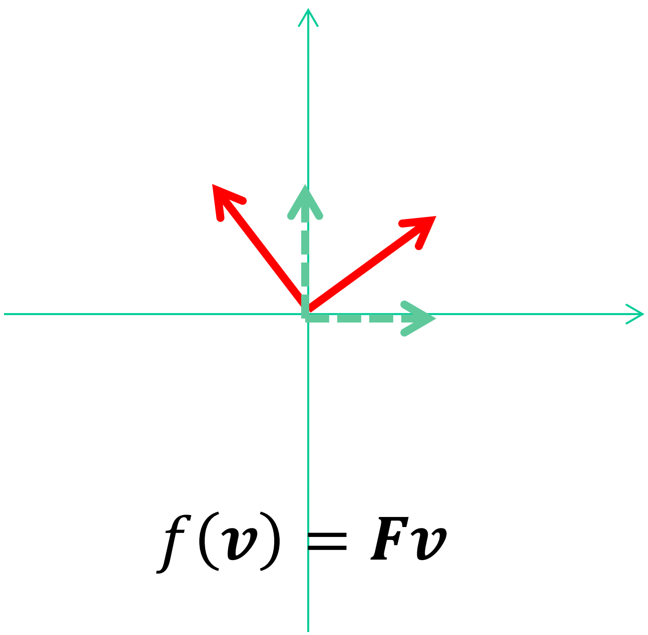
  - Vectors $\boldsymbol{v} \in \mathbb{R}^3$
  - Basis: $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$
  - Linear transformations $f(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{x}$
- Affine space
  - Vectors $\boldsymbol{v} \in \mathbb{R}^3$
  - Points $P \in \mathbb{R}^3$
    - point+vector = point
  - Frame: $(O, \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\})$
  - Affine transformations: $f(\boldsymbol{v}) = \boldsymbol{F}\boldsymbol{v}$ $f(P) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$

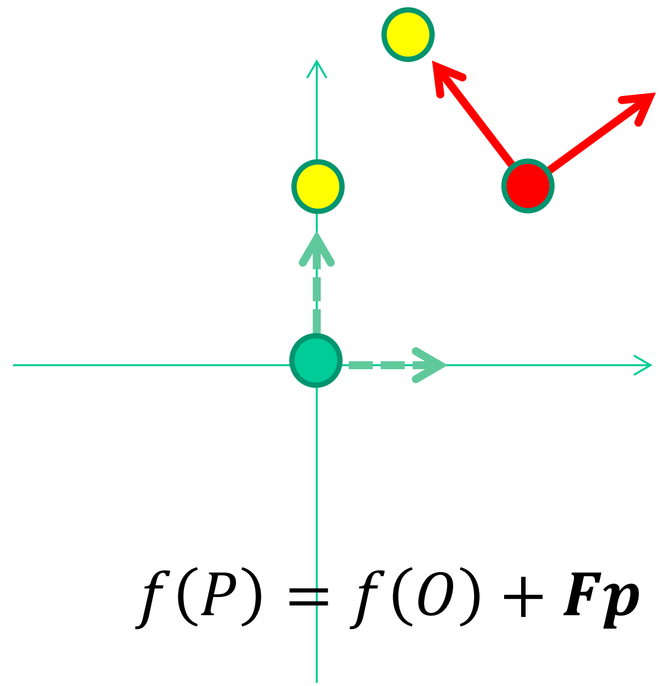
Affine space

- Vector space



$$f(\boldsymbol{v}) = \boldsymbol{F}\boldsymbol{v}$$

- Affine space



$$f(P) = f(O) + \boldsymbol{F}\boldsymbol{p}$$

Affine transformations

$$f(\boldsymbol{p}) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$$

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$$f(\boldsymbol{p}) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Affine transformations

$$f(\boldsymbol{p}) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$$

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$$\begin{pmatrix} q_1 \\ q_2 \\ 1 \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & t_1 \\ f_{21} & f_{22} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix}$$

Homogeneous coordinates

- We shall represent the **points** of an affine space using 3-dimensional vectors of the form $(p_1, p_2, 1)^T$
- We shall represent the **vectors** of an affine space using 3-dimensional vectors of the form $(v_1, v_2, 0)^T$
- Any affine transformation is a matrix

$$\left(\begin{array}{cc|c} f_{11} & f_{12} & t_1 \\ f_{21} & f_{22} & t_2 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Homogeneous coordinates

- Analogously, for 3D space we use 4-dimensional vectors and 4x4 matrices.
- E.g. the following transformation rotates around z axis and shifts along x axis by 0.5:

$$\left(\begin{array}{ccc|c} \cos\phi & -\sin\phi & 0 & 0.5 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Homogeneous coordinates

- Note how the representation implicitly enforces the rules:
  - vector + vector = vector
  - point + vector = point
  - point + point = undefined
  - convex combination of points = point

Homogeneous coordinates

- Rotation: $$\boldsymbol{R}(\alpha) = \left(\begin{array}{cc|c} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

- Scaling: $$\boldsymbol{S}(a, b) = \left(\begin{array}{cc|c} a & 0 & 0 \\ 0 & b & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

- Translation: $$\boldsymbol{T}(x, y) = \left(\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ \hline 0 & 0 & 1 \end{array}\right)$$

Quiz

- Construct a matrix, that performs a rotation by 10 degrees around the point (20, 30) in homogeneous coordinates.

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$$\boldsymbol{T}(20{,}30)\boldsymbol{R}(10)\boldsymbol{T}(-20,-30)$$

Quiz

- How to construct a matrix, that performs a rotation (in 3D) by 10 degrees around the axis given by the direction vector $(1, 2, 3)$

Mathematical background

- Matrices:
  - Linear transformations
  - Invertibility, rank, determinant
  - Orthogonal transformations
  - Affine transformations
  - Homogeneous coordinates