MTAT.03.015 Computer Graphics (Fall 2013) Lectures II & III: Math exercises

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Solution for every task gives 0.5 points. Solutions are accepted on paper or via e-mail (`kt@ut.ee`) until October 2, 2013.

1. Let s be a straight line in $\mathbb{R}^2$, passing through the origin. It can be described parametrically as
$$\mathbf{x} = \lambda\mathbf{s}, \qquad \lambda \in \mathbb{R},$$
or implicitly as
$$\mathbf{n}^T\mathbf{x} = 0\,.$$
Express the coordinates of the normal vector $\mathbf{n}$ via the coordinates of the direction vector $\mathbf{s}$.

2. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be points in $\mathbb{R}^2$. Find the coordinates of the intersection point of segments $[\mathbf{a}, \mathbf{b}]$ and $[\mathbf{c}, \mathbf{d}]$. Hint: Use the parametric representation.

3. Prove that the (Euclidean) norm $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$ satisfies the *triangle inequality*:
$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$
Derive from this inequality also the inequalities
$$\|\mathbf{x}\| - \|\mathbf{x}\| \leq \|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

4. Let $\mathbf{p}$ and $\mathbf{q}$ be orthonormal vectors in $\mathbb{R}^3$. What transformation does the matrix $\mathbf{p}\mathbf{p}^T + \mathbf{q}\mathbf{q}^T$ correspond to? Prove it.

5. Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ be an orthonormal basis in $\mathbb{R}^3$. Prove that $\mathbf{p}\mathbf{p}^T + \mathbf{q}\mathbf{q}^T + \mathbf{r}\mathbf{r}^T = \mathbf{I}$, where $\mathbf{I}$ denotes a unit matrix.

6. Orthogonalize the following set of vectors using the Gram-Schmidt algorithm:
$$\mathbf{e}_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0)^T$$
$$\mathbf{e}_2 = (-1, 1, -1)^T$$
$$\mathbf{e}_3 = (0, -2, -2)^T$$

7. Compute the area of a triangle given by vertices

$$\begin{aligned} \mathbf{a} &= (1, 2, 3)^T, \\ \mathbf{b} &= (-2, 2, 4)^T, \\ \mathbf{c} &= (7, -8, 0)^T. \end{aligned}$$

8. Points $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n \in \mathbb{R}^2$ are vertices of a simple polygon[1] listed in counter-clockwise order in a right-handed basis. Prove that the area of the polygon S can be computed as

$$S = \frac{1}{2}(|\mathbf{p}_1 \quad \mathbf{p}_2| + |\mathbf{p}_2 \quad \mathbf{p}_3| + \cdots + |\mathbf{p}_n \quad \mathbf{p}_1|)$$

9. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a continuous function that satisfies $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ for each $\mathbf{x}, \mathbf{y}$. Show that it then necessarily follows that for each $\alpha \in \mathbb{R}$ and each $\mathbf{x}$

$$f(\alpha \mathbf{x}) = \alpha f(\mathbf{x}),$$

i.e. f must be linear.

10. Consider a polyhedron with vertices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_k$. Let $\mathbf{n}_1$, $\mathbf{n}_2, \ldots, \mathbf{n}_l$ be the normals for the faces of the polyhedron. Let us apply a linear transformation $\mathbf{F}$ to all the vertices of the polyhedron. The vertices of the new polyhedron are thus $\mathbf{F}\mathbf{p}_1, \mathbf{F}\mathbf{p}_2, \ldots, \mathbf{F}\mathbf{p}_k$. Express the normals of the new polyhedron in terms of the original normals.

[1] A *simple polygon* is a polygon, whose edges do not intersect each other.