MTAT.03.015 Computer Graphics (Fall 2013)
Lectures XIII & XIV: Math exercises

Konstantin Tretyakov

Solution for every task gives 0.5 points. Solutions are accepted on paper or via e-mail (`kt@ut.ee`) until December 18, 2013.

1. Bring examples of a two-piece linear spline curve, which happens to be:
   (a) C^1-smooth at the connection point.
   (b) G^1-smooth, but not C^1-smooth at the connection point.
2. Construct the basis matrices for:
   (a) the linear Bézier' curve,
   (b) the quadratic Bézier' curve.
3. Construct the basis matrices for:
   (a) the linear Lagrange' curve,
   (b) the quadratic Lagrange' curve (assume the parameter vector for the control points to be $t = (0, 0.5, 1)$).
4. Construct a one-dimensional quadratic Lagrange' curve defined by control points $(0, 1, 0)$. Provide the answer as a polynomial in t.
5. Consider the curve in the previous exercise. Convert it to the Bézier' representation. That is, find the Bézier' control points for exactly the same curve.
6. Prove that degree n Bernstein polynomials $B_i^{(n)}, i \in \{0, 3\}$ sum to one, i.e.:

   $$\sum_{i=0}^{n} B_i^{(n)}(t) = 1, \text{ for all } n \in \mathbb{N}, t \in [0, 1].$$

   Hint: one way to show it is to note that the Bernstein polynomials are somehow related to a well-known probability distribution.
7. A Hermite' curve is a reparameterization of the cubic Bézier' curve, that is specified by its start and end points $\mathbf{p}_0$, $\mathbf{p}_3$ and its direction vectors (i.e. gradients) $\mathbf{s}_0$, $\mathbf{s}_3$ at those points. In other words, the geometry matrix for

the Hermite' curve is $\mathbf{G} = (\mathbf{p}_0, \mathbf{p}_3, \mathbf{s}_0, \mathbf{s}_3)$. Derive the basis matrix $\mathbf{M}_H$ of the Hermite curve.

8. Prove that a cubic B-spline will pass a control point, if it is repeated three times.

9. We derived the cubic B-spline to be a curve that is C^2-smooth at all points, no matter what the control points are. However, we also somewhy said that repeating control points "reduces smoothness" and even saw an example where repeating the control point three times results in a curve with a sharp corner. This seems like a contradiction. Explain this.

10. Prove that a (uniform) B-spline of degree k with n control points has $n + k + 1$ knots.

    Hint: count the number of curve segments and add the "virtual" segments on both sides, where the basis functions for the first and last control points are still nonzero.