Computer Graphics

Mathematical background

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Sep 11, 2013

In the previous episodes

- Computer graphics is useful and fun
- Computer graphics is about generating images
- Modeling, Rendering, Animation
- Raster vs Vector, 2D vs 3D
- Ad-hoc projection vs Light physics
- “Standard graphics pipeline”
- Matrix notation

Mathematical background

- Vectors:
  - Points, directions, vectors and matrices
  - Linear combinations, convex combinations
  - Norm, normalization
  - Inner product, orthogonality, orthogonalization
  - Box product, Cross product
  - Orientation
  - Representation of a straight line

Mathematical background

- Matrices:
  - Linear transformations
  - Invertibility, rank, determinant
  - Orthogonal transformations
  - Affine transformations
  - Homogeneous coordinates

Vectors

In general, vectors are elements of a _____________ space.

Vectors

In computer graphics, we primarily deal with vector spaces $\mathbb{R}^2, \mathbb{R}^3$ and $\mathbb{R}^4$.

Vectors

In computer graphics, we primarily deal with vector spaces $\mathbb{R}^2, \mathbb{R}^3$ and $\mathbb{R}^4$.

We use those vectors to denote __________ and ___________

Operations with vectors

- Linear combinations:

$$3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

Operations with vectors

- Linear combinations:

$$3 \cdot \binom{1}{2} - 2 \cdot \binom{2}{1} =$$

- A linear combination

$$\lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2 + \cdots + \lambda_n \boldsymbol{v}_n$$

is called *convex* if ______________________________

____

Operations with vectors

- Norm

$$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

Operations with vectors

- Norm

$$\left\| \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\| =$$

Operations with vectors

- Norm

$$\left\|\begin{pmatrix}3\\4\end{pmatrix}\right\| =$$

$$\left\|\begin{pmatrix}33\\44\end{pmatrix}\right\| =$$

Operations with vectors

- Normalization:

$$\text{normalize}(\boldsymbol{a}) := \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

Operations with vectors

- Normalization:

$$\text{normalize}\begin{pmatrix}0\\1\end{pmatrix} =$$

Operations with vectors

- Normalization:

$$\text{normalize}\begin{pmatrix}44\\33\end{pmatrix}=$$

Operations with vectors

- Inner product

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Operations with vectors

- Inner product

$$\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle =$$

Operations with vectors

- Inner product

$$\left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle =$$

Operations with vectors

- Inner product

$$\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle =$$

Operations with vectors

- Inner product

$$\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle =$$

Operations with vectors

- Inner product
  - $\langle \boldsymbol{a},\ \lambda \boldsymbol{b} + \boldsymbol{c} \rangle = \lambda \langle \boldsymbol{a}, \boldsymbol{b} \rangle + \langle \boldsymbol{a}, \boldsymbol{c} \rangle$

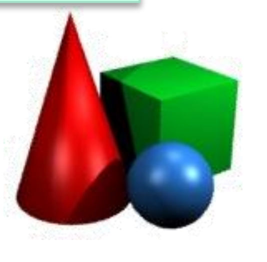
Linearity

  - $\langle \boldsymbol{a},\ \boldsymbol{a} \rangle = \|\boldsymbol{a}\|^2$
  - $\langle \boldsymbol{a},\ \boldsymbol{b} \rangle = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot \cos \alpha$

Relationship with the norm

  - $\langle \boldsymbol{a},\ \boldsymbol{b} \rangle = \boldsymbol{a}^T \boldsymbol{b}$

Relationship with the matrix product notation



Operations with vectors

- Inner product
  - $\boldsymbol{a}^T(\lambda\boldsymbol{b}+\boldsymbol{c}) = \lambda\boldsymbol{a}^T\boldsymbol{b}+\boldsymbol{a}^T\boldsymbol{c}$

Linearity

  - $\boldsymbol{a}^T\boldsymbol{a} = \|\boldsymbol{a}\|^2$
  - $\boldsymbol{a}^T\boldsymbol{b} = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot \cos\alpha$

Relationship with the norm

  - $\langle\boldsymbol{a},\ \boldsymbol{b}\rangle = \boldsymbol{a}^T\boldsymbol{b}$

Relationship with the matrix product notation

Operations with vectors

- Inner product
  - $\boldsymbol{p}^T \boldsymbol{a} = \|\boldsymbol{p}\| \cdot \|\boldsymbol{a}\| \cdot \cos\alpha$
  - If $\|\boldsymbol{p}\| = 1$,

$\boldsymbol{p}^T \boldsymbol{a}$ is the length of

____

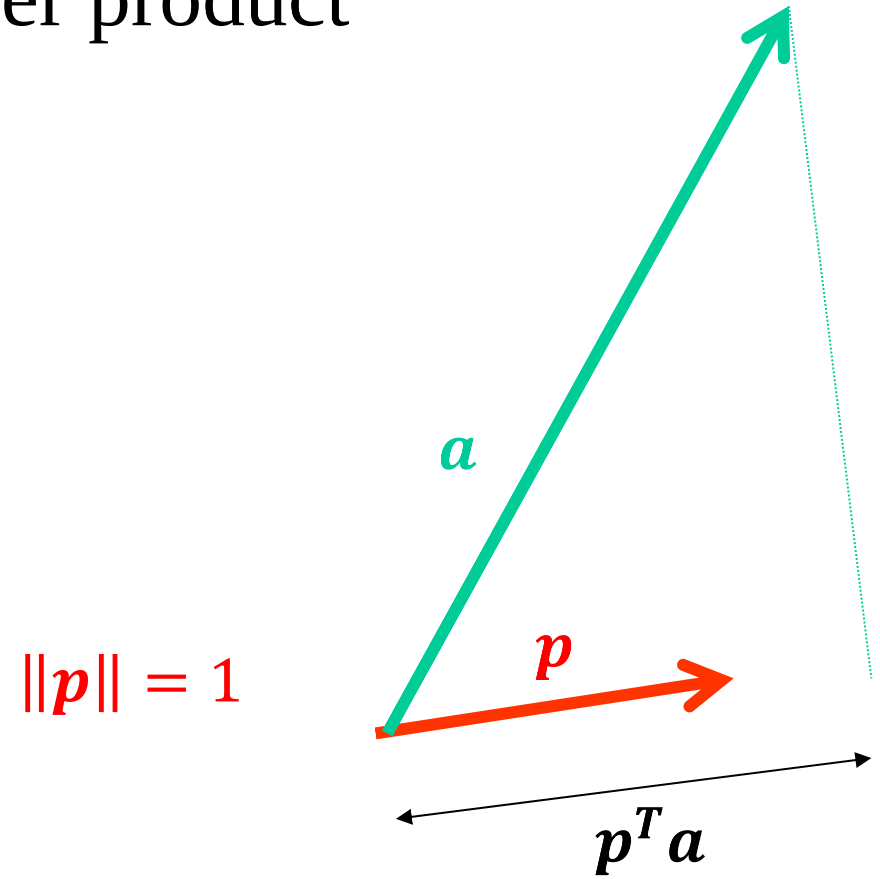
Operations with vectors

- Inner product
  - $\boldsymbol{p}^T \boldsymbol{a} = ||\boldsymbol{p}|| \cdot ||\boldsymbol{a}|| \cdot \cos\alpha$
  - If $||\boldsymbol{p}|| = 1$,

$\boldsymbol{p}^T \boldsymbol{a}$ is the **length of the projection** of $\boldsymbol{a}$ onto $\boldsymbol{p}$.

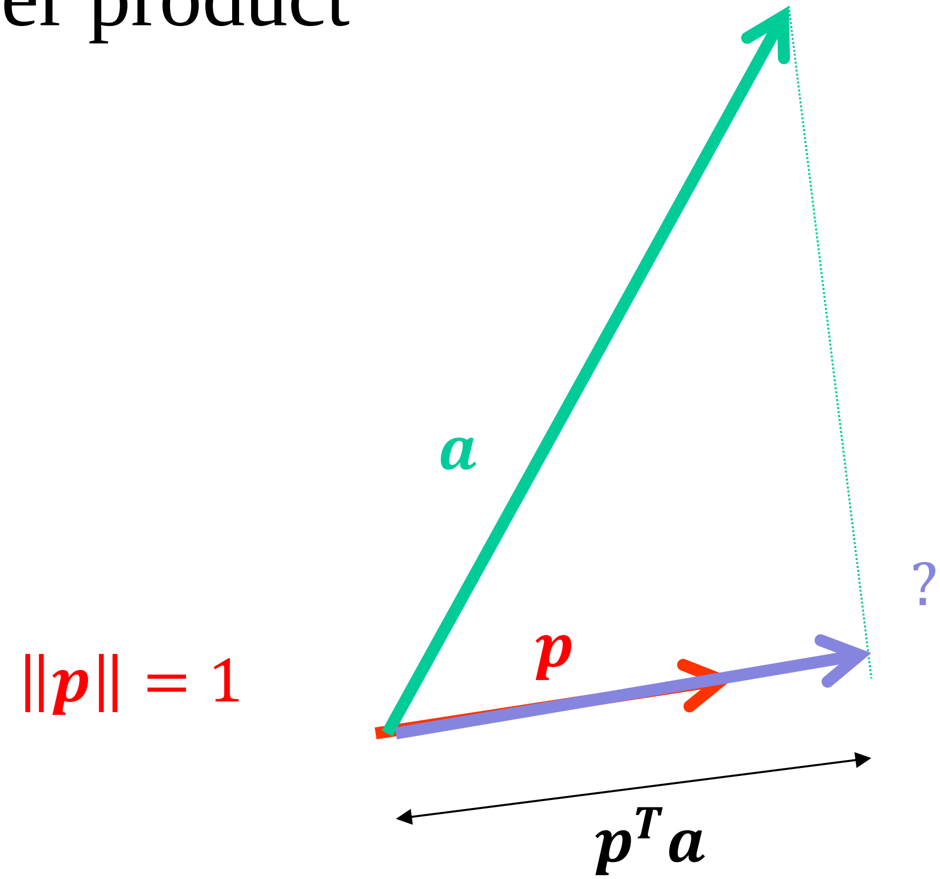
Operations with vectors

- Inner product



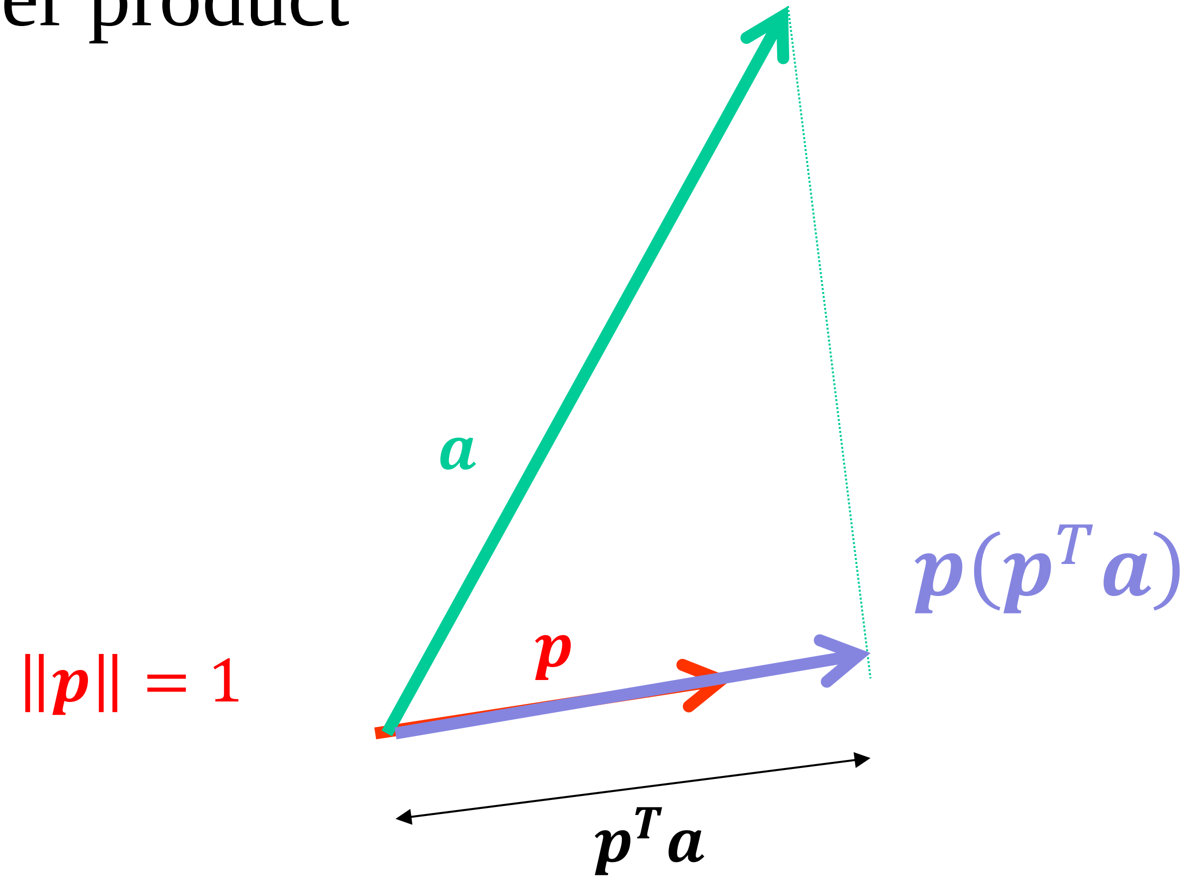
Operations with vectors

- Inner product



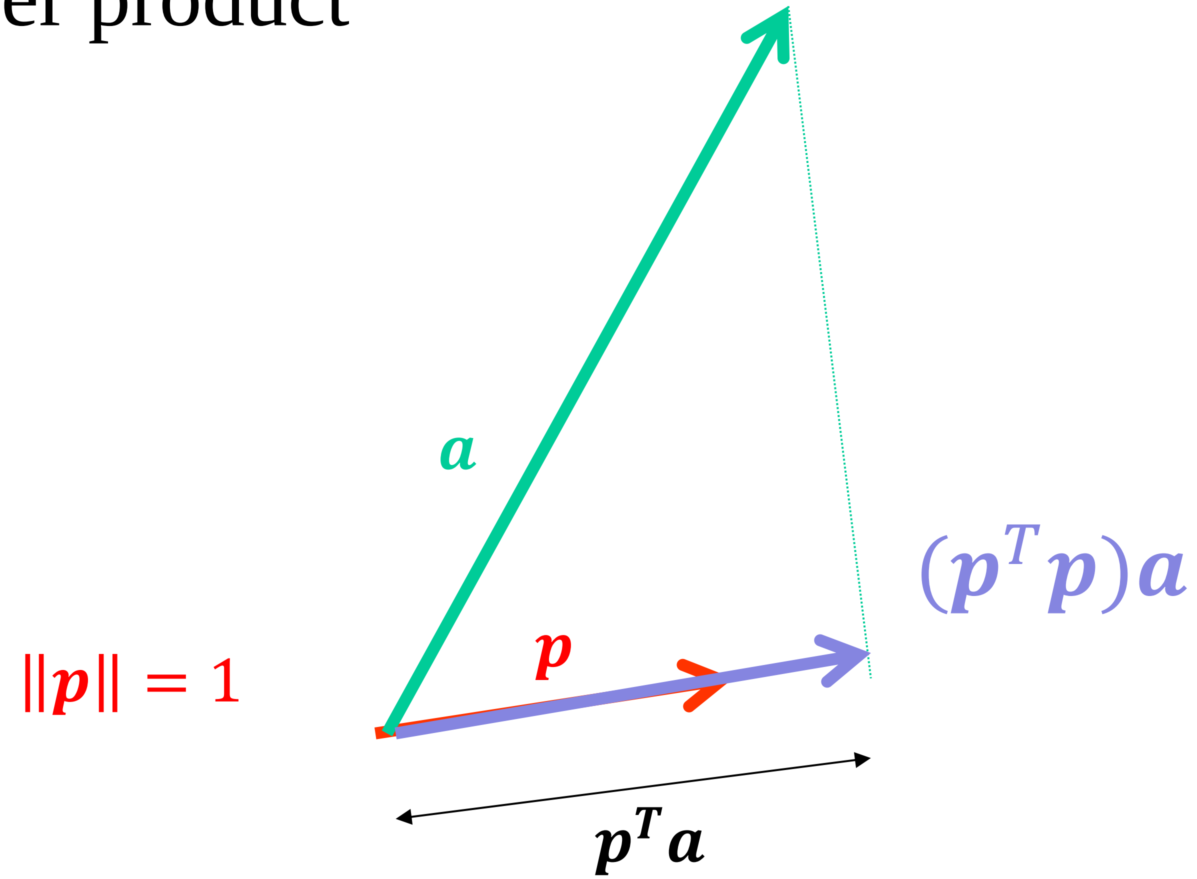
Operations with vectors

- Inner product



Operations with vectors

- Inner product



Operations with vectors

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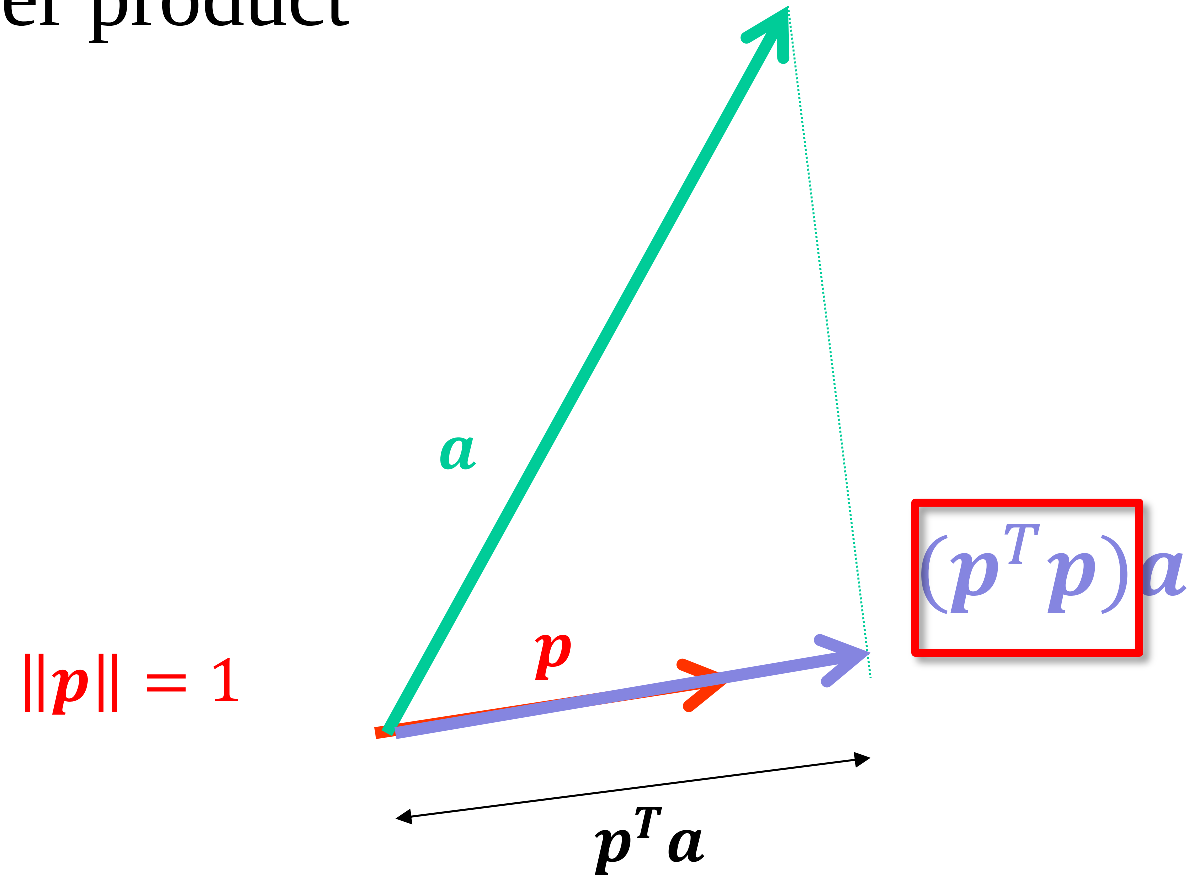

Operations with vectors

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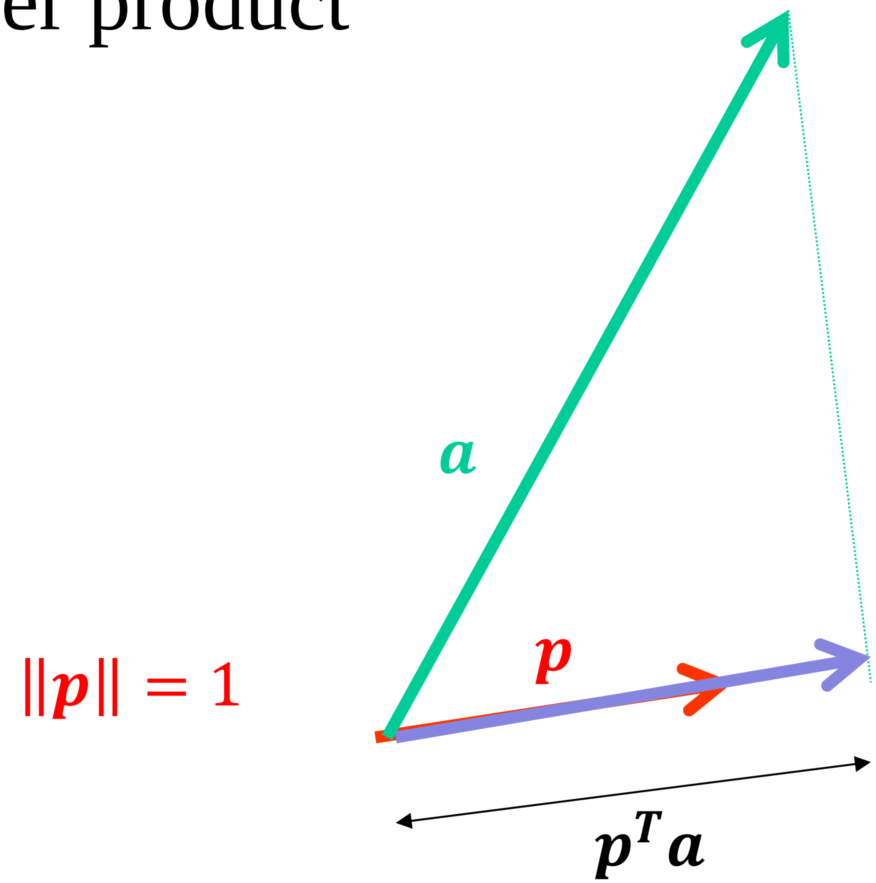


Projector matrix

$\boxed{(p^T p)}a$

Projector

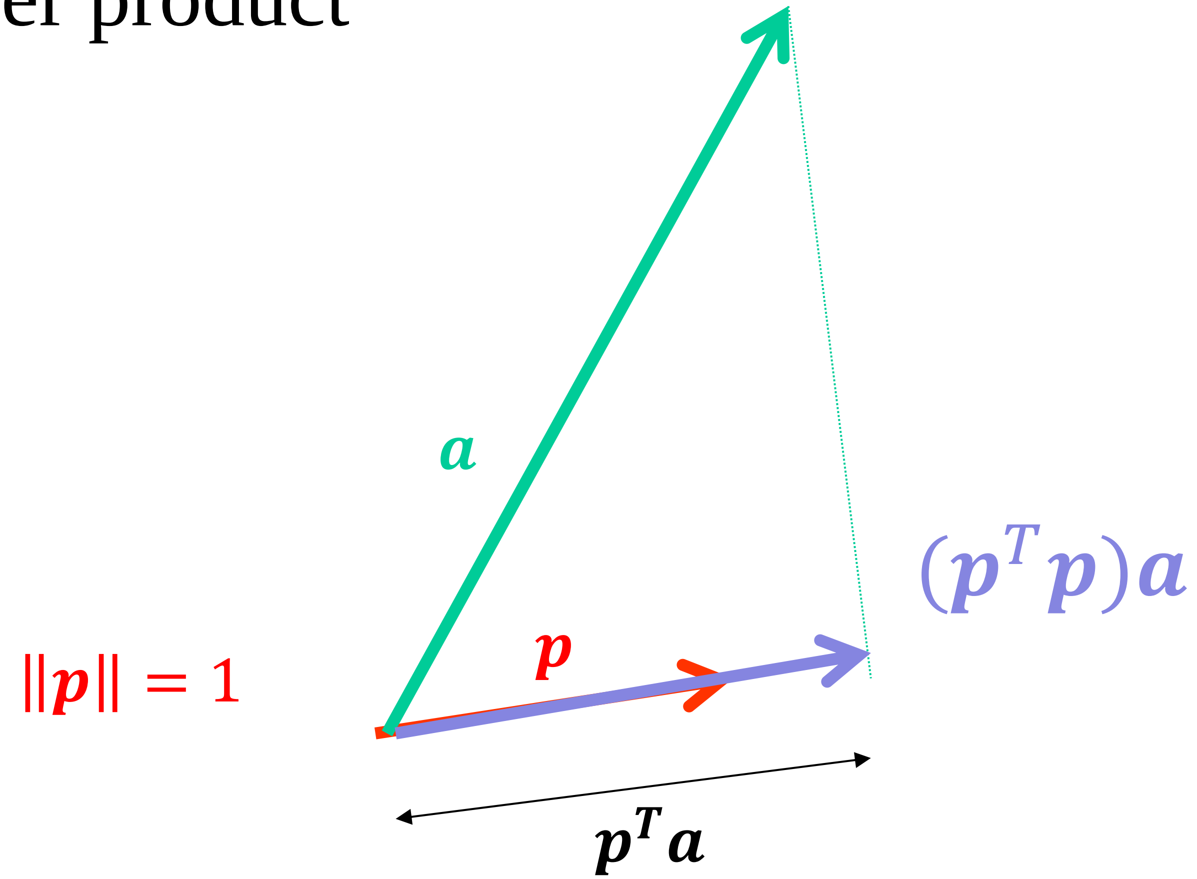
- For any nonzero vector $\boldsymbol{p}$ the matrix

$$\left(\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}\right)\left(\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}\right)^{T} = \frac{\boldsymbol{p}\boldsymbol{p}^{\boldsymbol{T}}}{\|\boldsymbol{p}\|^{2}} = \frac{\boldsymbol{p}\boldsymbol{p}^{T}}{\boldsymbol{p}^{T}\boldsymbol{p}}$$

is the *projector matrix* for $\boldsymbol{p}$.

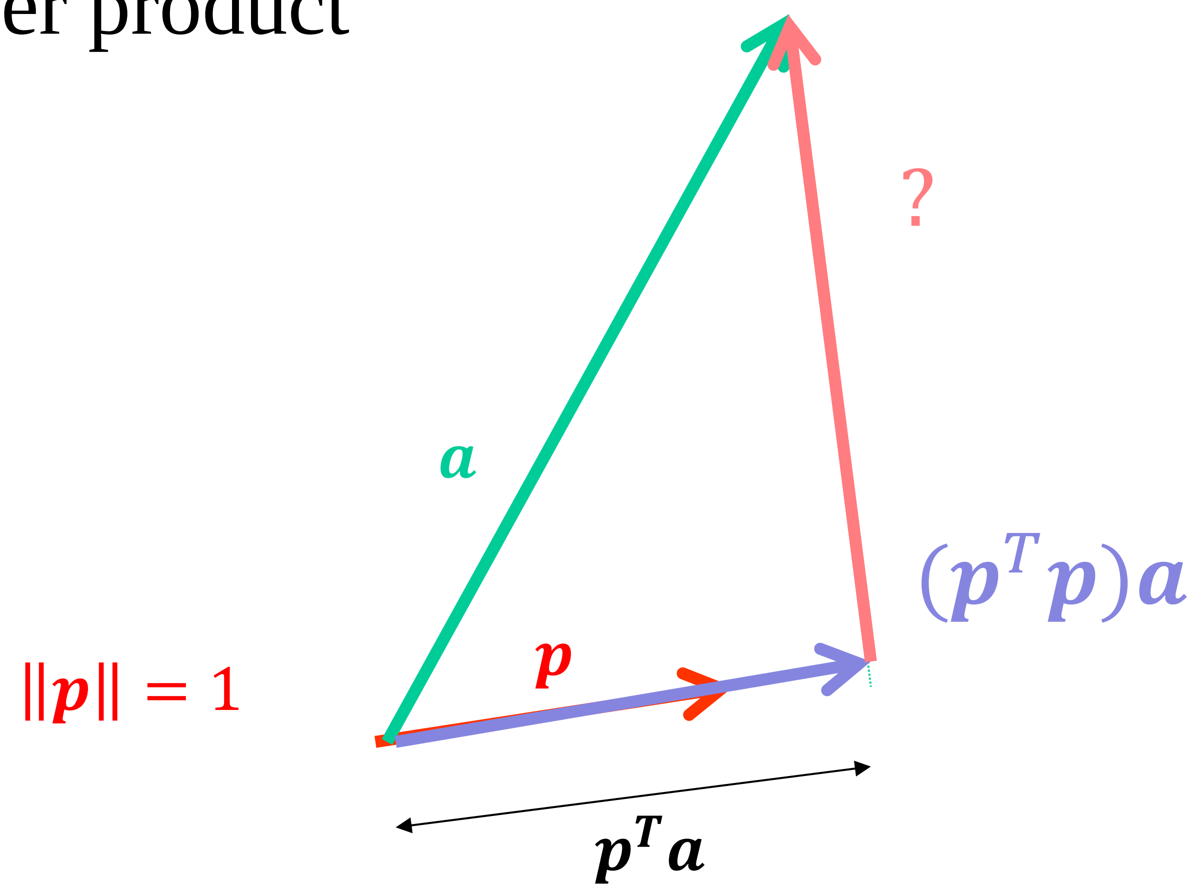
Operations with vectors

- Inner product



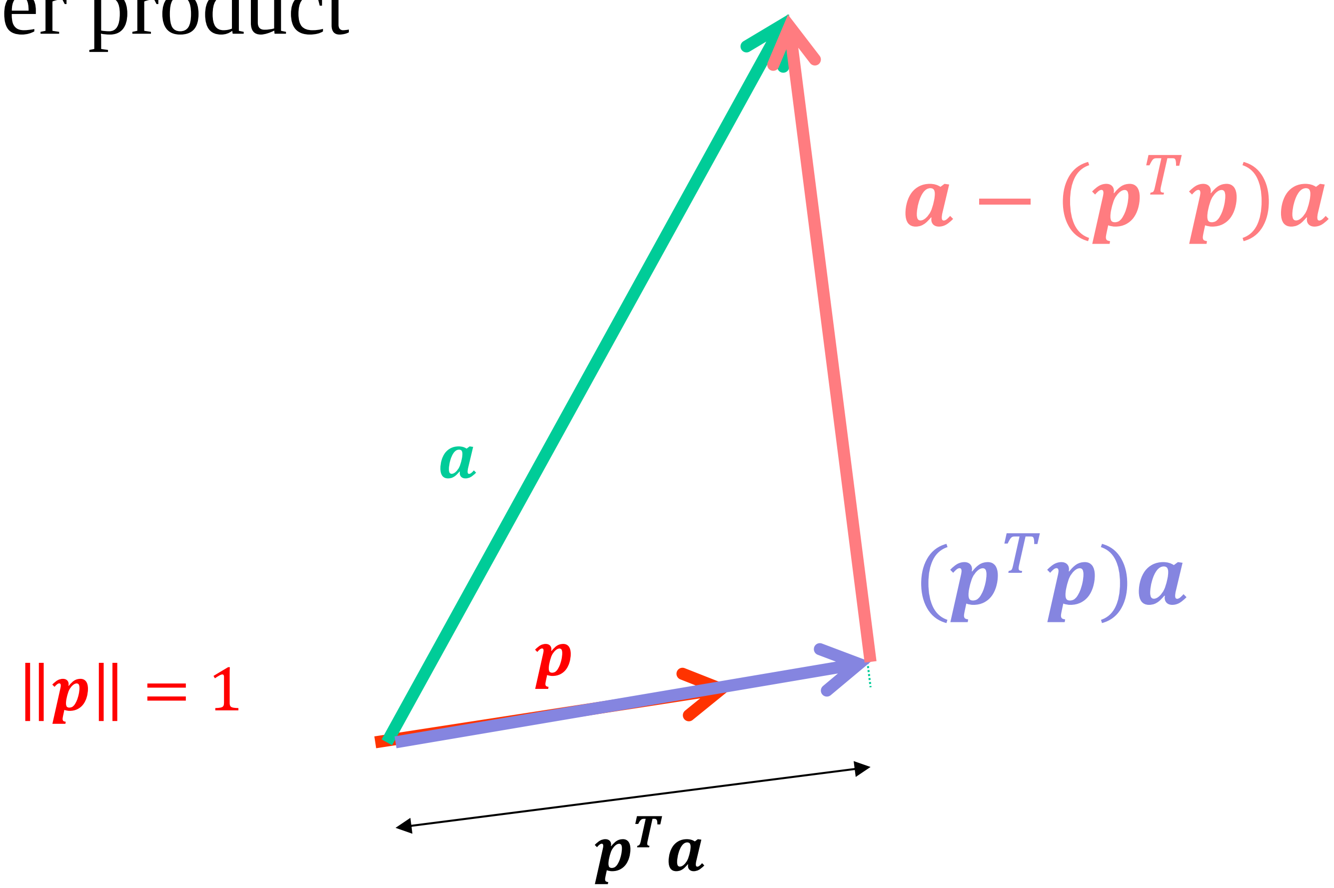
Operations with vectors

- Inner product



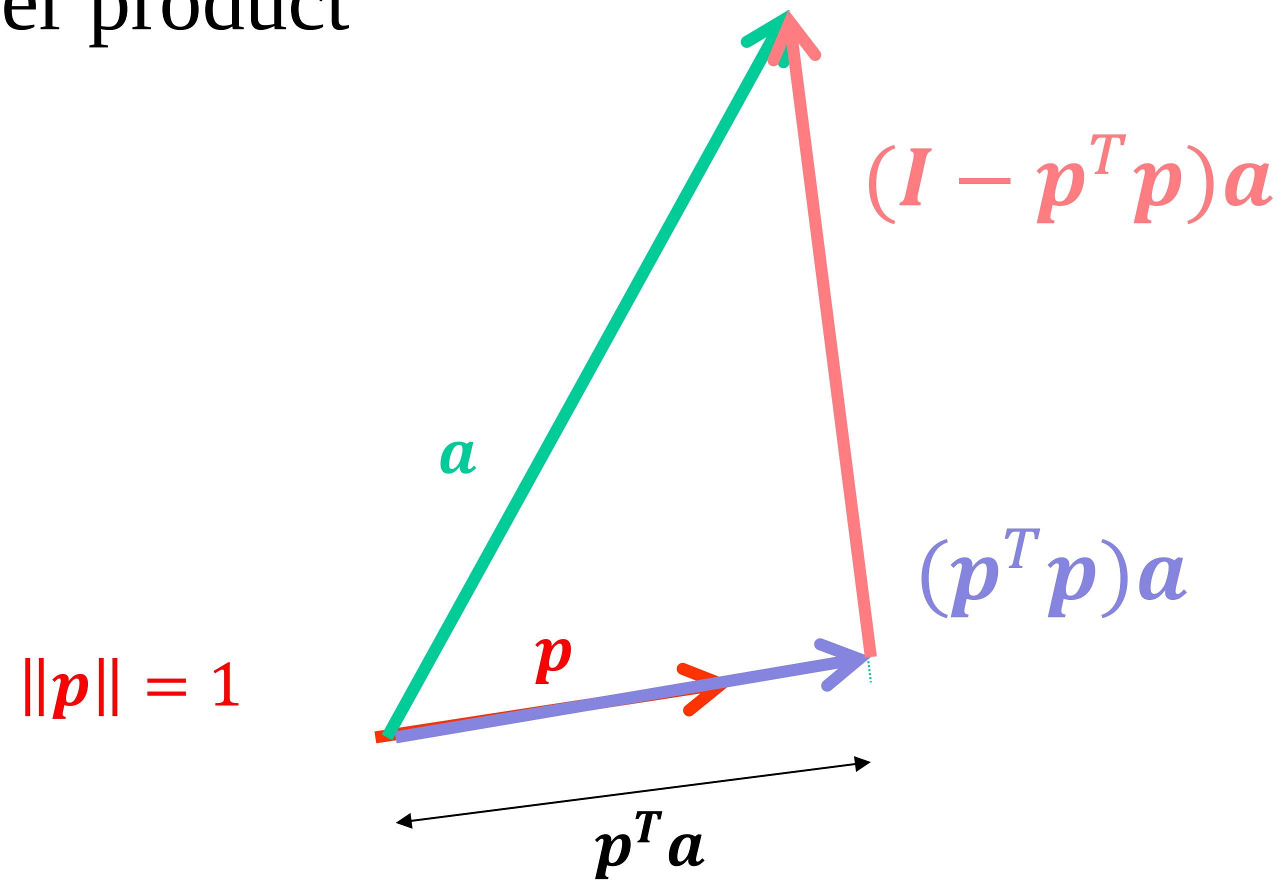
Operations with vectors

- Inner product



Operations with vectors

- Inner product

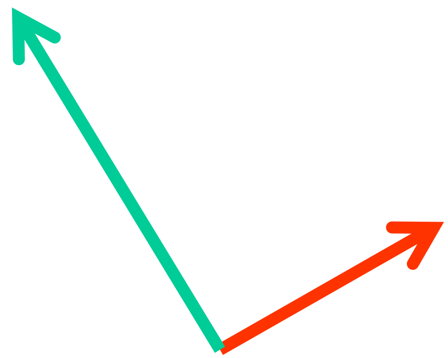


Operations with vectors

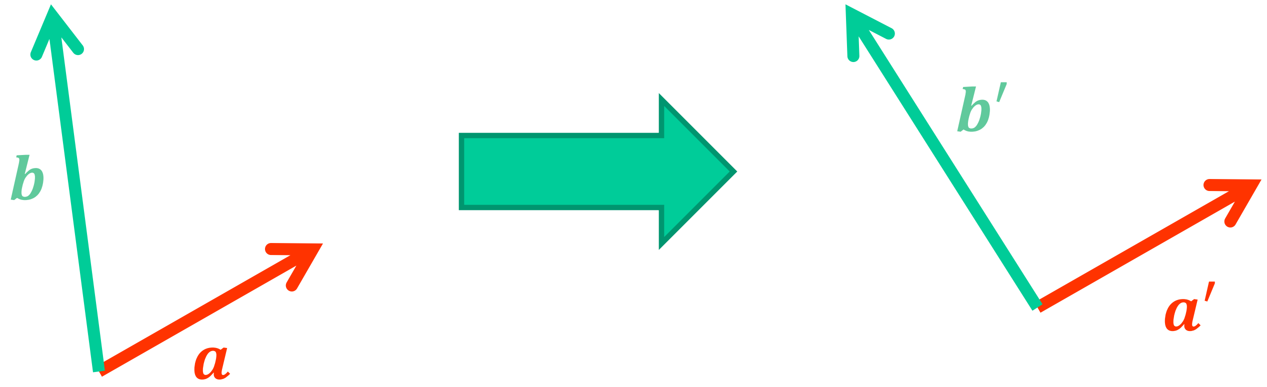
- Inner product
  - $\boldsymbol{a}^T \boldsymbol{b} = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot \cos \alpha$

Operations with vectors

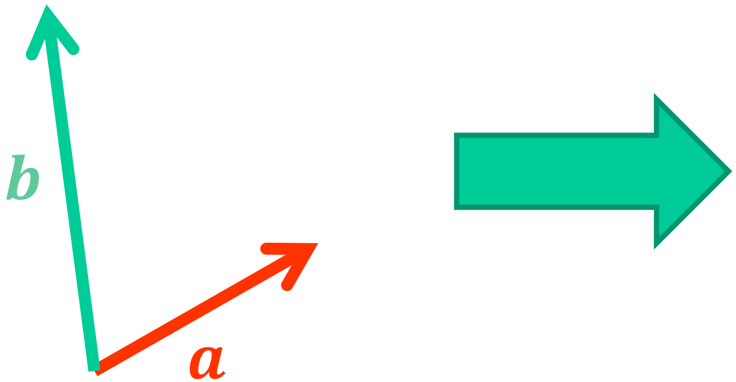
- Inner product
  - $\boldsymbol{a}^T\boldsymbol{b} = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot \cos\alpha$
  - $\boldsymbol{a}^T\boldsymbol{b} = 0 \iff \cos\alpha = 0$
  - In this case we say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are **orthogonal**.



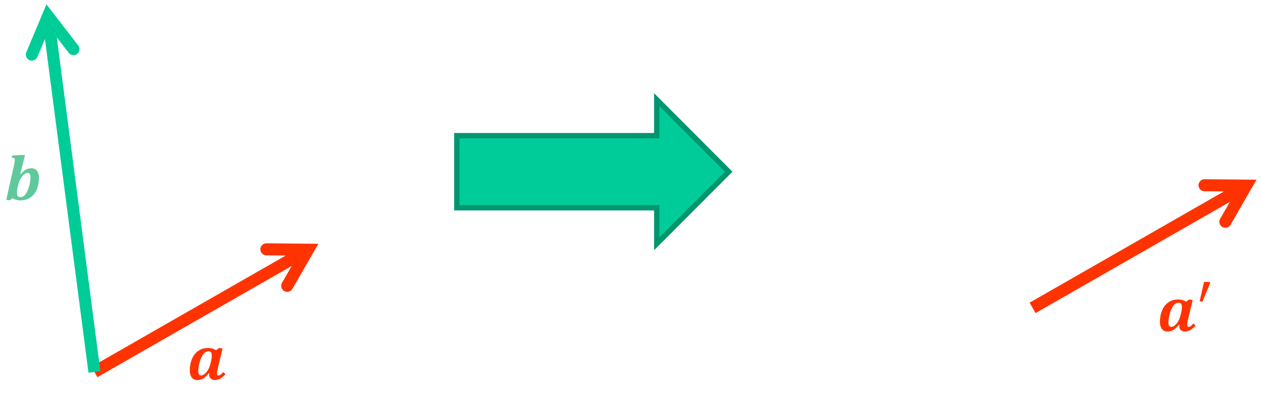
Orthogonalization



Orthogonalization

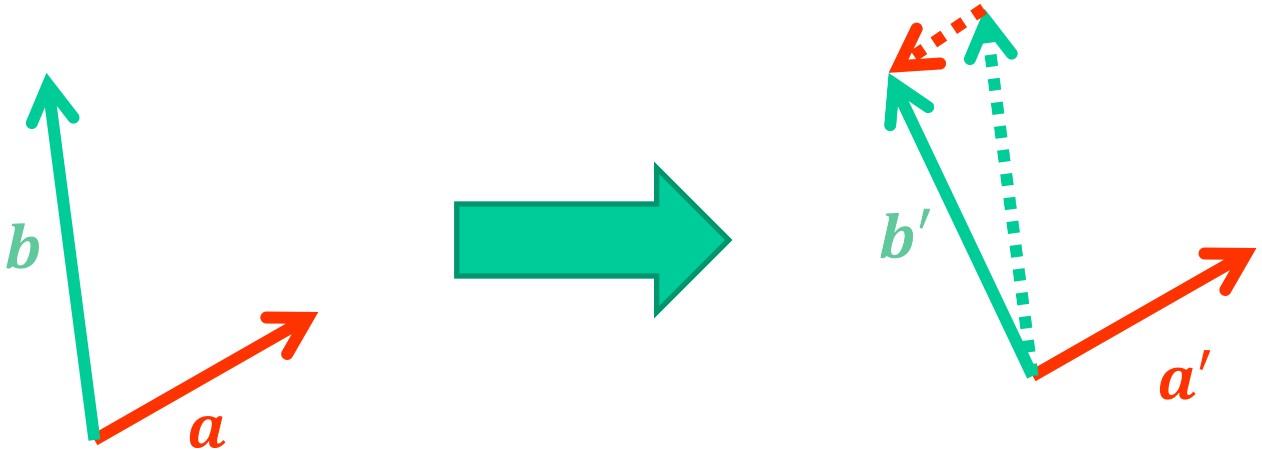


Orthogonalization



$$\boldsymbol{a}' = \boldsymbol{a}$$

Orthogonalization



$$\boldsymbol{a}' = \boldsymbol{a}$$

$$\boldsymbol{b}' = \boldsymbol{b} - \frac{\boldsymbol{a}'\boldsymbol{a}'^{T}}{\boldsymbol{a}'^{T}\boldsymbol{a}'}\boldsymbol{b}$$

Gram-Schmidt algorithm

$$\boldsymbol{a}' = \boldsymbol{a}$$

$$\boldsymbol{b}' = \boldsymbol{b} - \frac{\boldsymbol{a}'\boldsymbol{a}'^T}{\boldsymbol{a}'^T\boldsymbol{a}'}\boldsymbol{b}$$

$$\boldsymbol{c}' = \boldsymbol{c} - \frac{\boldsymbol{a}'\boldsymbol{a}'^T}{\boldsymbol{a}'^T\boldsymbol{a}'}\boldsymbol{c} - \frac{\boldsymbol{b}'\boldsymbol{b}'^T}{\boldsymbol{b}'^T\boldsymbol{b}'}\boldsymbol{c}$$

…

Orthonormality

- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal and unit-length, we say they are *ortho**normal***.
- A set of m orthonormal vectors in $\mathbb{R}^m$ is an *orthonormal basis* of $\mathbb{R}^m$.
- Give an example of an orthonormal basis for $\mathbb{R}^3$.

Operations with Vectors

- Box product
  - Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$.
  - The box product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is:

$$|\boldsymbol{a}\ \ \boldsymbol{b}| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$$

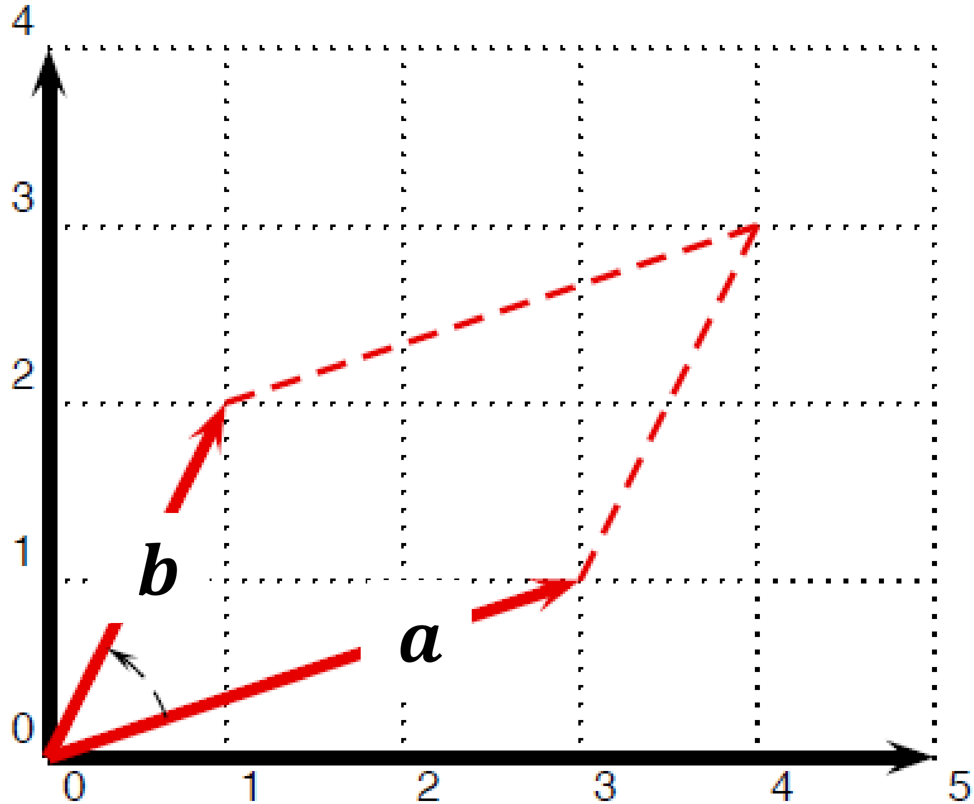
Operations with Vectors

- Box product
  - $|\boldsymbol{a}\ \ \boldsymbol{b}| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\alpha$
  - $|\boldsymbol{b}\ \ \boldsymbol{a}| = -|\boldsymbol{a}\ \ \boldsymbol{b}|$

Operations with Vectors

- Box product

$$|\boldsymbol{a}\ \ \boldsymbol{b}| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\alpha$$



Operations with Vectors

- Box product in 3D:

$$|\boldsymbol{a} \;\; \boldsymbol{b} \;\; \boldsymbol{c}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

  - Corresponds to the *signed volume* of a parallelepiped constructed on the three vectors

Operations with Vectors

- Box product in 3D:

$$|\boldsymbol{a}\ \ \boldsymbol{b}\ \ \boldsymbol{c}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

  - Corresponds to the *signed volume* of a parallelepiped constructed on the three vectors
  - The sign determines the *orientation* of the vectors.

Orientation

- *m* vectors in an *m*-dimensional space have an *orientation*.
- Orientations in 2D and 3D have conventional names: *right-handed* and *left-handed*.
- You can also speak about *positive* and *negative* orientation *relative to the basis*.

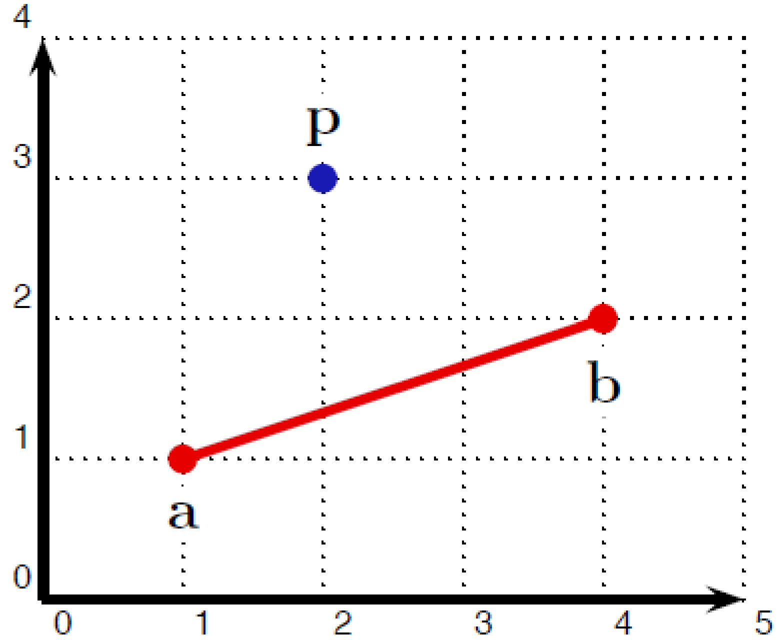
Right-handed basis

- In mathematics the right-handed basis is most often used.

In this basis any *positively* oriented pair is also a right-handed pair.

Quiz

- How to determine whether a given point lies to the left or to the right of a given segment?



Operations with Vectors

- Cross product

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Operations with Vectors

- Cross product

  - $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$
  - $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot |\sin \alpha|$
  - $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b})$ is *positively* oriented

Orthogonalization in 3D

- Orthogonalization of a right-handed basis in 3D using cross product:
  - $\boldsymbol{c}' = \boldsymbol{a} \times \boldsymbol{b}$
  - $\boldsymbol{b}' = \boldsymbol{c}' \times \boldsymbol{a}$
  - $\boldsymbol{a}' = \boldsymbol{a}$

Quiz

- A magical unicorn in your 3D world is flying in the direction given by vector $\boldsymbol{v}$.

- The user pushes the button “right”, which should give an impulse to the unicorn towards the right (wrt its current flight direction). Compute the vector pointing to the right.

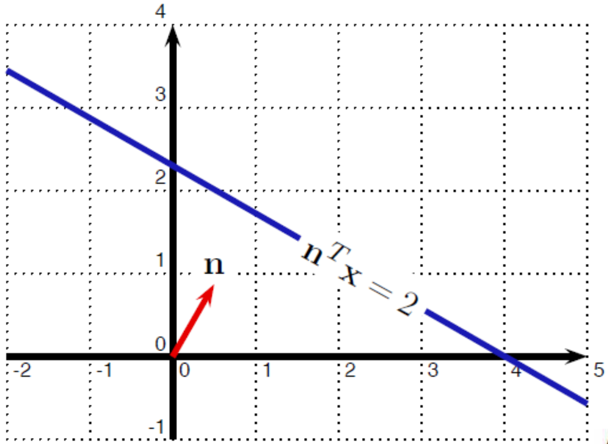
Mathematical background

- Vectors:
  - Points, directions, vectors and matrices
  - Linear combinations, convex combinations
  - Norm, normalization
  - Inner product, orthogonality, orthogonalization
  - Box product, Cross product
  - Orientation
  - Representation of a straight line

Straight line

- Parametric representation
  - $\boldsymbol{x} = \lambda \boldsymbol{a} + (1 - \lambda)\boldsymbol{b}$
  - $\boldsymbol{x} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$

- Implicit representation
  - $\boldsymbol{n}^T(\boldsymbol{x} - \boldsymbol{p}) = 0$
  - $\boldsymbol{n}^T\boldsymbol{x} = \boldsymbol{n}^T\boldsymbol{p}$
  - $n_1 x_1 + n_2 x_2 - b = 0$



Mathematical background

- Matrices:
  - Linear transformations
  - Invertibility, rank, determinant
  - Orthogonal transformations
  - Affine transformations
  - Homogeneous coordinates

Linear transformations

- A transformation $f: \mathcal{V}_1 \rightarrow \mathcal{V}_2$ is called *linear* (also *homomorphism*) if

$$f(\alpha \boldsymbol{x} + \boldsymbol{y}) = \alpha f(\boldsymbol{x}) + f(\boldsymbol{y})$$

- Examples of linear transformations are:
  - Rotation around origin, scaling, shear, reflection, projection or combinations of those.

Quiz

- Which of those are linear transformations?
  - $f(x) = x$
  - $f(x) = -4x$
  - $f(x) = 4x + 4$
  - $f(x) = x^2$
  - $f(x) = 3$
  - $f(x) = 0$

Quiz

- Which of those are linear transformations?
  - $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$
  - $f(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{x}$
  - $f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x}$
  - $f(\boldsymbol{x}) = |\boldsymbol{a} \;\; \boldsymbol{b} \;\; \boldsymbol{x}|$
  - $f(\boldsymbol{x}) = \boldsymbol{a} \times \boldsymbol{x}$
  - $f(x) = \boldsymbol{a}^T\boldsymbol{x} + |\boldsymbol{a} \;\; \boldsymbol{b} \;\; \boldsymbol{x}| + \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{A}\boldsymbol{x}$

Linear transformations

- Each linear transformation is uniquely defined by how it transforms the basis:

$$f\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} =$$

Linear transformations

- Each linear transformation is uniquely defined by how it transforms the basis:

$$f\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = f\left(2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Linear transformations

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$$f\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = f\left(2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$= 2f\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Linear transformations

- Each linear transformation is uniquely defined by how it transforms the basis:

$$f\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = f\left(2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$= 2\boldsymbol{f}_1 - 3\boldsymbol{f}_2 + 4\boldsymbol{f}_3$$

Linear transformations

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$$= |\boldsymbol{f}_1 \;\; \boldsymbol{f}_2 \;\; \boldsymbol{f}_3| \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

Linear transformations

- Each linear transformation is uniquely defined by how it transforms the basis:

$$f\begin{pmatrix}2\\-3\\4\end{pmatrix} = f\left(2\begin{pmatrix}1\\0\\0\end{pmatrix} - 3\begin{pmatrix}0\\1\\0\end{pmatrix} + 4\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$$

$$= \boldsymbol{F}\begin{pmatrix}2\\-3\\4\end{pmatrix}$$

Linear transformations

Each linear transformation corresponds to a matrix.

Linear transformations

Each linear transformation corresponds to a matrix.

Columns of a matrix show how it transforms the canonical basis

Quiz

- How does this matrix transform the (canonical) basis?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Quiz

- How does this matrix transform the (canonical) basis?

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Quiz

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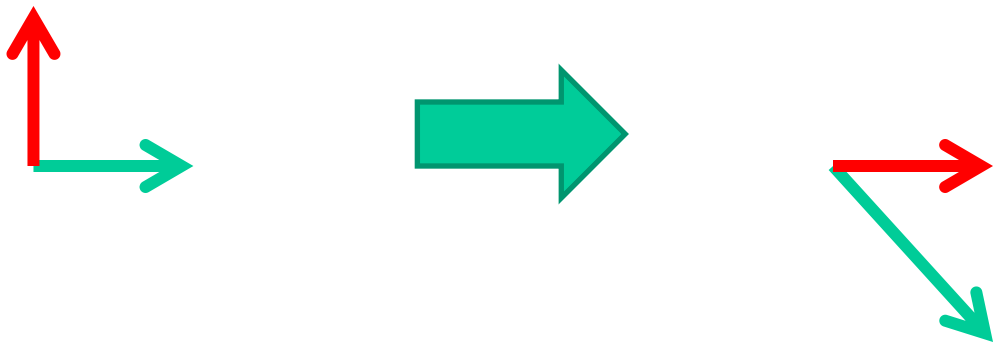
Quiz

- How does this matrix transform the (canonical) basis?

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Quiz

- Which matrix does the following?



Quiz

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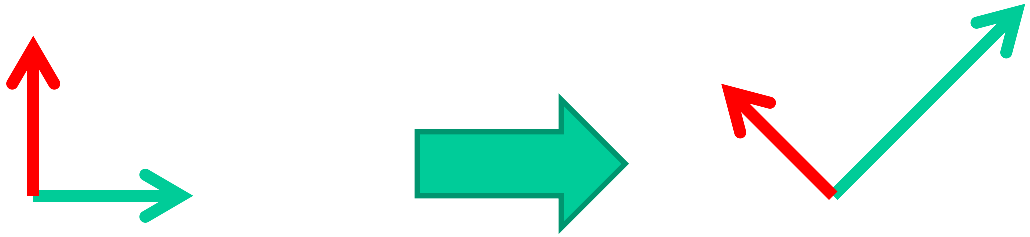

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Composition of transformations corresponds to matrix multiplication:

$$(f \otimes g)(\boldsymbol{x}) = f\big(g(\boldsymbol{x})\big) = \boldsymbol{F}\boldsymbol{G}\boldsymbol{x}$$

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Function composition is associative, hence matrix multiplications is too:

$$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$
$$(\boldsymbol{FG})\boldsymbol{H} = \boldsymbol{F}(\boldsymbol{GH})$$

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:
  - Sum of transformations corresponds to matrix sum:

$$(f + g)(\boldsymbol{x}) = f(\boldsymbol{x}) + g(\boldsymbol{x}) = (\boldsymbol{F} + \boldsymbol{G})\boldsymbol{x}$$

Linear transformations

- Let f, g, h be linear transformations and $\boldsymbol{F}, \boldsymbol{G}, \boldsymbol{H}$ the corresponding matrices, then:

  - Composition is distributive wrt sum:

$$(f + g) \otimes h = f \otimes h + g \otimes h$$
$$(\boldsymbol{F} + \boldsymbol{G})\boldsymbol{H} = \boldsymbol{F}\boldsymbol{H} + \boldsymbol{G}\boldsymbol{H}$$

Rank

- Consider a linear transformation $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ it will always either:
  - Map the whole 3D space to itself somehow
  - Project the whole 3D space to a plane
  - Project the whole 3D space to a line
  - Map all points to 0.

Rank

- The dimensionality of the resulting space is the ***rank*** of f.

  - If f is full rank (i.e. $\text{rank}(f) = 3$ in our case), it is *invertible*. Otherwise it is not.

  - f is invertible $\Leftrightarrow \det(\boldsymbol{F}) \neq 0$

Orthogonal transformations

- A transformation ***F*** is called orthogonal if it maps the canonical basis into an **orthonormal basis**.

Orthogonal transformations

- A transformation $\boldsymbol{F}$ is called orthogonal if it maps the canonical basis into an **orthonormal basis**.

  - It must keep lengths and angles intact, i.e. it is a **rotation** (possibly mirrored).

  - $\boldsymbol{F}^T \boldsymbol{F} = ?$

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  - It must keep lengths and angles intact, i.e. it is a **rotation** (possibly mirrored).

  - $\boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{I}$, because the columns are orthonormal
  - Hence, $\boldsymbol{F}^{-1} = \boldsymbol{F}^T$

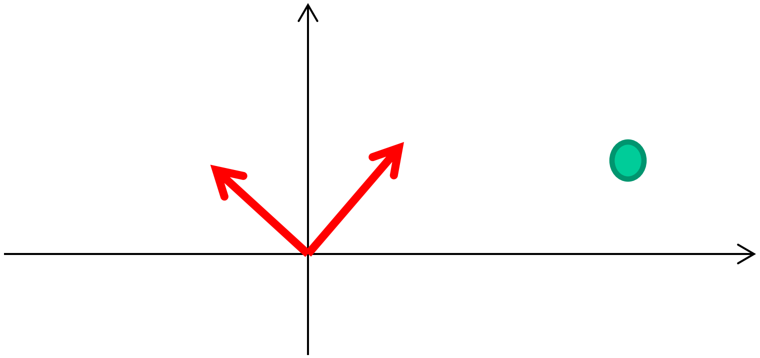
Orthogonal transformations

To compute the inverse of an orthogonal matrix, simply transpose it.

Quiz

- You are standing at the origin, rotated with respect to the coordinate system, looking in the direction $(0.3, 0.4)$ (your local "x" axis).

- At position $(7,2)$ there is an object. What are the coordinates of this object with respect to you?



Examples

- Rotation: $\boldsymbol{R}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$
- Scaling: $\boldsymbol{S}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
- Mirroring: $\mathbf{Mir}_{\boldsymbol{y}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Shear: $\mathbf{Sh}_{\mathbf{x}}(\mathrm{a}) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

Examples

- Rotation around z axis:

$$\boldsymbol{R_z}(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Rotation around y axis:

$$\boldsymbol{R_z}(\alpha) = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

Shift

- Shift (translation) is not a linear transformation.

- To deal with shifts we must introduce the notion of an *affine space* and *affine transformations*.

Affine space

- Vector space
- Affine space

Affine space

- Vector space
  - Vectors $\boldsymbol{v} \in \mathbb{R}^3$
  - Basis: $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$
  - Linear transformations $f(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{x}$
- Affine space

Affine space

- Vector space
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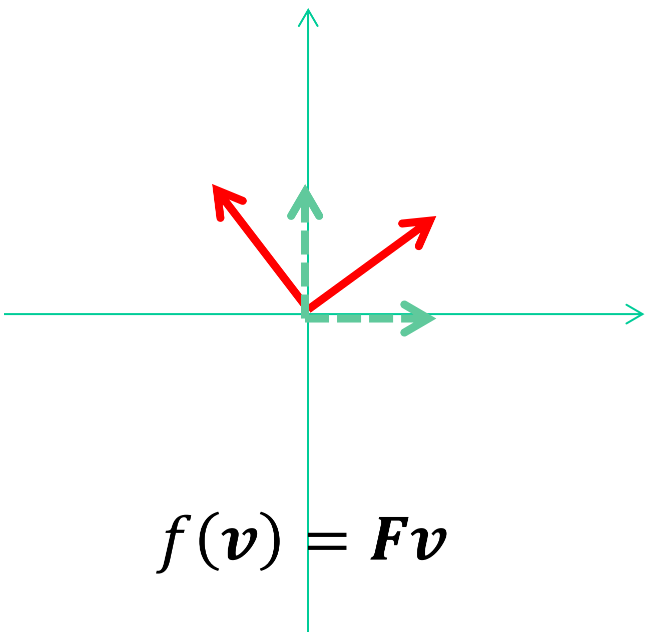
- Affine space
  - Vectors $\boldsymbol{v} \in \mathbb{R}^3$
  - Points $P \in \mathbb{R}^3$
    - point+vector = point
  - Frame: $(O, \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\})$
  - Affine transformations: $f(\boldsymbol{v}) = \boldsymbol{F}\boldsymbol{v}$ $f(P) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$

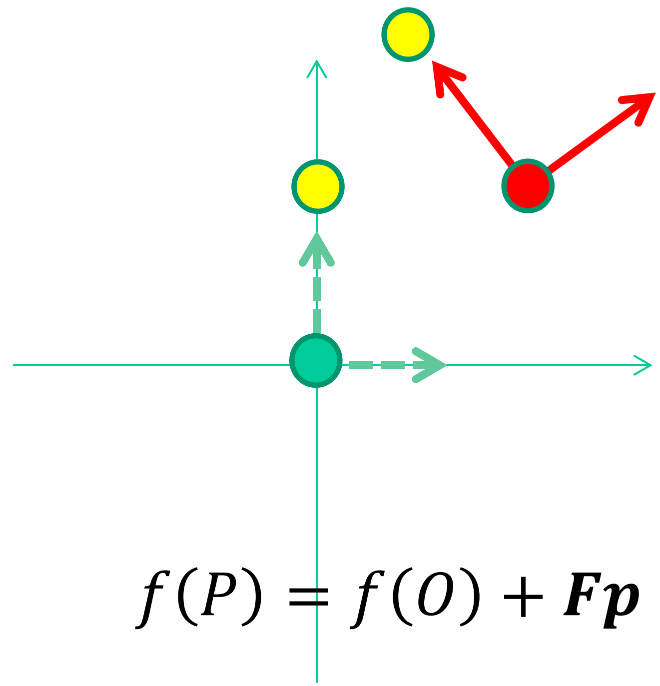
Affine space

- Vector space



$$f(\boldsymbol{v}) = \boldsymbol{F}\boldsymbol{v}$$

- Affine space



$$f(P) = f(O) + \boldsymbol{F}\boldsymbol{p}$$

Affine transformations

$$f(\boldsymbol{p}) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$$

Affine transformations

$$f(\boldsymbol{p}) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

Affine transformations

$$f(\boldsymbol{p}) = \boldsymbol{t} + \boldsymbol{F}\boldsymbol{p}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ 1 \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & t_1 \\ f_{21} & f_{22} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix}$$

Homogeneous coordinates

- We shall represent the **points** of an affine space using 3-dimensional vectors of the form $(p_1, p_2, 1)^T$
- We shall represent the **vectors** of an affine space using 3-dimensional vectors of the form $(v_1, v_2, 0)^T$
- Any affine transformation is a matrix

$$\left(\begin{array}{cc|c} f_{11} & f_{12} & t_1 \\ f_{21} & f_{22} & t_2 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Homogeneous coordinates

- Analogously, for 3D space we use 4-dimensional vectors and 4x4 matrices.
- E.g. the following transformation rotates around z axis and shifts along x axis by 0.5:

$$\left(\begin{array}{ccc|c} \cos\phi & -\sin\phi & 0 & 0.5 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

Homogeneous coordinates

- Note how the representation implicitly enforces the rules:
    - vector + vector = vector
    - point + vector = point
    - point + point = undefined
    - convex combination of points = point

Homogeneous coordinates

- Rotation: $$\boldsymbol{R}(\alpha) = \left(\begin{array}{cc|c} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

- Scaling: $$\boldsymbol{S}(a, b) = \left(\begin{array}{cc|c} a & 0 & 0 \\ 0 & b & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

- Translation: $$\boldsymbol{T}(x, y) = \left(\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ \hline 0 & 0 & 1 \end{array}\right)$$

Quiz

- Construct a matrix, that performs a rotation by 10 degrees around the point (20, 30) in homogeneous coordinates.

Quiz

- Construct a matrix, that performs a rotation by 10 degrees around the point (20, 30) in homogeneous coordinates.

$$\boldsymbol{T}(20,30)\boldsymbol{R}(10)\boldsymbol{T}(-20,-30)$$

Mathematical background

- Matrices:
  - Linear transformations
  - Invertibility, rank, determinant
  - Orthogonal transformations
  - Affine transformations
  - Homogeneous coordinates