# Computer Graphics

Mathematical background

Konstantin Tretyakov
kt@ut.ee

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# In the previous episodes

- Computer graphics is useful and fun
- Computer graphics is about generating images
- Modeling, Rendering, Animation
- Raster vs Vector, 2D vs 3D
- Ad-hoc projection vs Light physics
- “Standard graphics pipeline”
- Matrix notation

# Mathematical background

- Vectors:
  - Points, directions, vectors and matrices
  - Linear combinations, convex combinations
  - Norm, normalization
  - Inner product, orthogonality, orthogonalization
  - Box product, Cross product
  - Orientation
  - Representation of a straight line

# Mathematical background

- Matrices:
  - Linear transformations
  - Invertibility, rank, determinant
  - Orthogonal transformations
  - Affine transformations
  - Homogeneous coordinates

# Vectors

In general, vectors are elements of a _____________ space.

# Vectors

In computer graphics, we primarily deal with vector spaces $\mathbb{R}^2, \mathbb{R}^3$ and $\mathbb{R}^4$.

# Vectors

In computer graphics, we primarily deal with vector spaces $\mathbb{R}^2, \mathbb{R}^3$ and $\mathbb{R}^4$.

We use those vectors to denote __________ and ___________

# Operations with vectors

- Linear combinations:

$$3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

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$$3 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

- A linear combination

$$\lambda_1 \boldsymbol{v}_1 + \lambda_2 \boldsymbol{v}_2 + \cdots + \lambda_n \boldsymbol{v}_n$$

is called *convex* if ______________________________

______________________________________________

# Operations with vectors

- Norm

$$\|\boldsymbol{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

# Operations with vectors

- Norm

$$\left\| \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\| =$$

# Operations with vectors

- Norm

$$\left\|\begin{pmatrix}3\\4\end{pmatrix}\right\| =$$

$$\left\|\begin{pmatrix}33\\44\end{pmatrix}\right\| =$$

# Operations with vectors

- Normalization:

$$\mathrm{normalize}(\boldsymbol{a}) \coloneqq \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

# Operations with vectors

- Normalization:

$$\text{normalize}\begin{pmatrix}0\\1\end{pmatrix} =$$

# Operations with vectors

- Normalization:

$$\text{normalize}\begin{pmatrix}44\\33\end{pmatrix} =$$

# Operations with vectors

- Inner product

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

# Operations with vectors

- Inner product

$$\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle =$$

# Operations with vectors

- Inner product

$$\left\langle \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle =$$

# Operations with vectors

- Inner product

$$\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle =$$

# Operations with vectors

- Inner product

$$\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle =$$

# Operations with vectors

- Inner product
  - $\langle \boldsymbol{a},\ \lambda \boldsymbol{b} + \boldsymbol{c} \rangle = \lambda \langle \boldsymbol{a}, \boldsymbol{b} \rangle + \langle \boldsymbol{a}, \boldsymbol{c} \rangle$

Linearity

  - $\langle \boldsymbol{a},\ \boldsymbol{a} \rangle = ||\boldsymbol{a}||^2$
  - $\langle \boldsymbol{a},\ \boldsymbol{b} \rangle = ||\boldsymbol{a}|| \cdot ||\boldsymbol{b}|| \cdot \cos \alpha$

Relationship with the norm

  - $\langle \boldsymbol{a},\ \boldsymbol{b} \rangle = \boldsymbol{a}^T \boldsymbol{b}$

Relationship with the matrix product notation

# Operations with vectors

- Inner product
  - $\boldsymbol{a}^T(\lambda \boldsymbol{b} + \boldsymbol{c}) = \lambda \boldsymbol{a}^T \boldsymbol{b} + \boldsymbol{a}^T \boldsymbol{c}$

    Linearity

  - $\boldsymbol{a}^T \boldsymbol{a} = \|\boldsymbol{a}\|^2$
  - $\boldsymbol{a}^T \boldsymbol{b} = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot \cos\alpha$

    Relationship with the norm

  - $\langle \boldsymbol{a},\ \boldsymbol{b} \rangle = \boldsymbol{a}^T \boldsymbol{b}$

    Relationship with the matrix product notation

# Operations with vectors

- Inner product
  - $\boldsymbol{p}^T \boldsymbol{a} = ||\boldsymbol{p}|| \cdot ||\boldsymbol{a}|| \cdot \cos \alpha$
  - If $||\boldsymbol{p}|| = 1$,

$\boldsymbol{p}^T \boldsymbol{a}$ is the length of

______________________________

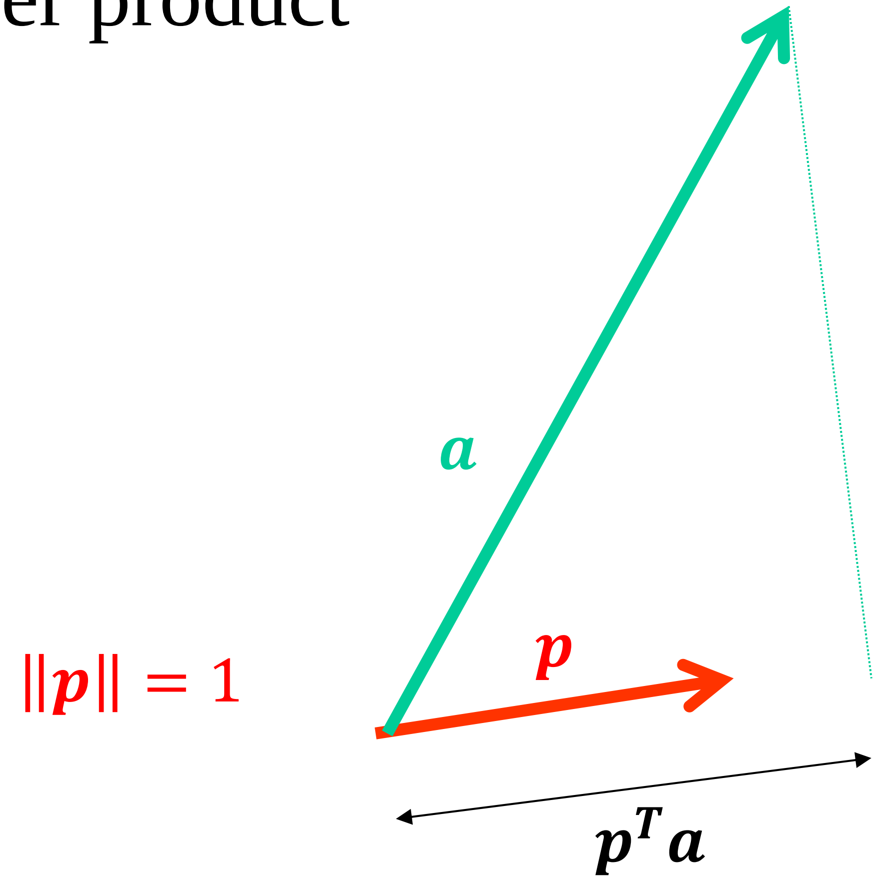
# Operations with vectors

- Inner product
  - $\boldsymbol{p}^T \boldsymbol{a} = ||\boldsymbol{p}|| \cdot ||\boldsymbol{a}|| \cdot \cos\alpha$
  - If $||\boldsymbol{p}|| = 1$,

$\boldsymbol{p}^T \boldsymbol{a}$ is the **length of the projection** of $\boldsymbol{a}$ onto $\boldsymbol{p}$.

# Operations with vectors

- Inner product



# Operations with vectors

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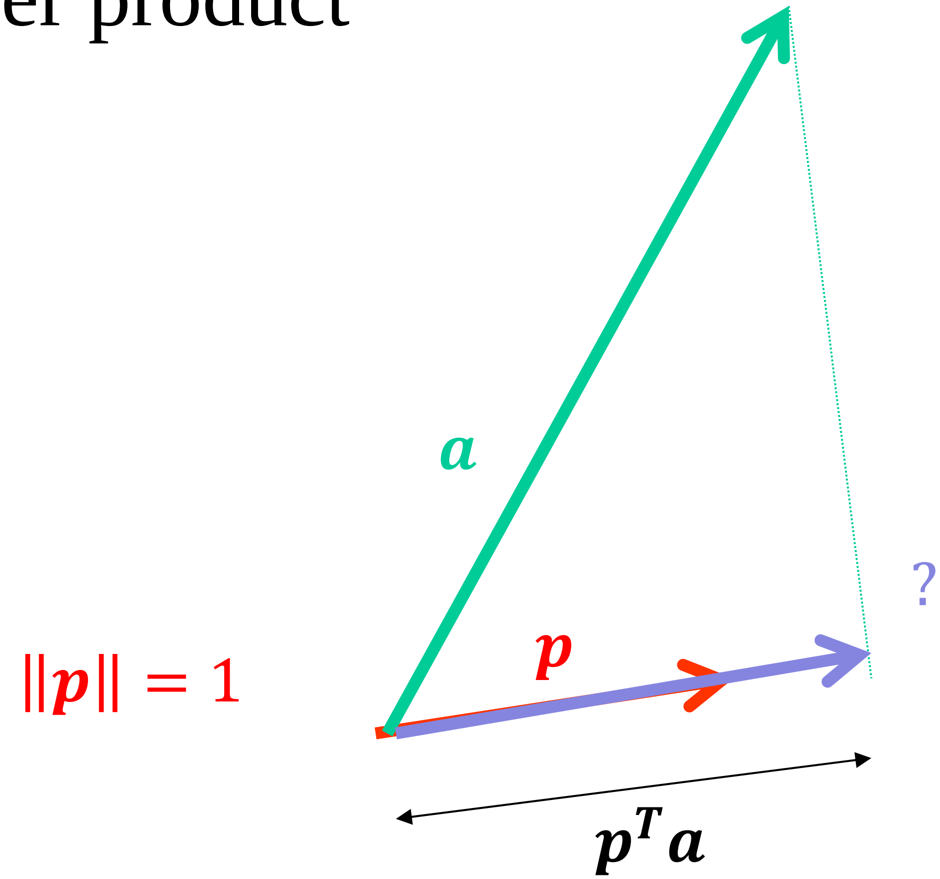

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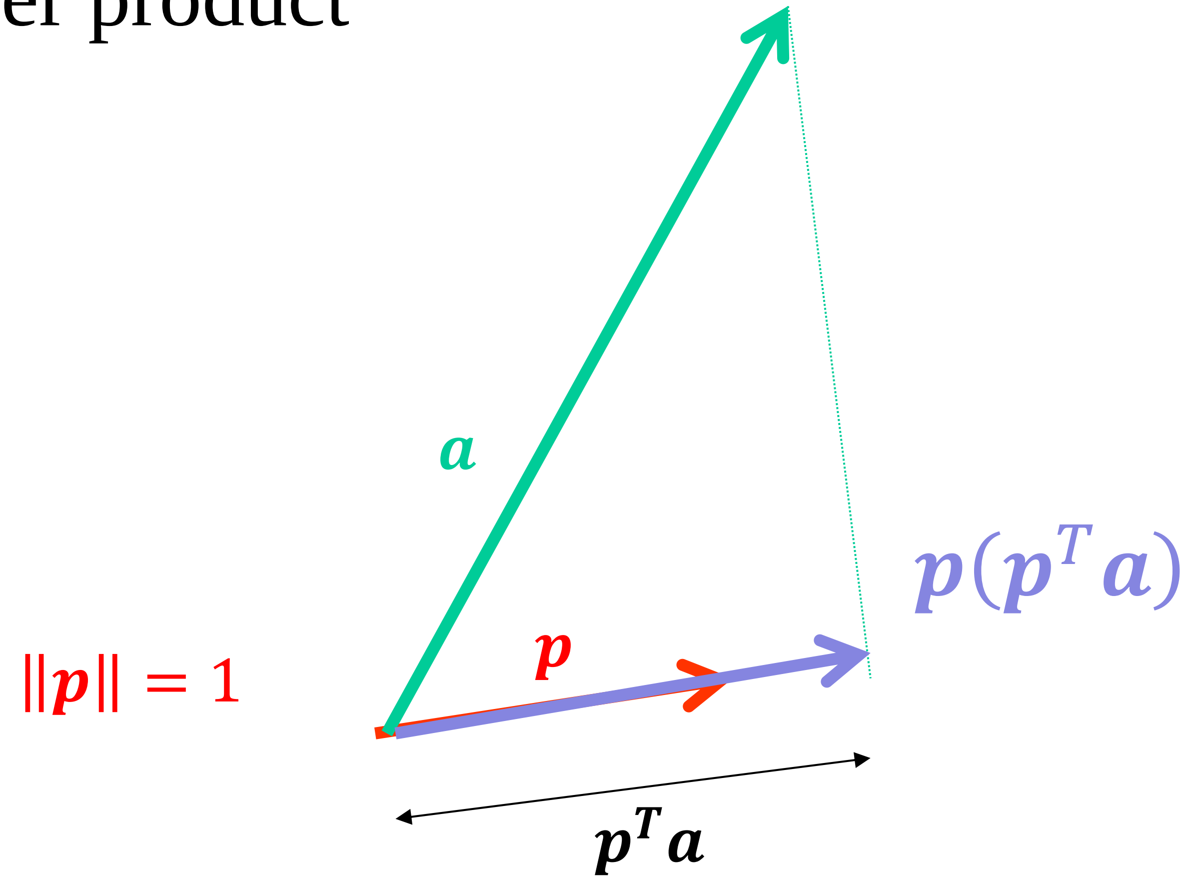

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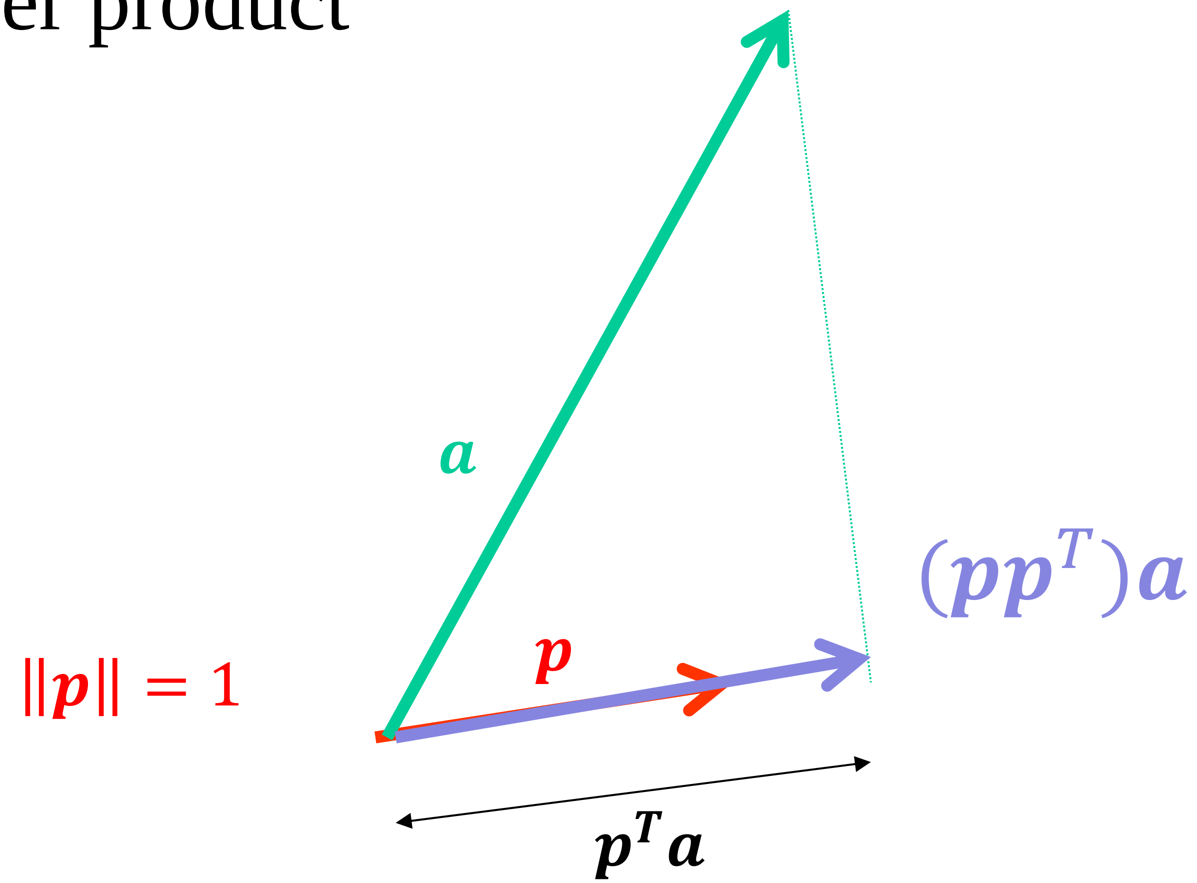

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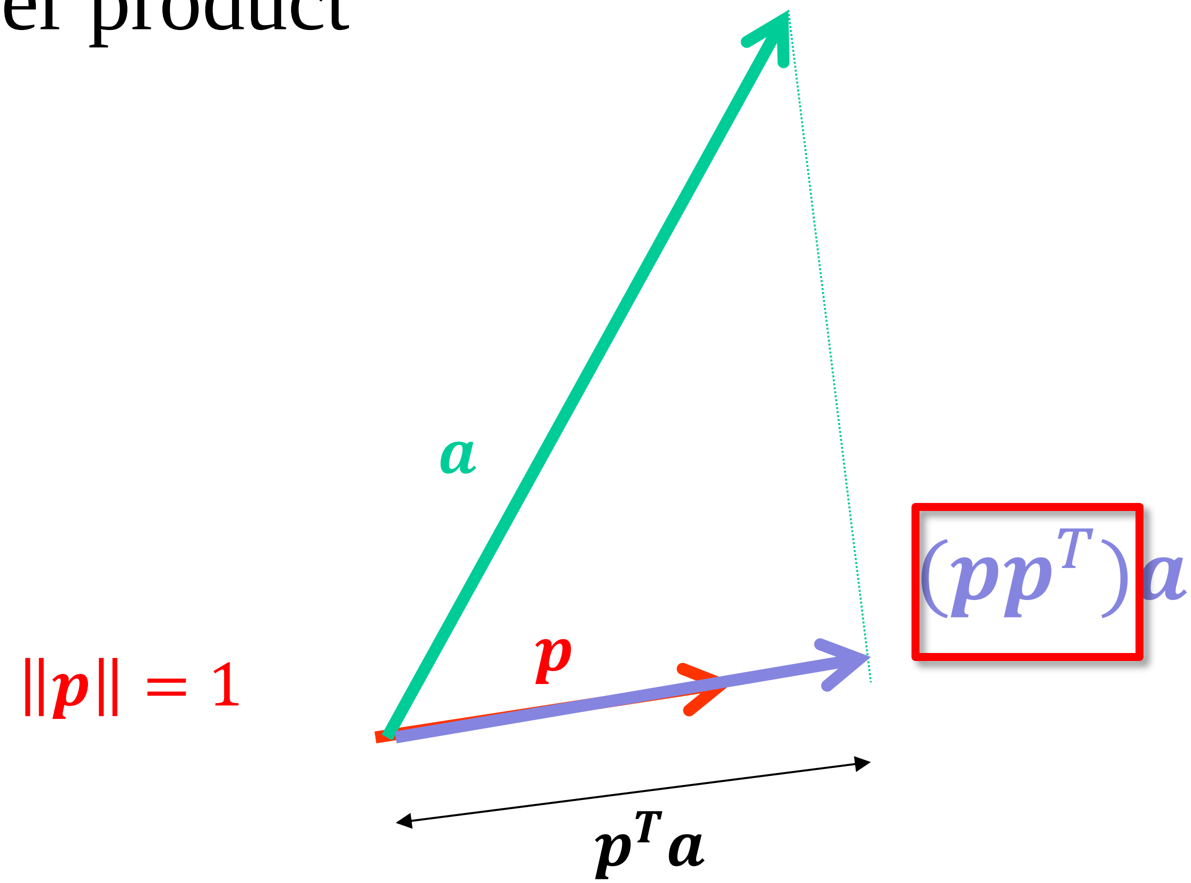

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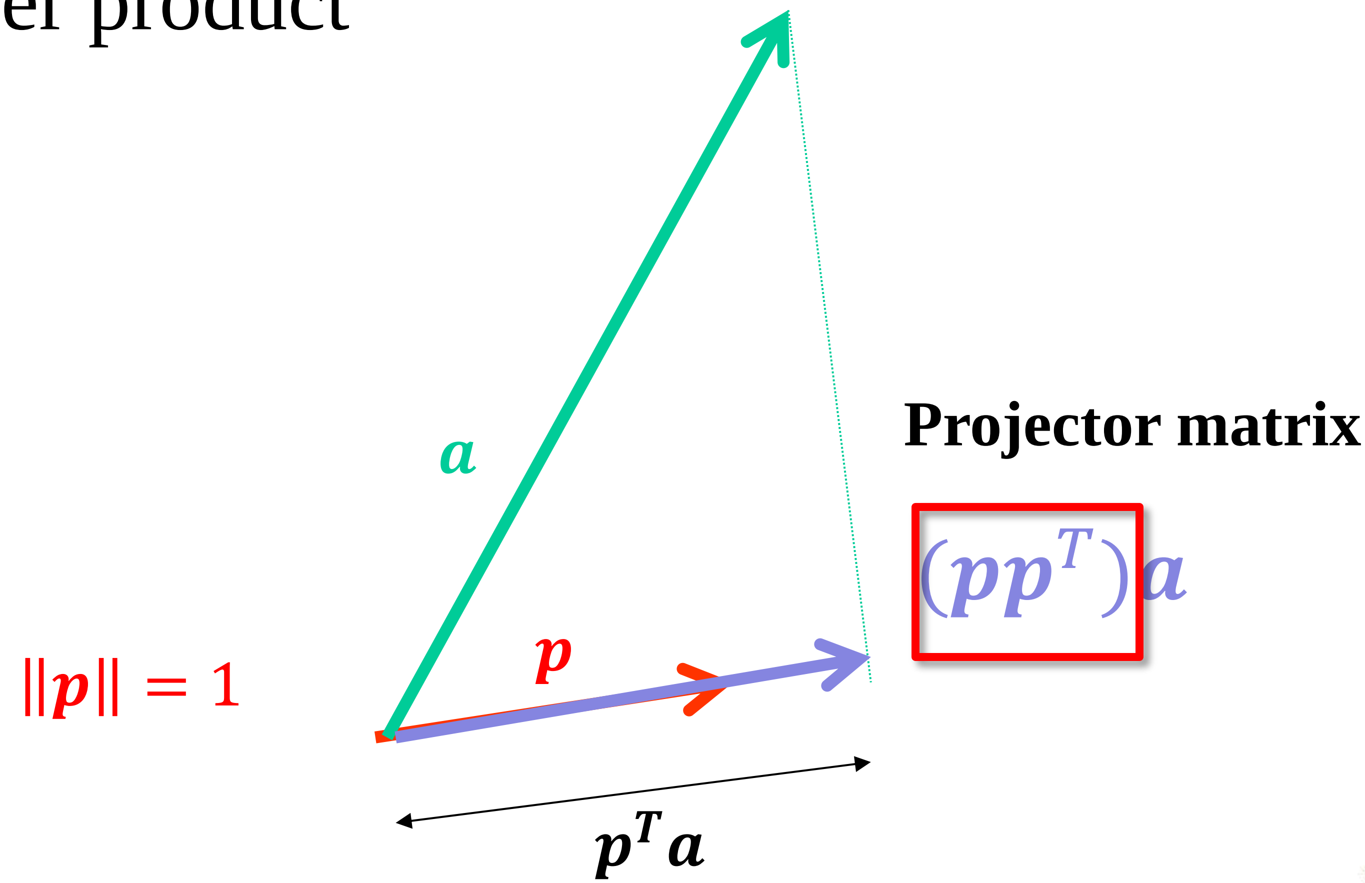


# Projector

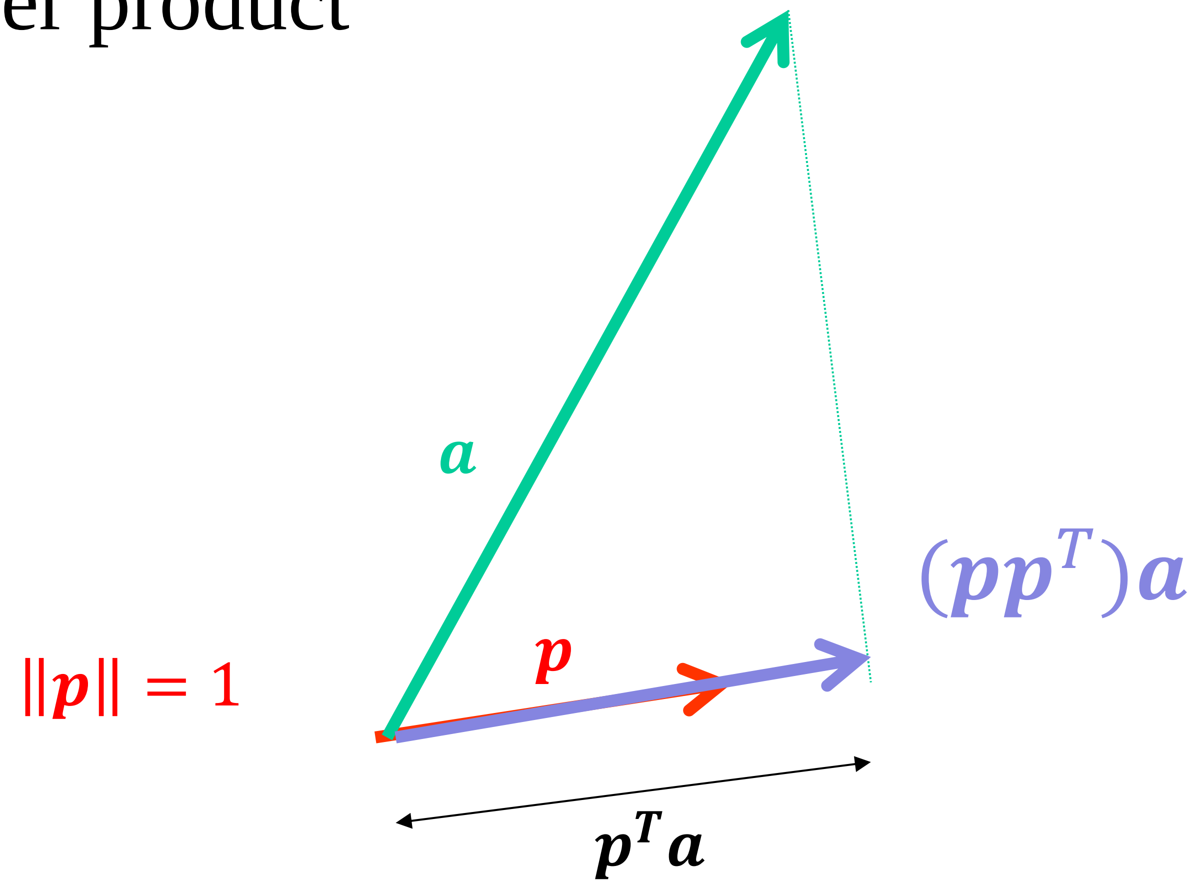
- For any nonzero vector $\boldsymbol{p}$ the matrix

$$\left(\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}\right)\left(\frac{\boldsymbol{p}}{\|\boldsymbol{p}\|}\right)^{T} = \frac{\boldsymbol{p}\boldsymbol{p}^{\boldsymbol{T}}}{\|\boldsymbol{p}\|^{2}} = \frac{\boldsymbol{p}\boldsymbol{p}^{T}}{\boldsymbol{p}^{T}\boldsymbol{p}}$$

is the *projector matrix* for $\boldsymbol{p}$.

# Operations with vectors

- Inner product



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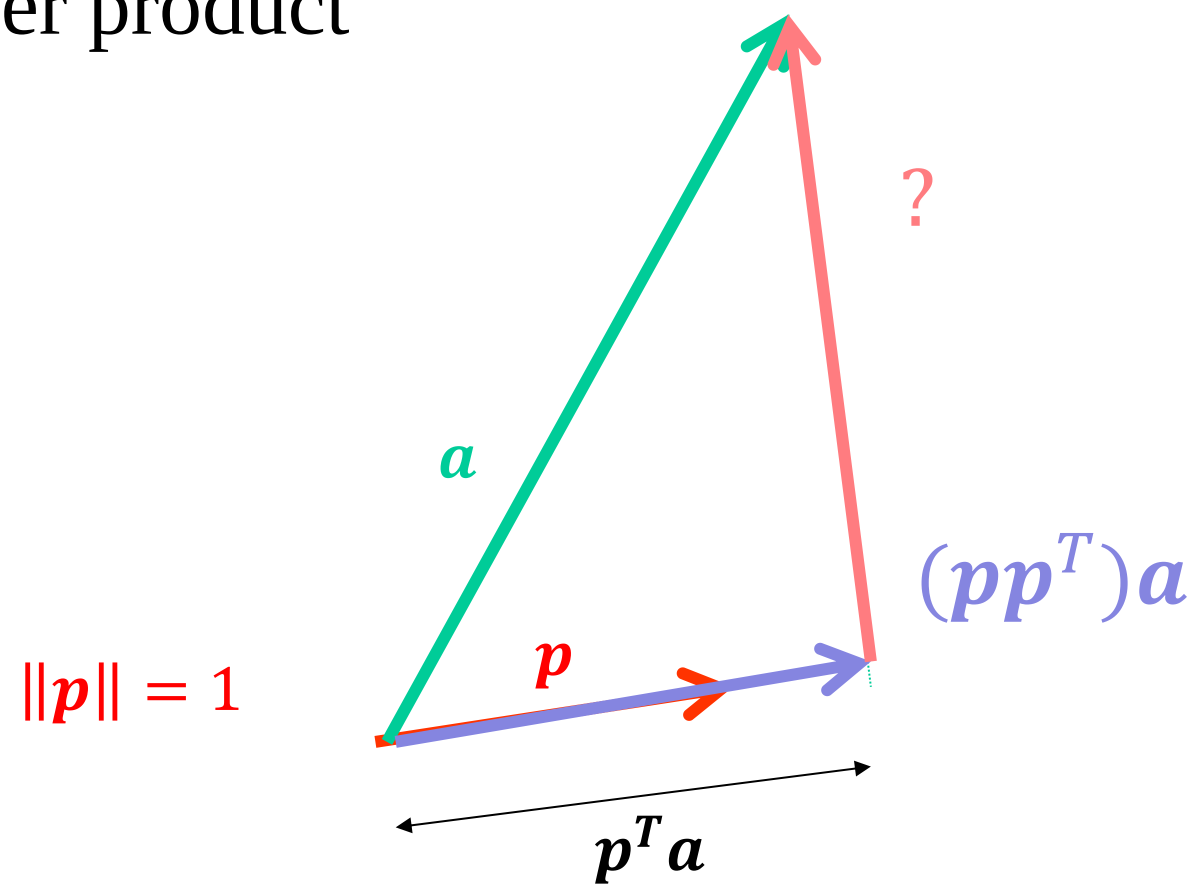

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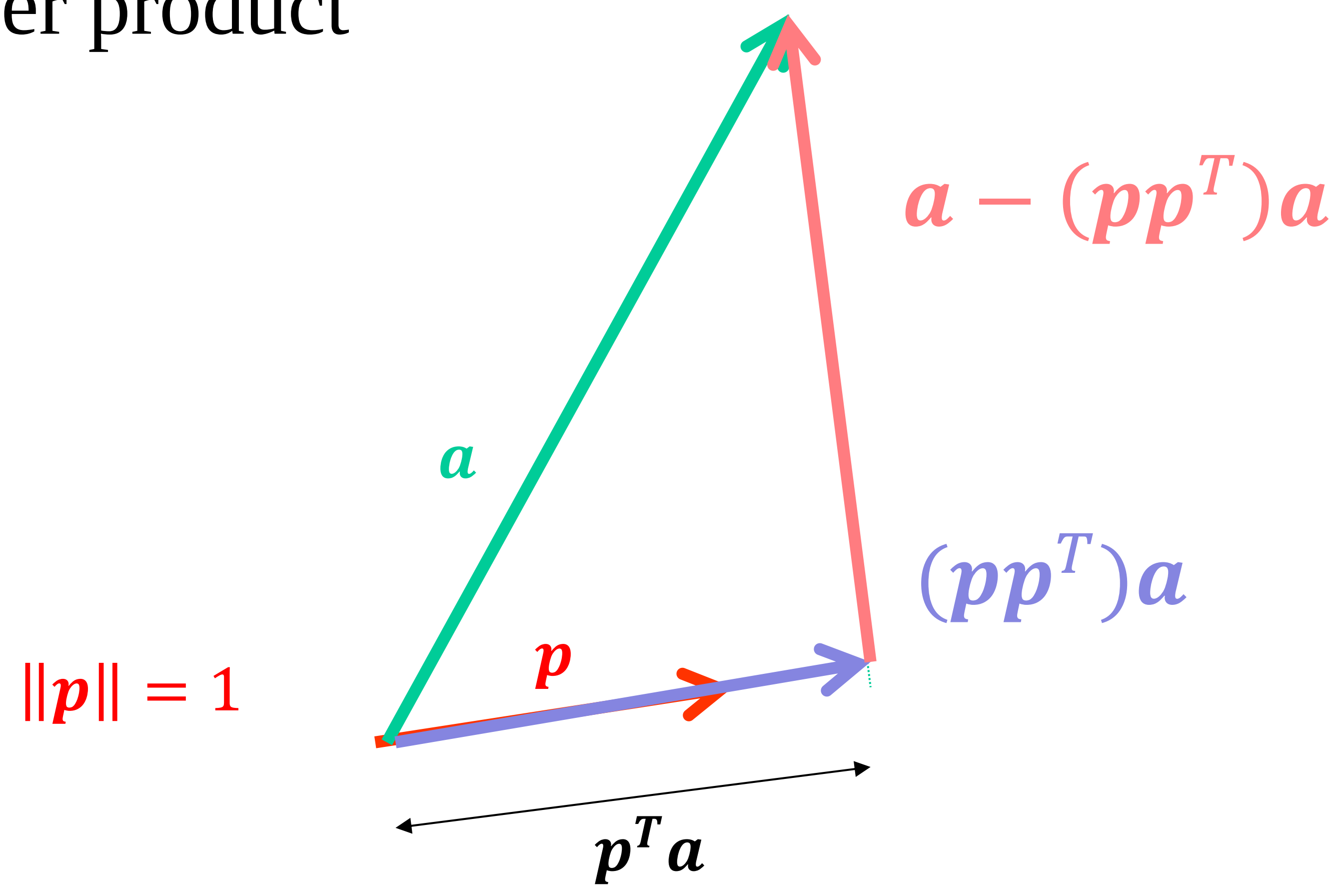

# Operations with vectors

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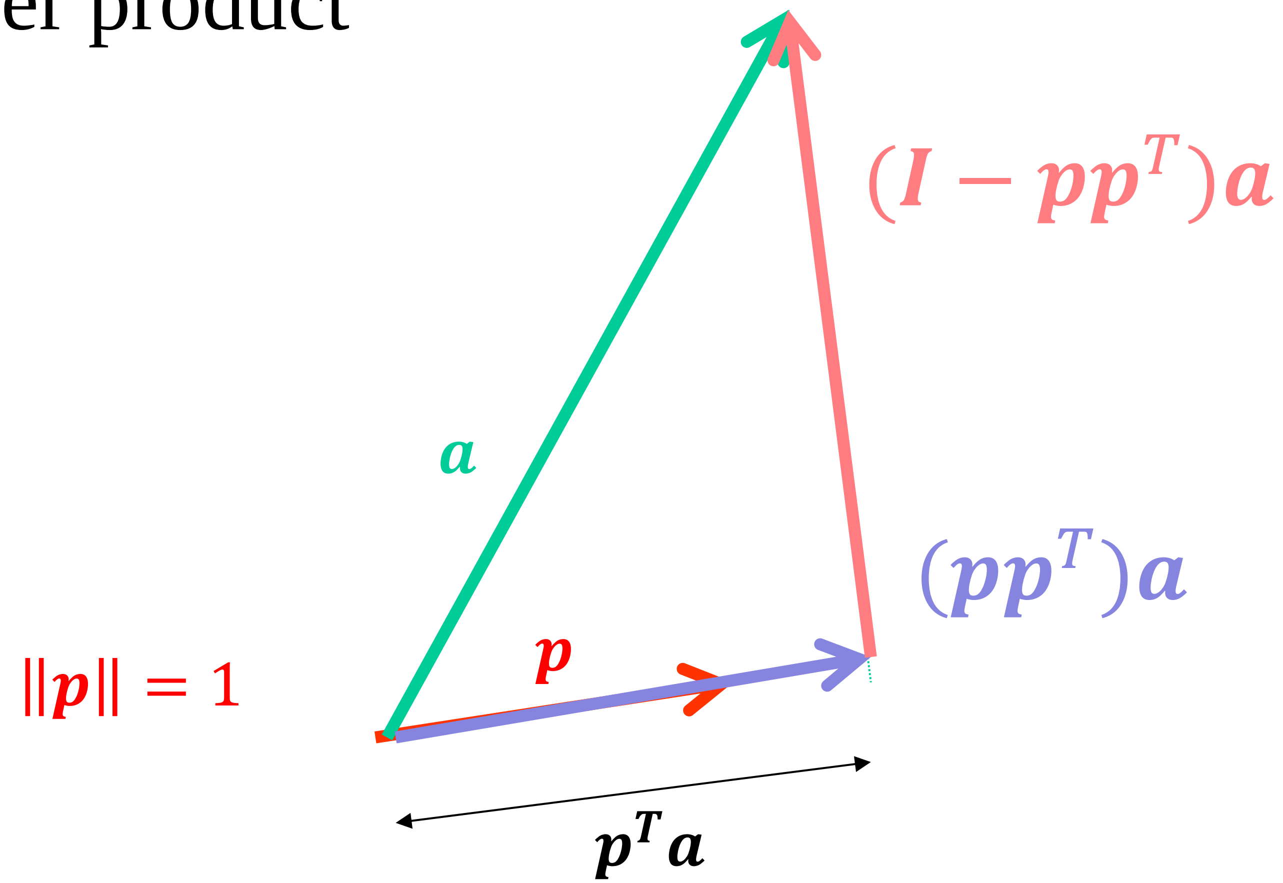
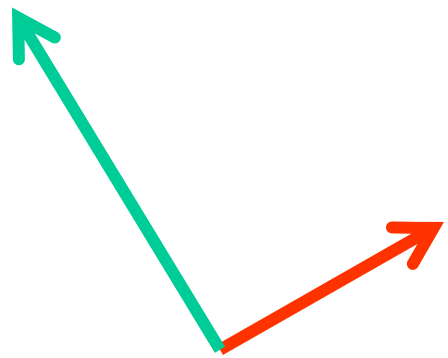


# Operations with vectors

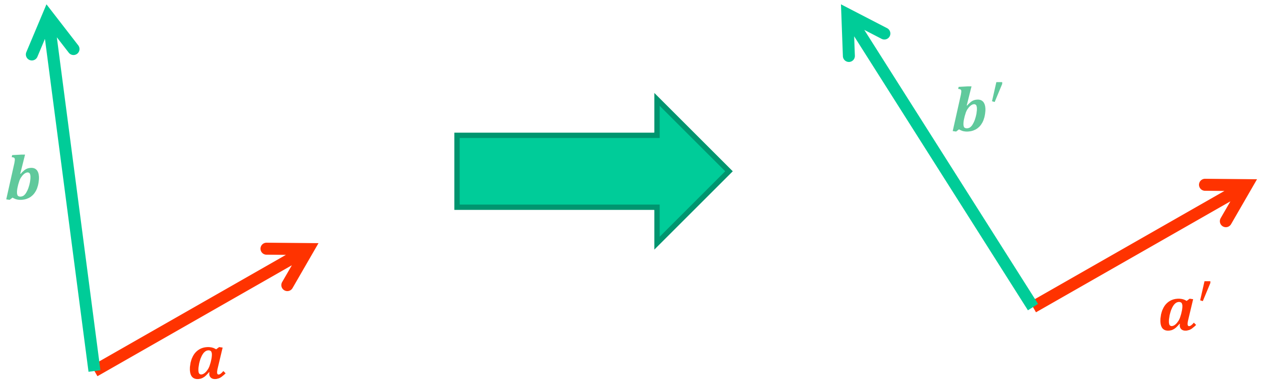
- Inner product
  - $\boldsymbol{a}^T \boldsymbol{b} = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot \cos \alpha$

# Operations with vectors

- Inner product
  - $\boldsymbol{a}^T\boldsymbol{b} = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot \cos\alpha$
  - $\boldsymbol{a}^T\boldsymbol{b} = 0 \;\Leftrightarrow\; \cos\alpha = 0$
  - In this case we say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are **orthogonal**.



# Orthogonalization



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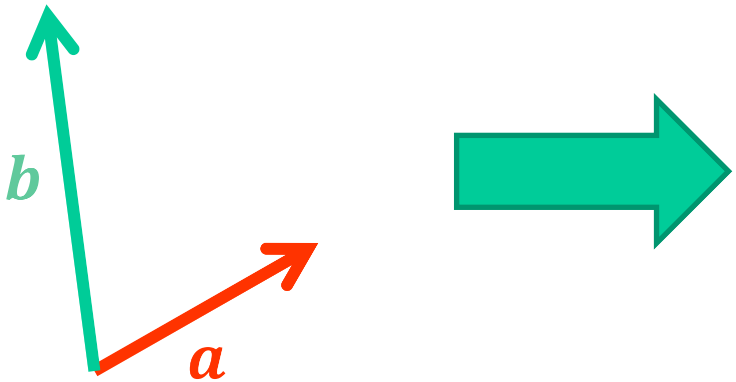

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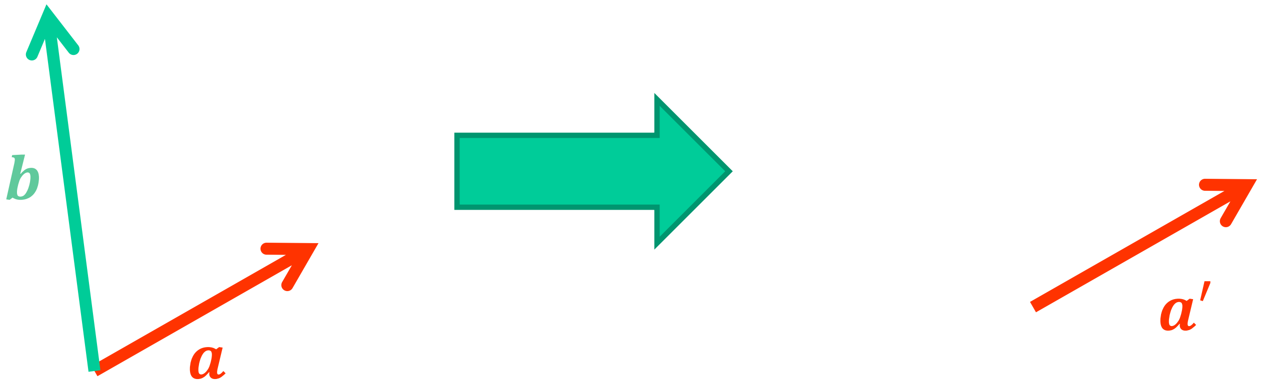
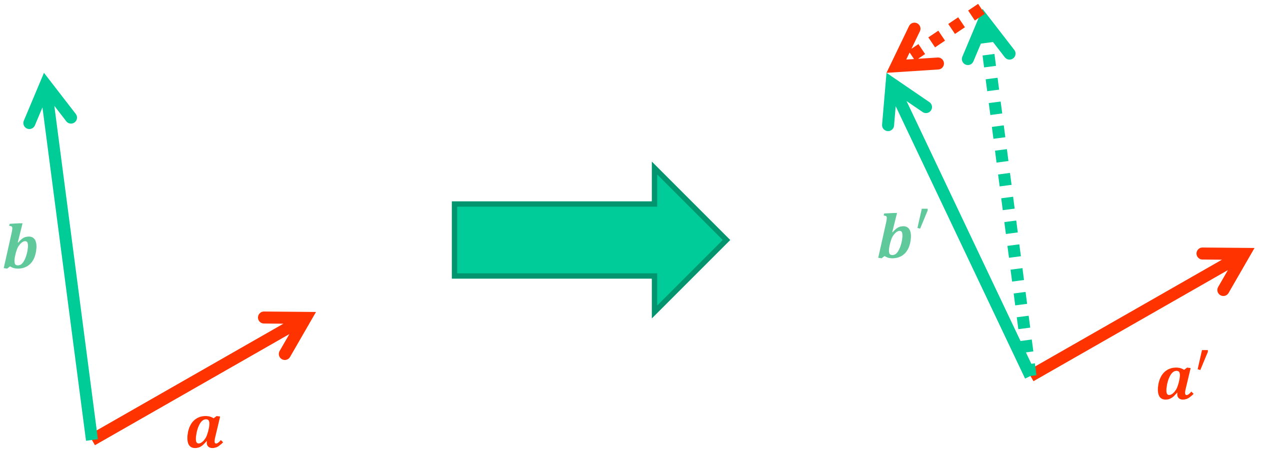



$$\boldsymbol{a}' = \boldsymbol{a}$$

# Orthogonalization




$$\boldsymbol{a}' = \boldsymbol{a}$$

$$\boldsymbol{b}' = \boldsymbol{b} - \frac{\boldsymbol{a}'\boldsymbol{a}'^T}{\boldsymbol{a}'^T\boldsymbol{a}'}\boldsymbol{b}$$

# Gram-Schmidt algorithm

$$\boldsymbol{a}' = \boldsymbol{a}$$

$$\boldsymbol{b}' = \boldsymbol{b} - \frac{\boldsymbol{a}'\boldsymbol{a}'^T}{\boldsymbol{a}'^T\boldsymbol{a}'}\boldsymbol{b}$$

$$\boldsymbol{c}' = \boldsymbol{c} - \frac{\boldsymbol{a}'\boldsymbol{a}'^T}{\boldsymbol{a}'^T\boldsymbol{a}'}\boldsymbol{c} - \frac{\boldsymbol{b}'\boldsymbol{b}'^T}{\boldsymbol{b}'^T\boldsymbol{b}'}\boldsymbol{c}$$

…

# Orthonormality

- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal and unit-length, we say they are *ortho**normal***.

- A set of $m$ orthonormal vectors in $\mathbb{R}^m$ is an *orthonormal basis* of $\mathbb{R}^m$.

- Give an example of an orthonormal basis for $\mathbb{R}^3$.

# Operations with Vectors

- Box product
  - Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$.
  - The box product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is:

$$|\boldsymbol{a}\ \ \boldsymbol{b}| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$$

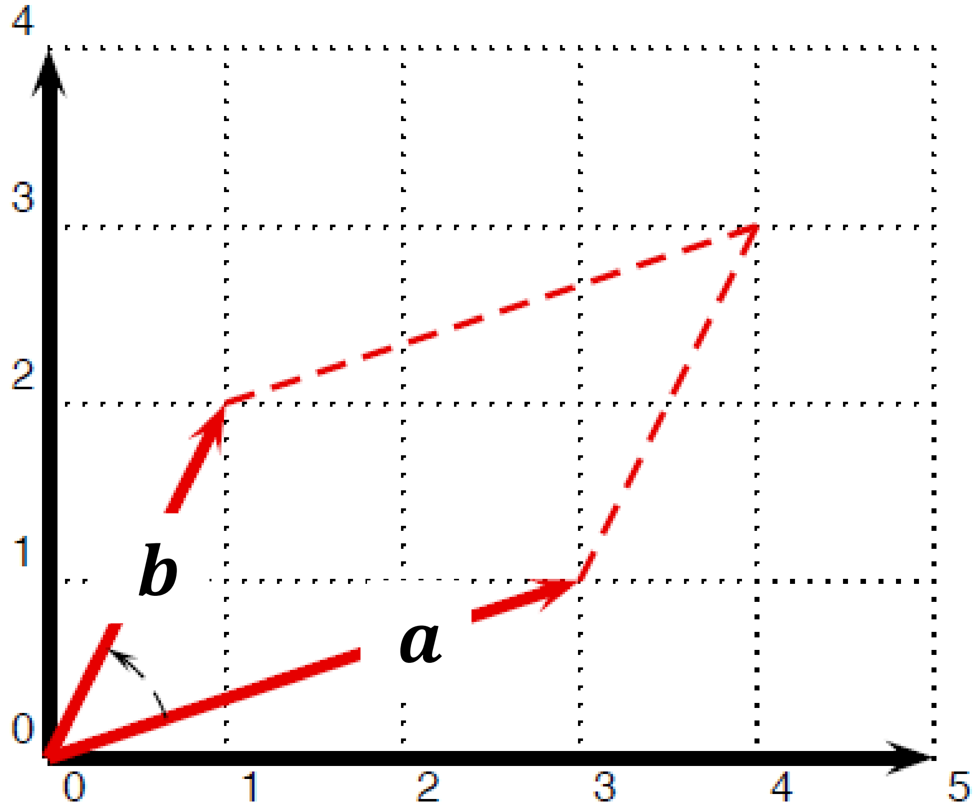
# Operations with Vectors

- Box product
  - $|\boldsymbol{a}\ \ \boldsymbol{b}| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin\alpha$
  - $|\boldsymbol{b}\ \ \boldsymbol{a}| = -|\boldsymbol{a}\ \ \boldsymbol{b}|$

# Operations with Vectors

- Box product

$$|\boldsymbol{a}\ \ \boldsymbol{b}| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| \sin \alpha$$



# Operations with Vectors

- Box product in 3D:

$$|\boldsymbol{a}\ \ \boldsymbol{b}\ \ \boldsymbol{c}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

  - Corresponds to the *signed volume* of a parallelepiped constructed on the three vectors

# Operations with Vectors

- Box product in 3D:

$$|\boldsymbol{a}\ \ \boldsymbol{b}\ \ \boldsymbol{c}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

  - Corresponds to the *signed volume* of a parallelepiped constructed on the three vectors
  - The sign determines the *orientation* of the vectors.

# Orientation

- *m* vectors in an *m*-dimensional space have an *orientation*.
- Orientations in 2D and 3D have conventional names: *right-handed* and *left-handed*.
- You can also speak about *positive* and *negative* orientation *relative to the basis*.

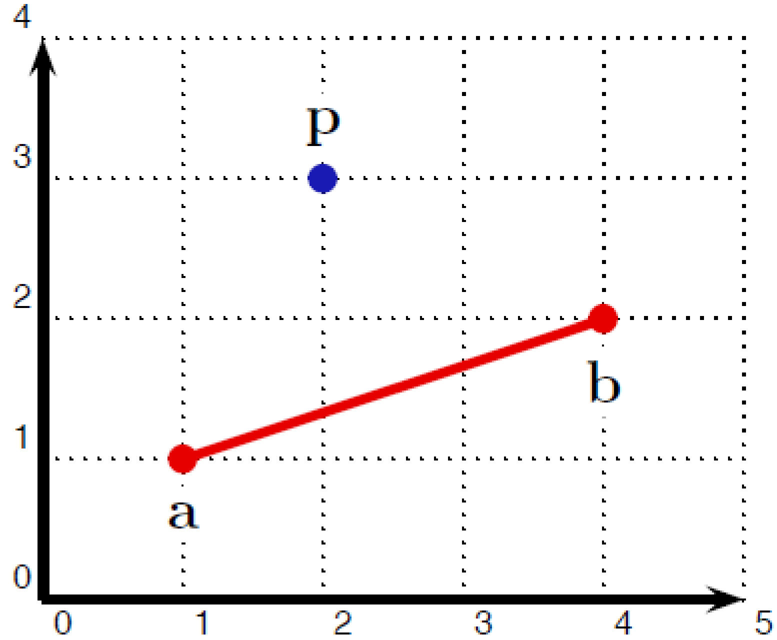
# Right-handed basis

- In mathematics the right-handed basis is most often used.


In this basis any *positively* oriented pair is also a right-handed pair.

# Quiz

- How to determine whether a given point lies to the left or to the right of a given segment?



# Operations with Vectors

- Cross product

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

# Operations with Vectors

- Cross product
  - $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$
  - $\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \cdot \|\boldsymbol{b}\| \cdot |\sin \alpha|$
  - $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b})$ is *positively* oriented

# Orthogonalization in 3D

- Orthogonalization of a right-handed basis in 3D using cross product:
  - $\boldsymbol{c}' = \boldsymbol{a} \times \boldsymbol{b}$
  - $\boldsymbol{b}' = \boldsymbol{c}' \times \boldsymbol{a}$
  - $\boldsymbol{a}' = \boldsymbol{a}$

# Quiz

- A magical unicorn in your 3D world is flying in the direction given by vector $\boldsymbol{v}$.

- The user pushes the button “right”, which should give an impulse to the unicorn towards the right (wrt its current flight direction). Compute the vector pointing to the right.

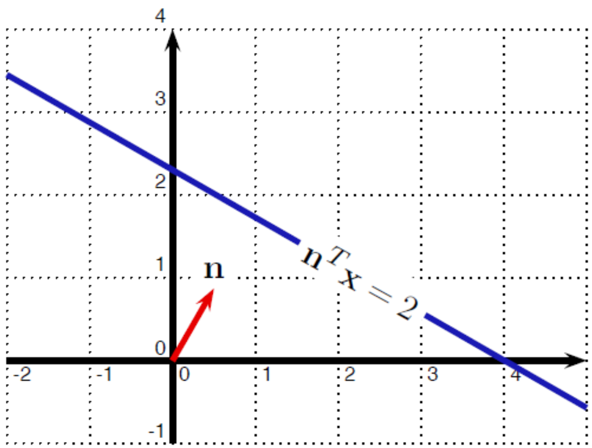
# Mathematical background

- Vectors:
  - Points, directions, vectors and matrices
  - Linear combinations, convex combinations
  - Norm, normalization
  - Inner product, orthogonality, orthogonalization
  - Box product, Cross product
  - Orientation
  - Representation of a straight line

# Straight line

- Parametric representation
  - $\boldsymbol{x} = \lambda \boldsymbol{a} + (1 - \lambda)\boldsymbol{b}$
  - $\boldsymbol{x} = \boldsymbol{a} + t(\boldsymbol{b} - \boldsymbol{a})$

- Implicit representation
  - $\boldsymbol{n}^T(\boldsymbol{x} - \boldsymbol{p}) = 0$
  - $\boldsymbol{n}^T\boldsymbol{x} = \boldsymbol{n}^T\boldsymbol{p}$
  - $n_1x_1 + n_2x_2 - b = 0$



# Mathematical background

- Matrices:
  - Linear transformations
  - Invertibility, rank, determinant
  - Orthogonal transformations
  - Affine transformations
  - Homogeneous coordinates

# Linear transformations

- A transformation $f: \mathcal{V}_1 \rightarrow \mathcal{V}_2$ is called *linear* (also *homomorphism*) if

$$f(\alpha \boldsymbol{x} + \boldsymbol{y}) = \alpha f(\boldsymbol{x}) + f(\boldsymbol{y})$$

- Examples of linear transformations are:
  - Rotation around origin, scaling, shear, reflection, projection or combinations of those.

# Quiz

- Which of those are linear transformations?
  - $f(x) = x$
  - $f(x) = -4x$
  - $f(x) = 4x + 4$
  - $f(x) = x^2$
  - $f(x) = 3$
  - $f(x) = 0$

# Quiz

- Which of those are linear transformations?
  - $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$
  - $f(\boldsymbol{x}) = \boldsymbol{x}^T\boldsymbol{x}$
  - $f(\boldsymbol{x}) = \boldsymbol{a}^T\boldsymbol{x}$
  - $f(\boldsymbol{x}) = |\boldsymbol{a}\ \ \boldsymbol{b}\ \ \boldsymbol{x}|$
  - $f(\boldsymbol{x}) = \boldsymbol{a} \times \boldsymbol{x}$
  - $f(x) = \boldsymbol{a}^T\boldsymbol{x} + |\boldsymbol{a}\ \ \boldsymbol{b}\ \ \boldsymbol{x}| + \boldsymbol{a} \times \boldsymbol{x} + \boldsymbol{A}\boldsymbol{x}$

# To be continued…