
Computer Graphics

Mathematical background: Transformations

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Sep 18, 2013

In the previous episode

- Vectors



In the previous episode

- Tools for working with vectors



In the previous episode

- Distances: _____
- Projections: _____
- Areas & Volumes: _____
- Perpendiculars: _____
- Orthogonalization: _____
- Represent straight line using:
 - _____
 - _____



In the previous episode

- Distances: **norm**
- Projections: **inner product**
- Areas & Volumes: **box product**
- Perpendiculars: **cross product**
- Orthogonalization: **inner/cross product**
- Represent straight line using:
 - **Linear combinations (parametric)**
 - **Inner product (implicit)**



Quiz

- Derive an implicit representation for a two-dimensional line using the box product.



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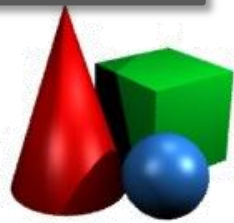


In the previous episode

- Distances: **norm**
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 - **Inner product** (implicit)



**(Bi)linear
operations**



(Bi)linearity

$$\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$$

$$|\alpha x + y \cdot z| = \alpha |x \cdot z| + |y \cdot z|$$

$$(\alpha x + y) \times z = \alpha(x \times z) + (y \times z)$$

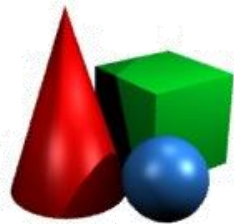
$$A(\alpha x + y) = \alpha Ax + Ay$$



(Bi)linearity

$$f(x + y) = f(x) + f(y)$$

$$f(\alpha x) = \alpha f(x)$$



Today



**Linear
transformations**



Quiz

- Which of those are **not** linear transformations?
 - $f(x) = x$
 - $f(x) = -4x$
 - $f(x) = 4x + 4$
 - $f(x) = x^2$
 - $f(x) = 3$
 - $f(x) = 0$



Quiz

- Which of those are **not** linear transformations?
 - $f(\mathbf{x}) = A\mathbf{x}$
 - $f(\mathbf{x}) = \mathbf{x}^T \mathbf{x}$
 - $f(\mathbf{x}) = \mathbf{a}^T \mathbf{x}$
 - $f(\mathbf{x}) = |\mathbf{a} \ \mathbf{b} \ \mathbf{x}|$
 - $f(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$
 - $f(\mathbf{x}) = \mathbf{a}^T \mathbf{x} + |\mathbf{a} \ \mathbf{b} \ \mathbf{x}| + \mathbf{a} \times \mathbf{x} + A\mathbf{x}$



Linear transformations

- Each linear transformation is uniquely defined by how it transforms the basis:

$$f \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} =$$



Linear transformations

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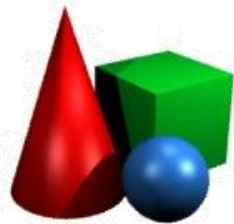
$$f\left(\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}\right) = f\left(2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$$



Linear transformations

- Each linear transformation is uniquely defined by how it transforms the basis:

$$\begin{aligned} f\left(\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}\right) &= f\left(2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \\ &= 2f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) - 3f\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) + 4f\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \end{aligned}$$



Linear transformations

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Linear transformations

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$$\begin{aligned} f \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} &= f \left(2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) \\ &= \mathbf{F} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \end{aligned}$$



Linear transformations

Each linear transformation corresponds to a matrix.



Linear transformations

Each linear transformation corresponds to a matrix.

Columns of a matrix show how it transforms the canonical basis



Quiz

- How does this matrix transform the (canonical) basis?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



Quiz

- How does this matrix transform the (canonical) basis?

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$



Quiz

- How does this matrix transform the (canonical) basis?

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$



Quiz

- How does this matrix transform the (canonical) basis?

$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

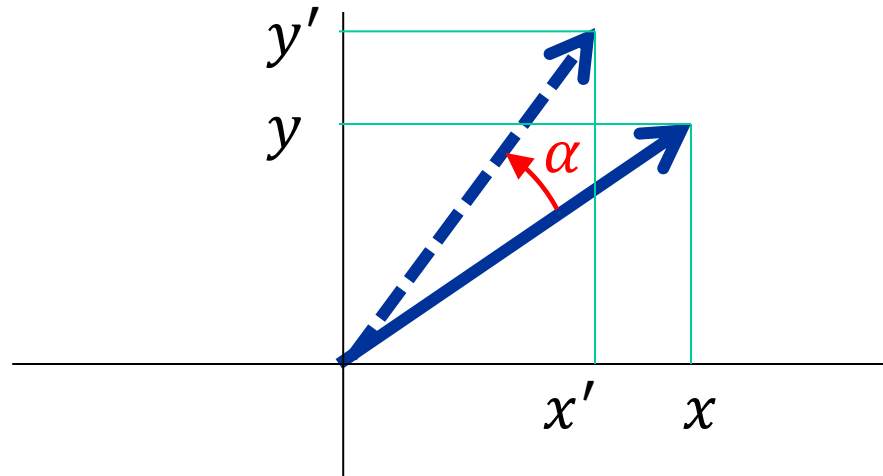


Quiz

$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

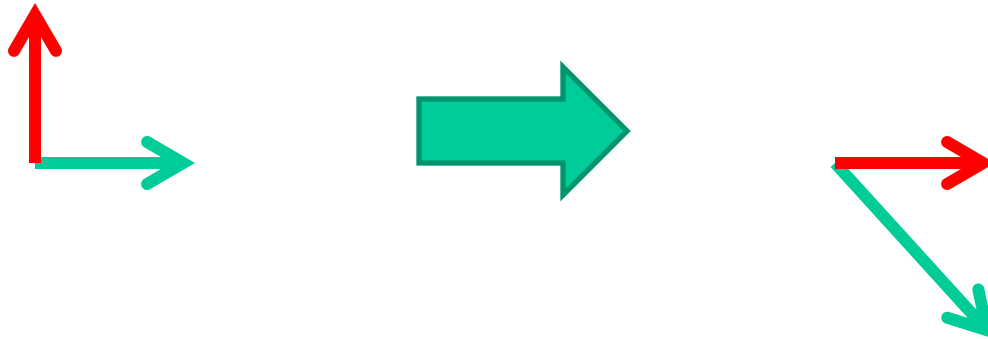
- Let (x, y) be a 2D vector

Let (x', y') be obtained from (x, y) via rotation by angle α . Express x' and y' in terms of x and y .



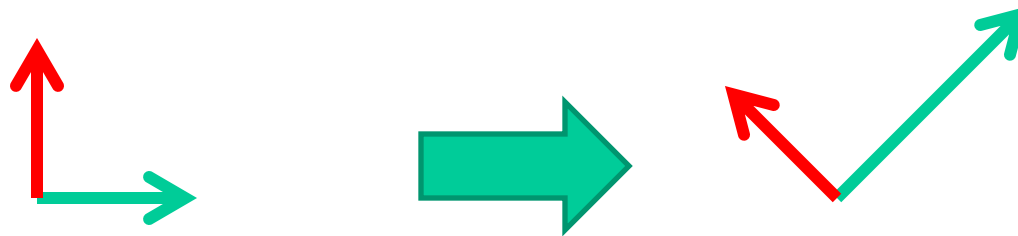
Quiz

- Which matrix does the following?



Quiz

- Which matrix does the following?



Linear transformations

- Let f, g, h be linear transformations and F, G, H the corresponding matrices, then:
 - Composition of transformations corresponds to matrix multiplication:

$$(f \circ g)(x) = f(g(x)) = FGx$$



Linear transformations

- Let f, g, h be linear transformations and F, G, H the corresponding matrices, then:
 - Function composition is associative, hence matrix multiplications is too:

$$(f \circ g) \circ h = f \circ (g \circ h)$$
$$(FG)H = F(GH)$$



Linear transformations

- Let f, g, h be linear transformations and F, G, H the corresponding matrices, then:
 - Sum of transformations corresponds to matrix sum:

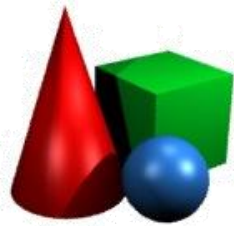
$$(f + g)(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x}) = (F + G)\mathbf{x}$$



Linear transformations

- Let f, g, h be linear transformations and F, G, H the corresponding matrices, then:
 - Composition is distributive wrt sum:

$$(f + g) \circ h = f \circ h + g \circ h$$
$$(F + G)H = FH + GH$$



Rank

- Consider a linear transformation $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$
it will always either:
 - Map the whole 3D space to itself somehow
 - Project the whole 3D space to a plane
 - Project the whole 3D space to a line
 - Map all points to 0.



Rank

- The dimensionality of the resulting space is the *rank* of f .
 - If f is full rank (i.e. $\text{rank}(f) = 3$ in our case), it is *invertible*. Otherwise it is not.
 - f is invertible $\Leftrightarrow \det(\mathbf{F}) \neq 0$



Orthogonal transformations

- A transformation F is called orthogonal if it maps the canonical basis into an **orthonormal basis**.



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 - $F^T F = ?$



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Orthogonal transformations

- A transformation \mathbf{F} is called orthogonal if it maps the canonical basis into an **orthonormal basis**.
 - It must keep lengths and angles intact, i.e. it is always a **rotation** (possibly mirrored).
 - $\mathbf{F}^T \mathbf{F} = \mathbf{I}$, because the columns are orthonormal
 - Hence, $\mathbf{F}^{-1} = \mathbf{F}^T$



Orthogonal transformations

To compute the inverse of an orthogonal matrix, simply transpose it.

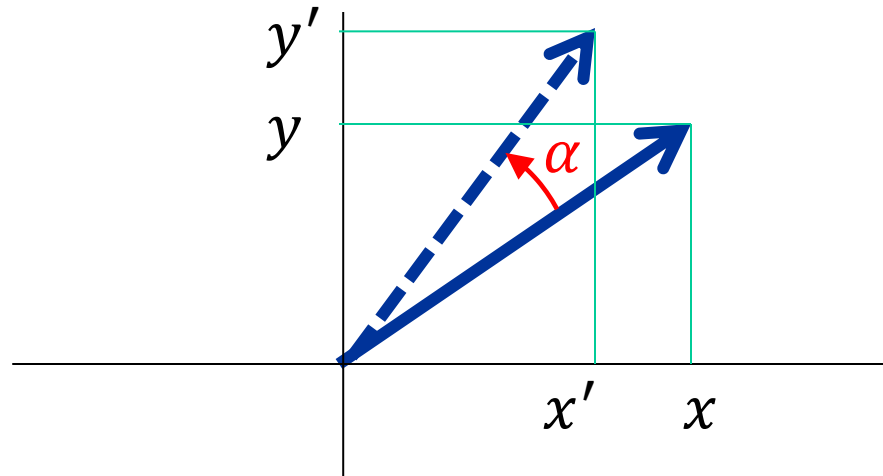


Quiz

$$\begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$$

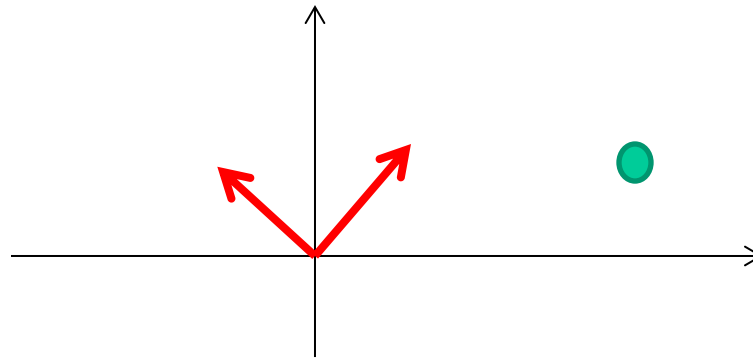
- Let (x, y) be a 2D vector

Let (x', y') be obtained from (x, y) via rotation by angle α . Express x and y in terms of x' and y' .



Quiz

- You are standing at the origin, rotated with respect to the coordinate system, looking in the direction $(0.6, 0.8)$ (your local “x” axis).
- At position $(7,2)$ there is an object. What are the coordinates of this object with respect to you?



Examples

- Rotation: $\mathbf{R}(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$
- Scaling: $\mathbf{S}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
- Mirroring: $\mathbf{Mir}_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
- Shear: $\mathbf{Sh}_x(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$



Examples

- Rotation around z axis:

$$\mathbf{R}_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Rotation around y axis:

$$\mathbf{R}_y(\alpha) = \begin{pmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix}$$



Shift

- Shift (translation) is not a linear transformation.
- To deal with shifts we must introduce the notion of an *affine space* and *affine transformations*.



Affine space

- Vector space
- Affine space



Affine space

- Vector space
 - Vectors $\boldsymbol{v} \in \mathbb{R}^3$
 - Basis: $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$
 - Linear transformations
$$f(\boldsymbol{x}) = \boldsymbol{F}\boldsymbol{x}$$
- Affine space



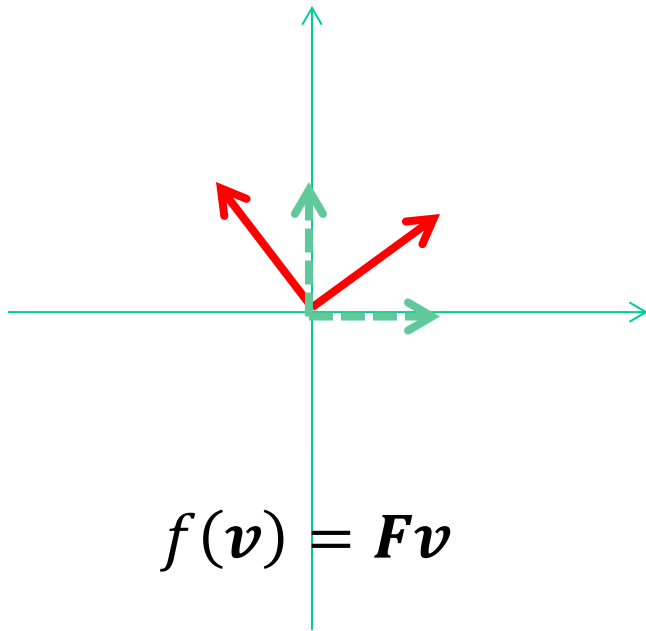
Affine space

- Vector space
 - Vectors $\mathbf{v} \in \mathbb{R}^3$
 - Basis: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
 - Linear transformations
$$f(\mathbf{x}) = \mathbf{F}\mathbf{x}$$
- Affine space
 - Vectors $\mathbf{v} \in \mathbb{R}^3$
 - Points $P \in \mathbb{R}^3$
 - ▶ point+vector = point
 - Frame:
$$(O, \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\})$$
 - Affine transformations:
$$f(\mathbf{v}) = \mathbf{F}\mathbf{v}$$
$$f(P) = \mathbf{t} + \mathbf{F}\mathbf{p}$$

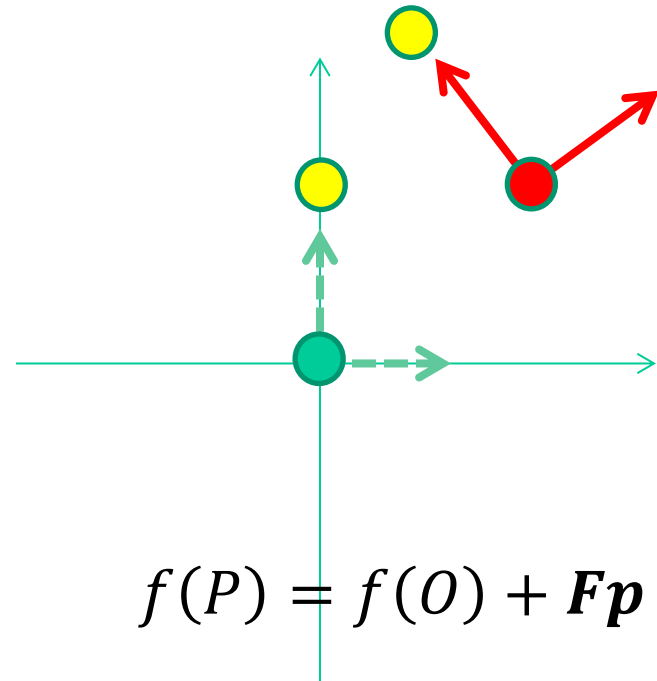


Affine space

- Vector space



- Affine space



Affine transformations

$$f(\mathbf{p}) = \mathbf{t} + \mathbf{F}\mathbf{p}$$



Affine transformations

$$f(\mathbf{p}) = \mathbf{t} + \mathbf{F}\mathbf{p}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$



Affine transformations

$$f(\mathbf{p}) = \mathbf{t} + \mathbf{F}\mathbf{p}$$

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} + \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

$$\begin{pmatrix} q_1 \\ q_2 \\ 1 \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & t_1 \\ f_{21} & f_{22} & t_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ 1 \end{pmatrix}$$



Homogeneous coordinates

- We shall represent the **points** of an affine space using 3-dimensional vectors of the form $(p_1, p_2, 1)^T$
- We shall represent the **vectors** of an affine space using 3-dimensional vectors of the form $(v_1, v_2, 0)^T$
- Any affine transformation is a matrix

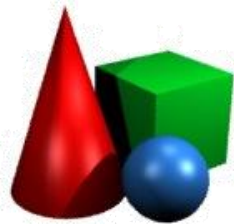
$$\left(\begin{array}{cc|c} f_{11} & f_{12} & t_1 \\ f_{21} & f_{22} & t_2 \\ \hline 0 & 0 & 1 \end{array} \right)$$



Homogeneous coordinates

- Analogously, for 3D space we use 4-dimensional vectors and 4x4 matrices.
- E.g. the following transformation rotates around z axis and shifts along x axis by 0.5:

$$\left(\begin{array}{ccc|c} \cos \phi & -\sin \phi & 0 & 0.5 \\ \sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \right)$$



Homogeneous coordinates

- Note how the representation implicitly enforces the rules:
 - $\text{vector} + \text{vector} = \text{vector}$
 - $\text{point} + \text{vector} = \text{point}$
 - $\text{point} + \text{point} = \text{undefined}$
 - $\text{convex combination of points} = \text{point}$



Homogeneous coordinates

- Rotation: $R(\alpha) = \left(\begin{array}{cc|c} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{array} \right)$

- Scaling: $S(a, b) = \left(\begin{array}{cc|c} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{array} \right)$

- Translation: $T(x, y) = \left(\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{array} \right)$



Quiz

- Construct a matrix, that performs a rotation by 10 degrees around the point $(20, 30)$ in homogeneous coordinates.



Quiz

- Construct a matrix, that performs a rotation by 10 degrees around the point (20, 30) in homogeneous coordinates.

$$\mathbf{T}(20,30)\mathbf{R}(10)\mathbf{T}(-20,-30)$$



Quiz

- How to construct a matrix, that performs a rotation (in 3D) by 10 degrees around the axis given by the direction vector $(1, 2, 3)$



Mathematical background

- Matrices:
 - Linear transformations
 - Invertibility, rank, determinant
 - Orthogonal transformations
 - Affine transformations
 - Homogeneous coordinates

