To what extent $f(x) = h - \frac{e^x + e^{-x}}{2}$ be the general equation of the hyperbolic function for different shapes of wheels?

December 17, 2020

Introduction And Rationale

Mathematics is the subject that I have always struggled with. However, after entering the International Baccalaureate program, I started to find my interest in mathematics as I was curious with the question “Can every problem be solved by math?” This question actually made me realize that mathematical theories and formulas that I have learned in class are meant to address not just academic problems but also real life situations like cars driven on different shaped wheels.

One of my hobbies is to watch science and math related videos on Youtube. Because of this, I often search for videos on cars and electronic devices to see how math and science actually applies to those fields. In one video that introduced new cars into the market, I observed how those cars moved. This somehow made me wonder whether cars can have differently-shaped wheels and not just run on circular wheels.

This question piqued my curiosity, so I began to search for cars with different shaped wheels which ultimately led me to find an interesting and hilarious video from “Mythbusters for the Impatient” on Youtube about whether a car with square wheels could move smoothly or not. In that video, they changed the shape of the wheels of a truck to squares and then tried driving it. Since the ride was not smooth, the makers tried to figure out if there was a way for them to make the ride more even with square wheels. They concluded that using square wheels would require further studies before these could be used as comparable circular wheel replacements.

One suggested solution to this problem was to change road surfaces where vehicles with differently shaped wheels were driven. In this process, several mathematical concepts were used to illustrate the solution. As I was searching for more information, I established that other mathematicians had already proposed a general equation for the road surface which made me further investigate and unpack if various road surfaces were relevant to differently shaped wheels on cars.

Aim and Approach

As mentioned in the introduction, this research aims to investigate under what circumstances wheels with different shapes like a pentagon could be used on vehicles. This was done by incorporating mathematical models that interpreted and analyzed the length of the surface of the ground. In order to construct a model for this research, I had to settle on a shape that I could use as a car wheel, aside from square shapes that the MythBusters had already explored. Considering the fact that the square is a common shape and that other mathematicians had already worked on it, I decided to think about other shapes and settled on the equilateral pentagon. My initial plan was to test the viability of the equilateral triangle; I found that I could not use a triangle as a wheel because the corners of the triangle would not fit the road surface (Vanderbilt University, 2011). Thus, having eliminated the square and now the triangle as options to test, the next logical shape to consider was the pentagon.

Having settled on exploring the feasibility of using pentagon shaped wheels, the next relevant factor was the road surface. I realized that the ground could not be flat because with wheels that had sides, the length of those sides would not physically be able to match up with a flat surface (“Riding on Square Wheels”). Consequently, the ground itself would have to be changed to correspond to the wheel shapes. This explanation is further illustrated in Figure 1, which shows the basic diagram of how square wheels move on a non flat surface that correspond to the square wheels’ sides.

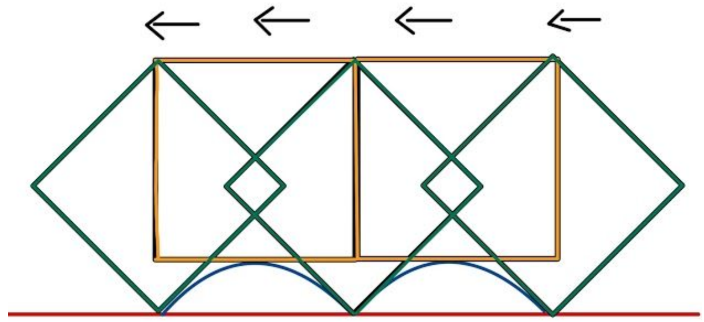

Figure 1. The illustration of how square wheels will move with corresponding ground (Derived from Mathematical Tourist, 2011) ; (Diagram not to scale)

After settling that road surfaces themselves could not be flat to accommodate the sides of the non circle wheels, the question on equilateral pentagon wheel viability was addressed by applying the general equation of the hyperbolic function with adjusted values to fit the dimensions of that equilateral pentagon as well as pythagorean theorem and trigonometric ratios were used to find the lengths of the applicable sides that are needed in calculation. Lastly, the concepts of derivative and integral calculus were applied to find the total length of the half arc of the ground.

Modellings And Mathematical Manipulations and results

Square Wheels

Considering that the MythBurster's video on square wheel did not include mathematical analysis in their exploration, I started my investigation from there, applying mathematical analyses to square wheels first.

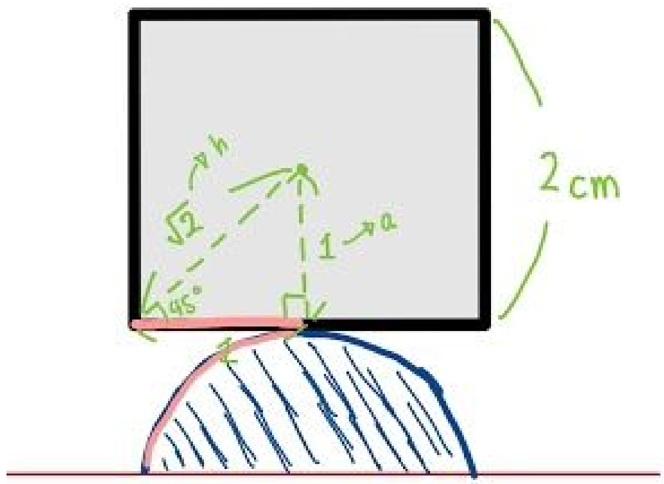

Figure 2. The illustration of the square wheel with corresponding measurements and ground.

Figure 2 illustrates the model of a square wheel, showing the length of one side as well as the lengths of the sides of the right triangle formed from the centre of the square to one of its vertices. The purpose of the right triangle is to calculate the values of h and a which are values applied to the Pythagorean theorem:

$$c^2 = a^2 + b^2$$

$$h^2 = 1^2 + 1^2$$

$$h = \pm\sqrt{2}$$

In Figure 2, half the side (Light pink) of the square corresponds with the half side of the ground (Light pink) in terms of length. It is also known that the half of the side of the square is 1 cm; this is considered 'b' of the right triangle (refer to Figure 2), while the hypotenuse is $h = \sqrt{2}$ considering the fact that distance cannot be negative - 'h' has positive sign. In order to prove that the half side of the ground is equivalent to 1, it is needed to discuss the hyperbolic function in relation to the square.

The original form of the the hyperbolic function is shown right below:

$$Cosh\ x = \frac{e^x+e^{-x}}{2}$$

From this equation, some changes were made so that the equation could be suitable to the square itself. This is illustrated through the transformation of the function. Considering the x-axis as the horizontal line where it represents the surface of the ground, the original function needs to be reflected by the x-axis by adding a negative sign in front of the function as shown below.

$$Cosh\ x = -\ \frac{e^x+e^{-x}}{2}$$

In addition, the height of the function should also match with the length of the line (centre to the vertex) so that the side and the curve (ground) can meet together. Thus, the function vertically translated up by scale factor of $k = h$ (refer to figure 2).

$$f(x) = h - \frac{e^x+e^{-x}}{2}$$

Therefore, with the information from the graph, integral calculus was incorporated to find the length of the half length of the ground. In getting the length of the half side of the ground, the concept of integral specifically in getting “area” is used to assess this problem. When the length of the curve needs to be calculated, it is impossible to measure it just by looking at it. That is why, it needs to form triangles in order to measure the length. This is further supported by Figure 4 where it shows how breaking down of the box into several pieces will eventually make the measurement closer to the original length of the curve.

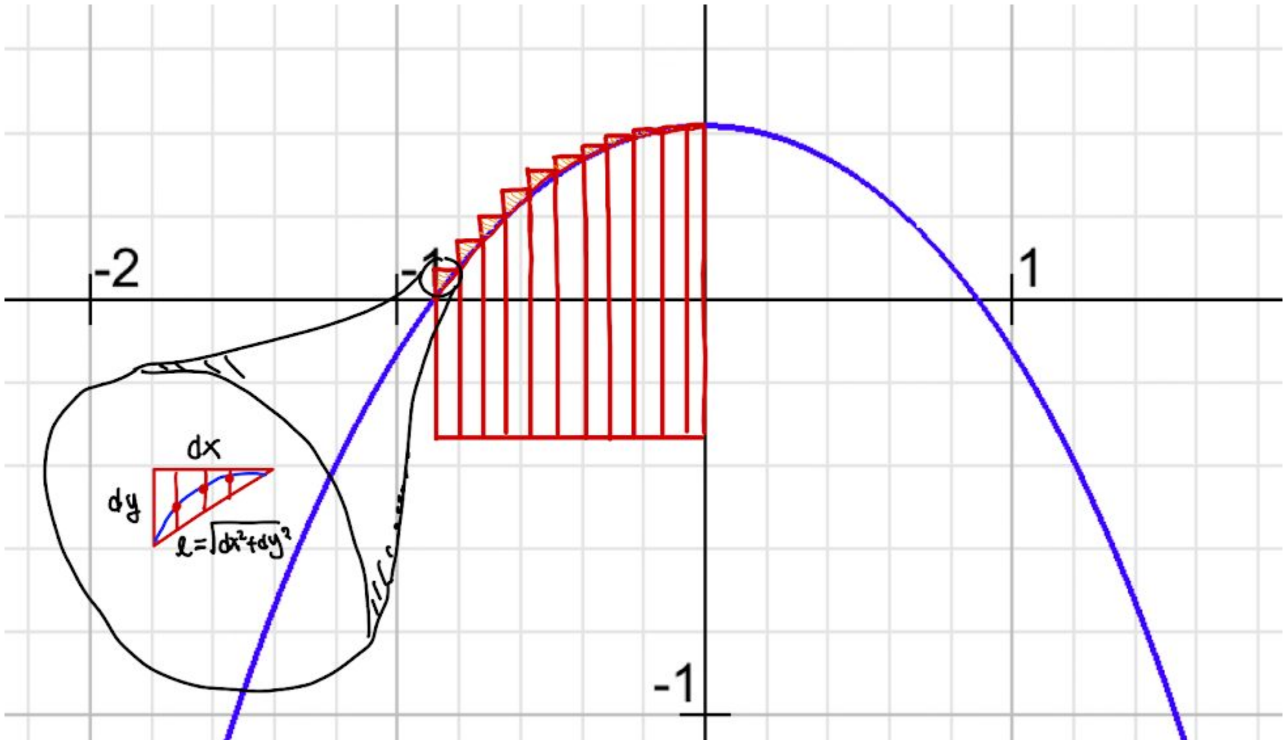

Figure 3. The illustration of how length of the curve is being measured ***(Diagram not to scale)****.*

When the triangle is expanded, the relationship between the triangle and the curve is clearly shown. The expanded view (Figure 3) indicates the box will be divided into several portions where *dx and dy* are labeled accordingly; *dx* and *dy* are used in this procedure because the length (*l) is* needed to be calculated using the pythagorean theorem. Also, the purpose of dividing those boxes into several portions is to get as close as possible to the actual length of the function (blue). This is similar to the concept of getting areas using integration or integrals. With all these in mind, the length of the curve is now transformed into an equation that uses the general formula of calculating the length of the curve using integrals.

Calculating the general equation of the length **:**

$$\text{Length} = \sqrt{(dx)^2 + (dy)^2}$$

$$= \sqrt{dx^2(1 + (\tfrac{dy}{dx})^2)}$$

$$= \sqrt{1 + (\tfrac{dy}{dx})^2}\,dx$$

$$= \sqrt{1 + f'(x)^2}\,dx$$

Altering hyperbolic equation to integral equation **:**

$$\lim_{dx\to 0} \sum \sqrt{1 + f'(x)^2}\,dx$$

$$\text{L} = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$

Therefore, from this formula, whether the half side of the curve is equal to 1 or not can be proven. However, in the process of the integration, ‘the yellow point’ with an arrow needs to be determined first in order to substitute in (refer to Figure 4). The ‘b’ and ‘c’ variables are both x-intercepts where they show the domain of the particular function - since a definite integral is utilized to find the area or length of the certain function (graph) only.

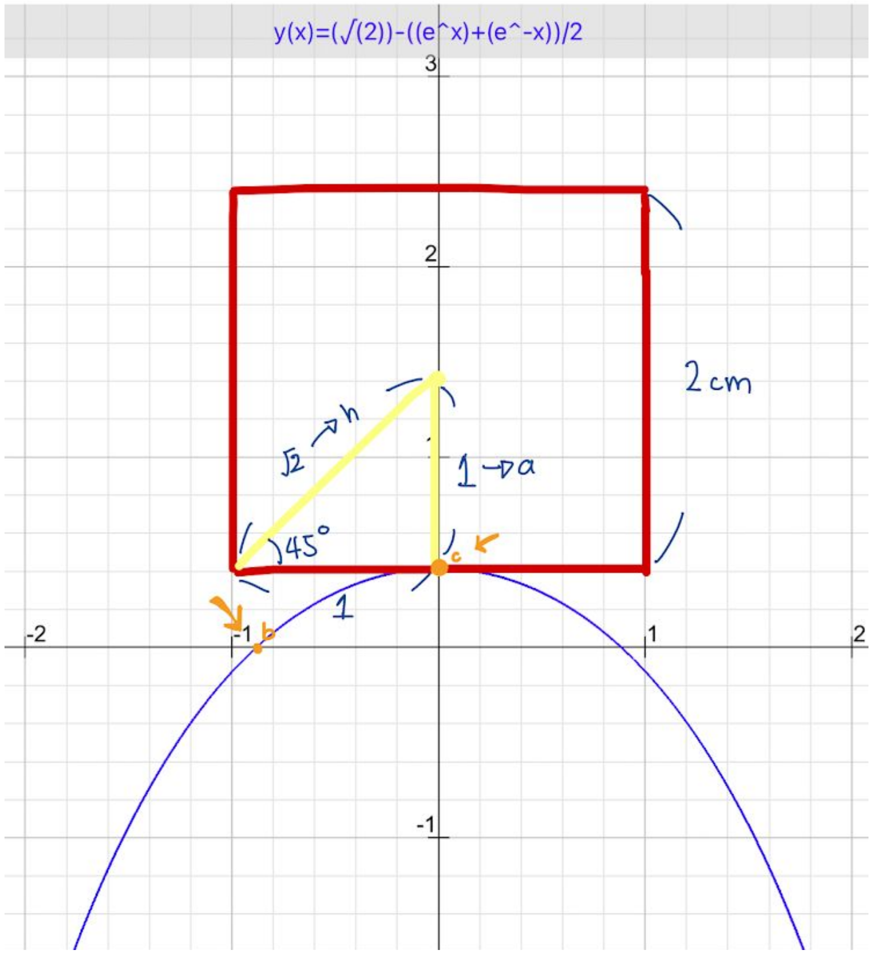

Figure 4. The illustration with the red dots where it will be plotted in integral formula.

In calculating the yellow dot with the arrow pointed, a graphing calculator was used to find the x-intercept which is the yellow dot with the arrow. Also, with the calculated x-intercept that are in the form of ‘b’ and ‘c’ will be calculated with the knowledge of $y = 0$. Letter b & c are located where the orange arrows are pointed at. This is because the equation is used in the process of getting the length of the arc from point ‘b’ to ‘c’. The picture below shows the calculation of the x-intercept from the graphing calculator.

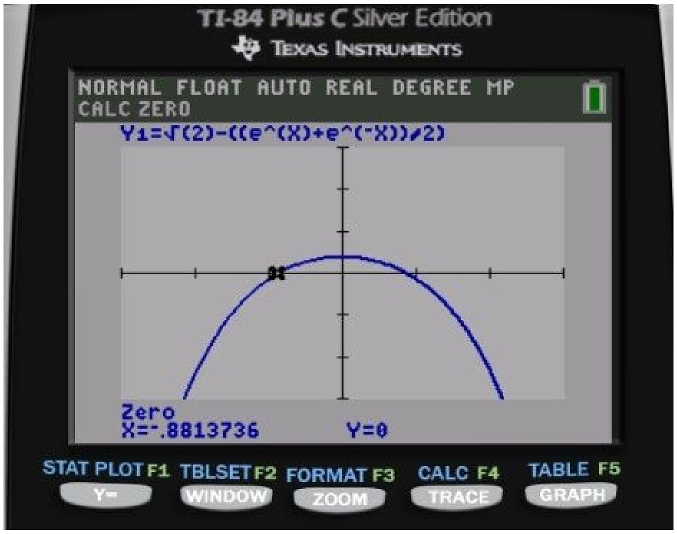

Figure 5. The left x-intercept of the function for square wheels.

After calculating the x-intercept, we can now proceed to the calculation of the curve itself. Also, before substituting the values, derivative of the f(x) is needed.

Derivative of f(x) for square wheel **:**

$$f(x) = \sqrt{2} - \frac{e^x + e^{-x}}{2}$$

$$f'(x) = 0 - \left(\frac{(e^x + e^{-x}) \cdot (-1)}{2} \right)$$

$$f'(x) = -\frac{e^x - e^{-x}}{2}$$

Substituting & Calculation of Integration**:**

$$\mathrm{L} = \int_b^c \sqrt{1 + f'(x)^2}\,dx$$

$$= \int_{-0.8813736}^{0} \sqrt{1 + f'(x)^2}\,dx$$

$$= \int_{-0.8813736}^{0} \sqrt{1 + (-\tfrac{e^{x} - e^{-x}}{2})}dx$$

$$= 1.063515814 \text{ cm}$$

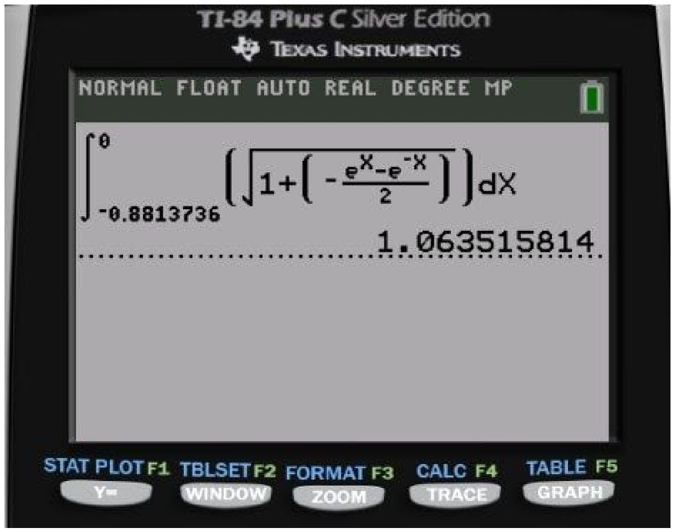

Figure 6. The calculated value of the half arc length of the ground.

Finally, this shows that the half length of the ground is equal to the half length of the square. Therefore, it can be concluded that the equation of $f(x) = h - \frac{e^{x}+e^{-x}}{2}$ can be used in square shaped wheels.

On the other hand, this research further analyzes the use of the general equation by repeating the same process done on the square wheel. Therefore, this time, pentagon is analyzed to assess the reliability of the equation.

Pentagon Wheels

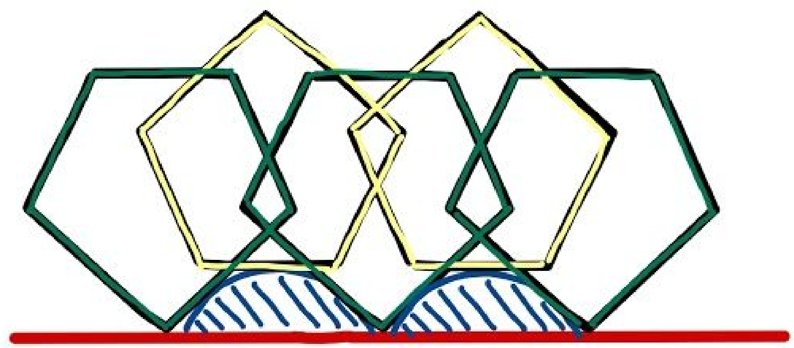

*Figure 7. The drawing of how pentagon wheels can move (**Diagram not to scale**)*

With the drawing of the pentagon wheels that moves along the new shape of the ground, it is clear that the analysis part will also be similar to the analysis of the square.

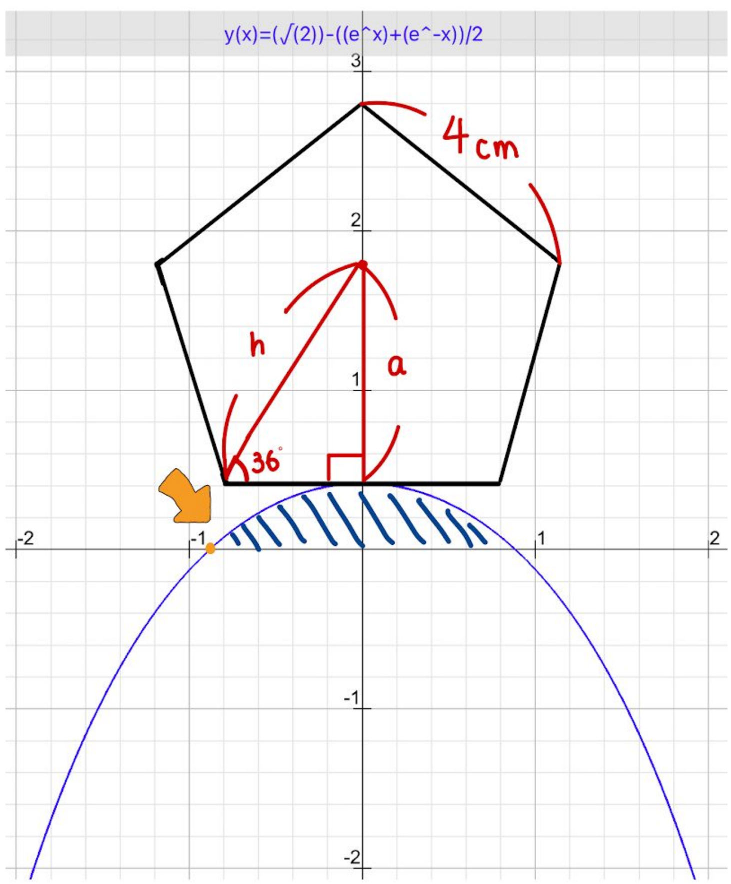

*Figure 8. The illustration of the pentagon wheel on the ground with its measurement of the one side (**Diagram not to scale**)*

To begin, as shown by the figure 9, the length of the pentagon is 4cm, and this pentagon is considered as an equilateral pentagon where all sides have the same measurements which is 4cm, and 2cm for the half length of the side. Furthermore, 'h' is calculated as it should be included in the equation of $f(x) = h - \frac{e^x + e^{-x}}{2}$. As a result, in getting the value of 'h', a trigonometric ratio was used to calculate. Since we know that each angle of the equilateral pentagon is $72°$, the angle shown in the drawing is $36°$. Thus, *h* is calculated with cosine principle.

<u>Trigonometric ratios</u> :

$$\text{Cos}36° = \frac{2}{h}$$

$$h = 2.47213\ldots$$

Thus, the equation of the ground for this equilateral pentagon wheel is $f(x) = 2.47213 - \frac{e^x + e^{-x}}{2}$. After getting the specific equation of the ground, the derivative of the f(x) is calculated, and the 'a' value or the x-intercept is also found so that the integral equation will be complete in getting the length of the half arc (*this should be equal to the half side of the pentagon, 2cm*).

<u>Derivative of f(x) for pentagon wheel</u> **:**

$$f(x) = 2.47213 - \frac{e^x + e^{-x}}{2}$$

$$f'(x) = 0 - \left(\frac{(e^x + e^{-x}) \cdot (-1)}{2}\right)$$

$$f'(x) = -\frac{e^x - e^{-x}}{2}$$

Calculating the x-intercept :

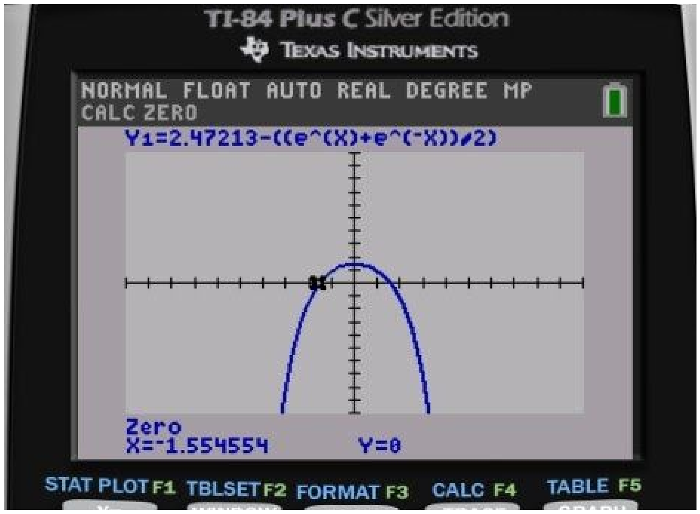

Figure 9. The left x-intercept of the function (for pentagon wheel).

Substituting & Calculation of Integration:

$$L = \int_{b}^{c} \sqrt{1 + f'(x)^2}\,dx$$

$$= \int^{0} \sqrt{1 + f'(x)^2}\,dx$$

$$= \int_{-1.554554}^{0} \sqrt{1 + (-\tfrac{e^x - e^{-x}}{2})}\,dx$$

$$= 2.140354697 \text{ cm}$$

The length of the half arc of the ground is 2.140354697 cm which is not equal to the half length of the side (2cm). This simply means that the general equation of the ground is not applicable for all polygons. However, I was wondering whether there is a missing link in the equation of $f(x) = h - \frac{e^x + e^{-x}}{2}$. Therefore, when I was gathering more information about this, I was able to find that the length of the ‘a’ in the triangle of the polygon needs to be added in the equation in order to adjust to the change (Vanderbilt University, 2011).

$$f(x) = h - \frac{e^{x}+e^{-x}}{2}$$

From this form, it now developed into another equation. By adding a in front of the cosh, the function will be able to either vertically stretched or compressed. Also, ‘x’ will be changed into x/2.

$$f(x) = h - a\ cosh(\frac{x}{a})$$

<u>Calculating ‘*a*’ in triangle</u> :

Based on figure 9, the value of *a* can be calculated.

$$\tan 36° = \frac{a}{2}$$

$$a = 1.453085$$

<u>Derivative of ‘new’ f(x) for pentagon wheel</u> :

$$f(x) = h - a\ cosh(\frac{x}{a})$$

$$f(x) = 2.47213 - (1.453085)\frac{e^{\frac{x}{1.4530}}+e^{\frac{-x}{1.4530}}}{2}$$

$$f'(x) = 0 - (1.453085)\frac{e^{\frac{x}{1.4530}} \cdot (\frac{1}{1.4530}) + e^{\frac{-x}{1.4530}} \cdot (-\frac{1}{1.4530})}{2}$$

$$f'(x) = -\frac{e^{\frac{x}{1.4530}} + e^{-\frac{x}{1.4530}}}{2}$$

The x-intercept of this function is also calculated by the graphic calculator, and the left x-intercept for this function was x = -1.633521.

<u>Substituting & calculation of integration</u>

$$L = \int_{b}^{c} \sqrt{1 + f'(x)^2}\,dx$$

$$= \int_{-1.633521}^{0} \sqrt{1 + \left(-\frac{e^{\frac{x}{1.4530}} - e^{-\frac{x}{1.4530}}}{2}\right)^2}\,dx$$

$$= 2.000038613 \text{ cm}$$

Thus, the length of the half side of the equilateral pentagon is now the same with the length of the half arc of the ground. Also, this implies that they can perfectly merge in terms of the measurement which pentagon shaped wheel will eventually move smoothly.

Conclusion and Evaluation

As I was expanding my investigation on this research paper, I was still wondering what other shapes will result in terms of the length of the ground and most importantly, if it can move smoothly or not. To go back to the main question of this research paper, $f(x) = h - \frac{e^x + e^{-x}}{2}$ may be applicable for square, but this equation might not be suitable for other shapes of wheels such as pentagon. Also, with the new equation formed, pentagon was able to be proven with different equations which can also be used for other shapes of wheels.

Moreover, in this research paper, several limitations exist. Firstly, considering the number of shapes that this research paper contains, the equation of the function may not be applicable for other shapes that have special conditions like different length of sides. Secondly, this research might also have other ways to compute and calculate whether different shapes of wheels can move smoothly or not. Lastly, this research does not have enough secondary sources to further support the analysis part.

To conclude, this paper can be improved by few recommendations. First, models and diagrams can be drawn more accurately and systematically to help readers understand the situation and the mathematical concepts better. Second, aside from the triangle, square, and pentagon, other shapes are strongly recommended so that the paper can have higher quality in terms of the accuracy of the results. Lastly, real models of different shapes of wheels can be constructed so that it can illustrate the real-life situation better and realistically.

Bibliography

Adam, and Jamie. "Square Wheels - Mythbusters for the Impatient." Mythbusters for the Impatient, 14 Feb. 2016, www.youtube.com/watch?v=CIN8Q_4iaxU.

Admin. "Why Are Wheels Circular?" Santa Ana Wheel, 28 Jan. 2019, www.santaanawheel.com/blog/why-are-wheels-circular/.

Smooth Rides on Square Wheels. 2011, bayanbox.ir/view/5588951697870640939/bike-with-square-wheels-s.pdf.

Stewart, James. *CALCULUS Early Transcendentals*. Seventh ed., McMaster University and University of Toronto, 2012.

Tourist, Math. *Riding on Square Wheels*, 25 May 2011, mathtourist.blogspot.com/2011/05/riding-on-square-wheels.html.