目　录

第一章　求极限 …… 1

§1.1　求极限的入门 …… 1

§1.2　利用泰勒公式求极限 …… 5

§1.3　利用无穷小替换求极限 …… 15

§1.4　利用洛必达法则求极限 …… 24

§1.5　幂指函数求极限 …… 28

§1.6　极限的应用之一:连续与间断点 …… 38

§1.7　极限的应用之二:求渐近线 …… 42

第二章　求导数 …… 47

§2.1　利用导数定义求导 …… 47

§2.2　利用导数定义求极限 …… 50

§2.3　复合函数求导 …… 54

§2.4　隐函数求导 …… 61

§2.5　参数方程求导 …… 67

§2.6　分段函数求导 …… 74

§2.7　变限积分函数求导 …… 82

§2.8　高阶导数 …… 87

第三章　求不定积分 …… 91

§3.1　最简单的积分 …… 91

§3.2　利用换元法求不定积分 …… 98

§3.3　利用分部积分法求不定积分 …… 122
§3.4　有理函数的积分 …… 134
§3.5　三角函数的积分 …… 140
§3.6　与递推有关的一组积分 …… 143

第一章 求 极 限

§1.1 求极限的入门

基础篇

1. $\lim\limits_{x\to 0}\dfrac{x^2-1}{2x^2-x-1}=\lim\limits_{x\to 0}\dfrac{(x+1)(x-1)}{(2x+1)(x-1)}=\lim\limits_{x\to 0}\dfrac{x+1}{2x+1}=1.$

2. $\lim\limits_{x\to+\infty}\dfrac{1}{2^x}=0$，$\lim\limits_{x\to-\infty}\dfrac{1}{2^x}=+\infty$，故原极限不存在.

3. $\lim\limits_{n\to\infty}(-1)^n\dfrac{1}{n}=0$. 本书中 n 指自然数，以下不一一指出.

4. $\lim\limits_{n\to\infty}\dfrac{3n+7}{2n+1}=\dfrac{3}{2}.$

5. $\lim\limits_{x\to 0+}\dfrac{|x|}{x}=\lim\limits_{x\to 0+}\dfrac{x}{x}=1.$

6. $\lim\limits_{x\to\infty}\mathrm{e}^{\left(\frac{1}{x}\right)^2}=1.$

7. $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=\lim\limits_{x\to\infty}\dfrac{1}{x}\cdot\sin x=0$（无穷小乘有界量仍为无穷小）.

8. $\lim\limits_{x\to\infty}\dfrac{x}{2x+1}=\lim\limits_{x\to\infty}\dfrac{1}{2+\dfrac{1}{x}}=\dfrac{1}{2}.$

9. $\lim\limits_{x\to\infty}\dfrac{\sin x}{\sqrt{x}}=\lim\limits_{x\to\infty}\dfrac{1}{\sqrt{x}}\sin x=0.$

10. $\lim\limits_{x\to 1}\left(\sin\dfrac{x}{2}\pi+\cos x\pi\right)=\sin\dfrac{\pi}{2}+\cos\pi=1-1=0.$

11. $\lim\limits_{x\to-1}\dfrac{x^3+1}{x+1}=\lim\limits_{x\to-1}\dfrac{(x+1)(x^2-x+1)}{x+1}=3.$

12. $\lim\limits_{x\to-1}\dfrac{x-\sqrt{3+2x}}{x^2+1}=\dfrac{-1-1}{(-1)^2+1}=-1.$

13. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}+1}{2x+3}=0.$

14. $\lim\limits_{x\to\infty}\dfrac{1+2x}{(x-3)^2}=0.$

15. $\lim\limits_{x\to\pi}\dfrac{\sqrt{1+\sin x}+\cos\dfrac{x}{2}}{x+1}=\dfrac{1+0}{\pi+1}=\dfrac{1}{\pi+1}.$

16. $\lim\limits_{x\to0}\dfrac{4x^3-2x^2+x}{3x^2+2x}=\dfrac{1}{2}.$

17. $\lim\limits_{x\to\sqrt{2}}\dfrac{x^2-3}{x^2+x+1}=\dfrac{-1}{3+\sqrt{2}}=\dfrac{\sqrt{2}-3}{7}.$

18. $\lim\limits_{x\to2}\dfrac{x^2+4}{x-3}=\dfrac{8}{-1}=-8.$

19. $\lim\limits_{x\to4}\dfrac{x^2-6x+8}{x^2-5x+4}=\lim\limits_{x\to4}\dfrac{x-2}{x-1}=\dfrac{2}{3}.$

20. $\lim\limits_{x\to\infty}\dfrac{1}{x}\left(2x-\dfrac{1}{x^2}\right)=2.$

21. $\lim\limits_{x\to2}\left(\dfrac{4}{x^2-4}-\dfrac{1}{x-2}\right)=\lim\limits_{x\to2}\left(\dfrac{4}{x^2-4}-\dfrac{x+2}{x^2-4}\right)=\lim\limits_{x\to2}\dfrac{2-x}{x^2-4}$

$$=-\lim_{x\to2}\frac{x-2}{(x-2)(x+2)}=-\frac{1}{4}.$$

强化篇

1. $\lim\limits_{x\to1}\dfrac{x^2-1}{2x^2-x-1}=\lim\limits_{x\to1}\dfrac{(x+1)(x-1)}{(2x+1)(x-1)}=\lim\limits_{x\to1}\dfrac{x+1}{2x+1}=\dfrac{2}{3}.$

2. $\lim\limits_{x\to\infty}\dfrac{x^2-1}{2x^2-x-1}=\lim\limits_{x\to\infty}\dfrac{1-\dfrac{1}{x^2}}{2-\dfrac{1}{x}-\dfrac{1}{x^2}}=\dfrac{1}{2}.$

3. $\lim\limits_{x\to\infty}\dfrac{(x-1)(x-2)(x-3)(x-4)(x-5)}{(5x-1)^5}$

$$= \lim_{x\to\infty} \frac{\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)\left(1-\frac{3}{x}\right)\left(1-\frac{4}{x}\right)\left(1-\frac{5}{x}\right)}{\left(5-\frac{1}{x}\right)^5}$$

$$= \frac{1}{5^5} = \frac{1}{3\ 125}.$$

4. $$\lim_{x\to\infty} \frac{(2x-3)^{20}(3x+2)^{30}}{(2x+1)^{50}} = \lim_{x\to\infty} \frac{\left(2-\frac{3}{x}\right)^{20}\left(3+\frac{2}{x}\right)^{30}}{\left(2+\frac{1}{x}\right)^{50}}$$

$$= \frac{2^{20}\times 3^{30}}{2^{50}} = \frac{3^{30}}{2^{30}} = \left(\frac{3}{2}\right)^{30}.$$

5. $$\lim_{x\to 2} \frac{x^2-4x+4}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x+2)(x-2)} = \frac{0}{4} = 0.$$

6. $$\lim_{x\to+\infty} \frac{\sqrt{6x+\sqrt{x+1}}}{\sqrt{3x+2}} = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}.$$

7. $$\lim_{x\to+\infty} \frac{\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}}{\sqrt{2x+1}} = \lim_{x\to+\infty} \frac{1+\frac{\sqrt[3]{x}+\sqrt[4]{x}}{\sqrt{x}}}{\sqrt{2+\frac{1}{x}}} = \frac{\sqrt{2}}{2}.$$

8. $$\lim_{x\to\infty} \frac{x+\sin x}{x-\sin x} = \lim_{x\to\infty} \frac{1+\frac{\sin x}{x}}{1-\frac{\sin x}{x}} = \frac{1+0}{1-0} = 1.$$

9. $$\lim_{x\to 1}\left(\frac{1}{1-x}-\frac{3}{1-x^3}\right) = \lim_{x\to 1}\left[\frac{1+x+x^2}{(1-x)(1+x+x^2)}-\frac{3}{1-x^3}\right]$$

$$= \lim_{x\to 1}\left(\frac{1+x+x^2}{1-x^3}-\frac{3}{1-x^3}\right)$$

$$= \lim_{x\to 1}\frac{x^2+x-2}{1-x^3} = \lim_{x\to 1}\frac{(x-1)(x+2)}{(1-x)(1+x+x^2)}$$

$$= -\lim_{x\to 1}\frac{x+2}{1+x+x^2} = -1.$$

10. $$\lim_{n\to\infty} \frac{(n+2)(2n+3)(3n+4)}{n^3} = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)\left(2+\frac{3}{n}\right)\left(3+\frac{4}{n}\right) = 6.$$

11. $$\lim_{x\to+\infty} \arccos(\sqrt{x^2+x}-x) = \lim_{x\to+\infty} \arccos\frac{x}{\sqrt{x^2+x}+x}$$

$$= \arccos\frac{1}{2} = \frac{\pi}{3}.$$

12. $\lim\limits_{x\to 0}\dfrac{e^{-\frac{1}{x^2}}}{x^{100}} \xlongequal[x^{100}=\frac{1}{t^{50}}]{t=\frac{1}{x^2}} \lim\limits_{t\to+\infty}\dfrac{e^{-t}}{\frac{1}{t^{50}}} = \lim\limits_{t\to+\infty}\dfrac{t^{50}}{e^t} = 0.$

13. $\lim\limits_{x\to-\infty}(\sqrt{x^2+x}-x) = \lim\limits_{x\to-\infty}\dfrac{x}{\sqrt{x^2+x}+x}$

$$= \lim_{x\to-\infty}\frac{x}{\sqrt{x^2(1+\frac{1}{x})}+x} = \lim_{x\to-\infty}\frac{x}{-x\sqrt{1+\frac{1}{x}}+x}$$

$$= \lim_{x\to-\infty}\frac{1}{-\sqrt{1+\frac{1}{x}}+1} = \frac{1}{-1+1} = \infty.$$

14. $\lim\limits_{x\to+\infty}(\sqrt{x^2+x}-x) = \lim\limits_{x\to+\infty}\dfrac{x}{\sqrt{x^2+x}+x} = \dfrac{1}{2}.$

注:在求极限中遇见偶次方根要特别注意.

15. $\lim\limits_{x\to1^-}\arctan\dfrac{1}{1-x} = \arctan(+\infty) = \dfrac{\pi}{2}.$

16. $\lim\limits_{x\to1^+}\arctan\dfrac{1}{1-x} = \arctan(-\infty) = -\dfrac{\pi}{2}.$

17. $\lim\limits_{x\to0^-}\dfrac{1}{1+e^{\frac{1}{x}}} = \dfrac{1}{1+0} = 1.$

18. $\lim\limits_{x\to0^+}\dfrac{1}{1+e^{\frac{1}{x}}} = \dfrac{1}{1+\infty} = 0.$

真题自测

(1992) 当 $x\to1$ 时,出现 $e^{\frac{1}{x-1}}$ 因子,讨论左右极限:

$$\lim_{x\to1^+}\frac{x^2-1}{x-1}e^{\frac{1}{x-1}} = +\infty,$$

$$\lim_{x\to1^-}\frac{x^2-1}{x-1}e^{\frac{1}{x-1}} = 0,$$

故当 $x\to1$ 时,函数 $\dfrac{x^2-1}{x-1}e^{\frac{1}{x-1}}$ 极限不存在但不为 ∞.

(2000) 当 $x\to 0$ 时,出现 $e^{\frac{1}{x}}$,$|x|$ 因子,考虑左右极限:

$$\lim_{x\to 0+}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)=\lim_{x\to 0+}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{x}\right)=0+1=1,$$

$$\lim_{x\to 0-}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)=\lim_{x\to 0-}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}-\frac{\sin x}{x}\right)=2-1=1,$$

原极限 $=1$.

§1.2　利用泰勒公式求极限

基础篇

1. $\lim\limits_{x\to 0}\dfrac{\cos x-e^{-\frac{x^2}{2}}}{x^4}$

$$=\lim_{x\to 0}\frac{1-\frac{1}{2}x^2+\frac{1}{4!}x^4+o(x^4)-\left[1-\frac{x^2}{2}+\frac{1}{2}\left(-\frac{x^2}{2}\right)^2+o(x^4)\right]}{x^4}$$

$$=\lim_{x\to 0}\frac{\left(\frac{1}{4!}-\frac{1}{2}\times\frac{1}{4}\right)x^4+o(x^4)}{x^4}=-\frac{1}{12}.$$

2. $\lim\limits_{x\to 0}\dfrac{x\cos x-\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{x\left[1-\frac{1}{2}x^2+o(x^2)\right]-\left[x-\frac{1}{6}x^3+o(x^3)\right]}{x^3}$

$$=-\frac{1}{3}.$$

3. $\lim\limits_{x\to 0}\left(\dfrac{1}{\ln(1+2x)}-\dfrac{1}{\sin 2x}\right)=\lim\limits_{x\to 0}\dfrac{\sin 2x-\ln(1+2x)}{4x^2}$

$$=\lim_{x\to 0}\frac{2x+o(x^2)-\left[2x-\frac{1}{2}\cdot(2x)^2+o(x^2)\right]}{4x^2}=\lim_{x\to 0}\frac{\frac{1}{2}\cdot 4x^2}{4x^2}=\frac{1}{2}.$$

4. $\lim\limits_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{x}-\cot x\right)=\lim\limits_{x\to 0}\dfrac{1}{x}\left(\dfrac{1}{x}-\dfrac{1}{\tan x}\right)=\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^3}=\dfrac{1}{3}.$

5. $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}=\lim\limits_{x\to 0}\dfrac{1+x+\frac{1}{2}x^2+o(x^2)-1-x}{x^2}=\dfrac{1}{2}.$

6. $\lim\limits_{x\to 0}\dfrac{e^{x^3}-1-x^3}{(\sin 2x)^6}=\lim\limits_{x\to 0}\dfrac{1+x^3+\frac{1}{2}(x^3)^2+o(x^6)-1-x^3}{(2x)^6}$

$$=\frac{\frac{1}{2}}{2^6}=\frac{1}{2^7}=\frac{1}{128}.$$

7. $\lim\limits_{x\to\infty}\left(x^2-x^3\tan\dfrac{1}{x}\right)=\lim\limits_{x\to\infty}\left\{x^2-x^3\left[\dfrac{1}{x}+\dfrac{1}{3x^3}+o\left(\dfrac{1}{x^3}\right)\right]\right\}=-\dfrac{1}{3}.$

8. $\lim\limits_{x\to 0^+}\dfrac{e^x-1-x}{\sqrt{1-x}-\cos\sqrt{x}}$

$$=\lim_{x\to 0^+}\frac{\frac{1}{2}x^2}{1-\frac{1}{2}x-\frac{1}{8}x^2+o(x^2)-\left[1-\frac{1}{2}x+\frac{1}{4!}x^2+o(x^2)\right]}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{2}x^2}{\left(-\frac{1}{8}-\frac{1}{4!}\right)x^2+o(x^2)}=-3.$$

9. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+2x}-1}{3x}=\lim\limits_{x\to 0}\dfrac{x}{3x}=\dfrac{1}{3}.$

10. $\lim\limits_{x\to 0}\left(\cot x-\dfrac{1}{x}\right)=\lim\limits_{x\to 0}\left(\dfrac{1}{\tan x}-\dfrac{1}{x}\right)=\lim\limits_{x\to 0}\dfrac{x-\tan x}{x^2}=0.$

11. $\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x^2+1-\sqrt{1+x^2}}{[x-\ln(1+x)](1-\cos x)}$

$$=\lim_{x\to 0}\frac{\frac{1}{2}x^2+1-\left[1+\frac{1}{2}x^2-\frac{1}{8}x^4+o(x^4)\right]}{\frac{1}{2}x^2\cdot\frac{1}{2}x^2}$$

$$=\lim_{x\to 0}\frac{\frac{1}{8}x^4+o(x^4)}{\frac{1}{4}x^4}=\frac{1}{2}.$$

12. $\lim\limits_{x\to 0}\dfrac{x(e^x+1)-2(e^x-1)}{x^2\sin x}=\lim\limits_{x\to 0}\dfrac{xe^x+x-2e^x+2}{x^3}$

$$= \lim_{x\to 0}\frac{e^x(x-2)+x+2}{x^3}$$

$$= \lim_{x\to 0}\frac{\left[1+x+\frac{1}{2}x^2+\frac{1}{3!}x^3+o(x^3)\right](x-2)+x+2}{x^3}$$

$$= \lim_{x\to 0}\frac{\left(\frac{1}{2}-\frac{2}{3!}\right)x^3+o(x^3)}{x^3} = \frac{1}{2}-\frac{2}{3!} = \frac{1}{6}.$$

13. $\lim\limits_{x\to 0}\dfrac{\cos x - e^{-\frac{x^2}{2}}}{x^2[x+\ln(1-x)]}$

$$= \lim_{x\to 0}\frac{1-\frac{1}{2}x^2+\frac{1}{4!}x^4+o(x^4)-\left[1-\frac{x^2}{2}+\frac{1}{2}\frac{x^4}{4}+o(x^4)\right]}{-\frac{1}{2}x^4}$$

$$= \lim_{x\to 0}\frac{\left(\frac{1}{4!}-\frac{1}{8}\right)x^4+o(x^4)}{-\frac{1}{2}x^4} = \frac{1}{6}.$$

14. $\lim\limits_{x\to 0}\dfrac{e^x+\ln(1-x)-1}{x-\arctan x}$

$$= \lim_{x\to 0}\frac{1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\left[-x-\frac{1}{2}(-x)^2+\frac{1}{3}(-x)^3+o(x^3)\right]-1}{\frac{1}{3}x^3}$$

$$= \lim_{x\to 0}\frac{\left(\frac{1}{6}-\frac{1}{3}\right)x^3+o(x^3)}{\frac{1}{3}x^3} = \frac{-\frac{1}{6}}{\frac{1}{3}} = -\frac{1}{2}.$$

15. $\lim\limits_{x\to 0}\dfrac{\frac{x^2}{2}+1-\sqrt{1+x^2}}{x^2\sin x^2} = \lim\limits_{x\to 0}\dfrac{\frac{x^2}{2}+1-\left[1+\frac{1}{2}x^2-\frac{1}{8}x^4+o(x^4)\right]}{x^4}$

$$= \lim_{x\to 0}\frac{\frac{1}{8}x^4+o(x^4)}{x^4} = \frac{1}{8}.$$

16. $\lim\limits_{x\to 0}\dfrac{e^{-\frac{x^3}{2}}-\cos\sqrt{x^3}}{x^6}$

$$= \lim_{x\to 0}\frac{1-\frac{x^3}{2}+\frac{1}{2}\left(-\frac{x^3}{2}\right)^2+o(x^6)-\left[1-\frac{1}{2}x^3+\frac{1}{4!}x^6+o(x^6)\right]}{x^6}$$

$$= \lim_{x\to 0}\frac{\left(\frac{1}{8}-\frac{1}{4!}\right)x^6+o(x^6)}{x^6}=\frac{1}{12}.$$

17. $\lim\limits_{x\to 0}\dfrac{e^x\sin x-x(1+x)}{x^2\sin x}$

$$= \lim_{x\to 0}\frac{\left[1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)\right]\left[x-\frac{1}{6}x^3+o(x^3)\right]-x(1+x)}{x^3}$$

$$= \lim_{x\to 0}\frac{\left(-\frac{1}{6}+\frac{1}{2}\right)x^3+o(x^3)}{x^3}=\frac{1}{3}.$$

18. $\lim\limits_{x\to 0}\dfrac{e^x\cos x-1-x}{x^3}$

$$= \lim_{x\to 0}\frac{\left[1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)\right]\left[1-\frac{1}{2}x^2+o(x^3)\right]-1-x}{x^3}$$

$$= \lim_{x\to 0}\frac{\left(\frac{1}{6}-\frac{1}{2}\right)x^3}{x^3}=-\frac{1}{3}.$$

强化篇

1. $\lim\limits_{x\to+\infty}(\sqrt[6]{x^6+x^5}-\sqrt[6]{x^6-x^5})$

$$= \lim_{x\to+\infty}\left[\sqrt[6]{x^6(1+\frac{1}{x})}-\sqrt[6]{x^6(1-\frac{1}{x})}\right]$$

$$= \lim_{x\to+\infty}\left[x\left(\sqrt[6]{1+\frac{1}{x}}-\sqrt[6]{1-\frac{1}{x}}\right)\right]$$

$$= \lim_{x\to+\infty}\left[x\left(\sqrt[6]{1+\frac{1}{x}}-1+1-\sqrt[6]{1-\frac{1}{x}}\right)\right]$$

$$= \lim_{x\to+\infty}\left[x\left(\frac{1}{6}\frac{1}{x}+\frac{1}{6}\frac{1}{x}\right)\right]$$

$$=\frac{1}{6}+\frac{1}{6}=\frac{1}{3}.$$

2. $\lim\limits_{x\to+\infty}\left[\left(\ ^{3}-\ ^{2}+\frac{}{2}\right)e^{\frac{1}{x}}-\sqrt{\ ^{6}+1}\right]$

$$\xlongequal{t=\frac{1}{x}}\lim_{t\to0}\left[\left(\frac{1}{\ _{3}}-\frac{1}{\ _{2}}+\frac{1}{2}\ \frac{1}{\ }\right)e^{t}-\sqrt{\frac{1}{\ _{6}}+1}\right]$$

$$=\lim_{t\to0}\frac{\left(1-\ +\frac{1}{2}\ ^{2}\right)e^{t}-\sqrt{1+\ ^{6}}}{\ _{3}}$$

$$=\lim_{t\to0}\frac{\left(1-\ +\frac{1}{2}\ ^{2}\right)\left[1+\ +\frac{1}{2}\ ^{2}+\frac{1}{6}\ ^{3}+\ (\ ^{3})\right]-\left[1+\frac{1}{2}\ ^{6}+\ (\ ^{6})\right]}{\ _{3}}$$

$$=\lim_{t\to0}\frac{\left(\frac{1}{6}-\frac{1}{2}+\frac{1}{2}\right)\ ^{3}+\ (\ ^{3})}{\ _{3}}=\frac{1}{6}.$$

3. $\lim\limits_{x\to0}\dfrac{\ln(1+x+x^2)-x}{x^2}$

$$=\lim_{x\to0}\frac{(x+x^2)-\frac{1}{2}(x+x^2)^2+o[(x+x^2)^2]-x}{x^2}$$

$$=\lim_{x\to0}\frac{(x+x^2)-\frac{1}{2}x^2-x}{x^2}=\frac{1}{2}.$$

4. $\lim\limits_{x\to0}\dfrac{\arcsin x-\sin x}{\arctan x-\tan x}=\lim\limits_{x\to0}\dfrac{x+\frac{1}{6}x^3-\left(x-\frac{1}{6}x^3\right)+o(x^3)}{x-\frac{1}{3}x^3-\left(x+\frac{1}{3}x^3\right)+o(x^3)}$

$$=\lim_{x\to0}\frac{\frac{1}{3}x^3+o(x^3)}{-\frac{2}{3}x^3+o(x^3)}=-\frac{1}{2}.$$

5. $\lim\limits_{x\to0}\dfrac{\int_0^x e^t\cos t\,dt-x-\frac{1}{2}x^2}{(x-\tan x)(\sqrt{1+x}-1)}=\lim\limits_{x\to0}\dfrac{\int_0^x e^t\cos t\,dt-x-\frac{1}{2}x^2}{-\frac{1}{3}x^3\cdot\frac{1}{2}x}\xlongequal{L'}$

$$\lim_{x\to0}\frac{e^x\cos x-1-x}{-\frac{2}{3}x^3}\xlongequal{\text{见基础篇最后一题}}\frac{1}{2}.$$

6. $\lim\limits_{x\to 0^+}\dfrac{e^{x^3}-1}{1-\cos\sqrt{x-\sin x}}=\lim\limits_{x\to 0^+}\dfrac{x^3}{\frac{1}{2}(x-\sin x)}=\dfrac{1}{\frac{1}{2}\times\frac{1}{6}}=12.$

7. $\lim\limits_{n\to\infty}n^2\ln\left(n\sin\dfrac{1}{n}\right)=\lim\limits_{n\to\infty}n^2\left(n\sin\dfrac{1}{n}-1\right)\xlongequal{x=\frac{1}{n}}\lim\limits_{x\to 0}\dfrac{1}{x^2}\left(\dfrac{1}{x}\sin x-1\right)$

$$=\lim_{x\to 0}\frac{\sin x-x}{x^3}=-\frac{1}{6}.$$

8. $\lim\limits_{x\to 0}\dfrac{\frac{x^2}{2}+1-\sqrt{1+x^2}}{(\cos x-e^{x^2})\sin x^2}=\lim\limits_{x\to 0}\dfrac{\frac{x^2}{2}+1-\left[1+\frac{1}{2}x^2-\frac{1}{8}x^4+o(x^4)\right]}{(\cos x-1+1-e^{x^2})\cdot x^2}$

$$=\lim_{x\to 0}\frac{\frac{1}{8}x^4+o(x^4)}{\left(-\frac{1}{2}-1\right)x^4+o(x^4)}=\frac{\frac{1}{8}}{-\frac{3}{2}}=-\frac{1}{12}.$$

9. $\lim\limits_{x\to 0}\dfrac{\sin(\tan x)-\tan(\tan x)}{x^3}=\lim\limits_{x\to 0}\dfrac{-\frac{1}{2}\tan^3 x}{x^3}=-\dfrac{1}{2}.$

10. 分母：$\tan x-\sin x=x+\dfrac{1}{3}x^3-\left(x-\dfrac{1}{6}x^3\right)+o(x^3)$

$$=\frac{1}{2}x^3+o(x^3).$$

分子：$\tan(\tan x)=\tan x+\dfrac{1}{3}\tan^3 x+o(\tan^3 x)$

$$=x+\frac{1}{3}x^3+\cdots+\frac{1}{3}\left(x+\frac{1}{3}x^3+\cdots\right)^3+o(x^3)$$

$$=x+\frac{2}{3}x^3+o(x^3),$$

$\sin(\sin x)=\sin x-\dfrac{1}{6}(\sin x)^3+o(\sin^3 x)$

$$=x-\frac{1}{6}x^3+\cdots-\frac{1}{6}\left(x-\frac{1}{6}x^3+\cdots\right)^3+o(x^3)$$

$$=x-\frac{1}{3}x^3+o(x^3),$$

则 $\tan(\tan x)-\sin(\sin x)=x+\dfrac{2}{3}x^3+o(x^3)-\left(x-\dfrac{1}{3}x^3\right)+o(x^3)$

$$=x^3+o(x^3).$$

所以$\lim\limits_{x\to 0}\dfrac{\tan(\tan x)-\sin(\sin x)}{\tan x-\sin x}=\lim\limits_{x\to 0}\dfrac{x^3+o(x^3)}{\frac{1}{2}x^3+o(x^3)}=2.$

11. $\lim\limits_{x\to 0}\dfrac{\ln\left(\dfrac{\sin x}{x}\right)}{x^2}=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin x}{x}-1}{x^2}=\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}=-\dfrac{1}{6}.$

12. $\lim\limits_{x\to 0}\dfrac{[\sin x-\sin(\sin x)]\sin x}{x^4}=\lim\limits_{x\to 0}\dfrac{\frac{1}{6}\sin^4 x}{x^4}=\dfrac{1}{6}.$

13. $\lim\limits_{x\to\infty}x\left[\left(\sin\dfrac{2}{x}+\cos\dfrac{1}{x}\right)^x-\mathrm{e}^2\right]\xlongequal{t=\frac{1}{x}}\lim\limits_{t\to 0}\dfrac{(\sin 2t+\cos t)^{\frac{1}{t}}-\mathrm{e}^2}{t}$

$=\lim\limits_{t\to 0}\dfrac{\mathrm{e}^{\frac{1}{t}\ln(\sin 2t+\cos t)}-\mathrm{e}^2}{t}=\mathrm{e}^2\lim\limits_{t\to 0}\dfrac{\mathrm{e}^{\frac{1}{t}\ln(\sin 2t+\cos t)-2}-1}{t}$

$=\mathrm{e}^2\lim\limits_{t\to 0}\dfrac{\ln(\sin 2t+\cos t)-2t}{t^2}=-\dfrac{5}{2}\mathrm{e}^2.$

14. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+2\sin x}-x-1}{x\ln(1+x)}$

$=\lim\limits_{x\to 0}\dfrac{1+2\sin x-(x+1)^2}{x^2}\dfrac{1}{\sqrt{1+2\sin x}+(x+1)}$

$=\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{1+2\sin x-(x^2+2x+1)}{x^2}$

$=\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{2(\sin x-x)-x^2}{x^2}$

$=\dfrac{1}{2}\times(0-1)=-\dfrac{1}{2}.$

15. $\lim\limits_{x\to 0}\dfrac{\mathrm{e}^{\frac{2}{x}\ln(1+x)}-\mathrm{e}^2[1-\ln(1+x)]}{x}=\lim\limits_{x\to 0}\dfrac{\mathrm{e}^{\frac{2}{x}\ln(1+x)}-\mathrm{e}^2+\mathrm{e}^2\ln(1+x)}{x}$

$=\lim\limits_{x\to 0}\dfrac{\mathrm{e}^{\frac{2}{x}\ln(1+x)}-\mathrm{e}^2}{x}+\mathrm{e}^2=\mathrm{e}^2\lim\limits_{x\to 0}\dfrac{\mathrm{e}^{\frac{2}{x}\ln(1+x)-2}-1}{x}+\mathrm{e}^2$

$=\mathrm{e}^2\lim\limits_{x\to 0}\dfrac{\frac{2}{x}\ln(1+x)-2}{x}+\mathrm{e}^2=2\mathrm{e}^2\lim\limits_{x\to 0}\dfrac{\ln(1+x)-x}{x^2}+\mathrm{e}^2$

$=-\mathrm{e}^2+\mathrm{e}^2=0.$

16. $\lim\limits_{x\to+\infty}\dfrac{\ln(x+\sqrt{x^2+1})-\ln(x+\sqrt{x^2-1})}{(e^{\frac{1}{x}}-1)^2}$

$$=\lim_{x\to+\infty}\frac{\ln\dfrac{x+\sqrt{x^2+1}}{x+\sqrt{x^2-1}}}{\dfrac{1}{x^2}}=\lim_{x\to+\infty}x^2\frac{x+\sqrt{x^2+1}-x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$$

$$=\lim_{x\to+\infty}x^2\frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$$

$$=\lim_{x\to+\infty}x^2\frac{2}{x+\sqrt{x^2-1}}\frac{1}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$=\frac{1}{2}.$$

17. $\lim\limits_{x\to+\infty}x^2(a^{\frac{1}{x}}-a^{\frac{1}{x+1}})\xlongequal{\text{Lagrange}}\lim\limits_{x\to+\infty}x^2a^{\xi}\ln a\left(\dfrac{1}{x}-\dfrac{1}{x+1}\right)\left(\dfrac{1}{x+1}<\xi<\dfrac{1}{x}\right)$

$$=\lim_{x\to+\infty}\frac{x^2}{x(1+x)}\ln a=\ln a.$$

18. $\lim\limits_{x\to0}\dfrac{\cos(\sin x)-\cos x}{(1-\cos x)\sin^2 x}\xlongequal{\text{Lagrange}}\lim\limits_{x\to0}\dfrac{-\sin\xi(\sin x-x)}{\dfrac{1}{2}x^4}\quad(\sin x<\xi<x)$

$$=\frac{1}{3}.$$

19. $\lim\limits_{x\to0}\dfrac{\ln^2(x+\sqrt{1+x^2})+e^{-x^2}-1}{x^4}$

$$=\lim_{x\to0}\frac{\left[x-\dfrac{1}{6}x^3+o(x^3)\right]^2+1+(-x^2)+\dfrac{1}{2}(-x^2)^2+o(x^4)-1}{x^4}$$

$$=\lim_{x\to0}\frac{x^2-\dfrac{1}{3}x^4+1-x^2+\dfrac{1}{2}x^4-1}{x^4}=\frac{1}{6}.$$

20. $\lim\limits_{x\to0}\dfrac{\sqrt{2+\tan x}-\sqrt{2+\sin x}}{x^3}$

$$=\lim_{x\to0}\frac{\tan x-\sin x}{x^3}\frac{1}{\sqrt{2+\tan x}+\sqrt{2+\sin x}}$$

$$= \frac{1}{2\sqrt{2}} \lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \frac{1}{2\sqrt{2}} \times \frac{1}{2} = \frac{1}{4\sqrt{2}}.$$

21. $\lim\limits_{x \to 0} \dfrac{\sqrt{1+\tan x} - \sqrt{1-\tan x}}{e^x - 1}$

$$= \lim_{x \to 0} \frac{\tan x + \tan x}{x} \frac{1}{\sqrt{1+\tan x} + \sqrt{1-\tan x}}$$

$$= 2 \times \frac{1}{2} = 1.$$

22. $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x} \right) = \lim\limits_{x \to 0} \dfrac{\sin^2 x - x^2}{x^2 \sin^2 x} = \lim\limits_{x \to 0} \dfrac{(\sin x - x)(x + \sin x)}{x^4}$

$$= \lim_{x \to 0} \frac{2x \cdot \left(-\frac{1}{6} x^3 \right)}{x^4} = -\frac{1}{3}.$$

23. $\lim\limits_{x \to 0} \dfrac{(1+x)^{\frac{1}{x}} - e}{x} = \lim\limits_{x \to 0} \dfrac{e^{\frac{1}{x}\ln(1+x)} - e}{x} = e \lim\limits_{x \to 0} \dfrac{e^{\frac{\ln(1+x)-x}{x}} - 1}{x}$

$$= e \lim_{x \to 0} \frac{\ln(1+x) - x}{x^2} = -\frac{1}{2} e.$$

24. $\lim\limits_{x \to 0} \dfrac{\int_0^x x \ln(1+t) \mathrm{d}t}{x - \arcsin x} = \lim\limits_{x \to 0} \dfrac{x \int_0^x \ln(1+t) \mathrm{d}t}{-\frac{1}{6} x^3} \overset{L'}{=\!=} \lim\limits_{x \to 0} \dfrac{\ln(1+x)}{-\frac{1}{3} x} = -3.$

25. $\lim\limits_{x \to 0} \left(\dfrac{\sin x}{x} \right)^{\frac{1}{x^2}} = \lim\limits_{x \to 0} \left(1 + \dfrac{\sin x - x}{x} \right)^{\frac{1}{x^2}} = e^{\lim\limits_{x \to 0} \frac{\sin x - x}{x^3}} = e^{-\frac{1}{6}}.$

26. $\lim\limits_{x \to 0} \dfrac{e^{x^4} - \cos^2 x - x^2}{x^4} = \lim\limits_{x \to 0} \dfrac{1 + x^4 - \left(1 - x^2 + \frac{1}{3} x^4 \right) - x^2 + o(x^4)}{x^4} =$

$$\lim_{x \to 0} \frac{\frac{2}{3} x^4}{x^4} = \frac{2}{3}.$$

真题自测

(1994) 由 $\lim\limits_{x \to 0} \dfrac{\ln(1+x) - (ax + bx^2)}{x^2} = 2$，则 $\ln(1+x) - (ax + bx^2)$ 与 x^2 是同阶无穷小.

由 $x \to 0$ 时，$\ln(1+x)-(ax+bx^2)$

$$= x-\frac{1}{2}x^2-(ax+bx^2)+o(x^2)$$

$$= (1-a)x+\left(-\frac{1}{2}-b\right)x^2+o(x^2),$$

则 $1-a=0$，$-\frac{1}{2}-b=2$，得 $a=1$，$b=-\frac{5}{2}$.

故选(A).

(1998) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}+\sqrt{1-x}-2}{x^2}$

$$=\lim_{x\to 0}\frac{1+\frac{1}{2}x-\frac{1}{8}x^2+1-\frac{1}{2}x-\frac{1}{8}x^2+o(x^2)-2}{x^2}$$

$$=\lim_{x\to 0}\frac{\left(-\frac{1}{8}-\frac{1}{8}\right)x^2+o(x^2)}{x^2}=-\frac{1}{4}.$$

(1999) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x\ln(1+x)-x^2}$

$$=\lim_{x\to 0}\frac{\tan x-\sin x}{-\frac{1}{2}x^3}\cdot\frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}$$

$$=-2\times\frac{1}{2}\times\frac{1}{2}=-\frac{1}{2}.$$

(2000) $\lim\limits_{x\to 0}\dfrac{\arctan x-x}{\ln(1+2x^3)}=\lim\limits_{x\to 0}\dfrac{-\frac{1}{3}x^3}{2x^3}=-\dfrac{1}{6}.$

(1987) $\lim\limits_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)=\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x(e^x-1)}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x^2}{x^2}=\dfrac{1}{2}.$

(1994) $\lim\limits_{x\to 0}\cot x\left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)=\lim\limits_{x\to 0}\dfrac{1}{\tan x}\cdot\dfrac{x-\sin x}{x\sin x}=\lim\limits_{x\to 0}\dfrac{x-\sin x}{x^3}=\dfrac{1}{6}.$

(1994) $\lim\limits_{x\to\infty}\left[x-x^2\ln\left(1+\dfrac{1}{x}\right)\right]\xlongequal{t=\frac{1}{x}}\lim\limits_{t\to 0}\left[\dfrac{1}{t}-\dfrac{1}{t^2}\ln(1+t)\right]$

$$=\lim_{t\to 0}\frac{t-\ln(1+t)}{t^2}=\frac{1}{2}.$$

(1999) $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{1}{x\tan x}\right)=\lim\limits_{x\to 0}\dfrac{\tan x-x}{x^3}=\dfrac{1}{3}$.

(2004) $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin^2 x}-\dfrac{\cos^2 x}{x^2}\right)=\lim\limits_{x\to 0}\dfrac{x^2-\sin^2 x\cos^2 x}{x^2\sin^2 x}=\lim\limits_{x\to 0}\dfrac{x^2-\dfrac{1}{4}\sin^2 2x}{x^4}$

$$=\lim_{x\to 0}\frac{4x^2-\sin^2 2x}{4x^4}$$

$$=\frac{1}{4}\lim_{x\to 0}\frac{(2x-\sin 2x)(2x+\sin 2x)}{x^3\cdot x}$$

$$=\frac{1}{4}\times\frac{1}{6}\times 2^3\times 4=\frac{4}{3}.$$

(2007) 原式 $=\lim\limits_{x\to 0}\dfrac{x-\dfrac{1}{3}x^3-x+\dfrac{1}{6}x^3+o(x^3)}{x^3}=-\dfrac{1}{6}$.

(2020) 原式 $=\lim\limits_{x\to 0}\dfrac{\ln(1+x)-(e^x-1)}{(e^x-1)\ln(1+x)}$

$$=\lim_{x\to 0}\frac{x-\dfrac{1}{2}x^2+o(x^2)-\left[x+\dfrac{x^2}{2}+o(x^2)\right]}{x^2}=-1.$$

§1.3　利用无穷小替换求极限

基础篇

1. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{x}=\lim\limits_{x\to 0}\dfrac{5x}{x}=5$.

2. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\dfrac{\dfrac{1}{2}x^2}{x^2}=\dfrac{1}{2}$.

3. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^3+x^2}=\lim\limits_{x\to 0}\dfrac{\dfrac{1}{2}x^2}{x^2}=\dfrac{1}{2}$.

4. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\sqrt{1+x^2}-1}=\lim\limits_{x\to 0}\dfrac{\dfrac{1}{2}x^2}{\dfrac{1}{2}x^2}=1$.

5. $\lim\limits_{x\to 0}\dfrac{\sqrt{1-\cos x^2}}{1-\cos x}=\lim\limits_{x\to 0}\dfrac{\sqrt{\frac{1}{2}x^4}}{\frac{1}{2}x^2}=\dfrac{\frac{\sqrt{2}}{2}}{\frac{1}{2}}=\sqrt{2}.$

6. $\lim\limits_{x\to\infty}x\ln\left(1+\dfrac{1}{x}\right)=1.$

7. $\lim\limits_{x\to 0}\dfrac{\sin x+x}{x}=2.$

8. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{\ln(1+x^2)}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x^2}{x^2}=\dfrac{1}{2}.$

9. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+3x}-1}{2x}=\lim\limits_{x\to 0}\dfrac{\frac{3}{2}x}{2x}=\dfrac{3}{4}.$

10. $\lim\limits_{n\to\infty}n\sin\dfrac{\pi}{n}=\lim\limits_{n\to\infty}n\dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}\cdot\dfrac{\pi}{n}=\lim\limits_{x\to\infty}n\cdot\dfrac{\pi}{n}\lim\limits_{x\to\infty}\dfrac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}=\pi\times 1=\pi.$

11. $\lim\limits_{x\to 1}\dfrac{\sin(x-1)}{1-x^2}=\lim\limits_{x\to 1}\dfrac{x-1}{1-x^2}=\lim\limits_{x\to 1}\dfrac{x-1}{(1-x)(1+x)}=-\dfrac{1}{2}.$

12. $\lim\limits_{x\to 0}\dfrac{1-\cos 4x}{x\sin x}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}(4x)^2}{x^2}=8.$

13. $\lim\limits_{x\to\infty}x^2\left(1-\cos\dfrac{1}{x}\right)=\lim\limits_{x\to 0}\left(x^2\cdot\dfrac{1}{2}\dfrac{1}{x^2}\right)=\dfrac{1}{2}.$

14. $\lim\limits_{x\to 0}\dfrac{x^2\sin x}{\arctan(x^3)}=\lim\limits_{x\to 0}\dfrac{x^2\cdot x}{x^3}=1.$

15. $\lim\limits_{x\to 0}\dfrac{\ln(1+x\sin x)}{1-\cos x}=\lim\limits_{x\to 0}\dfrac{x\sin x}{\frac{1}{2}x^2}=2.$

16. $\lim\limits_{x\to\infty}x^2\ln\left(1+\dfrac{3}{x^2}\right)=\lim\limits_{x\to\infty}\left(x^2\cdot\dfrac{3}{x^2}\right)=3.$

17. $\lim\limits_{x\to 0}\dfrac{\arcsin 2x}{\tan 3x}=\lim\limits_{x\to 0}\dfrac{2x}{3x}=\dfrac{2}{3}.$

18. $\lim\limits_{x\to 0}\dfrac{\ln\cos ax}{\ln\cos bx}=\lim\limits_{x\to 0}\dfrac{\cos ax-1}{\cos bx-1}=\lim\limits_{x\to 0}\dfrac{1-\cos ax}{1-\cos bx}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}(ax)^2}{\frac{1}{2}(bx)^2}=\dfrac{a^2}{b^2}.$

19. $\lim\limits_{x\to 0}\dfrac{x^2}{\sqrt{1+x^2}-1}=\lim\limits_{x\to 0}\dfrac{x^2}{\frac{1}{2}x^2}=2.$

20. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{1-2x}}{x}=\lim\limits_{x\to 0}\dfrac{-\frac{1}{2}(-2x)}{x}=1.$

21. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{e^{2x}-1}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x}{2x}=\dfrac{1}{4}.$

22. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2}-1}{x\sin 2x}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x^2}{2x^2}=\dfrac{1}{4}.$

23. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x\sin x}-1}{e^{x^2}-1}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x\sin x}{x^2}=\dfrac{1}{2}.$

强化篇

1. $\lim\limits_{x\to 0}\dfrac{2x-\sin x}{2x+\sin x}=\lim\limits_{x\to 0}\dfrac{2x-x}{2x+x}=\dfrac{1}{3}.$

2. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cos x}{\pi-2x}\xlongequal{u=\frac{\pi}{2}-x}\lim\limits_{u\to 0}\dfrac{\cos\left(\frac{\pi}{2}-u\right)}{\pi-2\left(\frac{\pi}{2}-u\right)}=\lim\limits_{u\to 0}\dfrac{\sin u}{2u}=\lim\limits_{u\to 0}\dfrac{u}{2u}=\dfrac{1}{2}.$

3. $\lim\limits_{n\to\infty}2^{n+1}\sin\dfrac{x}{2^n}(x\neq 0)=\lim\limits_{n\to\infty}2^{n+1}\dfrac{\sin\frac{x}{2^n}}{\frac{x}{2^n}}\cdot\dfrac{x}{2^n}=2x.$

4. $\lim\limits_{x\to 0}x\cot x=\lim\limits_{x\to 0}x\dfrac{1}{\tan x}=1.$

5. $\lim\limits_{x\to 0}\dfrac{\arcsin x-\sin x}{x^2(\sqrt{1+2x}-1)}=\lim\limits_{x\to 0}\dfrac{x+\frac{1}{6}x^3-x+\frac{1}{6}x^3+o(x^3)}{x^3}=\dfrac{1}{3}.$

6. $\lim\limits_{x\to\infty}x\ln\dfrac{x+2}{x}=\lim\limits_{x\to\infty}x\ln\left(1+\dfrac{2}{x}\right)=\lim\limits_{x\to\infty}\left(x\cdot\dfrac{2}{x}\right)=2.$

7. $\lim\limits_{x\to 1}\dfrac{\arcsin(1-x)}{\ln x}=\lim\limits_{x\to 1}\dfrac{1-x}{x-1}=-1.$

8. $\lim\limits_{x\to 0}\dfrac{\sin x-\tan x}{(\sqrt[3]{1+x^2}-1)(\sqrt{1+\sin x}-1)}=\lim\limits_{x\to 0}\dfrac{-\frac{1}{2}x^3}{\frac{1}{3}x^2\cdot\frac{1}{2}x}=-3.$

9. $\lim\limits_{x\to 0}\dfrac{\sin(\sin x)}{x}=\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1.$

10. $\lim\limits_{x\to\pi}\dfrac{\sin x}{x-\pi}\xlongequal{u=x-\pi}\lim\limits_{u\to 0}\dfrac{\sin(\pi+u)}{u}=\lim\limits_{u\to 0}\dfrac{-\sin u}{u}=-1.$

11. $\lim\limits_{x\to 0}\dfrac{\sin 5x-\sin 2x}{\sin 2x}=\lim\limits_{x\to 0}\dfrac{3x}{2x}=\dfrac{3}{2}.$

12. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}=\dfrac{1}{2}.$

13. $\lim\limits_{x\to e}\dfrac{\ln x-1}{x-e}=\lim\limits_{x\to e}\dfrac{\ln x-\ln e}{x-e}=\lim\limits_{x\to e}\dfrac{\ln\frac{x}{e}}{x-e}=\lim\limits_{x\to e}\dfrac{x-e}{e(x-e)}=\dfrac{1}{e}.$

14. $\lim\limits_{x\to 0}\dfrac{e^x-e^{\sin x}}{x-\sin x}=\lim\limits_{x\to 0}\dfrac{e^{\sin x}(e^{x-\sin x}-1)}{x-\sin x}=1.$

15. $\lim\limits_{t\to 1}(1-t)\tan\dfrac{\pi}{2}t=\lim\limits_{t\to 1}(1-t)\dfrac{\sin\frac{\pi}{2}t}{\cos\frac{\pi}{2}t}=\lim\limits_{t\to 1}\dfrac{1-t}{\cos\frac{\pi}{2}t}$

$$\xlongequal{u=1-t}\lim_{u\to 0}\frac{u}{\cos\frac{\pi}{2}(1-u)}=\lim_{u\to 0}\frac{u}{\sin\frac{\pi}{2}u}=\frac{2}{\pi}.$$

16. $\lim\limits_{x\to 0}\dfrac{\sec^2 x(x-\arcsin x)}{x\ln\cos x}=\lim\limits_{x\to 0}\dfrac{-\frac{1}{6}x^3+o(x^3)}{-\frac{1}{2}x^3}=\dfrac{1}{3}.$

17. $\lim\limits_{x\to 0}\dfrac{x-x\cos x}{\sin x-\tan x}=\lim\limits_{x\to 0}\dfrac{x(1-\cos x)}{-\frac{1}{2}x^3}=-2\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$

$$=-2\times\frac{1}{2}=-1.$$

18. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x\sin x}-1}{x(e^{-x}-1)}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x\sin x}{-x^2}=-\dfrac{1}{2}.$

19. $\lim\limits_{x\to 0}\dfrac{(\sqrt{1+\sin x}-1)\arctan^4 x}{\arcsin 2x\ (1-\cos x)(x^2+x^4)}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}\cdot x\cdot x^4}{2x\cdot\frac{1}{2}x^2\cdot x^2}=\dfrac{1}{2}.$

20. $\lim\limits_{x\to 5}\dfrac{1-\sqrt{x-4}}{x-5}=\lim\limits_{x\to 5}\dfrac{1-\sqrt{1+(x-5)}}{x-5}=\lim\limits_{x\to 5}\dfrac{-\frac{1}{2}(x-5)}{x-5}=-\dfrac{1}{2}.$

21. $\lim\limits_{x\to 2}\dfrac{\sqrt[4]{x+14}-2}{x^2-4}=\lim\limits_{x\to 2}\dfrac{\sqrt[4]{x-2+16}-2}{(x-2)(x+2)}=\lim\limits_{x\to 2}\dfrac{\sqrt[4]{16\left(\frac{x-2}{16}+1\right)}-2}{(x-2)(x+2)}$

$$=\frac{1}{4}\lim_{x\to 2}\frac{2\left(\sqrt[4]{\frac{x-2}{16}+1}-1\right)}{x-2}=\frac{1}{2}\lim_{x\to 2}\frac{\frac{1}{4}\,\frac{x-2}{16}}{x-2}$$

$$=\frac{1}{2}\times\frac{1}{4}\times\frac{1}{16}=\frac{1}{128}.$$

22. $\lim\limits_{x\to 0}\dfrac{1}{x^2}\ln\dfrac{\sin x}{x}=\lim\limits_{x\to 0}\dfrac{1}{x^2}\left(\dfrac{\sin x}{x}-1\right)=\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}=-\dfrac{1}{6}.$

23. $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x^3}{x^3}=\dfrac{1}{2}.$

24. $\lim\limits_{x\to 0}\dfrac{(\arctan x-\tan x)(1-\cos\sqrt{x})}{x^2(1-\sqrt{\cos x})}=\lim\limits_{x\to 0}\dfrac{-\frac{2}{3}x^3\cdot\frac{1}{2}x}{x^2\cdot\frac{1}{4}x^2}=-\dfrac{4}{3}.$

25. $\lim\limits_{x\to 0}\dfrac{\cos 2x-\cos x}{x^2}=\lim\limits_{x\to 0}\dfrac{\cos 2x-1+1-\cos x}{x^2}$

$$=\lim_{x\to 0}\frac{1-\cos x}{x^2}-\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$$

$$=\frac{1}{2}-\frac{1}{2}\times 2^2=\frac{1}{2}-2=-\frac{3}{2}.$$

26. $\lim\limits_{x\to 0}\dfrac{\ln(\sin^2 x+e^x)-x}{\ln(x^2+e^{2x})-2x}=\lim\limits_{x\to 0}\dfrac{\ln(\sin^2 x+e^x)-\ln e^x}{\ln(x^2+e^{2x})-\ln e^{2x}}$

$$=\lim_{x\to 0}\frac{\ln\left(1+\frac{\sin^2 x}{e^x}\right)}{\ln\left(1+\frac{x^2}{e^{2x}}\right)}=\lim_{x\to 0}\frac{\frac{\sin^2 x}{e^x}}{\frac{x^2}{e^{2x}}}=1.$$

27. $\lim\limits_{x\to 0}\dfrac{\ln(1+xe^x)}{\ln(x+\sqrt{1+x^2})}=\lim\limits_{x\to 0}\dfrac{xe^x}{x}=1.$

注：因为 $\left[\ln(x+\sqrt{1+x^2})\right]'=\dfrac{1}{\sqrt{1+x^2}}=(1+x^2)^{-\frac{1}{2}}$

$$=1-\frac{1}{2}x^2+\cdots,$$

所以 $\ln(x+\sqrt{1+x^2})=x-\dfrac{1}{6}x^3+\cdots$,

故 $\ln(x+\sqrt{1+x^2})=x+o(x)\sim x$.

28. $\lim\limits_{x\to 0}\dfrac{1-\sqrt{\cos x}}{1-\cos\sqrt{x}}=\lim\limits_{x\to 0}\dfrac{1-\sqrt{1+\cos x-1}}{\frac{1}{2}x}=\lim\limits_{x\to 0}\dfrac{-\frac{1}{2}(\cos x-1)}{\frac{1}{2}x}$

$$=\lim_{x\to 0}\frac{\frac{1}{4}x^2}{\frac{1}{2}x}=0.$$

29. $\lim\limits_{x\to 0}\dfrac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}$

$$=\lim_{x\to 0}\frac{x^2}{1+x\sin x-\cos x}(\sqrt{1+x\sin x}+\sqrt{\cos x})$$

$$=2\lim_{x\to 0}\frac{1}{\frac{1+x\sin x-\cos x}{x^2}}=2\frac{1}{\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}+\lim\limits_{x\to 0}\frac{x\sin x}{x^2}}$$

$$=\frac{2}{\frac{1}{2}+1}=\frac{2}{\frac{3}{2}}=\frac{4}{3}.$$

30. $\lim\limits_{x\to 0}\dfrac{\sqrt{\cos x}-\sqrt[3]{\cos x}}{\sin^2 x}=\lim\limits_{x\to 0}\dfrac{\sqrt{\cos x}-1+1-\sqrt[3]{\cos x}}{x^2}$

$$=\lim_{x\to 0}\frac{\sqrt{1+\cos x-1}-1}{x^2}+\lim_{x\to 0}\frac{1-\sqrt[3]{1+\cos x-1}}{x^2}$$

$$=\lim_{x\to 0}\frac{\frac{1}{2}(\cos x-1)}{x^2}+\lim_{x\to 0}\frac{-\frac{1}{3}(\cos x-1)}{x^2}$$

$$=\frac{1}{2}\times\left(-\frac{1}{2}\right)-\frac{1}{3}\times\left(-\frac{1}{2}\right)=-\frac{1}{4}+\frac{1}{6}$$

$$=\frac{-3+2}{12}=-\frac{1}{12}.$$

31. $\lim\limits_{h\to 0}\dfrac{a^{x+h}+a^{x-h}-2a^x}{h^2}=\lim\limits_{h\to 0}\dfrac{a^{x-h}(a^{2h}+1-2a^h)}{h^2}=a^x\lim\limits_{h\to 0}\left(\dfrac{a^h-1}{h}\right)^2$

$$=a^x\lim_{h\to 0}\left(\frac{e^{h\ln a}-1}{h}\right)^2=a^x\ln^2 a.$$

32. $\lim\limits_{n\to\infty}n(\sqrt[n]{x}-1)(x>0)=\lim\limits_{n\to\infty}n(e^{\frac{1}{n}\ln x}-1)=\lim\limits_{n\to\infty}\left(n\cdot\dfrac{1}{n}\ln x\right)=\ln x.$

33. $\lim\limits_{x\to 0}\dfrac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}=\lim\limits_{x\to 0}\dfrac{x^2+e^x-1}{x^4+e^{2x}-1}=\lim\limits_{x\to 0}\dfrac{x}{2x}=\dfrac{1}{2}.$

34. $\lim\limits_{x\to 0}\dfrac{\ln(x+\sqrt{1+x^2})-\ln(1+x)}{\ln^2(1+x)}=\lim\limits_{x\to 0}\dfrac{\frac{1}{\xi}(\sqrt{1+x^2}-1)}{x^2}$

$(x+1<\xi<x+\sqrt{1+x^2})=\dfrac{1}{2}.$

35. $\lim\limits_{x\to 0}\dfrac{\sin 2x-2\sin x}{x^3}=\lim\limits_{x\to 0}\dfrac{\sin 2x-2x+2(x-\sin x)}{x^3}$

$$=\lim_{x\to 0}\frac{\sin 2x-2x}{x^3}+2\lim_{x\to 0}\frac{x-\sin x}{x^3}$$

$$=-\frac{1}{6}\times 2^3+2\times\frac{1}{6}=-1.$$

36. $\lim\limits_{x\to 0}\dfrac{\ln(1+\sin^2 x)}{(1+\cos x)\tan^2 x}=\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x^2}=\dfrac{1}{2}.$

37. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\tan x}-\sqrt{1-\sin x}}{x}$

$=\lim\limits_{x\to 0}\dfrac{\tan x+\sin x}{x}\dfrac{1}{\sqrt{1+\tan x}+\sqrt{1-\sin x}}=2\times\dfrac{1}{2}=1.$

38. $\lim\limits_{x\to+\infty}\dfrac{x\sqrt{x}\sin\frac{1}{x}}{\sqrt{x}-1}=\lim\limits_{x\to+\infty}\dfrac{\sqrt{x}}{\sqrt{x}-1}=1.$

39. $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{1+2x}-1}{\ln(2-\cos x+\sin x)}=\lim\limits_{x\to 0}\dfrac{\frac{2}{3}x}{1-\cos x+\sin x}=\lim\limits_{x\to 0}\dfrac{\frac{2}{3}x}{x}=\dfrac{2}{3}.$

40. $\lim\limits_{x\to 0}\dfrac{\sqrt{\cos x}-\sqrt[3]{1+\sin^2 x}}{x^2}=\lim\limits_{x\to 0}\dfrac{\sqrt{\cos x}-1+1-\sqrt[3]{1+\sin^2 x}}{x^2}$

$$=\lim_{x\to 0}\frac{\sqrt{\cos x}-1}{x^2}+\lim_{x\to 0}\frac{1-\sqrt[3]{1+\sin^2 x}}{x^2}$$

$$= \lim_{x\to 0}\frac{\frac{1}{2}(\cos x-1)}{x^2}+\lim_{x\to 0}\frac{-\frac{1}{3}x^2}{x^2}$$

$$= \lim_{x\to 0}\frac{-\frac{1}{4}x^2}{x^2}-\frac{1}{3}=-\frac{1}{4}-\frac{1}{3}=-\frac{7}{12}.$$

41. $\lim\limits_{x\to 1}\frac{x^x-1}{\sin \pi x}=\lim\limits_{x\to 1}\frac{e^{x\ln x}-1}{\sin \pi x}=\lim\limits_{x\to 1}\frac{x\ln x}{\sin \pi x}=\lim\limits_{x\to 1}\frac{x-1}{\sin \pi x}\overset{L'}{=}\lim\limits_{x\to 1}\frac{1}{\pi\cos \pi x}$

$$=-\frac{1}{\pi}.$$

42. $\lim\limits_{x\to -\infty}\frac{\ln(1+3^x)}{\ln(1+2^x)}=\lim\limits_{x\to -\infty}\frac{3^x}{2^x}=\lim\limits_{x\to -\infty}\left(\frac{3}{2}\right)^x=0.$

43. $\lim\limits_{x\to +\infty}\frac{\ln(1+3^x)}{\ln(1+2^x)}=\lim\limits_{x\to +\infty}\frac{\ln 3^x\left(\frac{1}{3^x}+1\right)}{\ln 2^x\left(\frac{1}{2^x}+1\right)}=\lim\limits_{x\to +\infty}\frac{x\ln 3+\ln\left(1+\frac{1}{3^x}\right)}{x\ln 2+\ln\left(1+\frac{1}{2^x}\right)}$

$$=\lim_{x\to +\infty}\frac{\ln 3+\frac{\ln\left(1+\frac{1}{3^x}\right)}{x}}{\ln 2+\frac{\ln\left(1+\frac{1}{2^x}\right)}{x}}=\frac{\ln 3}{\ln 2}.$$

44. $\lim\limits_{x\to +\infty}\ln(1+2^x)\ln\left(1+\frac{3}{x}\right)=\lim\limits_{x\to +\infty}\frac{3\ln(1+2^x)}{x}=3\ln 2.$

真题自测

(1988) $\lim\limits_{x\to 1}\frac{x^x-1}{x\ln x}=\lim\limits_{x\to 1}\frac{e^{x\ln x}-1}{x\ln x}=1.$

(1991) $\lim\limits_{x\to 0}\frac{x-\sin x}{x^2(e^x-1)}=\lim\limits_{x\to 0}\frac{\frac{1}{6}x^3}{x^3}=\frac{1}{6}.$

(1992) $\lim\limits_{x\to 0}\frac{1-\sqrt{1-x^2}}{e^x-\cos x}=\lim\limits_{x\to 0}\frac{\frac{1}{2}x^2}{e^x-1+1-\cos x}=\lim\limits_{x\to 0}\frac{\frac{1}{2}x^2}{x+o(x)}=0.$

(1992) $\lim\limits_{x\to 0}\frac{e^x-\sin x-1}{1-\sqrt{1-x^2}}=\lim\limits_{x\to 0}\frac{e^x-\sin x-1}{\frac{1}{2}x^2}=1.$

(1994) 由 $\lim\limits_{x\to 0}\dfrac{a\tan x+b\,(1-\cos x)}{c\ln(1-2x)+d(1-\mathrm{e}^{-x^2})}=\lim\limits_{x\to 0}\dfrac{ax+o(x)}{-2cx+o(x)}=\dfrac{a}{-2c}=2$,得 $a=-4c$.故选(D).

(1995) $\lim\limits_{x\to 0^+}\dfrac{1-\sqrt{\cos x}}{x\,(1-\cos\sqrt{x})}=\lim\limits_{x\to 0}\dfrac{1-\sqrt{1+\cos x-1}}{x\cdot\frac{1}{2}x}$

$$=\lim_{x\to 0}\frac{-\frac{1}{2}(\cos x-1)}{\frac{1}{2}x^2}=\lim_{x\to 0}\frac{\frac{1}{4}x^2}{\frac{1}{2}x^2}=\frac{1}{2}.$$

(1997) $\lim\limits_{x\to 0}\dfrac{3\sin x+x^2\cos\frac{1}{x}}{(1+\cos x)\ln(1+x)}=\dfrac{1}{2}\lim\limits_{x\to 0}\dfrac{3x+o(x)}{x}=\dfrac{3}{2}$.

(2005) 原式 $=\lim\limits_{x\to\infty}\left(x\cdot\dfrac{2x}{x^2+1}\right)=2$.

(2006) 原式 $=\lim\limits_{x\to 0}\dfrac{x^2}{\frac{1}{2}x^2}=2$.

(2009) 原式 $=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x^2[x-\ln(1+\tan x)]}{x^4}=\lim\limits_{x\to 0}\dfrac{x-\ln(1+\tan x)}{2x^2}$

$$=\frac{1}{2}\lim_{x\to 0}\frac{x-\tan x+\frac{1}{2}\tan^2 x+o(\tan^2 x)}{x^2}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{x-\left[x+\frac{1}{3}x^3+o(x^3)\right]+\frac{1}{2}\left[x+\frac{1}{3}x^3+o(x^3)\right]^2}{x^2}$$

$$=\frac{1}{2}\lim_{x\to 0}\frac{\frac{1}{2}x^2}{x^2}=\frac{1}{4}.$$

(2009) 原式 $=\lim\limits_{x\to 0}\dfrac{\mathrm{e}(1-\mathrm{e}^{\cos x-1})}{\frac{1}{3}x^2}=3\mathrm{e}\cdot\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x^2}{x^2}=\dfrac{3}{2}\mathrm{e}$.

§1.4 利用洛必达法则求极限

基础篇

1. $\lim\limits_{x\to 0^+} x^x = e^{\lim\limits_{x\to 0^+} x\ln x} \overset{L'}{=\!=} e^{\lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}} = 1.$

2. $\lim\limits_{x\to 0} \dfrac{2x-\ln(1+2x)}{x^2} \overset{L'}{=\!=} \lim\limits_{x\to 0} \dfrac{2-\dfrac{2}{1+2x}}{2x} = \lim\limits_{x\to 0} \dfrac{4x}{2x(1+2x)} = 2.$

3. $\lim\limits_{x\to 0} \dfrac{e^x-e^{-x}}{\sin x} \overset{L'}{=\!=} \lim\limits_{x\to 0} \dfrac{e^x+e^{-x}}{\cos x} = 2.$

4. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\ln\sin x}{(\pi-2x)^2} \overset{L'}{=\!=} \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\cot x}{-4(\pi-2x)} \overset{L'}{=\!=} -\dfrac{1}{8}.$

5. $\lim\limits_{x\to 0} \dfrac{1+\sin x-\cos x}{1+\sin px+\cos px} = 0$(本题不能用法则).

6. $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2} = \lim\limits_{x\to 1}(1-x)\dfrac{\sin\frac{\pi}{2}x}{\cos\frac{\pi}{2}x} = \lim\limits_{x\to 1}\dfrac{1-x}{\cos\frac{\pi}{2}x}$

$$\overset{L'}{=\!=} \lim_{x\to 1}\frac{-1}{-\frac{\pi}{2}\sin\frac{\pi}{2}x} = \frac{2}{\pi}.$$

7. $\lim\limits_{x\to a} \dfrac{\sin x-\sin a}{x-a} \overset{L'}{=\!=} \lim\limits_{x\to a} \dfrac{\cos x}{1} = \cos a.$

8. $\lim\limits_{x\to a} \dfrac{\cos x-\cos a}{x-a} \overset{L'}{=\!=} \lim\limits_{x\to a} \dfrac{-\sin x}{1} = -\sin a.$

9. $\lim\limits_{x\to a} \dfrac{\tan x-\tan a}{x-a} \overset{L'}{=\!=} \lim\limits_{x\to a} \dfrac{\sec^2 x}{1} = \dfrac{1}{\cos^2 a}.$

10. $\lim\limits_{x\to a} \dfrac{\cot x-\cot a}{x-a} \overset{L'}{=\!=} \lim\limits_{x\to a} \dfrac{-\csc^2 x}{1} = -\dfrac{1}{\sin^2 a}.$

11. $\lim\limits_{x\to a} \dfrac{\sec x-\sec a}{x-a} \overset{L'}{=\!=} \lim\limits_{x\to a} \dfrac{\sec x\tan x}{1} = \dfrac{1}{\cos a}\dfrac{\sin a}{\cos a} = \dfrac{\sin a}{\cos^2 a}.$

12. $\lim\limits_{x\to a} \dfrac{\csc x-\csc a}{x-a} \overset{L'}{=\!=} \lim\limits_{x\to a} \dfrac{-\csc x\cot x}{1} = \dfrac{-1}{\sin a}\dfrac{\cos a}{\sin a} = -\dfrac{\cos a}{\sin^2 a}.$

13. $\lim\limits_{x\to\frac{\pi}{3}}\dfrac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x}\overset{L'}{=}\lim\limits_{x\to\frac{\pi}{3}}\dfrac{\cos\left(x-\frac{\pi}{3}\right)}{2\sin x}=\dfrac{1}{\sqrt{3}}=\dfrac{\sqrt{3}}{3}.$

14. $\lim\limits_{x\to\frac{\pi}{3}}\dfrac{\tan^3 x-3\tan x}{\cos\left(x+\frac{\pi}{6}\right)}\overset{L'}{=}\lim\limits_{x\to\frac{\pi}{3}}\dfrac{3\tan^2 x\sec^2 x-3\sec^2 x}{-\sin\left(x+\frac{\pi}{6}\right)}$

$$=\lim_{x\to\frac{\pi}{3}}\frac{3\sec^2 x(\tan^2 x-1)}{-\sin\left(x+\frac{\pi}{6}\right)}$$

$$=-3\lim_{x\to\frac{\pi}{3}}\sec^2 x(\tan^2 x-1)$$

$$=-3\times\frac{1}{\frac{1}{4}}\times\left[(\sqrt{3})^2-1\right]=-24.$$

15. $\lim\limits_{x\to 0}\dfrac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$

$$=\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}\ \frac{1}{\sqrt{1+\tan x}+\sqrt{1+\sin x}}$$

$=\dfrac{1}{2}\times\dfrac{1}{2}=\dfrac{1}{4}$(利用第 4 题的结果).

16. $\lim\limits_{x\to 0}\dfrac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}$

$$=\lim_{x\to 0}\frac{x^2}{1+x\sin x-\cos x}(\sqrt{1+x\sin x}+\sqrt{\cos x})$$

$$=2\lim_{x\to 0}\frac{x^2}{1+x\sin x-\cos x}\overset{L'}{=}2\lim_{x\to 0}\frac{2x}{\sin x+x\cos x+\sin x}$$

$$\overset{L'}{=}2\lim_{x\to 0}\frac{2}{2\cos x+\cos x-x\sin x}=2\times\frac{2}{2+1}=\frac{2}{3}\times 2=\frac{4}{3}.$$

17. $\lim\limits_{x\to 0}x\cot 2x=\lim\limits_{x\to 0}\dfrac{x}{\tan 2x}\overset{L'}{=}\lim\limits_{x\to 0}\dfrac{1}{2\sec^2 2x}=\dfrac{1}{2}.$

18. $\lim\limits_{x\to+\infty}x[\ln(x+1)-\ln x]=\lim\limits_{x\to+\infty}x\ln\left(1+\dfrac{1}{x}\right)\overset{t=\frac{1}{x}}{=\!=\!=}\lim\limits_{t\to 0}\dfrac{\ln(1+t)}{t}$

$$\overset{L'}{=}\lim_{t\to 0}\frac{1}{1+t}=1.$$

19. $\lim\limits_{x\to+\infty}\dfrac{\ln(x^2-x+1)}{\ln(x^{10}+x+1)}\overset{L'}{=}\lim\limits_{x\to+\infty}\dfrac{\dfrac{2x-1}{x^2-x+1}}{\dfrac{10x^9+1}{x^{10}+x+1}}$

$$=\lim_{x\to+\infty}\frac{(2x-1)(x^{10}+x+1)}{(x^2-x+1)(10x^9+1)}=\frac{2}{10}=\frac{1}{5}.$$

20. $\lim\limits_{x\to+\infty}\dfrac{\ln(2+e^{3x})}{\ln(3+e^{2x})}\overset{L'}{=}\lim\limits_{x\to+\infty}\dfrac{\dfrac{3e^{3x}}{2+e^{3x}}}{\dfrac{2e^{2x}}{3+e^{2x}}}=\dfrac{3}{2}.$

21. $\lim\limits_{x\to0}\dfrac{\ln\tan\left(\dfrac{\pi}{4}+ax\right)}{\sin bx}=\lim\limits_{x\to0}\dfrac{\tan\left(\dfrac{\pi}{4}+ax\right)-1}{bx}\overset{L'}{=}\lim\limits_{x\to0}\dfrac{a\sec^2\left(\dfrac{\pi}{4}+ax\right)}{b}$

$$=\frac{a}{b}\left(\frac{1}{\frac{\sqrt{2}}{2}}\right)^2=\frac{2a}{b}.$$

22. $\lim\limits_{x\to0}\dfrac{\ln\cos ax}{\ln\cos bx}=\lim\limits_{x\to0}\dfrac{\cos ax-1}{\cos bx-1}\overset{L'}{=}\lim\limits_{x\to0}\dfrac{-a\sin ax}{-b\sin bx}=\dfrac{a^2}{b^2}.$

23. $\lim\limits_{x\to a}\dfrac{x^m-a^m}{x^n-a^n}\overset{L'}{=}\lim\limits_{x\to a}\dfrac{mx^{m-1}}{nx^{n-1}}=\dfrac{m}{n}a^{m-n}.$

24. $\lim\limits_{x\to0}(x+e^x)^{\frac{1}{x}}=e^A.$

$A=\lim\limits_{x\to0}\dfrac{x+e^x-1}{x}\overset{L'}{=}2,$

故原极限 $=e^2$.

25. $\lim\limits_{x\to+\infty}\dfrac{x^k}{e^x}(k>0)\overset{L'}{=}\lim\limits_{x\to+\infty}\dfrac{kx^{k-1}}{e^x}\overset{L'}{=}\cdots\overset{L'}{=}0.$

26. $\lim\limits_{x\to\infty}x(a^{\frac{1}{x}}-b^{\frac{1}{x}})(a,b>0)\overset{t=\frac{1}{x}}{=\!=\!=}\lim\limits_{t\to0}\dfrac{1}{t}(a^t-b^t)$

$$\overset{L'}{=}\lim_{t\to0}(a^t\ln a-b^t\ln b)=\ln a-\ln b.$$

真题自测

(1993) $\lim\limits_{x\to\infty}\dfrac{3x^2+5}{5x+3}\sin\dfrac{2}{x}=\lim\limits_{x\to\infty}\dfrac{3x^2+5}{5x+3}\dfrac{\sin\dfrac{2}{x}}{\dfrac{2}{x}}\dfrac{2}{x}=\lim\limits_{x\to\infty}\dfrac{6x^2+10}{5x^2+3x}$

$$\overset{L'}{=}\lim_{x\to\infty}\frac{12x}{10x+3}\overset{L'}{=}\frac{12}{10}=\frac{6}{5}.$$

(1987) $\lim\limits_{x\to+\infty}\dfrac{\ln\left(1+\dfrac{1}{x}\right)}{\operatorname{arccot}x}=\lim\limits_{x\to+\infty}\dfrac{\dfrac{1}{x}}{\operatorname{arccot}x}\overset{L'}{=}\lim\limits_{x\to+\infty}\dfrac{-\dfrac{1}{x^2}}{-\dfrac{1}{1+x^2}}=1.$

(1988) $\lim\limits_{x\to1}\dfrac{x^x-1}{x\ln x}\overset{L'}{=}\lim\limits_{x\to1}\dfrac{x^x(1+\ln x)}{1+\ln x}=1.$

(1991) $\lim\limits_{x\to0}\dfrac{x-\sin x}{x^2(\mathrm{e}^x-1)}=\lim\limits_{x\to0}\dfrac{x-\sin x}{x^3}\overset{L'}{=}\lim\limits_{x\to0}\dfrac{1-\cos x}{3x^2}=\dfrac{1}{6}.$

(1992) $\lim\limits_{x\to0}\dfrac{1-\sqrt{1-x^2}}{\mathrm{e}^x-\cos x}=\lim\limits_{x\to0}\dfrac{\dfrac{1}{2}x^2}{\mathrm{e}^x-\cos x}\overset{L'}{=}\lim\limits_{x\to0}\dfrac{x}{\mathrm{e}^x+\sin x}=0.$

注意法则使用的条件.部分同学会有如下错误：

$$\lim_{x\to0}\frac{x}{\mathrm{e}^x+\sin x}\overset{L'}{=}\lim_{x\to0}\frac{1}{\mathrm{e}^x+\cos x}=\frac{1}{2}.$$

(1992) $\lim\limits_{x\to1}\dfrac{\ln\cos(x-1)}{1-\sin\dfrac{\pi}{2}x}=\lim\limits_{x\to1}\dfrac{\cos(x-1)-1}{1-\sin\dfrac{\pi}{2}x}\overset{L'}{=}\lim\limits_{x\to1}\dfrac{-\sin(x-1)}{-\dfrac{\pi}{2}\cos\dfrac{\pi}{2}x}$

$$=\lim_{x\to1}\frac{x-1}{\dfrac{\pi}{2}\cos\dfrac{\pi}{2}x}\overset{L'}{=}\lim_{x\to1}\frac{1}{-\dfrac{\pi^2}{4}\sin\dfrac{\pi}{2}x}=-\frac{4}{\pi^2}.$$

(1992) $\lim\limits_{x\to0}\dfrac{\mathrm{e}^x-\sin x-1}{1-\sqrt{1-x^2}}=\lim\limits_{x\to0}\dfrac{\mathrm{e}^x-x-1}{\dfrac{1}{2}x^2}\overset{L'}{=}\lim\limits_{x\to0}\dfrac{\mathrm{e}^x-1}{x}=1.$

(1993) $\lim\limits_{x\to0^+}x\ln x=\lim\limits_{x\to0+}\dfrac{\ln x}{\dfrac{1}{x}}\overset{L'}{=}\lim\limits_{x\to0^+}\dfrac{\dfrac{1}{x}}{-\dfrac{1}{x^2}}=0.$

(1988) 原式$=\lim\limits_{x\to1}(1+x)(1-x)\tan\dfrac{\pi}{2}x=2\lim\limits_{x\to1}(1-x)\tan\dfrac{\pi}{2}x$

$$=2\lim_{x\to1}\frac{1-x}{\cot\dfrac{\pi}{2}x}\overset{L'}{=}2\lim_{x\to1}\left(\frac{-1}{-\csc^2\dfrac{\pi}{2}x}\cdot\frac{2}{\pi}\right)$$

$$=\frac{4}{\pi}.$$

§1.5 幂指函数求极限

基础篇

1. $\lim\limits_{x\to 0}(1-x)^{\frac{1}{x}}=e^{\lim\limits_{x\to 0}(-x)\frac{1}{x}}=e^{-1}$.

2. $\lim\limits_{x\to 0}(1+2x)^{\frac{1}{x}}=e^{\lim\limits_{x\to 0}2x\frac{1}{x}}=e^{2}$.

3. $\lim\limits_{x\to\infty}\left(1-\dfrac{1}{x}\right)^{kx}=e^{\lim\limits_{x\to\infty}\left(-\frac{1}{x}\right)kx}=e^{-k}$.

4. $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}+\dfrac{1}{n^2}\right)^{n}=e^{\lim\limits_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right)n}=e$.

5. $\lim\limits_{x\to 0}(1+\tan x)^{3\cot x}=e^{\lim\limits_{x\to 0}(\tan x\cdot 3\cot x)}=e^{3}$.

6. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{2x}}=e^{\lim\limits_{x\to 0}x\frac{1}{2x}}=e^{\frac{1}{2}}$.

7. $\lim\limits_{x\to 0}(1+3x)^{\frac{1}{\sin x}}=e^{\lim\limits_{x\to 0}\frac{3x}{\sin x}}=e^{3}$.

8. $\lim\limits_{x\to 0}(1+3x)^{\frac{2}{x}}=e^{\lim\limits_{x\to 0}3x\frac{2}{x}}=e^{6}$.

9. $\lim\limits_{x\to+\infty}\left(1-\dfrac{1}{x}\right)^{\sqrt{x}}=e^{\lim\limits_{x\to+\infty}-\frac{\sqrt{x}}{x}}=e^{0}=1$.

10. $\lim\limits_{x\to 0}(1+e^{x}\sin^2 x)^{\frac{1}{1-\cos x}}=e^{\lim\limits_{x\to 0}\frac{e^{x}\sin^2 x}{1-\cos x}}=e^{\lim\limits_{x\to 0}\frac{x^2}{1-\cos x}}=e^{2}$.

11. $\lim\limits_{x\to 0}\sqrt[x]{1-2x}=\lim\limits_{x\to 0}(1-2x)^{\frac{1}{x}}=e^{\lim\limits_{x\to 0}\frac{-2x}{x}}=e^{-2}$.

12. $\lim\limits_{x\to 0}(1+x^2)^{\cot 2x}=e^{\lim\limits_{x\to 0}\frac{x^2}{\tan^2 x}}=e^{1}=e$.

13. $\lim\limits_{x\to 1}(1+\sin\pi x)^{\cot\pi x}=e^{\lim\limits_{x\to 1}\sin\pi x\frac{\cos\pi x}{\sin\pi x}}=e^{-1}$.

14. $\lim\limits_{x\to\infty}\left(\dfrac{1+x}{x}\right)^{2x}=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{2x}=e^{2}$.

15. $\lim\limits_{x\to\infty}\left(\dfrac{2x+3}{2x+1}\right)^{x+1}=\lim\limits_{x\to\infty}\left(1+\dfrac{2}{2x+1}\right)^{x+1}=e$.

16. $\lim\limits_{x\to+\infty}\left(\dfrac{ax+b_1}{ax+b_2}\right)^{x}=e^{A}$.

$$A=\lim_{x\to+\infty}\left(\frac{ax+b_1}{ax+b_2}-1\right)x=\lim_{x\to+\infty}\frac{ax+b_1-(ax+b_2)}{ax+b_2}x$$

$$=\lim_{x\to+\infty}\frac{(b_1-b_2)x}{ax+b_2}=\frac{b_1-b_2}{a},$$

故原极限 $=\mathrm{e}^{\frac{b_1-b_2}{a}}$.

17. $\lim\limits_{x\to0}\left(\dfrac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}=\mathrm{e}^A$.

$$A=\lim_{x\to0}\frac{a^x+b^x+c^x-3}{3}\frac{1}{x}\overset{L'}{=\!=}\lim_{x\to0}\frac{a^x\ln a+b^x\ln b+c^x\ln c}{3}$$

$$=\frac{\ln a+\ln b+\ln c}{3}=\frac{\ln abc}{3}=\ln\sqrt[3]{abc},$$

故原极限 $=\mathrm{e}^{\ln\sqrt[3]{abc}}=\sqrt[3]{abc}$.

18. $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\frac{1}{\cos x-\sin x}}=\mathrm{e}^A$.

$$A=\lim_{x\to\frac{\pi}{4}}\frac{\tan x-1}{\cos x-\sin x}=\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\cos x(\cos x-\sin x)}=-\sqrt{2},$$

故原极限 $=\mathrm{e}^{-\sqrt{2}}$.

19. $\lim\limits_{x\to0}(\cos 2x+2x\sin x)^{\frac{1}{x^4}}=\mathrm{e}^A$.

$$A=\lim_{x\to0}\frac{\cos 2x+2x\sin x-1}{x^4}=\lim_{x\to0}\frac{1-2\sin^2x+2x\sin x-1}{x^4}$$

$$=\lim_{x\to0}\frac{2\sin x(x-\sin x)}{x^4}=2\times\frac{1}{6}=\frac{1}{3},$$

故原极限 $=\mathrm{e}^{\frac{1}{3}}$.

20. $\lim\limits_{x\to0}\left[\dfrac{\ln(1+x)}{x}\right]^{\frac{1}{\mathrm{e}^x-1}}=\mathrm{e}^A$.

$$A=\lim_{x\to0}\frac{\ln(1+x)-x}{x}\frac{1}{\mathrm{e}^x-1}=-\frac{1}{2},$$

故原极限 $=\mathrm{e}^{-\frac{1}{2}}$.

21. $\lim\limits_{x\to\infty}\left[\dfrac{x^2}{(x-a)(x+b)}\right]^x=\mathrm{e}^A$.

$$A=\lim_{x\to\infty}\frac{x^2-(x-a)(x+b)}{(x-a)(x+b)}x=\lim_{x\to\infty}\frac{x^2-[x^2-(a-b)x-ab]}{(x-a)(x+b)}x$$

$$=a-b,$$

故原极限 $= e^{a-b}$.

22. $\lim\limits_{x\to 0}(\cos^2 x)^{\frac{1}{\sin 2x}} = e^A$.

$$A = \lim_{x\to 0}\frac{\cos^2 x - 1}{\sin^2 x} = \frac{-\sin^2 x}{\sin^2 x} = -1,$$

故原极限 $= e^{-1}$.

23. $\lim\limits_{n\to\infty}\left(\dfrac{n+\ln n}{n-\ln n}\right)^{\frac{n}{\ln n}} = e^A$.

$$A = \lim_{n\to\infty}\frac{n+\ln n-(n-\ln n)}{n-\ln n}\,\frac{n}{\ln n} = \lim_{n\to\infty}\frac{2\ln n\cdot n}{\ln n(n-\ln n)}$$

$$= 2\lim_{n\to\infty}\frac{n}{n-\ln n} = 2\times 1 = 2,$$

故原极限 $= e^2$.

24. $\lim\limits_{x\to 0}\left(\dfrac{1+\tan x}{1+\sin x}\right)^{\frac{1}{\sin x}} = e^A$.

$$A = \lim_{x\to 0}\frac{1+\tan x-(1+\sin x)}{(1+\sin x)\sin x} = \lim_{x\to 0}\frac{\tan x-\sin x}{x} = 0,$$

故原极限 $= e^0 = 1$.

25. $\lim\limits_{x\to a}\left(\dfrac{\sin x}{\sin a}\right)^{\frac{1}{x-a}} = e^A$.

$$A = \lim_{x\to a}\frac{\sin x-\sin a}{\sin a}\cdot\frac{1}{x-a} = \frac{1}{\sin a}\lim_{x\to a}\frac{\sin x-\sin a}{x-a} = \frac{\cos a}{\sin a},$$

故原极限 $= e^{\cot a}$.

强化篇

1. $\lim\limits_{x\to 0}\left(\dfrac{e^x+e^{2x}+\cdots+e^{nx}}{n}\right)^{\frac{1}{x}} = e^A$.

$$A = \lim_{x\to 0}\frac{e^x+e^{2x}+\cdots+e^{nx}-n}{nx} \overset{L'}{=} \lim_{x\to 0}\frac{e^x+2e^{2x}+\cdots+ne^{nx}}{n}$$

$$= \frac{1+2+\cdots+n}{n} = \frac{(1+n)n}{2n} = \frac{1+n}{2},$$

故原极限 $= e^{\frac{1+n}{2}}$.

2. $\lim\limits_{x\to 0}\left(\dfrac{a_1{}^x+a_2{}^x+\cdots+a_n{}^x}{n}\right)^{\frac{1}{x}} = e^A$.

$$A=\lim_{x\to 0}\frac{a_1{}^x+a_2{}^x+\cdots+a_n{}^x-n}{nx}$$

$$\overset{L'}{=\!=}\lim_{x\to 0}\frac{a_1{}^x\ln a_1+a_2{}^x\ln a_2+\cdots+a_n{}^x\ln a_n}{n}$$

$$=\frac{\ln a_1+\ln a_2+\cdots+\ln a_n}{n}=\frac{1}{n}\ln(a_1a_2\cdots a_n)$$

$$=\ln(a_1a_2\cdots a_n)^{\frac{1}{n}},$$

故原极限 $=\mathrm{e}^{\ln(a_1a_2\cdots a_n)^{\frac{1}{n}}}=\sqrt[n]{a_1a_2\cdots a_n}$.

3. $$\lim_{x\to+\infty}(x^{\frac{1}{x}}-1)^{\frac{1}{\ln x}}=\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{1}{\ln x}\ln\left(x^{\frac{1}{x}}-1\right)}=\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{\ln\left(\mathrm{e}^{\frac{1}{x}\ln x}-1\right)}{\ln x}}$$

$$=\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{\ln\left(\frac{1}{x}\ln x\right)}{\ln x}}=\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{\ln\ln x-\ln x}{\ln x}}\overset{L'}{=\!=}\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{\frac{1}{\ln x}\cdot\frac{1}{x}-\frac{1}{x}}{\frac{1}{x}}}$$

$$=\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{1}{\ln x}-1}=\mathrm{e}^{-1}.$$

4. $$\lim_{x\to-\infty}\left(\frac{\pi}{2}+\arctan x\right)^{\frac{1}{x}}=\mathrm{e}^{\lim\limits_{x\to-\infty}\frac{1}{x}\ln\left(\frac{\pi}{2}+\arctan x\right)}\overset{L'}{=\!=}\mathrm{e}^{\lim\limits_{x\to-\infty}\frac{1}{\frac{\pi}{2}+\arctan x}\frac{1}{1+x^2}}$$

$$\overset{L'}{=\!=}\mathrm{e}^{\lim\limits_{x\to-\infty}\frac{-\frac{2x}{(1+x^2)^2}}{\frac{1}{1+x^2}}}=\mathrm{e}^{\lim\limits_{x\to-\infty}\left(-\frac{2x}{1+x^2}\right)}=\mathrm{e}^0=1.$$

5. $$\lim_{x\to0^+}(\cot x)^{\sin x}=\mathrm{e}^{\lim\limits_{x\to0^+}\sin x\ln\cot x}=\mathrm{e}^{\lim\limits_{x\to0^+}(-x\ln\tan x)}=\mathrm{e}^{-\lim\limits_{x\to0^+}\frac{\ln\tan x}{\frac{1}{x}}}$$

$$\overset{L'}{=\!=}\mathrm{e}^{\lim\limits_{x\to0^+}\frac{\frac{1}{\tan x}\sec^2x}{\frac{1}{x^2}}}=\mathrm{e}^{\lim\limits_{x\to0^+}x\sec^2x}=\mathrm{e}^0=1.$$

6. $$\lim_{x\to+\infty}\left(\frac{\pi}{2}-\arctan x\right)^{\frac{1}{\ln x}}=\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{\ln\left(\frac{\pi}{2}-\arctan x\right)}{\ln x}}\overset{L'}{=\!=}\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{-\frac{1}{\frac{\pi}{2}-\arctan x}\cdot\frac{1}{1+x^2}}{\frac{1}{x}}}$$

$$=\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{\frac{x}{1+x^2}}{\arctan x-\frac{\pi}{2}}}\overset{L'}{=\!=}\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{\frac{1+x^2-2x^2}{(1+x^2)^2}}{\frac{1}{1+x^2}}}$$

$$=\mathrm{e}^{\lim\limits_{x\to+\infty}\frac{1-x^2}{1+x^2}}=\mathrm{e}^{-1}.$$

7. $$\lim_{x\to1^+}(x-1)^2\mathrm{e}^{\frac{1}{x-1}}\overset{t=\frac{1}{x-1}}{=\!=\!=\!=}\lim_{t\to+\infty}\frac{\mathrm{e}^t}{t^2}=+\infty.$$

$$\lim_{x\to1^-}(x-1)^2\mathrm{e}^{\frac{1}{x-1}}\overset{t=\frac{1}{x-1}}{=\!=\!=\!=}\lim_{t\to-\infty}\frac{\mathrm{e}^t}{t^2}=0.$$

故原极限不存在.

8. $\lim\limits_{x\to 0}\left(\dfrac{\arctan x}{x}\right)^{\frac{1}{x^2}} = e^{\frac{1}{x^2}\left(\frac{\arctan x - x}{x}\right)} = e^{-\frac{1}{3}}$.

9. $\lim\limits_{x\to 1} x^{\frac{1}{1-x}} = e^{\lim\limits_{x\to 1}\frac{1}{1-x}\ln x} = e^{\lim\limits_{x\to 1}\frac{x-1}{1-x}} = e^{-1}$.

10. $\lim\limits_{x\to 1}(2-x)^{\tan\frac{\pi}{2}x} = e^{\lim\limits_{x\to 1}\tan\frac{\pi}{2}x\ln(2-x)} = e^{\lim\limits_{x\to 1}\frac{\sin\frac{\pi}{2}x}{\cos\frac{\pi}{2}x}\ln(2-x)} = e^{\lim\limits_{x\to 1}\frac{1-x}{\cos\frac{\pi}{2}x}}$

$$\overset{L'}{=} e^{\lim\limits_{x\to 1}\frac{-1}{-\frac{\pi}{2}\sin\frac{\pi}{2}x}} = e^{\frac{2}{\pi}}.$$

11. $\lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x} = e^{A}$.

$$A = \lim_{x\to\frac{\pi}{4}}(\tan x - 1)\tan 2x = \lim_{x\to\frac{\pi}{4}}\frac{\sin x - \cos x}{\cos x}\,\frac{\sin 2x}{\cos 2x}$$

$$= \lim_{x\to\frac{\pi}{4}}\left(-2\sin x\cdot\frac{1}{\cos x + \sin x}\right)$$

$$= -2\times\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}} = -2\times\frac{1}{2} = -1,$$

故原极限 $= e^{-1}$.

12. $\lim\limits_{x\to\infty}\left(\tan\dfrac{\pi x}{2x+1}\right)^{\frac{1}{x}} = e^{\lim\limits_{x\to\infty}\frac{1}{x}\ln\tan\frac{\pi x}{2x+1}} \overset{L'}{=} e^{\lim\limits_{x\to\infty}\frac{1}{\tan\frac{\pi x}{2x+1}}\sec^2\frac{\pi x}{2x+1}\frac{\pi(2x+1)-2\pi x}{(2x+1)^2}}$

$$= e^{\lim\limits_{x\to\infty}\frac{\cos\frac{\pi x}{2x+1}}{\sin\frac{\pi x}{2x+1}}\frac{1}{\cos^2\frac{\pi x}{2x+1}}\frac{\pi}{(2x+1)^2}} = e^{\lim\limits_{x\to\infty}\frac{\frac{\pi}{(2x+1)^2}}{\cos\frac{\pi x}{2x+1}}}$$

$$\overset{L'}{=} e^{-2\pi\lim\limits_{x\to\infty}\frac{\frac{2}{(2x+1)^3}}{-\sin\frac{\pi x}{2x+1}\cdot\frac{\pi}{(2x+1)^2}}} = e^{-4\lim\limits_{x\to\infty}\frac{\frac{1}{2x+1}}{-\sin\frac{\pi x}{2x+1}}}$$

$$= e^0 = 1.$$

13. $\lim\limits_{x\to a}\left(\dfrac{\tan x}{\tan a}\right)^{\cot(x-a)} = e^{A}$.

$$A = \lim_{x\to a}\frac{\tan x - \tan a}{\tan a}\,\frac{1}{\tan(x-a)} \overset{L'}{=} \frac{1}{\tan a}\lim_{x\to a}\frac{\sec^2 x}{\sec^2(x-a)}$$

$$= \frac{\sec^2 a}{\tan a} = \frac{\frac{1}{\cos^2 a}}{\frac{\sin a}{\cos a}} = \frac{1}{\sin a\cos a},$$

故原极限 $= e^{\frac{1}{\sin a\cos a}}$.

14. $\lim\limits_{x\to 0}\dfrac{(1+x)^x-1}{x^2}=\lim\limits_{x\to 0}\dfrac{e^{x\ln(1+x)}-1}{x^2}=\lim\limits_{x\to 0}\dfrac{x\ln(1+x)}{x^2}=1.$

15. $\lim\limits_{x\to 0+}(\arcsin x)^{\tan x}=e^{\lim\limits_{x\to 0+}\tan x\ln(\arcsin x)}=e^{\lim\limits_{x\to 0+}x\ln(\arcsin x)}$

$\xlongequal{t=\arcsin x}e^{\lim\limits_{t\to 0+}(\sin t\ln t)}=e^{\lim\limits_{t\to 0+}\frac{\ln t}{\frac{1}{t}}}=e^0=1.$

16. $\lim\limits_{x\to+\infty}\left(\dfrac{2}{\pi}\arctan x\right)^x=e^A.$

$$A=\lim_{x\to+\infty}\left(\frac{2}{\pi}\arctan x-1\right)x=\lim_{x\to+\infty}\frac{2\arctan x-\pi}{\pi}x$$

$$=\lim_{x\to+\infty}\frac{2\arctan x-\pi}{\frac{\pi}{x}}\overset{L'}{=\!=}\lim_{x\to+\infty}\frac{\frac{2}{1+x^2}}{-\frac{\pi}{x^2}}=-\frac{2}{\pi},$$

故原极限 $=e^{-\frac{2}{\pi}}$.

17. $\lim\limits_{x\to 0}\left(\dfrac{\arcsin x}{x}\right)^{\frac{1}{x^2}}=e^A.$

$$A=\lim_{x\to 0}\left(\frac{\arcsin x}{x}-1\right)\frac{1}{x^2}=\lim_{x\to 0}\frac{\arcsin x-x}{x^3}\overset{L'}{=\!=}\lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}-1}{3x^2}$$

$$=\lim_{x\to 0}\frac{1-\sqrt{1-x^2}}{3x^2\sqrt{1-x^2}}=\lim_{x\to 0}\frac{\frac{1}{2}x^2}{3x^2\sqrt{1-x^2}}=\frac{1}{6}\lim_{x\to 0}\frac{1}{\sqrt{1-x^2}}=\frac{1}{6},$$

故原极限 $=e^{\frac{1}{6}}$.

18. $\lim\limits_{x\to 0}\left[\dfrac{(1+x)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}}=e^A.$

$$A=\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{ex}=\frac{1}{e}\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e}{x}=\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)-1}-1}{x}$$

$$=\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}\overset{L'}{=\!=}-\frac{1}{2},$$

故原极限 $=e^{-\frac{1}{2}}$.

19. $\lim\limits_{x\to 0}\left(\dfrac{2}{\pi}\arccos x\right)^{\frac{1}{x}}=e^A.$

$$A=\lim_{x\to 0}\left(\frac{2}{\pi}\arccos x-1\right)\frac{1}{x}=\lim_{x\to 0}\frac{2\arccos x-\pi}{\pi x}\overset{L'}{=}\lim_{x\to 0}\frac{-\frac{2}{\sqrt{1-x^2}}}{\pi}=$$

$$-\frac{2}{\pi},$$

故原极限 $=\mathrm{e}^{-\frac{2}{\pi}}$.

20. $\lim\limits_{x\to 0}\left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}}=\mathrm{e}^A$.

$$A=\lim_{x\to 0}\left(\frac{\cos x}{\cos 2x}-1\right)\frac{1}{x^2}=\lim_{x\to 0}\frac{\cos x-\cos 2x}{x^2\cos 2x}=\lim_{x\to 0}\frac{\cos x-\cos 2x}{x^2}$$

$$\overset{L'}{=}\lim_{x\to 0}\frac{-\sin x+2\sin 2x}{2x}\overset{L'}{=}\lim_{x\to 0}\frac{-\cos x+4\cos 2x}{2}$$

$$=\frac{-1+4}{2}=\frac{3}{2},$$

故原极限 $=\mathrm{e}^{\frac{3}{2}}$.

21. $\lim\limits_{x\to\frac{\pi}{2}}(\sin x)^{\tan x}=\mathrm{e}^A$.

$$A=\lim_{x\to\frac{\pi}{2}}(\sin x-1)\tan x=\lim_{x\to\frac{\pi}{2}}(\sin x-1)\frac{\sin x}{\cos x}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{\sin x-1}{\cos x}\overset{L'}{=}\lim_{x\to\frac{\pi}{2}}\frac{\cos x}{-\sin x}=0,$$

故原极限 $=\mathrm{e}^0=1$.

22. $\lim\limits_{x\to 0}\left[\tan\left(\frac{\pi}{4}-x\right)\right]^{\cot x}=\mathrm{e}^A$.

$$A=\lim_{x\to 0}\left[\tan\left(\frac{\pi}{4}-x\right)-1\right]\cot x=\lim_{x\to 0}\frac{\tan\left(\frac{\pi}{4}-x\right)-1}{\sin x}$$

$$\overset{L'}{=}\lim_{x\to 0}\frac{-\sec^2\left(\frac{\pi}{4}-x\right)}{\cos x}=-\left(\frac{1}{\frac{\sqrt{2}}{2}}\right)^2=-2,$$

故原极限 $=\mathrm{e}^{-2}$.

23. $\lim\limits_{x\to 0+}\sqrt[x]{\cos\sqrt{x}}=\lim\limits_{x\to 0^+}(\cos\sqrt{x})^{\frac{1}{x}}=\mathrm{e}^A$.

$$A=\lim_{x\to 0^+}\frac{\cos\sqrt{x}-1}{x}=-\frac{1}{2},$$

故原极限 $= e^{-\frac{1}{2}}$.

24. $\lim\limits_{n\to\infty}\cos^n\dfrac{x}{\sqrt{n}} = e^A$.

$$A = \lim_{n\to\infty}\left(\cos\frac{x}{\sqrt{n}} - 1\right)n = \lim_{n\to\infty}\left(-\frac{1}{2}\frac{x^2}{n}\cdot n\right) = -\frac{x^2}{2},$$

故原极限 $= e^{-\frac{x^2}{2}}$.

25. $\lim\limits_{x\to 0}\left(\dfrac{1+x\cdot 2^x}{1+x\cdot 3^x}\right)^{\frac{1}{x^2}} = e^A$.

$$A = \lim_{x\to 0}\left(\frac{1+x\cdot 2^x}{1+x\cdot 3^x} - 1\right)\frac{1}{x^2} = \lim_{x\to 0}\frac{1+x\cdot 2^x-(1+x\cdot 3^x)}{(1+x\cdot 3^x)\cdot x^2}$$

$$= \lim_{x\to 0}\frac{(2^x-3^x)\cdot x}{(1+x\cdot 3^x)\cdot x^2} = \lim_{x\to 0}\frac{2^x-3^x}{x}$$

$$\overset{L'}{=} \lim_{x\to 0}(2^x\ln 2 - 3^x\ln 3) = \ln 2 - \ln 3,$$

故原极限 $= e^{\ln 2-\ln 3} = e^{\ln\frac{2}{3}} = \dfrac{2}{3}$.

26. $\lim\limits_{n\to\infty}\tan^n\left(\dfrac{\pi}{4}+\dfrac{1}{n}\right) = e^A$.

$$A = \lim_{n\to\infty}\left[\tan\left(\frac{\pi}{4}+\frac{1}{n}\right)-1\right]n \xlongequal{x=\frac{1}{n}} \lim_{x\to 0}\frac{\tan\left(\frac{\pi}{4}+x\right)-1}{x}$$

$$\overset{L'}{=} \lim_{x\to 0}\sec^2\left(\frac{\pi}{4}+x\right) = \frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2} = 2,$$

故原极限 $= e^2$.

27. $\lim\limits_{x\to\infty}\dfrac{(x+a)^{x+a}(x+b)^{x+b}}{(x+a+b)^{2x+a+b}} = \lim\limits_{x\to\infty}\left(\dfrac{x+a}{x+a+b}\right)^{x+a}\left(\dfrac{x+b}{x+a+b}\right)^{x+b}$

$$= \lim_{x\to\infty}\left(\frac{x+a+b}{x+a}\right)^{-(x+a)}\lim_{x\to\infty}\left(\frac{x+a+b}{x+b}\right)^{-(x+b)} = e^{-b}\cdot e^{-a} = e^{-(a+b)}.$$

注:求极限时适当的初等数学变形能大大减少计算量.

28. $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt[n]{a}+\sqrt[n]{b}}{2}\right)^n = e^A$.

$$A = \lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b}-2}{2}\right)n \xlongequal{x=\frac{1}{n}} \lim_{x\to 0}\frac{a^x+b^x-2}{2x}$$

$$\overset{L'}{=}\lim_{x\to 0}\frac{a^x\ln a+b^x\ln b}{2}=\frac{\ln a+\ln b}{2}=\ln\sqrt{ab},$$

故原极限 $=\mathrm{e}^{\ln\sqrt{ab}}=\sqrt{ab}$.

29. $\lim\limits_{x\to 0}\left(\dfrac{a^{x+1}+b^{x+1}+c^{x+1}}{a+b+c}\right)^{\frac{1}{x}}=\mathrm{e}^A$.

$$A=\lim_{x\to 0}\frac{a^{x+1}+b^{x+1}+c^{x+1}-(a+b+c)}{(a+b+c)x}$$

$$\overset{L'}{=}\lim_{x\to 0}\frac{a^{x+1}\ln a+b^{x+1}\ln b+c^{x+1}\ln c}{a+b+c}$$

$$=\frac{a\ln a+b\ln b+c\ln c}{a+b+c},$$

故原极限 $=\mathrm{e}^{\frac{a\ln a+b\ln b+c\ln c}{a+b+c}}$.

30. $\lim\limits_{x\to 0}\left(\dfrac{a^{x^2}+b^{x^2}}{a^x+b^x}\right)^{\frac{1}{x}}=\mathrm{e}^A$.

$$A=\lim_{x\to 0}\frac{a^{x^2}+b^{x^2}-(a^x+b^x)}{(a^x+b^x)x}=\frac{1}{2}\lim_{x\to 0}\frac{a^{x^2}+b^{x^2}-(a^x+b^x)}{x}$$

$$\overset{L'}{=}\frac{1}{2}\lim_{x\to 0}(2xa^{x^2}\ln a+2xb^{x^2}\ln b-a^x\ln a-b^x\ln b)$$

$$=\frac{1}{2}(-\ln a-\ln b)=-\frac{\ln a+\ln b}{2}$$

$$=-\frac{1}{2}\ln ab=\ln(ab)^{-\frac{1}{2}},$$

故原极限 $=\mathrm{e}^{\ln(ab)^{-\frac{1}{2}}}=\dfrac{1}{\sqrt{ab}}$.

真题自测

(1987) $\lim\limits_{x\to 0}(1+x\mathrm{e}^x)^{\frac{1}{x}}=\mathrm{e}^{\lim\limits_{x\to 0}\frac{x\mathrm{e}^x}{x}}=\mathrm{e}$.

(1989) $\lim\limits_{x\to\infty}\left(\sin\dfrac{1}{x}+\cos\dfrac{1}{x}\right)^x=\mathrm{e}^A$.

$$A=\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}-1\right)x\xlongequal{t=\frac{1}{x}}\lim_{t\to 0}\frac{\sin t+\cos t-1}{t}=1,$$

故原极限 $=\mathrm{e}$.

(1989) $\lim\limits_{x\to 0}(2\sin x+\cos x)^{\frac{1}{x}}=e^{A}$.

$$A=\lim_{x\to 0}\frac{2\sin x+\cos x-1}{x}=2,$$

故原极限 $=e^{2}$.

(1991) $\lim\limits_{x\to 0^{+}}(\cos\sqrt{x})^{\frac{\pi}{x}}=e^{A}$.

$$A=\lim_{x\to 0^{+}}(\cos\sqrt{x}-1)\frac{\pi}{x}=\lim_{x\to 0^{+}}\frac{-\pi\cdot\frac{1}{2}x}{x}=-\frac{\pi}{2},$$

故原极限 $=e^{-\frac{\pi}{2}}$.

(1990) $\lim\limits_{x\to\infty}\left(\frac{x+a}{x-a}\right)^{x}=e^{A}$.

$$A=\lim_{x\to\infty}\frac{x+a-(x-a)}{x-a}x=2a,$$

故原极限 $=e^{2a}$.

(1992) $\lim\limits_{x\to\infty}\left(\frac{3+x}{6+x}\right)^{\frac{x-1}{2}}=e^{A}$.

$$A=\lim_{x\to\infty}\frac{3+x-(6+x)}{6+x}\frac{x-1}{2}=\lim_{x\to\infty}\frac{-3(x-1)}{2(x+6)}=-\frac{3}{2},$$

故原极限 $=e^{-\frac{3}{2}}$.

(1993) $\lim\limits_{x\to\infty}\left(\sin\frac{2}{x}+\cos\frac{1}{x}\right)^{x}=e^{A}$.

$$A=\lim_{x\to\infty}\left(\sin\frac{2}{x}+\cos\frac{1}{x}-1\right)x\xlongequal{t=\frac{1}{x}}\lim_{t\to 0}\frac{\sin 2t+\cos t-1}{t}=2,$$

故原极限 $=e^{2}$.

(1996) $\lim\limits_{x\to\infty}\left(\frac{x+2a}{x-a}\right)^{x}=e^{A}$.

$$A=\lim_{x\to\infty}\frac{x+2a-(x-a)}{x-a}x=3a,$$

故原极限 $=e^{3a}=8$，$3a=\ln 8=3\ln 2$，故 $a=\ln 2$.

(1995) $\lim\limits_{x\to 0}(1+3x)^{\frac{2}{\sin x}}=e^{\lim\limits_{x\to 0}\frac{6x}{\sin x}}=e^{6}$.

(2003) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{\ln(1+x^{2})}}=e^{A}$.

$$A=\lim_{x\to 0}\frac{\cos x-1}{\ln(1+x^{2})}=-\frac{1}{2},$$

故原极限 $=e^{-\frac{1}{2}}$.

(1988) $\lim\limits_{x\to0^+}\left(\frac{1}{\sqrt{x}}\right)^{\tan x}=e^{\lim\limits_{x\to0^+}\tan x\ln\frac{1}{\sqrt{x}}}=e^{\lim\limits_{x\to0^+}x\ln\frac{1}{\sqrt{x}}}=e^{\lim\limits_{x\to0^+}\left(-\frac{1}{2}x\ln x\right)}=e^0=1.$

(1989) $\lim\limits_{x\to+\infty}(x+e^x)^{\frac{1}{x}}=e^{\lim\limits_{x\to+\infty}\frac{\ln(x+e^x)}{x}}=e.$

(1990) $\lim\limits_{x\to\infty}\left(\frac{x+a}{x-a}\right)^x=e^A.$

$A=\lim\limits_{x\to\infty}\frac{x+a-(x-a)}{x-a}x=2a,$

故原极限 $=e^{2a}=9,2a=\ln 9=2\ln 3$,故 $a=\ln 3$.

(1987) 原式 $=\lim\limits_{n\to\infty}\left(1+\frac{-3}{n+1}\right)^n=e^{\lim\limits_{n\to\infty}\left(n\cdot\frac{-3}{n+1}\right)}=e^{-3}.$

(2011) 原式$=\lim\limits_{x\to0}\left(1+\frac{2^x-1}{2}\right)^{\frac{1}{x}}=e^{\lim\limits_{x\to0}\frac{2^x-1}{2x}}=e^{\lim\limits_{x\to0}\frac{x\ln 2}{2x}}=e^{\frac{1}{2}\ln 2}=\sqrt{2}.$

(2013) 原式 $=\lim\limits_{x\to0}\left[1+\frac{x-\ln(1+x)}{x}\right]^{\frac{1}{x}}=e^{\lim\limits_{x\to0}\frac{x-\ln(1+x)}{x^2}}=e^{\frac{1}{2}}.$

(2018) 左式$=\lim\limits_{x\to0}\left(1+\frac{-2\tan x}{1+\tan x}\right)^{\frac{1}{\sin x}}=e^{\lim\limits_{x\to0}\frac{\ln\left(1-\frac{2\tan x}{1+\tan x}\right)}{\sin kx}}$

$=e^{\lim\limits_{x\to0}\left(\frac{1}{\sin kx}\cdot\frac{-2\tan x}{1+\tan x}\right)}=e^{\lim\limits_{x\to0}\frac{-2\tan x}{\sin kx}}=e^{-\frac{2}{k}}=e,$

故 $-\frac{2}{k}=1\Rightarrow k=-2.$

(2019) 原式$=\lim\limits_{x\to0}(1+x+2^x-1)^{\frac{2}{x}}=e^{\lim\limits_{x\to0}\frac{2\cdot(x+2^x-1)}{x}}\overset{L'}{=\!=}e^{\lim\limits_{x\to0}2(1+2^x\ln 2)}=e^{2+2\ln 2}$

$=4e^2.$

(2022) 原式 $=\lim\limits_{x\to0}\left(1+\frac{e^x-1}{2}\right)^{\cot x}=e^{\lim\limits_{x\to0}\left(\cot x\cdot\frac{e^x-1}{2}\right)}=e^{\frac{1}{2}\lim\limits_{x\to0}x\cot x}=e^{\frac{1}{2}}.$

§1.6 极限的应用之一:连续与间断点

基础篇

1. (1) $x^2-3x+2=0\Rightarrow(x-2)(x-1)=0\Rightarrow x=1,\ x=2.$

$\lim\limits_{x\to 1} y=\lim\limits_{x\to 1}\dfrac{(x+1)(x-1)}{(x-2)(x-1)}=-2$，$x=1$ 为可去间断点.

$\lim\limits_{x\to 2} y=\lim\limits_{x\to 2}\dfrac{(x+1)(x-1)}{(x-2)(x-1)}=\infty$，$x=2$ 为无穷间断点.

(2) $x=k\pi(k\neq 0,\ k\in\mathbf{Z})$；$x=0$；$x=\dfrac{\pi}{2}+k\pi(k\in\mathbf{Z})$.

$\lim\limits_{x\to 0}\dfrac{x}{\tan x}=1$，$x=0$ 为可去间断点.

$\lim\limits_{x\to k\pi}\dfrac{x}{\tan x}=\infty$，$x=k\pi(k\neq 0,\ k\in\mathbf{Z})$ 为无穷间断点.

$\lim\limits_{x\to\frac{\pi}{2}+k\pi}\dfrac{x}{\tan x}=0$，$x=\dfrac{\pi}{2}+k\pi(k\in\mathbf{Z})$ 为可去间断点.

(3) $y=\dfrac{1}{2}\left(1+\cos\dfrac{2}{x}\right)$，$x=0$.

$\lim\limits_{x\to 0} y=\lim\limits_{x\to 0}\dfrac{1}{2}\left(1+\cos\dfrac{2}{x}\right)$ 极限不存在.

$x=0$ 为振荡间断点.

(4) $\lim\limits_{x\to 1^-}(x-1)=0$，$\lim\limits_{x\to 1^+}(3-x)=2$，$x=1$ 为跳跃间断点.

(5) $x=0$.

$$\lim_{x\to 0^+} y=\lim_{x\to 0^+}\frac{1-2^{-\frac{1}{x}}}{1+2^{-\frac{1}{x}}}=\frac{1-0}{1+0}=1,$$

$$\lim_{x\to 0^-} y=\lim_{x\to 0^-}\frac{2^{\frac{1}{x}}-1}{2^{\frac{1}{x}}+1}=\frac{0-1}{0+1}=-1,$$

$x=0$ 为跳跃间断点.

(6) $y=\begin{cases}\cos\dfrac{\pi}{2}x, & -1\leqslant x\leqslant 1,\\ x-1, & x>1,\\ 1-x, & x<-1.\end{cases}$

$\lim\limits_{x\to -1^-} y=\lim\limits_{x\to -1^-}(1-x)=2$，$\lim\limits_{x\to -1^+} y=\cos\dfrac{\pi}{2}=0$，$x=-1$ 为跳跃间断点.

$\lim\limits_{x\to 1^+} y=\lim\limits_{x\to 1^+}(x-1)=0$，$\lim\limits_{x\to 1^-} y=\cos\dfrac{\pi}{2}=0$.

$\lim\limits_{x\to1^+}y=\lim\limits_{x\to1^-}y=y(1)=0$，$x=1$ 为连续点.

2. $x=0$，$x=1$，$x=\pm\dfrac{\pi}{2}$.

$\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}\dfrac{e^x+e}{e^{\frac{1}{x}}-e}=0$，$\lim\limits_{x\to0^-}f(x)=\lim\limits_{x\to0^-}\dfrac{e^x+e}{e^{\frac{1}{x}}-e}=-\dfrac{1+e}{e}$，

$x=0$ 为跳跃间断点(第一类).

$\lim\limits_{x\to\pm\frac{\pi}{2}}f(x)=\lim\limits_{x\to\pm\frac{\pi}{2}}\left(\dfrac{e^x+e}{e^{\frac{1}{x}}-e}\cdot\dfrac{\tan x}{x}\right)=\infty$，

$\lim\limits_{x\to1}f(x)=\infty$，

$x=\pm\dfrac{\pi}{2}$，$x=1$ 为第二类间断点.

3. $f(x)=\dfrac{x}{x^2}=\dfrac{1}{x}$，$x=0$.

$\lim\limits_{x\to0}f(x)=\infty$，$x=0$ 为无穷间断点.

强化篇

1. $f(x)$ 有三个间断点，$x=0$，$x=\pm1$.

在 $x=0$ 处，$\lim\limits_{x\to0}f(x)=-\lim\limits_{x\to0}x\ln|x|\sin\dfrac{1}{x}\left(\lim\limits_{x\to0}\dfrac{x+1}{x^2-1}=-1\right)$.

$\lim\limits_{x\to0}x\ln|x|=\lim\limits_{x\to0}\dfrac{\ln|x|}{\dfrac{1}{x}}=\lim\limits_{x\to0}\dfrac{\dfrac{1}{x}}{-\dfrac{1}{x^2}}=0$(无穷小量).

$\sin\dfrac{1}{x}$ 是有界变量，则 $\lim\limits_{x\to0}f(x)=0$，$x=0$ 为可去间断点.

在 $x=-1$ 处，$\lim\limits_{x\to-1}f(x)=\lim\limits_{x\to-1}\dfrac{x\ln|x|\sin\dfrac{1}{x}}{x-1}=0$，则 $x=-1$ 为可去间断点.

在 $x=1$ 处，

$$\lim_{x\to1}f(x)=\lim_{x\to1}\frac{x\ln|x|\sin\frac{1}{x}}{x-1}=\sin1\lim_{x\to1}\frac{\ln x}{x-1}(x>0)$$

$$= \sin 1 \lim_{x\to 1} \frac{\ln[1+(x-1)]}{x-1} = \sin 1 \lim_{x\to 1} \frac{x-1}{x-1} = \sin 1,$$

则 $x=1$ 为可去间断点.

故选(D).

2. $f(x) = \dfrac{|x|^x - 1}{x(x+1)\ln|x|}$ 在 $x=-1, 0, 1$ 处没定义.

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{e^{x\ln|x|}-1}{x(x+1)\ln|x|} = \lim_{x\to -1} \frac{x\ln|x|}{x(x+1)\ln|x|} = \lim_{x\to -1} \frac{1}{x+1} = \infty,$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{x\ln|x|}-1}{x(x+1)\ln|x|} = \lim_{x\to 0} \frac{x\ln|x|}{x(x+1)\ln|x|} = \lim_{x\to 0} \frac{1}{x+1} = 1,$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{e^{x\ln|x|}-1}{x(x+1)\ln|x|} = \lim_{x\to 1} \frac{x\ln|x|}{x(x+1)\ln|x|} = \lim_{x\to 1} \frac{1}{x+1} = \frac{1}{2},$$

则 $x=0$ 和 $x=1$ 为可去间断点,故应选(C).

真题自测

(2009) $x = 0, \pm 1, \pm 2, \cdots$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{\pi x} = \frac{1}{\pi},\ x=0 \text{ 为可去间断点.}$$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x(1-x)(1+x)}{\sin \pi x} = 2\lim_{x\to 1} \frac{1-x}{\sin \pi x} \overset{L'}{=\!=} 2\lim_{x\to 1} \frac{-1}{\pi\cos \pi x} = \frac{2}{\pi},$$

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x(1-x)(1+x)}{\sin \pi x} = -2\lim_{x\to -1} \frac{1+x}{\sin \pi x} \overset{L'}{=\!=} -2\lim_{x\to -1} \frac{1}{\pi\cos \pi x} = \frac{2}{\pi},$$

$x = \pm 1$ 为可去间断点.

$x = k(k = \pm 2, \pm 3, \cdots)$,

$\lim\limits_{x\to k} f(x) = \infty$.

故选(C).

(1998) $x=\frac{\pi}{4}$, $\frac{3}{4}\pi$, $\frac{5}{4}\pi$, $\frac{7}{4}\pi$.

$\lim\limits_{x\to\frac{\pi}{4}^+}f(x)=+\infty$, $x=\frac{\pi}{4}$ 为无穷间断点.

$\lim\limits_{x\to\frac{3}{4}\pi}f(x)=\left(1+\frac{3}{4}\pi\right)^0=1$, $x=\frac{3}{4}\pi$ 为可去间断点.

$\lim\limits_{x\to\frac{5}{4}\pi^+}f(x)=+\infty$, $x=\frac{5}{4}\pi$ 为无穷间断点.

$\lim\limits_{x\to\frac{7}{4}\pi}f(x)=\left(1+\frac{7}{4}\pi\right)^0=1$, $x=\frac{7}{4}\pi$ 为可去间断点.

§1.7 极限的应用之二:求渐近线

基础篇

1. 找分母为零的点,$x=0$,

$\lim\limits_{x\to0}x\sin\frac{1}{x}=0$,故 $x=0$ 不是垂直渐近线.

$$\lim_{x\to\infty}x\sin\frac{1}{x}=\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}\xlongequal{t=\frac{1}{x}}\lim_{t\to0}\frac{\sin t}{t}=1,$$

故 $y=1$ 为水平渐近线.选(A).

2. $k=\lim\limits_{x\to\infty}\frac{y}{x}=\lim\limits_{x\to\infty}\frac{x}{2x+1}=\frac{1}{2}$,

$$b=\lim_{x\to\infty}\left(\frac{x^2}{2x+1}-\frac{1}{2}x\right)=\lim_{x\to\infty}\frac{-x}{2(2x+1)}=-\frac{1}{4},$$

故斜渐近线方程为 $y=\frac{1}{2}x-\frac{1}{4}$.

另解:$y=x\cdot\frac{x}{2x+1}=x\cdot\frac{1}{2+\frac{1}{x}}=\frac{1}{2}x\cdot\frac{1}{1+\frac{1}{2x}}=\frac{1}{2}x\left(1+\frac{1}{2x}\right)^{-1}$

$$=\frac{1}{2}x\left[1-\frac{1}{2x}+o\left(\frac{1}{x}\right)\right]=\frac{1}{2}x-\frac{1}{4}+x\cdot o\left(\frac{1}{x}\right)$$

（线性组合＋无穷小量），

故斜渐近线方程为 $y=\frac{1}{2}x-\frac{1}{4}$.

3. $\lim\limits_{x\to\infty}\frac{x+4\sin x}{5x-2\cos x}=\lim\limits_{x\to\infty}\frac{1+4\frac{\sin x}{x}}{5-2\frac{\cos x}{x}}=\frac{1}{5}$,

故水平渐近线为 $y=\frac{1}{5}$.

4. $k=\lim\limits_{x\to\infty}\frac{(2x-1)e^{\frac{1}{x}}}{x}=2\left(因\lim\limits_{x\to\infty}\frac{2x-1}{x}=2,\ \lim\limits_{x\to\infty}e^{\frac{1}{x}}=1,\right)$

$$b=\lim_{x\to\infty}(2x-1)e^{\frac{1}{x}}-2x$$

$$\xlongequal{t=\frac{1}{x}}\lim_{t\to0}\left(\frac{2}{t}-1\right)e^{t}-\frac{2}{t}=\lim_{t\to0}\frac{2e^{t}-te^{t}-2}{t}$$

$$=\lim_{t\to0}\frac{2(e^{t}-1)}{t}-\lim_{t\to0}e^{t}=2-1=1,$$

故斜渐近线方程为 $y=2x+1$.

另解：当 $x\to\infty$ 时，$e^{\frac{1}{x}}=1+\frac{1}{x}+o\left(\frac{1}{x}\right)$，故

$$y=(2x-1)\left[1+\frac{1}{x}+o\left(\frac{1}{x}\right)\right]$$

$$=2x+2-1-\frac{1}{x}+o\left(\frac{1}{x}\right)=2x+1-\frac{1}{x}+o\left(\frac{1}{x}\right),$$

故斜渐近线方程为 $y=2x+1$.

强化篇

1. 垂直渐近线：$\lim\limits_{x\to1^+}y(x)=\lim\limits_{x\to1^+}\sqrt{\frac{x^3}{x-1}}=+\infty$,

所以 $x=1$ 是 $y=y(x)$ 的垂直渐近线.

水平渐近线：$\lim\limits_{x\to\infty}\sqrt{\dfrac{x^3}{x-1}}=\infty$，没有水平渐近线.

斜渐近线：$k=\lim\limits_{x\to\infty}\dfrac{y}{x}=\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\cdot\sqrt{\dfrac{x^3}{x-1}}\right)=\pm1\left(x\to\infty,\sqrt{\dfrac{x^3}{x-1}}\right.$ 等价于无穷大 $\left.|x|\right)$.

(1) $x\to+\infty$：$b=\lim\limits_{x\to+\infty}\left(\sqrt{\dfrac{x^3}{x-1}}-x\right)$

$$=\lim_{x\to+\infty}\left(\sqrt{\frac{x^2\cdot x}{x-1}}-x\right)=\lim_{x\to+\infty}\left(x\sqrt{\frac{x}{x-1}}-x\right)$$

$$=\lim_{x\to+\infty}x\left(\sqrt{\frac{x}{x-1}}-1\right)=\lim_{x\to+\infty}x\left[\left(1-\frac{1}{x}\right)^{-\frac{1}{2}}-1\right]$$

$$=\lim_{x\to+\infty}\left[x\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{1}{x}\right)\right]=\frac{1}{2},$$

故 $x\to+\infty$ 时的斜渐近线为 $y=x+\dfrac{1}{2}$.

(2) $x\to-\infty$：$k=-1$，

$$b=\lim_{x\to-\infty}\left(\sqrt{\frac{x^3}{x-1}}+x\right)=\lim_{x\to-\infty}\left(-x\sqrt{\frac{x}{x-1}}+x\right)$$

$$=\lim_{x\to-\infty}(-x)\left[\left(1-\frac{1}{x}\right)^{-\frac{1}{2}}-1\right]$$

$$=\lim_{x\to-\infty}\left[(-x)\cdot\left(-\frac{1}{2}\right)\left(-\frac{1}{x}\right)\right]=-\frac{1}{2},$$

故 $y=-x-\dfrac{1}{2}$ 是 $x\to-\infty$ 时的斜渐近线.

2. $k_1=\lim\limits_{x\to+\infty}\dfrac{y}{x}=\lim\limits_{x\to+\infty}\dfrac{\sqrt{4x^2+x}\ln\left(2+\dfrac{1}{x}\right)}{x}$

$$=\lim_{x\to+\infty}\sqrt{4+\frac{1}{x}}\ln\left(2+\frac{1}{x}\right)=2\ln 2,$$

$$b_1=\lim_{x\to+\infty}(y-2\ln 2\cdot x)=\lim_{x\to+\infty}\left[\sqrt{4x^2+x}\ln\left(2+\frac{1}{x}\right)-2\ln 2\cdot x\right]$$

$$= \lim_{x\to+\infty} x\left[\sqrt{4+\frac{1}{x}}\ln\left(2+\frac{1}{x}\right)-2\ln 2\right]$$

$$\xlongequal{令\, t=\frac{1}{x}} \lim_{t\to 0^+}\frac{\sqrt{4+t}\ln(2+t)-2\ln 2}{t}$$

$$\xlongequal{L'} \lim_{t\to 0^+}\left[\frac{1}{2\sqrt{4+t}}\ln(2+t)+\frac{\sqrt{4+t}}{2+t}\right]=\frac{1}{4}\ln 2+1,$$

故斜渐近线为 $y=2\ln 2\cdot x+\frac{1}{4}\ln 2+1$.

同理，当 $x\to-\infty$ 时，斜渐近线为 $y=-2\ln 2\cdot x-\frac{1}{4}\ln 2-1$.

快捷解法：利用泰勒展开把函数写成线性组合与无穷小量之和.

当 $x\to+\infty$ 时，$\sqrt{4x^2+x}=\sqrt{4x^2\left(1+\frac{1}{4x}\right)}=2x\sqrt{1+\frac{1}{4x}}$

$$=2x\left[1+\frac{1}{8}\frac{1}{x}+o\left(\frac{1}{x}\right)\right],$$

$$\ln\left(2+\frac{1}{x}\right)=\ln\left[2\left(1+\frac{1}{2x}\right)\right]=\ln 2+\ln\left(1+\frac{1}{2x}\right)=\ln 2+\frac{1}{2x}+o\left(\frac{1}{x}\right),$$

$$y=\sqrt{4x^2+x}\ln\left(2+\frac{1}{x}\right)$$

$$=2x\left[1+\frac{1}{8}\frac{1}{x}+o\left(\frac{1}{x}\right)\right]\left[\ln 2+\frac{1}{2x}+o\left(\frac{1}{x}\right)\right]$$

$$=2\ln 2\cdot x+\frac{1}{4}\ln 2+1+o(1),$$

故当 $x\to+\infty$ 时，斜渐近线方程为 $y=2\ln 2\cdot x+\frac{1}{4}\ln 2+1$.

当 $x\to-\infty$ 时，令 $t=-x$，则 $t\to+\infty$，

$$y=\sqrt{4x^2+x}\ln\left(2+\frac{1}{x}\right)=\sqrt{4t^2-t}\ln\left(2-\frac{1}{t}\right).$$

当 $t\to+\infty$ 时，$\sqrt{4t^2-t}=\sqrt{4t^2\left(1-\frac{1}{4t}\right)}=2t\left(1-\frac{1}{4t}\right)^{\frac{1}{2}}$

$$=2t\left[1-\frac{1}{8}\frac{1}{t}+o\left(\frac{1}{t}\right)\right],$$

$$\ln\left(2-\frac{1}{t}\right)=\ln\left[2\left(1-\frac{1}{2t}\right)\right]=\ln 2+\ln\left(1-\frac{1}{2t}\right)$$
$$=\ln 2+\left[-\frac{1}{2t}+o\left(\frac{1}{t}\right)\right],$$
$$\sqrt{4t^2-t}\ln\left(2-\frac{1}{t}\right)=2t\left[1-\frac{1}{8t}+o\left(\frac{1}{t}\right)\right]\left[\ln 2-\frac{1}{2t}+o\left(\frac{1}{t}\right)\right]$$
$$=2\ln 2\cdot t-1-\frac{1}{4}\ln 2+o(1)$$
$$=-2\ln 2\cdot x-1-\frac{1}{4}\ln 2+o(1),$$

故当 $x\to-\infty$ 时,斜渐近线方程为 $y=-2\ln 2\cdot x-1-\frac{1}{4}\ln 2$.

真题自测

(2017) $k=\lim\limits_{x\to\infty}\dfrac{x\left(1+\arcsin\frac{2}{x}\right)}{x}=\lim\limits_{x\to\infty}\left(1+\arcsin\frac{2}{x}\right)=1$,

$$b=\lim_{x\to\infty}\left[x\left(1+\arcsin\frac{2}{x}\right)-x\right]=\lim_{x\to\infty}x\arcsin\frac{2}{x}=\lim_{x\to\infty}\frac{\arcsin\frac{2}{x}}{\frac{1}{x}}=2,$$

故斜渐近线方程为 $y=x+2$.

(2016) $k=\lim\limits_{x\to\infty}\dfrac{y}{x}=\lim\limits_{x\to\infty}\left[\dfrac{x^2}{1+x^2}+\dfrac{\arctan(1+x^2)}{x}\right]=1+0=1$

$\left(\text{因为}\frac{1}{x}\text{ 为无穷小量,}\arctan(1+x^2)\text{ 为有界变量,}\lim\limits_{x\to\infty}\dfrac{\arctan(1+x^2)}{x}=0\right)$,

$$b=\lim_{x\to\infty}\left[\frac{x^3}{1+x^2}+\arctan(1+x^2)-x\right]$$
$$=\lim_{x\to\infty}\left(\frac{x^3}{1+x^2}-x\right)+\lim_{x\to\infty}\arctan(1+x^2)$$
$$=\lim_{x\to\infty}\frac{x^3-x-x^3}{1+x^2}+\frac{\pi}{2}=0+\frac{\pi}{2}=\frac{\pi}{2},$$

故斜渐近线方程为 $y=x+\frac{\pi}{2}$.

第二章　求　导　数

§2.1　利用导数定义求导

基础篇

1. $f'(1)=\lim\limits_{x\to1}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to1}\dfrac{(x-1)(x-2)^2(x-3)^3}{x-1}$

$=\lim\limits_{x\to1}(x-2)^2(x-3)^3=(-2)^3=-8.$

$f'(2)=\lim\limits_{x\to2}\dfrac{f(x)-f(2)}{x-2}=\lim\limits_{x\to2}\dfrac{(x-1)(x-2)^2(x-3)^3}{x-2}$

$=\lim\limits_{x\to2}(x-1)(x-2)(x-3)^3=0.$

$f'(3)=\lim\limits_{x\to3}\dfrac{f(x)-f(3)}{x-3}=\lim\limits_{x\to3}\dfrac{(x-1)(x-2)^2(x-3)^3}{x-3}$

$=\lim\limits_{x\to3}(x-1)(x-2)^2(x-3)^2=0.$

2. 由导数定义，$f'(2)=\lim\limits_{x\to2}\dfrac{f(x)-f(2)}{x-2}=\lim\limits_{x\to2}\dfrac{x^2\sin(x-2)}{x-2}=4.$

3. 由导数定义，$f'(1)=\lim\limits_{x\to1}\dfrac{x+(x-1)\arcsin\sqrt{\dfrac{x}{x+1}}-1}{x-1}$

$=\lim\limits_{x\to1}\dfrac{x-1}{x-1}+\lim\limits_{x\to1}\dfrac{(x-1)\arcsin\sqrt{\dfrac{x}{x+1}}}{x-1}=1+\dfrac{\pi}{4}.$

4. $f'(0)=\lim\limits_{x\to0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0}\dfrac{(e^x-1)\sqrt[3]{\dfrac{1-x^2+x^3}{1+x-x^3}}-0}{x-0}$

$=\lim\limits_{x\to0}\sqrt[3]{\dfrac{1-x^2+x^3}{1+x-x^3}}=1.$

5. $f'(1)=\lim\limits_{x\to 1}\dfrac{f(x)-f(1)}{x-1}$

$$=\lim_{x\to 1}\frac{\left(\tan\frac{\pi x}{4}-1\right)\left(\tan\frac{\pi x^2}{4}-2\right)\cdots\left(\tan\frac{\pi x^{100}}{4}-100\right)}{x-1}$$

$$=\lim_{x\to 1}\frac{\tan\frac{\pi x}{4}-1}{x-1}\lim_{x\to 1}\frac{\left(\tan\frac{\pi x^2}{4}-2\right)\cdots\left(\tan\frac{\pi x^{100}}{4}-100\right)}{1}$$

$$\overset{L'}{=}\lim_{x\to 1}\frac{\frac{\pi}{4}\sec^2\frac{\pi x}{4}}{1}\times(-1)\times(-2)\times\cdots\times(-99)$$

$$=\frac{\pi}{4}\times 2\times(-1)^{99}99!\ =-\frac{99!}{2}\pi.$$

真题自测

(1989) $f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{x(x+1)(x+2)\cdots(x+n)}{x}$

$=\lim\limits_{x\to 0}(x+1)(x+2)\cdots(x+n)$

$=n!.$

(1995) 先求分段函数的导数，再讨论其连续性.

当 $x\neq 0$ 时，$f'(x)=\arctan\dfrac{1}{x^2}+x\cdot\dfrac{1}{1+\frac{1}{x^4}}\cdot\dfrac{-2}{x^3}$

$$=\arctan\frac{1}{x^2}-\frac{2x^2}{x^4+1}.$$

当 $x=0$ 时，对分段点使用导数定义：

$$f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x\arctan\frac{1}{x^2}}{x}=\frac{\pi}{2}.$$

故 $f'(x)=\begin{cases}\arctan\dfrac{1}{x^2}-\dfrac{2x^2}{x^4+1}, & x\neq 0,\\ \dfrac{\pi}{2}, & x=0.\end{cases}$

由 $\lim\limits_{x\to 0}f'(x)=\lim\limits_{x\to 0}\left(\arctan\dfrac{1}{x^2}-\dfrac{2x}{x^4+1}\right)=\dfrac{\pi}{2}=f'(0)$，故 $f'(x)$ 在 $x=0$ 处连续.

(1990) 由 $f'(0)=b$,则 $f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{\frac{1}{a}f(1+x)-f(0)}{x-0}$

$$=\lim_{x\to 0}\frac{\frac{1}{a}f(x+1)-\frac{1}{a}f(1)}{x}$$

$$=\frac{1}{a}\lim_{x\to 0}\frac{f(x+1)-f(1)}{x}=\frac{1}{a}f'(1)=b,$$

故 $f'(1)=ab$.选(D).

(1993) 连续性:$\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}\sqrt{|x|}\sin\dfrac{1}{x^2}=0=f(0)$.

可导性:$\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{\sqrt{|x|}\sin\frac{1}{x^2}}{x}$,不存在.

故 $f(x)$ 在 $x=0$ 处连续但不可导.选(C).

(1995) $f(x)$ 在 $x=0$ 处的连续性:

$\lim\limits_{x\to 0^-}f(x)=\lim\limits_{x\to 0^-}\dfrac{2(1-\cos x)}{x^2}=1=f(0)$,左连续;

$\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^+}\dfrac{\int_0^x\cos t^2\mathrm{d}t}{x}\overset{L'}{=}1=f(0)$,右连续.

故 $f(x)$ 在 $x=0$ 处连续.

$f(x)$ 在 $x=0$ 处的可导性:

右导数 $f'_+(0)=\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^+}\dfrac{\frac{1}{x}\int_0^x\cos t^2\mathrm{d}t-1}{x}$

$$=\lim_{x\to 0^+}\frac{\int_0^x\cos t^2\mathrm{d}t-x}{x^2}\overset{L'}{=}\lim_{x\to 0^+}\frac{\cos x^2-1}{2x}=0;$$

左导数 $f'_-(0)=\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^-}\dfrac{\frac{2}{x^2}(1-\cos x)-1}{x}$

$$=\lim_{x\to 0^-}\frac{2(1-\cos x)-x^2}{x^3}\overset{L'}{=}\lim_{x\to 0^-}\frac{2\sin x-2x}{3x^2}$$

$$=\frac{2}{3}\times 0=0.$$

故 $f'(0)=0$,$f(x)$ 在 $x=0$ 处可导.

(1996) (1) 当 $x \neq 0$ 时，$f'(x)=\dfrac{[g'(x)+e^{-x}]x-[g(x)-e^{-x}]}{x^2}$；

当 $x=0$ 时，$f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{\dfrac{g(x)-e^{-x}}{x}-0}{x-0}$

$$=\lim_{x\to 0}\frac{g(x)-e^{-x}}{x^2}\overset{L'}{=}\lim_{x\to 0}\frac{g'(x)+e^{-x}}{2x}$$

$$\overset{L'}{=}\lim_{x\to 0}\frac{g''(x)-e^{-x}}{2}=\frac{1}{2}[g''(0)-1].$$

故 $f'(x)=\begin{cases}\dfrac{[g'(x)+e^{-x}]x-g(x)+e^{-x}}{x^2}, & x\neq 0,\\ \dfrac{1}{2}[g''(0)-1], & x=0.\end{cases}$

(2) $\lim\limits_{x\to 0}f'(x)=\lim\limits_{x\to 0}\dfrac{[g'(x)+e^{-x}]x-[g(x)-e^{-x}]}{x^2}$

$$=\lim_{x\to 0}\frac{g'(x)+e^{-x}}{x}-\lim_{x\to 0}\frac{g(x)-e^{-x}}{x^2}$$

$$\overset{L'}{=}\lim_{x\to 0}[g''(x)-e^{-x}]-\lim_{x\to 0}\frac{g'(x)+e^{-x}}{2x}$$

$$=g''(0)-1-\frac{1}{2}[g''(0)-1]$$

$$=\frac{1}{2}[g''(0)-1]=f'(0),$$

故 $f'(x)$ 在 $(-\infty,+\infty)$ 上连续.

(2012) $f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{(e^x-1)(e^{2x}-2)\cdots(e^{nx}-n)-0}{x}$

$=\lim\limits_{x\to 0}\dfrac{e^x-1}{x}\cdot\lim\limits_{x\to 0}[(e^{2x}-2)\cdots(e^{nx}-n)]=(-1)^{n-1}(n-1)!$.选(A).

§2.2 利用导数定义求极限

基础篇

1. $\lim\limits_{x\to 1}\dfrac{f^2(x)-f^2(1)}{x^2-1}=\lim\limits_{x\to 1}\dfrac{f^2(x)-f^2(1)}{x-1}\cdot\dfrac{x-1}{x^2-1}$

$$= \frac{1}{2}\lim_{x\to 1}\frac{f^2(x)-f^2(1)}{x-1} = \frac{1}{2}[f^2(x)]'\Big|_{x=1}$$

$$= \frac{1}{2}\times 2f(x)f'(x)\Big|_{x=1} = f(1)f'(1)$$

$$= 1\times 2 = 2.$$

2. $\lim\limits_{x\to 0}\dfrac{f(1-\cos 2x)}{x\sin x} = \lim\limits_{x\to 0}\dfrac{f(0+1-\cos 2x)-f(0)}{1-\cos 2x}\,\dfrac{1-\cos 2x}{x\sin x}$

$$= 2\lim_{x\to 0}\frac{f(0+1-\cos 2x)-f(0)}{1-\cos 2x} = 2f'(0);$$

$$\lim_{x\to 0}\frac{f(x^2)}{\sin^2\left(\frac{x}{3}\right)} = \lim_{x\to 0}\frac{f(0+x^2)-f(0)}{x^2}\,\frac{x^2}{\sin^2\left(\frac{x}{3}\right)}$$

$$= \lim_{x\to 0}\frac{f(0+x^2)-f(0)}{x^2}\,\frac{x^2}{\frac{x^2}{9}} = 9f'(0).$$

3. $\lim\limits_{x\to a}\dfrac{x^x-a^a}{x-a} = (x^x)'\Big|_{x=a} = x^x(1+\ln x)\Big|_{x=a} = a^a(1+\ln a).$

4. $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a} = \cos x\Big|_{x=a} = \cos a.$

5. 构造导数定义.

$$\lim_{\Delta x\to 0}\frac{f(x+\alpha\Delta x)-f(x-\beta\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{[f(x+\alpha\Delta x)-f(x)]-[f(x-\beta\Delta x)-f(x)]}{\Delta x}$$

$$= \alpha\lim_{\Delta x\to 0}\frac{f(x+\alpha\Delta x)-f(x)}{\alpha\Delta x}+\beta\lim_{\Delta x\to 0}\frac{f(x-\beta\Delta x)-f(x)}{-\beta\Delta x}$$

$$= (\alpha+\beta)f'(x).$$

强化篇

1. $\lim\limits_{h\to 0}\dfrac{f(a+mh)-f(a-nh)}{h} = \lim\limits_{h\to 0}\dfrac{f(a+mh)-f(a-nh)}{(m+n)h}\cdot(m+n)$

$$= (m+n)f'(a).$$

2. $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{g(x)\cos\frac{1}{x}}{x}$

$$= \lim_{x\to 0}\frac{g(x)}{x}\cos\frac{1}{x} = 0(\text{无穷小乘有界量}).$$

3. $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{2\sin x + x^2\sin\dfrac{1}{x}}{x} = 2.$

4. $\lim\limits_{\Delta x\to 0}\dfrac{f(x_0-\Delta x)-f(x_0)}{\Delta x} = -\lim\limits_{\Delta x\to 0}\dfrac{f(x_0-\Delta x)-f(x_0)}{-\Delta x}$

$$= -f'(x_0) = -3.$$

5. $\lim\limits_{n\to\infty} n\left[\left(1+\dfrac{1}{n}\right)^{\frac{1}{a}}-1\right]$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{\frac{1}{a}}-1}{\frac{1}{n}} \xlongequal{x=\frac{1}{n}} \lim_{x\to 0^+}\frac{(1+x)^{\frac{1}{a}}-1}{x}$$

$$= \left[\left(1+x\right)^{\frac{1}{a}}\right]'\Big|_{x=0} = \frac{1}{a}(1+x)^{\frac{1}{a}-1}\Big|_{x=0} = \frac{1}{a}.$$

6. $\lim\limits_{x\to 1}\dfrac{f(\ln x)-1}{1-x} = \lim\limits_{x\to 1}\dfrac{f(0+\ln x)-f(0)}{\ln x}\dfrac{\ln x}{1-x} = f'(0)\lim\limits_{x\to 1}\dfrac{\ln x}{1-x}$

$$= \lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{x-1}{x-1} = 1.$$

$$\lim_{x\to 0}\frac{2^x f(x)-1}{x} = \lim_{x\to 0}\frac{2^{0+x}f(0+x)-2^0 f(0)}{x} = \left[2^x f(x)\right]'\Big|_{x=0}$$

$$= \left[2^x\ln 2 f(x) + 2^x f'(x)\right]\Big|_{x=0} = \ln 2 - 1.$$

7. $\lim\limits_{h\to 0}\dfrac{f(x+2h)-2f(x+h)+f(x)}{h^2} \overset{L'}{=} \lim\limits_{h\to 0}\dfrac{2f'(x+2h)-2f'(x+h)}{2h}$

$$= \lim_{h\to 0}\frac{f'(x+2h)-f'(x+h)}{h}$$

$$= \lim_{h\to 0}\frac{f'(x+2h)-f'(x+h)}{x+2h-(x+h)}$$

$$= f''(x).$$

8. 由 $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0}\dfrac{\frac{1}{2}f(x+1)-\frac{1}{2}f(1)}{x}$

$$= \frac{1}{2}\lim_{x\to 0}\frac{f(x+1)-f(1)}{x} = \frac{1}{2}f'(1) = c,$$

故 $f'(1)=2c$.

9. $\lim\limits_{x\to 0}\dfrac{f(1+x)+f(1+2\sin x)-2f(1-3\tan x)}{x}$

$=\lim\limits_{x\to 0}\dfrac{f(1+x)-f(1-3\tan x)}{x}+\lim\limits_{x\to 0}\dfrac{f(1+2\sin x)-f(1-3\tan x)}{x}$

$=\lim\limits_{x\to 0}\dfrac{f(1+x)-f(1-3\tan x)}{x+3\tan x}\dfrac{x+3\tan x}{x}$

$+\lim\limits_{x\to 0}\dfrac{f(1+2\sin x)-f(1-3\tan x)}{2\sin x+3\tan x}\dfrac{2\sin x+3\tan x}{x}$

$=4f'(1)+5f'(1)=9f'(1)=9.$

10. 由 $y-x=\mathrm{e}^{x(1-y)}$，方程两边同时对 x 求导得

$y'-1=\mathrm{e}^{x(1-y)}(1-y-xy')$，$y'-1=(1-y)\mathrm{e}^{x(1-y)}-xy'\mathrm{e}^{x(1-y)}$，

$y'=\dfrac{(1-y)\mathrm{e}^{x(1-y)}+1}{1+x\mathrm{e}^{x(1-y)}}$，$y'(0)=1.$

故 $\lim\limits_{n\to\infty}\left[f\left(\dfrac{1}{n}\right)-1\right]n=\lim\limits_{n\to\infty}\dfrac{f\left(\dfrac{1}{n}\right)-f(0)}{\dfrac{1}{n}}=1.$

11. $\lim\limits_{n\to\infty}\left[\dfrac{f\left(a+\dfrac{1}{n}\right)}{f(a)}\right]^n=\mathrm{e}^A$（$1^\infty$ 未定式）.

$A=\lim\limits_{n\to\infty}\left[\dfrac{f\left(a+\dfrac{1}{n}\right)}{f(a)}-1\right]n=\lim\limits_{n\to\infty}\dfrac{f\left(a+\dfrac{1}{n}\right)-f(a)}{f(a)}n$

$\xlongequal{\text{令 } x=\frac{1}{n}}\dfrac{1}{f(a)}\lim\limits_{x\to 0}\dfrac{f(a+x)-f(a)}{x}=\dfrac{f'(a)}{f(a)}$，

故原极限 $=\mathrm{e}^{\frac{f'(a)}{f(a)}}$.

真题自测

(1987) $\lim\limits_{x\to 0}\dfrac{f(a+x)-f(a-x)}{x}=\lim\limits_{x\to 0}\dfrac{f(a+x)-f(a-x)}{2x}\cdot 2=2f'(a)$，

故选(B).

(1989) $\lim\limits_{h\to 0}\dfrac{f(3-h)-f(3)}{2h}=\lim\limits_{h\to 0}\dfrac{f(3-h)-f(3)}{-h}\dfrac{-h}{2h}=-\dfrac{1}{2}f'(3)$

$$=-\frac{1}{2}\times 2=-1.$$

(1994) 判断极限的类型:1^{∞} 未定式.则原极限 $=\mathrm{e}^{A}$.

$$A=\lim_{n\to\infty}\left[\tan\left(\frac{\pi}{4}+\frac{2}{n}\right)-1\right]n\xlongequal{x=\frac{1}{n}}2\lim_{x\to 0}\frac{\tan\left(\frac{\pi}{4}+2x\right)-\tan\frac{\pi}{4}}{2x}$$

$$=2(\tan x)'\Big|_{x=\frac{\pi}{4}}=2\sec^2 x\Big|_{x=\frac{\pi}{4}}=2\times 2=4,$$

故原极限 $=\mathrm{e}^{4}$.

(2011) 根据导数定义:$f'(0)=\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h}$,有

$$\lim_{x\to 0}\frac{x^2 f(x)-2f(x^3)}{x^3}=\lim_{x\to 0}\left[\frac{f(x)}{x}-2\frac{f(x^3)}{x^3}\right]$$

$$=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}-2\lim_{x\to 0}\frac{f(x^3)-f(0)}{x^3-0}=f'(0)-2f'(0)=$$

$-f'(0)$,因此选(B).

(2013) 由题知,$f(1)=0$, $f'(1)=(2x-1)\Big|_{x=1}=1$,故

$$\lim_{n\to\infty}nf\left(\frac{n}{n+2}\right)=\lim_{n\to\infty}\left[\frac{-2n}{n+2}\cdot\frac{f\left(1+\frac{-2}{n+2}\right)-f(1)}{\frac{-2}{n+2}}\right]$$

$$=-2f'(1)=-2.$$

§2.3 复合函数求导

基础篇

1. $y'=\dfrac{1}{1+x}\cdot\dfrac{1}{2\sqrt{x}}=\dfrac{1}{2\sqrt{x}(1+x)}$.

2. $y'=\dfrac{1}{2\sqrt{x^2+a^2}}\cdot 2x=\dfrac{x}{\sqrt{x^2+a^2}}$.

3. $y'=\dfrac{1}{1+(\mathrm{e}^{2x}-1)}\cdot\dfrac{2\mathrm{e}^{2x}}{2\sqrt{\mathrm{e}^{2x}-1}}=\dfrac{1}{\sqrt{\mathrm{e}^{2x}-1}}$.

4. $y' = \dfrac{1}{\sqrt{1-\left(\dfrac{x}{a}\right)^2}} \cdot \dfrac{1}{a} = \dfrac{1}{\sqrt{a^2-x^2}}.$

5. $y' = \dfrac{1}{1+\left(\dfrac{x}{a}\right)^2} \cdot \dfrac{1}{a} = \dfrac{a}{a^2+x^2}.$

6. $y' = 1-\dfrac{e^x}{1+e^x} = \dfrac{1}{1+e^x}.$

7. $y' = \dfrac{\sec x\tan x+\sec^2 x}{\sec x+\tan x} = \sec x.$

8. $y' = \dfrac{-\csc x\cot x+\csc^2 x}{\csc x-\cot x} = \csc x.$

9. $y' = \dfrac{1}{2}(\sec x\tan^2 x+\sec^3 x)+\dfrac{1}{2}\sec x$

$= \dfrac{1}{2}[\sec x(\sec^2 x-1)+\sec^3 x]+\dfrac{1}{2}\sec x$

$= \sec^3 x.$

10. $y' = -\dfrac{1}{2}(-\csc x\cot^2 x-\csc^3 x)+\dfrac{1}{2}\csc x$

$= -\dfrac{1}{2}[-\csc x(\csc^2 x-1)-\csc^3 x]+\dfrac{1}{2}\csc x$

$= \csc^3 x.$

11. $y' = \dfrac{1}{x+\sqrt{x^2+a^2}} \cdot \left(1+\dfrac{2x}{2\sqrt{x^2+a^2}}\right)$

$= \dfrac{1}{x+\sqrt{x^2+a^2}} \cdot \dfrac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}$

$= \dfrac{1}{\sqrt{x^2+a^2}}.$

12. $y' = \dfrac{1}{x+\sqrt{x^2-a^2}} \cdot \left(1+\dfrac{2x}{2\sqrt{x^2-a^2}}\right)$

$= \dfrac{1}{x+\sqrt{x^2-a^2}} \cdot \dfrac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}}$

$$= \frac{1}{\sqrt{x^2-a^2}}.$$

13. $y' = \frac{1}{2}\sqrt{x^2+a^2} + \frac{x}{2}\cdot\frac{1}{2}\frac{2x}{\sqrt{x^2+a^2}} + \frac{a^2}{2}\cdot\frac{1}{\sqrt{x^2+a^2}}$

$$= \frac{1}{2}\sqrt{x^2+a^2} + \frac{(x^2+a^2)-a^2}{2\sqrt{x^2+a^2}} + \frac{a^2}{2}\frac{1}{\sqrt{x^2+a^2}}$$

$$= \sqrt{x^2+a^2}.$$

14. $y' = \frac{1}{2}\sqrt{x^2-a^2} + \frac{x}{2}\cdot\frac{2x}{2\sqrt{x^2-a^2}} - \frac{a^2}{2}\cdot\frac{1}{\sqrt{x^2-a^2}}$

$$= \frac{1}{2}\sqrt{x^2-a^2} + \frac{(x^2-a^2)+a^2}{2\sqrt{x^2-a^2}} - \frac{a^2}{2\sqrt{x^2-a^2}}$$

$$= \sqrt{x^2-a^2}.$$

15. $y' = \frac{1}{2}\sqrt{a^2-x^2} + \frac{x}{2}\cdot\frac{-2x}{2\sqrt{a^2-x^2}} + \frac{a^2}{2}\cdot\frac{1}{\sqrt{a^2-x^2}}$

$$= \frac{1}{2}\sqrt{a^2-x^2} + \frac{(a^2-x^2)-a^2}{2\sqrt{a^2-x^2}} + \frac{a^2}{2}\frac{1}{\sqrt{a^2-x^2}}$$

$$= \sqrt{a^2-x^2}.$$

16. $y' = -\frac{1}{4}\frac{4x^3}{(1+x^4)^2} + \frac{1}{4}\left(\frac{4x^3}{x^4} - \frac{4x^3}{1+x^4}\right)$

$$= -\frac{x^3}{(1+x^4)^2} + \frac{1}{4}\left(\frac{4}{x} - \frac{4x^3}{1+x^4}\right)$$

$$= -\frac{x^3}{(1+x^4)^2} + \frac{1}{x} - \frac{x^3}{1+x^4}.$$

17. $y' = \frac{1}{2}\left(\frac{-\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x}\right) = -\frac{1}{\cos x}.$

18. $y' = -\frac{1}{x^2}(\ln^3 x + 3\ln^2 x + 6\ln x + 6) + \frac{1}{x}\left(\frac{3}{x}\ln^2 x + \frac{6}{x}\ln x + \frac{6}{x}\right)$

$$= -\frac{\ln^3 x}{x^2}.$$

19. $y' = \frac{1}{\frac{1}{x} + \ln\left(\frac{1}{x} + \ln\frac{1}{x}\right)}\left[-\frac{1}{x^2} + \frac{1}{\frac{1}{x} + \ln\frac{1}{x}}\left(-\frac{1}{x^2} - \frac{1}{x}\right)\right]$

$$= -\frac{1+x+\frac{1}{x}+\ln\frac{1}{x}}{\left(1+x\ln\frac{1}{x}\right)\left[1+x\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right]}.$$

强化篇

1. $y' = \cos(\cos^2 x)\cdot 2\cos x\cdot(-\sin x)\cdot\cos(\sin^2 x)$

$+\sin(\cos^2 x)[-\sin(\sin^2 x)]\cdot 2\sin x\cdot\cos x$

$= -\sin 2x[\cos(\cos^2 x)\cos(\sin^2 x)+\sin(\cos^2 x)\sin(\sin^2 x)]$

$= -\sin 2x\cos(\cos^2 x-\sin^2 x) = -\sin 2x\cos(\cos 2x).$

2. $y' = \cos x\cdot\cos(\sin x)\cdot\cos[\sin(\sin x)].$

3. $y' = \dfrac{3^x(\ln 3\cdot\cos x-\sin x)-3^x\ln 3(\ln 3\cdot\sin x+\cos x)}{3^{2x}}$

$= -\dfrac{(1+\ln^2 3)\sin x}{3^x}.$

4. $y' = \dfrac{1}{\sqrt{a^2+b^2}}e^{ax}[a(a\sin bx-b\cos bx)+(ab\cos bx+b^2\sin bx)]$

$= \sqrt{a^2+b^2}\,e^{ax}\sin bx.$

5. $y' = \dfrac{1}{x\ln x\ln\ln x}.$

6. $y' = \dfrac{1}{\ln^2(\ln^3 x)}\cdot 2\ln(\ln^3 x)\dfrac{1}{\ln^3 x}\cdot 3\ln^2 x\cdot\dfrac{1}{x} = \dfrac{6}{x\ln x\ln(\ln^3 x)}.$

7. $y' = \ln(x+\sqrt{1+x^2})+\dfrac{x}{\sqrt{1+x^2}}-\dfrac{x}{\sqrt{1+x^2}} = \ln(x+\sqrt{1+x^2}).$

8. $y' = \dfrac{1}{e^x+\sqrt{1+e^{2x}}}\left(e^x+\dfrac{e^{2x}}{\sqrt{1+e^{2x}}}\right) = \dfrac{e^x}{\sqrt{1+e^{2x}}}.$

9. $y' = \dfrac{1}{2}(x+\sqrt{x+\sqrt{x}})^{-\frac{1}{2}}\cdot\left[1+\dfrac{1}{2}(x+\sqrt{x})^{-\frac{1}{2}}\cdot(1+\dfrac{1}{2}x^{-\frac{1}{2}})\right]$

10. $y' = \dfrac{1}{3}(1+\sqrt[3]{1+\sqrt[3]{x}})^{-\frac{2}{3}}\cdot\dfrac{1}{3}(1+\sqrt[3]{x})^{-\frac{2}{3}}\cdot\dfrac{1}{3}x^{-\frac{2}{3}}$

$= \dfrac{1}{27}\left[(1+\sqrt[3]{1+\sqrt[3]{x}})(1+\sqrt[3]{x})x\right]^{-\frac{2}{3}}.$

11. $y' = e^x[1+e^{e^x}(1+e^{e^{e^x}})].$

12. $y' = a^a x^{a^a-1}+ax^{a-1}a^{x^a}\ln a+a^x a^{a^x}\ln^2 a.$

13. $y' = x^{x^a}\left(ax^{a-1}\ln x + \frac{x^a}{x}\right) + x^{a^x}\left(a^x \ln a \cdot \ln x + \frac{a^x}{x}\right)$

$+ a^{x^x} \cdot \ln a \cdot x^x(1+\ln x)$

$= x^{a-1}x^{x^a}(1+a\ln x) + a^x x^{a^x}\left(\frac{1}{x} + \ln a\ \ln x\right)$

$+ x^x a^{x^x}\ln a(1+\ln x)$.

14. 两边取对数,得

$$\ln y = x\ \ln\frac{a}{b} + a(\ln b - \ln x\ ) + b(\ln x - \ln a).$$

两边同时对 x 求导数,得

$$\frac{y'}{y} = \ln\frac{a}{b} - \frac{a}{x} + \frac{b}{x}.$$

于是,$y' = \left(\frac{a}{b}\right)^x\left(\frac{b}{x}\right)^a\left(\frac{x}{a}\right)^b\left(\ln\frac{a}{b} - \frac{a}{x} + \frac{b}{x}\right)$.

15. $y' = \dfrac{-1}{1+\left(\dfrac{\sin x + \cos x}{\sin x - \cos x}\right)^2} \cdot$

$\dfrac{(\cos x - \sin x)(\sin x - \cos x) - (\cos x + \sin x)^2}{(\sin x - \cos x)^2} = 1.$

16. 由于$\dfrac{\mathrm{d}f(\sin x)}{\mathrm{d}x} = f'(\sin x)\cos x$,

$\dfrac{\mathrm{d}f^2(\sin x)}{\mathrm{d}x} = 2f(\sin x)f'(\sin x)\cos x$,

令 $x = 0$ 得 $f'(0) = 2f(0)f'(0)$,故 $f(0) = \dfrac{1}{2}$.

真题自测

(1988) 由 $f(t) = \lim\limits_{x\to\infty} t\left(1+\frac{1}{x}\right)^{2tx} = t\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{2tx} = t\mathrm{e}^{2t}$,

$f'(t) = \mathrm{e}^{2t} + 2t\mathrm{e}^{2t} = (1+2t)\mathrm{e}^{2t}$.

(1989) $y' = \dfrac{1}{\sqrt{1-\mathrm{e}^{-2\sqrt{x}}}} \cdot \mathrm{e}^{-\sqrt{x}} \cdot \dfrac{-1}{2\sqrt{x}} = \dfrac{-\mathrm{e}^{-\sqrt{x}}}{2\sqrt{x(1-\mathrm{e}^{-2\sqrt{x}})}}$.

(1990) $y' = e^{\tan\frac{1}{x}} \cdot \sec^2\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) \cdot \sin\frac{1}{x} + e^{\tan\frac{1}{x}} \cdot \cos\frac{1}{x} \cdot \left(-\frac{1}{x^2}\right)$

$= -\frac{1}{x^2}e^{\tan\frac{1}{x}}\left(\sec^2\frac{1}{x} \cdot \sin\frac{1}{x} + \cos\frac{1}{x}\right)$

$= -\frac{1}{x^2}e^{\tan\frac{1}{x}} \cdot \cos\frac{1}{x}\left(\sec^3\frac{1}{x} \cdot \sin\frac{1}{x} + 1\right)$.

(1991) $y' = -\frac{1}{1+3^{-x}} \cdot 3^{-x}\ln 3$.

(1993) $\frac{dy}{dx} = \cos[f(x^2)] \cdot f'(x^2) \cdot 2x$,

$\frac{d^2y}{dx^2} = -\sin[f(x^2)] \cdot [f'(x^2)]^2 \cdot 4x^2 + \cos[f(x^2)] \cdot$

$f''(x^2) \cdot 4x^2 + 2\cos[f(x^2)]f'(x^2)$.

(1993) 由 $y = f\left(\frac{3x-2}{3x+2}\right)$，$y' = f'\left(\frac{3x-2}{3x+2}\right) \cdot \frac{3(3x+2)-3(3x-2)}{(3x+2)^2}$

$= f'\left(\frac{3x-2}{3x+2}\right) \cdot \frac{12}{(3x+2)^2} = \arctan\left(\frac{3x-2}{3x+2}\right)^2 \cdot \frac{12}{(3x+2)^2}$,

$y'\Big|_{x=0} = \frac{\pi}{4} \times \frac{12}{4} = \frac{3}{4}\pi$.

(1996) $y' = \frac{2}{3}(x+e^{-\frac{x}{2}})^{-\frac{1}{3}} \cdot \left[1+e^{-\frac{x}{2}} \cdot \left(-\frac{1}{2}\right)\right]$，故 $y'\Big|_{x=0} = \frac{1}{3}$.

(1997) 由 $y = \ln\sqrt{\frac{1-x}{1+x^2}} = \frac{1}{2}[\ln(1-x) - \ln(1+x^2)]$,

$y' = \frac{1}{2}\left(\frac{-1}{1-x} - \frac{2x}{1+x^2}\right)$，$y'' = \frac{1}{2}\left(\frac{1}{x-1} - \frac{2x}{1+x^2}\right)'$

$= \frac{1}{2}\left[-\frac{1}{(x-1)^2} - \frac{2(1+x^2)-4x^2}{(1+x^2)^2}\right]$

$= \frac{1}{2}\left[-\frac{1}{(x-1)^2} - \frac{2(1-x^2)}{(1+x^2)^2}\right]$,

故 $y''\Big|_{x=0} = \frac{1}{2}(-1-2) = -\frac{3}{2}$.

(2006) 由 $h'(x) = e^{1+g(x)} \cdot g'(x)$，$h'(1) = e^{1+g(1)} \cdot g'(1) = e^{1+g(1)} \cdot 2 = 1$，$e^{1+g(1)} = \frac{1}{2}$，$1+g(1) = -\ln 2$，$g(1) = -\ln 2 - 1$，选(C).

(1987) 由 $y=\ln\dfrac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}=\ln\dfrac{x^2}{(\sqrt{1+x^2}+1)^2}$

$$=\ln(x^2)-\ln(\sqrt{1+x^2}+1)^2$$

$$=2\ln x-2\ln(\sqrt{1+x^2}+1),$$

故 $y'=2\left[\dfrac{1}{x}-\dfrac{x}{(\sqrt{1+x^2}+1)\sqrt{1+x^2}}\right]$

$$=\frac{2\left[\sqrt{1+x^2}(\sqrt{1+x^2}+1)-x^2\right]}{x(\sqrt{1+x^2}+1)\sqrt{1+x^2}}$$

$$=\frac{2}{x\sqrt{1+x^2}}.$$

(2005) 解法 1 $y'=[\mathrm{e}^{x\ln(1+\sin x)}]'=(1+\sin x)^x\left[\ln(1+\sin x)+\dfrac{x\cos x}{1+\sin x}\right]$,

则 $y'|_{x=\pi}=-\pi$,故 $\mathrm{d}y|_{x=\pi}=-\pi\mathrm{d}x$.

解法 2 $\ln y=x\ln(1+\sin x)$,两边对 x 求导得

$$\frac{y'}{y}=\ln(1+\sin x)+\frac{x\cos x}{1+\sin x},$$

于是得

$$y'=(1+\sin x)^x\left[\ln(1+\sin x)+\frac{x\cos x}{1+\sin x}\right],$$

$$y'|_{x=\pi}=-\pi,$$

故

$$\mathrm{d}y|_{x=\pi}=-\pi\mathrm{d}x.$$

(2012) $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=\mathrm{e}}=f'[f(\mathrm{e})]\cdot f'(\mathrm{e})=f'\left(\dfrac{1}{2}\right)\cdot f'(\mathrm{e})$

$$=2\cdot\frac{1}{2\mathrm{e}}=\frac{1}{\mathrm{e}}.$$

(2021) 由复合函数求导公式,得

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin\mathrm{e}^{-\sqrt{x}}\cdot\mathrm{e}^{-\sqrt{x}}\cdot\frac{1}{-2\sqrt{x}},$$

代入 $x=1$，得 $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=1}=\frac{\sin\left(\frac{1}{\mathrm{e}}\right)}{2\mathrm{e}}$.

§2.4　隐函数求导

基础篇

1. 两边同时对 x 求导：

$$2yy'=2p\Rightarrow y'=\frac{p}{y}.$$

2. 两边同时对 x 求导：

$$\frac{2x}{a^2}+\frac{2yy'}{b^2}=0\Rightarrow y'=-\frac{b^2x}{a^2y}.$$

3. 两边同时对 x 求导：

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot y'=0\Rightarrow y'=-\sqrt{\frac{y}{x}}.$$

4. 两边同时对 x 求导：

$$\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}y'=0\Rightarrow y'=-\sqrt[3]{\frac{y}{x}}.$$

5. 两边同时对 x 求导：

$$\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{xy'-y}{x^2}=\frac{x+yy'}{x^2+y^2}\Rightarrow y'=\frac{x+y}{x-y}.$$

6. 两边同时对 x 求导：

$$y'-\cos x+\sin(x-y)\cdot(1-y')=0,$$

$$y'-y'\sin(x-y)+\sin(x-y)-\cos x=0,$$

则
$$y'=\frac{\cos x-\sin(x-y)}{1-\sin(x-y)}.$$

7. 由 $\frac{x}{y}=\ln(xy)$，变形得 $x=y(\ln x+\ln y)$.

两边同时对 x 求导：$1=y'(\ln x+\ln y)+y\left(\frac{1}{x}+\frac{y'}{y}\right)$，

$$1=y'(\ln x+\ln y)+y'+\frac{y}{x},$$

则
$$y'=\frac{1-\frac{y}{x}}{1+\ln x+\ln y}=\frac{x-y}{x(1+\ln x+\ln y)}.$$

8. 先求斜率.

两边同时对 x 求导：

$$\frac{x}{8}+\frac{2y\cdot y'}{9}=0,$$

则
$$y'=\frac{9}{2y}\cdot\left(-\frac{x}{8}\right)=-\frac{9}{16}\cdot\frac{x}{y},$$

则
$$k=y'\Big|_{\left(2,\frac{3}{2}\sqrt{3}\right)}=-\frac{9}{16}\cdot\frac{x}{y}\Big|_{\left(2,\frac{3}{2}\sqrt{3}\right)}=-\frac{9}{16}\cdot\frac{2}{\frac{3}{2}\sqrt{3}}$$

$$=-\frac{3}{4\sqrt{3}}=-\frac{\sqrt{3}}{4},$$

则切线方程为 $y-\frac{3}{2}\sqrt{3}=-\frac{\sqrt{3}}{4}(x-2)$.

9. 两边同时对 x 求导：

$$1-y'+\frac{1}{2}\cos y\cdot y'=0\quad(1)\Rightarrow y'=\frac{1}{1-\frac{1}{2}\cos y}.$$

对(1)式继续求导：

$$-y''-\frac{1}{2}\sin y\cdot(y')^2+\frac{1}{2}\cos y\cdot y''=0,$$

则
$$y''=\frac{\frac{1}{2}(y')^2\cdot\sin y}{\frac{1}{2}\cos y-1}=\frac{(y')^2\sin y}{\cos y-2}=\frac{4\sin y}{(\cos y-2)^3}.$$

10. 两边同时对 x 求导：

$$y'=\sec^2(x+y)\cdot(1+y'),$$

$$y'-\sec^2(x+y)\cdot y'=\sec^2(x+y)$$

$$\Rightarrow y'=\frac{\sec^2(x+y)}{1-\sec^2(x+y)}=-\csc^2(x+y).$$

对 y' 继续求导：

$$\begin{aligned} y'' &= -2\csc(x+y)[-\csc(x+y)\cot(x+y)(1+y')] \\ &= -2\frac{1}{\sin^2(x+y)}\cdot\cot^3(x+y) \\ &= -\frac{4\cot^4(x+y)}{\sin(2x+2y)}. \end{aligned}$$

强化篇

1. 当 $x=0$ 时，由原方程得 $y(0)=0$，在 $xy+e^y=x+1$ 两边对 x 求导得 $y+xy'+e^y\cdot y'=1$，再求导得

$$2y'+xy''+e^y\cdot y''+e^y(y')^2=0.$$

代入 $x=0$，$y(0)=0$，$y'(0)=1$，得 $y''(0)=-3$.

2. 当 $x=0$ 时，由原方程得 $y(0)=0$.在 $xy+\int_0^x e^{t^2}dt+1=x+e^y$ 两边对 x 求导得 $y+xy'+e^{x^2}=1+e^y\cdot y'$.

代入 $x=0$，$y(0)=0$ 得 $y'(0)=0$，再求导得

$$2y'+xy''+2xe^{x^2}=e^y(y')^2+e^y y''.$$

代入 $x=0$，$y(0)=0$，$y'(0)=0$，求得 $y''(0)=0$.

3. 曲线 $\tan\left(x+y+\frac{\pi}{4}\right)=e^y$ 的两端对 x 求导得

$$\sec^2\left(x+y+\frac{\pi}{4}\right)(1+y')=e^y\cdot y'.$$

将 $x=0$，$y=0$ 代入上式，有 $\frac{1+y'}{\cos^2\frac{\pi}{4}}=y'$，解得 $y'\Big|_{x=0}=-2$，故切线方程为 $y=-2x$.

4. 方程两边分别对 x 求导，得 $(y+xy')\cos(xy)+\frac{y'-1}{y-x}=1$，将 $x=0$，$y=1$ 代入得 $y'(0)=1$，故所求的切线方程为 $y=x+1$.

5. 由 $\ln\frac{x^2}{y}-xy^2=1$，$2\ln|x|-\ln y-xy^2=1$，两边取微分：

$$\frac{2}{x}dx-\frac{1}{y}dy-y^2dx-2xydy=0,$$

则
$$\left(\frac{2}{x}-y^2\right)dx=\left(\frac{1}{y}+2xy\right)dy,$$

故 $$dy=\frac{\frac{2}{x}-y^2}{\frac{1}{y}+2xy}dx,$$

即 $$dy=\frac{2y-xy^3}{x+2x^2y^2}dx.$$

6. 由 $x^{2y}+y^{2x}=x$，$e^{2y\ln x}+e^{2x\ln y}=x$，两边同时取微分：

$$x^{2y}\cdot\frac{2y}{x}dx+x^{2y}\cdot 2\ln x dy+y^{2x}2\ln y dx+y^{2x}\cdot\frac{2x}{y}dy=dx,$$

$$\left(x^{2y}\cdot\frac{2y}{x}+y^{2x}\cdot 2\ln y\right)dx+\left(x^{2y}\cdot 2\ln x+y^{2x}\cdot\frac{2x}{y}\right)dy=dx,$$

则 $$dy=\frac{1-\left(x^{2y}\cdot\frac{2y}{x}+y^{2x}\cdot 2\ln y\right)}{x^{2y}\cdot 2\ln x+y^{2x}\cdot\frac{2x}{y}}dx.$$

7. 两边同时对 x 求导：

$$e^{x+y}(1+y')-y-xy'=1,\quad (1)$$

对(1)式继续求导：

$$e^{x+y}(1+y')^2+e^{x+y}\cdot y''-y'-y'-xy''=0,\quad (2)$$

将 $x=0$，$y=0$，$y'=0$ 代入(2)式：

$$1+y''=0,\quad 解得\ y''=-1.$$

8. 两边同时对 x 求导：

$$e^y\cdot y'-y-xy'=0,\quad (1)$$

由(1)得 $$y'=\frac{y}{e^y-x}.$$

继续对(1)求导：

$$e^y\cdot(y')^2+e^y\cdot y''-y'-y'-xy''=0,$$

解得 $$y''=\frac{2y'-e^y\cdot(y')^2}{e^y-x}.$$

真题自测

(1988) $y'=e^{xy}(x^2y'+xy+1)$，

$$y''=e^{xy}(x^2y''+2xy'+xy'+y)+e^{xy}(x^2y'+xy+1)(xy'+y).$$

将 $x=0$ 代入，得 $y'\Big|_{x=0}=1$，$y''\Big|_{x=0}=2$.

(1990) $2\mathrm{d}y-\mathrm{d}x=(\mathrm{d}x-\mathrm{d}y)\ln(x-y)+(x-y)\dfrac{\mathrm{d}x-\mathrm{d}y}{x-y}$，整理得 $\mathrm{d}y=\dfrac{x}{2x-y}\mathrm{d}x$.

(1992) 方程两侧同时对 x 求导，得

$$y'-\mathrm{e}^y-x\mathrm{e}^y\cdot y'=0,$$
$$y''-\mathrm{e}^y\cdot y'-[\mathrm{e}^y\cdot y'+x\mathrm{e}^y\cdot(y')^2+x\mathrm{e}^y\cdot y'']=0.$$

将 $x=0$ 代入，得 $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\Big|_{x=0}=2\mathrm{e}^2$.

(1992) 方程两侧同时对 x 求导，得 $\mathrm{e}^{x+y}(1+y')-\sin xy\cdot(y+xy')=0$，整理得

$$y'=\frac{y\sin xy-\mathrm{e}^{x+y}}{\mathrm{e}^{x+y}-x\sin xy}.$$

(1993) 方程两侧同时对 x 求导，得

$$2\cos(x^2+y^2)(x+yy')+\mathrm{e}^x-y^2-2xyy'=0,$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y^2-2x\cos(x^2+y^2)-\mathrm{e}^x}{2y\cos(x^2+y^2)-2xy}.$$

(1994) 方程两侧同时对 x 求导，得 $\mathrm{e}^{xy}(y+xy')+2yy'=-\sin x$，整理得

$$y'=-\frac{y\mathrm{e}^{xy}+\sin x}{x\mathrm{e}^{xy}+2y}.$$

(1994) $y'=(1+y')f'(x+y)$，$y''=(1+y')^2f''(x+y)+y''f'(x+y)$，整理得

$$y'=\frac{f'(x+y)}{1-f'(x+y)},\ y''=\frac{(1+y')^2f''(x+y)}{1-f'(x+y)}=\frac{f''(x+y)}{[1-f'(x+y)]^3}.$$

(1995) 化简方程，$\ln x+f(y)=y$，方程两侧同时对 x 求导，得

$$\frac{1}{x}+f'(y)y'=y',\ -\frac{1}{x^2}+f''(y)\cdot(y')^2+f'(y)y''=y'',$$

整理得　$y'=\dfrac{1}{x[1-f'(y)]}$，$y''=\dfrac{f''(y)-[1-f'(y)]^2}{x^2[1-f'(y)]^3}$.

(1999) 方程两侧同时对 x 求导，得

$$\frac{2x+y'}{x^2+y}=3x^2y+x^3y'+\cos x.$$

将 $x=0$ 代入，得 $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}=1$.

(2000) 方程两侧同时对 x 求导，得

$$2^{xy}\cdot\ln 2(y+xy')=1+y'.$$

将 $x=0$ 代入，整理得 $\left.\mathrm{d}y\right|_{x=0}=(\ln 2-1)\mathrm{d}x$.

(2002) 当 $x=0$ 时，$y=0$，方程两侧同时对 x 求导，得

$$\mathrm{e}^{y}\cdot y'+6xy'+6y+2x=0.$$

当 $x=0,y=0$ 时，$y'(0)=0$.

再次方程两侧同时对 x 求导，得

$$\mathrm{e}^{y}\cdot y''+\mathrm{e}^{y}(y')^{2}+6xy''+12y'+2=0,$$

可解得 $y''(0)=-2$.

(2007) 当 $x=0$ 时，$y=1$.

方程 $y-x\mathrm{e}^{y-1}=1$ 两侧同时对 x 求导，得

$$y'-\mathrm{e}^{y-1}-x\mathrm{e}^{y-1}y'=0,$$

$$y''-\mathrm{e}^{y-1}y'-\mathrm{e}^{y-1}y'-x\mathrm{e}^{y-1}(y')^{2}-x\mathrm{e}^{y-1}y''=0.$$

将 $x=0$，$y=1$ 代入，得 $\left.y'\right|_{x=0}=1$，$\left.y''\right|_{x=0}=2$.

$$\frac{\mathrm{d}z}{\mathrm{d}x}=f'(\ln y-\sin x)\left(\frac{y'}{y}-\cos x\right),\ \left.\frac{\mathrm{d}z}{\mathrm{d}x}\right|_{x=0}=0.$$

$$\frac{\mathrm{d}^{2}z}{\mathrm{d}x^{2}}=f''(\ln y-\sin x)\left(\frac{y'}{y}-\cos x\right)^{2}$$

$$+f'(\ln y-\sin x)\left[\frac{y''}{y}-\frac{(y')^{2}}{y^{2}}+\sin x\right],$$

$$\left.\frac{\mathrm{d}^{2}z}{\mathrm{d}x^{2}}\right|_{x=0}=1.$$

(2012) 将 $x=0$ 代入 $x^{2}-y+1=\mathrm{e}^{y}$，可得到 $\left.y\right|_{x=0}=0$.

在 $x^{2}-y+1=\mathrm{e}^{y}$ 两边对 x 求导，得

$$2x-\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{y}\frac{\mathrm{d}y}{\mathrm{d}x},\qquad(*)$$

将 $x=0$，$y=0$ 代入上式，可得 $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}=0$.

在(*)式两边再对 x 求导，得

$$2-\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{e}^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\mathrm{e}^y\frac{\mathrm{d}^2y}{\mathrm{d}x^2},$$

将 $x=0$，$y=0$，$\frac{\mathrm{d}y}{\mathrm{d}x}=0$ 代入上式，则可解得 $\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0}=1$.

(2022) 原方程两边对 x 求导得 $2x+y+xy'+3y^2y'=0$，再次求导得 $2+2y'+xy''+6y(y')^2+3y^2y''=0$，当 $x=1$ 时，$y=1$，$y'=-\frac{3}{4}$，$y''=-\frac{31}{32}$.

§2.5　参数方程求导

基础篇

1. $y'=\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{-2\sin 2t}{\cos t}=-4\sin t$.

2. $y'=\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{\left(\frac{3at^2}{1+t^2}\right)'}{\left(\frac{3at}{1+t^2}\right)'}=\frac{\frac{6at(1+t^2)-6at^3}{(1+t^2)^2}}{\frac{3a(1+t^2)-6at^2}{(1+t^2)^2}}$

$$=\frac{\frac{6at}{(1+t^2)^2}}{\frac{3a-3at^2}{(1+t^2)^2}}=\frac{6at}{3a-3at^2}=\frac{2t}{1-t^2}.$$

3. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y'_t}{x'_t}=-\frac{1}{t}$，$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y'}{\mathrm{d}x}=\frac{\mathrm{d}y'}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{t^3}$.

4. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y'_t}{x'_t}=\frac{f'(t)+tf''(t)-f'(t)}{f''(t)}=t$，

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{(y'_x)'_t}{x'_t}=\frac{(t)'}{x'_t}=\frac{(t)'}{f''(t)}=\frac{1}{f''(t)}$.

5. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y'_t}{x'_t}=\frac{1-\frac{1}{1+t^2}}{\frac{2t}{1+t^2}}=\frac{t}{2}$，$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{(y'_x)'_t}{x'_t}=\frac{1+t^2}{4t}$.

6. $\dfrac{dy}{dx}=\dfrac{(\ln\sqrt{1+t^2})'_t}{(\arctan t)'_t}=\dfrac{\dfrac{1}{2}\cdot\dfrac{2t}{1+t^2}}{\dfrac{1}{1+t^2}}=t$,

$\dfrac{d^2y}{dx^2}=\dfrac{dy'}{dx}=\dfrac{dy'}{dt}\cdot\dfrac{dt}{dx}=1\cdot\dfrac{1}{\dfrac{1}{1+t^2}}=1+t^2$.

7. $\dfrac{dy}{dx}=\dfrac{(t^2+4t+1)'_t}{(t^2+7)'_t}=\dfrac{2t+4}{2t}$,则$\left.\dfrac{dy}{dx}\right|_{t=1}=\dfrac{2+4}{2}=3$.

当 $t=1$ 时,$x=8$,$y=1+4+1=6$,则曲线在 $t=1$ 对应点的切线方程为 $y-6=3(x-8)$,即 $y=3x-18$.

8. 由 $x=\sqrt{3}t-1$,$\dfrac{dx}{dt}=\sqrt{3}$.

由 $te^y-y=0$,方程两边同时对 t 求导:

$e^y+te^y\cdot y'_t-y'_t=0$,则$\dfrac{dy}{dt}=\dfrac{e^y}{-te^y+1}$.

所以$\dfrac{dy}{dx}=\dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}=\dfrac{\dfrac{e^y}{1-te^y}}{\sqrt{3}}=\dfrac{e^y}{\sqrt{3}(1-te^y)}$.

当 $x=-1$ 时,$t=0$, $y=0$, $\left.\dfrac{dy}{dx}\right|_{t=0}=\dfrac{1}{\sqrt{3}}=\dfrac{\sqrt{3}}{3}$,

$$\frac{d^2y}{dx^2}=\frac{dy'}{dx}=\frac{dy'}{dt}\cdot\frac{dt}{dx}=\left[\frac{e^y}{\sqrt{3}(1-te^y)}\right]'_t\cdot\frac{1}{\sqrt{3}}=\frac{1}{3}\left(\frac{e^y}{1-te^y}\right)'_t$$

$$=\frac{1}{3}\cdot\frac{e^y\cdot y'_t(1-te^y)+(e^y+te^y\cdot y'_t)e^y}{(1-te^y)^2}.$$

将 $t=0$, $y=0$, $y'_t=1$ 代入:

$$\left.\frac{d^2y}{dx^2}\right|_{t=0}=\frac{1}{3}\cdot\frac{1\times(1-0)+(1+0)\times1}{(1-0)^2}=\frac{2}{3}.$$

强化篇

1. $x'_t=6t+2$,方程 $e^y\sin t-y+1=0$ 两边同时对 t 求导:

$$e^y\cdot y'_t\sin t+e^y\cdot\cos t-y'_t=0,\ y'_t=\frac{e^y\cos t}{1-e^y\cdot\sin t},\ \left.y'_t\right|_{t=0}=e,$$

则 $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y'_t}{x'_t}=\frac{\dfrac{\mathrm{e}^y\cos t}{1-\mathrm{e}^y\cdot\sin t}}{6t+2}=\frac{\mathrm{e}^y\cdot\cos t}{(6t+2)(1-\mathrm{e}^y\cdot\sin t)}.$$

则 $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y'}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}$$

$$=\frac{(\mathrm{e}^y y'_t\cos t-\mathrm{e}^y\sin t)(6t+2)(1-\mathrm{e}^y\sin t)}{(6t+2)^2(1-\mathrm{e}^y\sin t)^2(6t+2)}-$$

$$\frac{\mathrm{e}^y\cos t[6(1-\mathrm{e}^y\sin t)+(6t+2)(-\mathrm{e}^y y'_t\sin t-\mathrm{e}^y\cos t)]}{(6t+2)^2(1-\mathrm{e}^y\sin t)^2(6t+2)}.$$

将 $t=0$ 代入得 $\left.\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{t=0}=\dfrac{2\mathrm{e}^2-3\mathrm{e}}{4}$.

2. $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(\mathrm{e}^t\cos t)'_t}{(\mathrm{e}^t\sin t)'_t}=\frac{\mathrm{e}^t\cos t-\mathrm{e}^t\sin t}{\mathrm{e}^t\sin t+\mathrm{e}^t\cos t}$$

$$=\frac{\cos t-\sin t}{\sin t+\cos t},$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y'}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\left(\frac{\cos t-\sin t}{\sin t+\cos t}\right)'\cdot\frac{1}{\mathrm{e}^t(\sin t+\cos t)}$$

$$=\frac{(-\cos t-\sin t)\cdot(\sin t+\cos t)-(\cos t-\sin t)^2}{\mathrm{e}^t\cdot(\sin t+\cos t)^2\cdot(\sin t+\cos t)}$$

$$=\frac{-(\sin t+\cos t)^2-(\cos t-\sin t)^2}{\mathrm{e}^t\cdot(\sin t+\cos t)^3}.$$

$$\text{左边}=-\mathrm{e}^{2t}(\sin t+\cos t)^2\cdot\frac{(\sin t+\cos t)^2+(\cos t-\sin t)^2}{\mathrm{e}^t\cdot(\sin t+\cos t)^3}$$

$$=-\mathrm{e}^t\frac{(\sin t+\cos t)^2+(\cos t-\sin t)^2}{\sin t+\cos t}.$$

$$=-\mathrm{e}^t\frac{2+2\sin t\cos t-2\sin t\cos t}{\sin t+\cos t}$$

$$=\frac{-2\mathrm{e}^t}{\sin t+\cos t},$$

$$\text{右边}=2\left(\mathrm{e}^t\sin t\frac{\cos t-\sin t}{\sin t+\cos t}-\mathrm{e}^t\cos t\right)$$

$$=2\mathrm{e}^t\left[\frac{\sin t(\cos t-\sin t)-\cos t(\sin t+\cos t)}{\sin t+\cos t}\right]$$

$$=-2\mathrm{e}^t\frac{1}{\sin t+\cos t}=\frac{-2\mathrm{e}^t}{\sin t+\cos t}$$

$=$ 左边.

3. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(t^2\mathrm{e}^{2t})'}{[(t^2+1)\mathrm{e}^t]'}=\dfrac{2t\mathrm{e}^{2t}+2t^2\mathrm{e}^{2t}}{2t\mathrm{e}^t+(t^2+1)\mathrm{e}^t}=\dfrac{2t\mathrm{e}^t+2t^2\mathrm{e}^t}{2t+t^2+1}$

$$=\frac{2t\mathrm{e}^t(1+t)}{(1+t)^2}=\frac{2t\mathrm{e}^t}{1+t},$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y'}{\mathrm{d}x}=\frac{\mathrm{d}y'}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{2(1+t)^2\mathrm{e}^t-2t\mathrm{e}^t}{(1+t)^2[(t^2+1)\mathrm{e}^t]'}$$

$$=\frac{2\mathrm{e}^t[(1+t)^2-t]}{(1+t)^2[(t^2+1)\mathrm{e}^t]'},$$

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{t=0}=\left.\frac{2\mathrm{e}^t(1+t+t^2)}{(1+t)^2(t+1)^2\mathrm{e}^t}\right|_{t=0}=\left.\frac{2(1+t+t^2)}{(1+t)^4}\right|_{t=0}=\frac{2(1+0+0)}{(1+0)^4}=2.$$

4. 由 $r=2(1-\cos\theta)$,则 $\begin{cases}x=r\cos\theta=2(1-\cos\theta)\cos\theta,\\ y=r\sin\theta=2(1-\cos\theta)\sin\theta,\end{cases}$ 则

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y'_\theta}{x'_\theta}=\frac{2\sin^2\theta+2(1-\cos\theta)\cos\theta}{2\sin\theta\cos\theta-2(1-\cos\theta)\sin\theta},\ \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{\pi}{2}}=\frac{2\times1+0}{0-2\times1}=-1.$$

当 $\theta=\dfrac{\pi}{2}$ 时,$x=0$, $y=2$,故 $y=-x+2$ 为其切线方程.

5. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{[a(\sin t-t\cos t)]'_t}{[a(\cos t+t\ \sin t)]'_t}=\dfrac{a(\cos t-\cos t+t\sin t)}{a\ (-\sin t+\sin t+t\cos t)}$

$$=\frac{t\sin t}{t\cos t}=\tan t,$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y'}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=(\tan t)'\frac{1}{at\cos t}=\frac{1}{at\cos^3 t}.$$

6. $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y'_t}{x'_t}=\dfrac{\left(\arccos\dfrac{1}{\sqrt{1+t^2}}\right)'_t}{\left(\arcsin\dfrac{t}{\sqrt{1+t^2}}\right)'_t}$

$$=\frac{-\dfrac{1}{\sqrt{1-\dfrac{1}{1+t^2}}}\cdot\dfrac{t}{\sqrt{1+t^2}}\cdot\left(-\dfrac{1}{1+t^2}\right)}{\dfrac{1}{\sqrt{1-\dfrac{t^2}{1+t^2}}}\cdot\dfrac{\sqrt{1+t^2}-\dfrac{t^2}{\sqrt{1+t^2}}}{1+t^2}}=1.$$

7. 由 $x^x+tx-t^2=0$,方程两边同时对 t 求导:

$$x^x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\ln x+x\cdot\frac{1}{x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}\right)+x+t\frac{\mathrm{d}x}{\mathrm{d}t}-2t=0,$$

$$x^x(\ln x+1)\frac{\mathrm{d}x}{\mathrm{d}t}+t\frac{\mathrm{d}x}{\mathrm{d}t}=2t-x,$$

则
$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{2t-x}{x^x(\ln x+1)+t}.$$

由 $\arctan(ty)=\ln(1+t^2y)$，方程两边同时对 t 求导：

$$\frac{1}{1+(ty)^2}\left(y+t\frac{\mathrm{d}y}{\mathrm{d}t}\right)=\frac{1}{1+t^2y}\left(2ty+t^2\frac{\mathrm{d}y}{\mathrm{d}t}\right),$$

$$\frac{y}{1+(ty)^2}+\frac{t}{1+(ty)^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{2ty}{1+t^2y}+\frac{t^2}{1+t^2y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t},$$

$$\left[\frac{t}{1+(ty)^2}-\frac{t^2}{1+t^2y}\right]\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{2ty}{1+t^2y}-\frac{y}{1+(ty)^2},$$

则
$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{2ty[1+(ty)^2]-y(1+t^2y)}{t(1+t^2y)-t^2[1+(ty)^2]}.$$

故
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\Big/\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{2ty[1+(ty)^2]-y(1+t^2y)}{t(1+t^2y)-t^2[1+(ty)^2]}\cdot\frac{x^x(\ln x+1)+t}{2t-x}.$$

8. 切线斜率：

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(\frac{3}{2t^2}+\frac{1}{2t}\right)'}{\left(\frac{1+t}{t^3}\right)'}=\frac{\left(\frac{3}{2}\frac{1}{t^2}+\frac{1}{2}\frac{1}{t}\right)'}{\left(\frac{1}{t^3}+\frac{1}{t^2}\right)'}=\frac{-3\frac{1}{t^3}-\frac{1}{2}\frac{1}{t^2}}{\frac{-3}{t^4}-\frac{2}{t^3}},$$

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=1}=\frac{-3-\frac{1}{2}}{-3-2}=\frac{7}{10},$$

则法线斜率 $-\dfrac{1}{\mathrm{d}y/\mathrm{d}x}=-\dfrac{10}{7}$.

当 $t=1$ 时，$x=2$，$y=\dfrac{3}{2}+\dfrac{1}{2}=2$，故切线方程为 $y-2=\dfrac{7}{10}(x-2)$，法线方程为 $y-2=-\dfrac{10}{7}(x-2)$.

真题自测

(1992) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\dfrac{f'(\mathrm{e}^{3t}-1)\cdot 3\mathrm{e}^{3t}}{f'(t)}$,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=0}=3.$$

(1994) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=t$,故 $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=\sqrt{\frac{\pi}{2}}}=\sqrt{\dfrac{\pi}{2}}$,

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{t=\sqrt{\frac{\pi}{2}}}=\frac{\mathrm{d}y'}{\mathrm{d}t}\cdot\left.\frac{\mathrm{d}t}{\mathrm{d}x}\right|_{t=\sqrt{\frac{\pi}{2}}}=-\left.\frac{1}{2t\sin t^2}\right|_{t=\sqrt{\frac{\pi}{2}}}=-\frac{1}{\sqrt{2\pi}}.$$

(1994) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\dfrac{3t^2+2t}{1-\dfrac{1}{1+t}}=3t^2+5t+2$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y'}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=(6t+5)\frac{1}{1-\dfrac{1}{1+t}}=\frac{1}{t}(6t+5)(t+1).$$

(1996) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=4tf'(t^2)$,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y'}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{4[f'(t^2)+2t^2f''(t^2)]}{f(t^2)}.$$

(1997) $\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{1}{1+t^2}$.

$2\dfrac{\mathrm{d}y}{\mathrm{d}t}-y^2-2ty\cdot\dfrac{\mathrm{d}y}{\mathrm{d}t}+\mathrm{e}^t=0$,得$\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{y^2-\mathrm{e}^t}{2(1-ty)}$,则

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\frac{(y^2-\mathrm{e}^t)(1+t^2)}{2(1-ty)}.$$

(2003) $\dfrac{\mathrm{d}x}{\mathrm{d}t}=4t$, $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\mathrm{e}^{1+2\ln t}}{1+2\ln t}\cdot\dfrac{2}{t}=\dfrac{2\mathrm{e}t}{1+2\ln t}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}=\frac{\mathrm{e}}{2(1+2\ln t)},$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}y'}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{\mathrm{e}}{2}\cdot\frac{-1}{(1+2\ln t)^2}\cdot\frac{2}{t}\cdot\frac{1}{4t}=-\frac{\mathrm{e}}{4t^2(1+2\ln t)^2}.$$

当 $x=9$ 时,$t=2$, $\left.\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{t=2}=-\dfrac{\mathrm{e}}{16(1+2\ln 2)^2}$.

(2010) 解法 1　因为

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\dfrac{\mathrm{d}y}{\mathrm{d}t}}{\dfrac{\mathrm{d}x}{\mathrm{d}t}}=\frac{\ln(1+t^2)}{-\mathrm{e}^{-t}},\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\ln(1+t^2)}{-\mathrm{e}^{-t}}\right]\frac{1}{\dfrac{\mathrm{d}x}{\mathrm{d}t}}$$

$$=\mathrm{e}^{2t}\left[\frac{2t}{1+t^2}+\ln(1+t^2)\right],$$

所以$\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{t=0}=0$.

解法 2　由$\begin{cases} x=\mathrm{e}^{-t}, \\ y=\int_0^t \ln(1+u^2)\mathrm{d}u \end{cases}$得$y=\int_0^{-\ln x}\ln(1+u^2)\mathrm{d}u$,所以

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1}{x}\ln(1+\ln^2 x)$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\dfrac{1}{x^2}\left[\ln(1+\ln^2 x)-\dfrac{2\ln x}{1+\ln^2 x}\right]$.

又当$t=0$时,$x=1$,所以$\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{t=0}=0$.

(2013) 因为$\dfrac{\mathrm{d}x}{\mathrm{d}t}=\cos t$, $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\sin t+t\cos t-\sin t=t\cos t$,所以$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\dfrac{\mathrm{d}y}{\mathrm{d}t}}{\dfrac{\mathrm{d}x}{\mathrm{d}t}}=$

t, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{1}{\cos t}$,从而有$\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{t=\frac{\pi}{4}}=\dfrac{1}{\cos\dfrac{\pi}{4}}$

$=\sqrt{2}$.

(2015) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\dfrac{\mathrm{d}y}{\mathrm{d}t}}{\dfrac{\mathrm{d}x}{\mathrm{d}t}}=\dfrac{3+3t^2}{\dfrac{1}{1+t^2}}=3(1+t^2)^2$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\dfrac{\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)}{\dfrac{\mathrm{d}x}{\mathrm{d}t}}=\dfrac{12t(1+t^2)}{\dfrac{1}{1+t^2}}=$

$12t(1+t^2)^2$,于是$\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{t=1}=48$.

(2017) 由$\dfrac{\mathrm{d}x}{\mathrm{d}t}=1+\mathrm{e}^t$, $\dfrac{\mathrm{d}y}{\mathrm{d}t}=\cos t$,知$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\dfrac{\mathrm{d}y}{\mathrm{d}t}}{\dfrac{\mathrm{d}x}{\mathrm{d}t}}=\dfrac{\cos t}{1+\mathrm{e}^t}$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)=$

$\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot\dfrac{\mathrm{d}t}{\mathrm{d}x}$

$=\dfrac{\dfrac{-\sin t(1+\mathrm{e}^t)-\mathrm{e}^t\cos t}{(1+\mathrm{e}^t)^2}}{1+\mathrm{e}^t}=\dfrac{-\sin t-\mathrm{e}^t\sin t-\mathrm{e}^t\cos t}{(1+\mathrm{e}^t)^3}$,从而有

$$\left.\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{t=0}=\frac{-1}{(1+1)^3}=-\frac{1}{8}.$$

(2020) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\dfrac{\mathrm{d}y}{\mathrm{d}t}}{\dfrac{\mathrm{d}x}{\mathrm{d}t}}=\dfrac{\dfrac{1}{\sqrt{t^2+1}}}{\dfrac{t}{\sqrt{t^2+1}}}=\dfrac{1}{t}$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\left(\dfrac{1}{t}\right)'_x=\dfrac{-\dfrac{1}{t^2}}{\dfrac{t}{\sqrt{t^2+1}}}=$

$-\dfrac{\sqrt{t^2+1}}{t^3}$,所以$\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{t=1}=-\sqrt{2}$.

(2021) 由参数方程求导公式,得$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{4t\mathrm{e}^t+2t}{2\mathrm{e}^t+1}$,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\left(\frac{4t\mathrm{e}^t+2t}{2\mathrm{e}^t+1}\right)'_t\frac{\mathrm{d}t}{\mathrm{d}x}$$

$$=\frac{(4\mathrm{e}^t+4t\mathrm{e}^t+2)(2\mathrm{e}^t+1)-(4t\mathrm{e}^t+2t)2\mathrm{e}^t}{(2\mathrm{e}^t+1)^3},$$

直接代入 $t=0$,得$\left.\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right|_{t=0}=\dfrac{2}{3}$.

§2.6 分段函数求导

基础篇

1. $y=\mathrm{e}^{|x|}=\begin{cases}\mathrm{e}^x, & x\geqslant 0,\\ \mathrm{e}^{-x}, & x<0,\end{cases}$则 $y=f(x)$ 在 $x=0$ 处:

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{\mathrm{e}^x-1}{x}=1,$$

$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{\mathrm{e}^{-x}-1}{x}=-1,$$

故 $y=\mathrm{e}^{|x|}$ 在 $x=0$ 处不可导.

2. 当 $x>0$ 时,$f'(x)=\dfrac{1}{1+x}$;当 $x<0$ 时,$f'(x)=1$;

当 $x=0$ 时,$f'_+(0)=\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0^+}\dfrac{\ln(1+x)-0}{x}=1$,

$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{x-0}{x}=1.$$

故 $f'(x)=\begin{cases}\dfrac{1}{1+x}, & x>0,\\ 1, & x=0,\\ 1, & x<0,\end{cases}$ 即 $f'(x)=\begin{cases}\dfrac{1}{1+x}, & x>0,\\ 1, & x\leqslant 0.\end{cases}$

3. 连续性：$\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}x^2\sin\dfrac{1}{x}=0=f(0)$，

则 $f(x)$ 在 $x=0$ 处连续.

可导性：$f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{x^2\sin\frac{1}{x}-0}{x}=0$，

故 $f(x)$ 在 $x=0$ 处连续且可导.

4. 由 $f(x)=x\,|\,x(x-2)\,|$ 知，$f(x)$ 在 $x=2$ 不可导.

$$f(x)=\begin{cases}x^2(x-2), & x<0,\\ x^2(2-x), & 0\leqslant x<2,\\ x^2(x-2), & x\geqslant 2.\end{cases}$$

则当 $x<0$ 时，$f'(x)=3x^2-4x$；

当 $0<x<2$ 时，$f'(x)=4x-3x^2$；

当 $x>2$ 时，$f'(x)=3x^2-4x$.

而当 $x=0$ 时，$f'_+(0)=\lim\limits_{x\to 0^+}\dfrac{x^2(2-x)}{x}=0$，

$$f'_-(0)=\lim_{x\to 0^-}\frac{x^2(x-2)}{x}=0,$$

故 $f'(0)=0$.

综上所述，$f(x)=\begin{cases}3x^2-4x, & x<0,\\ 4x-3x^2, & 0\leqslant x<2,\\ 3x^2-4x, & x>2.\end{cases}$

5. $f(x)$ 在 $x=0$ 连续：$\lim\limits_{x\to 0^+}(a+be^x)=a+b=1$.

$f(x)$ 在 $x=0$ 可导：$f'_+(0)=\lim\limits_{x\to 0^+}\dfrac{a+be^x-1}{x}$，

$f'_-(0)=\lim\limits_{x\to 0^-}\dfrac{1+\ln(1-4x)-1}{x}=-4$，

则 $\lim\limits_{x\to 0^+}\dfrac{a+be^x-1}{x}=\lim\limits_{x\to 0^+}\dfrac{b(e^x-1)+a+b-1}{x}$

$$=\lim_{x\to 0^+}\frac{b(e^x-1)}{x}\quad(a+b=1)$$

$=b=-4$,故 $a=5$, $b=-4$, $f'(0)=-4$.

6. ① 当 $x\neq 0$ 时,$g'(x)=\dfrac{xf'(x)-f(x)}{x^2}$,则 $g'(x)$ 在 $(-\infty,0)\cup(0,+\infty)$ 均连续.

② 当 $x=0$ 时,

$$g'(0)=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}$$

$$=\lim_{x\to 0}\frac{\dfrac{f(x)}{x}-f'(0)}{x}=\lim_{x\to 0}\frac{f(x)-xf'(0)}{x^2}$$

$$\overset{L'}{=\!=}\lim_{x\to 0}\frac{f'(x)-f'(0)}{2x}=\frac{f''(0)}{2}.$$

又

$$\lim_{x\to 0}g'(x)=\lim_{x\to 0}\frac{xf'(x)-f(x)}{x^2}$$

$$\overset{L'}{=\!=}\lim_{x\to 0}\frac{xf''(x)}{2x}$$

$$=\frac{f''(0)}{2}=g'(0),$$

故 $g'(x)$ 在 $(-\infty,+\infty)$ 内连续.

7. $f(x)=\begin{cases}\sin^3 x, & 0<x<1,\\ 0. & x=0,\\ -\sin^3 x, & -1<x<0.\end{cases}$

由 $f'_+(0)=\lim\limits_{x\to 0^+}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0^+}\dfrac{\sin^3 x}{x}=0$,

$f'_-(0)=\lim\limits_{x\to 0^-}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to 0^-}\dfrac{-\sin^3 x}{x}=0$,

得 $f'(0)=0$,所以

$$f'(x)=\begin{cases}3\sin^2 x\cos x, & 0<x<1,\\ 0, & x=0,\\ -3\sin^2 x\cos x, & -1<x<0.\end{cases}$$

又

$$f''_+(0)=\lim_{x\to 0+}\frac{f'(x)-f'(0)}{x}=\lim_{x\to 0+}\frac{3\sin^2 x\cos x}{x}=0,$$

$$f''_-(0)=\lim_{x\to 0-}\frac{f'(x)-f'(0)}{x}=\lim_{x\to 0-}\frac{-3\sin^2 x\cos x}{x}=0,$$

故

$$f''(x)=\begin{cases}6\sin x\cos^2 x-3\sin^3 x, & 0<x<1,\\ 0, & x=0,\\ -6\sin x\cos^2 x+3\sin^3 x, & -1<x<0.\end{cases}.$$

强化篇

1. (1) $f(x)$ 在 $x=0$ 连续：$\lim\limits_{x\to 0-}f(x)=\lim\limits_{x\to 0-}(a+x+\sqrt{1-x})=a+1=1$.

$f(x)$ 在 $x=0$ 可导：

$$f'_+(0)=\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0+}\frac{1+b\ln(1+x)-1}{x}=b,$$

$$f'_-(0)=\lim_{x\to 0-}\frac{a+x+\sqrt{1-x}-1}{x}=\lim_{x\to 0-}\frac{x+\sqrt{1-x}-1}{x}$$

$$=\frac{1}{2}\quad(a=0),$$

故 $a=0$，$b=\dfrac{1}{2}$，$f'(0)=\dfrac{1}{2}$.

$$f'(x)=\begin{cases}1-\dfrac{1}{2\sqrt{1-x}}, & x\leqslant 0,\\ \dfrac{1}{2}\cdot\dfrac{1}{1+x}, & x>0.\end{cases}$$

(2) 当 $x=0$ 时，$y=f(0)=1$，$f'(0)=\dfrac{1}{2}$.

故切线方程为 $y-1=\dfrac{1}{2}(x-0)$，即 $y=\dfrac{1}{2}x+1$；

法线方程为 $y-1=-2(x-0)$，即 $y=-2x+1$.

2. 由 $f(x)$ 在 $(-\infty,+\infty)$ 连续，则只要证 $F(x)$ 在 $x=0$ 连续.

$$\begin{aligned}\lim_{x\to 0}F(x)&=\lim_{x\to 0}\frac{f(x)-e^x}{x}=\lim_{x\to 0}\frac{f(x)-f(0)+f(0)-e^x}{x}\\&=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}+\lim_{x\to 0}\frac{1-e^x}{x}\quad[f(0)=1]\\&=f'(0)-1=1-1=0=F(0),\end{aligned}$$

故 $F(x)$ 在 $(-\infty,+\infty)$ 连续.

当 $x\neq 0$ 时，$F'(x)=\dfrac{[f'(x)-e^x]x-[f(x)-e^x]}{x^2}$.

$$\begin{aligned}\text{当 } x=0 \text{ 时，}F'(0)&=\lim_{x\to 0}\frac{F(x)-F(0)}{x-0}=\lim_{x\to 0}\frac{\dfrac{f(x)-e^x}{x}-0}{x}\\&=\lim_{x\to 0}\frac{f(x)-e^x}{x^2}\overset{L'}{=}\lim_{x\to 0}\frac{f'(x)-e^x}{2x}\\&=\frac{1}{2}\lim_{x\to 0}\frac{f'(x)-f'(0)+f'(0)-e^x}{x}\\&=\frac{1}{2}\left[\lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0}+\lim_{x\to 0}\frac{1-e^x}{x}\right]\\&=\frac{1}{2}[f''(0)-1].\end{aligned}$$

$$\begin{aligned}\text{由}\lim_{x\to 0}F'(x)&=\lim_{x\to 0}\frac{[f'(x)-e^x]x-[f(x)-e^x]}{x^2}\\&=\lim_{x\to 0}\frac{f'(x)-e^x}{x}-\lim_{x\to 0}\frac{f(x)-e^x}{x^2}\\&=f''(0)-1-\left[\frac{1}{2}f''(0)-\frac{1}{2}\right]\\&=\frac{1}{2}f''(0)-\frac{1}{2}=\frac{1}{2}[f''(0)-1]\\&=F'(0),\end{aligned}$$

故 $F'(x)$ 连续，$F(x)$ 有一阶连续导数.

3. 当 $x>0$ 时，$f'(x)=\alpha x^{\alpha-1}\cos\dfrac{1}{x^\beta}+x^\alpha\left(-\sin\dfrac{1}{x^\beta}\right)(-\beta)x^{-\beta-1}$；

当 $x=0$ 时，$f'(0)=\lim\limits_{x\to 0^+}\dfrac{x^{\alpha}\cos\dfrac{1}{x^{\beta}}}{x}=\lim\limits_{x\to 0^+}x^{\alpha-1}\cos\dfrac{1}{x^{\beta}}=0.$　(1)

由(1)知 $\alpha-1>0$，$\alpha>1$.

$$f'(x)=\begin{cases}\alpha x^{\alpha-1}\cos\dfrac{1}{x^{\beta}}+\beta x^{\alpha}\sin\dfrac{1}{x^{\beta}}\cdot x^{-\beta-1}, & x>0,\\ 0, & x\leqslant 0.\end{cases}$$

由 $f'(x)$ 连续，

$$\lim_{x\to 0}\left(\alpha x^{\alpha-1}\cos\frac{1}{x^{\beta}}+\beta x^{\alpha-\beta-1}\sin\frac{1}{x^{\beta}}\right)=0,$$

则 $\alpha-1>0$，$\alpha-\beta-1>0$，故 $\alpha-\beta>1$，故选(A).

4. ① 当 $x>0$ 时，$f'(x)=2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}$.

② 当 $x<0$ 时，$f'(x)=2\sin x^2-\dfrac{1-\cos x^2}{x^2}$.

③ 当 $x=0$ 时，

$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{1-\cos x^2}{x^2}=0,$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{x^2\sin\dfrac{1}{x}}{x}=0,$$

所以

$$f'(x)=\begin{cases}2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}, & x>0,\\ 0, & x=0,\\ 2\sin x^2-\dfrac{1-\cos x^2}{x^2}, & x<0.\end{cases}$$

因为

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-}\left(2\sin x^2 - \frac{1-\cos x^2}{x^2}\right) = 0,$$

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right) \text{极限不存在},$$

故 $f'(x)$ 在 $x=0$ 处不连续.

5. 因为 $f_-(0)=f_+(0)=0$，$f(0)=c$，所以 $c=0$.

由

$$f'_-(0) = \lim_{x\to 0^-}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^-}\frac{ax^2+b\sin x-0}{x} = b,$$

$$f'_+(0) = \lim_{x\to 0^+}\frac{f(x)-f(0)}{x} = \lim_{x\to 0^+}\frac{\ln(1+x)-0}{x} = 1,$$

所以 $b=1$，且有

$$f'(x) = \begin{cases} 2ax+\cos x, & x<0, \\ 1, & x=0, \\ \dfrac{1}{1+x}, & x>0. \end{cases}$$

又因为 $\lim\limits_{x\to 0} f'(x) = f'(0) = 1$，所以当 $b=1$，$c=0$ 时 $f'(x)$ 在 $x=0$ 处连续.

又

$$f''(0) = \lim_{x\to 0}\frac{f'(x)-f'(0)}{x} = \begin{cases} \lim\limits_{x\to 0^-}\dfrac{2ax+\cos x-1}{x} = 2a, \\ \lim\limits_{x\to 0^+}\dfrac{\dfrac{1}{1+x}-1}{x} = -1, \end{cases}$$

则当 $2a\neq -1$，即 $a\neq -\dfrac{1}{2}$ 时，$f(x)$ 在 $x=0$ 处二阶不可导.

综上，$a\neq -\dfrac{1}{2}$，$b=1$，$c=0$ 为所求之值.

6. 由 $\lim\limits_{x\to 0}\dfrac{f(x)}{x} = A$，且 $f(x)$ 连续可知 $f(0)=0$，$\varphi(0)=0$.

令 $xt=u$，$x\neq 0$，则 $\varphi(x) = \dfrac{1}{x}\displaystyle\int_0^x f(u)\mathrm{d}u$.

① 当 $x \neq 0$ 时，$\varphi'(x) = \dfrac{xf(x) - \int_0^x f(u)\mathrm{d}u}{x^2}$.

② 当 $x = 0$ 时，$\varphi'(0) = \lim\limits_{x \to 0} \dfrac{\varphi(x) - \varphi(0)}{x} = \lim\limits_{x \to 0} \dfrac{\int_0^x f(u)\mathrm{d}u}{x^2} \overset{L'}{=\!=} \dfrac{A}{2}$.

所以

$$\varphi'(x) = \begin{cases} \dfrac{xf(x) - \int_0^x f(u)\mathrm{d}u}{x^2}, & x \neq 0, \\ \dfrac{A}{2}, & x = 0. \end{cases}$$

又

$$\begin{aligned} \lim_{x \to 0} \varphi'(x) &= \lim_{x \to 0} \frac{xf(x) - \int_0^x f(u)\mathrm{d}u}{x^2} \\ &= \lim_{x \to 0} \frac{f(x)}{x} - \lim_{x \to 0} \frac{\int_0^x f(u)\mathrm{d}u}{x^2} \\ &= A - \lim_{x \to 0} \frac{f(x)}{2x} \\ &= \frac{A}{2} = \varphi'(0), \end{aligned}$$

故 $\varphi'(x)$ 在 $x = 0$ 处连续.

7. 去绝对值得

$$f(x) = \begin{cases} \ln x, & x \geqslant 1, \\ -\ln x, & 0 < x < 1, \\ -\ln(-x), & -1 < x < 0, \\ \ln(-x), & x \leqslant -1. \end{cases}$$

显然，当 $x = 0$ 时，$f(x)$ 无定义.

① 当 $x > 1$ 时，$f'(x) = \dfrac{1}{x}$.

② 当 $0 < x < 1$ 时，$f'(x) = -\dfrac{1}{x}$.

③ 当 $x = 1$ 时，

$$f'_-(1)=\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^-}\frac{-\ln x-0}{x-1}$$
$$=-1,$$
$$f'_+(1)=\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1^+}\frac{\ln x-0}{x-1}$$
$$=1,$$

可知 $f(x)$ 在 $x=1$ 处导数不存在.

④ 当 $-1<x<0$ 时，$f'(x)=-\dfrac{1}{x}$.

⑤ 当 $x<-1$ 时，$f'(x)=\dfrac{1}{x}$.

⑥ 当 $x=-1$ 时，

$$f'_-(-1)=\lim_{x\to-1^-}\frac{f(x)-f(-1)}{x+1}=\lim_{x\to-1^-}\frac{\ln(-x)-0}{x+1}$$
$$=-1,$$
$$f'_+(-1)=\lim_{x\to-1^+}\frac{f(x)-f(-1)}{x+1}=\lim_{x\to-1^+}\frac{-\ln(-x)-0}{x+1}$$
$$=1,$$

可知 $f(x)$ 在 $x=-1$ 处导数不存在.

故 $f'(x)=\begin{cases}\dfrac{1}{x}, & |x|>1,\\ -\dfrac{1}{x}, & 0<|x|<1.\end{cases}$

§2.7 变限积分函数求导

基础篇

1. $f'(x)=e^{x^2}$.

2. $y'=2x\sqrt{1+x^4}$.

3. $y'=\dfrac{3x^2}{\sqrt{1+x^{12}}}-\dfrac{2x}{\sqrt{1+x^8}}$.

4. $y' = -\sqrt[3]{x}\ln(1+x^2)$.

5. $y = x\int_0^x g(t)\mathrm{d}t - \int_0^x tg(t)\mathrm{d}t$,

$$y' = \int_0^x g(t)\mathrm{d}t + xg(x) - xg(x) = \int_0^x g(t)\mathrm{d}t.$$

6. $\dfrac{\mathrm{d}}{\mathrm{d}b}\int_a^b \sin x^2\mathrm{d}x = \sin b^2$.

7. $F'(x) = -\mathrm{e}^{-x}f(\mathrm{e}^{-x}) - f(x)$.

8. $y' = x^2 f(x) - \sin^2 x f(\sin x)\cdot\cos x$.

9. 由 $y = x^2\int_0^{x^2} f(t)\mathrm{d}t$,则

$$y' = 2x\int_0^{x^2} f(t)\mathrm{d}t + 2x^3 f(x^2).$$

10. $y = x^2\int_0^x f(t)\mathrm{d}t - 2x\int_0^x tf(t)\mathrm{d}t + \int_0^x t^2 f(t)\mathrm{d}t$,

$$y' = 2x\int_0^x f(t)\mathrm{d}t + x^2 f(x) - 2\int_0^x tf(t)\mathrm{d}t - 2x^2 f(x) + x^2 f(x)$$

$$= 2x\int_0^x f(t)\mathrm{d}t - 2\int_0^x tf(t)\mathrm{d}t.$$

强化篇

1. $$\lim_{x\to+\infty}\frac{\int_0^x \frac{\mathrm{d}t}{\sqrt[3]{t^3+5t^2+2}}}{\ln x} \overset{L'}{=\!=} \lim_{x\to+\infty}\frac{\frac{1}{\sqrt[3]{x^3+5x^2+2}}}{\frac{1}{x}} = 1.$$

2. $$\lim_{x\to0}\frac{\int_0^{x^2} t\mathrm{e}^t\sin t\,\mathrm{d}t}{x^6\mathrm{e}^x} = \lim_{x\to0}\frac{\int_0^{x^2} t\mathrm{e}^t\sin t\,\mathrm{d}t}{x^6} \overset{L'}{=\!=} \lim_{x\to0}\frac{x^2\cdot\mathrm{e}^{x^2}\sin x^2\cdot 2x}{6x^5}$$

$$= \lim_{x\to0}\frac{x^2\cdot1\cdot x^2\cdot2x}{6x^5} = \frac{1}{3}.$$

3. $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(\int_{t^2}^1 u^2\ln u\,\mathrm{d}u\right)'_t}{\left(\int_1^{t^2} u\ln u\,\mathrm{d}u\right)'_t} = \frac{-t^4\ln t^2\cdot2t}{t^2\ln t^2\cdot2t} = -t^2.$$

4. $$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^{\cos x}\cos(\pi t^2)\mathrm{d}t - \int_0^{\sin x}\cos(\pi t^2)\mathrm{d}t\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\cos x}\cos(\pi t^2)\mathrm{d}t - \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\sin x}\cos(\pi t^2)\mathrm{d}t$$

$$= \cos[\pi(\cos x)^2](-\sin x) - \cos[\pi(\sin x)^2]\cos x$$

$$= -\cos[\pi(\cos x)^2]\sin x - \cos[\pi(\sin x)^2]\cos x.$$

5. $$\lim_{x\to+\infty}\frac{\mathrm{e}^{-x^2}\int_0^x \sin^2 t\,\mathrm{e}^{t^2}\,\mathrm{d}t}{x} = \lim_{x\to+\infty}\frac{\int_0^x \sin^2 t\,\mathrm{e}^{t^2}\,\mathrm{d}t}{x\,\mathrm{e}^{x^2}} \overset{L'}{=\!=} \lim_{x\to+\infty}\frac{\sin^2 x\,\mathrm{e}^{x^2}}{\mathrm{e}^{x^2}+x\cdot 2x\,\mathrm{e}^{x^2}}$$

$$= \lim_{x\to+\infty}\frac{1}{1+2x^2}\sin^2 x = 0.$$

6. $F(x) = \int_0^x (x^2-t^2)f(t)\mathrm{d}t = x^2\int_0^x f(t)\mathrm{d}t - \int_0^x t^2 f(t)\mathrm{d}t$,

$F'(x) = 2x\int_0^x f(t)\mathrm{d}t + x^2 f(x) - x^2 f(x) = 2x\int_0^x f(t)\mathrm{d}t.$

由 $F'(x)$ 与 x^k 是同阶无穷小,得

$$\lim_{x\to 0}\frac{F'(x)}{x^k} = c \neq 0,$$

而 $$\lim_{x\to 0}\frac{2x\int_0^x f(t)\mathrm{d}t}{x^k} = \lim_{x\to 0}\frac{2\int_0^x f(t)\mathrm{d}t}{x^{k-1}} \overset{L'}{=\!=} \lim_{x\to 0}\frac{2f(x)}{(k-1)x^{k-2}}.$$

由 $f(0)=0$, $f'(0)\neq 0$, 知$\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}$ 存在,也即$\lim\limits_{x\to 0}\frac{f(x)}{x}$ 存在.若要$\lim\limits_{x\to 0}\frac{2f(x)}{(k-1)x^{k-2}}$ 存在,则要 $k-2=1$,即 $k=3$.

7. $F(x) = \int_0^x (2t-x)f(t)\mathrm{d}t = \int_0^x 2tf(t)\mathrm{d}t - x\int_0^x f(t)\mathrm{d}t$,

$F'(x) = 2xf(x) - \int_0^x f(t)\mathrm{d}t - xf(x) = xf(x) - \int_0^x f(t)\mathrm{d}t$,

$F''(x) = f(x) + xf'(x) - f(x) = xf'(x).$

令 $F''(x)=0$,即 $xf'(x)=0$.由于 $f'(x)>0$,因此 $x=0$.

当 $x>0$ 时,$F''(x)>0$,凹;

当 $x<0$ 时,$F''(x)<0$,凸.

故$(0, F(0))$ 为拐点,得证.

8. $$\int_{t^2}^{t} xf(t^2-x^2)\mathrm{d}x \xlongequal[\mathrm{d}u=-2x\mathrm{d}x]{u=t^2-x^2} -\frac{1}{2}\int_{t^2-t^4}^{0} f(u)\mathrm{d}u$$

$$= \frac{1}{2}\int_0^{t^2-t^4} f(u)\mathrm{d}u,$$

故$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t^2}^{t} xf(t^2-x^2)\mathrm{d}x=\frac{1}{2}[f(t^2-t^4)\cdot(2t-4t^3)]$

$$=(t-2t^3)f(t^2-t^4).$$

9. $\lim\limits_{x\to 0}\frac{f(x)}{x}=2$，$f(x)$续连$\Rightarrow f(0)=0$，$f'(0)=2$.

$$F'_+(0)=\lim_{x\to 0^+}\frac{F(x)-F(0)}{x-0}=\lim_{x\to 0^+}\frac{\int_0^1 f(tx)\mathrm{d}t-0}{x-0}$$

$$\xlongequal[\mathrm{d}u=x\mathrm{d}t]{\text{令 } u=tx}\lim_{x\to 0^+}\frac{\int_0^x f(u)\frac{1}{x}\mathrm{d}u}{x}$$

$$=\lim_{x\to 0^+}\frac{\int_0^x f(u)\mathrm{d}u}{x^2}\xlongequal{L'}\lim_{x\to 0^+}\frac{f(x)}{2x}=1,$$

$$F'_-(0)=\lim_{x\to 0^-}\frac{F(x)-F(0)}{x-0}=\lim_{x\to 0^-}\frac{\int_0^x \ln(1+2t)\mathrm{d}t}{x^2}$$

$$\xlongequal{L'}\lim_{x\to 0^-}\frac{\ln(1+2x)}{2x}=1.$$

故 $F'(0)=F'_+(0)=F'_-(0)=1$.

真题自测

(1987) $\lim\limits_{x\to 0}\frac{1}{bx-\sin x}\int_0^x\frac{t^2}{\sqrt{a+t^2}}\mathrm{d}t=\lim\limits_{x\to 0}\frac{\frac{x^2}{\sqrt{a+x^2}}}{b-\cos x}$

$$=\lim_{x\to 0}\frac{x^2}{b-\cos x}\cdot\frac{1}{\sqrt{a+x^2}}=\frac{1}{\sqrt{a}}\lim_{x\to 0}\frac{x^2}{b-\cos x}=1,$$

故$\lim\limits_{x\to 0}(b-\cos x)=0$，$b=1$；

$\lim\limits_{x\to 0}\frac{x^2}{1-\cos x}=2$，$a=4$.

(1993) $\lim\limits_{x\to 0}\frac{f(x)}{g(x)}=\lim\limits_{x\to 0}\frac{\int_0^{\sin x}\sin t^2\mathrm{d}t}{x^3+x^4}=\lim\limits_{x\to 0}\frac{\sin(\sin x)^2\cdot\cos x}{3x^2}$

$$=\lim_{x\to 0}\frac{x^2}{3x^2}=\frac{1}{3},$$

故当 $x \to 0$ 时，$f(x)$ 与 $g(x)$ 是同阶但非等价无穷小.选(B).

(1998) $\lim\limits_{x\to 0}(ax-\sin x)=0$ 且 $\lim\limits_{x\to 0}\dfrac{ax-\sin x}{\int_b^x \frac{\ln(1+t^3)}{t}\mathrm{d}t}=c$，

故必有 $\lim\limits_{x\to 0}\int_b^x \dfrac{\ln(1+t^3)}{t}\mathrm{d}t=0$，$b=0$.

$$\lim_{x\to 0}\frac{ax-\sin x}{\int_0^x \frac{\ln(1+t^3)}{t}\mathrm{d}t}=\lim_{x\to 0}\frac{a-\cos x}{\frac{\ln(1+x^3)}{x}}=c,$$

故有 $\lim\limits_{x\to 0}(a-\cos x)=0$，$a=1$.

$$\lim_{x\to 0}\frac{1-\cos x}{\frac{\ln(1+x^3)}{x}}=\lim_{x\to 0}\frac{\frac{1}{2}x^2}{x^2}=\frac{1}{2},\ c=\frac{1}{2}.$$

(2004) 若 $\lim\limits_{x\to 0}\dfrac{\alpha}{x^k}=\lim\limits_{x\to 0}\dfrac{\int_0^x \cos t^2\mathrm{d}t}{x^k}=\lim\limits_{x\to 0}\dfrac{\cos x^2}{kx^{k-1}}$ 存在，则 $k=1$，故 α 与 x 同阶.

若 $\lim\limits_{x\to 0}\dfrac{\beta}{x^m}=\lim\limits_{x\to 0}\dfrac{\int_0^{x^2}\tan\sqrt{t}\,\mathrm{d}t}{x^m}=\lim\limits_{x\to 0}\dfrac{\tan x\cdot 2x}{mx^{m-1}}=\lim\limits_{x\to 0}\dfrac{2x^2}{mx^{m-1}}$ 存在，则 $m-1=2$，故 β 与 x^3 同阶.

若 $\lim\limits_{x\to 0}\dfrac{\gamma}{x^l}=\lim\limits_{x\to 0}\dfrac{\int_0^{\sqrt{x}}\sin t^3\mathrm{d}t}{x^l}=\lim\limits_{x\to 0}\dfrac{\sin(\sqrt{x})^3\cdot\frac{1}{2\sqrt{x}}}{lx^{l-1}}=\lim\limits_{x\to 0}\dfrac{\frac{1}{2}x}{lx^{l-1}}$ 存在，

则 $l-1=1$，故 γ 与 x^2 同阶.

按阶数由低到高，正确的排序为 α，γ，β.

故选(B).

(1990) $F'(x)=f(\mathrm{e}^{-x})\cdot(-\mathrm{e}^{-x})-f(x)=-\mathrm{e}^{-x}f(\mathrm{e}^{-x})-f(x)$，

故选(A).

(1992) $\int_0^{x^2}(x^2-t)f(t)\mathrm{d}t=x^2\int_0^{x^2}f(t)\mathrm{d}t-\int_0^{x^2}tf(t)\mathrm{d}t$，故

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2}(x^2-t)f(t)\mathrm{d}t&=2x\int_0^{x^2}f(t)\mathrm{d}t+x^2f(x^2)\cdot 2x-x^2f(x^2)\cdot 2x\\&=2x\int_0^{x^2}f(t)\mathrm{d}t.\end{aligned}$$

(1995) $\int_{x^2}^{0} x\cos t^2\,\mathrm{d}t = -x\int_{0}^{x^2}\cos t^2\,\mathrm{d}t$,故

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{x^2}^{0} x\cos t^2\,\mathrm{d}t = -\int_{0}^{x^2}\cos t^2\,\mathrm{d}t - 2x^2\cos x^4.$$

(1998) 令 $u = x^2 - t^2$，则 $\int_0^x tf(x^2-t^2)\mathrm{d}t = \int_{x^2}^{0}\left[-\frac{1}{2}f(u)\right]\mathrm{d}u = \frac{1}{2}\int_0^{x^2} f(u)\mathrm{d}u$,故

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x tf(x^2-t^2)\mathrm{d}t = xf(x^2).$$

(1999) 令 $u = x - t$，$\int_0^x \sin(x-t)^2\mathrm{d}t = \int_0^x \sin u^2\mathrm{d}u$,故

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \sin(x-t)^2\mathrm{d}t = \sin x^2.$$

§2.8　高阶导数

基础篇

1. 方法一:归纳法.$y = (2x+3)^{-1}$,

$$y' = -1\cdot(2x+3)^{-2}\cdot 2,$$
$$y'' = -1\cdot(-2)(2x+3)^{-3}\cdot 2^2,$$
$$\cdots\cdots$$
$$y^{(n)} = (-1)^n\cdot n!\,(2x+3)^{-(n+1)}\cdot 2^n,$$

所以 $y^{(n)}(0) = \frac{1}{3}(-1)^n n!\left(\frac{2}{3}\right)^n$.

方法二:由 $\left(\frac{1}{ax+b}\right)^{(n)} = \frac{(-1)^n a^n n!}{(ax+b)^{n+1}}$,得

$$y^{(n)}(0) = \frac{1}{3}(-1)^n n!\left(\frac{2}{3}\right)^n.$$

2. $y = \frac{1}{2}\left(\frac{1}{1-x} - \frac{1}{1+x}\right)$,所以 $y^{(n)} = \frac{1}{2}\left[\frac{n!}{(1-x)^{n+1}} - \frac{(-1)^n n!}{(1+x)^{n+1}}\right]$.

3. $y = 4 + \frac{3}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$,所以

$$y^{(n)}=\frac{3}{2}\cdot(-1)^n n!\left[\frac{1}{(x-1)^{n+1}}-\frac{1}{(x+1)^{n+1}}\right].$$

4. $f'(x)=4\sin^3 x\cdot\cos x-4\cos^3 x\cdot\sin x$

$=4\sin x\cos x(\sin^2 x-\cos^2 x)$

$=2\sin 2x(-\cos 2x)=-\sin 4x.$

由公式得

$$f^{(n)}(x)=4^{n-1}\cdot\cos\left(4x+\frac{n\pi}{2}\right).$$

5. 方法一:应用莱布尼茨公式,并注意到幂函数 x^n 在 $x=0$ 处为 0,于是

$$y^{(10)}(0)=C_{10}^3(x^3)'''(\sin x)^7\Big|_{x=0}=\frac{10\times9\times8}{3!}\times3!\ \sin\left(x+\frac{7\pi}{2}\right)\Big|_{x=0}$$

$=-720.$

方法二:泰勒公式.

当 $x=0$ 时,

$$y=x^3\sin x=x^3\left[x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+o(x^8)\right]$$

$$=x^4-\frac{1}{3!}x^6+\frac{1}{5!}x^8-\frac{1}{7!}x^{10}+o(x^{11}).\qquad ①$$

又由于当 $x=0$ 时,$f(x)$ 的麦克劳林公式为

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\cdots+\frac{f^{(10)}(0)}{10!}x^{10}+\cdots,\qquad ②$$

由 ① = ② 知

$\dfrac{y^{(10)}(0)}{10!}=-\dfrac{1}{7!}$,故 $y^{(10)}(0)=-720.$

补充:考生可用该方法求 $y^{(8)}(0)$, $y^{(7)}(0)$.

6. 当 $x=0$ 时,$y=\arctan x=x-\dfrac{1}{3}x^3+\dfrac{1}{5}x^5+\cdots$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1}.\qquad ①$$

又由于 $f(x)=f(0)+f'(0)x+\cdots+\dfrac{f^{(n)}(0)}{n!}x^n+\cdots,\qquad ②$

由 ① = ② 知

$$y^{(n)}(0)=\begin{cases}(-1)^k(2k)!, & n=2k+1,\\ 0, & n=2k\end{cases}(k=0,1,\cdots).$$

强化篇

1. $y'=e^x\cos x+e^x\cos\left(x+\frac{\pi}{2}\right)=e^x\left[\cos x+\cos\left(x+\frac{\pi}{2}\right)\right]$

$=\sqrt{2}e^x\cos\left(x+\frac{\pi}{4}\right)$,

$y''=\sqrt{2}e^x\left[\cos\left(x+\frac{\pi}{4}\right)+\cos\left(x+\frac{2}{4}\pi\right)\right]=(\sqrt{2})^2e^x\cos\left(x+\frac{2}{4}\pi\right)$

……

$y^{(n)}=2^{\frac{n}{2}}e^x\cos\left(x+\frac{n}{4}\pi\right)$.

2. $y=\frac{x^3-1}{x^2-2x-3}=\frac{(x^3+1)-2}{(x-3)(x+1)}$

$=\frac{(x+1)(x^2-x+1)}{(x-3)(x+1)}-\frac{2}{(x-3)(x+1)}$

$=\frac{x(x-3)+2x+1}{x-3}-\frac{1}{2}\left(\frac{1}{x-3}-\frac{1}{x+1}\right)$

$=x+2+\frac{7}{x-3}-\frac{1}{2}\left(\frac{1}{x-3}-\frac{1}{x+1}\right)$,

故 $y^{(n)}=\frac{1}{2}\cdot(-1)^n\cdot n!\left[\frac{13}{(x-3)^{n+1}}+\frac{1}{(x+1)^{n+1}}\right]$.

3. 利用泰勒公式,有

$$f(x)=\frac{x^4}{1+x^3}=f(0)+f'(0)x+\cdots+\frac{f^{(n)}(0)}{n!}x^n+\cdots$$

$$=x^4\sum_{n=0}^{\infty}(-1)^n(x^3)^n=x^4\sum_{n=0}^{\infty}(-1)^nx^{3n}$$

$$=\sum_{n=0}^{\infty}(-1)^nx^{3n+4},$$

故 $y^{(3n+4)}(0)=(3n+4)!\cdot(-1)^n$.

4. 由泰勒公式知

$$f(x)=\frac{\sin x}{x}=\frac{1}{x}\left[x-\frac{x^3}{3!}+\cdots+\frac{(-1)^nx^{2n+1}}{(2n+1)!}+\cdots\right]$$

$$= 1 - \frac{x^2}{3!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots. \quad ①$$

又

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots. \quad ②$$

由 ① = ② 得 $f^{(n)}(0) = \begin{cases} 0, & n = 2k-1, \\ \dfrac{(-1)^k}{2k+1}, & n = 2k. \end{cases}$

5. 由泰勒公式知

$$f(x) = x^2 \ln(1+x)$$

$$= x^2\left[x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^{n-2}}{n-2} + o(x^{n-2})\right]$$

$$= x^3 - \frac{x^4}{2} + \frac{x^5}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n-2} + o(x^n). \quad ①$$

又

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots. \quad ②$$

由 ① = ② 得 $f^{(n)}(0) = \dfrac{(-1)^{n-1} \cdot n!}{n-2}$.

6. $f(x) = \arctan x - \dfrac{x}{1+ax^2} = \left(x - \dfrac{x^3}{3} + \cdots\right) - x(1 - ax^2 + \cdots)$

$$= \left(a - \frac{1}{3}\right)x^3 + \cdots. \quad ①$$

又

$$f(x) = f(0) + f'(0)x + \cdots + \frac{f'''(0)}{3!}x^3 + \cdots. \quad ②$$

由 ① = ② 得 $a - \dfrac{1}{3} = \dfrac{f'''(0)}{3!} = \dfrac{1}{6}$，故 $a = \dfrac{1}{2}$.

第三章 求不定积分

§3.1 最简单的积分

基础篇

. $\int(5x^3+4x+1)\mathrm{d}x=\frac{5}{4}x^4+2x^2+x+C.$

2. $\int\left(\frac{3}{\sqrt{x}}+\frac{5}{x^2}+2x\right)\mathrm{d}x=6\sqrt{x}-\frac{5}{x}+x^2+C.$

3. $\int\left(\frac{a}{x}+\frac{a^2}{x^2}+\frac{a^3}{x^3}+\frac{a^4}{x^4}\right)\mathrm{d}x=a\ln|x|-\frac{a^2}{x}-\frac{a^3}{2}\frac{1}{x^2}-\frac{a^4}{3}\frac{1}{x^3}+C.$

4. $\int(x+x^2+x^3+x^4)\mathrm{d}x=\frac{1}{2}x^2+\frac{1}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5+C.$

5. $\int\sqrt[m]{x^n}\,\mathrm{d}x=\int x^{\frac{n}{m}}\mathrm{d}x=\frac{m}{m+n}x^{\frac{n+m}{m}}+C.$

6. $\int(\sqrt{x}+1)(\sqrt{x^3}-1)\mathrm{d}x=\int(x^2-\sqrt{x}+\sqrt{x^3}-1)\mathrm{d}x$

$$=\frac{1}{3}x^3-\frac{2}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}-x+C.$$

7. $\int\frac{(1-x)^2}{\sqrt{x}}\mathrm{d}x=\int\frac{1-2x+x^2}{\sqrt{x}}\mathrm{d}x=2\sqrt{x}-2\times\frac{2}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+C$

$$=2\sqrt{x}-\frac{4}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+C.$$

8. $\int(x^2+1)^2\mathrm{d}x=\int(x^4+2x^2+1)\mathrm{d}x=\frac{1}{5}x^5+\frac{2}{3}x^3+x+C.$

9. $\int(\sqrt{x}+x\sqrt{x}+\sqrt{x\sqrt{x}})\mathrm{d}x=\int(x^{\frac{1}{2}}+x^{\frac{3}{2}}+x^{\frac{3}{4}})\mathrm{d}x$

$$=\frac{2}{3}x^{\frac{3}{2}}+\frac{2}{5}x^{\frac{5}{2}}+\frac{4}{7}x^{\frac{7}{4}}+C.$$

10. $\int x^2\sqrt[3]{x}\,\mathrm{d}x=\int x^{\frac{7}{3}}\mathrm{d}x=\frac{3}{10}x^{\frac{10}{3}}+C.$

11. $\int\frac{1}{x^2\sqrt{x}}\mathrm{d}x=\int x^{-\frac{5}{2}}\mathrm{d}x=-\frac{2}{3}x^{-\frac{3}{2}}+C.$

12. $\int\sqrt[3]{x^2}\,\mathrm{d}x=\int x^{\frac{2}{3}}\mathrm{d}x=\frac{3}{5}x^{\frac{5}{3}}+C.$

13. $\int\sqrt{x}(x^2-5)\mathrm{d}x=\int(x^{\frac{5}{2}}-5\sqrt{x})\mathrm{d}x=\frac{2}{7}x^{\frac{7}{2}}-\frac{10}{3}x^{\frac{3}{2}}+C.$

14. $\int\frac{\mathrm{d}h}{\sqrt{2gh}}=\frac{1}{\sqrt{2g}}\int\frac{1}{\sqrt{h}}\mathrm{d}h=\frac{2\sqrt{h}}{\sqrt{2g}}=\frac{\sqrt{2h}}{\sqrt{g}}.$

15. $\int(1-x^2)\left(1-\frac{1}{x}\right)\mathrm{d}x=\int\left(1-\frac{1}{x}-x^2+x\right)\mathrm{d}x$

$$=x-\ln|x|-\frac{1}{3}x^3+\frac{1}{2}x^2+C.$$

16. $\int\left(2\mathrm{e}^x+\frac{3}{x}\right)\mathrm{d}x=2\mathrm{e}^x+3\ln|x|+C.$

17. $\int\left(\frac{3}{1+x^2}-\frac{2}{\sqrt{1-x^2}}\right)\mathrm{d}x=3\arctan x-2\arcsin x+C.$

18. $\int 2^x\,\mathrm{d}x=\frac{2^x}{\ln 2}+C.$

19. $\int\sin x\,\mathrm{d}x=-\cos x+C$, $\int\cos x\,\mathrm{d}x=\sin x+C.$

20. $\int\frac{3}{\sqrt{1-x^2}}\mathrm{d}x=3\arcsin x+C.$

21. $\int(1+\sin x+\cos x)\mathrm{d}x=x-\cos x+\sin x+C.$

22. $\int\left(\frac{1}{x^2+1}-\frac{1}{x^2-1}\right)\mathrm{d}x=\arctan x-\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C.$

强化篇

1. $\int (x^2-1)^3 \mathrm{d}x = \int (x^6-3x^4+3x^2-1)\mathrm{d}x = \frac{1}{7}x^7 - \frac{3}{5}x^5 + x^3 - x + C.$

2. $\int (x^2+1)^3 \mathrm{d}x = \int (x^6+3x^4+3x^2+1)\mathrm{d}x = \frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + C.$

3. $\int \left(\frac{1+x}{x}\right)^2 \mathrm{d}x = \int \frac{1+2x+x^2}{x^2}\mathrm{d}x = \int \left(\frac{1}{x^2}+\frac{2}{x}+1\right)\mathrm{d}x$

$$= -\frac{1}{x} + 2\ln|x| + x + C.$$

4. $\int \frac{\mathrm{d}x}{x^2+2x+1} = \int \frac{\mathrm{d}x}{(x+1)^2} = -\frac{1}{x+1} + C.$

5. $\int \frac{1-x^2}{1+x^2}\mathrm{d}x = \int \frac{1+x^2-2x^2}{1+x^2}\mathrm{d}x = \int \left(1-\frac{2x^2}{1+x^2}\right)\mathrm{d}x$

$$= \int 1\mathrm{d}x - 2\int \frac{x^2}{1+x^2}\mathrm{d}x = x - 2\int \frac{x^2+1-1}{1+x^2}\mathrm{d}x$$

$$= x - 2\int \left(1-\frac{1}{1+x^2}\right)\mathrm{d}x = x - 2(x - \arctan x) + C$$

$$= x - 2x + 2\arctan x + C = -x + 2\arctan x + C.$$

6. $\int \frac{x^2}{1+x^2}\mathrm{d}x = \int \frac{x^2+1-1}{1+x^2}\mathrm{d}x = \int \left(1-\frac{1}{1+x^2}\right)\mathrm{d}x = x - \arctan x + C.$

7. $\int \frac{x-1}{x^2-1}\mathrm{d}x = \int \frac{1}{x+1}\mathrm{d}x = \ln|x+1| + C.$

8. $\int \frac{x-1}{\sqrt{x}+1}\mathrm{d}x = \int \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}+1}\mathrm{d}x = \int (\sqrt{x}-1)\mathrm{d}x$

$$= \frac{2}{3}x^{\frac{3}{2}} - x + C.$$

9. $\int \frac{\sqrt{1+x^2}}{\sqrt{1-x^4}}\mathrm{d}x = \int \frac{\sqrt{1+x^2}}{\sqrt{(1+x^2)(1-x^2)}}\mathrm{d}x = \int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

$$= \arcsin x + C.$$

10. $\int \frac{x^4}{1+x^2}\mathrm{d}x = \int \frac{x^4-1+1}{1+x^2}\mathrm{d}x = \int \left[\frac{(x^2-1)(x^2+1)}{x^2+1} + \frac{1}{1+x^2}\right]\mathrm{d}x$

$$= \int \left(x^2 - 1 + \frac{1}{1+x^2}\right)\mathrm{d}x = \frac{1}{3}x^3 - x + \arctan x + C.$$

11. $\int \sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x = \int \frac{1-x}{\sqrt{1-x^2}}\mathrm{d}x$（这一步是解题的关键，后续的一些题目中也会用到，如$\int \sqrt{\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}}\,\mathrm{d}x$，需要积累）

$$= \int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x - \int \frac{x}{\sqrt{1-x^2}}\mathrm{d}x = \arcsin x + \sqrt{1-x^2} + C.$$

12. $\int \frac{3}{\sqrt{4-4x^2}}\mathrm{d}x = \int \frac{3}{\sqrt{4(1-x^2)}}\mathrm{d}x = \frac{3}{2}\int \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \frac{3}{2}\arcsin x + C.$

13. $\int \frac{3x^4+2x^2}{x^2+1}\mathrm{d}x = 3\int \frac{x^4}{x^2+1}\mathrm{d}x + 2\int \frac{x^2}{x^2+1}\mathrm{d}x$

$$= 3\int \frac{x^4-1+1}{x^2+1}\mathrm{d}x + 2\int \frac{x^2+1-1}{x^2+1}\mathrm{d}x$$

$$= 3\int \left(x^2 - 1 + \frac{1}{x^2+1}\right)\mathrm{d}x + 2\int \left(1 - \frac{1}{1+x^2}\right)\mathrm{d}x$$

$$= 3\left(\frac{1}{3}x^3 - x + \arctan x\right) + 2(x - \arctan x) + C$$

$$= x^3 - 3x + 3\arctan x + 2x - 2\arctan x + C$$

$$= x^3 - x + \arctan x + C.$$

本题就是第 6 题和第 10 题的组合.

14. $\int \frac{1+2x^2}{x^2(1+x^2)}\mathrm{d}x = \int \frac{1}{x^2(1+x^2)}\mathrm{d}x + \int \frac{2x^2}{x^2(1+x^2)}\mathrm{d}x$

$$= \int \left(\frac{1}{x^2} - \frac{1}{1+x^2}\right)\mathrm{d}x + \int \frac{2}{1+x^2}\mathrm{d}x$$

$$= -\frac{1}{x} - \arctan x + 2\arctan x + C$$

$$= -\frac{1}{x} + \arctan x + C.$$

15. $$\int \frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1-x^4}}\mathrm{d}x = \int\left(\frac{\sqrt{1+x^2}}{\sqrt{1-x^4}}+\frac{\sqrt{1-x^2}}{\sqrt{1-x^4}}\right)\mathrm{d}x$$
$$= \int\left(\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1+x^2}}\right)\mathrm{d}x$$
$$= \arcsin x + \ln(x+\sqrt{1+x^2})+C.$$

第 6，7，8，9，10，11，13，14，15，17 题这组题的目的是希望考生做完后在脑中有一个意识，以后遇到很多积分要把它化成分子的项数比分母多的“倒三角”型，而不是分母的项数比分子多，进而拆分成多项积分.

赘言：第 15 题最后一步能否马上写答案？还是需要翻《高等数学超详解》或笔记？

16. $$\int 2^x\cdot \mathrm{e}^x\,\mathrm{d}x = \int(2\mathrm{e})^x\,\mathrm{d}x = \frac{(2\mathrm{e})^x}{\ln 2\mathrm{e}}+C = \frac{(2\mathrm{e})^x}{\ln 2+1}+C.$$

17. $$\int \frac{2\cdot 3^x-5\cdot 2^x}{3^x}\mathrm{d}x = 2\int 1\mathrm{d}x - 5\int\left(\frac{2}{3}\right)^x\mathrm{d}x = 2x-\frac{5\cdot\left(\frac{2}{3}\right)^x}{\ln\frac{2}{3}}+C.$$

18. $$\int(\mathrm{e}^{x+1}+3^x4^{-x})\mathrm{d}x = \int\left(\mathrm{e}\cdot\mathrm{e}^x+\frac{3^x}{4^x}\right)\mathrm{d}x = \mathrm{e}\int\mathrm{e}^x\,\mathrm{d}x+\int\left(\frac{3}{4}\right)^x\mathrm{d}x$$
$$= \mathrm{e}^{x+1}+\frac{\left(\frac{3}{4}\right)^x}{\ln\frac{3}{4}}+C.$$

19. $$\int \mathrm{e}^x\left(1-\frac{\mathrm{e}^{-x}}{\sqrt{x}}\right)\mathrm{d}x = \int\left(\mathrm{e}^x-\frac{1}{\sqrt{x}}\right)\mathrm{d}x = \mathrm{e}^x-2\sqrt{x}+C.$$

20. $$\int \frac{\sqrt{x}-x^3\mathrm{e}^x+x^2}{x^3}\mathrm{d}x = \int\left(\frac{1}{x^{\frac{5}{2}}}-\mathrm{e}^x+\frac{1}{x}\right)\mathrm{d}x$$
$$= -\frac{2}{3}x^{-\frac{3}{2}}-\mathrm{e}^x+\ln|x|+C.$$

21. $$\int \frac{\mathrm{e}^{3x}+1}{\mathrm{e}^x+1}\mathrm{d}x = \int\frac{(\mathrm{e}^x+1)(\mathrm{e}^{2x}-\mathrm{e}^x+1)}{\mathrm{e}^x+1}\mathrm{d}x = \int(\mathrm{e}^{2x}-\mathrm{e}^x+1)\mathrm{d}x$$
$$= \frac{1}{2}\mathrm{e}^{2x}-\mathrm{e}^x+x+C.$$

本题要用到的公式：

$$a^3+b^3=(a+b)(a^2-ab+b^2),$$

$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

22. $\int\frac{2^{x+1}-3^{x+1}}{6^x}\mathrm{d}x=\int\left(\frac{2^{x+1}}{6^x}-\frac{3^{x+1}}{6^x}\right)\mathrm{d}x=2\int\left(\frac{2}{6}\right)^x\mathrm{d}x-3\int\left(\frac{3}{6}\right)^x\mathrm{d}x$

$$=2\int\left(\frac{1}{3}\right)^x\mathrm{d}x-3\int\left(\frac{1}{2}\right)^x\mathrm{d}x$$

$$=2\frac{\left(\frac{1}{3}\right)^x}{\ln\frac{1}{3}}-3\frac{\left(\frac{1}{2}\right)^x}{\ln\frac{1}{2}}+C.$$

23. $\int(2^x+3^x)^2\mathrm{d}x=\int(2^{2x}+2\cdot2^x\cdot3^x+3^{2x})\mathrm{d}x=\int(4^x+2\cdot6^x+9^x)\mathrm{d}x$

$$=\frac{4^x}{\ln 4}+2\cdot\frac{6^x}{\ln 6}+\frac{9^x}{\ln 9}+C.$$

24. $\int3\mathrm{e}^{x+5}\mathrm{d}x=3\mathrm{e}^{x+5}+C.$

25. $\int\frac{2^{x+1}-5^{x-1}}{10^x}\mathrm{d}x=2\int\frac{2^x}{10^x}\mathrm{d}x-\frac{1}{5}\int\left(\frac{5}{10}\right)^x\mathrm{d}x$

$$=2\int\left(\frac{1}{5}\right)^x\mathrm{d}x-\frac{1}{5}\int\left(\frac{1}{2}\right)^x\mathrm{d}x$$

$$=2\cdot\frac{\left(\frac{1}{5}\right)^x}{\ln\frac{1}{5}}-\frac{1}{5}\frac{\left(\frac{1}{2}\right)^x}{\ln\frac{1}{2}}+C.$$

26. $\int\sin^2x\,\mathrm{d}x=\frac{1}{2}\int(1-\cos 2x)\mathrm{d}x=\frac{1}{2}\left(x-\frac{1}{2}\sin 2x\right)+C.$

27. $\int\cos^2x\,\mathrm{d}x=\frac{1}{2}\int(1+\cos 2x)\mathrm{d}x=\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C.$

28. $\int\sqrt{1-\sin 2x}\,\mathrm{d}x=\int\sqrt{\sin^2x+\cos^2x-\sin 2x}\,\mathrm{d}x$

$$=\int\sqrt{(\cos x-\sin x)^2}\,\mathrm{d}x=\int|\cos x-\sin x|\,\mathrm{d}x$$

$$=\int\sqrt{2}\left|\cos\left(x+\frac{\pi}{4}\right)\right|\mathrm{d}x$$

$$=\sqrt{2}\sin\left[(k+1)\pi+\frac{\pi}{4}+x\right]+C_k,\ x\in\left[k\pi+\frac{\pi}{4},\ k\pi+\frac{5}{4}\pi\right),\ k\in\mathbf{Z}.$$

由原函数在 $k\pi+\frac{5\pi}{4}$ 处的连续性可知，$C_{k+1}=C_k+2\sqrt{2}$，故原函数为

$$\sqrt{2}\sin\left[(k+1)\pi+\frac{\pi}{4}+x\right]+C+2\sqrt{2}k,\ x\in\left[k\pi+\frac{\pi}{4},\ k\pi+\frac{5}{4}\pi\right),\ k\in\mathbf{Z}.$$

29. $\int\sqrt{1+\cos 2x}\,\mathrm{d}x=\int\sqrt{2\cos^2 x}\,\mathrm{d}x=\sqrt{2}\int|\cos x|\,\mathrm{d}x$

$$=\sqrt{2}\sin[(k+1)\pi+x]+C+2\sqrt{2}k,\ x\in\left[k\pi+\frac{\pi}{2},\ k\pi+\frac{3}{2}\pi\right),\ k\in\mathbf{Z}.$$

30. $\int\frac{1}{1+\cos 2x}\mathrm{d}x=\int\frac{1}{2\cos^2 x}\mathrm{d}x=\frac{1}{2}\int\sec^2 x\,\mathrm{d}x=\frac{1}{2}\tan x+C.$

31. $\int\tan x\,\mathrm{d}x=\int\frac{\sin x}{\cos x}\mathrm{d}x=-\int\frac{\mathrm{d}\cos x}{\cos x}=-\ln|\cos x|+C.$

32. $\int\cot x\,\mathrm{d}x=\int\frac{\cos x}{\sin x}\mathrm{d}x=\int\frac{\mathrm{d}\sin x}{\sin x}=\ln|\sin x|+C.$

33. $\int\tan^2 x\,\mathrm{d}x=\int(\sec^2 x-1)\mathrm{d}x=\tan x-x+C.$

34. $\int\cot^2 x\,\mathrm{d}x=\int(\csc^2 x-1)\mathrm{d}x=-\cot x-x+C.$

35. $\int\csc x(\csc x+1)\mathrm{d}x=\int\csc^2 x\,\mathrm{d}x+\int\csc x\,\mathrm{d}x$

$$=-\cot x+\ln|\csc x-\cot x|+C.$$

36. $\int\frac{\cos 2x}{\cos^2 x\sin^2 x}\mathrm{d}x=\int\frac{\cos^2 x-\sin^2 x}{\cos^2 x\sin^2 x}\mathrm{d}x=\int(\csc^2 x-\sec^2 x)\mathrm{d}x$

$$=-\cot x-\tan x+C.$$

37. $\int\frac{\cos 2x}{\cos x-\sin x}\mathrm{d}x=\int\frac{\cos^2 x-\sin^2 x}{\cos x-\sin x}\mathrm{d}x=\int(\cos x+\sin x)\mathrm{d}x$

$$=\sin x-\cos x+C.$$

38. $\int\cos x(\tan x+\sec x)\mathrm{d}x=\int(\sin x+1)\mathrm{d}x=-\cos x+x+C.$

39. $\int\frac{1}{16-x^4}\mathrm{d}x=\int\frac{1}{(4-x^2)(4+x^2)}\mathrm{d}x=\frac{1}{8}\int\left(\frac{1}{4-x^2}+\frac{1}{4+x^2}\right)\mathrm{d}x$

$$=\frac{1}{8}\left(\frac{1}{4}\ln\left|\frac{2+x}{2-x}\right|+\frac{1}{2}\arctan\frac{x}{2}\right)+C.$$

通过第 26 ～ 38 题，考生要掌握几个中学学过的三角函数公式，并在解题中会运用.

倍角公式：

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1,\end{aligned}$$

$$\sin 2x = 2\sin x \cos x.$$

降幂公式：

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x),$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x),$$

$$\sin^2 x + \cos^2 x = 1,$$

$$1 + \tan^2 x = \sec^2 x,$$

$$1 + \cot^2 x = \csc^2 x.$$

§3.2　利用换元法求不定积分

基础篇

1. $\int \frac{1}{x+a}\mathrm{d}x = \int \frac{1}{x+a}\mathrm{d}(x+a) = \ln|x+a| + C.$

2. $\int (2x-3)^{10}\mathrm{d}x = \frac{1}{2}\int (2x-3)^{10}\mathrm{d}(2x-3) = \frac{1}{2} \times \frac{1}{11}(2x-3)^{11} + C$

$$= \frac{1}{22}(2x-3)^{11} + C.$$

3. $\int \sqrt[3]{1-3x}\,\mathrm{d}x = -\frac{1}{3}\int \sqrt[3]{1-3x}\,\mathrm{d}(1-3x) = -\frac{1}{3} \times \frac{3}{4}(1-3x)^{\frac{4}{3}} + C$

$$= -\frac{1}{4}(1-3x)^{\frac{4}{3}} + C.$$

4. $\int \frac{\mathrm{d}x}{\sqrt{2-5x}} = -\frac{1}{5}\int \frac{\mathrm{d}(2-5x)}{\sqrt{2-5x}} = -\frac{2}{5}\sqrt{2-5x} + C.$

5. $\int \frac{\mathrm{d}x}{\sqrt[3]{2-3x}} = -\frac{1}{3} \times \frac{3}{2}(2-3x)^{\frac{2}{3}} + C = -\frac{1}{2}(2-3x)^{\frac{2}{3}} + C.$

6. $\int \frac{\mathrm{d}x}{(5x-2)^{\frac{5}{2}}} = \frac{1}{5} \times \left(-\frac{2}{3}\right)(5x-2)^{-\frac{3}{2}} + C = -\frac{2}{15}(5x-2)^{-\frac{3}{2}} + C.$

7. $\int \mathrm{e}^{5x}\,\mathrm{d}x = \frac{1}{5}\int \mathrm{e}^{5x}\,\mathrm{d}(5x) = \frac{1}{5}\mathrm{e}^{5x} + C.$

8. $\int \frac{1}{1-2x}\mathrm{d}x = -\frac{1}{2}\ln|1-2x| + C.$

9. $\int (\sin ax - \mathrm{e}^{\frac{x}{b}})\,\mathrm{d}x = -\frac{1}{a}\cos ax - b\mathrm{e}^{\frac{x}{b}} + C.$

10. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\mathrm{d}x = -2\cos\sqrt{x} + C.$

11. $\int x\mathrm{e}^{-x^2}\,\mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-x^2} + C.$

12. $\int \frac{1}{2+3x^2}\mathrm{d}x = \frac{1}{\sqrt{3}}\int \frac{1}{2+(\sqrt{3}x)^2}\mathrm{d}(\sqrt{3}x) = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{3}x}{\sqrt{2}} + C$

$= \frac{1}{\sqrt{6}}\arctan\frac{\sqrt{6}}{2}x + C.$

13. $\int \frac{1}{2-3x^2}\mathrm{d}x = \frac{1}{\sqrt{3}}\int \frac{1}{2-3x^2}\mathrm{d}(\sqrt{3}x) = \frac{1}{2\sqrt{2}\times\sqrt{3}}\ln\left|\frac{\sqrt{2}+\sqrt{3}x}{\sqrt{2}-\sqrt{3}x}\right| + C$

$= \frac{1}{2\sqrt{6}}\ln\left|\frac{\sqrt{2}+\sqrt{3}x}{\sqrt{2}-\sqrt{3}x}\right| + C.$

14. $\int \frac{1}{\sqrt{2-3x^2}}\mathrm{d}x = \frac{1}{\sqrt{3}}\int \frac{\mathrm{d}(\sqrt{3}x)}{\sqrt{2-(\sqrt{3}x)^2}} = \frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{3}x}{\sqrt{2}} + C.$

15. $\int \frac{1}{\sqrt{3x^2-2}}\mathrm{d}x = \frac{1}{\sqrt{3}}\int \frac{\mathrm{d}(\sqrt{3}x)}{\sqrt{(\sqrt{3}x)^2-(\sqrt{2})^2}}$

$= \frac{1}{\sqrt{3}}\ln\left|\sqrt{3}x + \sqrt{3x^2-2}\right| + C.$

16. $\int (\mathrm{e}^{-x} + \mathrm{e}^{-2x})\mathrm{d}x = -\mathrm{e}^{-x} - \frac{1}{2}\mathrm{e}^{-2x} + C.$

17. $\int (\sin 5x - \sin 5\alpha)\mathrm{d}x = -\frac{1}{5}\cos 5x - x \cdot \sin 5\alpha + C.$

18. $\int \sec x\,\mathrm{d}x = \int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\mathrm{d}x = \int \frac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\mathrm{d}x$

$$= \int \frac{\mathrm{d}(\sec x+\tan x)}{\sec x+\tan x} = \ln|\sec x+\tan x|+C.$$

19. $\int \sin(ax+b)\mathrm{d}x = -\frac{1}{a}\cos(ax+b)+C.$

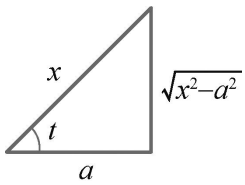
20. $\int \tan^4 x\sec^2 x\,\mathrm{d}x = \int \tan^4 x\,\mathrm{d}\tan x = \frac{1}{5}\tan^5 x+C.$

21. $\int \frac{1}{a^2-x^2}\mathrm{d}x = \int \frac{\mathrm{d}x}{(a-x)(a+x)} = \frac{1}{2a}\int\left(\frac{1}{a-x}+\frac{1}{a+x}\right)\mathrm{d}x$

$$= \frac{1}{2a}(-\ln|a-x|+\ln|a+x|)+C$$

$$= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|+C.$$

22. $\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} \xlongequal[\mathrm{d}x=a\sec t\tan t\mathrm{d}t]{x=a\sec t} \int \frac{a\sec t\tan t\,\mathrm{d}t}{a\cdot\tan t}$



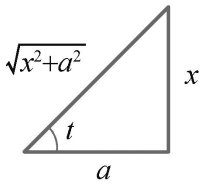
$$= \int \sec t\,\mathrm{d}t$$

$$= \ln|\sec t+\tan t|+C$$

$$= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right|+C.$$

23. $\int \frac{1}{\sqrt{x^2+a^2}}\mathrm{d}x \xlongequal[\mathrm{d}x=a\sec^2 t\mathrm{d}t]{x=a\tan t} \int \frac{a\sec^2 t\,\mathrm{d}t}{a\cdot\sec t} = \int \sec t\,\mathrm{d}t$



$$= \ln|\sec t+\tan t|+C$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a}+\frac{x}{a}\right|+C$$

$$= \ln(x+\sqrt{x^2+a^2})+C.$$

24. $\int \frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x \xlongequal[\mathrm{d}x=a\cos t\mathrm{d}t]{x=a\sin t} \int \frac{a\cos t\,\mathrm{d}t}{a\cos t} = t+C = \arcsin\frac{x}{a}+C.$

25. $\int \frac{\mathrm{d}x}{x^2+a^2} \xlongequal[\mathrm{d}x=a\sec^2 t\mathrm{d}t]{x=a\tan t} \int \frac{a\sec^2 t\,\mathrm{d}t}{a^2\sec^2 t} = \frac{1}{a}t+C = \frac{1}{a}\arctan\frac{x}{a}+C.$

26. $\int \frac{1}{x^2-a^2}\mathrm{d}x = \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)\mathrm{d}x$

$= \frac{1}{2a}(\ln|x-a|-\ln|x+a|)+C$

$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C.$

27. $\int (x^2+1)^{2\,020}\cdot x\,\mathrm{d}x = \frac{1}{2}\int (x^2+1)^{2\,020}\mathrm{d}(x^2+1)$

$= \frac{1}{2}\times\frac{1}{2\,021}(x^2+1)^{2\,021}+C$

$= \frac{1}{4\,042}(x^2+1)^{2\,021}+C.$

28. $\int (2x-1)^{2\,020}\mathrm{d}x = \frac{1}{2}\int (2x-1)^{2\,020}\mathrm{d}(2x-1)$

$= \frac{1}{2}\times\frac{1}{2\,021}(2x-1)^{2\,021}+C$

$= \frac{1}{4\,042}(2x-1)^{2\,021}+C.$

29. $\int (\sin 2x+\cos 2x)\mathrm{d}x = -\frac{1}{2}\cos 2x+\frac{1}{2}\sin 2x+C.$

30. $\int x\cdot\cos(1+x^2)\mathrm{d}x = \frac{1}{2}\int\cos(1+x^2)\mathrm{d}(1+x^2)$

$= \frac{1}{2}\sin(1+x^2)+C.$

31. $\int \frac{1}{1+2x}\mathrm{d}x = \frac{1}{2}\int\frac{1}{1+2x}\mathrm{d}(1+2x) = \frac{1}{2}\ln|1+2x|+C.$

32. $\int \frac{1+\cos x}{x+\sin x}\mathrm{d}x = \int\frac{\mathrm{d}(x+\sin x)}{x+\sin x} = \ln|x+\sin x|+C.$

33. $\int \frac{2x+4}{x^2+4x+6}\mathrm{d}x = \int\frac{\mathrm{d}(x^2+4x+6)}{x^2+4x+6} = \ln(x^2+4x+6)+C.$

34. $\int \frac{2x-1}{\sqrt{1-x^2}}\mathrm{d}x = \int\frac{2x}{\sqrt{1-x^2}}\mathrm{d}x-\int\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

$= -2\sqrt{1-x^2}-\arcsin x+C.$

35. $\int \frac{2x-1}{\sqrt{x^2-x}}\mathrm{d}x = \int \frac{\mathrm{d}(x^2-x)}{\sqrt{x^2-x}} = 2\sqrt{x^2-x}+C.$

36. $\int x\mathrm{e}^{x^2+1}\mathrm{d}x = \frac{1}{2}\int \mathrm{e}^{x^2+1}\mathrm{d}(x^2+1) = \frac{1}{2}\mathrm{e}^{x^2+1}+C.$

37. $\int \mathrm{e}^{3-x}\mathrm{d}x = -\mathrm{e}^{3-x}+C.$

38. $\int \mathrm{e}^x\sin(\mathrm{e}^x)\mathrm{d}x = \int \sin(\mathrm{e}^x)\mathrm{d}(\mathrm{e}^x) = -\cos(\mathrm{e}^x)+C.$

39. $\int \frac{\mathrm{d}x}{x(1+\ln^2 x)} = \int \frac{\mathrm{d}\ln x}{1+\ln^2 x} = \arctan\ln x + C.$

40. $\int \frac{(\arctan x)^2}{1+x^2}\mathrm{d}x = \int (\arctan x)^2\mathrm{d}\arctan x = \frac{1}{3}(\arctan x)^3+C.$

41. $\int \frac{\mathrm{e}^x}{1+\mathrm{e}^x}\mathrm{d}x = \int \frac{\mathrm{d}(\mathrm{e}^x+1)}{\mathrm{e}^x+1} = \ln(\mathrm{e}^x+1)+C.$

42. $\int \frac{x}{\sqrt{2-3x^2}}\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(x^2)}{\sqrt{2-3x^2}} = -\frac{1}{6}\int \frac{\mathrm{d}(2-3x^2)}{\sqrt{2-3x^2}}$

$$= -\frac{1}{3}\sqrt{2-3x^2}+C.$$

43. $\int \frac{\sin x+\cos x}{\sqrt[3]{\sin x-\cos x}}\mathrm{d}x = \int \frac{\mathrm{d}(\sin x-\cos x)}{\sqrt[3]{\sin x-\cos x}} = \frac{3}{2}(\sin x-\cos x)^{\frac{2}{3}}+C.$

44. $\int \frac{\mathrm{d}x}{(\arcsin x)^2\sqrt{1-x^2}} = \int \frac{\mathrm{d}\arcsin x}{(\arcsin x)^2} = -\frac{1}{\arcsin x}+C.$

45. $\int \frac{10^{2\arccos x}}{\sqrt{1-x^2}}\mathrm{d}x = -\int 10^{2\arccos x}\mathrm{d}\arccos x = -\frac{10^{2\arccos x}}{\ln 10^2}+C$

$$= -\frac{10^{2\arccos x}}{2\ln 10}+C.$$

强化篇

1. $\int \frac{\sqrt[5]{1-2x+x^2}}{1-x}\mathrm{d}x = \int \frac{\sqrt[5]{(1-x)^2}}{1-x}\mathrm{d}x = \int (1-x)^{-\frac{3}{5}}\mathrm{d}x$

$$= -\frac{5}{2}(1-x)^{\frac{2}{5}}+C.$$

2. $\int \frac{\mathrm{d}x}{\sin^2\left(2x+\frac{\pi}{4}\right)} = \frac{1}{2}\int \csc^2\left(2x+\frac{\pi}{4}\right)\mathrm{d}\left(2x+\frac{\pi}{4}\right)$

$$= -\frac{1}{2}\cot\left(2x+\frac{\pi}{4}\right)+C.$$

3. $\int \frac{1}{1+\cos x}\mathrm{d}x = \int \frac{\mathrm{d}x}{1+2\cos^2\frac{x}{2}-1} = \int \sec^2\frac{x}{2}\mathrm{d}\frac{x}{2} = \tan\frac{x}{2}+C.$

4. $\int \frac{\mathrm{d}x}{1-\cos x} = \int \frac{\mathrm{d}x}{1-\left(1-2\sin^2\frac{x}{2}\right)} = \int \csc^2\frac{x}{2}\mathrm{d}\frac{x}{2} = -\cot\frac{x}{2}+C.$

5. $\int \frac{\mathrm{d}x}{1+\sin x} = \int \frac{1-\sin x}{(1+\sin x)(1-\sin x)}\mathrm{d}x = \int \frac{1-\sin x}{\cos^2 x}\mathrm{d}x$

$$= \int \sec^2 x\,\mathrm{d}x - \int \frac{\sin x}{\cos^2 x}\mathrm{d}x = \tan x - \frac{1}{\cos x}+C.$$

6. $\int \sin^3 x\,\mathrm{d}x = -\int (1-\cos^2 x)\mathrm{d}\cos x = -\cos x + \frac{1}{3}\cos^3 x + C.$

7. $\int \cos^3 x\,\mathrm{d}x = \int (1-\sin^2 x)\mathrm{d}\sin x = \sin x - \frac{1}{3}\sin^3 x + C.$

8. $\int \frac{1}{\cos x\sin x}\mathrm{d}x = \int \frac{\mathrm{d}x}{\cos^2 x\cdot\tan x} = \int \frac{\mathrm{d}\tan x}{\tan x} = \ln|\tan x|+C.$

9. $\int \frac{\ln\tan x}{\cos x\sin x}\mathrm{d}x = \int \ln\tan x\,\mathrm{d}\ln\tan x = \frac{1}{2}(\ln\tan x)^2+C.$

注：以后在题中见到$\frac{1}{\cos x\sin x}\mathrm{d}x$ 因子，要想到凑微分成 $\mathrm{d}\ln\tan x$，避免复杂的三角函数计算.

10. $\int \frac{\mathrm{d}x}{x+\sqrt{1-x^2}} \xlongequal[\mathrm{d}x=\cos t\,\mathrm{d}t]{x=\sin t} \int \frac{\cos t\,\mathrm{d}t}{\sin t+\cos t}$

$$= \int \frac{\frac{1}{2}(\sin t+\cos t)+\frac{1}{2}(\cos t-\sin t)}{\sin t+\cos t}\mathrm{d}t$$

$$= \frac{1}{2}t+\frac{1}{2}\ln|\sin t+\cos t|+C$$

$$= \frac{1}{2}\arcsin x + \frac{1}{2}\ln|x+\sqrt{1-x^2}|+C.$$

11. $\int x^2\sqrt{2+x^3}\,dx = \frac{1}{3}\int\sqrt{2+x^3}\,d(2+x^3) = \frac{1}{3}\times\frac{2}{3}(2+x^3)^{\frac{3}{2}}+C$

$$= \frac{2}{9}(2+x^3)^{\frac{3}{2}}+C.$$

12. $\int\frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\int\frac{d(3x)}{\sqrt{4-9x^2}} = \frac{1}{3}\arcsin\frac{3x}{2}+C.$

13. $\int\frac{1}{\sqrt{x}}e^{3\sqrt{x}}\,dx = 2\int e^{3\sqrt{x}}\,d\sqrt{x} = \frac{2}{3}e^{3\sqrt{x}}+C.$

14. $\int\frac{(\arctan\sqrt{x})^2}{\sqrt{x}(1+x)}dx = 2\int(\arctan\sqrt{x})^2\frac{1}{1+(\sqrt{x})^2}d\sqrt{x}$

$$= 2\int(\arctan\sqrt{x})^2\,d\arctan\sqrt{x}$$

$$= \frac{2}{3}(\arctan\sqrt{x})^3+C.$$

15. $\int\sqrt{\frac{\ln(x+\sqrt{1+x^2})}{1+x^2}}dx = \int\sqrt{\ln(x+\sqrt{1+x^2})}\,d\ln(x+\sqrt{1+x^2})$

$$= \frac{2}{3}[\ln(x+\sqrt{1+x^2})]^{\frac{3}{2}}+C.$$

16. $\int\frac{1}{1+e^x}dx = \int\frac{1+e^x-e^x}{1+e^x}dx = \int\left(1-\frac{e^x}{1+e^x}\right)dx$

$$= x-\ln(1+e^x)+C.$$

17. $\int\frac{1}{(1+e^x)^2}dx = \int\frac{1+e^x-e^x}{(1+e^x)^2}dx = \int\left[\frac{1}{1+e^x}-\frac{e^x}{(1+e^x)^2}\right]dx$

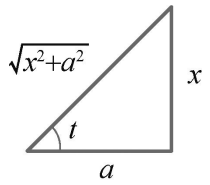
$$= x-\ln(1+e^x)+\frac{1}{1+e^x}+C.$$

18. $\int\frac{1}{e^x+e^{-x}}dx = \int\frac{dx}{e^{-x}(e^{2x}+1)} = \int\frac{d(e^x)}{1+e^{2x}} = \arctan(e^x)+C.$

19. $\int\frac{1}{e^x-e^{-x}}dx = \int\frac{dx}{e^{-x}(e^{2x}-1)} = \int\frac{d(e^x)}{e^{2x}-1} = \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right|+C.$

20. $\int\frac{1}{\sqrt{x(1+x)}}dx = 2\int\frac{d\sqrt{x}}{\sqrt{1+(\sqrt{x})^2}} = 2\ln(\sqrt{x}+\sqrt{1+x})+C.$

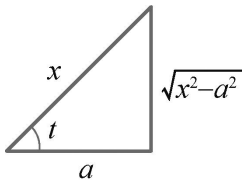
21. $$\int \frac{\mathrm{d}x}{\sqrt{(x^2+a^2)^3}} \xlongequal[\mathrm{d}x = a\sec^2 t\,\mathrm{d}t]{x = a\tan t} \int \frac{a\sec^2 t\,\mathrm{d}t}{a^3 \cdot \sec^3 t}$$



$$= \frac{1}{a^2}\int \cos t\,\mathrm{d}t = \frac{1}{a^2}\sin t + C$$

$$= \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}} + C.$$

22. $$\int \frac{\mathrm{d}x}{\sqrt{(x^2-a^2)^3}} \xlongequal[\mathrm{d}x = a\sec t\tan t\,\mathrm{d}t]{x = a\sec t} \int \frac{a\sec t\tan t\,\mathrm{d}t}{a^3\tan^3 t}$$

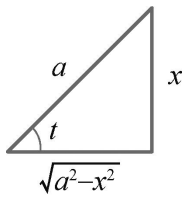


$$= \frac{1}{a^2}\int \frac{\sec t}{\tan^2 t}\mathrm{d}t = \frac{1}{a^2}\int \frac{\cos t}{\sin^2 t}\mathrm{d}t$$

$$= -\frac{1}{a^2}\frac{1}{\sin t} + C$$

$$= -\frac{1}{a^2}\frac{x}{\sqrt{x^2-a^2}} + C.$$

23. $$\int \frac{1}{\sqrt{(a^2-x^2)^3}}\mathrm{d}x \xlongequal[\mathrm{d}x = a\cos t\,\mathrm{d}t]{x = a\sin t} \int \frac{a\cos t\ \mathrm{d}t}{a^3\cos^3 t}$$



$$= \frac{1}{a^2}\int \sec^2 t\,\mathrm{d}t = \frac{1}{a^2}\tan t + C$$

$$= \frac{1}{a^2}\frac{x}{\sqrt{a^2-x^2}} + C.$$

24. $$\int \frac{\sqrt{x^2-4}}{x}\mathrm{d}x = \int \frac{x}{\sqrt{x^2-4}}\mathrm{d}x - 4\int \frac{1}{x\sqrt{x^2-4}}\mathrm{d}x$$

$$= \sqrt{x^2-4} + \frac{4}{2}\int \frac{\mathrm{d}\frac{2}{x}}{\sqrt{1-\left(\frac{2}{x}\right)^2}}$$

$$= \sqrt{x^2-4} + 2\arcsin\frac{2}{x} + C.$$

25. $$\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\mathrm{d}x \xlongequal[\mathrm{d}x = 4t^3\mathrm{d}t]{x = t^4} \int \frac{4t^3\mathrm{d}t}{t^2+t} = 4\int \frac{t^2}{t+1}\mathrm{d}t = 4\int\left(t-1+\frac{1}{t+1}\right)\mathrm{d}t$$

$$= 4\left(\frac{1}{2}t^2 - t + \ln|1+t|\right) + C$$

$$= 2\sqrt{x} - 4\sqrt[4]{x} + 4\ln(1+\sqrt[4]{x}) + C.$$

26. $$\int \tan\sqrt{1+x^2}\,\frac{x}{\sqrt{1+x^2}}\mathrm{d}x = \int \tan\sqrt{1+x^2}\,\mathrm{d}\sqrt{1+x^2}$$

$$= -\ln|\cos\sqrt{1+x^2}| + C.$$

27. $$\int \sqrt{\frac{x}{1-x\sqrt{x}}}\mathrm{d}x = \int \frac{\sqrt{x}}{\sqrt{1-x^{\frac{3}{2}}}}\mathrm{d}x = \frac{2}{3}\int \frac{\mathrm{d}(x^{\frac{3}{2}})}{\sqrt{1-x^{\frac{3}{2}}}}$$

$$= -\frac{4}{3}\sqrt{1-x^{\frac{3}{2}}} + C.$$

28. $$\int \frac{x}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}\mathrm{d}x = \int \frac{x}{\sqrt{(1+x^2)(1+\sqrt{1+x^2})}}\mathrm{d}x$$

$$= \int \frac{\mathrm{d}\sqrt{1+x^2}}{\sqrt{1+\sqrt{1+x^2}}}$$

$$= 2\sqrt{1+\sqrt{1+x^2}} + C.$$

29. $$\int \frac{1-x}{\sqrt{9-4x^2}}\mathrm{d}x = \int \frac{1}{\sqrt{9-4x^2}}\mathrm{d}x - \int \frac{x}{\sqrt{9-4x^2}}\mathrm{d}x$$

$$= \frac{1}{2}\arcsin\frac{2x}{3} + \frac{1}{8}\times 2\sqrt{9-4x^2} + C$$

$$= \frac{1}{2}\arcsin\frac{2}{3}x + \frac{1}{4}\sqrt{9-4x^2} + C.$$

30. $$\int \frac{x^2\mathrm{d}x}{\sqrt{a^2-x^2}} = -\int x\,\mathrm{d}\sqrt{a^2-x^2} = -x\sqrt{a^2-x^2} + \int \sqrt{a^2-x^2}\,\mathrm{d}x$$

$$= -x\sqrt{a^2-x^2} + \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C$$

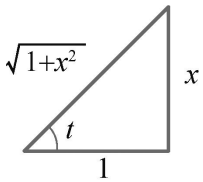
$$= -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C.$$

31. $$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = \int \frac{\mathrm{d}x}{x^2\sqrt{1-\frac{1}{x^2}}} = -\int \frac{\mathrm{d}\frac{1}{x}}{\sqrt{1-\frac{1}{x^2}}} = -\arcsin\frac{1}{x} + C.$$

32. $$\int \frac{\mathrm{d}x}{x\sqrt{x^2+1}} = \int \frac{\mathrm{d}x}{x^2\sqrt{1+\frac{1}{x^2}}} = -\int \frac{\mathrm{d}\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}}$$

$$= -\ln\left(\frac{1}{x}+\sqrt{1+\frac{1}{x^2}}\right)+C.$$

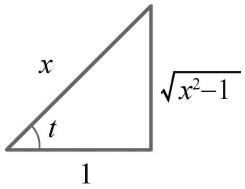
33. $$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+1}} \xlongequal[\mathrm{d}x=\sec^2 t\,\mathrm{d}t]{x=\tan t} \int \frac{\sec^2 t\,\mathrm{d}t}{\tan^2 t\cdot\sec t} = \int \frac{\cos t}{\sin^2 t}\mathrm{d}t$$



$$= \int \frac{\mathrm{d}\sin t}{\sin^2 t} = -\frac{1}{\sin t}+C$$

$$= -\frac{\sqrt{1+x^2}}{x}+C.$$

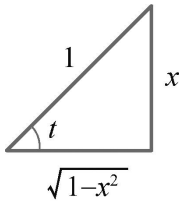
34. $$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} \xlongequal[\mathrm{d}x=\sec t\tan t\,\mathrm{d}t]{x=\sec t} \int \frac{\sec t\tan t\,\mathrm{d}t}{\sec^2 t\cdot\tan t}$$



$$= \int \cos t\,\mathrm{d}t = \sin t + C$$

$$= \frac{\sqrt{x^2-1}}{x}+C.$$

35. $$\int \frac{\mathrm{d}x}{x^2\sqrt{1-x^2}} \xlongequal[\mathrm{d}x=\cos t\,\mathrm{d}t]{x=\sin t} \int \frac{\cos t\,\mathrm{d}t}{\sin^2 t\cdot\cos t}$$



$$= -\cot t + C$$

$$= -\frac{\sqrt{1-x^2}}{x}+C.$$

36. $$\int \frac{x^3}{9+x^2}\mathrm{d}x = \frac{1}{2}\int \frac{x^2}{9+x^2}\mathrm{d}(x^2) = \frac{1}{2}\int \frac{9+x^2-9}{9+x^2}\mathrm{d}(x^2)$$

$$= \frac{1}{2}\int\left(1-\frac{9}{9+x^2}\right)\mathrm{d}(x^2) = \frac{1}{2}[x^2-9\ln(9+x^2)]+C.$$

37. $$\int \frac{\mathrm{d}x}{2x^2-1} = \int \frac{\mathrm{d}x}{(\sqrt{2}x)^2-1} = \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}(\sqrt{2}x)}{(\sqrt{2}x)^2-1} = \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}x-1}{\sqrt{2}x+1}\right|+C.$$

38. $$\int \frac{\mathrm{d}x}{(x+1)(x-2)} = -\frac{1}{3}\int\left(\frac{1}{x+1}-\frac{1}{x-2}\right)\mathrm{d}x$$

$$= -\frac{1}{3}(\ln|x+1| - \ln|x-2|) + C.$$

39. $$\int \frac{x}{x^2-x-2}\mathrm{d}x = \int \frac{\frac{1}{2}(2x-1)+\frac{1}{2}}{x^2-x-2}\mathrm{d}x$$

$$= \frac{1}{2}\ln|x^2-x-2| + \frac{1}{2}\int \frac{\mathrm{d}x}{x^2-x-2}$$

$$= \frac{1}{2}\ln|x^2-x-2| + \frac{1}{2}\int \frac{\mathrm{d}x}{\left(x-\frac{1}{2}\right)^2-\frac{9}{4}}$$

$$= \frac{1}{2}\ln|x^2-x-2| + \frac{1}{2}$$

$$\times \frac{1}{2\times\frac{3}{2}}\ln\left|\frac{x-\frac{1}{2}-\frac{3}{2}}{x-\frac{1}{2}+\frac{3}{2}}\right| + C$$

$$= \frac{1}{2}\ln|x^2-x-2| + \frac{1}{6}\ln\left|\frac{x-2}{x+1}\right| + C.$$

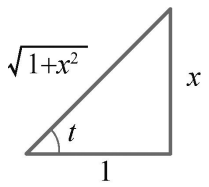
40. $$\int \frac{x^3+1}{(x^2+1)^2}\mathrm{d}x = \int \frac{x^3}{(x^2+1)^2}\mathrm{d}x + \int \frac{1}{(x^2+1)^2}\mathrm{d}x$$

$$= \frac{1}{2}\int \frac{x^2}{(x^2+1)^2}\mathrm{d}(x^2) + \int \frac{1}{(x^2+1)^2}\mathrm{d}x$$

$$= \frac{1}{2}\int \left[\frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}\right]\mathrm{d}(x^2) + I$$

$$= \frac{1}{2}\left[\ln(x^2+1) + \frac{1}{x^2+1}\right] + I.$$

下面单独计算 $I = \int \frac{1}{(x^2+1)^2}\mathrm{d}x$.令 $x = \tan t$,



$$I = \int \frac{\sec^2 t\,\mathrm{d}t}{\sec^4 t} = \int \cos^2 t\,\mathrm{d}t = \frac{1}{2}\int (1+\cos 2t)\mathrm{d}t$$

$$= \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right) + C = \frac{1}{2}\left(\arctan x + \frac{x}{\sqrt{1+x^2}}\,\frac{1}{\sqrt{1+x^2}}\right) + C$$

$$= \frac{1}{2}\left(\arctan x + \frac{x}{1+x^2}\right) + C.$$

故原积分 $= \frac{1}{2}\left[\ln(x^2+1) + \frac{1}{x^2+1} + \frac{x}{x^2+1} + \arctan x\right] + C$

$$= \frac{1}{2}\left[\ln(x^2+1) + \frac{x+1}{x^2+1} + \arctan x\right] + C.$$

41. $\int \frac{\mathrm{d}x}{\sqrt{\mathrm{e}^{2x}+1}} = \int \frac{\mathrm{d}x}{\sqrt{\mathrm{e}^{2x}(1+\mathrm{e}^{-2x})}} = \int \frac{\mathrm{d}x}{\mathrm{e}^x\sqrt{1+\mathrm{e}^{-2x}}} = -\int \frac{\mathrm{d}(\mathrm{e}^{-x})}{\sqrt{1+\mathrm{e}^{-2x}}}$

$$= -\ln(\mathrm{e}^{-x} + \sqrt{1+\mathrm{e}^{-2x}}) + C.$$

42. $\int \frac{\mathrm{d}x}{\sqrt{\mathrm{e}^{2x}-1}} = -\int \frac{\mathrm{d}(\mathrm{e}^{-x})}{\sqrt{1-\mathrm{e}^{-2x}}} = -\arcsin(\mathrm{e}^{-x}) + C.$

43. $\int \frac{\mathrm{d}x}{\sqrt{1-\mathrm{e}^{2x}}} = -\int \frac{\mathrm{d}(\mathrm{e}^{-x})}{\sqrt{\mathrm{e}^{-2x}-1}} = -\ln(\mathrm{e}^{-x} + \sqrt{\mathrm{e}^{-2x}-1}) + C.$

44. $\int \sqrt{\frac{\mathrm{e}^x-1}{\mathrm{e}^x+1}}\,\mathrm{d}x = \int \frac{\mathrm{e}^x-1}{\sqrt{\mathrm{e}^{2x}-1}}\mathrm{d}x = \int \frac{\mathrm{e}^x}{\sqrt{\mathrm{e}^{2x}-1}}\mathrm{d}x - \int \frac{1}{\sqrt{\mathrm{e}^{2x}-1}}\mathrm{d}x$

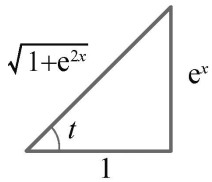
$$= \int \frac{\mathrm{d}(\mathrm{e}^x)}{\sqrt{\mathrm{e}^{2x}-1}} + \int \frac{\mathrm{d}(\mathrm{e}^{-x})}{\sqrt{1-\mathrm{e}^{-2x}}}$$

$$= \ln(\mathrm{e}^x + \sqrt{\mathrm{e}^{2x}-1}) + \arcsin(\mathrm{e}^{-x}) + C.$$

45. $\int \frac{1}{\mathrm{e}^x(1+\mathrm{e}^{2x})}\mathrm{d}x = \int \frac{1+\mathrm{e}^{2x}-\mathrm{e}^{2x}}{\mathrm{e}^x(1+\mathrm{e}^{2x})}\mathrm{d}x = \int\left(\frac{1}{\mathrm{e}^x} - \frac{\mathrm{e}^x}{1+\mathrm{e}^{2x}}\right)\mathrm{d}x$

$$= -\mathrm{e}^{-x} - \arctan(\mathrm{e}^x) + C.$$

46. $\int \sqrt{1+\mathrm{e}^{2x}}\,\mathrm{d}x \xlongequal[x=\ln\tan t]{\mathrm{e}^x=\tan t} \int \sec t\,\frac{1}{\sin t\cos t}\mathrm{d}t$



$$\mathrm{d}x = \frac{1}{\sin t\cos t}\mathrm{d}t$$

$$= \int \frac{1}{\sin t\cos^2 t}\mathrm{d}t = \int \frac{\sin^2 t + \cos^2 t}{\sin t\cos^2 t}\mathrm{d}t$$

$$= \int\left(\frac{\sin t}{\cos^2 t} + \frac{1}{\sin t}\right)\mathrm{d}t = \frac{1}{\cos t} + \ln|\csc t - \cot t| + C$$

$$= \sqrt{1+\mathrm{e}^{2x}} + \ln\left|\frac{\sqrt{1+\mathrm{e}^{2x}}}{\mathrm{e}^x} - \frac{1}{\mathrm{e}^x}\right| + C$$

$$= \sqrt{1+\mathrm{e}^{2x}} + \ln(\sqrt{1+\mathrm{e}^{2x}} - 1) - x + C.$$

47. $\int e^x\sqrt{1+e^{2x}}\,dx=\int\sqrt{1+e^{2x}}\,d(e^x)$

$$=\frac{e^x}{2}\sqrt{1+e^{2x}}+\frac{1}{2}\ln(e^x+\sqrt{1+e^{2x}})+C.$$

48. $\int\frac{dx}{\sqrt{e^x+1}}\xlongequal[\substack{e^x=t^2-1\\x=\ln(t^2-1)}]{\sqrt{e^x+1}=t}\int\frac{\frac{2t}{t^2-1}}{t}dt=2\int\frac{dt}{t^2-1}=\ln\left|\frac{t-1}{t+1}\right|+C$

$$=\ln\left(\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right)+C.$$

49. $\int\frac{2^x\cdot 3^x}{9^x+4^x}dx=\int\frac{\left(\frac{2}{3}\right)^x}{1+\left(\frac{2}{3}\right)^{2x}}dx=\frac{1}{\ln\frac{2}{3}}\int\frac{\left(\frac{2}{3}\right)^x\ln\frac{2}{3}}{1+\left(\frac{2}{3}\right)^{2x}}dx$

$$=\frac{1}{\ln\frac{2}{3}}\int\frac{d\left[\left(\frac{2}{3}\right)^x\right]}{1+\left(\frac{2}{3}\right)^{2x}}=\frac{1}{\ln\frac{2}{3}}\arctan\left[\left(\frac{2}{3}\right)^x\right]+C.$$

50. $\int\frac{1+\ln x}{x^{-x}+x^x}dx=\int\frac{1+\ln x}{e^{-x\ln x}+e^{x\ln x}}dx=\int\frac{1+\ln x}{e^{-x\ln x}(1+e^{2x\ln x})}dx$

$$=\int\frac{e^{x\ln x}(1+\ln x)dx}{1+e^{2x\ln x}}=\int\frac{d(e^{x\ln x})}{1+e^{2x\ln x}}$$

$$=\arctan(e^{x\ln x})+C=\arctan(x^x)+C.$$

51. $\int\frac{\sqrt{\ln\tan x}}{\sin 2x}dx=\frac{1}{2}\int\sqrt{\ln\tan x}\,d\ln\tan x=\frac{1}{2}\times\frac{2}{3}(\ln\tan x)^{\frac{3}{2}}+C$

$$=\frac{1}{3}(\ln\tan x)^{\frac{3}{2}}+C.$$

52. $\int\frac{(\ln\tan x)^2}{\cos x\sin x}dx=\int(\ln\tan x)^2d\ln\tan x=\frac{1}{3}(\ln\tan x)^3+C.$

53. $\int\frac{dx}{x\ln x\ln\ln x}=\int\frac{\frac{1}{x\ln x}dx}{\ln\ln x}=\int\frac{d\ln\ln x}{\ln\ln x}=\ln|\ln\ln x|+C.$

54. $\int\frac{1+\ln x}{(x\ln x)^2}dx=\int\frac{d(x\ln x)}{(x\ln x)^2}=-\frac{1}{x\ln x}+C.$

55. $\int \frac{(1+2x^2)e^{x^2}}{2-3xe^{x^2}}dx = \int \frac{d(xe^{x^2})}{2-3xe^{x^2}} = -\frac{1}{3}\int \frac{d(2-3xe^{x^2})}{2-3xe^{x^2}}$

$$= -\frac{1}{3}\ln|2-3xe^{x^2}|+C.$$

56. $\int e^{e^x\cos x}(\cos x-\sin x)e^x dx = \int e^{e^x\cos x}d(e^x\cos x) = e^{e^x\cos x}+C.$

57. $\int \frac{\cos x+x\sin x}{(x+\cos x)^2}dx = \int \frac{\cos x+x\sin x}{\cos^2 x(x\sec x+1)^2}dx$

$$= \int \frac{\sec x+x\sec x\tan x}{(x\sec x+1)^2}dx$$

$$= \int \frac{d(x\sec x+1)}{(x\sec x+1)^2} = -\frac{1}{x\sec x+1}+C.$$

58. $\int \frac{x+\sin x\cos x}{(x\sin x+\cos x)^2}dx = \int \frac{x+\sin x\cos x}{\cos^2 x(x\tan x+1)^2}dx$

$$= \int \frac{x\sec^2 x+\tan x}{(x\tan x+1)^2}dx = \int \frac{d(x\tan x+1)}{(x\tan x+1)^2}$$

$$= -\frac{1}{x\tan x+1}+C.$$

59. $\int \frac{dx}{(a\sin x+b\cos x)^2} = \int \frac{dx}{\cos^2 x(a\tan x+b)^2} = \frac{1}{a}\int \frac{d(a\tan x+b)}{(a\tan x+b)^2}$

$$= -\frac{1}{a}\frac{1}{a\tan x+b}+C.$$

60. $\int \frac{1-\ln x}{(x-\ln x)^2}dx = \int \frac{\frac{1-\ln x}{x^2}}{\left(\frac{x-\ln x}{x}\right)^2}dx = \int \frac{d\frac{\ln x}{x}}{\left(1-\frac{\ln x}{x}\right)^2}$

$$= -\int \frac{d\left(1-\frac{\ln x}{x}\right)}{\left(1-\frac{\ln x}{x}\right)^2} = \frac{1}{1-\frac{\ln x}{x}}+C$$

$$= \frac{x}{x-\ln x}+C.$$

61. $\int \frac{1+x}{x(1+xe^x)}dx = \int \frac{(1+x)e^x}{xe^x(1+xe^x)}dx = \int \frac{d(xe^x)}{xe^x(1+xe^x)}$

$$= \int \left(\frac{1}{x\mathrm{e}^x} - \frac{1}{1+x\mathrm{e}^x} \right) \mathrm{d}(x\mathrm{e}^x)$$

$$= \ln|x\mathrm{e}^x| - \ln|1+x\mathrm{e}^x| + C$$

$$= \ln\left|\frac{x\mathrm{e}^x}{1+x\mathrm{e}^x}\right| + C = x + \ln\left|\frac{x}{1+x\mathrm{e}^x}\right| + C.$$

62. $\displaystyle\int \frac{\ln x + 2}{x\ln x(1+x\ln^2 x)}\mathrm{d}x = \int \frac{\ln^2 x + 2\ln x}{x\ln^2 x(1+x\ln^2 x)}\mathrm{d}x$

$$= \int \frac{\mathrm{d}(x\ln^2 x)}{x\ln^2 x(1+x\ln^2 x)}$$

$$= \int \left(\frac{1}{x\ln^2 x} - \frac{1}{1+x\ln^2 x} \right) \mathrm{d}(x\ln^2 x)$$

$$= \ln\left|\frac{x\ln^2 x}{x\ln^2 x + 1}\right| + C.$$

63. 由$[(1+x^2)\arctan x]' = 2x\arctan x + 1$，

原积分$= \dfrac{1}{2}\displaystyle\int [(2x\arctan x + 1)\ln(1+x^2) - \ln(1+x^2)]\mathrm{d}x$

$$= \frac{1}{2}\left\{ \int \ln(1+x^2)\mathrm{d}[(1+x^2)\arctan x] - \int \ln(1+x^2)\mathrm{d}x \right\}$$

$$= \frac{1}{2}\Big[(1+x^2)\ln(1+x^2)\arctan x$$

$$- \int 2x\arctan x\,\mathrm{d}x - \int \ln(1+x^2)\mathrm{d}x \Big]$$

$$= \frac{1}{2}\Big[(1+x^2)\ln(1+x^2)\arctan x - x^2\arctan x$$

$$+ \int \frac{x^2}{1+x^2}\mathrm{d}x - x\ln(1+x^2) + \int \frac{2x^2}{1+x^2}\mathrm{d}x \Big]$$

$$= \frac{1}{2}[(1+x^2)\ln(1+x^2)\arctan x - x^2\arctan x + 3x - 3\arctan x -$$

$$x\ln(1+x^2)] + C.$$

64. $\displaystyle\int \sqrt{(x^2+x)\mathrm{e}^x}\,(x^2+3x+1)\mathrm{e}^x\,\mathrm{d}x = \int \sqrt{(x^2+x)\mathrm{e}^x}\,\mathrm{d}[(x^2+x)\mathrm{e}^x]$

$$= \frac{2}{3}[(x^2+x)\mathrm{e}^x]^{\frac{3}{2}} + C.$$

65. $\int\left\{\frac{f(x)}{f'(x)}-\frac{f^2(x)f''(x)}{[f'(x)]^3}\right\}\mathrm{d}x=\int\frac{f(x)}{f'(x)}\left\{1-\frac{f(x)f''(x)}{[f'(x)]^2}\right\}\mathrm{d}x$

$=\int\frac{f(x)}{f'(x)}\cdot\frac{[f'(x)]^2-f(x)f''(x)}{[f'(x)]^2}\mathrm{d}x$

$=\int\frac{f(x)}{f'(x)}\mathrm{d}\frac{f(x)}{f'(x)}=\frac{1}{2}\left[\frac{f(x)}{f'(x)}\right]^2+C.$

66. $\int(\sin 2x-\mathrm{e}^{\frac{x}{3}})\,\mathrm{d}x=-\frac{1}{2}\cos 2x-3\mathrm{e}^{\frac{x}{3}}+C.$

67. $\int\cos^2(\omega t+\varphi)\sin(\omega t+\varphi)\mathrm{d}t=-\frac{1}{\omega}\int\cos^2(\omega t+\varphi)\mathrm{d}\cos(\omega t+\varphi)$

$=-\frac{1}{3\omega}\cos^3(\omega t+\varphi)+C.$

68. $\int\frac{\sin x}{\cos^3 x}\mathrm{d}x=-\int\frac{\mathrm{d}\cos x}{\cos^3 x}=\frac{1}{2}\frac{1}{\cos^2 x}+C.$

69. $\int\frac{\arcsin(1-x)}{\sqrt{2x-x^2}}\mathrm{d}x=\int\frac{\arcsin(1-x)}{\sqrt{1-(1-x)^2}}\mathrm{d}x$

$=-\int\frac{\arcsin(1-x)}{\sqrt{1-(1-x)^2}}\mathrm{d}(1-x)$

$=-\int\arcsin(1-x)\mathrm{d}\arcsin(1-x)$

$=-\frac{1}{2}\arcsin^2(1-x)+C.$

70. $\int\frac{2^{\tan\frac{1}{x}}}{x^2}\sec^2\frac{1}{x}\mathrm{d}x=-\int 2^{\tan\frac{1}{x}}\sec^2\frac{1}{x}\mathrm{d}\frac{1}{x}=-\int 2^{\tan\frac{1}{x}}\mathrm{d}\tan\frac{1}{x}=-\frac{2^{\tan\frac{1}{x}}}{\ln 2}+C.$

71. $\int\tan^3 x\sec x\,\mathrm{d}x=\int\tan^2\cdot\tan x\sec x\,\mathrm{d}x=\int(\sec^2 x-1)\mathrm{d}\sec x$

$=\frac{1}{3}\sec^3 x-\sec x+C.$

72. $\int\sin^2 x\cos^3 x\,\mathrm{d}x=\int\sin^2 x\cos^2 x\,\mathrm{d}\sin x=\int\sin^2 x(1-\sin^2 x)\mathrm{d}\sin x$

$=\int(\sin^2 x-\sin^4 x)\mathrm{d}\sin x=\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x+C.$

73. 对于被积函数是 $\sin\triangle\sin\square$，$\sin\triangle\cos\square$，$\cos\triangle\cos\square$ 的积分，要用到中学学过的积化和差公式.

由 $\cos(7x+5x)=\cos 7x\cos 5x-\sin 7x\sin 5x$,

$\cos(7x-5x)=\cos 7x\cos 5x+\sin 7x\sin 5x$,

得 $\sin 7x\sin 5x=\frac{1}{2}(\cos 2x-\cos 12x)$.

故 $\int\sin 5x\sin 7x\,\mathrm{d}x=\frac{1}{2}\int(\cos 2x-\cos 12x)\mathrm{d}x$

$$=\frac{1}{2}\left(\frac{1}{2}\sin 2x-\frac{1}{12}\sin 12x\right)+C$$

$$=\frac{1}{4}\left(\sin 2x-\frac{1}{6}\sin 12x\right)+C.$$

74. $\int\cos x\cos\frac{x}{2}\mathrm{d}x=\frac{1}{2}\int\left(\cos\frac{3}{2}x+\cos\frac{x}{2}\right)\mathrm{d}x$

$$=\frac{1}{2}\left(\frac{2}{3}\sin\frac{3}{2}x+2\sin\frac{x}{2}\right)+C$$

$$=\frac{1}{3}\sin\frac{3}{2}x+\sin\frac{x}{2}+C.$$

75. $\int\sin 5x\cos 7x\,\mathrm{d}x=\frac{1}{2}\int(\sin 12x-\sin 2x)\mathrm{d}x$

$$=\frac{1}{2}\left(-\frac{1}{12}\cos 12x+\frac{1}{2}\cos 2x\right)+C$$

$$=\frac{1}{4}\left(\cos 2x-\frac{1}{6}\cos 12x\right)+C.$$

76. $\int\frac{\cot x}{\sqrt{\sin x}}\mathrm{d}x=\int\frac{\cos x}{\sin^{\frac{3}{2}}x}\mathrm{d}x=\frac{-2}{\sqrt{\sin x}}+C.$

做完这道题要把$\int\frac{1}{x^{\frac{3}{2}}}\mathrm{d}x=\frac{-2}{\sqrt{x}}+C$ 积累下来.

77. $\int\frac{\tan x}{\cos^3 x}\mathrm{d}x=\int\frac{\sin x}{\cos^4 x}\mathrm{d}x=-\int\frac{\mathrm{d}\cos x}{\cos^4 x}=\frac{1}{3}\frac{1}{\cos^3 x}+C.$

背后利用的是$\int\frac{1}{x^4}\mathrm{d}x=-\frac{1}{3}\frac{1}{x^3}+C$ 基本积分.

78. $\int\sec^3 x\,\mathrm{d}x=\int\sec x\cdot\sec^2 x\,\mathrm{d}x=\int\sec x\,\mathrm{d}\tan x$

$$=\sec x\tan x-\int\tan^2 x\sec x\,\mathrm{d}x$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx,$$

故原积分 $= \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$

79. $\int \sec^4 x \, dx = \int \sec^2 x (1 + \tan^2 x) dx = \int (\sec^2 x + \sec^2 x \tan^2 x) dx$

$$= \tan x + \frac{1}{3} \tan^3 x + C.$$

80. $\int \sec^5 x \, dx = \int \sec^3 x \, d\tan x = \sec^3 x \tan x - 3\int \tan^2 x \sec^3 x \, dx$

$$= \sec^3 x \tan x - 3\int (\sec^2 x - 1) \sec^3 x \, dx,$$

故原积分 $= \frac{1}{4} \sec^3 x \tan x + \frac{3}{4} \times \frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) + C$

$$= \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} (\sec x \tan x + \ln |\sec x + \tan x|) + C.$$

81. $\int \sec^6 x \, dx = \int \sec^4 x \cdot \sec^2 x \, dx = \int (1 + \tan^2 x)^2 d\tan x$

$$= \int (1 + 2\tan^2 x + \tan^4 x) d\tan x$$

$$= \tan x + \frac{2}{3} \tan^3 x + \frac{1}{5} \tan^5 x + C.$$

82. $\int \csc^3 x \, dx = -\int \csc x \, d\cot x = -\csc x \cot x - \int \cot^2 x \csc x \, dx$

$$= -\csc x \cot x - \int (\csc^2 x - 1) \csc x \, dx,$$

故原积分 $= -\frac{1}{2} \csc x \cot x + \frac{1}{2} \ln |\csc x - \cot x| + C.$

83. $\int \csc^4 x \, dx = -\int (1 + \cot^2 x) d\cot x = -\cot x - \frac{1}{3} \cot^3 x + C.$

84. $\int \csc^5 x \, dx = -\int \csc^3 x \, d\cot x = -\csc^3 x \cot x - 3\int \cot^2 x \csc^3 x \, dx$

$$= -\csc^3 x \cot x - 3\int (\csc^2 x - 1) \csc^3 x \, dx,$$

故原积分 $= -\frac{1}{4} \csc^3 x \cot x + \frac{3}{4} \left(-\frac{1}{2} \csc x \cot x + \right.$

$$\frac{1}{2}\ln|\csc x-\cot x|\Big)+C$$

$$=-\frac{1}{4}\csc^3 x\cot x-\frac{3}{8}(\csc x\cot x-\ln|\csc x-\cot x|)+C.$$

85. $\int\tan x(\tan x+1)\mathrm{d}x=\int(\tan^2 x+\tan x)\mathrm{d}x$

$$=\tan x-x-\ln|\cos x|+C.$$

86. $\int\sqrt{\tan x}\,\mathrm{d}x\xlongequal[\substack{x=\arctan u^2\\ \mathrm{d}x=\frac{2u}{1+u^4}\mathrm{d}u}]{u=\sqrt{\tan x}}\int u\cdot\frac{2u}{1+u^4}\mathrm{d}u=\int\frac{2u^2}{1+u^4}\mathrm{d}u$

$$=\int\frac{u^2+1+u^2-1}{1+u^4}\mathrm{d}u=\int\frac{u^2+1}{1+u^4}\mathrm{d}u+\int\frac{u^2-1}{1+u^4}\mathrm{d}u$$

$$=\int\frac{1+\frac{1}{u^2}}{\frac{1}{u^2}+u^2}\mathrm{d}u+\int\frac{1-\frac{1}{u^2}}{\frac{1}{u^2}+u^2}\mathrm{d}u$$

$$=\int\frac{\mathrm{d}\left(u-\frac{1}{u}\right)}{\left(u-\frac{1}{u}\right)^2+2}+\int\frac{\mathrm{d}\left(u+\frac{1}{u}\right)}{\left(u+\frac{1}{u}\right)^2-2}$$

$$=\frac{1}{\sqrt{2}}\arctan\frac{u-\frac{1}{u}}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln\left|\frac{u+\frac{1}{u}-\sqrt{2}}{u+\frac{1}{u}+\sqrt{2}}\right|+C$$

$$=\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{\tan x}-\frac{1}{\sqrt{\tan x}}}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}-\sqrt{2}}{\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}+\sqrt{2}}\right)+C.$$

87. $\int\tan^4 x\,\mathrm{d}x=\int(\sec^2 x-1)\tan^2 x\,\mathrm{d}x=\int(\sec^2 x\tan^2 x-\tan^2 x)\mathrm{d}x$

$$=\frac{1}{3}\tan^3 x-\tan x+x+C.$$

88. $\int\tan^5 x\,\mathrm{d}x=\int(\sec^2 x-1)\tan^3 x\,\mathrm{d}x=\int(\sec^2 x\tan^3 x-\tan^3 x)\mathrm{d}x$

$$=\int\tan^3 x\,\mathrm{d}\tan x+\int\frac{1-\cos^2 x}{\cos^3 x}\mathrm{d}\cos x$$

$$=\frac{1}{4}\tan^4 x-\frac{1}{2}\frac{1}{\cos^2 x}-\ln|\cos x|+C.$$

89. $\int x e^x dx = x e^x - e^x + C.$

90. $\int (1+x)e^x dx = \int (x e^x)' dx = x e^x + C.$

91. $\int e^x [f(x) + f'(x)] dx = \int [e^x f(x)]' dx = e^x f(x) + C.$

92. $\int e^x \frac{1-x}{x^2} dx = -\int \frac{e^x(x-1)}{x^2} dx = -\int \left(\frac{e^x}{x}\right)' dx = -\frac{e^x}{x} + C.$

93. $\int \frac{x^3}{x^8-2} dx = \frac{1}{4}\int \frac{d(x^4)}{x^8-2} = \frac{1}{4} \times \frac{1}{2\sqrt{2}} \ln \left| \frac{x^4-\sqrt{2}}{x^4+\sqrt{2}} \right| + C$

$$= \frac{1}{8\sqrt{2}} \ln \left| \frac{x^4-\sqrt{2}}{x^4+\sqrt{2}} \right| + C.$$

94. $\int \frac{dx}{\sqrt{x}(1+x)} = 2\int \frac{d\sqrt{x}}{1+(\sqrt{x})^2} = 2\arctan\sqrt{x} + C.$

95. $\int \frac{\sin x \cos x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} dx (|a| \neq |b|)$

$$= \frac{1}{2}\int \frac{d\sin^2 x}{\sqrt{a^2 \sin^2 x + b^2 \cos^2 x}} = \frac{1}{2}\int \frac{d\sin^2 x}{\sqrt{a^2 \sin^2 x + b^2(1-\sin^2 x)}}$$

$$= \frac{1}{2}\int \frac{d\sin^2 x}{\sqrt{(a^2-b^2)\sin^2 x + b^2}} = \frac{1}{2} \frac{1}{a^2-b^2} 2\sqrt{(a^2-b^2)\sin^2 x + b^2} + C$$

$$= \frac{\sqrt{(a^2-b^2)\sin^2 x + b^2}}{a^2-b^2} + C.$$

96. $\int \frac{dx}{\sin^2 x \sqrt[4]{\cot x}} = \int \frac{\csc^2 x dx}{\sqrt[4]{\cot x}} = -\int \frac{d\cot x}{\sqrt[4]{\cot x}} = -\frac{4}{3}(\cot x)^{\frac{3}{4}} + C.$

97. $\int \frac{dx}{\sin x + \cos x} = \int \frac{dx}{\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)}$

$$= \frac{1}{\sqrt{2}} \ln \left| \csc\left(x + \frac{\pi}{4}\right) - \cot\left(x + \frac{\pi}{4}\right) \right| + C.$$

98. 设$\int \frac{2\sin x + 3\cos x}{5\sin x + 7\cos x} dx = \int \frac{A(5\sin x + 7\cos x) + B(5\cos x - 7\sin x) dx}{5\sin x + 7\cos x}.$

由待定系数法，$\begin{cases} 5A - 7B = 2, \\ 7A + 5B = 3, \end{cases}$解得 $A = \frac{31}{74}$，$B = \frac{1}{74}$.

故原积分 $= \frac{31}{74}x + \frac{1}{74}\ln|5\sin x + 7\cos x| + C.$

99. $\int \frac{\mathrm{d}x}{\sin^2 x+2\cos^2 x}=\int \frac{\mathrm{d}\tan x}{\tan^2 x+2}=\frac{1}{\sqrt{2}}\arctan\frac{\tan x}{\sqrt{2}}+C.$

100. $\int \frac{x^2-1}{x^4+1}\mathrm{d}x=\int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\mathrm{d}x=\int \frac{\mathrm{d}\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2}$

$$=\frac{1}{2\sqrt{2}}\ln\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+C.$$

101. $\int \frac{\cos x}{\sqrt{2+\cos 2x}}\mathrm{d}x=\int \frac{\mathrm{d}\sin x}{\sqrt{3-2\sin^2 x}}=\frac{1}{\sqrt{2}}\arcsin\frac{\sqrt{6}\sin x}{3}+C.$

102. $\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\mathrm{d}x=\frac{1}{2}\int \frac{\mathrm{d}\sin^2 x}{\sin^4 x+\cos^4 x}=\frac{1}{2}\int \frac{\mathrm{d}\sin^2 x}{\sin^4 x+(1-\sin^2 x)^2}$

$$=\frac{1}{2}\int \frac{\mathrm{d}\sin^2 x}{\sin^4 x+1-2\sin^2 x+\sin^4 x}$$

$$\xlongequal{u=\sin^2 x}\frac{1}{2}\int \frac{\mathrm{d}u}{2u^2-2u+1}=\frac{1}{4}\int \frac{\mathrm{d}u}{u^2-u+\frac{1}{2}}$$

$$=\frac{1}{4}\int \frac{\mathrm{d}u}{\left(u-\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$=\frac{1}{4}\times\frac{1}{\frac{1}{2}}\arctan\frac{u-\frac{1}{2}}{\frac{1}{2}}+C$$

$$=\frac{1}{2}\arctan\left[2(\sin^2 x-\frac{1}{2})\right]+C$$

$$=-\frac{1}{2}\arctan\cos 2x+C.$$

103. $\int \frac{\arctan x}{1+x^2}\mathrm{d}x=\int \arctan x\,\mathrm{d}\arctan x=\frac{1}{2}(\arctan x)^2+C.$

104. $\int \frac{\mathrm{d}x}{\sqrt{1-x^2}\arcsin x}=\int \frac{\mathrm{d}\arcsin x}{\arcsin x}=\ln|\arcsin x|+C.$

105. $\int \frac{1}{1-x^2}\ln\frac{1+x}{1-x}\mathrm{d}x=\frac{1}{2}\int \ln\frac{1+x}{1-x}\mathrm{d}\ln\frac{1+x}{1-x}$

$$= \frac{1}{4}\left(\ln\frac{1+x}{1-x}\right)^2 + C.$$

106. $$\int \frac{\sin x\cos x}{\sin x+\cos x}\mathrm{d}x = \frac{1}{2}\int \frac{(\sin x+\cos x)^2-1}{\sin x+\cos x}\mathrm{d}x$$

$$= \frac{1}{2}\int\left(\sin x+\cos x-\frac{1}{\sin x+\cos x}\right)\mathrm{d}x$$

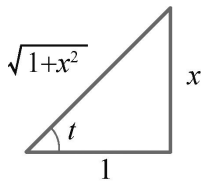
$$= \frac{1}{2}\left[\sin x-\cos x-\frac{1}{\sqrt{2}}\ln\left|\csc\left(x+\frac{\pi}{4}\right)\right.\right.$$

$$\left.\left.-\cot\left(x+\frac{\pi}{4}\right)\right|\right]+C.$$

107. $$\int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x \xlongequal[\mathrm{d}x=\sec^2 t\,\mathrm{d}t]{x=\tan t} \int \frac{\ln\tan t}{\sec^3 t}\sec^2 t\,\mathrm{d}t$$

$$= \int \cos t\ln\tan t\,\mathrm{d}t$$

$$= \int \ln\tan t\,\mathrm{d}\sin t$$



$$= \sin t\ln\tan t-\int \sin t\,\frac{\mathrm{d}t}{\cos t\sin t}$$

$$= \sin t\ln\tan t-\ln|\sec t+\tan t|+C$$

$$= \frac{x}{\sqrt{1+x^2}}\ln x-\ln(\sqrt{1+x^2}+x)+C.$$

注：对于凑微分法的题目，解法往往不唯一，读者可以自行完成.

真题自测

(1989) $$\int \frac{\mathrm{d}x}{x\ln^2 x} = \int \frac{\mathrm{d}\ln x}{\ln^2 x} = -\frac{1}{\ln x}+C.$$

(1992) $$\int \frac{x^3}{\sqrt{1+x^2}}\mathrm{d}x = \frac{1}{2}\int \frac{x^2}{\sqrt{1+x^2}}\mathrm{d}(x^2) = \int x^2\,\mathrm{d}\sqrt{1+x^2}$$

$$= x^2\sqrt{1+x^2}-\int \sqrt{1+x^2}\,\mathrm{d}(x^2+1)$$

$$= x^2\sqrt{1+x^2}-\frac{2}{3}(1+x^2)^{\frac{3}{2}}+C.$$

(1995) 由 $f(x^2-1)=\ln\dfrac{x^2}{x^2-2}$，令 $t=x^2-1$，$x^2=1+t$，则

$$f(t)=\ln\frac{1+t}{t-1},\ f[\varphi(x)]=\ln\frac{\varphi(x)+1}{\varphi(x)-1}=\ln x,$$

所以$\dfrac{\varphi(x)+1}{\varphi(x)-1}=x$，反解 $\varphi(x)=\dfrac{x+1}{x-1}$.

$$\int\varphi(x)\mathrm{d}x=\int\frac{x+1}{x-1}\mathrm{d}x=\int\frac{x-1+2}{x-1}\mathrm{d}x=\int\left(1+\frac{2}{x-1}\right)\mathrm{d}x$$
$$=x+2\ln|x-1|+C.$$

(1997) $\displaystyle\int\frac{\mathrm{d}x}{\sqrt{x(4-x)}}=2\int\frac{\mathrm{d}\sqrt{x}}{\sqrt{4-(\sqrt{x})^2}}=2\arcsin\frac{\sqrt{x}}{2}+C.$

(1995) $$\int\frac{x+5}{x^2-6x+13}\mathrm{d}x=\int\frac{\frac{1}{2}(2x-6)+8}{x^2-6x+13}\mathrm{d}x$$
$$=\frac{1}{2}\ln|x^2-6x+13|+8\int\frac{\mathrm{d}x}{(x-3)^2+4}$$
$$=\frac{1}{2}\ln(x^2-6x+13)+4\arctan\frac{x-3}{2}+C.$$

(2004) 令 $\mathrm{e}^x=t$，$x=\ln t$，$\mathrm{e}^{-x}=\dfrac{1}{t}$，则 $f'(t)=\dfrac{\ln t}{t}$，

$$f(t)=\int f'(t)\mathrm{d}t=\int\frac{\ln t}{t}\mathrm{d}t=\frac{1}{2}(\ln t)^2+C,$$

即 $f(x)=\dfrac{1}{2}(\ln x)^2+C$. 由 $f(1)=0$，$f(1)=0+C=0$，故 $f(x)=\dfrac{1}{2}(\ln x)^2$.

(1992) $$I=\int\frac{\operatorname{arccot}(\mathrm{e}^x)}{\mathrm{e}^x}\mathrm{d}x=-\int\operatorname{arccot}(\mathrm{e}^x)\mathrm{d}(\mathrm{e}^{-x})$$
$$=-\mathrm{e}^{-x}\operatorname{arccot}(\mathrm{e}^x)-\int\mathrm{e}^{-x}\frac{\mathrm{e}^x}{1+\mathrm{e}^{2x}}\mathrm{d}x$$
$$=-\mathrm{e}^{-x}\operatorname{arccot}(\mathrm{e}^x)-\int\frac{1+\mathrm{e}^{2x}-\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}\mathrm{d}x$$
$$=-\mathrm{e}^{-x}\operatorname{arccot}(\mathrm{e}^x)-x+\frac{1}{2}\ln(1+\mathrm{e}^{2x})+C.$$

(1998) $$\int\frac{\ln x-1}{x^2}\mathrm{d}x=-\int(\ln x-1)\mathrm{d}\frac{1}{x}=-\frac{\ln x-1}{x}+\int\frac{1}{x^2}\mathrm{d}x$$
$$=\frac{1-\ln x}{x}-\frac{1}{x}=\frac{-\ln x}{x}+C.$$

(1996) 由$\int xf(x)\mathrm{d}x=\arcsin x+C$，则

$$xf(x)=(\arcsin x+C)'=\frac{1}{\sqrt{1-x^2}},\ f(x)=\frac{1}{x\sqrt{1-x^2}},$$

则 $$\int\frac{1}{f(x)}\mathrm{d}x=\int x\sqrt{1-x^2}\,\mathrm{d}x=-\frac{1}{2}\times\frac{2}{3}(1-x^2)^{\frac{3}{2}}+C$$

$$=-\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C.$$

(1993) $$\int\frac{x\mathrm{e}^x}{\sqrt{\mathrm{e}^x-1}}\mathrm{d}x=\int\frac{x\mathrm{d}(\mathrm{e}^x)}{\sqrt{\mathrm{e}^x-1}}=2\int x\mathrm{d}\sqrt{\mathrm{e}^x-1}$$

$$=2(x\sqrt{\mathrm{e}^x-1}-\int\sqrt{\mathrm{e}^x-1}\,\mathrm{d}x).$$

令 $\sqrt{\mathrm{e}^x-1}=t$，$\mathrm{e}^x=t^2+1$，$x=\ln(t^2+1)$，$\mathrm{d}x=\dfrac{2t}{t^2+1}\mathrm{d}t$，

$$\int\sqrt{\mathrm{e}^x-1}\,\mathrm{d}x=\int t\frac{2t}{t^2+1}\mathrm{d}t=2(t-\arctan t)+C$$

$$=2(\sqrt{\mathrm{e}^x-1}-\arctan\sqrt{\mathrm{e}^x-1})+C,$$

原积分 $=2\left[x\sqrt{\mathrm{e}^x-1}-2(\sqrt{\mathrm{e}^x-1}-\arctan\sqrt{\mathrm{e}^x-1})\right]+C$

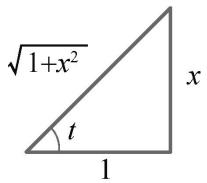
$$=2x\sqrt{\mathrm{e}^x-1}-4\sqrt{\mathrm{e}^x-1}+4\arctan\sqrt{\mathrm{e}^x-1}+C.$$

(2001) $$\int\frac{\mathrm{d}x}{(2x^2+1)\sqrt{1+x^2}}\xlongequal[\mathrm{d}x=\sec^2 t\mathrm{d}t]{x=\tan t}\int\frac{\sec^2 t\mathrm{d}t}{(2\tan^2 t+1)\cdot\sec t}$$

$$=\int\frac{\sec t\mathrm{d}t}{2\tan^2 t+1}=\int\frac{\cos t\mathrm{d}t}{2\sin^2 t+\cos^2 t}$$

$$=\int\frac{\mathrm{d}\sin t}{1+\sin^2 t}=\arctan\sin t+C$$

$$=\arctan\frac{x}{\sqrt{1+x^2}}+C.$$



(2002) 令 $\sin^2 x=t$，$\sin x=\sqrt{t}$，$x=\arcsin\sqrt{t}$，

则 $f(x)=\dfrac{\arcsin\sqrt{x}}{\sqrt{x}}$，代入原积分：

$$\int\frac{\sqrt{x}}{\sqrt{1-x}}f(x)\mathrm{d}x=\int\frac{\sqrt{x}}{\sqrt{1-x}}\frac{\arcsin\sqrt{x}}{\sqrt{x}}\mathrm{d}x$$

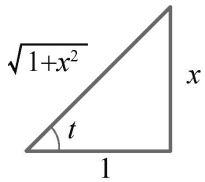
$$=\int\frac{\arcsin\sqrt{x}}{\sqrt{1-x}}\mathrm{d}x=-2\int\arcsin\sqrt{x}\,\mathrm{d}\sqrt{1-x}$$

$$= -2\left(\sqrt{1-x}\arcsin\sqrt{x} - \int \sqrt{1-x}\cdot\frac{1}{\sqrt{1-x}}\cdot\frac{1}{2\sqrt{x}}\mathrm{d}x\right)$$

$$= -2\sqrt{1-x}\arcsin\sqrt{x} + 2\sqrt{x} + C.$$

(2003) 令 $x = \tan t$，$\mathrm{d}x = \sec^2 t\,\mathrm{d}t$，代入得

$$\int \frac{x\mathrm{e}^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x = \int \frac{\tan t\,\mathrm{e}^t}{\sec^3 t}\cdot\sec^2 t\,\mathrm{d}t$$



$$= \int \frac{\tan t\cdot\mathrm{e}^t}{\sec t}\mathrm{d}t$$

$$= \int \sin t\cdot\mathrm{e}^t\,\mathrm{d}t$$

$$= \frac{1}{2}(\mathrm{e}^t\sin t - \mathrm{e}^t\cos t) + C$$

$$= \frac{1}{2}\left(\mathrm{e}^{\arctan x}\frac{x}{\sqrt{1+x^2}} - \mathrm{e}^{\arctan x}\frac{1}{\sqrt{1+x^2}}\right) + C$$

$$= \frac{1}{2}\mathrm{e}^{\arctan x}\frac{x-1}{\sqrt{1+x^2}} + C.$$

(2006) $$\int \frac{\arcsin(\mathrm{e}^x)}{\mathrm{e}^x}\mathrm{d}x = -\int \arcsin(\mathrm{e}^x)\mathrm{d}(\mathrm{e}^{-x})$$

$$= -\arcsin(\mathrm{e}^x)\cdot\mathrm{e}^{-x} + \int \mathrm{e}^{-x}\frac{\mathrm{e}^x}{\sqrt{1-\mathrm{e}^{2x}}}\mathrm{d}x$$

$$= -\mathrm{e}^{-x}\arcsin(\mathrm{e}^x) - \int \frac{\mathrm{d}(\mathrm{e}^{-x})}{\sqrt{\mathrm{e}^{-2x}-1}}$$

$$= -\mathrm{e}^{-x}\arcsin(\mathrm{e}^x) - \ln\left(\mathrm{e}^{-x} + \sqrt{\mathrm{e}^{-2x}-1}\right) + C.$$

§3.3 利用分部积分法求不定积分

基础篇

1. $\int x\sin x\,\mathrm{d}x = -x\cos x + \sin x + C.$

2. $\int x\cos x\,\mathrm{d}x = x\sin x + \cos x + C.$

3. $\int x\mathrm{e}^x\,\mathrm{d}x=\int x\,\mathrm{d}(\mathrm{e}^x)=x\mathrm{e}^x-\int \mathrm{e}^x\,\mathrm{d}x=x\mathrm{e}^x-\mathrm{e}^x+C.$

4. $\int x\ln x\,\mathrm{d}x=\frac{1}{2}\int \ln x\,\mathrm{d}(x^2)=\frac{1}{2}\left(x^2\ln x-\int x^2\cdot\frac{1}{x}\mathrm{d}x\right)$

$$=\frac{1}{2}(x^2\ln x-\frac{1}{2}x^2)+C.$$

5. $\int x^2\ln x\,\mathrm{d}x=\frac{1}{3}\int \ln x\,\mathrm{d}(x^3)=\frac{1}{3}\left(x^3\ln x-\int x^3\cdot\frac{1}{x}\mathrm{d}x\right)$

$$=\frac{1}{3}\left(x^3\ln x-\frac{1}{3}x^3\right)+C.$$

6. $\int x\mathrm{e}^{-x}\,\mathrm{d}x=-\int x\,\mathrm{d}(\mathrm{e}^{-x})=-x\mathrm{e}^{-x}+\int \mathrm{e}^{-x}\,\mathrm{d}x=-x\mathrm{e}^{-x}-\mathrm{e}^{-x}+C.$

7. $\int \mathrm{e}^x\sin x\,\mathrm{d}x=\frac{1}{2}\begin{vmatrix}(\mathrm{e}^x)' & (\sin x)'\\ \mathrm{e}^x & \sin x\end{vmatrix}+C=\frac{1}{2}(\mathrm{e}^x\sin x-\mathrm{e}^x\cos x)+C.$

第 8 ～ 11 题做法相同.

8. $\int \mathrm{e}^x\cos x\,\mathrm{d}x=\frac{1}{2}(\mathrm{e}^x\cos x+\mathrm{e}^x\sin x)+C.$

9. $\int \mathrm{e}^{2x}\sin 3x\,\mathrm{d}x=\frac{1}{13}(2\mathrm{e}^{2x}\sin 3x-3\mathrm{e}^{2x}\cos 3x)+C.$

10. $\int \mathrm{e}^{ax}\sin bx\,\mathrm{d}x=\frac{1}{a^2+b^2}(a\mathrm{e}^{ax}\sin bx-b\mathrm{e}^{ax}\cos bx)+C.$

11. $\int \mathrm{e}^{ax}\cos bx\,\mathrm{d}x=\frac{1}{a^2+b^2}(a\mathrm{e}^{ax}\cos bx+b\mathrm{e}^{ax}\sin bx)+C.$

第 10、11 题请读者自行完成推导，其结果最好能加以记忆.

12. $\int x^2\arctan x\,\mathrm{d}x=\frac{1}{3}\int \arctan x\,\mathrm{d}(x^3)$

$$=\frac{1}{3}(x^3\arctan x-\int\frac{x^3}{1+x^2}\mathrm{d}x)$$

$$=\frac{1}{3}\left[x^3\arctan x-\frac{1}{2}\int\frac{1+x^2-1}{1+x^2}\mathrm{d}(x^2)\right]$$

$$=\frac{1}{3}\left\{x^3\arctan x-\frac{1}{2}[x^2-\ln(1+x^2)]\right\}+C$$

$$=\frac{1}{3}x^3\arctan x-\frac{1}{6}x^2+\frac{1}{6}\ln(1+x^2)+C.$$

13. 由表格计算法：

$$\begin{array}{ccccccc}
x^2 & \longrightarrow & 2x & \longrightarrow & 2 & \longrightarrow & 0 \\
 & + & & - & & + & \\
\sin x & \longrightarrow & -\cos x & \longrightarrow & -\sin x & \longrightarrow & \cos x
\end{array}$$

原积分 $= -x^2\cos x + 2x\sin x + 2\cos x + C$.

14. 由表格计算法：

$$\begin{array}{ccccccc}
x^2 & \longrightarrow & 2x & \longrightarrow & 2 & \longrightarrow & 0 \\
 & + & & - & & + & \\
\cos x & \longrightarrow & \sin x & \longrightarrow & -\cos x & \longrightarrow & -\sin x
\end{array}$$

原积分 $= x^2\sin x + 2x\cos x - 2\sin x + C$.

15. 由表格计算法：

$$\begin{array}{ccccccc}
x^2 & \longrightarrow & 2x & \longrightarrow & 2 & \longrightarrow & 0 \\
 & + & & - & & + & \\
\mathrm{e}^x & \longrightarrow & \mathrm{e}^x & \longrightarrow & \mathrm{e}^x & \longrightarrow & \mathrm{e}^x
\end{array}$$

原积分 $= x^2\mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x + C$.

16. 由表格计算法：

$$\begin{array}{ccccccc}
x^2 & \longrightarrow & 2x & \longrightarrow & 2 & \longrightarrow & 0 \\
 & + & & - & & + & \\
\mathrm{e}^{-2x} & \longrightarrow & -\frac{1}{2}\mathrm{e}^{-2x} & \longrightarrow & \frac{1}{4}\mathrm{e}^{-2x} & \longrightarrow & -\frac{1}{8}\mathrm{e}^{-2x}
\end{array}$$

原积分 $= -\frac{1}{2}x^2\mathrm{e}^{-2x} - \frac{1}{2}x\mathrm{e}^{-2x} - \frac{1}{4}\mathrm{e}^{-2x} + C$.

17. 由表格计算法：

$$\begin{array}{ccccccc}
x^2 & \longrightarrow & 2x & \longrightarrow & 2 & \longrightarrow & 0 \\
 & + & & - & & + & \\
\sin 3x & \longrightarrow & -\frac{1}{3}\cos 3x & \longrightarrow & -\frac{1}{9}\sin 3x & \longrightarrow & \frac{1}{27}\cos 3x
\end{array}$$

原积分 $= -\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x + C.$

18. $\int \ln^2 x\,\mathrm{d}x = x\ln^2 x - 2\int x\,\frac{\ln x}{x}\mathrm{d}x$

$= x\ln^2 x - 2(x\ln x - x) + C.$

19. $\int x\ln(x-1)\mathrm{d}x = \frac{1}{2}\int \ln(x-1)\mathrm{d}(x^2)$

$= \frac{1}{2}\left[x^2\ln(x-1) - \int \frac{x^2}{x-1}\mathrm{d}x\right]$

$= \frac{1}{2}\left[x^2\ln(x-1) - \int\left(x+1+\frac{1}{x-1}\right)\mathrm{d}x\right]$

$= \frac{1}{2}\left[x^2\ln(x-1) - \frac{1}{2}x^2 - x - \ln|x-1|\right] + C$

$= \frac{1}{2}x^2\ln(x-1) - \frac{1}{4}x^2 - \frac{1}{2}x - \frac{1}{2}\ln(x-1) + C.$

20. $\int \frac{\ln^3 x}{x^2}\mathrm{d}x \xlongequal[\substack{x=\mathrm{e}^t \\ \mathrm{d}x=\mathrm{e}^t\mathrm{d}t}]{\ln x = t} \int \frac{t^3}{\mathrm{e}^{2t}}\mathrm{e}^t\,\mathrm{d}t = \int t^3\mathrm{e}^{-t}\,\mathrm{d}t.$

由表格计算法：

$$\begin{array}{ccccccccc}
t^3 & \longrightarrow & 3t^2 & \longrightarrow & 6t & \longrightarrow & 6 & \longrightarrow & 0 \\
 & + & & - & & + & & - & \\
\mathrm{e}^{-t} & \longrightarrow & -\mathrm{e}^{-t} & \longrightarrow & \mathrm{e}^{-t} & \longrightarrow & -\mathrm{e}^{-t} & \longrightarrow & \mathrm{e}^{-t}
\end{array}$$

原积分 $= -t^3\mathrm{e}^{-t} - 3t^2\mathrm{e}^{-t} - 6t\mathrm{e}^{-t} - 6\mathrm{e}^{-t} + C$

$= -(\ln x)^3\,\frac{1}{x} - 3(\ln x)^2\,\frac{1}{x} - 6\ln x\,\frac{1}{x} - \frac{6}{x} + C.$

21. $\int\left(\frac{\ln x}{x}\right)^3\mathrm{d}x = \int \frac{\ln^3 x}{x^3}\mathrm{d}x \xrightarrow[\substack{x=\mathrm{e}^t \\ \mathrm{d}x=\mathrm{e}^t\mathrm{d}t}]{\ln x = t} \int \frac{t^3}{\mathrm{e}^{3t}}\mathrm{e}^t\,\mathrm{d}t = \int t^3\mathrm{e}^{-2t}\,\mathrm{d}t.$

由表格计算法：

$$\begin{array}{ccccccccc}
t^3 & \longrightarrow & 3t^2 & \longrightarrow & 6t & \longrightarrow & 6 & \longrightarrow & 0 \\
 & + & & - & & + & & - & \\
\mathrm{e}^{-2t} & \longrightarrow & -\frac{1}{2}\mathrm{e}^{-2t} & \longrightarrow & \frac{1}{4}\mathrm{e}^{-2t} & \longrightarrow & -\frac{1}{8}\mathrm{e}^{-2t} & \longrightarrow & \frac{1}{16}\mathrm{e}^{-2t}
\end{array}$$

原积分 $= -\frac{1}{2}t^3\mathrm{e}^{-2t} - \frac{3}{4}t^2\mathrm{e}^{-2t} - \frac{3}{4}t\mathrm{e}^{-2t} - \frac{3}{8}\mathrm{e}^{-2t} + C$

$= -\frac{1}{2}(\ln x)^3\,\frac{1}{x^2} - \frac{3}{4}(\ln x)^2\,\frac{1}{x^2} - \frac{3}{4}\ln x\,\frac{1}{x^2} - \frac{3}{8}\,\frac{1}{x^2} + C.$

22. $\int x\sin x\cos x\,\mathrm{d}x = \frac{1}{2}\int x\sin 2x\,\mathrm{d}x = -\frac{1}{4}\int x\,\mathrm{d}\cos 2x$

$= -\frac{1}{4}(x\cos 2x - \frac{1}{2}\sin 2x) + C.$

23. $\int(\arcsin x)^2\mathrm{d}x = x(\arcsin x)^2 - 2\int x\frac{\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x$

$= x(\arcsin x)^2 + 2\int\arcsin x\,\mathrm{d}\sqrt{1-x^2}$

$= x(\arcsin x)^2 + 2(\sqrt{1-x^2}\arcsin x - x) + C.$

24. $\int \mathrm{e}^{\sqrt{3x+9}}\mathrm{d}x \xlongequal[3x+9=t^2]{\sqrt{3x+9}=t} \frac{2}{3}\int \mathrm{e}^t\cdot t\,\mathrm{d}t = \frac{2}{3}(t\mathrm{e}^t - \mathrm{e}^t) + C$

$\mathrm{d}x = \frac{2}{3}t\,\mathrm{d}t$

$= \frac{2}{3}(\sqrt{3x+9}\,\mathrm{e}^{\sqrt{3x+9}} - \mathrm{e}^{\sqrt{3x+9}}) + C.$

25. $\int x(\ln x)^2\mathrm{d}x = \frac{1}{2}\int(\ln x)^2\mathrm{d}(x^2) = \frac{1}{2}\left(x^2\ln^2 x - 2\int x^2\frac{\ln x}{x}\mathrm{d}x\right)$

$= \frac{1}{2}\left[x^2\ln^2 x - \int\ln x\,\mathrm{d}(x^2)\right]$

$= \frac{1}{2}\left[x^2\ln^2 x - \left(x^2\ln x - \frac{1}{2}x^2\right)\right] + C.$

26. $\int\arctan x\,\mathrm{d}x = x\arctan x - \int\frac{x}{1+x^2}\mathrm{d}x$

$= x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$

27. $\int\arcsin x\,\mathrm{d}x = x\arcsin x - \int\frac{x}{\sqrt{1-x^2}}\mathrm{d}x$

$= x\arcsin x + \sqrt{1-x^2} + C.$

28. $\int\frac{\arcsin x}{x^2}\mathrm{d}x = -\int\arcsin x\,\mathrm{d}\frac{1}{x} = -\frac{\arcsin x}{x} + \int\frac{1}{x}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

$= -\frac{\arcsin x}{x} - \ln\left|\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}\right| + C.$

29. $\int x^2\arcsin x\,\mathrm{d}x = \frac{1}{3}\int\arcsin x\,\mathrm{d}(x^3)$

$= \frac{1}{3}\left(x^3\arcsin x - \int\frac{x^3}{\sqrt{1-x^2}}\mathrm{d}x\right)$

$= \frac{1}{3}\left(x^3\arcsin x + \int x^2\,\mathrm{d}\sqrt{1-x^2}\right)$

$$= \frac{1}{3}\left[x^3\arcsin x + x^2\sqrt{1-x^2} + \frac{2}{3}(1-x^2)^{\frac{3}{2}}\right] + C.$$

30. $\int \ln(x+\sqrt{1+x^2})\mathrm{d}x = x\ln(x+\sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}}\mathrm{d}x$

$$= x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C.$$

31. $\int x^5 \mathrm{e}^{x^3}\mathrm{d}x = \int x^2 \cdot x^3 \cdot \mathrm{e}^{x^3}\mathrm{d}x = \frac{1}{3}\int x^3 \mathrm{d}(\mathrm{e}^{x^3})$

$$= \frac{1}{3}(x^3\mathrm{e}^{x^3} - \mathrm{e}^{x^3}) + C.$$

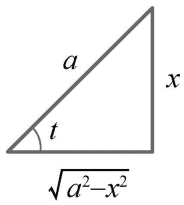
32. $\int x(\arctan x)^2\mathrm{d}x = \frac{1}{2}\int(\arctan x)^2\mathrm{d}(x^2)$

$$= \frac{1}{2}\left[x^2(\arctan x)^2 - 2\int\frac{x^2\arctan x}{1+x^2}\mathrm{d}x\right]$$

$$= \frac{1}{2}\left[x^2(\arctan x)^2 - 2\int\left(1-\frac{1}{1+x^2}\right)\arctan x\,\mathrm{d}x\right]$$

$$= \frac{1}{2}\left\{x^2\arctan^2 x - 2\left[x\arctan x - \frac{1}{2}\ln(1+x^2) - \frac{1}{2}\arctan^2 x\right]\right\} + C.$$

33. $\int\sqrt{a^2-x^2}\,\mathrm{d}x \xlongequal[\mathrm{d}x = a\cos t\,\mathrm{d}t]{x = a\sin t} \int a\cos t \cdot a\cos t\,\mathrm{d}t$



$$= a^2 \cdot \frac{1}{2}\int(1+\cos 2t)\mathrm{d}t$$

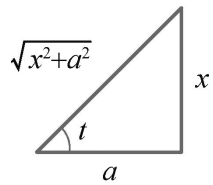
$$= \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right) + C$$

$$= \frac{a^2}{2}\left(\arcsin\frac{x}{a} + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right) + C$$

$$= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C.$$

34. $\int\sqrt{x^2+a^2}\,\mathrm{d}x \xlongequal[\mathrm{d}x = a\sec^2 t\,\mathrm{d}t]{x = a\tan t} \int a\sec t \cdot a\sec^2 t\,\mathrm{d}t$

$$= a^2\left(\frac{1}{2}\sec t\tan t + \frac{1}{2}\ln|\sec t + \tan t|\right) + C$$

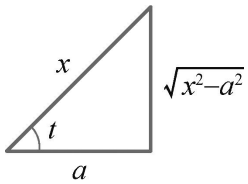


$$= a^2\left(\frac{1}{2}\frac{\sqrt{x^2+a^2}}{a}\frac{x}{a} + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right|\right) + C$$

$$= \frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)+C.$$

35. $\displaystyle\int\sqrt{x^2-a^2}\,\mathrm{d}x \xlongequal[\mathrm{d}x=a\sec t\tan t\,\mathrm{d}t]{x=a\sec t} \int a\tan t\cdot a\sec t\tan t\,\mathrm{d}t$

$$= a^2\int\tan^2 t\cdot\sec t\,\mathrm{d}t = a^2\int(\sec^2 t-1)\sec t\,\mathrm{d}t$$



$$= a^2\left[\int(\sec^3 t-\sec t)\,\mathrm{d}t\right]$$

$$= a^2\left(\frac{1}{2}\sec t\tan t-\frac{1}{2}\ln|\sec t+\tan t|\right)+C$$

$$= a^2\left(\frac{1}{2}\,\frac{x}{a}\,\frac{\sqrt{x^2-a^2}}{a}-\frac{1}{2}\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|\right)+C$$

$$= \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C.$$

36. 由 34 题，$\displaystyle\int\sqrt{x^2+a}\,\mathrm{d}x = \frac{x}{2}\sqrt{x^2+a}+\frac{a}{2}\ln\left|x+\sqrt{x^2+a}\right|+C.$

37. $\displaystyle\int e^{\sqrt{x}}\,\mathrm{d}x \xlongequal[\substack{x=t^2\\ \mathrm{d}x=2t\,\mathrm{d}t}]{\sqrt{x}=t} \int e^t\cdot 2t\,\mathrm{d}t = 2(te^t-e^t)+C$

$$= 2(\sqrt{x}\,e^{\sqrt{x}}-e^{\sqrt{x}})+C.$$

38. $\displaystyle\int x\sin\sqrt{x}\,\mathrm{d}x \xlongequal[\substack{x=t^2\\ \mathrm{d}x=2t\,\mathrm{d}t}]{\sqrt{x}=t} \int t^2\sin t\cdot 2t\,\mathrm{d}t = 2\int t^3\sin t\,\mathrm{d}t.$

由表格计算法：

$$\begin{array}{ccccccccc}
t^3 & \longrightarrow & 3t^2 & \longrightarrow & 6t & \longrightarrow & 6 & \longrightarrow & 0 \\
 & + & & - & & + & & - & \\
\sin t & \longrightarrow & -\cos t & \longrightarrow & -\sin t & \longrightarrow & \cos t & \longrightarrow & \sin t
\end{array}$$

原积分 $= 2(-t^3\cos t+3t^2\sin t+6t\cos t-6\sin t)+C$

$= 2(-\sqrt{x^3}\cos\sqrt{x}+3x\sin\sqrt{x}+6\sqrt{x}\cos\sqrt{x}-6\sin\sqrt{x})+C.$

39. $\displaystyle\int x\sin^2 x\,\mathrm{d}x = \frac{1}{2}\int x(1-\cos 2x)\,\mathrm{d}x$

$$= \frac{1}{2}\left(\int x\,\mathrm{d}x-\int x\cos 2x\,\mathrm{d}x\right)$$

$$= \frac{1}{2}\left[\frac{1}{2}x^2-\frac{1}{2}\left(x\sin 2x+\frac{1}{2}\cos 2x\right)\right]+C$$

$$= \frac{1}{4}x^2-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x+C.$$

强化篇

1. $\int x\mathrm{e}^x\cos x\,\mathrm{d}x=\frac{1}{2}\int x\,\mathrm{d}(\mathrm{e}^x\cos x+\mathrm{e}^x\sin x)$

$$=\frac{1}{2}\left[x(\mathrm{e}^x\cos x+\mathrm{e}^x\sin x)-\int(\mathrm{e}^x\cos x+\mathrm{e}^x\sin x)\mathrm{d}x\right]$$

$$=\frac{1}{2}\left[x(\mathrm{e}^x\cos x+\mathrm{e}^x\sin x)-\frac{1}{2}(\mathrm{e}^x\cos x+\mathrm{e}^x\sin x)\right.$$

$$\left.-\frac{1}{2}(\mathrm{e}^x\sin x-\mathrm{e}^x\cos x)\right]+C$$

$$=\frac{1}{2}[x(\mathrm{e}^x\cos x+\mathrm{e}^x\sin x)-\mathrm{e}^x\sin x]+C.$$

2. $\int \mathrm{e}^{\sqrt[3]{x}}\,\mathrm{d}x\xlongequal[\substack{x=t^3\\ \mathrm{d}x=3t^2\mathrm{d}t}]{\sqrt[3]{x}=t}\int \mathrm{e}^t\cdot 3t^2\,\mathrm{d}t=3\int t^2\cdot\mathrm{e}^t\,\mathrm{d}t.$

由表格计算法：

$$\begin{array}{ccccccc}
t^2 & \longrightarrow & 2t & \longrightarrow & 2 & \longrightarrow & 0\\
 & + & & - & & + & \\
\mathrm{e}^t & \longrightarrow & \mathrm{e}^t & \longrightarrow & \mathrm{e}^t & \longrightarrow & \mathrm{e}^t
\end{array}$$

原积分$=3(t^2\mathrm{e}^t-2t\mathrm{e}^t+2\mathrm{e}^t)+C$

$$=3\sqrt[3]{x^2}\,\mathrm{e}^{\sqrt[3]{x}}-6\sqrt[3]{x}\,\mathrm{e}^{\sqrt[3]{x}}+6\mathrm{e}^{\sqrt[3]{x}}+C.$$

3. $\int\cos\ln x\,\mathrm{d}x=x\cos\ln x+\int x\sin\ln x\,\frac{1}{x}\mathrm{d}x$

$$=x\cos\ln x+x\sin\ln x-\int x\cos\ln x\,\frac{1}{x}\mathrm{d}x,$$

故原积分$=\frac{1}{2}(x\cos\ln x+x\sin\ln x)+C.$

4. $\int\sin\ln x\,\mathrm{d}x=x\sin\ln x-\int x\cos\ln x\,\frac{1}{x}\mathrm{d}x$

$$=x\sin\ln x-x\cos\ln x-\int x\sin\ln x\,\frac{1}{x}\mathrm{d}x,$$

故原积分$=\frac{1}{2}(x\sin\ln x-x\cos\ln x)+C.$

5. $\int\mathrm{e}^x\sin^2 x\,\mathrm{d}x=\frac{1}{2}\int\mathrm{e}^x(1-\cos 2x)\mathrm{d}x$

$$= \frac{1}{2}\left[e^x - \frac{1}{5}(e^x\cos 2x + 2e^x\sin 2x)\right] + C$$

$$= \frac{1}{2}e^x - \frac{1}{10}(e^x\cos 2x + 2e^x\sin 2x) + C.$$

6. $\int x^n \ln x\,dx = \frac{1}{n+1}\int \ln x\,d(x^{n+1}) = \frac{1}{n+1}\left(x^{n+1}\ln x - \frac{1}{n+1}x^{n+1}\right) + C.$

7. $\int x\ln\frac{1+x}{1-x}dx = \frac{1}{2}\int \ln\frac{1+x}{1-x}d(x^2)$

$$= \frac{1}{2}\left(x^2\ln\frac{1+x}{1-x} - \int x^2\,\frac{2}{1-x^2}dx\right)$$

$$= \frac{1}{2}\left(x^2\ln\frac{1+x}{1-x} + 2\int\frac{1-x^2-1}{1-x^2}dx\right)$$

$$= \frac{1}{2}\left[x^2\ln\frac{1+x}{1-x} + 2\left(x - \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right)\right] + C$$

$$= \frac{1}{2}x^2\ln\frac{1+x}{1-x} + x - \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C.$$

8. $\int \arctan\sqrt{x}\,dx = x\arctan\sqrt{x} - \int x\,\frac{1}{1+x}\,\frac{1}{2\sqrt{x}}dx$

$$= x\arctan\sqrt{x} - \frac{1}{2}\int\frac{\sqrt{x}}{1+x}dx$$

$$\xlongequal[\substack{x=t^2\\ dx=2t\,dt}]{\sqrt{x}=t} x\arctan\sqrt{x} - \frac{1}{2}\int\frac{t}{1+t^2}\cdot 2t\,dt$$

$$= x\arctan\sqrt{x} - t + \arctan t + C$$

$$= x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C.$$

9. $\int x^2\ln\frac{1-x}{1+x}dx = \frac{1}{3}\int\ln\frac{1-x}{1+x}d(x^3) = -\frac{1}{3}\int\ln\frac{1+x}{1-x}d(x^3)$

$$= -\frac{1}{3}\left(x^3\ln\frac{1+x}{1-x} - \int x^3\cdot\frac{2}{1-x^2}dx\right)$$

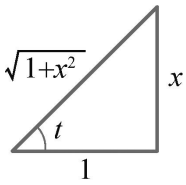
$$= -\frac{1}{3}\left[x^3\ln\frac{1+x}{1-x} + \int\frac{1-x^2-1}{1-x^2}d(x^2)\right]$$

$$= -\frac{1}{3}\left(x^3\ln\frac{1+x}{1-x} + x^2 + \ln|1-x^2|\right) + C.$$

10. $\int\frac{x\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}dx = \int\ln(x+\sqrt{1+x^2})\,d\sqrt{1+x^2}$

$$= \sqrt{1+x^2}\ln(x+\sqrt{1+x^2}) - x + C.$$

11. $\int \frac{x^2}{(1+x^2)^2}\mathrm{d}x \xlongequal[\mathrm{d}x=\sec^2 t\,\mathrm{d}t]{x=\tan t} \int \frac{\tan^2 t}{\sec^4 t}\sec^2 t\,\mathrm{d}t$



$$= \int \sin^2 t\,\mathrm{d}t = \frac{1}{2}\int (1-\cos 2t)\,\mathrm{d}t$$

$$= \frac{1}{2}\left(t - \frac{1}{2}\sin 2t\right) + C$$

$$= \frac{1}{2}\left(\arctan x - \frac{x}{\sqrt{1+x^2}}\ \frac{1}{\sqrt{1+x^2}}\right) + C$$

$$= \frac{1}{2}\left(\arctan x - \frac{x}{1+x^2}\right) + C$$

$$= \frac{1}{2}\arctan x - \frac{1}{2}\ \frac{x}{1+x^2} + C.$$

12. $\int \frac{1}{(1+x^2)^2}\mathrm{d}x \xlongequal[\mathrm{d}x=\sec^2 t\,\mathrm{d}t]{x=\tan t} \int \frac{\sec^2 t\,\mathrm{d}t}{\sec^4 t} = \int \cos^2 t\,\mathrm{d}t$

$$= \frac{1}{2}\int (1+\cos 2t)\,\mathrm{d}t$$

$$= \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right) + C$$

$$= \frac{1}{2}\arctan x + \frac{1}{2}\ \frac{x}{1+x^2} + C.$$

13. $\int \arctan(x^2)\,\mathrm{d}x = x\arctan(x^2) - \int \frac{2x^2}{1+x^4}\mathrm{d}x$

$$= x\arctan(x^2) - \int \frac{x^2+1+x^2-1}{1+x^4}\mathrm{d}x$$

$$= x\arctan(x^2) - \left(\int \frac{1+\frac{1}{x^2}}{\frac{1}{x^2}+x^2}\mathrm{d}x + \int \frac{1-\frac{1}{x^2}}{\frac{1}{x^2}+x^2}\mathrm{d}x\right)$$

$$= x\arctan(x^2) - \int \frac{\mathrm{d}\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2} - \int \frac{\mathrm{d}\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2}$$

$$= x\arctan(x^2) - \frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\ln\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right) + C.$$

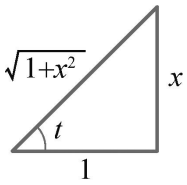
14. $\int x^2\sqrt{x^2+a^2}\,\mathrm{d}x = \frac{1}{2}\times\frac{2}{3}\int x\,\mathrm{d}\left[(x^2+a^2)^{\frac{3}{2}}\right]$

$$= \frac{1}{3}\left[x(x^2+a^2)^{\frac{3}{2}} - \int (x^2+a^2)\sqrt{x^2+a^2}\,\mathrm{d}x\right],$$

故原积分 $= \frac{3}{4} \times \frac{1}{3}\left[x(x^2+a^2)^{\frac{3}{2}} - a^2\int \sqrt{x^2+a^2}\,\mathrm{d}x\right]$

$$= \frac{1}{4}\left\{x(x^2+a^2)^{\frac{3}{2}} - a^2\left[\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right)\right]\right\} + C.$$

15. 令 $x = \tan t$，$\mathrm{d}x = \sec^2 t\,\mathrm{d}t$，



$$\int \frac{\mathrm{e}^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x = \int \frac{\mathrm{e}^t}{\sec^3 t}\sec^2 t\,\mathrm{d}t = \int \cos t\,\mathrm{e}^t\,\mathrm{d}t$$

$$= \frac{1}{2}(\mathrm{e}^t\cos t + \mathrm{e}^t\sin t) + C$$

$$= \frac{1}{2}\left(\mathrm{e}^{\arctan x}\frac{1}{\sqrt{1+x^2}} + \mathrm{e}^{\arctan x}\frac{x}{\sqrt{1+x^2}}\right) + C$$

$$= \frac{1}{2}\mathrm{e}^{\arctan x}\frac{1+x}{\sqrt{1+x^2}} + C.$$

16. $\int \mathrm{e}^{2x}\sin^2 x\,\mathrm{d}x = \frac{1}{2}\int \mathrm{e}^{2x}(1-\cos 2x)\mathrm{d}x$

$$= \frac{1}{2}\left[\frac{1}{2}\mathrm{e}^{2x} - \frac{1}{8}(2\mathrm{e}^{2x}\cos 2x + 2\mathrm{e}^{2x}\sin 2x)\right] + C$$

$$= \frac{1}{4}\mathrm{e}^{2x} - \frac{1}{8}(\mathrm{e}^{2x}\cos 2x + \mathrm{e}^{2x}\sin 2x) + C.$$

17. $\int (\mathrm{e}^x - \cos x)^2\mathrm{d}x = \int (\mathrm{e}^{2x} - 2\mathrm{e}^x\cos x + \cos^2 x)\,\mathrm{d}x$

$$= \int \mathrm{e}^{2x}\mathrm{d}x - 2\int \mathrm{e}^x\cos x\,\mathrm{d}x + \int \cos^2 x\,\mathrm{d}x$$

$$= \frac{1}{2}\mathrm{e}^{2x} - (\mathrm{e}^x\cos x + \mathrm{e}^x\sin x) + \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C.$$

18. $\int \frac{\arctan(\mathrm{e}^x)}{\mathrm{e}^x}\mathrm{d}x = -\int \arctan(\mathrm{e}^x)\mathrm{d}(\mathrm{e}^{-x})$

$$= -\arctan(\mathrm{e}^x)\mathrm{e}^{-x} + \int \mathrm{e}^{-x}\frac{\mathrm{e}^x}{1+\mathrm{e}^{2x}}\mathrm{d}x$$

$$= -\frac{\arctan(\mathrm{e}^x)}{\mathrm{e}^x} + x - \frac{1}{2}\ln(1+\mathrm{e}^{2x}) + C.$$

19. $\int \frac{x}{\cos^2 x}\mathrm{d}x = \int x\,\mathrm{d}\tan x = x\tan x + \ln|\cos x| + C.$

20. $\int \frac{x e^x}{(x+1)^2} dx = -\int x e^x d\frac{1}{x+1} = -\frac{x e^x}{x+1} + e^x + C.$

真题自测

(1990) $\int \frac{\ln x}{(1-x)^2} dx = \int \ln x d\frac{1}{1-x} = \frac{\ln x}{1-x} + \ln|1-x| - \ln|x| + C.$

(1994) $\int x^3 e^{x^2} dx = \frac{1}{2}\int x^2 d(e^{x^2}) = \frac{1}{2}(x^2 e^{x^2} - e^{x^2}) + C.$

(1996)
$$\begin{aligned}\int \frac{\arctan x}{x^2(1+x^2)} dx &= \int \arctan x\left(\frac{1}{x^2} - \frac{1}{1+x^2}\right) dx \\ &= -\int \arctan x \, d\frac{1}{x} - \int \arctan x d \arctan x \\ &= -\frac{\arctan x}{x} + \frac{1}{2}\ln\left(\frac{x^2}{1+x^2}\right) - \frac{1}{2}(\arctan x)^2 + C.\end{aligned}$$

(1997)
$$\begin{aligned}\int e^{2x}(\tan x + 1)^2 dx &= \int e^{2x}(\tan^2 x + 2\tan x + 1) dx \\ &= \int e^{2x}(1+\tan^2 x) dx + 2\int e^{2x}\tan x dx \\ &= \int e^{2x} d\tan x + 2\int e^{2x}\tan x dx \\ &= e^{2x}\tan x - 2\int \tan x e^{2x} dx + 2\int e^{2x}\tan x dx \\ &= e^{2x}\tan x + C.\end{aligned}$$

(2001)
$$\begin{aligned}\int \frac{\arctan(e^x)}{e^{2x}} dx &= -\frac{1}{2}\int \arctan(e^x) d(e^{-2x}) \\ &= -\frac{1}{2}\left[\frac{\arctan(e^x)}{e^{2x}} - \int e^{-2x}\frac{e^x}{1+e^{2x}} dx\right] \\ &= -\frac{1}{2}\left[\frac{\arctan(e^x)}{e^{2x}} - \int \frac{1}{e^x(1+e^{2x})} dx\right] \\ &= -\frac{1}{2}\left[\frac{\arctan(e^x)}{e^{2x}} - \int\left(\frac{1}{e^{2x}} - \frac{1}{1+e^{2x}}\right) d(e^x)\right] \\ &= -\frac{1}{2}\left\{\frac{\arctan(e^x)}{e^{2x}} - \left[-\frac{1}{e^x} - \arctan(e^x)\right]\right\} + C \\ &= -\frac{1}{2}\left[\frac{\arctan(e^x)}{e^{2x}} + \frac{1}{e^x} + \arctan(e^x)\right] + C.\end{aligned}$$

(2004) 令 $e^x = t$，$x = \ln t$，则 $f'(t) = \frac{\ln t}{t}$，即 $f'(x) = \frac{\ln x}{x}$，所以

$$f(x)=\int\frac{\ln x}{x}\mathrm{d}x=\frac{1}{2}(\ln x)^2+C.$$

由初始条件，$f(1)=C=0$，故 $f(x)=\frac{1}{2}(\ln x)^2$，$x>0$.

(2000) 令 $\ln x=t$，$x=\mathrm{e}^t$，则 $f(t)=\frac{\ln(1+\mathrm{e}^t)}{\mathrm{e}^t}$，则

$$\begin{aligned}\int f(x)\mathrm{d}x&=\int\frac{\ln(1+\mathrm{e}^x)}{\mathrm{e}^x}\mathrm{d}x=-\int\ln(1+\mathrm{e}^x)\mathrm{d}(\mathrm{e}^{-x})\\&=-\frac{\ln(1+\mathrm{e}^x)}{\mathrm{e}^x}+\int\mathrm{e}^{-x}\frac{\mathrm{e}^x}{1+\mathrm{e}^x}\mathrm{d}x\\&=-\frac{\ln(1+\mathrm{e}^x)}{\mathrm{e}^x}+x-\ln(1+\mathrm{e}^x)+C.\end{aligned}$$

§3.4 有理函数的积分

基础篇

1. $$\begin{aligned}\int\frac{2x+3}{(x-2)(x+5)}\mathrm{d}x&=\int\left(\frac{A}{x-2}+\frac{B}{x+5}\right)\mathrm{d}x\\&=\int\frac{A(x+5)+B(x-2)}{(x-2)(x+5)}\mathrm{d}x,\end{aligned}$$

$A(x+5)+B(x-2)=2x+3.$

令 $x=-5$，$-7B=-7$，$B=1$.

令 $x=2$，$7A=7$，$A=1$.

上式 $=\int\left(\frac{1}{x-2}+\frac{1}{x+5}\right)\mathrm{d}x=\ln|x-2|+\ln|x+5|+C.$

2. $$\int\frac{x\,\mathrm{d}x}{(x+1)(x+2)(x+3)}=\int\left(\frac{A}{x+1}+\frac{B}{x+2}+\frac{C}{x+3}\right)\mathrm{d}x$$

$$=\int\frac{A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)}{(x+1)(x+2)(x+3)}\mathrm{d}x,$$

$A(x+2)(x+3)+B(x+1)(x+3)+C(x+1)(x+2)=x.$

令 $x=-2$，$-B=-2$，$B=2$.

令 $x=-3$，$2C=-3$，$C=-\frac{3}{2}$.

令 $x=-1$，$2A=-1$，$A=-\frac{1}{2}$.

$$上式=\int\left(\frac{-\frac{1}{2}}{x+1}+\frac{2}{x+2}+\frac{-\frac{3}{2}}{x+3}\right)\mathrm{d}x$$

$$=-\frac{1}{2}\ln|x+1|+2\ln|x+2|-\frac{3}{2}\ln|x+3|+C.$$

3. $\int\frac{x^2+1}{(x+1)^2(x-1)}\mathrm{d}x=\int\left[\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x-1}\right]\mathrm{d}x.$

由待定系数法，

$A(x+1)(x-1)+B(x-1)+C(x+1)^2=x^2+1.$

令 $x=-1$，$-2B=2$，$B=-1$.

令 $x=1$，$4C=2$，$C=\frac{1}{2}$.

令 $x=0$，$-A-B+C=1$，$A=-B+C-1=1+\frac{1}{2}-1=\frac{1}{2}$.

$$上式=\int\left(\frac{\frac{1}{2}}{x+1}+\frac{-1}{(x+1)^2}+\frac{\frac{1}{2}}{x-1}\right)\mathrm{d}x$$

$$=\frac{1}{2}\ln|x+1|+\frac{1}{x+1}+\frac{1}{2}\ln|x-1|+C.$$

4. $\int\frac{\mathrm{d}x}{(x+1)(x^2+1)}=\int\left(\frac{A}{x+1}+\frac{Bx+C}{x^2+1}\right)\mathrm{d}x$

$$=\int\frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}\mathrm{d}x,$$

$A(x^2+1)+(Bx+C)(x+1)=1.$

由待定系数法：

比较 x^2 前系数，$A+B=0\Rightarrow B=-A$.

比较 x 前系数，$B+C=0\Rightarrow C=A$.

比较常数，$A+C=1\Rightarrow A=\frac{1}{2}$.

则 $A=\frac{1}{2}$，$B=-\frac{1}{2}$，$C=\frac{1}{2}$.

$$上式=\int\left(\frac{\frac{1}{2}}{x+1}+\frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}\right)\mathrm{d}x$$

$$=\frac{1}{2}\ln|x+1|-\frac{1}{2}\int\frac{x-1}{x^2+1}\mathrm{d}x$$

$$=\frac{1}{2}\ln|x+1|-\frac{1}{2}\left(\int\frac{x}{x^2+1}\mathrm{d}x-\int\frac{1}{x^2+1}\mathrm{d}x\right)$$

$$=\frac{1}{2}\ln|x+1|-\frac{1}{2}\times\frac{1}{2}\ln|x^2+1|+\frac{1}{2}\arctan x+C$$

$$=\frac{1}{2}\ln|x+1|-\frac{1}{4}\ln(x^2+1)+\frac{1}{2}\arctan x+C.$$

5. $$\int\frac{\mathrm{d}x}{x^4-1}=-\frac{1}{2}\int\left(\frac{1}{x^2+1}-\frac{1}{x^2-1}\right)\mathrm{d}x$$

$$=-\frac{1}{2}\left(\arctan x-\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\right)+C$$

$$=-\frac{1}{2}\arctan x+\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|+C.$$

6. $$\int\frac{\mathrm{d}x}{1+\sqrt[3]{x+1}}\xlongequal[\substack{x=t^3-1\\ \mathrm{d}x=3t^2\mathrm{d}t}]{\sqrt[3]{x+1}=t}\int\frac{3t^2\mathrm{d}t}{1+t}=3\int\left(t-1+\frac{1}{t+1}\right)\mathrm{d}t$$

$$=3\left(\frac{1}{2}t^2-t+\ln|1+t|\right)+C$$

$$=3\left(\frac{1}{2}\sqrt[3]{(x+1)^2}-\sqrt[3]{x+1}+\ln\left|1+\sqrt[3]{x+1}\right|\right)+C.$$

7. $$\int\frac{(\sqrt{x})^3-1}{\sqrt{x}+1}\mathrm{d}x\xlongequal[\substack{x=t^2\\ \mathrm{d}x=2t\mathrm{d}t}]{\sqrt{x}=t}\int\frac{t^3-1}{t+1}\cdot 2t\mathrm{d}t$$

$$=2\int\frac{t^4-t}{t+1}\mathrm{d}t=2\left\{\int\left[(t-1)(t^2+1)+\frac{1}{t+1}\right]\mathrm{d}t-\int\frac{t}{t+1}\mathrm{d}t\right\}$$

$$=2\left[\int\left(t^3+t-t^2-1+\frac{1}{t+1}\right)\mathrm{d}t-\int\frac{t}{t+1}\mathrm{d}t\right]$$

$$=2\left(\frac{1}{4}t^4+\frac{1}{2}t^2-\frac{1}{3}t^3-t+\ln|1+t|-t+\ln|1+t|\right)+C$$

$$=2\left[\frac{1}{4}x^2+\frac{1}{2}x-\frac{1}{3}\sqrt{x^3}-2\sqrt{x}+2\ln(1+\sqrt{x})\right]+C.$$

强化篇

1. $\int \frac{1}{1+x^3}\mathrm{d}x = \int \frac{1}{(1+x)(1-x+x^2)}\mathrm{d}x = \int \left(\frac{A}{1+x} + \frac{Bx+C}{1-x+x^2}\right)\mathrm{d}x.$

由待定系数法，

$$A(1-x+x^2)+(Bx+C)(1+x)=1.$$

比较 x^2 前系数，$A+B=0 \Rightarrow B=-A$.

比较 x 前系数，$-A+B+C=0 \Rightarrow C=2A$.

比较常数，$A+C=1 \Rightarrow A=\frac{1}{3}$.

则 $A=\frac{1}{3}$，$B=-\frac{1}{3}$，$C=\frac{2}{3}$.

$$\text{上式}=\int\left(\frac{\frac{1}{3}}{1+x}+\frac{-\frac{1}{3}x+\frac{2}{3}}{1-x+x^2}\right)\mathrm{d}x$$

$$=\frac{1}{3}\ln|1+x|-\frac{1}{3}\int\frac{x-2}{1-x+x^2}\mathrm{d}x$$

$$=\frac{1}{3}\ln|1+x|-\frac{1}{3}\int\frac{\frac{1}{2}(-1+2x)-\frac{3}{2}}{1-x+x^2}\mathrm{d}x$$

$$=\frac{1}{3}\ln|1+x|-\frac{1}{3}\left[\frac{1}{2}\ln|1-x+x^2|-\frac{3}{2}\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}\mathrm{d}x\right]$$

$$=\frac{1}{3}\ln|1+x|-\frac{1}{6}\ln|1-x+x^2|+\frac{1}{2}\times\frac{1}{\frac{\sqrt{3}}{2}}\arctan\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}+C$$

$$=\frac{1}{3}\ln|1+x|-\frac{1}{6}\ln(1-x+x^2)+\frac{\sqrt{3}}{3}\arctan\frac{2\sqrt{3}\left(x-\frac{1}{2}\right)}{3}+C.$$

2. $\int \frac{x\,\mathrm{d}x}{x^3-1} = \int \frac{x}{(x-1)(x^2+x+1)}\mathrm{d}x = \int\left(\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}\right)\mathrm{d}x.$

由待定系数法，

$A(x^2+x+1)+(Bx+C)(x-1)=x.$

比较 x^2 前系数，$A+B=0\Rightarrow B=-A$.

比较 x 前系数，$A-B+C=1\Rightarrow 2A+C=1$，$C=1-2A$.

比较常数，$A-C=0\Rightarrow 3A-1=0$，$A=\frac{1}{3}$.

则 $A=\frac{1}{3}$，$B=-\frac{1}{3}$，$C=1-2A=1-\frac{2}{3}=\frac{1}{3}$.

$$\text{上式}=\int\left(\frac{\frac{1}{3}}{x-1}+\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}\right)\mathrm{d}x$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{x-1}{x^2+x+1}\mathrm{d}x$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{3}\int\frac{\frac{1}{2}(2x+1)-\frac{3}{2}}{x^2+x+1}\mathrm{d}x$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln|x^2+x+1|+\frac{1}{2}\int\frac{\mathrm{d}x}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$=\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2+x+1)+\frac{\sqrt{3}}{3}\arctan\frac{2\sqrt{3}\left(x+\frac{1}{2}\right)}{3}+C.$$

3. $\int\frac{\mathrm{d}x}{x^4+1}=\frac{1}{2}\int\frac{1+x^2+1-x^2}{x^4+1}\mathrm{d}x=\frac{1}{2}\left(\int\frac{1+x^2}{x^4+1}\mathrm{d}x+\int\frac{1-x^2}{1+x^4}\mathrm{d}x\right)$

$$=\frac{1}{2}\left(\int\frac{\frac{1}{x^2}+1}{x^2+\frac{1}{x^2}}\mathrm{d}x+\int\frac{\frac{1}{x^2}-1}{x^2+\frac{1}{x^2}}\mathrm{d}x\right)$$

$$=\frac{1}{2}\left[\int\frac{\mathrm{d}\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2}-\int\frac{\mathrm{d}\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2-2}\right]$$

$$=\frac{1}{2\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}}-\frac{1}{4\sqrt{2}}\ln\left(\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right)+C.$$

4. $\int\sqrt{\frac{1-x}{1+x}}\,\frac{\mathrm{d}x}{x}=\int\frac{1-x}{\sqrt{1-x^2}}\,\frac{\mathrm{d}x}{x}=\int\frac{1}{x\sqrt{1-x^2}}\mathrm{d}x-\int\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

$$= -\ln\left|\frac{1}{x}+\sqrt{\left(\frac{1}{x}\right)^2-1}\right|-\arcsin x+C.$$

5. $\displaystyle\int\frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2(x-1)^4}}=\int\frac{\mathrm{d}x}{(x+1)(x-1)\sqrt[3]{\dfrac{x-1}{x+1}}}.$

令$\sqrt[3]{\dfrac{x-1}{x+1}}=t$，$\dfrac{x-1}{x+1}=t^3$，$x-1=xt^3+t^3$，$x=\dfrac{1+t^3}{1-t^3}$，

$$\mathrm{d}x=\frac{3t^2(1-t^3)+3t^2(1+t^3)}{(1-t^3)^2}\mathrm{d}t=\frac{6t^2}{(1-t^3)^2}\mathrm{d}t.$$

$$\text{上式}=\int\frac{1}{\left(\dfrac{1+t^3}{1-t^3}+1\right)\left(\dfrac{1+t^3}{1-t^3}-1\right)\cdot t}\cdot\frac{6t^2}{(1-t^3)^2}\mathrm{d}t$$

$$=\int\frac{1}{\dfrac{2}{1-t^3}\cdot\dfrac{2t^3}{1-t^3}\cdot t}\cdot\frac{6t^2}{(1-t^3)^2}\mathrm{d}t$$

$$=\frac{3}{2}\int\frac{1}{t^2}\mathrm{d}t=-\frac{3}{2}\ \frac{1}{t}+C=-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C.$$

6. $\displaystyle\int\frac{1}{x(1+x^2)}\mathrm{d}x=\int\frac{x}{x^2(1+x^2)}\mathrm{d}x=\frac{1}{2}\int\frac{\mathrm{d}(x^2)}{x^2(1+x^2)}$

$$=\frac{1}{2}\int\left(\frac{1}{x^2}-\frac{1}{1+x^2}\right)\mathrm{d}(x^2)=\frac{1}{2}\ln\frac{x^2}{1+x^2}+C.$$

7. $\displaystyle\int\frac{x^3}{1+x}\mathrm{d}x=\int\frac{x^3+1-1}{1+x}\mathrm{d}x=\int\frac{(1+x)(x^2-x+1)-1}{1+x}\mathrm{d}x$

$$=\int\left(x^2-x+1-\frac{1}{1+x}\right)\mathrm{d}x$$

$$=\frac{1}{3}x^3-\frac{1}{2}x^2+x-\ln|1+x|+C.$$

8. $\displaystyle\int\frac{1}{x^4(1+x^2)}\mathrm{d}x=\int\frac{1+x^2-x^2}{x^4(1+x^2)}\mathrm{d}x=\int\left[\frac{1}{x^4}-\frac{1}{x^2(1+x^2)}\right]\mathrm{d}x$

$$=-\frac{1}{3}\ \frac{1}{x^3}-\left(-\frac{1}{x}-\arctan x\right)+C$$

$$=-\frac{1}{3}\ \frac{1}{x^3}+\frac{1}{x}+\arctan x+C.$$

9. $\displaystyle\int\frac{1}{x(1+x^4)}\mathrm{d}x=\frac{1}{4}\int\frac{\mathrm{d}(x^4)}{x^4(1+x^4)}=\frac{1}{4}\ln\frac{x^4}{1+x^4}+C.$

10. $\int \frac{x^2+1}{x^4+1}\mathrm{d}x = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\mathrm{d}x = \int \frac{\mathrm{d}\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2}$

$$= \frac{1}{\sqrt{2}}\arctan\frac{x-\frac{1}{x}}{\sqrt{2}}+C.$$

§3.5 三角函数的积分

基础篇

1. $\int \frac{\mathrm{d}x}{3+\sin^2 x} = \int \frac{\mathrm{d}x}{4\sin^2 x+3\cos^2 x} = \frac{1}{2}\int \frac{\mathrm{d}(2\tan x)}{4\tan^2 x+3}$

$$= \frac{1}{2\sqrt{3}}\arctan\frac{2\tan x}{\sqrt{3}}+C.$$

2. $\int \frac{\mathrm{d}x}{3+\cos x} = \int \frac{\mathrm{d}x}{2+2\cos^2\frac{x}{2}} = \frac{1}{2}\int \frac{\mathrm{d}x}{\sin^2\frac{x}{2}+2\cos^2\frac{x}{2}}$

$$= \int \frac{\mathrm{d}\tan\frac{x}{2}}{\tan^2\frac{x}{2}+2} = \frac{1}{\sqrt{2}}\arctan\frac{\tan\frac{x}{2}}{\sqrt{2}}+C.$$

3. $\int \frac{\mathrm{d}x}{1+\sin x+\cos x} \xlongequal{u=\tan\frac{x}{2}} \int \frac{\frac{2}{1+u^2}\mathrm{d}u}{1+\frac{2u}{1+u^2}+\frac{1-u^2}{1+u^2}}$

$$= \int \frac{\frac{2}{1+u^2}\mathrm{d}u}{\frac{1+u^2+2u+1-u^2}{1+u^2}}$$

$$= \int \frac{2}{2+2u}\mathrm{d}u = \ln\left|1+\tan\frac{x}{2}\right|+C.$$

4. $\int \sqrt{1-x^2}\arcsin x\,\mathrm{d}x \xlongequal[\mathrm{d}x=\cos t\,\mathrm{d}t]{x=\sin t} \int \cos t\cdot t\cdot\cos t\,\mathrm{d}t$

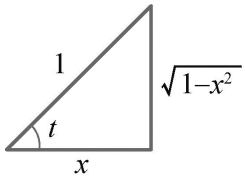
$$= \int t\cdot\cos^2 t\,\mathrm{d}t = \frac{1}{2}\int t(1+\cos 2t)\,\mathrm{d}t$$

$$= \frac{1}{2}\left[\frac{1}{2}t^2 + \frac{1}{2}\left(t\sin 2t + \frac{1}{2}\cos 2t\right)\right] + C$$

$$= \frac{1}{2}\left\{\frac{1}{2}\arcsin^2 x + \frac{1}{2}\left[\arcsin x \cdot 2x \cdot \sqrt{1-x^2} + \frac{1}{2}(1-2x^2)\right]\right\} + C$$

$$= \frac{1}{4}\arcsin^2 x + \frac{1}{4}\left(2x \cdot \arcsin x \cdot \sqrt{1-x^2} + \frac{1}{2} - x^2\right) + C.$$

5. $\displaystyle\int \frac{\arccos x}{\sqrt{(1-x^2)^3}}\mathrm{d}x \xlongequal[\mathrm{d}x = -\sin t\,\mathrm{d}t]{x = \cos t} \int \frac{t}{\sin^3 t}(-\sin t)\mathrm{d}t$



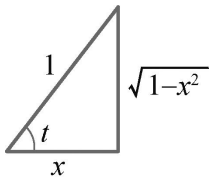
$$= -\int t \cdot \csc^2 t\,\mathrm{d}t = \int t\,\mathrm{d}\cot t$$

$$= t\cot t - \int \cot t\,\mathrm{d}t$$

$$= t\cot t - \ln|\sin t| + C$$

$$= \arccos x \cdot \frac{x}{\sqrt{1-x^2}} - \ln\sqrt{1-x^2} + C.$$

6. $\displaystyle\int \frac{x\arccos x}{\sqrt{1-x^2}}\mathrm{d}x \xlongequal[\mathrm{d}x = -\sin t\,\mathrm{d}t]{x = \cos t} \int \frac{\cos t \cdot t}{\sin t}(-\sin t)\mathrm{d}t$



$$= -\int t \cdot \cos t\,\mathrm{d}t = -t\sin t - \cos t + C$$

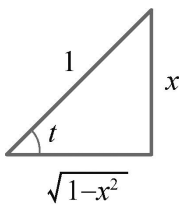
$$= -\arccos x \cdot \sqrt{1-x^2} - x + C.$$

7. $\displaystyle\int \frac{\arcsin x}{x^2}\,\frac{1+x^2}{\sqrt{1-x^2}}\mathrm{d}x = \int\left(\frac{\arcsin x}{x^2\sqrt{1-x^2}} + \frac{\arcsin x}{\sqrt{1-x^2}}\right)\mathrm{d}x$

$$= \int \frac{\arcsin x}{x^2\sqrt{1-x^2}}\mathrm{d}x + \frac{1}{2}(\arcsin x)^2.$$

下面单独计算$\displaystyle\int \frac{\arcsin x}{x^2\sqrt{1-x^2}}\mathrm{d}x$：

$$\int \frac{\arcsin x}{x^2\sqrt{1-x^2}}\mathrm{d}x \xlongequal[\mathrm{d}x = \cos t\,\mathrm{d}t]{x = \sin t} \int \frac{t}{\sin^2 t \cdot \cos t} \cdot \cos t\,\mathrm{d}t$$



$$= -\int t\,\mathrm{d}\cot t = -t\cot t + \int \cot t\,\mathrm{d}t$$

$$= -t\cot t + \ln|\sin t| + C$$

$$= -\arcsin x \cdot \frac{\sqrt{1-x^2}}{x} + \ln|x| + C.$$

故原积分 $= -\arcsin x \cdot \dfrac{\sqrt{1-x^2}}{x} + \ln|x| + \dfrac{1}{2}(\arcsin x)^2 + C.$

8. $\displaystyle\int \frac{x^3 \arccos x}{\sqrt{1-x^2}}\mathrm{d}x \xlongequal[\mathrm{d}x = -\sin t\,\mathrm{d}t]{x=\cos t} \int \frac{\cos^3 t \cdot t}{\sin t}(-\sin t)\mathrm{d}t$

$$= -\int t \cdot \cos^3 t\ \mathrm{d}t = -\int t\,\mathrm{d}\left(\sin t - \frac{1}{3}\sin^3 t\right)$$

$$= -t\left(\sin t - \frac{1}{3}\sin^3 t\right) + \int\left(\sin t - \frac{1}{3}\sin^3 t\right)\mathrm{d}t$$

$$= -t\left(\sin t - \frac{1}{3}\sin^3 t\right) - \cos t + \frac{1}{3}\left(\cos t - \frac{1}{3}\cos^3 t\right) + C$$

$$= -\arccos x\left[\sqrt{1-x^2} - \frac{1}{3}\sqrt{(1-x^2)^3}\right] - x + \frac{1}{3}\left(x - \frac{1}{3}x^3\right) + C$$

$$= \arccos x\left[\frac{1}{3}\sqrt{(1-x^2)^3} - \sqrt{1-x^2}\right] - \frac{2}{3}x - \frac{1}{9}x^3 + C.$$

真题自测

(1987) $\displaystyle\int \frac{1}{a^2\sin^2 x + b^2\cos^2 x}\mathrm{d}x\ (a, b \neq 0)$

$$= \frac{1}{a}\int \frac{\mathrm{d}(a\tan x)}{a^2\tan^2 x + b^2} = \frac{1}{ab}\arctan\frac{a\tan x}{b} + C.$$

(1993) $\displaystyle\int \frac{\tan x}{\sqrt{\cos x}}\mathrm{d}x = \int \frac{\sin x}{\cos^{\frac{3}{2}} x}\mathrm{d}x = -\int \frac{\mathrm{d}\cos x}{\cos^{\frac{3}{2}} x} = \frac{2}{\sqrt{\cos x}} + C.$

(1994) $\displaystyle\int \frac{\mathrm{d}x}{\sin 2x + 2\sin x} = \int \frac{\mathrm{d}x}{2\sin x\ \cos x + 2\sin x}$

$$= \frac{1}{2}\int \frac{\mathrm{d}x}{\sin x(\cos x + 1)}$$

$$= \frac{1}{2}\int \frac{\mathrm{d}x}{2\sin\frac{x}{2}\cos\frac{x}{2} \cdot 2\cos^2\frac{x}{2}}$$

$$= \frac{1}{8}\int \frac{\mathrm{d}x}{\sin\frac{x}{2}\cos^3\frac{x}{2}}$$

$$= \frac{1}{8}\left(\int \frac{\sin\frac{x}{2}}{\cos^3\frac{x}{2}}\mathrm{d}x + \int \frac{\mathrm{d}x}{\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= \frac{1}{8}\left(-2\int \frac{\mathrm{d}\cos\frac{x}{2}}{\cos^3\frac{x}{2}} + 2\int \frac{1}{\sin x}\mathrm{d}x\right)$$

$$= \frac{1}{8}\left(\frac{1}{\cos^2\frac{x}{2}} + 2\ln|\csc x - \cot x|\right) + C.$$

(1998) $\int \frac{\ln\sin x}{\sin^2 x}\mathrm{d}x = -\int \ln\sin x\ \mathrm{d}\cot x$

$$= -\ln\sin x \cdot \cot x + \int \cot^2 x\,\mathrm{d}x$$

$$= -\ln\sin x \cdot \cot x - \cot x - x + C.$$

§3.6　与递推有关的一组积分

1. $I_n = \int \sin^n x\,\mathrm{d}x = -\int \sin^{n-1}x\ \mathrm{d}\cos x$

$$= -\sin^{n-1}x \cdot \cos x + (n-1)\int \cos^2 x \sin^{n-2}x\,\mathrm{d}x$$

$$= -\sin^{n-1}x \cdot \cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2}x\,\mathrm{d}x$$

$$= -\sin^{n-1}x \cdot \cos x + (n-1)\left(\int \sin^{n-2}x\,\mathrm{d}x - I_n\right),$$

故 $I_n = -\frac{1}{n}\sin^{n-1}x \cdot \cos x + \frac{n-1}{n}I_{n-2}$.

2. $I_n = \int \cos^n x\,\mathrm{d}x = \int \cos^{n-1}x\,\mathrm{d}\sin x$

$$= \sin x \cdot \cos^{n-1}x + (n-1)\int \sin^2 x \cos^{n-2}x\,\mathrm{d}x$$

$$= \sin x \cdot \cos^{n-1}x + (n-1)\int (1-\cos^2 x)\cos^{n-2}x\,\mathrm{d}x,$$

故 $I_n = \frac{1}{n}\sin x \cdot \cos^{n-1}x + \frac{n-1}{n}I_{n-2}$.

3. $I_n=\int\tan^n x\,\mathrm{d}x=\int(\sec^2 x-1)\tan^{n-2}x\,\mathrm{d}x$

$$=\frac{1}{n-1}\tan^{n-1}x-I_{n-2}.$$

4. $I_n=\int\csc^n x\,\mathrm{d}x=-\int\csc^{n-2}x\,\mathrm{d}\cot x$

$$=-\csc^{n-2}x\cot x-(n-2)\int\cot^2 x\csc^{n-2}x\,\mathrm{d}x$$

$$=-\csc^{n-2}x\cot x-(n-2)\int(\csc^2 x-1)\csc^{n-2}x\,\mathrm{d}x,$$

故 $I_n=-\dfrac{1}{n-1}\csc^{n-2}x\cot x+\dfrac{n-2}{n-1}I_{n-2}.$

5. $I_n=\int\sec^n x\,\mathrm{d}x=\int\sec^{n-2}x\,\mathrm{d}\tan x$

$$=\tan x\cdot\sec^{n-2}x-(n-2)\int\tan^2 x\sec^{n-2}x\,\mathrm{d}x$$

$$=\tan x\cdot\sec^{n-2}x-(n-2)\int(\sec^2 x-1)\sec^{n-2}x\,\mathrm{d}x,$$

故 $I_n=\dfrac{1}{n-1}\tan x\cdot\sec^{n-2}x+\dfrac{n-2}{n-1}I_{n-2}.$

6. $I_n=\int\dfrac{1}{x^n\sqrt{1+x^2}}\mathrm{d}x=\int\dfrac{x}{x^{n+1}\sqrt{1+x^2}}\mathrm{d}x=\int\dfrac{1}{x^{n+1}}\mathrm{d}\sqrt{1+x^2}$

$$=\frac{\sqrt{1+x^2}}{x^{n+1}}+(n+1)\int\frac{\sqrt{1+x^2}}{x^{n+2}}\mathrm{d}x$$

$$=\frac{\sqrt{1+x^2}}{x^{n+1}}+(n+1)\int\frac{1+x^2}{x^{n+2}\sqrt{1+x^2}}\mathrm{d}x$$

$$=\frac{\sqrt{1+x^2}}{x^{n+1}}+(n+1)\left(\int\frac{1}{x^{n+2}\sqrt{1+x^2}}\mathrm{d}x+\int\frac{1}{x^n\sqrt{1+x^2}}\mathrm{d}x\right)$$

$$=\frac{\sqrt{1+x^2}}{x^{n+1}}+(n+1)(I_{n+2}+I_n),$$

故 $I_n=-\dfrac{1}{n}\dfrac{\sqrt{1+x^2}}{x^{n+1}}-\dfrac{n+1}{n}I_{n+2}.$

进一步整理：$I_{n+2}=-\dfrac{1}{n+1}\dfrac{\sqrt{1+x^2}}{x^{n+1}}-\dfrac{n}{n+1}I_n$，即

$$I_n=-\frac{1}{n-1}\frac{\sqrt{1+x^2}}{x^{n-1}}-\frac{n-2}{n-1}I_{n-2}.$$