Topic 2 - Vector Space

Vector Space is also known as a linear space.

Definition:

A vector space over $\mathbb{R}$ (in the real domain) is a set V together with to operations

(i) Vector addition $\oplus: (V, V) \to V$ (i.e. $\forall x, y \in X: x+y \in V$)

(ii) Scalar multipl. $\odot: (\mathbb{R}, V) \to V$ (i.e. $\forall \alpha \in \mathbb{R}, x \in V: \alpha x \in V$)

that satisfy:

1. Associativity of addition: $x + (y+z) = (x+y) + z$
2. Commutativity of addition: $x + y = y + x$
3. Identity Element: $\exists\, 0 \in V: x + 0 = 0 + x = x$
4. Inverse Element $\exists$ inverse Element $-x \in V: x + (-x) = (-x) + x = 0$
5. Neutral Element $1 \cdot x = x$
6. Commutativity of multiplication: $\alpha(\beta x) = (\alpha\beta) x$
7. Distributivity of multipl.: $(\alpha+\beta) x = \alpha x + \beta x$
8. Distributivity of multipl.: $\alpha(x+y) = \alpha x + \alpha y$

Remark:

We can extend this definition to vector space in the complex number domain $\mathbb{C}$.

In machine learning the data we have is always in the real domain.

Example 1:

$\mathbb{R}$ with "$+$" (the standard addition of real numbers) and "$\cdot$" (the standard multiplication of real numbers) is a vector space.

Example 2:

$\mathbb{R}^n$ the n-dim. Euclidean Space with

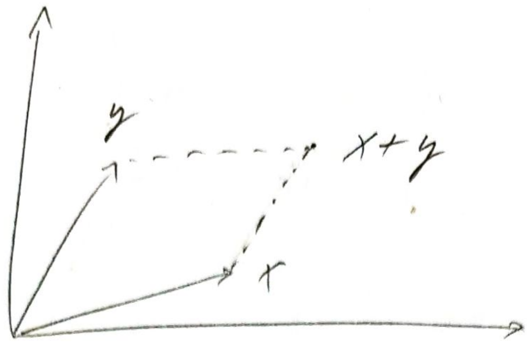
$$+ : \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad y \in \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n$$

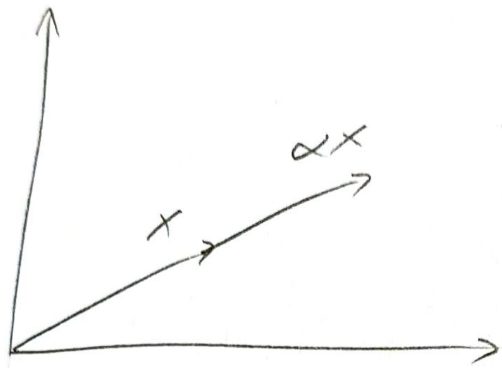
$$x + y \overset{\text{def.}}{=} \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix} \in \mathbb{R}^n$$

$$\cdot : \quad \begin{matrix} \alpha \in \mathbb{R} \\ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \end{matrix} \qquad \alpha \cdot x \overset{\text{def.}}{=} \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

is a Vector Space.

In $\mathbb{R}^2$:





Many input data can be modeled by vectors in $\mathbb{R}^n$.

- Digital signal of length n
- Stock price of length n
- n different features / attributes of one single object

Example 3:

All real $m \times n$ matrices (dentoted by $\mathbb{R}^{m\times n}$) with:

$+$: $X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & & & x_{2n} \\ \vdots & & & \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$

$$Y = \begin{bmatrix} y_{11} & y_{21} & \cdots & y_{1n} \\ \vdots & \vdots & & \vdots \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

$$X + Y \overset{\text{def.}}{=} \begin{bmatrix} x_{11}+y_{11} & x_{12}+y_{12} & \cdots & x_{1n} \; y_{1n} \\ x_{21}+y_{21} & y_{22}+y_{22} & & x_{2n}+y_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1}+y_{m1} & x_{m2}+y_{m2} & \cdots & x_{mn}+y_{mn} \end{bmatrix}$$

$\cdot$: $\alpha \in \mathbb{R}$, $X \in \mathbb{R}^{m\times n}$

$$\alpha \cdot X = \begin{bmatrix} \alpha x_{11} & \cdots & \alpha x_{1n} \\ \alpha x_{21} & \cdots & \alpha x_{2n} \\ \vdots & & \\ \alpha x_{m1} & \cdots & \alpha x_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

is a Vector Space.

- This vector space is the same as $\mathbb{R}^{mn}$:

$$\underset{\mathbb{R}^{m\times n}}{\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}} \quad \overset{\text{Vectorization}}{\rightsquigarrow} \quad \underset{\mathbb{R}^{mn}}{\begin{bmatrix} x_{11} \\ \vdots \\ x_{m1} \\ x_{12} \\ \vdots \\ x_{m2} \\ \vdots \\ x_{1n} \\ \vdots \\ x_{mn} \end{bmatrix}} \begin{array}{l} \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{1-st col} \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{2nd column} \\ \\ \left.\begin{array}{c} \\ \\ \\ \end{array}\right\} \text{nth column} \end{array}$$

This matrix vector space is also very useful in modeling data in machine learning!

<u>Example 4:</u>

All real 3-array of size $m\times n\times l$ (denoted by $\mathbb{R}^{m\times n\times l}$) with

"+": $X, Y \in \mathbb{R}^{m\times n\times l}$ $\qquad X = [x_{ijk}]_{i,j,k}$

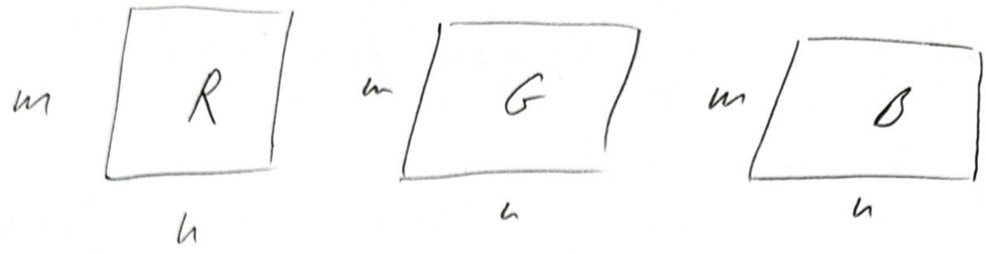
$\qquad\qquad Y = [y_{ijk}]_{i,j,k}$

$$X + Y \overset{\text{def.}}{=} [x_{ijk} + y_{ijk}]_{i,j,k} \in \mathbb{R}^{m\times n\times l}$$

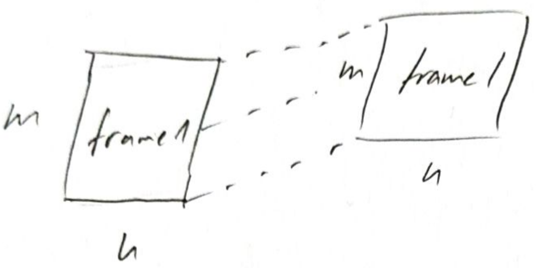
"$\cdot$": $\alpha \in \mathbb{R}$, $X \in \mathbb{R}^{m\times n\times l}$

$\alpha \cdot X = [\alpha x_{ijk}]_{i,j,k}$ is a Vector Space.

- Color images? Can be modeled by $m \times n \times 3$



- Black-White videos: Can be model by $m \times n \times l$ array.



- Color video: $m \times n \times 3 \times l$ array.

3-array, 4-array ... are called <u>"tensors"</u>

<u>Example 5:</u>

Consider the set of all strings.

Define "+" by

$$'I' + 'am' = 'I\ am'$$

and some scalar multiplication.

This will be <u>NOT</u> be a Vector Space! $'I' + 'am' \neq 'am' + 'I'$

- How can we "vectorize" the text data?
  → This is a fundamental problem in text data analysis.

Example 6:

The set of all continuous function on $[a,b]$.
Denoted by

$$C[a,b] := \{ f \mid f \text{ is a continuous function on } [a,b] \}$$

with

"+" : $\forall f, g \in C[a,b]$, define $f+g$ by

$$(f+g)(t) = f(t) + g(t) \qquad \forall t \in [a,b]$$

"$\cdot$" : $\forall \alpha \in \mathbb{R}, f \in C[a,b]$ define αf by

$$(\alpha f)(t) = \alpha \cdot f(t) \qquad \forall t \in [a,b]$$

is a Vector Space.

- $C[a,b]$ is referred as a <u>function space</u>.
- $C[a,b]$ might be a hypothesis space for some machine learning task.

Example 7:

The infinite sequences

$$\ell_\infty = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ \vdots \end{pmatrix} \;\middle|\; \begin{array}{l} \exists \text{ a finite number } C \\ \text{s.t. } |a_i| \leq C \;\forall i \end{array} \right\}$$

with

"+": $a, b \in \ell_\infty$, define

$$a + b = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \\ \vdots \end{pmatrix} \in \ell_\infty$$

"$\cdot$" $\alpha \in \mathbb{R}$
$a \in \ell_\infty$, define

$$\alpha \cdot a = \begin{pmatrix} \alpha a_1 \\ \alpha a_2 \\ \vdots \\ \alpha a_n \\ \vdots \end{pmatrix} \in \ell_\infty$$

is a Vector Space.

$\rightsquigarrow$ Can be used to model stock prices with a very fine time resolution.