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STATISTICAL THEORY OF TWO-SIDED MULTIPACTOR

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We develop a statistical theory of secondary-emission discharge (SED) taking the energy distribution of secondary electrons into account. The theory allows one to describe quantitatively the initial stage of development of a two-sided multipactor. For an arbitrary probability density of normal components of the ejection velocity and an arbitrary distance between the walls enclosing the microwave discharge plasma, we construct an analytical solution for the electron distribution function over transit times. The performed analysis is based on the results of a detailed study of conditions under which an electron reaches the opposite side. With allowance for the spread in thermal velocities, we derive a recurrence relation between the electron distribution functions over emission phases and formulate a general integral equation from which the resulting stationary distribution and the threshold of SED onset are determined.

1. INTRODUCTION

The phenomenon of resonant secondary electron emission known as multipactor was found and described for the first time by Farnsworth [1] in 1934. It consists in an avalanche increase in the number of free electrons between two emitting metal plates under the action of an external alternating electric field. In this case, the discharge is contributed only by the electrons whose transit times through the gap are equal to an odd number of half-periods of a high-frequency field. Since such a discharge is, as a rule, a factor impeding the operation of many electronic devices and systems (e.g., powerful microwave oscillators, accelerators, and space communication systems), it became the subject of intense research during the last 50 years, as is evident from review papers [2, 3].

A great number of publications (see, e.g., [4–7]) are devoted to studying resonant secondary-emission discharge (SED). Analysis performed in these studies is based on the key assumption that the initial ejection velocity of secondary electrons is either constant or amounts to a fixed fraction $1/k$ of the impact velocity of primary electrons. However, within the framework of deterministic models for calculating multipactor discharge, it is not possible to describe the experimentally observed overlapping of resonance zones [5].[1] For correct explanation and interpretation of experimental results, it is necessary to take into account the thermal spread in initial ejection velocities of electrons.

It is clear form qualitative reasons that the presence of a random component in the starting velocity results in appearance of transit-time fluctuations, which can lead to quenching of the resonant regime. The effects of the thermal spread are the most pronounced for large transit times of electrons when relatively small transit-time fluctuations can be comparable with the period of a high-frequency field. In this respect,

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[1] Note that along with purely theoretical interest, the problem of a structure of resonance zones is of great practical importance for, e.g., space communication systems. Thus, if the zones of the usual multipactor discharge are separated, then using modulated radiation can suppress the undesirable discharge under certain conditions (see, e.g., [8]). Otherwise, this cannot be done.

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we refer to papers [9–11] in which a model of the polyphase stage of multipactor discharge was developed for the above-mentioned limiting case. In particular, it was found within the framework of such a theory that the very large coefficient of secondary electron emission, which exceeds 1.96, is required to sustain the SED with large transit times.

The problem of thermal spread for small transit times was also discussed many times in the literature [12–17]. However, in view of its complexity, particular results were mainly obtained by methods of numerical simulation performed for the very limited set of the problem parameters.

Thus, the consistent statistical theory of multipactor discharge, which takes into account the energy distribution of secondary electrons and allows one to describe the initial stage of SED development for an arbitrary distance between the walls enclosing the discharge, does not exist at present. This paper is devoted to development of such a theory. It should be noted that an analytical model developed in what follows is also of great interest for numerical simulations of SED since it allows one to diminish significantly the complexity of detailed calculations of the conditions of multipactor onset.

2. INITIAL EQUATIONS

Let us consider the motion of an electron under the action of a high-frequency electric field $E = E_0 \sin(\omega t)$ between two parallel-plate electrodes separated by distance L. At the initial stage when the action of space charge can be neglected, the motion of electrons in the gap can be described by the equation

$$m\ddot{x} = eE_0 \sin(\omega t), \tag{1}$$

where m and e are the electron mass and charge, respectively (the x axis is perpendicular to the emitting surface and is directed along the electric field).

Assume that

$$x|_{t=t_\mathrm{s}} = 0, \qquad \dot{x}|_{t=t_\mathrm{s}} = v_\perp \tag{2}$$

at the starting time $t = t_\mathrm{s}$, where $v_\perp$ is the normal component of the initial velocity. Integrating Eq. (1) with allowance for Eq. (2), one can easily find that

$$\dot{x}(t) = v_\perp + v_\omega \left[\cos(\omega t_\mathrm{s}) - \cos(\omega t)\right],$$

$$x(t) = \left[v_\perp + v_\omega \cos(\omega t_\mathrm{s})\right](t - t_\mathrm{s}) + \frac{v_\omega}{\omega}\left[\sin(\omega t_\mathrm{s}) - \sin(\omega t)\right], \tag{3}$$

where $v_\omega = eE_0/(m\omega)$.

The trajectory $x(t)$ crosses the boundary $x = L$ at the time t_i determined as the least root of the transcendental equation

$$L = \left[v_\perp + v_\omega \cos(\omega t_\mathrm{s})\right](t_\mathrm{i} - t_\mathrm{s}) + \frac{v_\omega}{\omega}\left[\sin(\omega t_s) - \sin(\omega t_\mathrm{i})\right]. \tag{4}$$

We introduce the following dimensionless variables:

$$\xi = \omega x/v_\omega, \qquad \varphi = \omega t, \qquad \varphi_\mathrm{s} = \omega t_\mathrm{s}, \qquad \varphi_\mathrm{i} = \omega t_\mathrm{i}.$$

In terms of these variables, the electron trajectory is written as follows:

$$\xi(\varphi, \varphi_\mathrm{s}, u) = (u + \cos\varphi_\mathrm{s})(\varphi - \varphi_\mathrm{s}) + \sin\varphi_\mathrm{s} - \sin\varphi, \tag{5}$$

whereas Eq. (4) takes the form

$$\lambda = (u + \cos\varphi_\mathrm{s})(\varphi_\mathrm{i} - \varphi_\mathrm{s}) + \sin\varphi_\mathrm{s} - \sin\varphi_\mathrm{i}, \tag{6}$$

where $\lambda = \omega L/v_\omega$ and $u = v_\perp/v_\omega$. If we introduce the variable $\tau = \varphi_\mathrm{i} - \varphi_\mathrm{s}$ having the meaning of the dimensionless transit time, then Eq. (6) yields

$$\lambda = (u + \cos\varphi_\mathrm{s})\,\tau + \sin\varphi_\mathrm{s} - \sin(\varphi_\mathrm{s} + \tau). \tag{7}$$

Equations (5)–(7) are initial equations for the further analysis.

3. TRAJECTORY ANALYSIS

First of all, we reveal conditions under which electrons appearing at the first wall can leave it and reach the opposite surface. The corresponding constraints on the problem parameters can be found by using trajectory analysis.

It is evident that only the electrons emitted with velocity u in a certain "favorable" range of starting phases φ_s can contribute to the discharge. Thus, if the total constant component $u + \cos\varphi_\mathrm{s}$ of the electron velocity is directed out of the emitting surface for the chosen φ_s and trajectory (5) is everywhere nonnegative, then such a value of φ_s necessarily belongs to the desired range. Formally, this criterion can be formulated as the set of inequalities

$$\xi(\varphi, \varphi_\mathrm{s}, u) = (u + \cos\varphi_\mathrm{s})\,(\varphi - \varphi_\mathrm{s}) + \sin\varphi_\mathrm{s} - \sin\varphi \geq 0, \tag{8a}$$

$$u + \cos\varphi_\mathrm{s} > 0. \tag{8b}$$

We need to find such φ_s, depending on u as a parameter, that inequalities (8a) and (8b) hold for any $\varphi > \varphi_\mathrm{s}$.

Let us introduce the new variable ϕ related to φ by the formula $\phi = \varphi - \varphi_\mathrm{s}$. Then, allowing for the fact that $\sin\varphi = \sin(\varphi_\mathrm{s} + \phi) = \sin\varphi_\mathrm{s}\cos\phi + \cos\varphi_\mathrm{s}\cos\phi$ and $\phi > 0$, inequality (8a) can equivalently be rewritten as

$$u + a(\phi)\cos\varphi_\mathrm{s} + b(\phi)\sin\varphi_\mathrm{s} \geq 0, \tag{9}$$

where

$$a(\phi) = 1 - \frac{\sin\phi}{\phi}, \qquad b(\varphi) = \frac{1 - \cos\phi}{\phi}\,.$$

We emphasize that the inequalities $a(\phi) > 0$ and $b(\phi) \geq 0$ are valid for $\phi > 0$.

First of all, we notice that φ_s enters Eqs. (8b) and (9) as the argument of trigonometric functions. Therefore, the segment $[0, 2\pi]$ can be chosen as the variation range of φ_s without any loss of generality. We also note that if the conditions $\cos\varphi_\mathrm{s} > 0$ and $\sin\varphi_\mathrm{s} \geq 0$ are satisfied simultaneously, then inequality (9) along with Eq. (8b) is valid for all $u \geq 0$. Hence, for any nonnegative u, the range $[0, \pi/2)$ of starting phases is always "favorable."

For $\varphi_\mathrm{s} = \pi/2$ and $u = 0$, when the constant component of the velocity is zero, the trajectory is a periodic function of ϕ, and $\max\xi = 2$. In this case, the electron can reach the upper boundary if $\lambda < 2$. For $\lambda > 2$ and $\varphi_\mathrm{s} = \pi/2$, the electrons with $u > 0$ reach the opposite surface.

We now analyze inequality (9) assuming that φ_s belongs to the interval $(\pi/2, 2\pi)$. At first, we find the maximum velocity u_max above which this inequality holds for any φ_s. To do this, we use the following formula:

$$a(\phi)\cos\varphi_\mathrm{s} + b(\phi)\sin\varphi_\mathrm{s} = A(\phi)\cos[\varphi_\mathrm{s} - \theta(\phi)],$$

where

$$A(\phi) = \sqrt{a^2(\phi) + b^2(\phi)}, \qquad \tan\theta(\phi) = b(\phi)/a(\phi).$$

Then Eq. (9) takes the form

$$u + A(\phi)\cos[\varphi_\mathrm{s} - \theta(\phi)] \geq 0.$$

This inequality is necessarily valid if $u > u_\mathrm{max} = \max A(\phi)$. By straightforward calculation, it can easily be

verified that the maximum[2] of $A(\phi)$ is attained for $\phi = 4.0856$. In this case, $A(\phi = 4.0856) = 1.2596$. At the point of the maximum, the quantity φ_s lying in the interval $(\pi/2, 2\pi)$ is equal to

$$\varphi_s = \theta(\phi)|_{\phi=4.0856} = \arctan\left[\frac{a(\phi)}{b(\phi)}\right]\Bigg|_{\phi=4.0856} + \pi = 3.455 \text{ rad} \approx 197.957^\circ.$$

Thus, if $u > u_{max} = 1.2596$, then all the starting phases φ_s belonging to the interval $[0, 2\pi]$ are favorable.

We now consider the case where $0 < u < u_{max}$. It is evident that the trajectory

$$\xi(\phi, \varphi_s, u) = (u + \cos\varphi_s)\,\phi + \sin\varphi_s - \sin(\varphi_s + \phi)$$

going from the point $\xi = 0$ does not cross the level $\xi = 0$ once again if the first maximum of the function ξ with respect to ϕ is nonnegative (see Fig. 1). At the point of the minimum, where $\phi = \phi_{min}$, the electron velocity is zero:

$$u + \cos\varphi_s - \cos(\varphi_s + \phi_{min}) = 0,$$

whence

$$\phi_{min} = -\varphi_s + 2\pi + \arccos(u + \cos\varphi_s).$$

Therefore, the interval of favorable starting phases is determined from the joint solution of the inequality $\xi(\phi_{min}, \varphi_s, u) \geq 0$ and Eq. (8b) as follows:

$$(u + \cos\varphi_s)\,[2\pi + \arccos(u+\cos\varphi_s) - \varphi_s] + \sin\varphi_s - \sin[\arccos(u + \cos\varphi_s)] \geq 0,$$

$$u + \cos\varphi_s > 0. \tag{10}$$

Simple analysis shows that the range

$$\varphi_s \in (\pi/2, \varphi_s^+(u)] \cup [\varphi_s^-(u), 2\pi] \tag{11}$$

is a solution of this system. Here, $\varphi_s^+(u)$ and $\varphi_s^-(u)$ correspond to such values of φ_s for which the left-hand side of the first inequality in Eq. (10) is zero. The results of numerical calculations of $\varphi_s^+(u)$ and $\varphi_s^-(u)$ are shown in Fig. 2. They can be interpreted as follows. For each $\varphi_s \in (\pi/2, 2\pi)$, there exists a minimum ejection velocity $u = u_{min}(\varphi_s)$ such that an electron can reach the opposite electrode if its velocity is no less than this minimum value. Note that if $u < 0.6$, then the dependence of u_{min} on φ_s can be approximated with a fairly good accuracy by

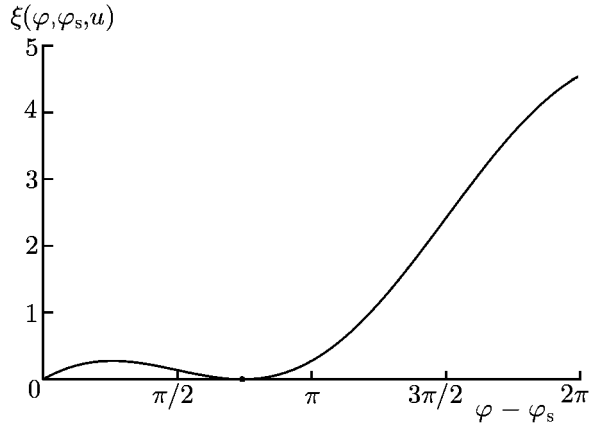

Fig. 1. Electron trajectory which touches the lower boundary for $\varphi_s = 3\pi/2$ and $u = 0.7246$.

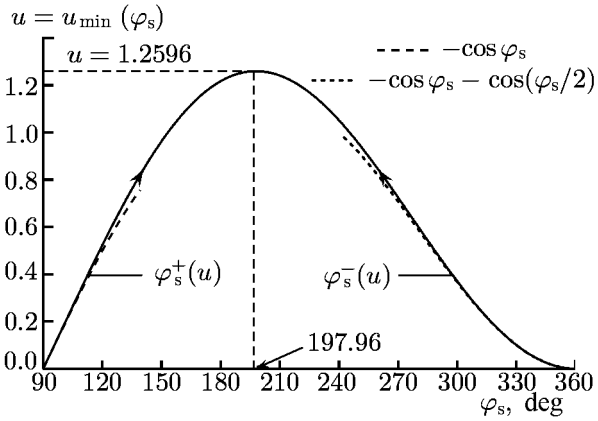

Fig. 2. Dependences of φ_s^+ and φ_s^- on the dimensionless electron ejection velocity u and their approximations.

[2]This value corresponds to the root of the equation $A'_\phi(\phi) = 0$, or $2\,(\phi\sin\phi + \cos\phi - 1) - \phi^2\cos\phi = 0$.

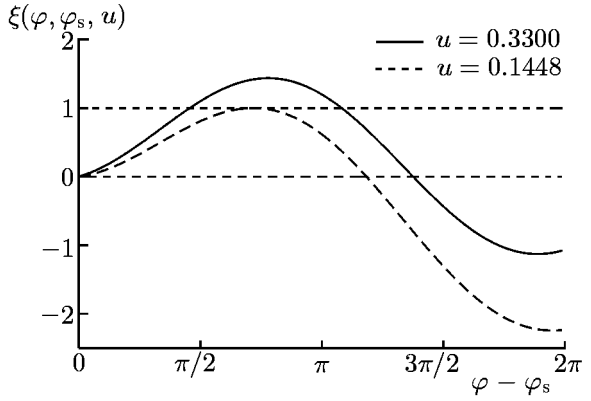

Fig. 3. Trajectory crossing the given level $\lambda = 1$ in the region of its first maximum (solid line) and then entering the region $\xi < 0$. The dashed line corresponds to the trajectory touching the indicated level.

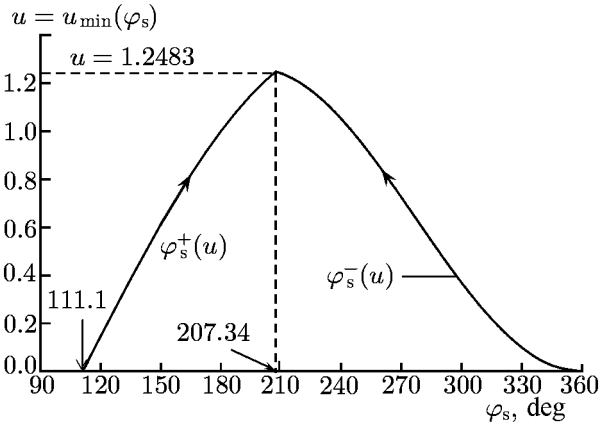

Fig. 4. Dependence of $u_{\min}$ on φ_s for $\lambda = 1$.

$$u_{\min}(\varphi_s) = \begin{cases} -\cos\varphi_s, & \varphi_s \in [90^\circ, 125^\circ]; \\ -\cos\varphi_s - \cos(\varphi_s/2), & \varphi_s \in [280^\circ, 360^\circ]. \end{cases}$$

The inverse representation of φ_s via u is as follows:

$$\varphi_s^+(u) = \pi - \arccos(u), \qquad \varphi_s^-(u) = 2\pi - \arccos[(1 + \sqrt{9 - 8u})/4 - u]. \tag{12}$$

If $u < 0.2$, then the further simplification of Eq. (12) is possible by expanding inverse trigonometric functions into power series over the small parameter u. In this case, it is easy to obtain $\varphi_s^+(u) = \pi/2 + u$ and $\varphi_s^-(u) = 2\pi - \sqrt{8u/3}$ for $\varphi_s^+(u)$ and $\varphi_s^-(u)$. These dependences are in agreement with those found earlier in [8].

Condition (8a) can turn out to be unnecessarily strict for certain parameters of the problem. For example, for relatively small values of λ (for $\lambda < 2$), the case is possible where the trajectory of an electron with $\varphi_s \in (\pi/2, 2\pi)$, starting from the point $\xi = 0$, crosses the level $\xi = \lambda$ in the region of its first maximum, although the analytic continuation of this trajectory formally enters the region in which $\xi < 0$ (see Fig. 3). In the considered case, the minimum ejection velocity can be found by solving the transcendental equation $\xi(\phi_{\max}, \varphi_s, u_{\min}) = \lambda$, where $\phi_{\max}$ corresponds to the first maximum of the trajectory and is equal to

$$\phi_{\max} = -\varphi_s + 2\pi - \arccos(u + \cos\varphi_s).$$

Thus, if the equation

$$(u + \cos\varphi_s)\,[2\pi - \arccos(u + \cos\varphi_s) - \varphi_s] + \sin\varphi_s + \sin[\arccos(u + \cos\varphi_s)] = \lambda \tag{13}$$

for given $\varphi_s \in (\pi/2, 2\pi)$ has a positive solution such that $0 < u = u_{\min} < 1.2596$ and $\min\xi \le 0$, then electrons can reach the upper boundary and also contribute to the discharge for all $u > u_{\min}$.

As an example, Fig. 4 shows the results of calculations of the dependence of $u_{\min}$ on φ_s for $\lambda = 1$. For this λ, the solution of Eq. (13), which satisfies the condition $\min\xi \le 0$, exists in the interval of φ_s ranging from 111.1° to 207.34°.[3] In the range $\varphi_s \in (207.34^\circ, 360^\circ)$, the minimum ejection velocity is determined from Eq. (10).

[3]The value $\varphi_s = 111.1^\circ$ is a root of Eq. (13) for $u = 0$ and $\lambda = 1$. For $\varphi_s = 207.34^\circ$ and $u_{\min} = 1.2483$, the first maximum of the trajectory reaches the level $\xi = 1$, while the first minimum of the trajectory touches the boundary $\xi = 0$. The indicated

The above analysis can be generalized to determine a favorable interval of impact times φ_i. Neglecting time spread of secondary-electron appearance, i.e., assuming that t_i coincides with the appearance time of secondary electrons emitted with a certain velocity u_1 which differs from u in the general case, it is not difficult to find

$$\xi(\varphi, \varphi_\mathrm{i}, u_1) = \lambda - [(u_1 - \cos\varphi_\mathrm{i})(\varphi - \varphi_\mathrm{i}) + \sin\varphi - \sin\varphi_\mathrm{i}]$$

for the corresponding trajectory going from the point $\xi = \lambda$ at $\varphi = \varphi_\mathrm{i}$. In this case, the necessary condition under which the secondary electrons appearing in the impact can leave the second wall is the set of inequalities $\xi(\varphi, \varphi_\mathrm{i}, u_1) \le \lambda$ and $\cos\varphi_\mathrm{i} - u_1 < 0$. Assuming that $\phi = \varphi - \varphi_\mathrm{i}$ and $\psi_\mathrm{s} = \varphi_\mathrm{i} - \pi N$, where $N = 2n - 1$ and n is a positive integer, one can rewrite the corresponding condition as follows:

$$(u_1 + \cos\psi_\mathrm{s})\,\phi + \sin\psi_\mathrm{s} - \sin(\psi_\mathrm{s} + \phi) \ge 0, \qquad \phi > 0;$$

$$u_1 + \cos\psi_\mathrm{s} > 0.$$

This set of inequalities is identical to that studied above. The difference consists in the replacements $\varphi_\mathrm{s} \to \psi_\mathrm{s}$ and $u \to u_1$. Consequently, when determining the favorable (for secondary emission) interval of possible values of the parameter ψ_s having the meaning of the starting phase of ejection of an electron from the opposite surface, one can use the results obtained for φ_s. As far as the favorable impact time φ_i is concerned, it is related to ψ_s by $\varphi_\mathrm{i} = \psi_\mathrm{s} + \pi N$.

4. RESONANT TRANSIT OF ELECTRONS THROUGH THE GAP

Before proceeding to study fluctuation phenomena related to the spread in initial velocities, we describe briefly the basic conclusions of the classical theory of secondary-emission discharge, according to which the electron ejection velocity is assumed to be a constant deterministic quantity $u_1 = u = \mathrm{const}$.

We denote the starting and impact phases of an electron in the case of its lth passage through the gap by $\varphi_\mathrm{s}^{(l)}$ and $\varphi_\mathrm{i}^{(l)}$, respectively. Then, based on Eq. (6), we have

$$\lambda = (u + \cos\varphi_\mathrm{s}^{(l)})\,(\varphi_\mathrm{i}^{(l)} - \varphi_\mathrm{s}^{(l)}) + \sin\varphi_\mathrm{s}^{(l)} - \sin\varphi_\mathrm{i}^{(l)}. \tag{14a}$$

In turn, as follows from trajectory analysis, in order that the secondary electrons appearing anew with the phase $\varphi_\mathrm{s}^{(l+1)}$ turn out to be in the same conditions as the preceding electrons, the transit time should be equal to the odd number of the field half-periods:

$$\varphi_\mathrm{i}^{(l)} = \varphi_\mathrm{s}^{(l+1)} + \pi N, \qquad N = 1, 3, 5, \ldots . \tag{14b}$$

Thus, combining Eqs. (14a) and (14b), we obtain

$$\lambda = (u + \cos\varphi_\mathrm{s}^{(l)})\,(\varphi_\mathrm{s}^{(l+1)} - \varphi_\mathrm{s}^{(l)} + \pi N) + \sin\varphi_\mathrm{s}^{(l)} + \sin\varphi_\mathrm{s}^{(l+1)}. \tag{15}$$

Equation (15) can be considered a mapping relating $\varphi_\mathrm{s}^{(l+1)}$ to $\varphi_\mathrm{s}^{(l)}$. The fixed point $\varphi_\mathrm{s}^{(l+1)} = \varphi_\mathrm{s}^{(l)} = \varphi^\mathrm{st}$ of this mapping corresponds to a steady-state monophase distribution and is found from the relation

$$\lambda = (u + \cos\varphi^\mathrm{st})\,\pi N + 2\sin\varphi^\mathrm{st} \tag{16}$$

following from Eq. (15). Such a monophase regime can exist only if it is stable. Of course, the value of φ^st should also belong to the favorable range of starting phases.

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parameters are found by solving the following system of equations:

$$(u + \cos\varphi_\mathrm{s})\,[2\pi - \arccos(u + \cos\varphi_\mathrm{s}) - \varphi_\mathrm{s}] + \sin\varphi_\mathrm{s} + \sin[\arccos(u + \cos\varphi_\mathrm{s})] = 1,$$
$$(u + \cos\varphi_\mathrm{s})\,[2\pi + \arccos(u + \cos\varphi_\mathrm{s}) - \varphi_\mathrm{s}] + \sin\varphi_\mathrm{s} - \sin[\arccos(u + \cos\varphi_\mathrm{s})] = 0.$$

Putting $\varphi_{\rm s}^{(l+1)} = \varphi^{\rm st} + \delta\varphi_{\rm s}^{(l+1)}$ and $\varphi_{\rm s}^{(l)} = \varphi^{\rm st} + \delta\varphi_{\rm s}^{(l)}$, it is easy to find from Eq. (15) (see, e.g., [15, 18]) that $\varphi^{\rm st}$ is stable with respect to small perturbations if

$$K = \left| \frac{\delta\varphi_{\rm s}^{(l+1)}}{\delta\varphi_{\rm s}^{(l)}} \right| = \left| \frac{\pi N \sin\varphi^{\rm st} + u}{2\cos\varphi^{\rm st} + u} \right| < 1. \tag{17}$$

The conditions $K = 1$ and $K = -1$ determine the limiting values of the starting phases $\varphi_+^{\rm st}$ and $\varphi_-^{\rm st}$, for which the system can be in stationary state. Thus, for $K = 1$, we have from Eq. (17) that

$$\varphi_+^{\rm st} = \arctan(2/(\pi N)) . \tag{18}$$

Substituting Eq. (18) into Eq. (16) yields $\lambda = \lambda_{\max} = \sqrt{(\pi N)^2 + 4} + \pi N u$. The last formula describes the location of the upper boundary of the discharge zone.

Next, for $K = -1$, it is easy to find from Eq. (17) that $\varphi_-^{\rm st}$ is given by

$$\varphi_-^{\rm st} = 2\pi - \arcsin\Bigl(2u/\sqrt{(\pi N)^2 + 4}\Bigr) - \arctan(2/\pi N)) . \tag{19}$$

The parameter λ corresponding to limiting value (19) is equal [17] to

$$\lambda = \lambda\bigl(\varphi_-^{\rm st}\bigr) = \frac{(\pi N)^2 - 4}{(\pi N)^2 + 4} \left[\sqrt{(\pi N)^2 + 4\,(1 - u^2)} + \pi N u \right] .$$

However, the above expression for λ is an estimate for the location of the lower discharge boundary only in a certain range of u. This is related to the fact that the phase $\varphi_-^{\rm st}$, in contrast to $\varphi_+^{\rm st}$, depends explicitly on u, so that it can be outside the favorable range of starting phases for certain ejection velocities. Recall that the electron emitted with the phase $\varphi_{\rm s}$ close to 2π reaches the opposite electrode if $\varphi_{\rm s}$ exceeds $\varphi_{\rm s}^-(u)$. Therefore, for given u, the stationary phase $\varphi^{\rm st}$ determining the location of the lower boundary of the discharge zone should be found from the condition $\varphi^{\rm st} = \max\bigl(\varphi_{\rm s}^-(u), \varphi_-^{\rm st}\bigr)$. In this case, $\lambda_{\min} = \max\bigl(\lambda(\varphi_{\rm s}^-(u)), \lambda(\varphi_-^{\rm st})\bigr)$. In particular, if we use approximate formula (12) for $\varphi_{\rm s}^-(u)$, then it is easy to obtain

$$\lambda(\varphi_{\rm s}^-(u)) = \frac{1}{4}(1 + \sqrt{9 - 8u}) \left[\pi N - \sqrt{6 + 8u - 2\sqrt{9 - 8u}} \right]$$

for $\lambda(\varphi_{\rm s}^-(u))$.

Thus, in the absence of a fluctuating component of the ejection velocity, the discharge is possible in certain discrete zones of λ whose boundaries lie in the ranges

$$\max \left\{ \begin{array}{l} \dfrac{1}{4}(1 + \sqrt{9 - 8u}) \left[\pi N - \sqrt{6 + 8u - 2\sqrt{9 - 8u}} \right] , \\ \dfrac{(\pi N)^2 - 4}{(\pi N)^2 + 4} \left[\sqrt{(\pi N)^2 + 4\,(1 - u^2)} + \pi N u \right] \end{array} \right\} < \lambda < \sqrt{(\pi N)^2 + 4} + \pi N u, \tag{20}$$

where $N = 1, 3, 5, \ldots$.

Under actual conditions, as was emphasized in Sec. 1, the initial ejection velocity of the electron is a random quantity. Therefore, the above-described resonant approach is valid if the time of transit between the walls is so small that the thermal spread in starting velocities does not lead to a large spread in impact phases. In the general case, one should allow for the presence of a fluctuating component of the electron ejection velocity. This component can significantly affect the SED development.

5. STATISTICS OF TRANSIT TIMES

We assume that the dimensionless ejection velocity u is characterized by a given distribution function

$F(u)$. Since u is random, the electron trajectory crosses the given level $\xi = \lambda$ at random time τ. The problem consists in expressing the probability density of the least root of Eq. (7) via the known distribution $F(u)$. We denote this probability density by $G(\tau \mid \varphi_\mathrm{s}; \lambda)$.

The formulated problem is solved if we find a function $g(\tau \mid \varphi_\mathrm{s}; \lambda)$ relating the random quantity u to the dimensionless transit time τ:

$$u = g(\tau \mid \varphi_\mathrm{s}; \lambda), \qquad u \geq 0. \tag{21}$$

If this function is known, then the desired probability density is

$$G(\tau \mid \varphi_\mathrm{s}; \lambda) = \left| \frac{\mathrm{d}g(\tau \mid \varphi_\mathrm{s}; \lambda)}{\mathrm{d}\tau} \right| F[g(\tau \mid \varphi_\mathrm{s}; \lambda)]. \tag{22}$$

To construct $g(\tau \mid \varphi_\mathrm{s}; \lambda)$, we take an auxiliary function $g_0(\tau \mid \varphi_\mathrm{s}; \lambda)$ which is a formal solution of Eq. (7) with respect to u:

$$u = g_0(\tau \mid \varphi_\mathrm{s}; \lambda) = \frac{\lambda - \sin\varphi_\mathrm{s} + \sin(\varphi_\mathrm{s} + \tau)}{\tau} - \cos\varphi_\mathrm{s}. \tag{23}$$

In the general case, the function $g(\tau \mid \varphi_\mathrm{s}; \lambda)$ is obtained from Eq. (23) by removing the nonmonotonicity intervals and eliminating the intervals in which $g_0(\tau \mid \varphi_\mathrm{s}; \lambda) < u_\mathrm{min}$ for given φ_s and λ. Recall that if $\varphi_\mathrm{s} \in [0, \pi/2)$, then $u_\mathrm{min} = 0$. For $\varphi_\mathrm{s} \in [\pi/2, 2\pi)$, u_min is found from either Eq. (10) or (13), depending on λ.

Let τ_max denote the value of τ for which $g_0(\tau \mid \varphi_\mathrm{s}; \lambda)$ reaches $u_\mathrm{min}(\varphi_\mathrm{s}, \lambda)$ for the first time. The quantity τ_max having the meaning of the maximum possible transit time of an electron is the least root of the equation

$$g_0(\tau_\mathrm{max} \mid \varphi_\mathrm{s}; \lambda) = u_\mathrm{min}(\varphi_\mathrm{s}, \lambda).$$

Next, in the interval $[0, \tau_\mathrm{max}]$, we need to find the points of minima of the function $g_0(\tau \mid \varphi_\mathrm{s}; \lambda)$ if they exist. We denote these points by $\tau_1, \tau_2, \ldots, \tau_m$ ($\tau_m \leq \tau_\mathrm{max}$). Then we should draw horizontal segments to the right from them to the first intersection with the curve $g_0(\tau \mid \varphi_\mathrm{s}; \lambda)$. We denote the corresponding intersection points by $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(m)}$ ($\tau^m \leqslant \tau_\mathrm{max}$). Specifying the function $g(\tau \mid \varphi_\mathrm{s}; \lambda)$ to be constant and equal to

$$g(\tau \mid \varphi_\mathrm{s}; \lambda) = \begin{cases} \dfrac{\lambda - \sin\varphi_\mathrm{s} + \sin(\varphi_\mathrm{s} + \tau_k)}{\tau_k} - \cos\varphi_\mathrm{s}, & \tau_k \in \Delta_k; \\ \dfrac{\lambda - \sin\varphi_\mathrm{s} + \sin(\varphi_\mathrm{s} + \tau_\mathrm{max})}{\tau_\mathrm{max}} - \cos\varphi_\mathrm{s}, & \tau > \tau_\mathrm{max} \end{cases} \tag{24a}$$

in the intervals $\tau \in \Delta_k = (\tau_k, \tau^{(k)})$, where $k = 1, 2, \ldots, m$, and for $\tau \geqslant \tau_\mathrm{max}$ and assuming that

$$g(\tau \mid \varphi_\mathrm{s}; \lambda) = \frac{\lambda - \sin\varphi_\mathrm{s} + \sin(\varphi_\mathrm{s} + \tau)}{\tau} - \cos\varphi_\mathrm{s} \tag{24b}$$

outside the intervals Δ_k ($\tau \notin \bigcup\limits_k \Delta_k, \tau \leq \tau_\mathrm{max}$), we complete constructing the function $g(\tau \mid \varphi_\mathrm{s}; \lambda)$.

As a results, from Eq. (22) we obtain

$$G(\tau \mid \varphi_\mathrm{s}; \lambda) = \left| \frac{\lambda - \sin\varphi_\mathrm{s} + \sin(\varphi_\mathrm{s} + \tau) - \tau\cos(\varphi_\mathrm{s} + \tau)}{\tau^2} \right| \times F\left[\frac{\lambda - \sin\varphi_\mathrm{s} + \sin(\varphi_\mathrm{s} + \tau)}{\tau} - \cos\varphi_\mathrm{s} \right] \chi\left(\tau; \bigcup_k \Delta_k \right) \theta(\tau; \tau_\mathrm{max}(\varphi_\mathrm{s})) \tag{25}$$

for the desired probability density, where

$$\chi\left(\tau;\bigcup_k \Delta_k\right) = \begin{cases} 0, & \tau \in \bigcup\limits_k \Delta_k; \\ 1, & \tau \notin \bigcup\limits_k \Delta_k, \end{cases} \qquad \theta(\tau;\tau_{\max}) = \begin{cases} 0, & \tau > \tau_{\max}; \\ 1, & \tau \le \tau_{\max}. \end{cases}$$

Let us illustrate Eq. (25) by a particular example. For definiteness, we choose $F(u)$ to be the Maxwellian distribution

$$F(u) = \frac{2v_\omega}{\sqrt{2\pi}\,v_T}\,\exp\left(-\frac{v_\omega^2 u^2}{2v_T^2}\right), \qquad u \ge 0, \qquad \int\limits_0^\infty F(u)\,\mathrm{d}u = 1, \tag{26}$$

where v_T is the electron thermal velocity.

Consider the case where $\lambda = \pi$ and $\varphi_\mathrm{s} = 75^\circ = 5\pi/12$ rad. For given φ_s, the minimum ejection velocity for which the electron can reach the opposite electrode is zero. For the indicated parameters of the problem, the maximum transit time $\tau_{\max}$, determined by solving the equation

$$\frac{\lambda - \sin(5\pi/12) + \sin(5\pi/12 + \tau_{\max})}{\tau_{\max}} - \cos(5\pi/12) = 0,$$

is equal to 8.1755. The function $g_0(\tau \mid 5\pi/12;\pi)$ has one nonmonotonicity interval $\Delta_1 = (3.739; 7.745)$ for $\tau \in [0, 8.1755]$ (see the dashed line in Fig. 5). Hence, to construct the function $g(\tau \mid 5\pi/12;\pi)$, we need to replace $g_0(\tau \mid 5\pi/12;\pi)$ by $g_0(3.739 \mid 5\pi/12;\pi)$ and 0 on the interval Δ_1 and for $\tau > 8.1755$, respectively. The function reconstructed in such a way is shown in Fig. 5 by the solid line.

To obtain a plot of the desired probability density, we should plot the auxiliary function

$$G_0(\tau \mid 5\pi/12;\pi) = \frac{\pi - \sin(5\pi/12) + \sin(5\pi/12+\tau) - \tau\cos(5\pi/12+\tau)}{\tau^2} \times F\left[\frac{\pi - \sin(5\pi/12) + \sin(5\pi/12+\tau)}{\tau} - \cos(5\pi/12)\right]$$

and then put it equal to zero for $\tau \in \Delta_1$ and $\tau > \tau_{\max}$ and leave it unchanged outside these intervals. For a Maxwellian distribution function, the results of such a procedure are shown in Fig. 6 for different values of v_T/v_ω.

In conclusion of this section, we make the following remarks.

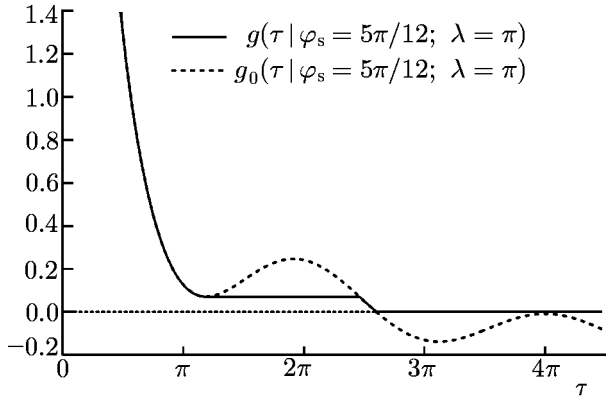

Fig. 5. Function $g_0(\tau \mid \varphi_\mathrm{s};\lambda)$ (dashed line) and the reconstructed function $g(\tau \mid \varphi_\mathrm{s};\lambda)$ for $\varphi_\mathrm{s} = 5\pi/12$ and $\lambda = \pi$.

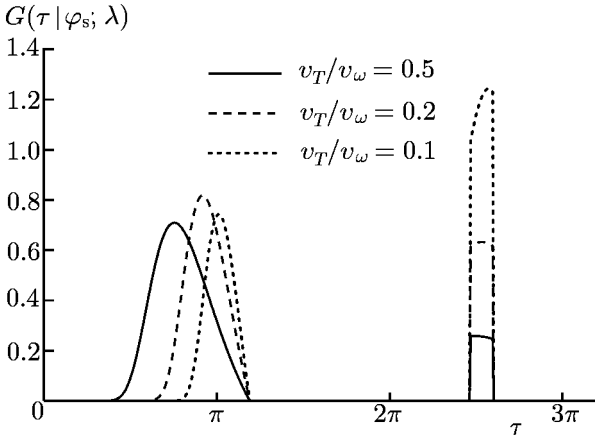

Fig. 6. Function $G(\tau \mid \varphi_\mathrm{s};\lambda)$ for $\varphi_\mathrm{s} = 5\pi/12$, $\lambda = \pi$, and different values of v_T/v_ω.

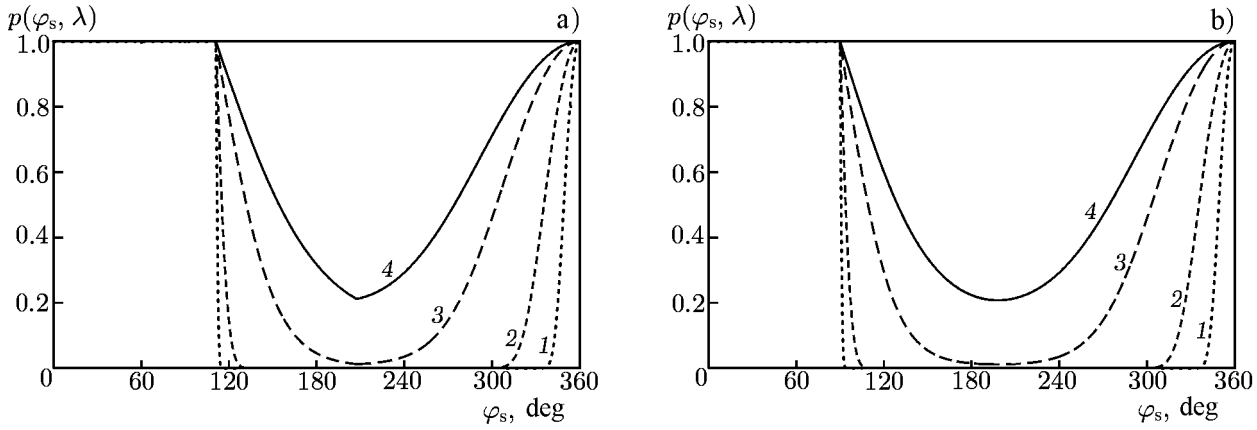

Fig. 7. Probability of the fact that the electron reaches the opposite electrode as a function of the starting phase for $\lambda = 1$ (*a*), $\lambda > 2$ (*b*), and different values of v_T/v_ω. Curves *1*, *2*, *3*, and *4* correspond to $v_T/v_\omega = 0.02$, $v_T/v_\omega = 0.1$, $v_T/v_\omega = 0.5$, and $v_T/v_\omega = 1$, respectively.

1. First of all, we note that the quantity

$$p(\varphi_s, \lambda) = \int\limits_0^{\tau_{\max}(\varphi_s,\lambda)} G(\tau \mid \varphi_s; \lambda)\, d\tau \equiv \int\limits_{u_{\min}(\varphi_s,\lambda)}^{\infty} F(u)\, du \tag{27}$$

has the meaning of the probability that the electron reaches the upper boundary for given φ_s, λ, and v_T/v_ω. In particular, for distribution function (26), the last integral on the right-hand side of Eq. (27) is reduced to the complementary error function:

$$p(\varphi_s, \lambda) = \operatorname{erfc}\left(\frac{v_\omega}{\sqrt{2}\, v_T}\, u_{\min}(\varphi_s, \lambda) \right),$$

where

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int\limits_z^{\infty} \exp(-x^2)\, dx.$$

The behavior of the function $p(\varphi_s, \lambda)$ is determined by the dependence $u_{\min}(\varphi_s, \lambda)$. According to the above discussion, if $\lambda > 2$, then $u_{\min}$ and, hence, p do not depend on the problem geometry (the corresponding probability is an explicit function of λ only for $\lambda < 2$).

As an example, Fig. 7 shows the plots of the dependence of p on φ_s for $\lambda = 1$, $\lambda > 2$, and different values of v_T/v_ω. It is seen in Fig. 7 that with decreasing spread in initial ejection velocities, the corresponding curves tend asymptotically to a rectangular distribution (with $p = 1$) localized in the range of starting phases in which $u_{\min}(\varphi_s, \lambda) = 0$.

2. If the initial phase φ_s is also a random quantity characterized by a certain distribution function $f(\varphi_s)$, then probability density (25) found above can be considered the conditional probability. The joint distribution function over starting phases and transit times is equal to $G(\tau \mid \varphi_s; \lambda) f(\varphi_s)$ in this case. Knowledge of this function allows one to determine the fraction

$$\eta(\lambda) = \int\limits_0^{2\pi} d\varphi_s\, f(\varphi_s) \int\limits_0^{\infty} G(\tau \mid \varphi_s; \lambda)\, d\tau \tag{28}$$

of primary electrons reaching the opposite electrode.

6. STATISTICAL CHARACTERISTICS OF SED

We now calculate the distribution function $f_{\rm i}(v_{\rm i};\lambda)$ of primary electrons over the impact velocities $v_{\rm i}$. In accordance with Eq. (3),

$$v_{\rm i} = \dot{x}(t)\big|_{t=t_{\rm i}} = v_\omega\,[u + \cos\varphi_{\rm s} - \cos(\varphi_{\rm s}+\tau)]$$

at the time $t = t_{\rm i}$ of impact against the wall. In this case, the desired function can be represented as

$$f_{\rm i}(v_{\rm i};\lambda) = \int\limits_0^{2\pi} {\rm d}\varphi_{\rm s} \int\limits_0^\infty \int\limits_0^\infty w(\tau,\varphi_{\rm s},u;\lambda)\,\delta[v_{\rm i} - v_\omega\,(u + \cos\varphi_{\rm s} - \cos(\varphi_{\rm s}+\tau))]\,{\rm d}\tau\,{\rm d}u, \tag{29}$$

where $\delta(v)$ is a Dirac delta function and $w(\tau,\varphi_{\rm s},u;\lambda)$ is the joint electron distribution function over transit times, ejection velocities, and starting phases. It is equal to

$$w(\tau,\varphi_{\rm s},u;\lambda) = G(\tau \mid \varphi_{\rm s};\lambda)\,\delta[u - g(\tau \mid \varphi_{\rm s};\lambda)]f(\varphi_{\rm s}).$$

Note that the function $f_{\rm i}(v_{\rm i};\lambda)$ is normalized in accordance with the condition $\int_0^\infty f_{\rm i}(v_{\rm i};\lambda)\,{\rm d}v_{\rm i} = \eta(\lambda)$, where $\eta(\lambda)$ is determined from Eq. (28) allowing for the fact that only a fraction of ejected electrons reaches the opposite wall.

Based on the results obtained above, we can calculate the effective coefficient of secondary electron emission, which is defined as the ratio of the number of electrons appearing on the surface to the number of electrons reaching this surface, i.e.,

$$\sigma_{\rm eff}(\lambda) = \frac{\int_0^\infty \sigma(v_{\rm i})f(v_{\rm i};\lambda)\,{\rm d}v_{\rm i}}{\int_0^\infty f(v_{\rm i};\lambda)\,{\rm d}v_{\rm i}} \equiv \frac{1}{\eta(\lambda)} \int\limits_0^{2\pi} {\rm d}\varphi_{\rm s}\, f(\varphi_{\rm s}) \int\limits_0^\infty {\rm d}\tau\, G(\tau \mid \varphi_{\rm s};\lambda)\sigma(\tau \mid \varphi_{\rm s};\lambda), \tag{30}$$

where the function $\sigma(v_{\rm i})$ describes the dependence of the secondary-emission coefficient on the electron-impact velocity and is determined by the properties of the plate material, whereas

$$\sigma(\tau \mid \varphi_{\rm s};\lambda) = \sigma(v_{\rm i})\big|_{v_{\rm i}=v_{\rm i}(\tau|\varphi_{\rm s};\lambda)}, \qquad v_{\rm i}(\tau \mid \varphi_{\rm s};\lambda) = v_\omega\,\frac{\lambda - \sin\varphi_{\rm s} + \sin(\varphi_{\rm s}+\tau) - \tau\cos(\varphi_{\rm s}+\tau)}{\tau}.$$

For an arbitrary given distribution function over starting velocities and emission phases, Eq. (30) makes it possible to find the condition of discharge onset, which is reduced to the requirement $\eta\sigma_{\rm eff} > 1$. In this case, the number of electrons emitted from the second surface is greater than the number of electrons starting from the first surface.

We emphasize that, in contrast to a resonant approach assuming that all electrons have the same phase and energy, electrons with a wide energy range contribute to the discharge if the spread over initial velocities is taken into account. As a result of a large number of interactions with the walls, a certain stationary distribution function over emission phases, which has a finite width, can be formed in the system. Finding this function is the necessary stage of studying SED.

Let N_0 electrons start from the lower plate at the initial time. Let their starting phases be described by a given distribution function $f_0(\varphi_{\rm s})$ normalized to unity. In accordance with the above analysis, the number of electrons reaching the upper electrode is equal to $N_0\eta_1$ and the number N_1 of secondary electrons appearing on this electrode amounts to $N_0\eta_1\sigma_{\rm eff}^{(1)}$. The values of η_1 and $\sigma_{\rm eff}^{(1)}$ are determined by Eqs. (28) and (30), respectively, for $f(\varphi_{\rm s}) \equiv f_0(\varphi_{\rm s})$. It is easy to show that the number N_l of electrons emitted during the lth interaction is expressed in terms of N_{l-1} as follows:

$$N_l = N_{l-1}\,\eta_l(\lambda)\,\sigma_{\text{eff}}^{(l)}(\lambda) \equiv N_{l-1}\int\limits_0^{2\pi} \mathrm{d}\varphi_\mathrm{s}\, f_{l-1}(\varphi_\mathrm{s}) \int\limits_0^{\infty} \mathrm{d}\tau\, G_l(\tau \mid \varphi_\mathrm{s};\lambda)\sigma_l(\tau \mid \varphi_\mathrm{s};\lambda). \tag{31}$$

Here, $l = 1, 2, \ldots$, $f_{l-1}(\varphi_\mathrm{s})$ is the orthonormalized distribution function over phases at the $(l-1)$th transit, $G_l(\tau \mid \varphi_\mathrm{s};\lambda) = G(\tau \mid \mathrm{mod}(\varphi_\mathrm{s} + \pi\,(l-1); 2\pi);\lambda)$, and $\eta_l(\lambda)$ and $\sigma_{\text{eff}}^{(l)}(\lambda)$ are calculated from Eqs. (28) and (30), respectively, in which we should make the replacements $f(\varphi_\mathrm{s}) \to f_{l-1}(\varphi_\mathrm{s})$ and $\sigma(\tau \mid \varphi_\mathrm{s};\lambda) \to \sigma_l(\tau \mid \varphi_\mathrm{s};\lambda) \equiv \sigma(\tau \mid \mathrm{mod}(\varphi_\mathrm{s} + \pi\,(l-1); 2\pi), \lambda)$.

The electron distribution $f_l(\varphi_\mathrm{s})$ over the emission phase for the lth transit is related to $f_{l-1}(\varphi_\mathrm{s})$ by the recurrence formula

$$N_l f_l(\varphi_\mathrm{s}) = N_{l-1}\left[\int\limits_0^{\varphi_\mathrm{s}} W_l(\varphi_\mathrm{s} \mid \varphi_\mathrm{s}';\lambda) f_{l-1}(\varphi_\mathrm{s}')\,\mathrm{d}\varphi_\mathrm{s}' + \sum_{n=1}^{\infty}\int\limits_0^{2\pi} W_l(\varphi_\mathrm{s} + 2\pi n \mid \varphi_\mathrm{s}';\lambda) f_{l-1}(\varphi_\mathrm{s}')\,\mathrm{d}\varphi_\mathrm{s}'\right],$$

$$W_l(\varphi_\mathrm{s} \mid \varphi_\mathrm{s}';\lambda) = G_l(\varphi_\mathrm{s} - \varphi_\mathrm{s}' \mid \varphi_\mathrm{s}';\lambda)\sigma_l(\varphi_\mathrm{s} - \varphi_\mathrm{s}' \mid \varphi_\mathrm{s}';\lambda) \tag{32}$$

derived in detail in Appendix. Equation (32) allows one to calculate all subsequent values $f_l(\varphi_\mathrm{s})$, where $l = 1, 2, \ldots$, in quadratures for any given initial distribution $f_0(\varphi_\mathrm{s})$, e.g., for the uniform distribution in the interval $[0, 2\pi]$.

A stationary electron distribution over starting phases can be found from Eq. (32). In a steady state, $f_{l-1}(\varphi_\mathrm{s}) = f_l(\mathrm{mod}(\varphi_\mathrm{s} + \pi; 2\pi)) \equiv f_{\mathrm{st}}(\varphi_\mathrm{s})$. In this case, η_l and $\sigma_{\text{eff}}^{(l)}$ do not depend on the transit number l, while the number N_l of electrons increases according to the geometric-progression law: $N_l = N_0\,(\eta\sigma_{\text{eff}})^l$. The threshold value of σ_{eff} for which the discharge occurs is found from the condition $\sigma_{\text{eff}} = \eta^{-1}$. In this case, the stationary functions $f_{\mathrm{st}}(\varphi_\mathrm{s})$ are solutions of the second-kind homogeneous Fredholm integral equation

$$f_{\mathrm{st}}(\varphi_\mathrm{s}) = \int\limits_0^{2\pi} K(\varphi_\mathrm{s}|\varphi_\mathrm{s}';\lambda) f_{\mathrm{st}}(\mathrm{mod}(\varphi_\mathrm{s}' + \pi; 2\pi))\,\mathrm{d}\varphi_\mathrm{s}', \tag{33}$$

where

$$K(\varphi_\mathrm{s} \mid \varphi_\mathrm{s}';\lambda) = W_1(\varphi_\mathrm{s} \mid \varphi';\lambda)\theta(\varphi;\varphi_\mathrm{s}') + \sum_{n=1}^{\infty} W_1(\varphi_\mathrm{s} + 2\pi n \mid \varphi_\mathrm{s}';\lambda).$$

7. SIMULATION RESULTS

Equations (31)–(33) given above were used as the basis of a numerical algorithm. The main purpose of simulation was to analyze the influence of the thermal spread on the formation of the distribution function over the electron starting phases and to find the threshold value of the secondary-emission coefficient as a function of the parameter λ. The results presented in what follows refer to the case where Maxwellian distribution (26) was chosen for $F(u)$ and the simplest approximation $\sigma(v_\mathrm{i}) = \sigma = \mathrm{const}$ was used for $\sigma(v_\mathrm{i})$ (in this case, $\sigma_{\text{eff}} = \sigma$).

In the case considered, Eq. (33) takes the form

$$f_{\mathrm{st}}(\varphi_\mathrm{s}) = \sigma\int\limits_0^{2\pi} Q(\varphi_\mathrm{s} \mid \varphi_\mathrm{s}';\lambda) f_{\mathrm{st}}(\mathrm{mod}(\varphi_\mathrm{s}' + \pi; 2\pi)\,\mathrm{d}\varphi_\mathrm{s}',$$

$$Q(\varphi_\mathrm{s} \mid \varphi_\mathrm{s}';\lambda) = G(\varphi_\mathrm{s} - \varphi_\mathrm{s}' \mid \varphi_\mathrm{s}';\lambda) + \sum_{n=1}^{\infty} G(\varphi_\mathrm{s} - \varphi_\mathrm{s}' + 2\pi n \mid \varphi_\mathrm{s}';\lambda). \tag{34}$$

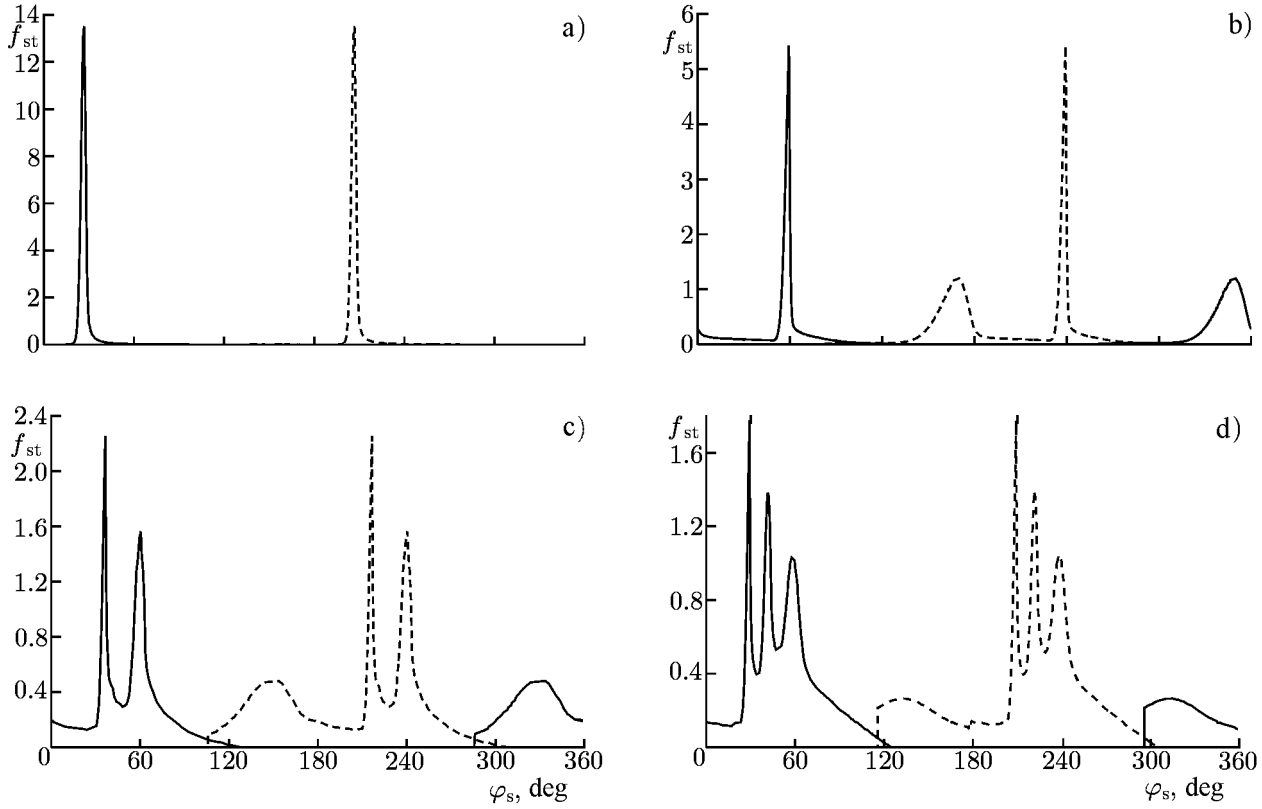

Fig. 8. Stationary distribution functions over emission phases for even (dashed lines) and odd (solid lines) transits for $v_T/v_\omega = 0.02$ and $\lambda = 3.72$ (*a*), $\lambda = 5.05$ (*b*), $\lambda = 4.4$ (*c*), and $\lambda = 4.15$ (*d*).

It is evident that an avalanche increase in the number of electrons is possible only if the secondary-emission coefficient σ exceeds the threshold $\sigma_{\text{th}} = \min \sigma$, where $\min \sigma$ is the least characteristic value of Eq. (34).

As an example, Fig. 8 shows the stationary electron distributions over emission phases for odd (solid lines) and even (dashed lines) transits. The corresponding functions are solutions of Eq. (34) and differ by a 180° shift. Calculations were performed for $v_T/v_\omega = 0.02$ and four values of λ. The values of λ are chosen such that the number of maxima of these functions vary from one to four. One maximum (for $\lambda = 3.72$) corresponds to realization of the classical resonant regime of SED, which was discussed in Sec. 4. In this case, a small thermal spread in velocities leads only to a slight broadening of the main peak (see Fig. 8*a*). The presence of a few maxima of the stationary distribution functions shown in Figs. 8*b*–8*d* corresponds to realization of hybrid resonances predicted in [17]. We note that in this parameter region, even a relatively small thermal spread can lead to the formation of such a fairly wide distribution function over emission phases. We also note that, as the transit number l increases, the function $f_l(\varphi_s)$ calculated on the basis of Eq. (32) tends asymptotically to $f_{st}(\varphi_s)$, i.e., to the solution of Eq. (34).

The dependences of the threshold secondary-emission coefficient $\sigma_{\text{th}} = \min \sigma$ on λ are shown by solid lines in Fig. 9 for different values of v_T/v_ω. The shaded regions correspond to the values of σ for which SED develops. For comparison, the dots in the same figure show the results of calculation of σ_{th} by the Monte-Carlo method based on direct statistical simulation of SED with the use of $5 \cdot 10^7$ emitted electron trajectories (in both cases, the step with respect to λ amounted to 0.01). It is seen from comparison of the corresponding calculations that in the considered case, both algorithms yield almost identical results. However, application of the Monte-Carlo method requires much greater computational effort compared with the analytical approach developed above.

When interpreting the obtained dependences, we recall that within the framework of the classical theory of resonant SED, which ignores transit-time fluctuations (see Sec. 4), the threshold secondary-emission coefficient is equal to 1 if the parameter λ belongs to the narrow region determined by Eq. (20) and tends

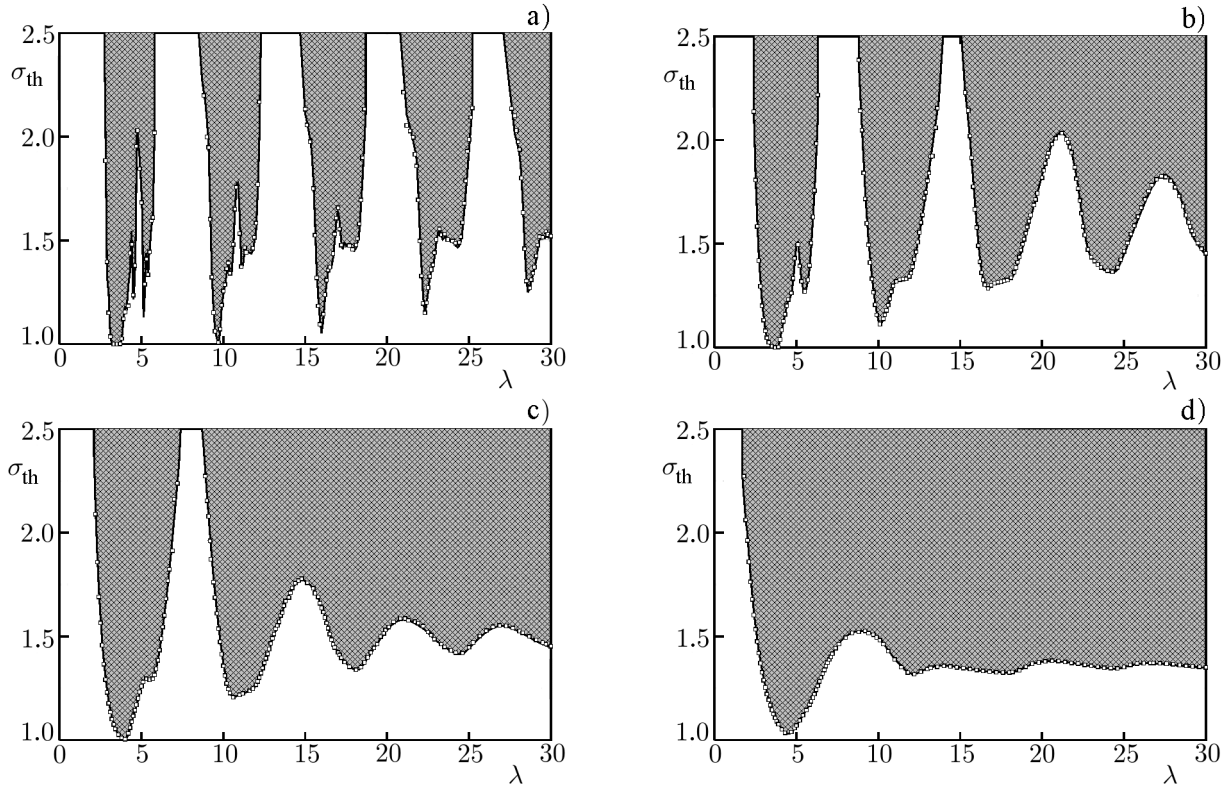

Fig. 9. Threshold secondary-emission coefficient σ_{th} as a function of λ for $v_T/v_\omega = 0.02$ (*a*), $v_T/v_\omega = 0.1$ (*b*), $v_T/v_\omega = 0.2$ (*c*), and $v_T/v_\omega = 0.5$ (*d*). The results of calculations by the Monte-Carlo methods are shown by dots.

to infinity outside this region. The results of presented calculations show that even a small thermal spread ($v_T/v_\omega \sim 0.02$) noticeably increases σ_{th} in the region of high-order resonance zones, although it does not result in their overlapping. At the same time, the quantity σ_{th} also turns out to be finite in relatively wide regions of hybrid resonances (on the right of each zone of the classical resonance). An increase in the thermal spread leads to successive broadening and overlapping of the corresponding regions. We should note that, in contrast to the approximate theory of the polyphase regime of SED, which was developed in [11] for the case of sufficiently large λ, the presented calculations show that the polyphase regime of SED is possible for a smaller threshold secondary-emission coefficient which amounts to 1.6 for the considered model in which $\sigma(v_{\text{i}}) = \text{const}$.

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APPENDIX

To find the recurrence formula relating $f_l(\varphi_{\text{s}})$ to $f_{l-1}(\varphi_{\text{s}})$, it is convenient to introduce the function $\Phi_l(\varphi_{\text{i}} \mid \varphi_{\text{s}}; \lambda) \equiv G_l(\varphi_{\text{i}} - \varphi_{\text{s}} \mid \varphi_{\text{s}}; \lambda)$ describing the distribution of electrons emitted with phases φ_{s} over impact times φ_{i} and to introduce the function $W_l(\varphi_{\text{i}} \mid \varphi_{\text{s}}; \lambda) = \Phi_l(\varphi_{\text{i}} \mid \varphi_{\text{s}}; \lambda)\sigma_l(\varphi_{\text{i}} - \varphi_{\text{s}} \mid \varphi_{\text{s}}; \lambda)$. We note that the function $W_l(\varphi_{\text{i}} \mid \varphi_{\text{s}}; \lambda)$ has the property

$$W_l(\varphi_{\text{i}} + 2\pi n \mid \varphi_{\text{s}} + 2\pi m; \lambda) = W_l(\varphi_{\text{i}} + 2\pi\,(n-m) \mid \varphi_{\text{s}}; \lambda) \tag{A1}$$

which is used in what follows.

Let $l = 2$. In this case, the number N_2 of secondary electrons appearing during the second interaction

with the wall is

$$N_2(\lambda) = \int_0^{2\pi} \mathrm{d}\varphi_\mathrm{s}\, f_0(\varphi_\mathrm{s}) \int_{\varphi_\mathrm{s}}^{\infty} \mathrm{d}\varphi_\mathrm{i}\, \Phi_1(\varphi_\mathrm{i} \mid \varphi_\mathrm{s}; \lambda)\sigma_1(\varphi_\mathrm{i} - \varphi_\mathrm{s} \mid \varphi_\mathrm{s}; \lambda) \int_0^{\infty} \mathrm{d}\tau\, G_2(\tau \mid \varphi_\mathrm{i}; \lambda)\sigma_2(\tau \mid \varphi_\mathrm{i}; \lambda). \tag{A2}$$

Changing the order of integration over φ_s and φ_i in Eq. (A2), we obtain

$$\int_0^{2\pi} \mathrm{d}\varphi_\mathrm{s} \int_{\varphi_\mathrm{s}}^{\infty} \mathrm{d}\varphi_\mathrm{i}\, \{\ldots\} = \int_0^{2\pi} \mathrm{d}\varphi_\mathrm{i} \int_0^{\varphi_\mathrm{s}} \mathrm{d}\varphi_\mathrm{s}\, \{\ldots\} + \int_{2\pi}^{\infty} \mathrm{d}\varphi_\mathrm{i} \int_0^{2\pi} \mathrm{d}\varphi_\mathrm{s}\, \{\ldots\}.$$

Taking into account that

$$\int_{2\pi}^{\infty} \mathrm{d}\varphi_\mathrm{i}\, W(\varphi_\mathrm{i}) = \sum_{n=1}^{\infty} \int_{2\pi n}^{2\pi(n+1)} \mathrm{d}\varphi_\mathrm{i}\, W(\varphi_\mathrm{i}) = \sum_{n=1}^{\infty} \int_0^{2\pi} \mathrm{d}\varphi'\, W(\varphi' + 2\pi n), \tag{A3}$$

we rewrite Eq. (A2) in the final form

$$N_2(\lambda) = N_1 \int_0^{2\pi} \mathrm{d}\varphi_\mathrm{s}\, f_1(\varphi_\mathrm{s}) \int_0^{\infty} \mathrm{d}\tau\, G_2(\tau \mid \varphi_\mathrm{s}; \lambda)\sigma_2(\tau \mid \varphi_\mathrm{s}; \lambda),$$

where

$$f_1(\varphi_\mathrm{s}) = \frac{1}{N_1} \left[\int_0^{\varphi_\mathrm{s}} W_1(\varphi_\mathrm{s} \mid \varphi'; \lambda) f_0(\varphi')\, \mathrm{d}\varphi' + \sum_{n=1}^{\infty} \int_0^{2\pi} W_1(\varphi_\mathrm{s} + 2\pi n \mid \varphi'; \lambda) f_0(\varphi')\, \mathrm{d}\varphi' \right] \tag{A4}$$

has the meaning of the distribution function over emission phases after the first interaction with the opposite electrode. The factor $1/N_1$ on the right-hand side of Eq. (A4) is introduced in order that $f_1(\varphi_\mathrm{s})$ be normalized to unity.

Let now $l = 3$. Straightforward calculation of N_3 yields

$$\begin{aligned} N_3(\lambda) = \int_0^{2\pi} \mathrm{d}\varphi_\mathrm{s}\, f_0(\varphi_\mathrm{s}) \int_{\varphi_\mathrm{s}}^{\infty} \mathrm{d}\varphi_\mathrm{i}\, \Phi_1(\varphi_\mathrm{i} \mid \varphi_\mathrm{s}; \lambda)\sigma_1(\varphi_\mathrm{i} - \varphi_\mathrm{s} \mid \varphi_\mathrm{s}; \lambda) \\ \times \int_{\varphi_\mathrm{i}}^{\infty} \mathrm{d}\varphi_\mathrm{i}'\, \Phi_2(\varphi_\mathrm{i}' \mid \varphi_\mathrm{i}; \lambda)\sigma_2(\varphi_\mathrm{i}' - \varphi_\mathrm{i} \mid \varphi_\mathrm{i}; \lambda) \int_0^{\infty} \mathrm{d}\tau\, G_3(\tau \mid \varphi_\mathrm{i}'; \lambda)\sigma_3(\tau \mid \varphi_\mathrm{i}'; \lambda). \end{aligned} \tag{A5}$$

We transform the triple integral over φ_s, φ_i, and φ_i' in Eq. (A5) by changing the order of integration:

$$\begin{aligned} \int_0^{2\pi} \mathrm{d}\varphi_\mathrm{s} \int_{\varphi_\mathrm{s}}^{\infty} \mathrm{d}\varphi_\mathrm{i} \int_{\varphi_\mathrm{i}}^{\infty} \mathrm{d}\varphi_\mathrm{i}'\, \{\ldots\} = \int_0^{2\pi} \mathrm{d}\varphi_\mathrm{i}' \int_0^{\varphi_\mathrm{i}'} \mathrm{d}\varphi_\mathrm{i} \int_0^{\varphi_\mathrm{i}} \mathrm{d}\varphi_\mathrm{s}\, \{\ldots\} \\ + \int_{2\pi}^{\infty} \mathrm{d}\varphi_\mathrm{i}' \int_0^{2\pi} \mathrm{d}\varphi_\mathrm{i} \int_0^{\varphi_\mathrm{i}} \mathrm{d}\varphi_\mathrm{s}\, \{\ldots\} + \int_{2\pi}^{\infty} \mathrm{d}\varphi_\mathrm{i}' \int_{2\pi}^{\varphi_\mathrm{i}'} \mathrm{d}\varphi_\mathrm{i} \int_0^{2\pi} \mathrm{d}\varphi_\mathrm{s}\, \{\ldots\}. \end{aligned} \tag{A6}$$

Then we note that the limits of integration over φ_i' in the second term on the right-hand side of Eq. (A6)

can be reduced to the interval $[0, 2\pi]$ by using Eq. (A3).

We now transform the last term on the right-hand side of Eq. (A.6):

$$\int\limits_{2\pi}^{\infty} \mathrm{d}\varphi_{\mathrm{i}}' \int\limits_{2\pi}^{\infty} W(\varphi_{\mathrm{i}}', \varphi_{\mathrm{i}})\, \mathrm{d}\varphi_{\mathrm{i}} = \sum_{n=1}^{\infty} \int\limits_{0}^{2\pi} \mathrm{d}\psi \int\limits_{27\pi}^{2\pi n+\psi} W(\psi + 2\pi n, \varphi_{\mathrm{i}})\, \mathrm{d}\varphi_{\mathrm{i}}. \tag{A7}$$

In turn, we have

$$\begin{aligned}\int\limits_{2\pi}^{2\pi n+\psi} W(\psi + 2\pi n, \varphi_{\mathrm{i}})\, \mathrm{d}\varphi_{\mathrm{i}} &= \int\limits_{2\pi}^{2\pi n} W(\psi + 2\pi n, \varphi_{\mathrm{i}})\, \mathrm{d}\varphi_{\mathrm{i}} + \int\limits_{2\pi n}^{2\pi n+\psi} W(\psi + 2\pi n, \varphi_{\mathrm{i}})\, \mathrm{d}\varphi_{\mathrm{i}} \\ &= \int\limits_{2\pi}^{2\pi n} W(\psi + 2\pi n, \varphi_{\mathrm{i}})\, \mathrm{d}\varphi_{\mathrm{i}} + \int\limits_{0}^{\psi} W(\psi + 2\pi n, \varphi' + 2\pi n)\, \mathrm{d}\varphi'\end{aligned}$$

for the internal integral in Eq. (A7). It is easy to verify straightforwardly that

$$\begin{aligned}\sum_{n=1}^{\infty} \int\limits_{2\pi}^{2\pi n} W(\psi + 2\pi n, \varphi_{\mathrm{i}})\, \mathrm{d}\varphi_{\mathrm{i}} &= \sum_{k=1}^{\infty} \sum_{n=k+1}^{\infty} \int\limits_{0}^{2\pi} W(\psi + 2\pi n, \varphi' + 2\pi k)\, \mathrm{d}\varphi' \\ &= \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \int\limits_{0}^{2\pi} W(\psi + 2\pi\,(m+k), \varphi' + 2\pi k)\, \mathrm{d}\varphi'.\end{aligned}$$

Combining the obtained results and taking Eq. (A1) into account, we obtain Eq. (31) with $l = 3$ for N_3, in which

$$f_2(\varphi_{\mathrm{s}}) = \frac{N_1}{N_2} \left[\int\limits_{0}^{\varphi_{\mathrm{s}}} W_2(\varphi_{\mathrm{s}} \mid \varphi_{\mathrm{s}}'; \lambda) f_1(\varphi_{\mathrm{s}}')\, \mathrm{d}\varphi_{\mathrm{s}}' + \sum_{n=1}^{\infty} \int\limits_{0}^{2\pi} W_2(\varphi_{\mathrm{s}} + 2\pi n \mid \varphi_{\mathrm{s}}'; \lambda) f_1(\varphi_{\mathrm{s}}')\, \mathrm{d}\varphi_{\mathrm{s}}' \right]$$

and the function $f_1(\varphi_{\mathrm{s}})$ is defined by Eq. (A4). It can be proved by induction that

$$f_l(\varphi) = \frac{N_{l-1}}{N_l} \left[\int\limits_{0}^{\varphi} W_l(\varphi \mid \varphi_{\mathrm{s}}'; \lambda) f_{l-1}(\varphi_{\mathrm{s}}')\, \mathrm{d}\varphi_{\mathrm{s}}' + \sum_{n=1}^{\infty} \int\limits_{0}^{2\pi} W_l(\varphi + 2\pi n \mid \varphi_{\mathrm{s}}'; \lambda) f_{l-1}(\varphi_{\mathrm{s}}')\, \mathrm{d}\varphi_{\mathrm{s}}' \right], \tag{A8}$$

whence we have Eq. (32).

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