Note on space charge fields of 2D round Gaussian distribution

PACS numbers:

For the 2D round beam with Gaussian distribution, the charge density can be expressed by:

$$\rho(r,\theta,z) = \begin{cases} \frac{\lambda}{\sqrt{2\pi}\sigma} e^{-\frac{r^2}{2\sigma^2}} & r \le a \\ 0 & r > a \end{cases} \tag{1}$$

Let's consider a cylinder along the axis of the DC beam. Its height is equal to the volicity of beam v. we can get the the total charge in this cylinder by volume integral.

$$Q(r) = \int_V \rho(r,\theta,z) dV \tag{2}$$

If $r \le a$

$$\begin{aligned} Q(r) &= \int_0^v \int_0^r \frac{\lambda}{\sqrt{2\pi}\sigma} e^{-\frac{r^2}{2\sigma^2}} 2\pi r dr dz \\ &= \sqrt{2\pi}\lambda v\sigma \left(1 - e^{-\frac{r^2}{2\sigma^2}}\right) \end{aligned} \tag{3}$$

and if $r > a$

$$\begin{aligned} Q(r) &= \int_0^v \int_0^a \frac{\lambda}{\sqrt{2\pi}\sigma} e^{-\frac{r^2}{2\sigma^2}} 2\pi r dr dz \\ &= \sqrt{2\pi}\lambda v\sigma \left(1 - e^{-\frac{a^2}{2\sigma^2}}\right) \end{aligned} \tag{4}$$

According to the defination of beam current, the beam current can be expressed as

$$I_0 = Q(a) = \sqrt{2\pi}\lambda v\sigma \left(1 - e^{-\frac{a^2}{2\sigma^2}}\right) \tag{5}$$

Apply Gauss Law of electric field for any closed surface

$$\oint_A \mathbf{E} \cdot d\mathbf{A} = \frac{Q(r)}{\varepsilon_0} \tag{6}$$

Considering the 2D axisymetric distribution, There is only radial component $E_{sc}(r)$ exists, and on the side face $E_{sc}(r)$ is constant. Apply Eq.6 onto the cylinder surface, we get

$$E_{sc}(r) \cdot 2\pi r v = \frac{\sqrt{2\pi}\lambda v\sigma}{\varepsilon_0} \left(1 - e^{-\frac{r^2}{2\sigma^2}}\right) \tag{7}$$

By combining Eq.2 to 7, the electric field generated by such a single axisymmetric DC beam with Gaussian charge distribution can be expressed by

$$\mathbf{E}_{\rm sc}(r) = \begin{cases} \frac{I_0}{2\pi\varepsilon_0 v r} \frac{1-e^{-\frac{r^2}{2\sigma^2}}}{1-e^{-\frac{a^2}{2\sigma^2}}} \mathbf{e}_r & r \le a \\ \frac{I_0}{2\pi\varepsilon_0 v r} \mathbf{e}_r & r > a \end{cases} \tag{8}$$

where a is truncated radius, I_0 beam current. $\mathbf{e}_r$ unit vector on radial direction.

Note that relativsitic effects is not included above.