Designing Frozen and Repeat Ground Track Reference Orbits for Near-Equatorial LEOs

Open Space Singapore Rolling Workshop, June 2021

Photo courtesy of Peter Komka/MTI, via Associated Press

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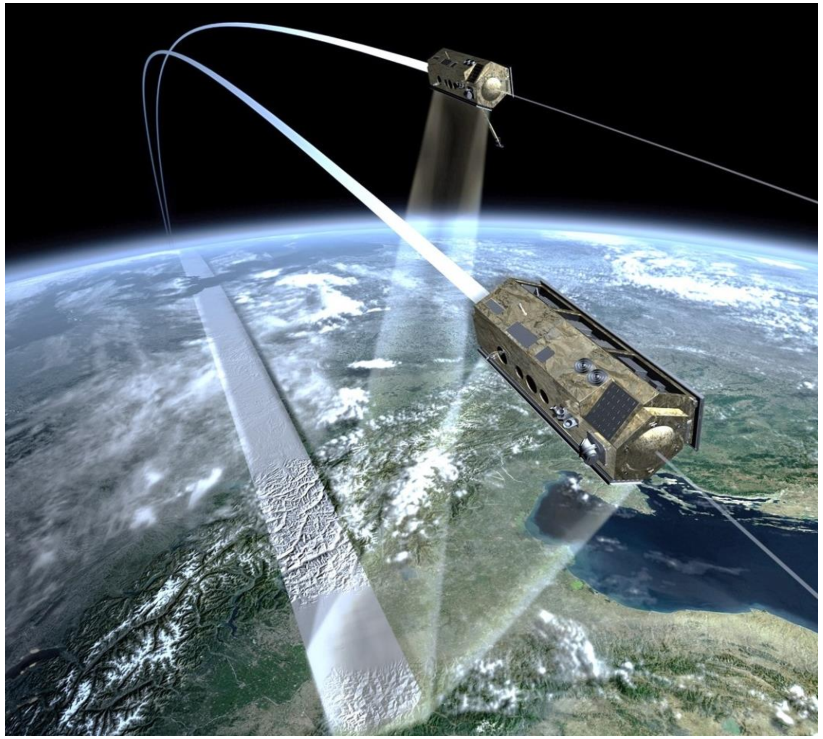
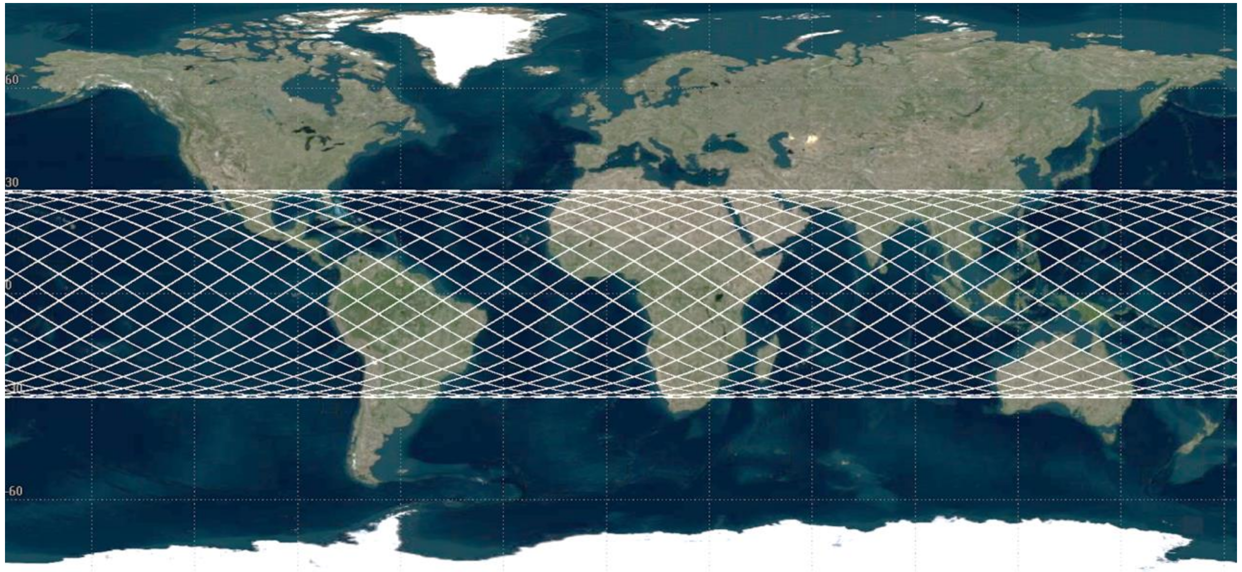
Repeat Ground Track: A ground track, with a longitude of ascending node L in the first orbit, that will ascend the equator again at L, after N orbits.

Frozen Orbit: An orbit with a carefully selected set of initial orbital elements chosen such that it minimizes the rates of change of altitude.

Reference Orbit: The trajectory of frozen and repeat ground track orbit, generated with all known and modelled perturbing forces, less drag effects, to be used a reference for the actual spacecraft to follow with propulsive manoeuvres.



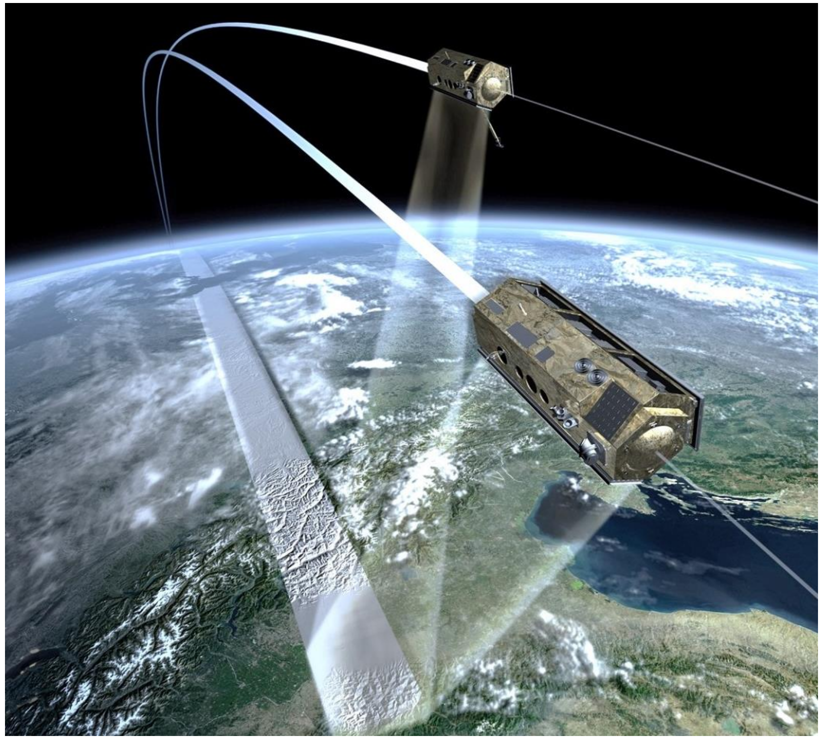
Projecting one ground track in Earth-fixed 2D, with the ground track shift $\Delta\lambda$



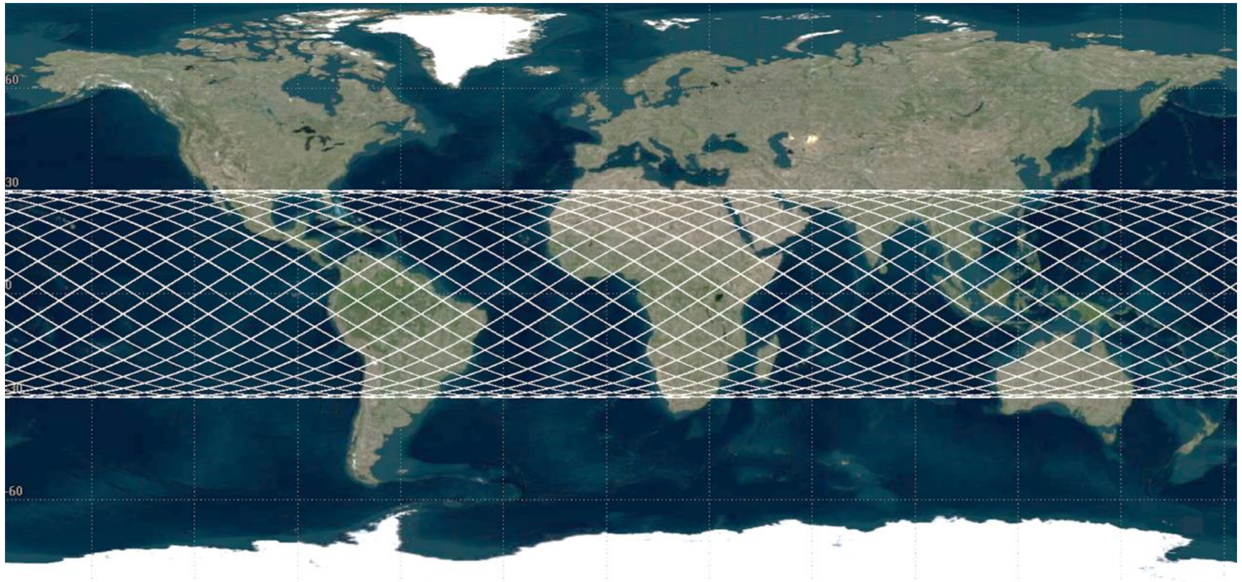
Projecting all ground tracks in the same example frozen repeat orbit

Key Contributions of this Research Study:

1) Frozen repeat ground track reference trajectories have not been assessed for performance and feasibility at low inclinations.
2) The end-cycle orbit state discontinuity problem for a “seamless” frozen repeat ground track orbit has yet to be fully resolved.

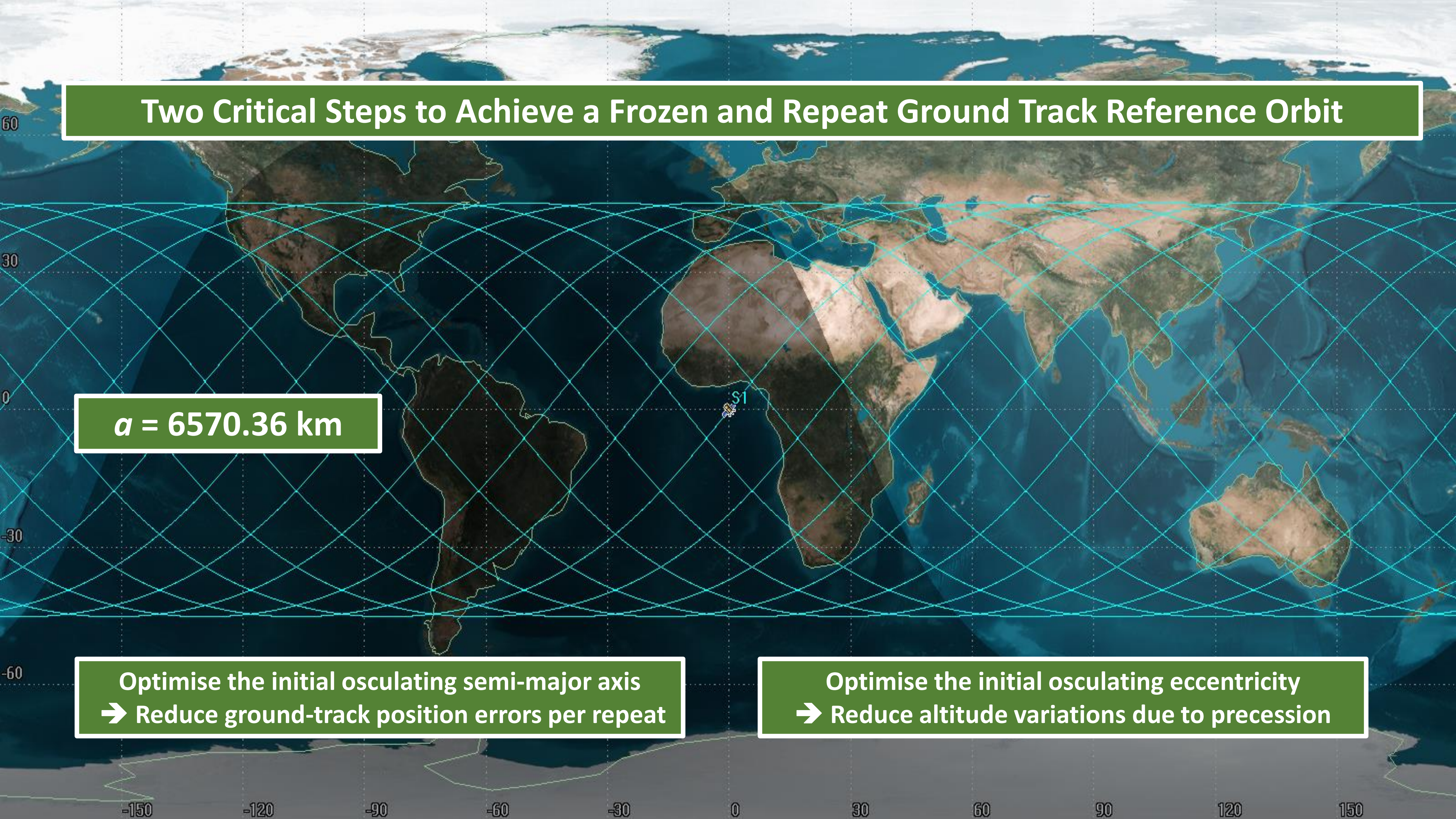


Projecting one ground track in Earth-fixed 2D, with the ground track shift Δλ



Projecting all ground tracks in the same example frozen repeat orbit

Two Critical Steps to Achieve a Frozen and Repeat Ground Track Reference Orbit



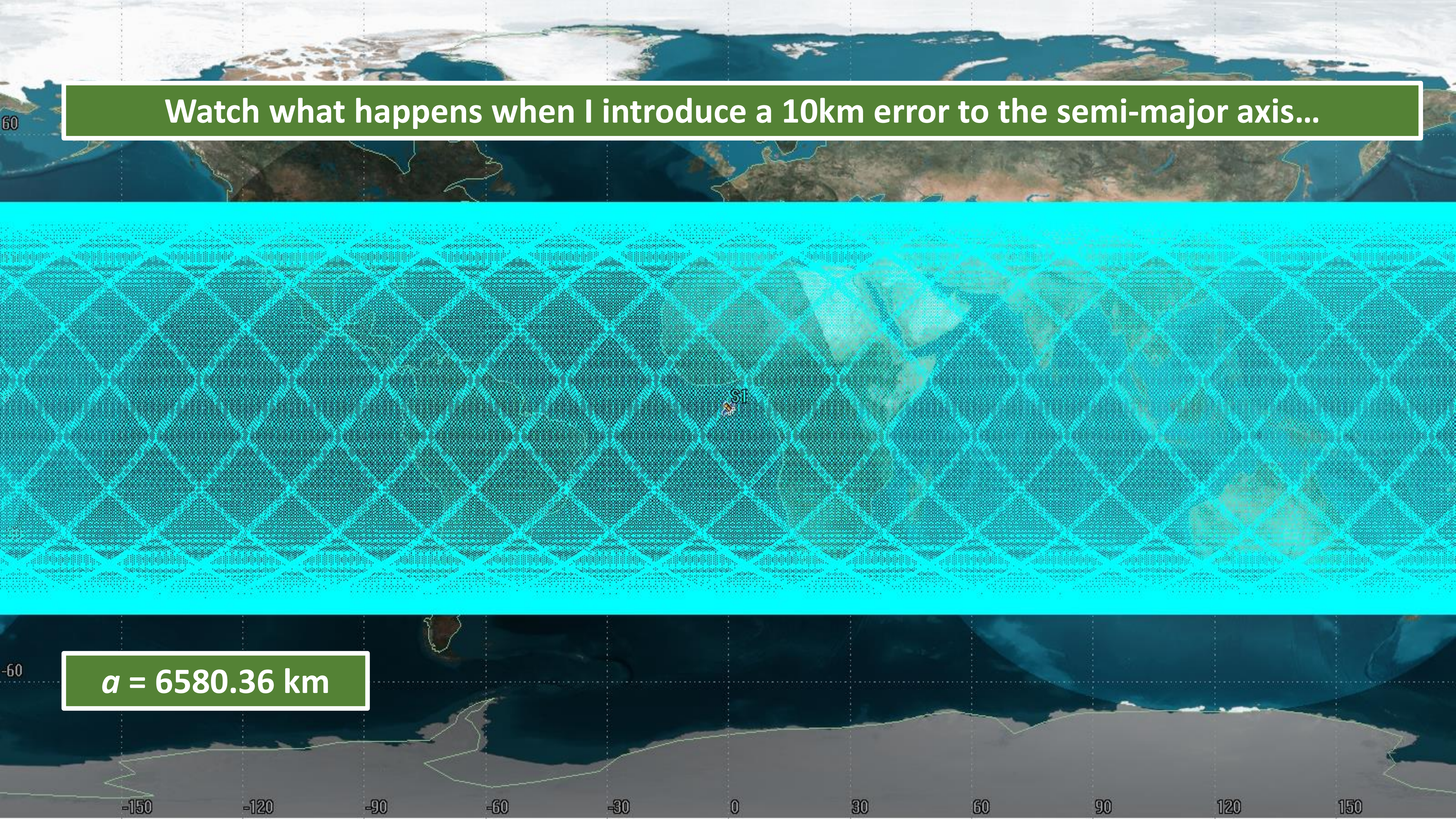
a = 6570.36 km

Optimise the initial osculating semi-major axis
➔ Reduce ground-track position errors per repeat

Optimise the initial osculating eccentricity
➔ Reduce altitude variations due to precession

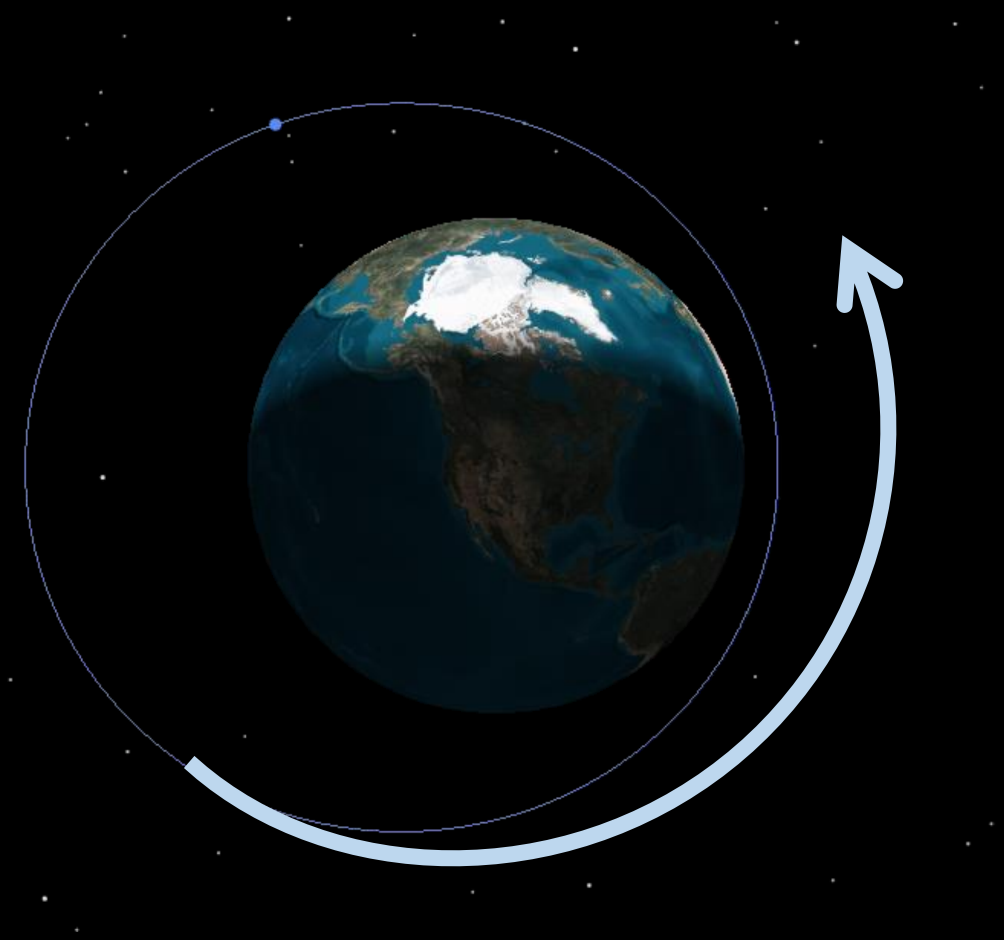
Watch what happens when I introduce a 2km error to the semi-major axis…

a = 6572.36 km


Watch what happens when I introduce a 10km error to the semi-major axis…
a = 6580.36 km
S1
60
-60
-150
-120
-90
-60
-30
0
30
60
90
120
150

Precession of the Orbit Argument of Perigee

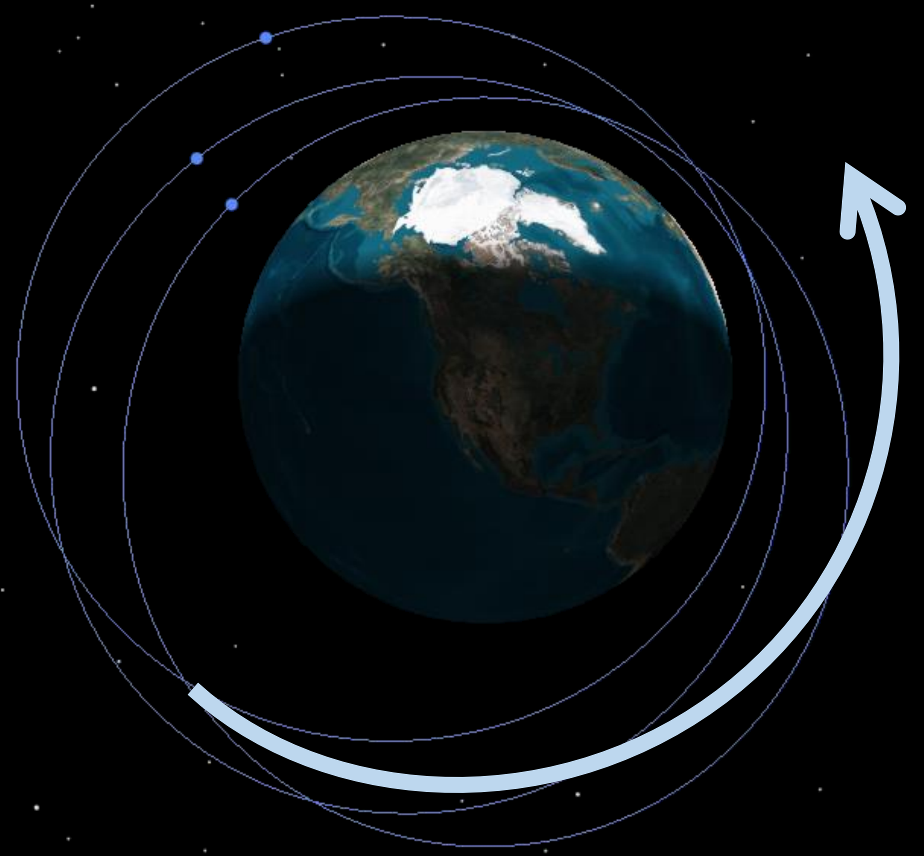
Argument of Perigee (ω): Precession causes it to re-orient the ellipse about its own orbital plane.



A single 45^{o} inclined orbit with 0^{o} ARGP. Notice where the apogee is. In a repeat ground track orbit, what do you think the observer on Earth sees if the orbit plane starts rotating about itself?

Precession of the Orbit Argument of Perigee

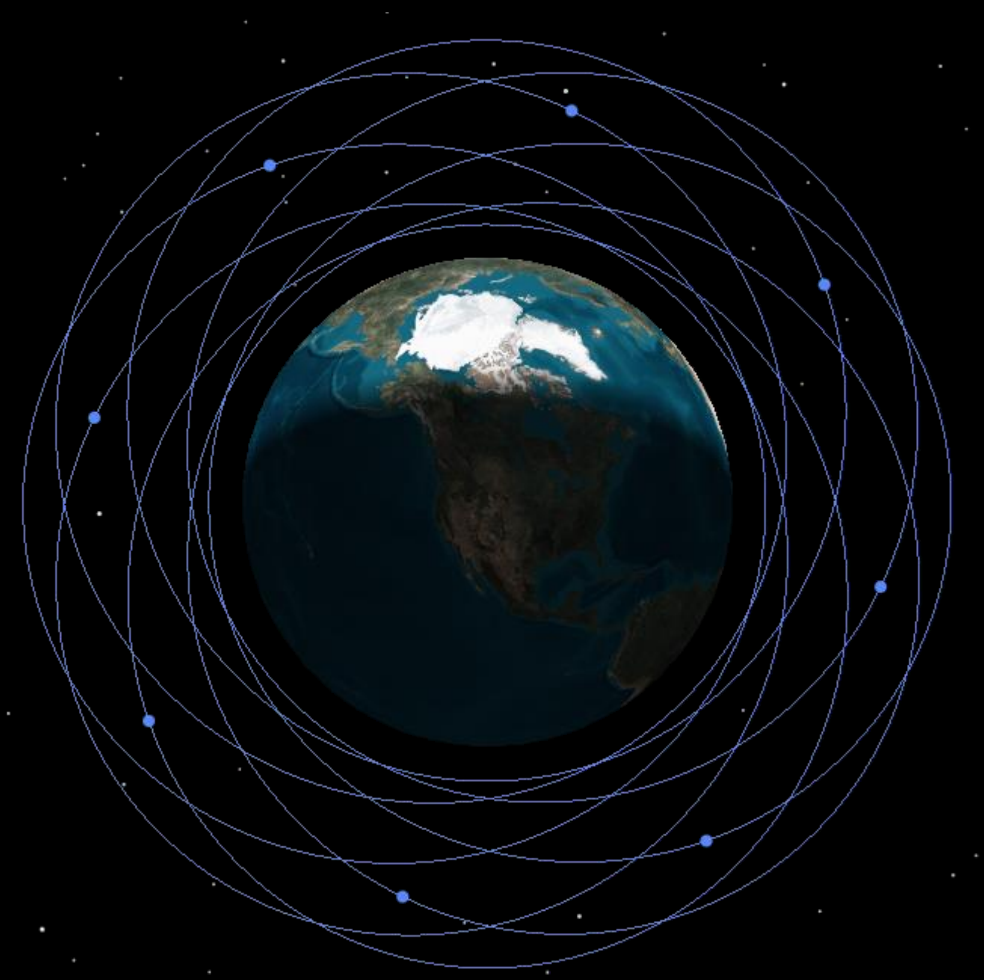
Argument of Perigee (ω): Precession causes it to re-orient the ellipse about its own orbital plane.



When it precesses over time, from an observer on the Earth, it looks like the altitude is varying! Undesirable!

Precession of the Orbit Argument of Perigee

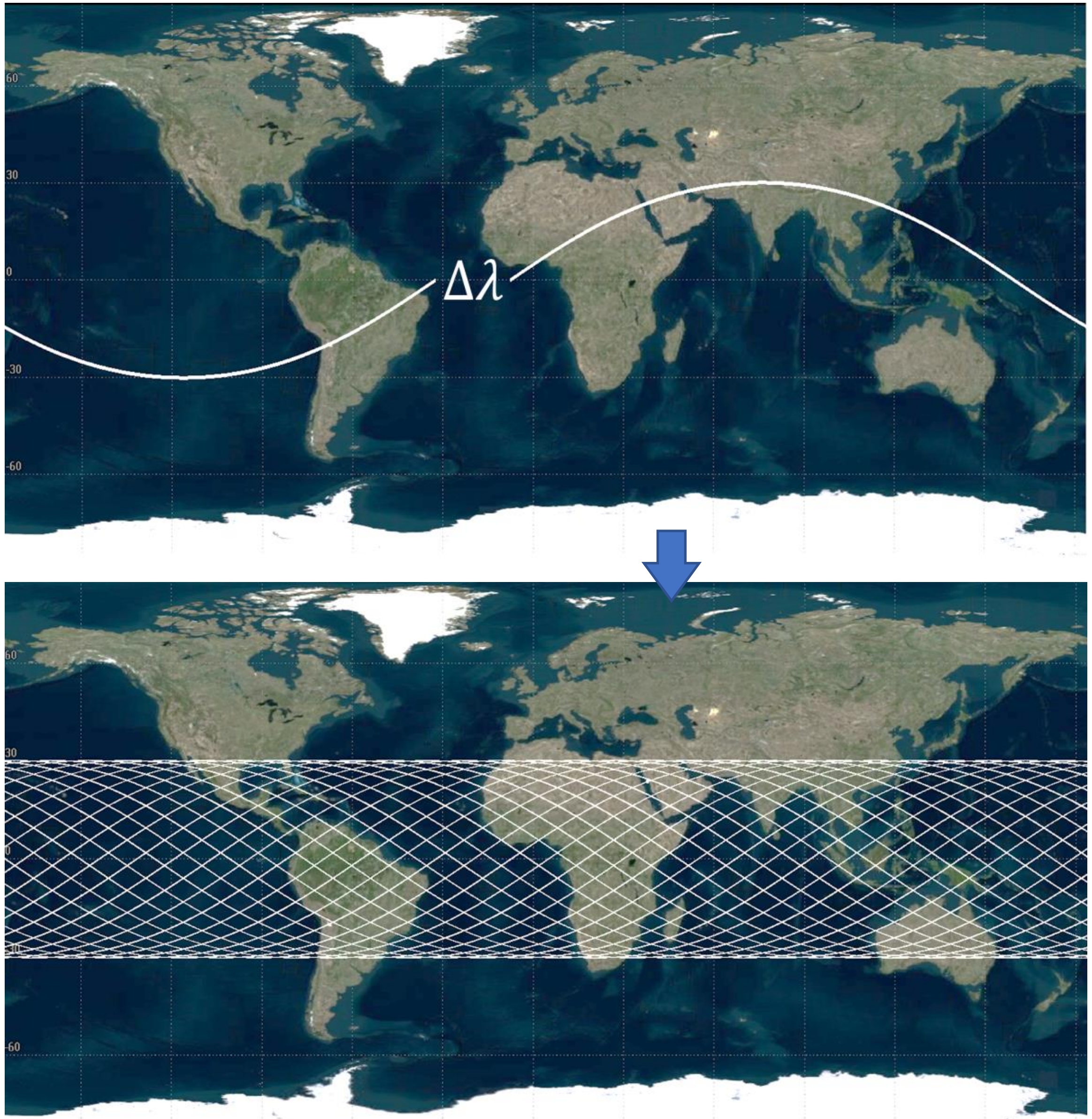
Argument of Perigee (ω): Precession causes it to re-orient the ellipse about its own orbital plane.



Let us see what the argument of perigee rotation looks like when it precesses a full 360⁰ (in 45⁰ steps)

Design Stage I: Preliminary Orbit Design

- **Overview of Repeat Ground Track Orbits**
- The key condition that makes a repeated orbit, is that after a deliberately designed P number of nodal periods, the orbiter must ascend the same longitude of ascending node again as per the first epoch.
- This can be thought of as a "phase unwrapping" problem: given some ground track shift Δλ caused by a perturbing RAAN and Earth's rotation during the TOF of one orbit, we want an integer P number of ground track shifts to "fit into" Q, a multiple of 360^{O} (one equator's circumference).



$$Nodal\ Period = \tau = \frac{2\pi}{\dot{M} + \dot{\omega}}$$

$$Ground\ \text{Track Shift} = \Delta\lambda = \left(\sigma_{\oplus} - \frac{d\Omega}{dt}\right) \cdot \tau$$

$$\boxed{\frac{\Delta\lambda}{2\pi} = \frac{\sigma_{\oplus} - \dot{\Omega}}{\dot{M} + \dot{\omega}} = \frac{Q}{P} \quad = \frac{Days\ in\ a\ Repeat\ Cycle}{Orbits\ in\ a\ Repeat\ Cycle}}$$

Representing $\sigma_{\oplus}$ as the Earth's rotation rate, M as the mean anomaly, ω as the argument of perigee, and Ω as the RAAN

Design Stage I: Preliminary Orbit Design

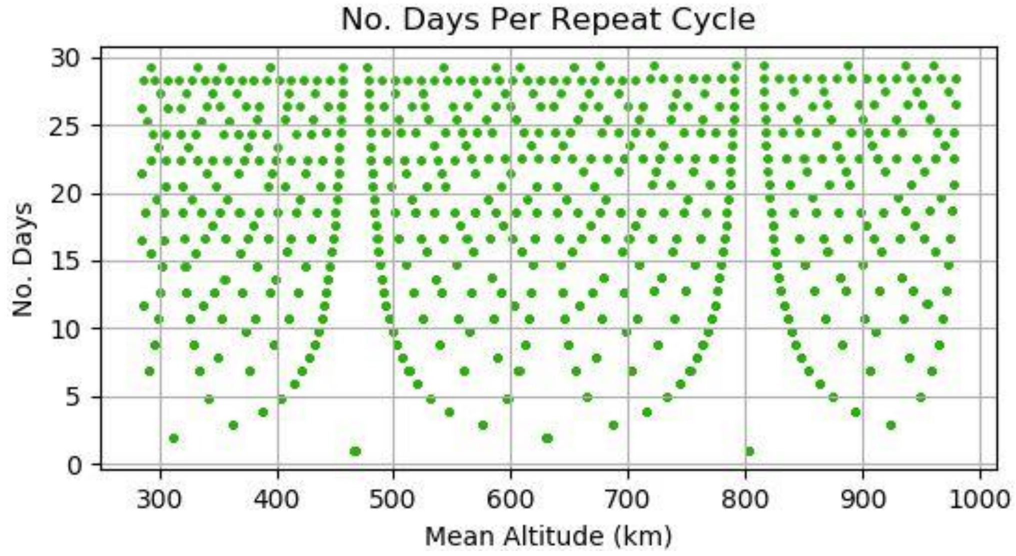
- **Overview of Repeat Ground Track Orbits**
- From the previous equations, we saw that Δλ depended on Earth's rotation rate, and the rate of Ω. The nodal period depended on of the rates of ω (precession) and M, the latter rate being the dominant term and a function of the semi-major axis. Each unique semi-major axis gives a unique nodal period and repeat ground track solution. The semi-major axis can be converted into the mean ECF altitudes. We use the J2 model for rates of ω, Ω, and M as follows:

$$\dot{\Omega} = \frac{-3J_2 n}{2}\left(\frac{R_\oplus}{p}\right)^2 \cdot \cos i$$

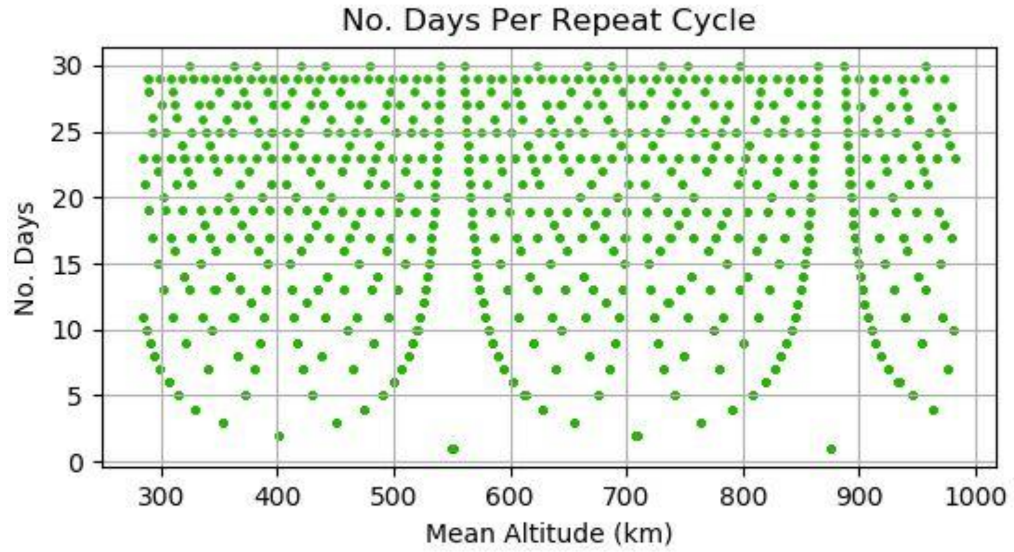
$$\dot{\omega} = \frac{3}{2}J_2 n\left(\frac{R_e}{p}\right)^2\left(2 - \frac{5}{2}\sin^2 i\right)$$

$$\dot{M} = n\left\{1 + \frac{3}{2}J_2\left(\frac{R_e}{p}\right)^2\sqrt{1-e^2}\left(1 - \frac{3}{2}\sin^2 i\right)\right\}$$

- Solutions (in mean altitudes) are shown for a near-equatorial orbit (top) and polar orbit (bottom). Note that these solutions are valid only for a simple two-body propagation with perturbations only up to the J2 term. We pick the solutions of ~4 days repeat.



Valid repeat ground track solutions for i = 10.0°



Valid repeat ground track solutions for i = 97.44°

Design Stage I: Preliminary Orbit Design

- **Preliminary Solution for Frozen Parameters**
- Altitude variations persist if either ω or e experiences secular variation. An estimate to this max variation is $\Delta R = a \cdot e$. Thus, we solve for frozen elements of ω and e that minimises these variations. Again we use the J2-orbit model, and forcefully set the rates of ω and e to be zero:

$$\dot{e} \cong \frac{-3nJ_3}{2}\left(\frac{R_\oplus}{a}\right)^3 \left(\sin i - \frac{5}{4}\sin^3(i)\right)\cos\omega = \mathbf{0}$$

$$\dot{\omega} = 3J_2 n\left(\frac{R_\oplus}{a}\right)^2 \left(1 - \frac{5}{4}\sin^2(i)\right)\cdot\gamma = \mathbf{0}$$

- Where the auxiliary gamma term is a term for:

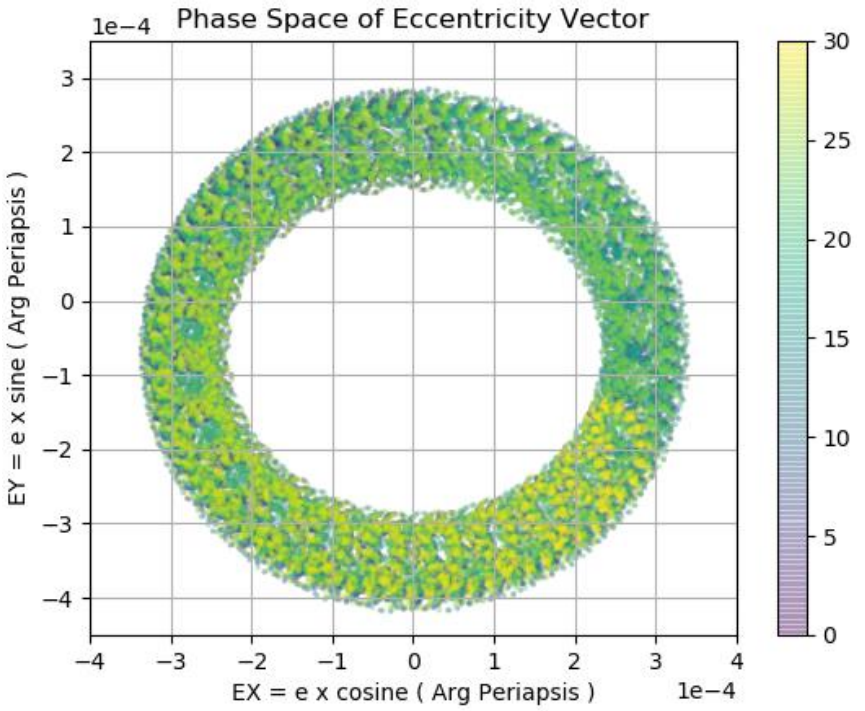
$$\gamma \cong 1 + \frac{J_3}{2J_2}\left(\frac{R_\oplus}{a}\right)\left(\frac{\sin^2(i) - e^2\cos^2(i)}{\sin i}\right)\frac{\sin\omega}{e}$$

- Ostensibly, setting $\omega = 90^O$ keeps the eccentricity rate zero in this simplified model for both near-equatorial and polar orbits. The general solution for e would be:

$$e_o = -\frac{J_3}{2J_2}\left(\frac{R_\oplus}{p}\right)\sin(i)\sin(\omega)$$

- We now solve for ω and e and propagate the phase space variations of the orbit eccentricity vector across 30 repeat cycles.

Results of Phase Space Variations of the Eccentricity Vector using the preliminary estimates, for some arbitrary Starlink satellite at 550km.



Near-equatorial orbit eccentricity vector phase space **pre-optimisation**, colour map represents repeat cycles.

$$e_x(t,e) = e_{mean}(t, e)\cos\omega(t,e)$$

$$e_y(t,e) = e_{mean}(t, e)\sin\omega(t,e)$$

Design Stage I: Preliminary Orbit Design

- In the optimisation stage, the mean elements are converted to initial osculating elements, and two elements are further optimised; this time using a high precision orbit propagator (70x70 geopotential model, with sun and moon third-body perturbations). The two elements are:
- Osculating semi-major axis at $t = 0$ ➔ to correct for ground track errors at ascending nodes.
- Initial osculating eccentricity at $t = 0$ ➔ to minimise phase space variations and hence minimise altitude variations.
- During optimisation, the orbit propagator must be as precise as possible, exempting non-linear effects such as manoeuvres and drag forces, so that the reference trajectory is realistic enough to prevent wasting ΔV compensating for modelling errors during actual flight. Nevertheless, even after optimisation, it is expected that state vectors at the first and final ascending nodes will not be identical, which means a discontinuity will be encountered.

Design Stage II: Optimisation of Orbit Elements

- **The Optimisation Problem**
- Optimising the osculating semi-major axis to minimise repeat ground track errors at the first ascending nodes.

$$a^* = arg\, min_a(|\Theta_2 - \Theta_1|)$$
$$= arg\, min_a(\Delta\Theta)$$

Ground track error at first ascending node $\Delta\Theta$ < 1m?

No

Update values according to a first order Taylor expansion:

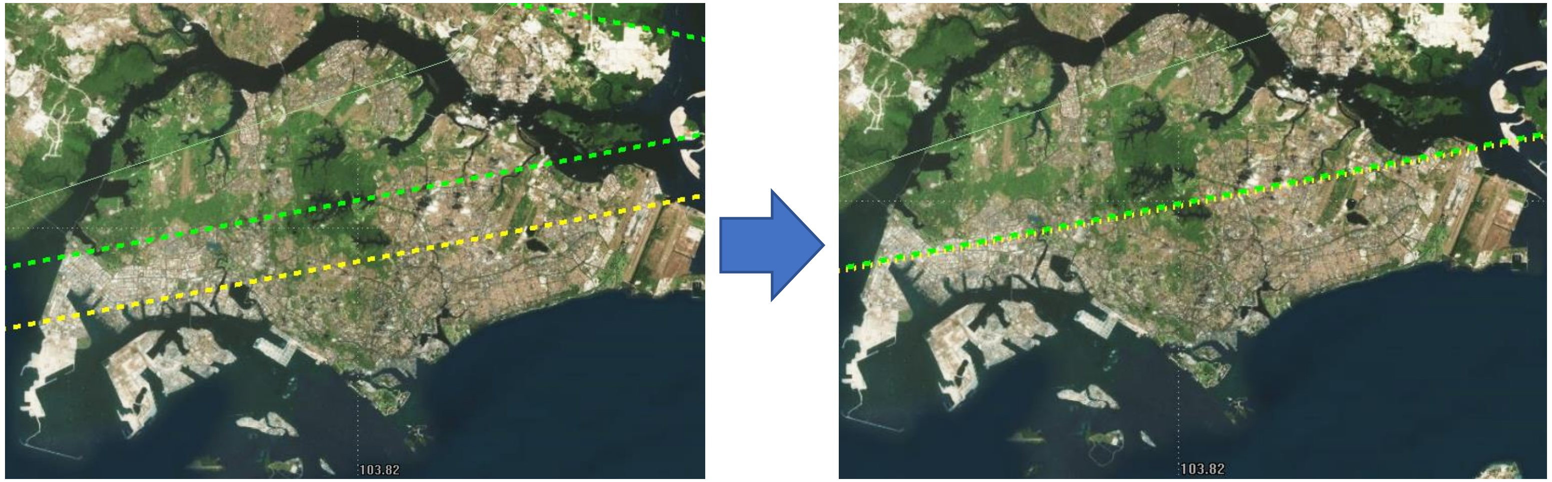
$$\Delta\Theta_{N+1} = \Delta\Theta_N + J_N(X_{N+1} - X_N)$$

Where J_N is the Jacobian with respect to the control vector X_N.

Yes

Stop running the differential corrector.

- Straightforward optimisation – can be done easily using in-built STK features, or Broyden’s Method for numerical optimisation. For both near-equatorial and polar, ground track error converged to ~10E-2 meters in both cases. Note that ground track errors are local horizontal errors, it does not ensure that altitude variations are stopped (on to that in the next section).



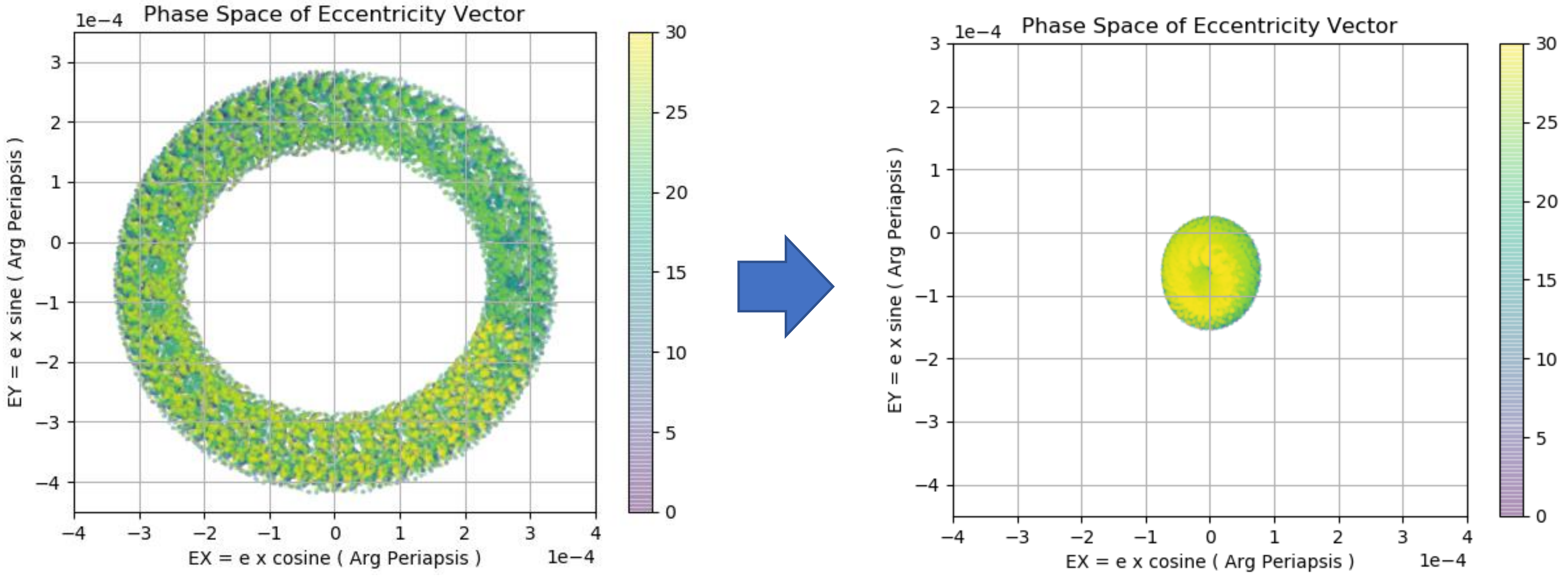
Design Stage II: Optimisation of Orbit Elements

- **The Optimisation Problem**
- Optimising the osculating eccentricity to minimise phase space (and thus altitude) variations across long term.

$$e^* = \arg\min_e \sum \left[(e_x - e_{x0})^2 + \left(e_y - e_{y0}\right)^2\right]$$

- Where we define the eccentricity vector as ➔ $e_x = e_{mean}\cos\omega\,,\ e_y = e_{mean}\sin\omega$. The central idea is that there exists an optimum frozen value, and any initial values that deviate from it will exhibit long-periodic oscillations about it. It is approximated as the "centre of the circle" traced by the phase space across 30 repeat cycles. Thus, we aim to reduce the diameter of the "circle" through iterative corrections in the initial osculating eccentricity.

Results for Eccentricity Phase Space for the Near-Equatorial Orbit



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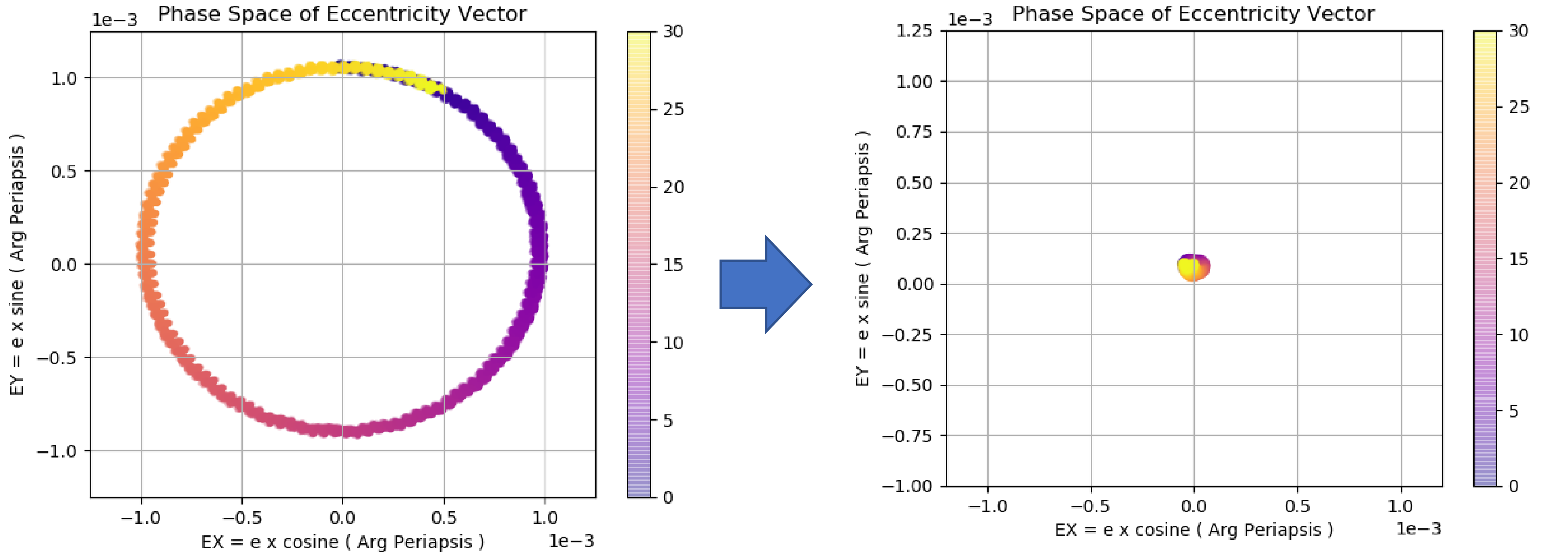
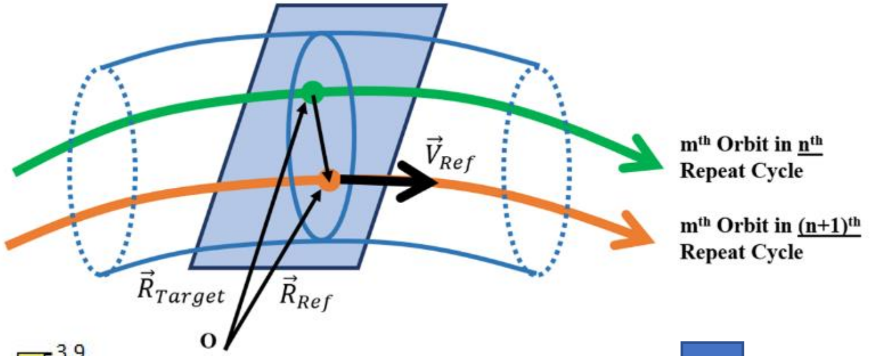
Results for Eccentricity Phase Space for the Polar Orbit

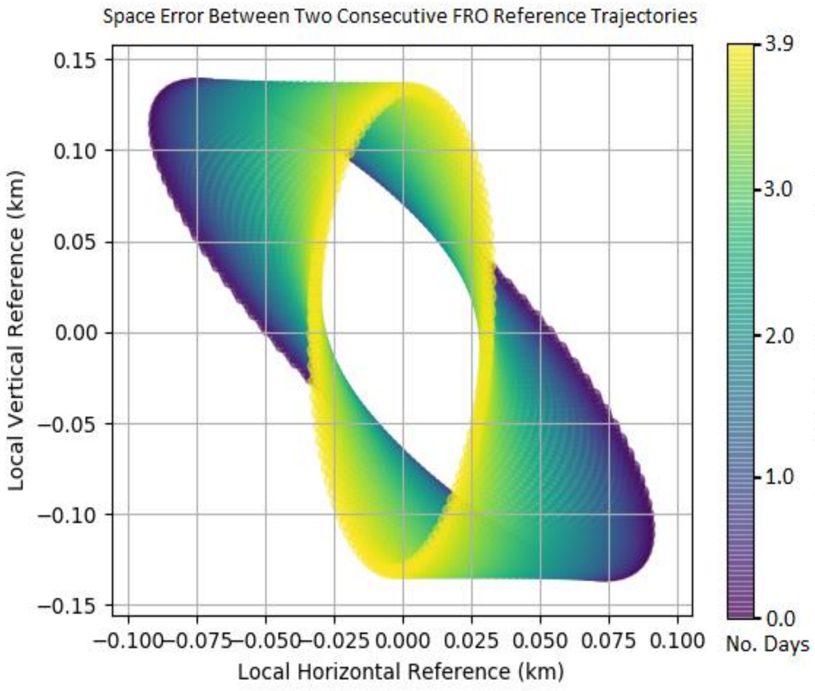


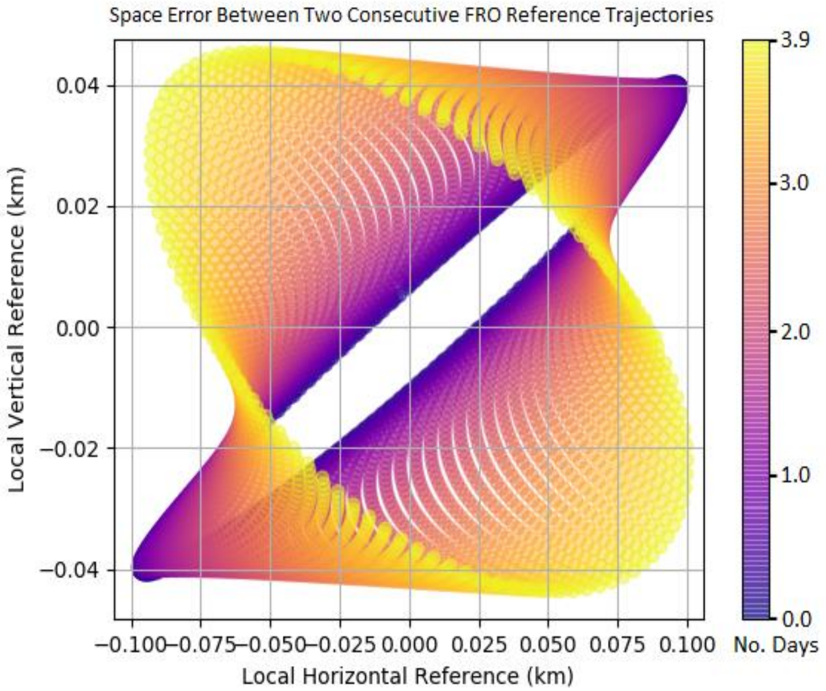
Design Stage II: Optimisation of Orbit Elements

- Optimisation Space Error Results
- We compute the radial and cross-track error below, between two consecutive repeat cycles of a reference trajectory (e.g. for a repeat cycle of 59 orbits, we compare space errors in ECEF between orbit sets 1 to 59, and orbit sets 60 to 118). We interpolate both orbits with a sliding quadratic interpolant taken at 1ms interpolated step size.

	Near-Equatorial		Polar Orbit	
	Mean (m)	STD (m)	Mean (m)	STD (m)
Radial	-0.012	128.473	-0.231	29.207
Off-Track	-0.677	44.447	-2.444	58.752





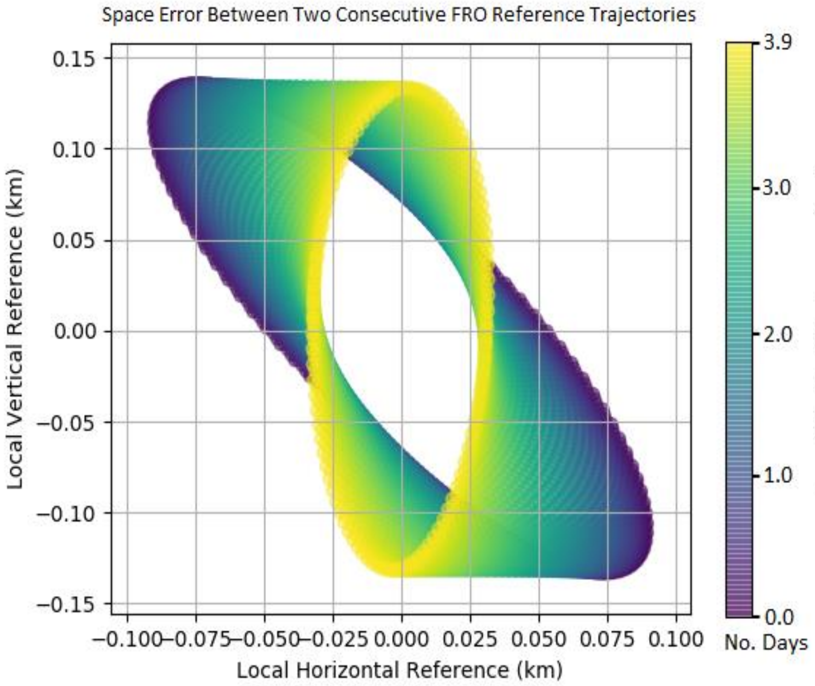


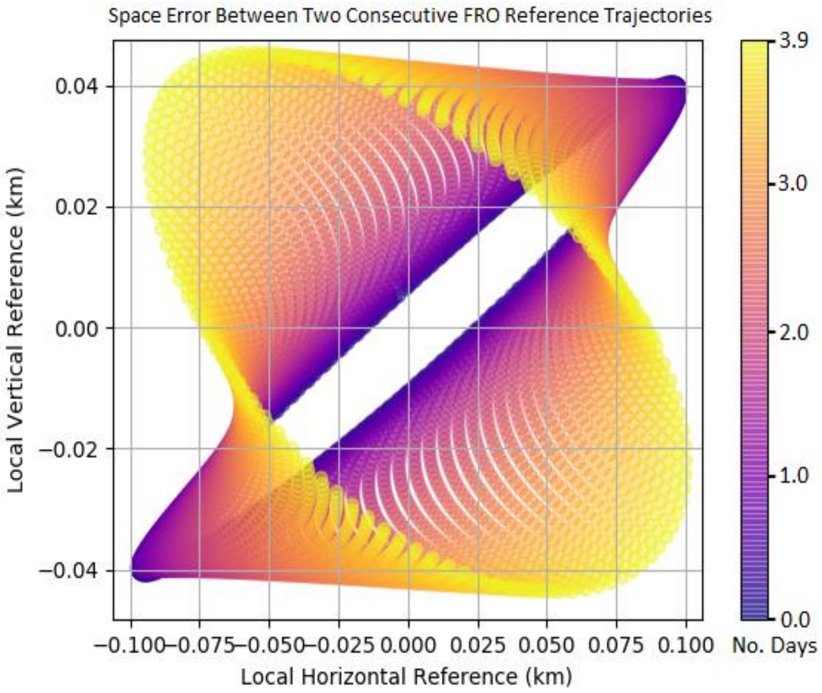
Checkpoints for computing space errors

$$\left[\vec{V}_{Ref} \cdot \left(\vec{R}_{Target} - \vec{R}_{Ref}\right)\right]_{ECEF} \approx 0$$

Design Stage II: Optimisation of Orbit Elements

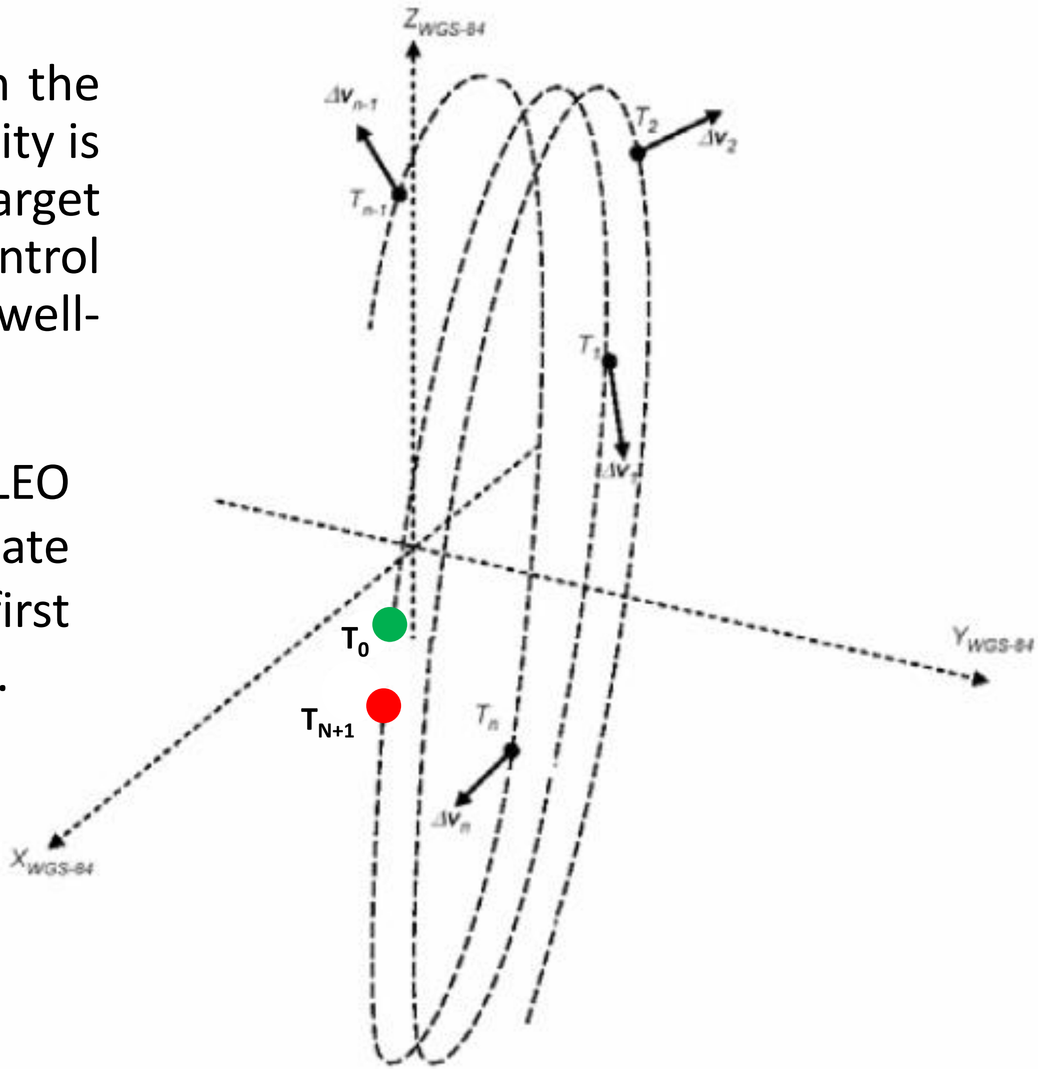
- We have thus arrived at a realistic reference trajectory propagated with high-fidelity.
- One caveat though, is that even after optimisation, state vectors (or equivalently the osculating elements) at the first and last ascending node are not exactly the same for both orbits in the Earth-centred fixed frame coordinates. This issue on orbit continuity will be the third stage of our design process, addressed in the next slide.





Design Stage III: Ensuring Orbit Continuity

- **Overview of FRO Orbit Discontinuity Problem**
- A discontinuity describes the repeat cycle transition from the final to initial ascending node at this stage. This discontinuity is undesirable because the reference orbit is the target trajectory for the orbit control system, and from a control systems point of view, the system response may not be well-defined in face of a discontinuous input.
- The discontinuity exists as it is nearly impossible for a LEO with full perturbations to exactly arrive at the identical state vector (position and velocity) between the exiting first ascending node and the entering the final ascending node.
- Two possible methods are explored to resolve the orbit discontinuity:
  - **1. Method of Virtual Manoeuvres**
  - **2. Method of Average Filtering**



Design Stage III: Ensuring Orbit Continuity

- **The Method of Virtual Manoeuvres**

The method of virtual maneuvers proposed by D'Amico *et al* [7] was applied to both orbits, forcing the final state vector to match the initial.

The desired goal for the final epoch state vector is the initial state epoch state vector (so that they match). Broyden's method was iterated, updating the virtual ΔV with a max step size of 1cm/s.

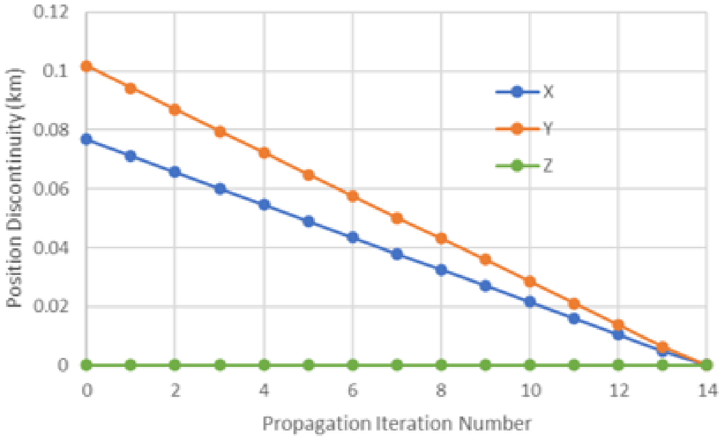


Figure 2.7 – Convergence of Position Discontinuities in the Near-Equatorial Orbit

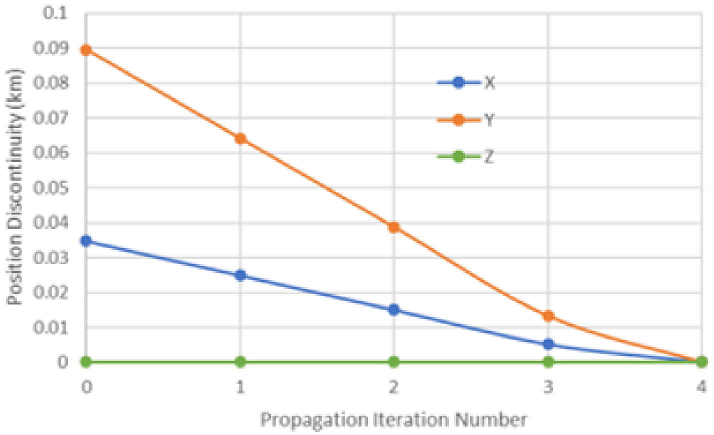


Figure 2.8 – Convergence of Position Discontinuities in the Polar Orbit

But… this introduces a velocity vector discontinuity in the reference states instead! Also, the addition of a virtual manoeuvre "disrupts" the carefully selected initial osculating elements solved through optimisation!

Design Stage III: Ensuring Orbit Continuity

- **The Method of Virtual Manoeuvres**

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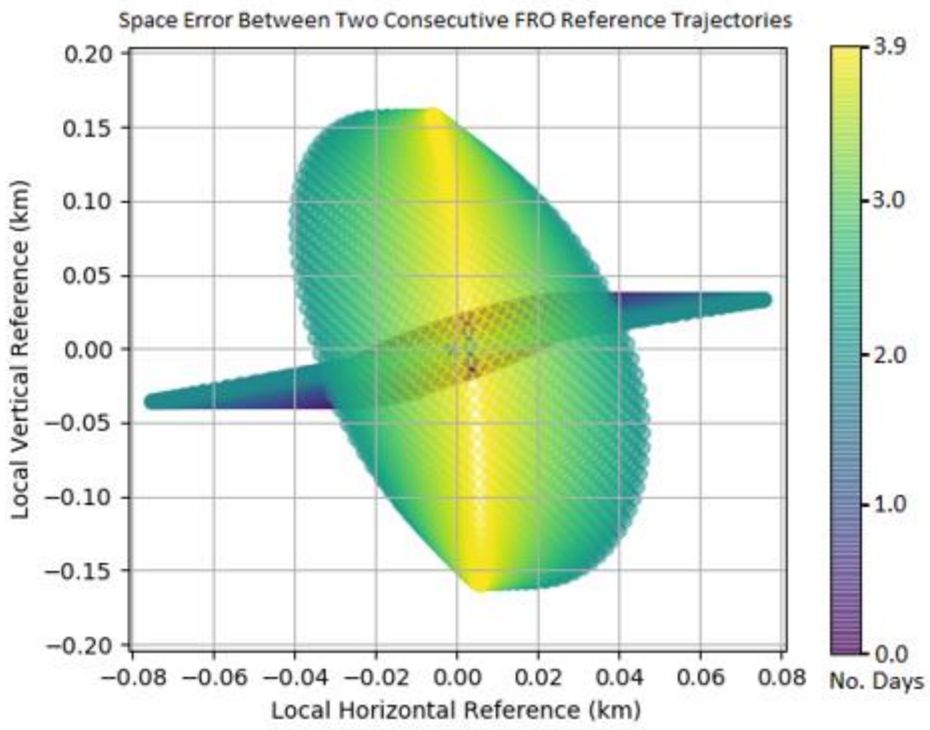


Figure 3.2 – Spatial error evolution for $i = 10°$ with virtual ΔV applied on t = 2.0 days

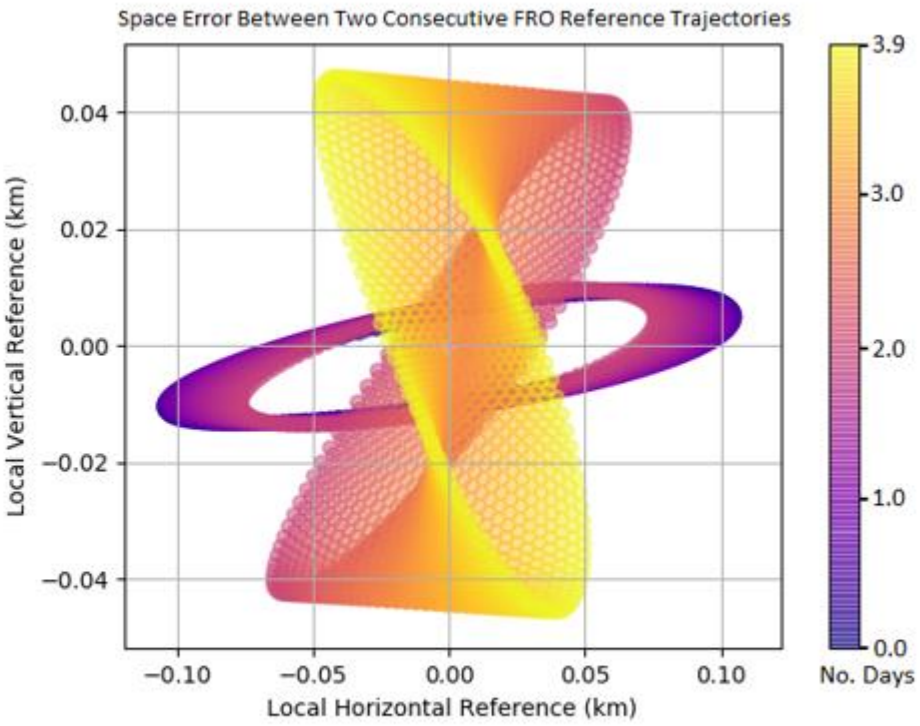
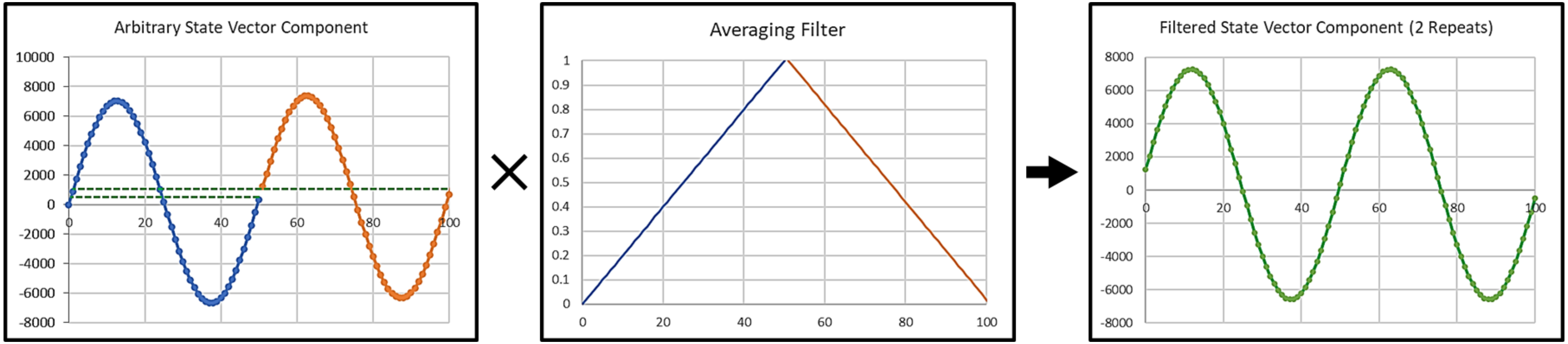


Figure 3.3 – Spatial error evolution for $i = 97.4°$ with virtual ΔV applied on t = 2.0 days

Design Stage III: Ensuring Orbit Continuity

- **The Method of Averaging Filters**
- For the near-equatorial orbit, we propose a simple "averaging filter" that works in the following way: we propagate orbit elements forwards and backwards one full repeat cycle using optimised osculating elements in Table 3.4.1, and apply a weighted average with the weight on the forward propagation decreasing linearly and the backwards propagation increasing linearly.
- Below illustrates how the averaging filter works, on an arbitrary state vector component where for simplicity of illustration, we set 1 period to be equal to one full repeat cycle.



An example of the use case of the averaging filter; blue plots represent the backward propagated state vector component; orange plots represent the forward propagated state vector component; and green plots represent the output of the filter.

- The result, in Fig 4.3.1, is that across the epochs $t \in [0,100]$, the state vector component value at $t = 0$ and $t = 100$, will be forcibly equalised to the state vector component value at $t = 50$. Again, these epoch values are used as arbitrary examples for some state vector component.

Design Stage III: Ensuring Orbit Continuity

- **The Method of Averaging Filters**
- The averaging filter will be described The averaged reference trajectory states R_Avg (t), V_Avg (t) are obtained as a linear combination of the forward and backward propagated trajectories, where weights on backward propagated states increase with time, and weights on forward propagated states decrease with time.

$$\vec{R}_{Avg}(t) = \left[1 - \frac{t}{T_{repeat}}\right] \times \vec{R}_{Prop}(t) + \left[\frac{t}{T_{repeat}}\right] \times \vec{R}_{Prop}\left(t - T_{repeat}\right)$$

$$\vec{V}_{Avg}(t) = \left[1 - \frac{t}{T_{repeat}}\right] \times \vec{V}_{Prop}(t) + \left[\frac{t}{T_{repeat}}\right] \times \vec{V}_{Prop}\left(t - T_{repeat}\right)$$

- For 0 ≤ t ≤ T_repeat , where t is the time elapsed since the beginning of the repeat cycle, T_repeat is the repeat cycle period, and R_Prop (t), V_Prop (t) are the position and velocity trajectories obtained by numerically propagating the state vectors at the beginning of the repeat cycle. This trivially satisfies the continuity constraints R_Ref (0) = R_Ref (T_repeat) and V_Ref (0) = V_Ref (T_repeat).
- A simplified visualization of the filter's effects on a toy trajectory is illustrated in Figure 4.2A and 4.2B.

Design Stage III: Ensuring Orbit Continuity

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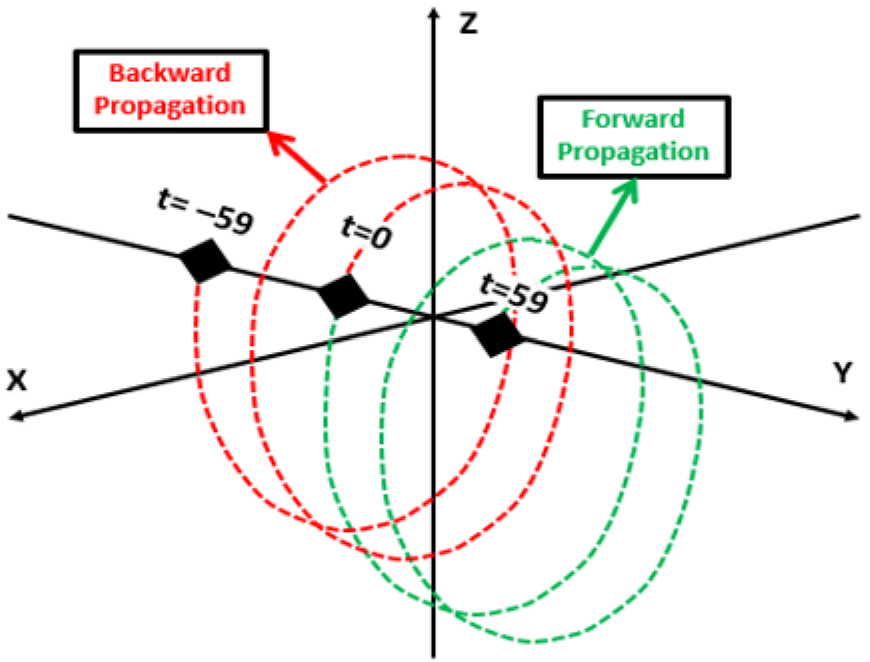
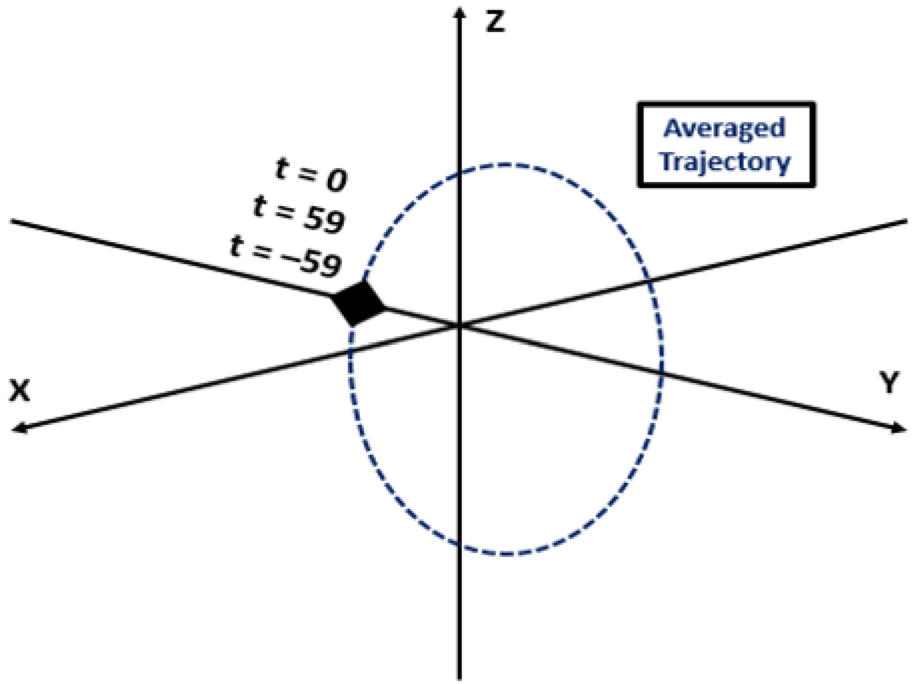




Validation Stage: Simulating Orbit Maintenance

Motivation for Orbit Maintenance Validation

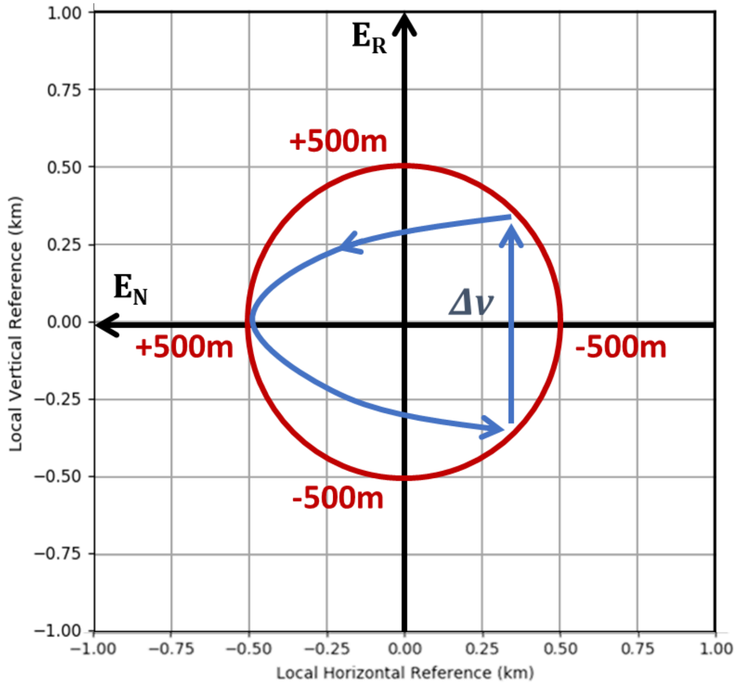
- Reference trajectory was designed with high-fidelity perturbations, excluding drag effects, so that the orbit maintenance of the actual trajectory only has to compensate for that and not modelling errors.

- To simulate that the reference trajectory design is realistic enough, we perform periodic orbit maintenance using all perturbing forces plus drag effects in the actual orbit propagation. If we can successfully maintain the orbit and keep within the torus without including Delta-V for modelling errors in the reference, then the continuous and optimised reference is a realistic trajectory for guidance.

Summary of Orbit Maintenance Process

- Drag effects decay the semi-major axis and thus altitude, which shortens the nodal period, changes ground track shift Δλ, and introduces an eastward drift in the ground track. With orbit raising, the semi-major axis rises, causing an increase in nodal period, and inducing a westward ground track shift instead. The balance between maintaining westward and eastward drifts is central to keeping the actual orbit trajectory within the 1km torus.

- Key idea is to perform periodic thrusts that minimise changes to the mean altitude and mean eccentricity, only in periodic apogee-perigee thrust pairs. Orbit altitude is always raised beyond the reference altitude by same magnitude at which it had decayed.



Ideal space error evolution.

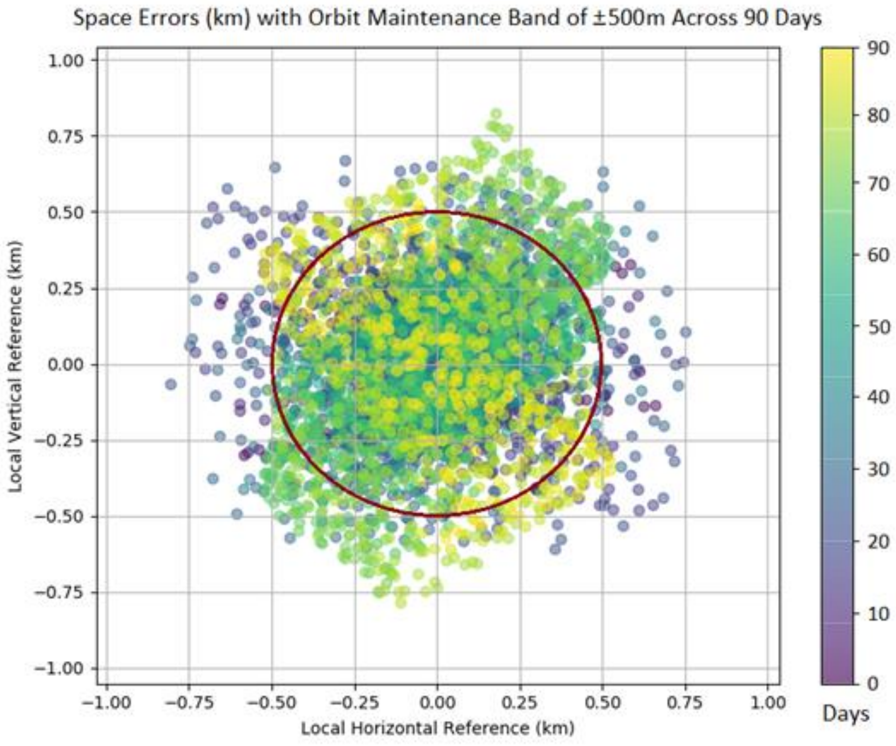
Taking v as the initial velocity, ΔR = 500m (half the torus), each prograde thrust Δv at perigee-apogee of the transfer orbit is approximated with a first order Taylor expansion of the vis-visa equation:

$$\Delta v_1 \approx \Delta v_2 \approx \frac{1}{4} V \left(\frac{\Delta R}{R}\right) = 13.665\ cm/s$$

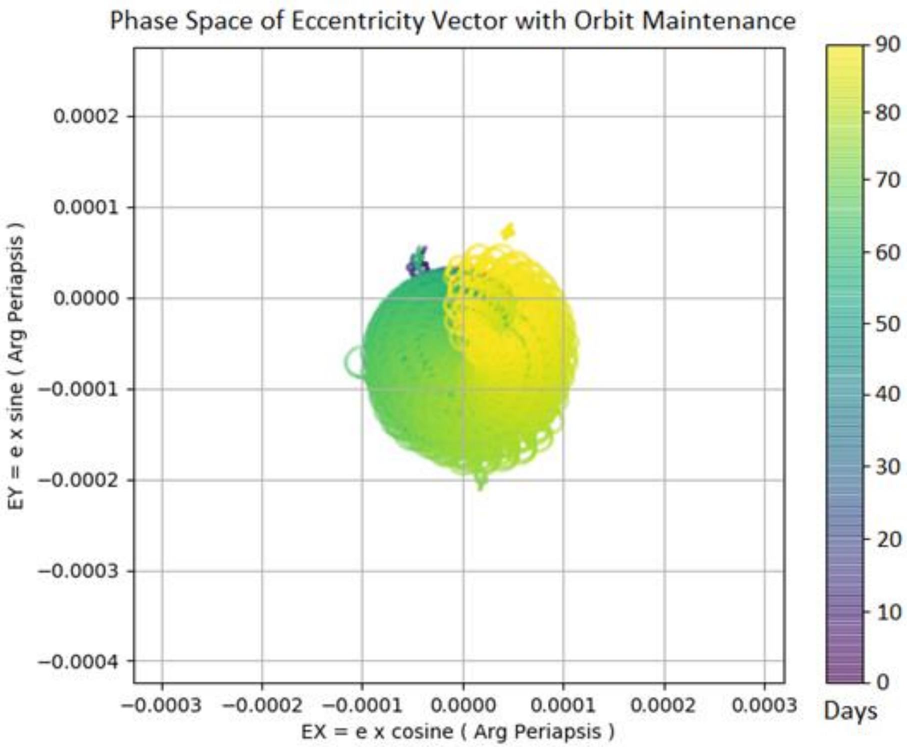
Validation Stage: Simulating Orbit Maintenance

Orbit Maintenance Results

- Below results are that of the near-equatorial orbit. Each thrust pair was performed every 5.625 days, the rough time taken for the mean semi-major axis to decay 500m or so, using a Jacchia-Roberts atmospheric density model. In total, 3-months of simulated station-keeping was recorded.
- The resultant space error analysis between the actual trajectory and the optimal-averaged trajectory with eccentricity vector phase space variations are below.



Scatter plot of space errors taken at 30-minute intervals across 90 days (time line by colour bar)

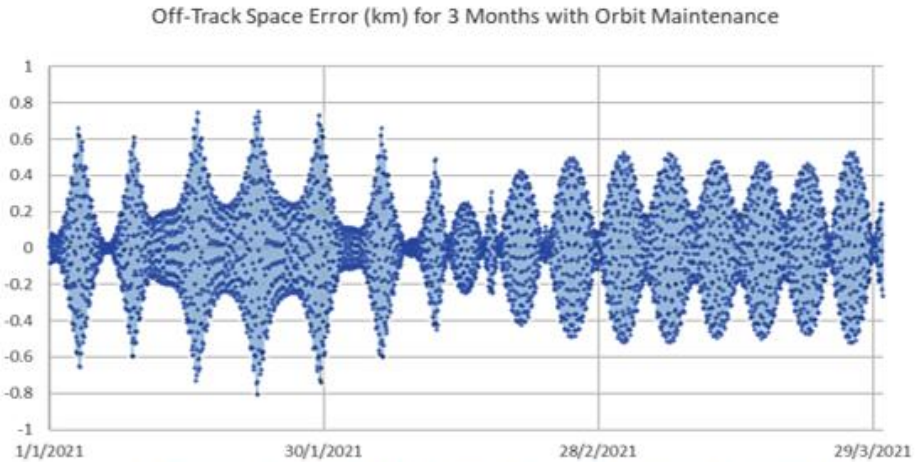


Eccentricity phase space evolution taken at 30-minute intervals across 90 days (time line by colour bar)

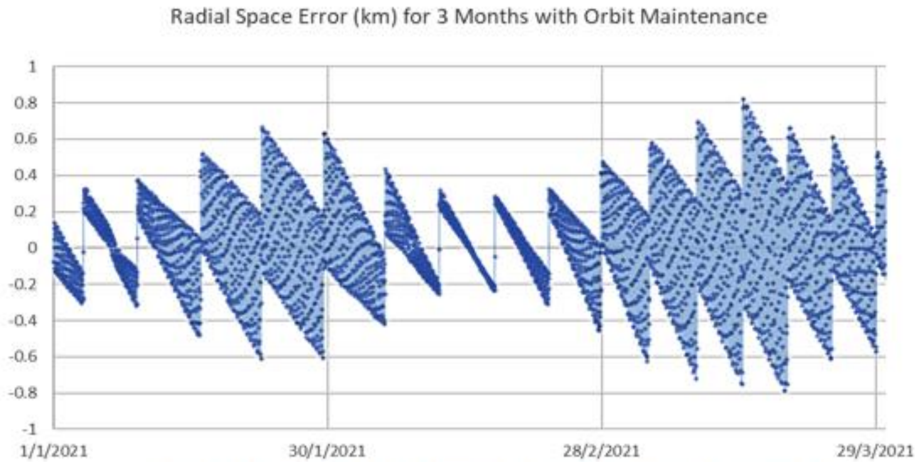
Validation Stage: Simulating Orbit Maintenance

Orbit Maintenance Results

- The decomposition of space error components and sigma values are graphed and tabulated below. In summary, we maintain the mean eccentricity about the frozen value with a variation radius of 1E-4. The orbit control scheme keeps the space craft within 1-sigma of 394.539m, and expectedly consumes a total ΔV ~ 17.7m/s per year.



Off-track errors, 16 thrust-pairs across 3 months.



Radial errors, 16 thrust-pairs across 3 months.

Mean Space Error	1-Sigma (68%)	2-Sigma (95%)	3-Sigma (99.7%)
320.979m	394.539m	631.756m	799.456m