The Poisson problem

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May 9, 2017

1 Introduction

This document describes the strong- and associated weak form of the Poisson problem, which forms the theoretical foundation of this *IFEM* application. Refer to the class `Poisson` of the source code for the actual integrand implementation. To better see the correspondence with the implementation, we here employ the tensor notation assuming Einstein's summation convention, i.e., double indices automatically implies a sum, e.g., $a_i b_i := \sum_{j=1}^{n_\mathrm{d}} a_j b_j$, where n_d denotes the number of spatial dimensions (1, 2 or 3). Furthermore, partial derivation with respect to a given coordinate direction is indicated by a subscript comma, i.e., $a_{,i} := \frac{\partial a}{\partial x_i}$.

2 Strong form

Given a heat source function $f(x_i)$ defined over a domain $\Omega \in \mathbb{R}^{n_\mathrm{d}}$, a heat flux function $h(x_i)$ defined over the boundary $\partial\Omega_h$, and a function $g(x_i)$ defined over the boundary $\partial\Omega_g = \partial\Omega \setminus \partial\Omega_h$, find the scalar function $u(x_i) \in \mathcal{U}(\Omega)$ satisfying

$$\left.\begin{aligned} q_{i,i} &= f \\ q_i &= -\kappa_{ij} u_{,j} \end{aligned}\right\} \quad \forall \quad x_i \in \overline{\Omega} \tag{1}$$

$$q_i n_i = h \quad \forall \quad x_i \in \partial\Omega_h \tag{2}$$

$$u = g \quad \forall \quad x_i \in \partial\Omega_g \tag{3}$$

where κ_{ij} is the conductivity tensor and n_i is the outward-directed unit normal vector on $\partial\Omega_h$. If the conductivity tensor equals the identity tensor, $\kappa_{ij} = \delta_{ij}$, we have $q_i = -u_{,i}$ and Equation (1) reduces to

$$-u_{,ii} = f \quad \forall \quad x_i \in \overline{\Omega} \tag{4}$$

The solution space $\mathcal{U}(\Omega)$ defines the set of all admissible solution functions on the domain Ω, with the additional constraint $u(x_i) = g(x_i)\ \forall x_i \in \partial\Omega_g$.

3 Weak form

The weak form is obtained by multiplying Equation (1) by a test function $v(x_i) \in \mathcal{V}(x_i)$ and integrating over the domain Ω, viz.

$$-\int_\Omega \kappa_{ij}\, u_{,ji}\, v \,\mathrm{d}V = \int_\Omega f\, v \,\mathrm{d}V \tag{5}$$

where the test space $\mathcal{V}(x_i)$ is the same as $\mathcal{U}(x_i)$, except that their functions $v(x_i)$ have the constraint $v(x_i) = 0\ \forall x_i \in \Omega_g$. By applying the Green's identity (integration by parts), this is transformed to

$$\int_\Omega \kappa_{ij}\, u_{,j}\, v_{,i} \,\mathrm{d}V - \int_{\partial\Omega} \kappa_{ij}\, u_{,j}\, n_i\, v \,\mathrm{d}A = \int_\Omega f\, v \,\mathrm{d}V \tag{6}$$

The boundary integral of the second term can be further transformed by using that $v(x_i) = 0\ \forall x_i \in \Omega_g$ and substituting Equation (2), resulting in

$$\int_\Omega \kappa_{ij}\, u_{,j}\, v_{,i} \,\mathrm{d}V = \int_\Omega f\, v \,\mathrm{d}V - \int_{\partial\Omega_h} h\, v \,\mathrm{d}A \tag{7}$$

or simply

$$a(u, v) = l(v) \tag{8}$$

where we introduce the bilinear form (a, v) and the linear functional $l(v)$ as

$$\begin{aligned} a(u, v) &:= \int\limits_{\Omega} \kappa_{ij} \, u_{,j} \, v_{,i} \, \mathrm{d}V &(9) \\ l(v) &:= \int\limits_{\Omega} f \, v \, \mathrm{d}V - \int\limits_{\partial\Omega_h} h \, v \, \mathrm{d}A &(10) \end{aligned}$$

4 Energy norms

The computed finite element solution can be assessed by evaluating some norms. For a given finite element solution u^h we therefore define its energy norm as

$$U^h = \sqrt{a(u^h, u^h)} \tag{11}$$

and the corresponding external energy is

$$U^h_{ext} = \sqrt{l(u^h)} \tag{12}$$

The implementation can therefore be verified by always asserting that $U^h = U^h_{ext}$ for any problem setup.