Project 5, FYS 3150 / 4150, fall 2013

Student #

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1 Introduction

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2 Theory

2.1 Derivation of the expression for τ_{crunch}

Start with Elgarøy's notes (where $t = 0$ at the Big Bang singularity for our sub-universe) and get

$$R(\psi) = a(\psi)R_0 = \frac{R_0\Omega_{m0}}{2(\Omega_{m0} - 1)}(1 - \cos\psi) \tag{1}$$

where $a(\psi)$ is the dimensionless scale factor and

$$t(\psi) = \frac{\Omega_{m0}}{2H_0(\Omega_{m0} - 1)^{3/2}}(\psi - \sin\psi). \tag{2}$$

From these results we see that $t_{max} = t(\psi = \pi)$ and $t_{crunch} = t(\psi = 2\pi)$, thus the elapsed time between these events is

$$\tau_{crunch} = t_{crunch} - t_{max} = \frac{\pi\Omega_{m0}}{2H_0(\Omega_{m0} - 1)^{3/2}}.$$

The mass parameter is defined by

$$\Omega_{m0} = \frac{8\pi G\rho_0}{3H_0^2}$$

and for readability we will make the substitution

$$u^2 = 8\pi G\rho_0 - 3H_0^2$$

thus

$$(\Omega_{m0} - 1)^{-3/2} = (\frac{u^2}{3H_0^2})^{-3/2} = 3\sqrt{3}H_0^3 u^{-3}$$

$$\frac{\Omega_{m0}}{(\Omega_{m0} - 1)^{3/2}} = \frac{8\pi G\rho_0}{3H_0^2} 3\sqrt{3}H_0^3 u^{-3} = 8\sqrt{3}\pi G\rho_0 H_0 u^{-3}$$

$$\tau_{crunch} = \frac{\pi}{2H_0} 8\sqrt{3}\pi G\rho_0 H_0 u^{-3} = 4\sqrt{3}\pi^2 G\rho_0 u^{-3}$$

Now we remember that at the time when $\rho = \rho_0$, everything is at rest, so we have $H_0 = (\frac{\dot{a}}{a})_{\tau=0} = 0$. Inserting this, we get $u^2 = 8\pi G\rho_0$, and

$$\tau_{crunch} = 4\sqrt{3}\pi^2 G\rho_0 (8\pi G\rho_0)^{-3/2} = \sqrt{\frac{4^2 3\pi^4 G^2 \rho_0^2}{8^3 \pi^3 G^3 \rho_0^3}} = \sqrt{\frac{3\pi}{32G\rho_0}} \tag{3}$$

which is what we wanted to show.

2.2 Lack of a singularity in our model

TODO: Use Elgarøy II and Peacock as sources. (We can also check our virialization radius based on this.)

The reason we do not see a singularity in our model is that we have assumed pressureless (i.e. collisionless) matter in the Friedmann equations. In our simulation we will see internal pressure as kinetic energy from the collapse is turned into random motions by near-collisions between particles. This will act as a kind of pressure in our simulation and will halt the collapse, causing the gravitationally bound particles to form more or less stable orbits around the center of mass. Keep in mind that the kinetic energy is not evenly distributed, so occasionally particles that receive more than their fair share will become unbound and may escape from the system before they have time to lose their energy. (SOURCE: Given) predicts that this should happen to roughly ?? % of the particles.

We say that the system is stable when the virial theorem (...) is satisfied. Elgarøy II shows that his happens at time $\tau_{vir} = 0.81\tau_{crunch}$ when the sphere has collapsed to half its initial size, so we see that τ_{crunch} is a natural time scale for virialization to occur (some sources do in fact use τ_{crunch} to mark the point when the system is virialized).

2.3 G in units of ly, $M_{\odot}$ and τ_{crunch}

With τ_{crunch} given in years, we can rewrite equation 3 as

$$G_{yr} = \frac{3\pi}{32\tau_{crunch}^2 \rho_0}.$$

Switching time units to τ_{crunch}, we get that $\tau_{crunch} = 1$ in these units, hence

$$G = \frac{3\pi}{32\rho_0} = \frac{\pi^2 R_0^3}{8\mu N}$$

where for the latter equality we have used the definitions of average mass $\mu = \frac{M}{N}$ and initial mass density for a sphere $\rho_0 = \frac{M}{V_0} = \frac{\mu N}{\frac{4}{3}\pi R_0^3} = \frac{3\mu N}{4\pi R_0^3}$.

This means our gravitational constant and our time unit both depend on N, R_o and μ. We can verify that the units for G are now correct with R_0 given in light years and μgiven in solar masses.

2.4 Gravitational potential with modified gravity

Starting with the magnitude of the force

$$F = \frac{GMm}{r^2 + \epsilon^2} = \frac{GMm}{\epsilon^2} \frac{1}{(\frac{r}{\epsilon})^2 + 1} = \frac{GMm}{\epsilon^2} \frac{1}{u^2 + 1}$$

where we have used the subsitution $u = \frac{r}{\epsilon}$ which gives $\frac{du}{dr} = \frac{1}{\epsilon}$, hence $dr = \epsilon \cdot du$, and so

$$E_p = m\Phi = \int F dr = \frac{GMm}{\epsilon^2} \int \frac{1}{u^2 + 1} \epsilon du = \frac{GMm}{\epsilon} (\arctan(u) + C)$$

We want $E_p \to 0$ as $r \to \infty$, and since $\arctan(u) \to \frac{\pi}{2}$ as $u \to \infty$, we achieve this by choosing $C = -\frac{\pi}{2}$:

$$E_p = \frac{GMm}{\epsilon} (\arctan(\frac{r}{\epsilon}) - \frac{\pi}{2}) \qquad (\epsilon > 0)$$

3 Results and analysis

3.1 Benchmarks

Reproduce project 3 2-body problem.

Center of mass conserved.

Energy mostly conserved, especially with gravitational correction.

4 Conclusion

What we learned:

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4.1 Critique

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