Supplement to the paper: Distributed Reinforcement Learning via Aggregative Actor-Critic

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Appendix

This external appendix contains the proofs of the theoretical statements of the paper.

A. Proof of Lemma 3.1

For the sake of reading, we report the statement of the lemma.

Lemma 3.1: There exist $K^\star \in \mathbb{R}^{n\times m}$ and $v^\star \in \mathbb{R}^m$ such that $u_k = K^\star x_k + v^\star$, $k \geq 0$, is the optimal solution of problem (11).

Proof: We first reformulate problem (11) as a standard linear quadratic optimal control problem. By denoting as $A = \text{blkdiag}(A_1, \ldots, A_N)$ and $B = \text{blkdiag}(B_1, \ldots, B_N)$ the overall system matrices (here blkdiag is the block diagonal operator) and by $w_k = (w_{1,k}, \ldots, w_{N,k})$ the overall disturbance vector, the system dynamics can be rewritten as

$$x_{k+1} = Ax_k + Bu_k + w_k.$$

Let us define $H = [H_1, \ldots, H_N] \in \mathbb{R}^{s\times n}$ and let us define the following matrices and vectors

$$\begin{aligned}
\mathbb{R}^{n\times n} \ni Q &= \text{blkdiag}(Q_1, \ldots, Q_N) + \frac{1}{N^2}\sum_{i=1}^{N} H^\top F_i H,\\
\mathbb{R}^{m\times m} \ni R &= \text{blkdiag}(R_1, \ldots, R_N),\\
\mathbb{R}^{n} \ni q &= [q_1^\top, \ldots, q_N^\top]^\top + \frac{1}{N}\sum_{i=1}^{N} H^\top f,\\
\mathbb{R}^{m} \ni r &= [r_1^\top, \ldots, r_N^\top]^\top.
\end{aligned}$$

Moreover, let us define the augmented state $\tilde{x} = \left[\begin{smallmatrix} 1 \\ x \end{smallmatrix}\right] \in \mathbb{R}^{n+1}$ with system matrices $\tilde{A} = \text{blkdiag}(1, A)$, $\tilde{B} = \left[\begin{smallmatrix} 0 \\ B \end{smallmatrix}\right]$, and the augmented cost matrices $\tilde{Q} = \left[\begin{smallmatrix} 0 & q^\top \\ q & Q \end{smallmatrix}\right]$ and $\tilde{S} = \left[\begin{smallmatrix} r^\top \\ 0 \end{smallmatrix}\right]$. With these positions, problem (11) is seen to be equivalent to the optimal control problem

$$\begin{aligned}
\min_{u}\ & \mathbb{E}\Bigg[\frac{1}{2}\sum_{k=0}^{\infty}\alpha^k\big(\tilde{x}^\top\tilde{Q}\tilde{x} + u^\top R u + 2\tilde{x}^\top\tilde{S}u\big)\Bigg] \\
\text{subj. to }\ & \tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}u_k + \tilde{w}_k,
\end{aligned} \tag{36}$$

where $\tilde{w}_k = \left[\begin{smallmatrix} 0 \\ w_k \end{smallmatrix}\right]$. By using standard dynamic programming arguments (see, e.g., [18]), it can be seen that the optimal solution of problem (36) is a linear feedback $u = \tilde{K}\tilde{x}$, where $\tilde{K} = -(R + \alpha\tilde{B}^\top\tilde{P}\tilde{B})^{-1}(\tilde{S} + \alpha\tilde{B}^\top\tilde{P}\tilde{A})\tilde{x}$ and $\tilde{P}$ is the solution of a suitable Algebraic Riccati Equation. Let us write $\tilde{K} \in \mathbb{R}^{(n+1)\times m}$ as $\tilde{K} = [v^\star \ \ K^\star]$, with $v^\star \in \mathbb{R}^m$ and $K^\star \in \mathbb{R}^{n\times m}$. Then, the linear feedback becomes

$$u = \tilde{K}\tilde{x} = \begin{bmatrix} v^\star & K^\star \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = K^\star x + v,$$

and the proof follows. ■

B. Proof of Proposition 3.2

For the sake of reading, we report the statement of the proposition.

Proposition 3.2: Consider a policy π for fixed parameters K_i, v_i for all $i \in \mathbb{I}$. Then, there exist matrices $\tilde{P}_i, \tilde{S}_i$ and vectors $\tilde{p}_i, \tilde{s}_i$ and scalars $\tilde{\rho}_i$ for all $i \in \mathbb{I}$ such that the value function satisfies

$$J_\pi(\bar{x}) = \sum_{i=1}^{N}\Big(\bar{x}_i^\top\tilde{P}_i\bar{x}_i + \tilde{\sigma}(\bar{x})^\top\tilde{S}_i\tilde{\sigma}(\bar{x}) + \tilde{p}_i^\top\bar{x}_i + \tilde{s}_i\tilde{\sigma}(\bar{x}) + \tilde{\rho}_i\Big),$$

where $\tilde{\sigma}(x) := \frac{1}{N}\sum_{i=1}^{N}\tilde{H}_i x_i$ and the matrices $\tilde{H}_i$ depend on the system matrices A_i, B_i.

Proof: Fix the initial states to $\bar{x}_i$ and the policy parameters of π_i to some K_i, v_i for all $i \in \mathbb{I}$. Since each system follows the policy π_i such that $u_{i,k} = K_i x_{i,k} + v_i + \eta_{i,k}$, it holds

$$\begin{aligned}
x_{i,k+1} &= A_i x_{i,k} + B_i(K_i x_{i,k} + v_i + \eta_{i,k}) + w_{i,k}\\
&= (A_i + B_i K_i)x_{i,k} + B_i v_i + B\eta_{i,k} + w_{i,k}.
\end{aligned} \tag{37}$$

The evolution of each system i can be thus written in closed form as

$$x_{i,k} = \underbrace{(A_i + B_i K_i)^k}_{:=\Phi_{i,k}}\bar{x}_i + \underbrace{\sum_{\tau=0}^{k-1}A_i^{k-\tau-1}(B_i v_i + B\eta_{i,\tau} + w_{i,\tau})}_{:=\xi_{i,k}}$$

Similarly we can also express the input as a function of the initial state and of the noise realizations:

$$u_{i,k} = K_i x_{i,k} + v_i + \eta_{i,k} = K_i\Phi_{i,k}\bar{x}_i + \underbrace{K_i\xi_{i,k} + \eta_{i,k}}_{:=\psi_{i,k}}$$

Notice that, since we suppose $\mathbb{E}[w_{i,k}] = 0, \mathbb{E}[\eta_{i,k}] = 0$ and both of them i.i.d. we have

$$\mathbb{E}[x_{i,k}] = \mathbb{E}[\Phi_{i,k}\bar{x}_i] + \mathbb{E}[\xi_{i,k}] = \Phi_{i,k}\bar{x}_i \tag{38}$$

and, similarly,

$$\mathbb{E}[u_{i,t}] = \mathbb{E}[K_i\Phi_{i,k}\bar{x}_i] + \mathbb{E}[\psi_{i,k}] = K_i\Phi_{i,k}\bar{x}_i. \tag{39}$$

For ease of exposition, let us assume that the linear terms in the cost are zero, i.e., that $q_i = 0, r_i = 0, f_i = 0$ (the derivations that follow are similar for the case in which the linear terms are nonzero). Thus we must consider

$$\begin{aligned}
J_\pi(x) = \sum_{i=1}^{N}\mathbb{E}\Bigg[\frac{1}{2}\sum_{k=0}^{\infty}\alpha^k\big(&x_{i,k}^\top Q_i x_{i,k} + u_{i,k}^\top R_i u_{i,k}\\
&+ \sigma(x_k)^\top F_i\sigma(x_k)\big)\Bigg].
\end{aligned} \tag{40}$$

Considering the closed form evolution of each system, exploiting the linearity of the expected value and using the definition of $\sigma(x)$, we obtain

$$
\begin{aligned}
J_\pi(x) = \sum_{i=1}^{N} \frac{1}{2} \sum_{k=0}^{\infty} \bigg\{ & \mathbb{E}\bigg[\bar{x}_i^\top \alpha^k \bigg(\Phi_{i,k}^\top Q_i \Phi_{i,k} \\
& \qquad + \Phi_{i,k}^\top K_i^\top R_i K_i \Phi_{i,k} \bigg) \bar{x}_i \bigg] \\
& + \mathbb{E}\bigg[2\alpha^k \bigg(\xi_{i,k}^\top Q_i \Phi_{i,k} + \psi_{i,k}^\top R_i K_i \Phi_{i,k} \bigg) \bar{x}_i \bigg] \\
& + \mathbb{E}\bigg[\alpha^k \bigg(\xi_{i,k}^\top Q_i \xi_{i,k} + \psi_{i,k}^\top R_i \psi_{i,k} \bigg) \bigg] \\
& + \frac{1}{N^2} \sum_{j=1}^{N} \sum_{\ell=1}^{N} \bigg(\mathbb{E}\bigg[\bar{x}_\ell^\top \alpha^k \bigg(\Phi_{\ell,k}^\top H_\ell^\top F_i H_j \Phi_{j,k} \bigg) \bar{x}_\ell \bigg] \\
& + \mathbb{E}\bigg[2\alpha^k \bigg(\xi_{\ell,k}^\top H_\ell^\top F_i H_j \Phi_{j,k} \bigg) \bar{x}_j \bigg] \\
& + \mathbb{E}\bigg[\alpha^k \bigg(\xi_{\ell,k}^\top H_\ell^\top F_i H_j \xi_{j,k} \bigg) \bigg] \bigg) \bigg\}.
\end{aligned}
$$

Then, in light of (38) and (39) and defining

$$
\begin{aligned}
\tilde{P}_i &:= \sum_{k=0}^{\infty} \alpha^k \left(\tilde{\Phi}_{i,k}^\top Q_i \tilde{\Phi}_{i,k} + \tilde{\Phi}_{i,k}^\top K_i^\top R_i K_i \tilde{\Phi}_{i,k} \right) \\
\tilde{S}_i &:= F_i \\
\tilde{\sigma}(\bar{x}) &:= \frac{1}{N} \sum_{i=0}^{N} \underbrace{\sum_{k=0}^{\infty} \sqrt{\alpha^k} H_i \Phi_{i,k}}_{:=\tilde{H}_i} \bar{x}_i \\
\zeta_i &:= \sum_{k=0}^{\infty} \alpha^k \left(\xi_{i,k}^\top Q_i \xi_{i,k} + \psi_{i,k}^\top R_i \psi_{i,k} \right) \\
\varsigma &:= \frac{1}{N} \sum_{i=0}^{N} \sum_{k=0}^{\infty} \sqrt{\alpha^k} H_i \xi_{i,k},
\end{aligned}
$$

we can finally write:

$$
J_\pi(x) = \frac{1}{2} \sum_{i=1}^{N} \left(\bar{x}_i^\top \tilde{P}_i \bar{x}_i + \tilde{\sigma}(\bar{x})^\top \tilde{S}_i \tilde{\sigma}(\bar{x}) + \tilde{\rho}_i \right), \tag{41}
$$

with $\tilde{\rho}_i = \mathbb{E}[\zeta_i] + \mathbb{E}[\varsigma^\top \tilde{S}_i \varsigma]$. For the case in which the linear terms are nonzero, there will be additional linear terms in (41). The proof follows. ∎

C. Consistency of Distributed Algorithm

For the sake of reading, we report the statement of the theorem.

Theorem 4.3: Let Assumption 4.1 hold and assume that, with probability $\varepsilon > 0$, it holds

$$
\limsup_{k\to\infty} \|x_k - \bar{x}\| \le \delta, \tag{42}
$$

$$
\limsup_{k\to\infty} \|u_k - \bar{u}\| \le \delta^u, \tag{43}
$$

$$
\limsup_{k\to\infty} \|\mu_k - \bar{\mu}\| \le \delta^\mu, \tag{44}
$$

for some $\bar{x} \in \mathbb{R}^n$, $\bar{u} \in \mathbb{R}^m$, $\bar{\mu} \in \mathbb{R}^\ell$ and $\delta, \delta^u, \delta^\mu \ge 0$, and there exist $B^x, B^u, B^y, B^\mu \ge 0$ such that

$$
\|x_{i,k}\| \le B^x, \;\; \|u_{i,k}\| \le B^u, \;\; \|y_{i,k}\| \le B^y, \;\; \|\mu_{i,k}\| \le B^x,
$$

for all $i \in \mathbb{I}$ and $k \ge 0$.

Then, with probability ε, for all $i \in \mathbb{I}$ there exist constants $L_1, L_2, L_3, L_4, L_5 \ge 0$ such that the following holds

(i) the local estimate of the temporal difference approaches the centralized temporal difference, i.e.,
$$
\limsup_{k\to\infty} |d_{i,k} - d_k| \le L_1,
$$

(ii) the distributed update of the eligibility trace vectors approaches the centralized one, i.e.,
$$
\limsup_{k\to\infty} \|z_{i,k} - z_{i,k}^c\| \le L_2,
$$

(iii) the descent direction of the policy evaluation step (19c) approaches the centralized one, i.e.,
$$
\limsup_{k\to\infty} \|\mu_{i,k} - \mu_{i,k}^c\| \le L_3,
$$

(iv) the descent direction of the policy gradient step (20) approaches the centralized one, i.e.,
$$
\limsup_{k\to\infty} \|K_{i,k} - K_{i,k}^c\| \le L_4,
$$
$$
\limsup_{k\to\infty} \|v_{i,k} - v_{i,k}^c\| \le L_5.
$$

In order to prove Theorem 4.3, we first present some preliminary results.

Lemma 1.1: Let $X \subseteq \mathbb{R}^n$ be bounded (i.e. there exists $B^x > 0$ such that $\|x\| \le B^x$ for all $x \in X$), and let $f : X \to \mathbb{R}^{n^2}$, be a function defined as $f(x) = \operatorname{vec}(xx^\top)$. Then, it holds $\|f(x) - f(x')\| \le \sqrt{n(3+n)} B^x \|x - x'\|$ for all $x, x' \in X$.

Proof: We have that

$$
\begin{aligned}
&\|f(x) - f(x')\|^2 = \|\operatorname{vec}(xx^\top) - \operatorname{vec}(x'x'^\top)\|^2 \\
&= \left\| \begin{bmatrix} x_1 x - x_1' x' \\ \vdots \\ x_n x - x_n' x' \end{bmatrix} \right\|^2 \\
&= \sum_{i=1}^{n} \|x_i x - x_i' x'\|^2 \\
&\overset{(a)}{\le} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \max_{z\in X} \|\nabla f_j^i(z)\|^2 \right) \|x - x'\|^2 \\
&= \sum_{i=1}^{n} \Bigg(4 \underbrace{\max_{z\in X} z_i^2}_{\le B^{x2}} + \sum_{j\ne i}^{n} \underbrace{\max_{z\in X} (z_i^2 + z_j^2)}_{\le 2B^{x2}} \Bigg) \|x - x'\|^2 \\
&\le \sum_{i=1}^{n} (4B^{x2} + (n-1)B^{x2}) \|x - x'\|^2 \\
&= n(3+n) B^{x2} \|x - x'\|^2
\end{aligned}
$$

where in (a) we used the mean value theorem and we used the notation $f_j^i(x) := x_i x_j$. The proof follows by taking the square root. ∎

Lemma 1.2: Let $X \subseteq \mathbb{R}^n$ be bounded (i.e. there exists $B^x > 0$ such that $\|x\| \leq B^x$ for all $x \in X$), and let $f : X \to \mathbb{R}$, be a function defined as $f(x) = x^\top Qx + q^\top x$ with $Q \succeq 0$. Then, it holds $\|f(x) - f(x')\| \leq \sqrt{2\lambda_{\max}(Q)B^x + \|q\|}\|x - x'\|$ for all $x, x' \in X$.

Proof: We have that

$$\begin{aligned}\|f(x) - f(x')\|^2 &= \|x^\top Qx + q^\top x - x'^\top Qx' - q^\top x'\|^2 \\ &\leq \|x^\top Qx - x'^\top Qx'\|^2 + \|q^\top x - q^\top x'\|^2. \end{aligned} \tag{45}$$

By the mean value theorem, for the first term we have

$$\begin{aligned}\|x^\top Qx - x'^{,\top} Qx'\| &\leq \Big(\max_{z\in X} \|2Qz\|\Big)\|x - x'\| \\ &\leq 2\lambda_{\max}(Q)\Big(\max_{z\in X}\|z\|\Big)\|x - x'\| \\ &= 2\lambda_{\max}(Q)B^x\|x - x'\|,\end{aligned}$$

while for the second term we have $\|q^\top x - q^\top x'\| \leq \|q\|\|x - x'\|$. Then, plugging these results in (45), we have

$$\|f(x) - f(x')\|^2 \leq (4\lambda^2_{\max}(Q)B^{x2} + \|q\|^2)\|x - x'\|^2,$$

and the proof follows by taking the square root. ■

Let us define $\mathcal{A} = A \otimes I$, where $\otimes$ is the Kronecker product, and let us define the following symbols

$$y_k = \begin{bmatrix} y_{1,k} \\ \vdots \\ y_{N,k}\end{bmatrix}, \ \ H = \text{blkdiag}(H_1, \dots, H_N) \tag{46}$$

and introduce the following symbols

$$h_k = \begin{bmatrix} h_{1,k} \\ \vdots \\ h_{N,k}\end{bmatrix} \quad \tilde{g}_k = . \begin{bmatrix} g_1(x_{1,k}, u_{1,k}, y_{1,k}) \\ \vdots \\ g_N(x_{N,k}, u_{N,k}, y_{N,k})\end{bmatrix}, \tag{47}$$

and

$$\psi_k = \begin{bmatrix} \psi_{1,k} \\ \vdots \\ \psi_{N,k}\end{bmatrix} \quad \tilde{J}_k = . \begin{bmatrix} \mu_{1,k}^\top \phi_1(x_{1,k+1}, y_{1,k+1}) \\ \vdots \\ \mu_{N,k}^\top \phi_N(x_{N,k+1}, y_{N,k+1})\end{bmatrix}, \tag{48}$$

and let us denote by $\mathbf{1} = [I, \dots, I]^\top$ the vector stacking N identity matrices. In the following lemma we show that $y_{i,k}$ asymptotically approaches $\sigma(x_k)$.

Lemma 1.3: Under the same assumptions of Theorem 4.3, it holds with probability ε

$$\limsup_{k\to\infty} \|y_{i,k} - \sigma(x_k)\| \leq \frac{\zeta\delta}{1-\rho}, \qquad \forall i \in \mathbb{I}, \tag{49}$$

where $\rho = \lambda_{\max}(\mathcal{A} - \frac{\mathbf{1}\mathbf{1}^\top}{N})$ and $\zeta = \lambda_{\max}((I - \frac{\mathbf{1}\mathbf{1}^\top}{N})H)$.

Proof: Let us restrict our attention to sample paths $\omega \in \bar{\Omega} \subset \Omega$ such that $\mathbb{P}\{\omega \in \bar{\Omega} : \text{condition (42) holds}\} = \varepsilon$. Let us define the average of $y_{i,k}$ as

$$\bar{y}_k = \frac{1}{N}\sum_{i=1}^N y_{i,k}, \qquad k \geq 0. \tag{50}$$

The evolution in matrix form of y_k is

$$y_{k+1} = \mathcal{A}y_k + Hx_{k+1} - Hx_k, \tag{51}$$

while the evolution of $\bar{y}_k$ is

$$\begin{aligned}\bar{y}_{k+1} &= \frac{1}{N}\mathbf{1}^\top y_{k+1} = \frac{1}{N}\Big(\mathbf{1}^\top \mathcal{A}y_k + \mathbf{1}^\top Hx_{k+1} - \mathbf{1}^\top Hx_k\Big) \\ &= \frac{1}{N}\mathbf{1}^\top y_k + \sigma(x_{k+1}) - \sigma(x_k) \\ &= \bar{y}_k + \sigma(x_{k+1}) - \sigma(x_k),\end{aligned} \tag{52}$$

where we used the fact that $\mathbf{1}^\top\mathcal{A} = \mathbf{1}^\top$ by Assumption 4.1. Thus, it holds

$$\begin{aligned}&\|y_{k+1} - \mathbf{1}\bar{y}_{k+1}\| = \\ &\left\|\mathcal{A}y_k + Hx_{k+1} - Hx_k - \frac{\mathbf{1}\mathbf{1}^\top}{N}y_k - \mathbf{1}\sigma(x_{k+1}) + \mathbf{1}\sigma(x_k)\right\| \\ &\leq \left\|(\mathcal{A} - \frac{\mathbf{1}\mathbf{1}^\top}{N})y_k\right\| \\ &\qquad + \|Hx_{k+1} - Hx_k - \mathbf{1}\sigma(x_{k+1}) + \mathbf{1}\sigma(x_k)\| \\ &\overset{(a)}{=} \left\|(\mathcal{A} - \frac{\mathbf{1}\mathbf{1}^\top}{N})(y_k - \mathbf{1}\bar{y}_k)\right\| \\ &\qquad + \|Hx_{k+1} - Hx_k - \mathbf{1}\sigma(x_{k+1}) + \mathbf{1}\sigma(x_k)\| \\ &\overset{(b)}{\leq} \rho\|y_k - \mathbf{1}\bar{y}_k\| \\ &\qquad + \left\|H(x_{k+1} - x_k) - \frac{\mathbf{1}\mathbf{1}^\top}{N}H(x_{k+1} - x_k)\right\| \\ &\leq \rho\|y_k - \mathbf{1}\bar{y}_k\| + \left\|(I - \frac{\mathbf{1}\mathbf{1}^\top}{N})H(x_{k+1} - x_k)\right\| \\ &\leq \rho\|y_k - \mathbf{1}\bar{y}_k\| + \underbrace{\lambda_{\max}\Big((I - \frac{\mathbf{1}\mathbf{1}^\top}{N})H\Big)}_{:=\zeta}\|x_{k+1} - x_k\|\end{aligned}$$

where in (a) we used the fact that $\mathbf{1} \in \ker(A - \frac{\mathbf{1}\mathbf{1}^\top}{N})$ and in (b) we defined $\rho < 1$ as the spectral radius of $A - \frac{\mathbf{1}\mathbf{1}^\top}{N}$ (cf. [21]). Using (52),

$$\bar{y}_{k+1} - \bar{y}_k = \sigma(x_{k+1}) - \sigma(x_k) \qquad \forall k \geq 0 \tag{53}$$

and in particular, for $k = 0$ we have $\bar{y}_1 - \bar{y}_0 = \sigma(x_1) - \sigma(x_0)$ and using the fact that $\bar{y}_0 = \frac{1}{N}\mathbf{1}^\top Hx_0 = \sigma(x_0)$ we obtain that $\bar{y}_k = \sigma(x_k)$ for all $k \geq 0$. Collecting with the previous conditions we obtain

$$\|y_{k+1} - \mathbf{1}\sigma(x_{k+1})\| \leq \rho\|y_k - \mathbf{1}\sigma(x_k)\| + \zeta\|x_{k+1} - x_k\|.$$

Defining the sequence $Z_k = \|y_k - \mathbf{1}\sigma(x_k)\| \geq 0$ the previous condition can be rewritten as

$$Z_{k+1} \leq \rho Z_k + \zeta\|x_{k+1} - x_k\|, \tag{54}$$

which applied recursively gives

$$Z_k \leq \rho^k Z_0 + \zeta\sum_{\tau=0}^{k-1}\rho^{k-\tau-1}\|x_{\tau+1} - x_\tau\|. \tag{55}$$

By condition (42), it holds

$$\limsup_{k\to\infty}\|x_{k+1} - x_k\| \leq \delta. \tag{56}$$

and calculating the $\limsup$ of both sides of (55) we obtain

$$
\begin{aligned}
\limsup_{k\to\infty} Z_k &\le Z_0 \underbrace{\limsup_{k\to\infty} \rho^k}_{=0} \\
&\quad + \zeta \limsup_{k\to\infty} \sum_{\tau=0}^{k-1} \rho^{k-\tau-1} \|x_{\tau+1} - x_\tau\| \\
&\le \frac{\zeta\delta}{1-\rho}
\end{aligned}
$$

and the proof follows. ∎

In the following lemma we prove that the distributed update of the eligibility trace tends to the centralized one.

Lemma 1.4: Under the same assumptions of Theorem 4.3, it holds with probability ε

$$
\limsup_{k\to\infty} \|z_{i,k}^c - z_{i,k}\| \le \frac{\zeta\delta}{1-\rho}(\bar{B}+1), \tag{57}
$$

for all $i \in \mathbb{I}$, where $\bar{B}$ is equal to

$$
\bar{B} = \sqrt{s(s+3)} \max\left\{B^y, \frac{1}{N}\lambda_{\max}\big(\mathbf{1}^\top H\big) B^x\right\}.
$$

Proof: Let us restrict our attention to sample paths $\omega \in \bar{\Omega} \subset \Omega$ such that $\mathbb{P}\{\omega \in \bar{\Omega} : \text{condition (42) holds}\} = \varepsilon$. By definition of $z_{i,k}$ and $z_{i,k}^c$, it holds

$$
\|z_{i,k}^c - z_{i,k}\| = \|\phi_i(x_{i,k}, \sigma(x_k)) - \phi_i(x_{i,k}, y_{i,k})\|. \tag{58}
$$

Recall that $\phi_i : \mathbb{R}^{n_i} \times \mathbb{R}^s \to \mathbb{R}^\ell$ with $\ell = n_i^2 + n_i + s^2 + s + 1$ is equal to

$$
\phi_i(x_i, \xi) = \begin{bmatrix} \operatorname{vec}(x_i x_i^\top) \\ x_i \\ 1 \\ \operatorname{vec}(\xi\xi^\top) \\ \xi \end{bmatrix}, \tag{59}
$$

from which it follows that

$$
\begin{aligned}
\|z_{i,k}^c - z_{i,k}\| &\le \underbrace{\|\operatorname{vec}(x_{i,k}x_{i,k}^\top) - \operatorname{vec}(x_{i,k}x_{i,k}^\top)\|}_{0} \\
&+ \underbrace{\|x_{i,k} - x_{i,k}\| + \|1-1\|}_{0} \\
&+ \|\operatorname{vec}(\sigma(x_k)\sigma(x_k)^\top) - \operatorname{vec}(y_{i,k}y_{i,k}^\top)\| + \|\sigma(x_k) - y_{i,k}\|.
\end{aligned} \tag{60}
$$

Note that it holds

$$
\begin{aligned}
\|\sigma(x_k)\| &= \frac{1}{N}\|\mathbf{1}^\top H x_k\| \\
&\le \frac{1}{N}\lambda_{\max}\big(\mathbf{1}^\top H\big)\|x_k\| \\
&\le \frac{1}{N}\lambda_{\max}\big(\mathbf{1}^\top H\big) B^x \\
&:= B^\sigma.
\end{aligned}
$$

Using Lemma 1.1, we obtain

$$
\|\operatorname{vec}(\sigma(x_k)\sigma(x_k)^\top) - \operatorname{vec}(y_{i,k}y_{i,k}^\top)\| \le \bar{B}\|\sigma(x_k) - y_{i,k}\|,
$$

where $\bar{B} := \sqrt{s(s+3)}\max\{B^\sigma, B^y\}$. Plugging the last inequality in (60) we obtain

$$
\begin{aligned}
\limsup_{k\to\infty} \|z_{i,k}^c - z_{i,k}\| &\le (\bar{B}+1) \limsup_{k\to\infty} \|\sigma(x_k) - y_{i,k}\| \\
&\le (\bar{B}+1)\frac{\zeta\delta}{1-\rho},
\end{aligned}
$$

where in the last inequality we used Lemma 1.3. ∎

Let us introduce the centralized quantities

$$
h_k^c = \frac{1}{N}\sum_{i=1}^N g_i(x_{i,k}, u_{i,k}, \sigma(x_k)) \tag{61}
$$

$$
\psi_k^c = \frac{1}{N}\sum_{i=1}^N \mu_{i,k}^\top \phi_i(x_{i,k}, \sigma(x_k)) \tag{62}
$$

$$
\varphi_k^c = \frac{1}{N}\sum_{i=1}^N \mu_{i,k}^\top \phi_i(x_{i,k-1}, \sigma(x_{k-1})) \tag{63}
$$

In the following lemma, we show that $h_{i,k}$ asymptotically approaches $\frac{1}{N}\mathbf{1}\mathbf{1}^\top \tilde{g}_k$.

Lemma 1.5: Under the same assumptions of Theorem 4.3, it holds with probability ε

$$
\limsup_{k\to\infty} \|h_{i,k} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top \tilde{g}_k\| \le B^h, \tag{64}
$$

for all $i \in \mathbb{I}$, where

$$
\begin{aligned}
B^h &= \frac{\zeta'}{1-\rho}\sqrt{\sum_{i=1}^N\left(\delta^2\left(B_i^{x\,2} + \frac{B_i^{y\,2}\zeta^2}{(1-\rho)^2}\right) + B_i^{u\,2}\delta^{u\,2}\right)}, \\
\zeta' &= \lambda_{\max}\left(I - \frac{\mathbf{1}\mathbf{1}^\top}{N}\right), \\
B_i^x &= \sqrt{2\lambda_{\max}(Q_i)B^x + \|q_i\|}, \\
B_i^u &= \sqrt{2\lambda_{\max}(R_i)B^u + \|r_i\|}, \\
B_i^y &= \sqrt{2\lambda_{\max}(F_i)B^y + \|f_i\|}.
\end{aligned}
$$

Proof: Let us define the average of $h_{i,k}$ as

$$
\bar{h}_k = \frac{1}{N}\sum_{i=1}^N h_{i,k} \tag{65}
$$

The evolution in matrix form of h_k is

$$
h_{k+1} = \mathcal{A}h_k + \tilde{g}_k - \tilde{g}_{k-1} \tag{66}
$$

and the evolution of $\bar{h}_k$, following the same arguments as in (52), is

$$
\begin{aligned}
\bar{h}_{k+1} &= \frac{1}{N}\mathbf{1}^\top h_{k+1} \\
&= \frac{1}{N}\mathbf{1}^\top h_k + \frac{1}{N}\mathbf{1}^\top(\tilde{g}_k - \tilde{g}_{k-1}) \\
&= \bar{h}_k + \frac{1}{N}\mathbf{1}^\top(\tilde{g}_k - \tilde{g}_{k-1}).
\end{aligned} \tag{67}
$$

Moreover, by following the same arguments as in Lemma 1.3, it holds for all $k \ge 0$

$$
\bar{h}_k = \frac{1}{N}\mathbf{1}^\top \tilde{g}_k. \tag{68}
$$

Thus, it holds,

$$\begin{aligned}
&\|h_{k+1} - \mathbf{1}\bar{h}_{k+1}\| = \\
&\left\|\mathcal{A}h_k + \tilde{g}_k - \tilde{g}_{k-1} - \frac{\mathbf{1}\mathbf{1}^\top}{N}(h_k + \tilde{g}_k - \tilde{g}_{k-1})\right\| \\
&\leq \left\|(\mathcal{A} - \frac{\mathbf{1}\mathbf{1}^\top}{N})h_k\right\| \\
&\qquad + \|\tilde{g}_k - \tilde{g}_{k-1} - \frac{\mathbf{1}\mathbf{1}^\top}{N}(\tilde{g}_k - \tilde{g}_{k-1})\| \\
&\leq \rho\|h_k - \mathbf{1}\bar{h}_k\| \\
&\qquad + \left\|\tilde{g}_k - \tilde{g}_{k-1} - \frac{1}{N}\mathbf{1}^\top\tilde{g}_k + \frac{1}{N}\mathbf{1}^\top\tilde{g}_{k-1}\right\| \\
&\leq \rho\|h_k - \mathbf{1}\bar{h}_k\| + \left\|(I - \frac{\mathbf{1}\mathbf{1}^\top}{N})\,(\tilde{g}_k - \tilde{g}_{k-1})\right\| \\
&\leq \rho\|h_k - \mathbf{1}\bar{h}_k\| + \underbrace{\lambda_{\max}\left(I - \frac{\mathbf{1}\mathbf{1}^\top}{N}\right)}_{:=\zeta'}\|\tilde{g}_k - \tilde{g}_{k-1}\|.
\end{aligned}$$

Collecting with the previous conditions we obtain

$$\|h_{k+1} - \mathbf{1}\bar{h}_{k+1}\| \leq \rho\|h_k - \mathbf{1}\bar{h}_k\| + \zeta'\|\tilde{g}_k - \tilde{g}_{k-1}\|. \quad (69)$$

Defining the state $Z_k = \|h_k - \mathbf{1}\bar{h}_k\| \geq 0$ the previous condition can be rewritten as

$$Z_{k+1} \leq \rho Z_k + \zeta\|\tilde{g}_k - \tilde{g}_{k-1}\|, \quad (70)$$

which applied recursively gives, for all $k \geq 1$,

$$Z_k \leq \rho^k Z_0 + \zeta' \sum_{\tau=1}^{k-1} \rho^{k-\tau-1}\|\tilde{g}_\tau - \tilde{g}_{\tau-1}\|. \quad (71)$$

Using Lemma 1.2, for all $i \in \mathbb{I}$ we have

$$\begin{aligned}
&\|g_i(x_{i,k}, u_{i,k}, y_{i,k}) - g_i(x_{i,k-1}, u_{i,k-1}, y_{i,k-1})\|^2 \\
&\qquad \leq B_i^{x\,2}\|x_{i,k} - x_{i,k-1}\|^2 + B_i^{u\,2}\|u_{i,k} - u_{i,k-1}\|^2 \\
&\qquad\quad + B_i^{y\,2}\|y_{i,k} - y_{i,k-1}\|^2.
\end{aligned}$$

By conditions (42) and (43), it holds

$$\limsup_{k\to\infty}\|x_{i,k} - x_{i,k-1}\| \leq \delta,$$
$$\limsup_{k\to\infty}\|u_{i,k} - u_{i,k-1}\| \leq \delta^u,$$

and similarly, by Lemma 1.3, it holds

$$\limsup_{k\to\infty}\|y_{i,k} - y_{i,k-1}\| \leq \frac{\zeta\delta}{1-\rho}.$$

Then, calculating the $\limsup$ of (71) we obtain

$$\limsup_{k\to\infty} Z_k \leq \zeta' \limsup_{k\to\infty}\sum_{\tau=1}^{k-1}\rho^{k-\tau-1}\|\tilde{g}_\tau - \tilde{g}_{\tau-1}\| \quad (72)$$

where, for each $i \in \{1, \ldots, N\}$ we have:

$$\limsup_{k\to\infty}\|g_{i,k} - g_{i,k-1}\|^2 \leq \delta^2\left(B_i^{x\,2} + \frac{B_i^{y\,2}\zeta^2}{(1-\rho)^2}\right) + B_i^{u\,2}\delta^{u\,2}.$$

Thus, we finally obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\|h_k - \frac{1}{N}\mathbf{1}\mathbf{1}^\top\tilde{g}_k\| = \limsup_{k\to\infty} Z_k \\
&\leq \underbrace{\frac{\zeta'}{1-\rho}\sqrt{\sum_{i=1}^N\left(\delta^2\left(B_i^{x\,2} + \frac{B_i^{y\,2}\zeta^2}{(1-\rho)^2}\right) + B_i^{u\,2}\delta^{u\,2}\right)}}_{B^h}.
\end{aligned}$$

■

In the following lemma, we show that $h_{i,k}$ asymptotically approaches h_k^c.

Lemma 1.6: Under the same assumptions of Theorem 4.3, it holds with probability ε

$$\limsup_{k\to\infty}\left\|\frac{1}{N}\mathbf{1}^\top\tilde{g}_k - h_k^c\right\| \leq \frac{\zeta\delta}{N(1-\rho)}\sum_{i=1}^N B_i^{y'}, \quad (73)$$

for all $i \in \mathbb{I}$, where

$$B_i^{y'} = \sqrt{2\lambda_{\max}(F_i)\max\{B^y, B^\sigma\} + \|f_i\|}. \quad (74)$$

Proof: Let us compute

$$\begin{aligned}
&\left\|\frac{1}{N}\mathbf{1}^\top\tilde{g}_k - h_k^c\right\| \\
&\leq \frac{1}{N}\left\|\sum_{i=1}^N(g_i(x_{i,k}, u_{i,k}, y_{i,k}) - g_i(x_{i,k}, u_{i,k}, \sigma(x_k)))\right\| \\
&\leq \frac{1}{N}\sum_{i=1}^N\|g_i(x_{i,k}, u_{i,k}, y_{i,k}) - g_i(x_{i,k}, u_{i,k}, \sigma(x_k))\|.
\end{aligned}$$

Using similar arguments as in Lemma 1.5, we obtain for all $i \in \{1, \ldots, N\}$

$$\begin{aligned}
&\|g_i(x_{i,k}, u_{i,k}, y_{i,k}) - g_i(x_{i,k}, u_{i,k}, \sigma(x_k))\| \\
&\leq B_i^{y'}\|y_{i,k} - \sigma(x_k)\|.
\end{aligned}$$

Computing the $\limsup$ as k goes to infinity we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\|g_i(x_{i,k}, u_{i,k}, y_{i,k}) - g_i(x_{i,k}, u_{i,k}, \sigma(x_k))\| \\
&= B_i^{y'}\limsup_{k\to\infty}\|y_{i,k} - \sigma(x_k)\| \\
&= B_i^{y'}\frac{\zeta\delta}{1-\rho}.
\end{aligned}$$

Combining with the above derivations we obtain

$$\limsup_{k\to\infty}\left\|\frac{1}{N}\mathbf{1}^\top\tilde{g}_k - h_k^c\right\| = \frac{\zeta\delta}{N(1-\rho)}\sum_{i=1}^N B_i^{y'}, \quad (75)$$

and the proof follows. ■

Corollary 1.7: Under the same assumptions of Theorem 4.3, it holds with probability ε

$$\limsup_{k\to\infty}\|h_{i,k} - h_k^c\| \leq B^h + \frac{\zeta\delta}{N(1-\rho)}\sum_{i=1}^N B_i^{y'}, \quad (76)$$

for all $i \in \mathbb{I}$. □

Lemma 1.8: Under the same assumptions of Theorem 4.3, it holds with probability ε

$$\limsup_{k\to\infty}\|\psi_{i,k} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top\tilde{J}_k\| \leq \frac{\zeta'}{1-\rho}B^J, \quad (77)$$

for all $i \in \mathbb{I}$, where

$$B^{J^2} = N\bigg(B^{\mu 2}\delta^2\Big[(\sqrt{n(n+3)}2B^x+1)^2 + (\sqrt{s(s+3)}2B^y+1)^2\frac{\zeta^2}{(1-\rho)^2}\Big] + B^{\phi^2}\delta^{\mu 2}\bigg).$$

Proof: Let us define the average of $\psi_{i,k}$ as

$$\bar{\psi}_k = \frac{1}{N}\sum_{i=1}^{N}\psi_{i,k}. \tag{78}$$

Following similar arguments as in Lemma 1.5, it holds

$$\begin{aligned}
&\|\psi_{k+1} - \mathbf{1}\bar{\psi}_{k+1}\| = &(79)\\
&\left\|\mathcal{A}\psi_k + \tilde{J}_k - \tilde{J}_{k-1} - \frac{\mathbf{1}\mathbf{1}^\top}{N}(\psi_k + \tilde{J}_k - \tilde{J}_{k-1})\right\| &(80)\\
&\le \left\|(\mathcal{A} - \frac{\mathbf{1}\mathbf{1}^\top}{N})\psi_k\right\| &(81)\\
&\quad + \|\tilde{J}_k - \tilde{J}_{k-1} - \frac{\mathbf{1}\mathbf{1}^\top}{N}(\tilde{J}_k - \tilde{J}_{k-1})\| &(82)\\
&\le \rho\|\psi_k - \mathbf{1}\bar{\psi}_k\| &(83)\\
&\quad + \left\|\tilde{J}_k - \tilde{J}_{k-1} - \frac{1}{N}\mathbf{1}^\top\tilde{J}_k + \frac{1}{N}\mathbf{1}^\top\tilde{J}_{k-1}\right\| &(84)\\
&\le \rho\|\psi_k - \mathbf{1}\bar{\psi}_k\| + \left\|(I - \frac{\mathbf{1}\mathbf{1}^\top}{N})\left(\tilde{J}_k - \tilde{J}_{k-1}\right)\right\| &(85)\\
&\le \rho\|\psi_k - \mathbf{1}\bar{\psi}_k\| + \underbrace{\lambda_{\max}\left(I - \frac{\mathbf{1}\mathbf{1}^\top}{N}\right)}_{\zeta'}\|\tilde{J}_k - \tilde{J}_{k-1}\| &(86)
\end{aligned}$$

Defining the state $Z_k = \|\psi_k - \mathbf{1}\bar{\psi}_k\| \ge 0$ the previous condition can be rewritten as

$$Z_{k+1} \le \rho Z_k + \zeta'\|\tilde{J}_k - \tilde{J}_{k-1}\| \tag{87}$$

which applied recursively gives, for $k \ge 1$,

$$Z_k \le \rho^k Z_0 + \zeta\sum_{\tau=1}^{k-1}\rho^{k-\tau-1}\|\tilde{J}_\tau - \tilde{J}_{\tau-1}\|. \tag{88}$$

We can see that, for each $i \in \{1,\dots,N\}$, we have

$$\begin{aligned}
&\|\mu_{i,k}^\top\phi_i(x_{i,k+1},y_{i,k+1}) - \mu_{i,k-1}^\top\phi_i(x_{i,k},y_{i,k})\|^2\\
&= \|\mu_{i,k}^\top\phi_i(x_{i,k+1},y_{i,k+1}) - \mu_{i,k}^\top\phi_i(x_{i,k},y_{i,k})\\
&\quad + \mu_{i,k}^\top\phi_i(x_{i,k},y_{i,k}) - \mu_{i,k-1}^\top\phi_i(x_{i,k},y_{i,k})\|^2\\
&\le \|\mu_{i,k}^\top(\phi_i(x_{i,k+1},y_{i,k+1}) - \phi_i(x_{i,k},y_{i,k}))\|^2 &(89)\\
&\quad + \|(\mu_{i,k} - \mu_{i,k-1})^\top\phi_i(x_{i,k},y_{i,k})\|^2 &(90)
\end{aligned}$$

where, for the first term we have

$$\begin{aligned}
&\|\mu_{i,k}^\top(\phi_i(x_{i,k+1},y_{i,k+1}) - \phi_i(x_{i,k},y_{i,k}))\|^2\\
&\le \|\mu_{i,k}\|^2\|\phi_i(x_{i,k+1},y_{i,k+1}) - \phi_i(x_{i,k},y_{i,k})\|^2. &(91)
\end{aligned}$$

Following similar arguments as in Lemma 1.4, we obtain

$$\begin{aligned}
&\|\mu_{i,k}^\top(\phi_i(x_{i,k+1},y_{i,k+1}) - \phi_i(x_{i,k},y_{i,k}))\|^2\\
&\le B^{\mu 2}\big[(\sqrt{n(n+3)}2B^x+1)^2\|x_{i,k+1} - x_{i,k}\|^2\\
&+ (\sqrt{s(s+3)}2B^y+1)^2\|y_{i,k+1} - y_{i,k}\|^2\big].
\end{aligned}$$

The second term reads:

$$\begin{aligned}
&\|(\mu_{i,k} - \mu_{i,k-1})^\top\phi_i(x_{i,k},y_{i,k})\|^2\\
&\le \|\mu_{i,k} - \mu_{i,k-1}\|^2\|\phi_i(x_{i,k},y_{i,k})\|^2 &(92)
\end{aligned}$$

then, we have

$$\begin{aligned}
\|\phi_i(x_{i,k},y_{i,k})\|^2 &= \|\operatorname{vec}(x_{i,k}x_{i,k}^\top)\|^2 + \|x_{i,k}\|^2 + 1\\
&\quad + \|\operatorname{vec}(y_{i,k}y_{i,k}^\top)\|^2 + \|y_{i,k}\|^2\\
&\le B^{x4} + B^{x2} + 1 + B^{y4} + B^{y2}\\
&:= B^{\phi^2}
\end{aligned}$$

Then we can write

$$\|(\mu_{i,k} - \mu_{i,k-1})^\top\phi_i(x_{i,k},y_{i,k})\|^2 \le B^{\phi^2}\|\mu_{i,k} - \mu_{i,k-1}\|^2.$$

Collecting the previous results we obtain

$$\begin{aligned}
&\|\mu_{i,k}^\top\phi_i(x_{i,k+1},y_{i,k+1}) - \mu_{i,k-1}^\top\phi_i(x_{i,k},y_{i,k})\|^2 &(93)\\
&\le B^{\mu 2}\big[(\sqrt{n(n+3)}2B^x+1)^2\|x_{i,k+1} - x_{i,k}\|^2 &(94)\\
&\quad +(\sqrt{s(s+3)}2B^y+1)^2\|y_{i,k+1} - y_{i,k}\|^2\big] &(95)\\
&\quad +B^{\phi^2}\|\mu_{i,k} - \mu_{i,k-1}\|^2 &(96)
\end{aligned}$$

Now we apply similar bounds as in Lemma 1.5, where we also note that by condition (44) it holds

$$\limsup_{k\to\infty}\|\mu_{i,k} - \mu_{i,k-1}\| \le \delta^\mu. \tag{97}$$

Then, calculating the limsup, we obtain

$$\limsup_{k\to\infty} Z_k \le \zeta'\limsup_{k\to\infty}\sum_{\tau=1}^{k-1}\rho^{k-\tau-1}\|\tilde{J}_\tau - \tilde{J}_{\tau-1}\|$$

where, for each $i \in \{1,\dots,N\}$ we have:

$$\begin{aligned}
&\limsup_{k\to\infty}\|\tilde{J}_k - \tilde{J}_{k-1}\|^2\\
&\le \sum_{i=1}^{N}\limsup_{k\to\infty}\|\mu_{i,k}^\top\phi_i(x_{i,k+1},y_{i,k+1}) - \mu_{i,k-1}^\top\phi_i(x_{i,k},y_{i,k})\|^2\\
&\le N\bigg(B^{\mu 2}\delta^2\Big[(\sqrt{n(n+3)}2B^x+1)^2\\
&+ (\sqrt{s(s+3)}2B^y+1)^2\frac{\zeta^2}{(1-\rho)^2}\Big] + B^{\phi^2}\delta^{\mu 2}\bigg)\\
&:= B^{J^2}.
\end{aligned}$$

Then,

$$\limsup_{k\to\infty} Z_k \le \frac{\zeta'}{1-\rho}B^J,$$

and the proof follows. ■

Lemma 1.9: Under the same assumptions of Theorem 4.3, it holds with probability ε

$$\limsup_{k\to\infty}\left\|\frac{1}{N}\mathbf{1}^\top\tilde{J}_k - \psi_k^c\right\| \le \frac{N\zeta\delta B^{J\prime}}{1-\rho} \tag{98}$$

for all $i \in \mathbb{I}$, where

$$B^{J\prime} = B^\mu(\sqrt{s(s+3)}2\max\{B^y, B^\sigma\}+1).$$

Proof: Let us compute

$$\begin{aligned}
&\left\|\frac{1}{N}\mathbf{1}^\top \tilde{J}_k - \psi_k^c\right\| \\
&\leq \frac{1}{N}\left\|\sum_{i=1}^N \left(\mu_{i,k}^\top(\phi_i(x_{i,k},y_{i,k}) - \phi_i(x_{i,k},\sigma(x_k)))\right)\right\| \\
&\leq \frac{1}{N}\sum_{i=1}^N \|\mu_{i,k}\|\,\|\phi_i(x_{i,k},y_{i,k}) - \phi_i(x_{i,k},\sigma(x_k))\| \\
&\leq \underbrace{B^\mu(\sqrt{s(s+3)}2\max\{B^y,B^\sigma\}+1)}_{B^{J\prime}}\sum_{i=1}^N \|y_{i,k}-\sigma(x_k)\|.
\end{aligned}$$

Computing the $\limsup$ as k goes to infinity we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\left\|\frac{1}{N}\mathbf{1}^\top \tilde{J}_k - \psi_k^c\right\| \\
&\leq B^{J\prime}\sum_{i=1}^N \limsup_{k\to\infty}\|y_{i,k}-\sigma(x_k)\| \\
&\leq \frac{N\zeta\delta B^{J\prime}}{1-\rho}.
\end{aligned}$$

■

Corollary 1.10: Under the same assumptions of Theorem 4.3, it holds with probability ε

$$\limsup_{k\to\infty}\|\psi_{i,k}-\psi_k^c\| \leq \frac{\zeta' B^J + N\zeta\delta B^{J\prime}}{1-\rho} \tag{99}$$

for all $i \in \mathbb{I}$. □

In a similar way, we can derive the following result.

Corollary 1.11: Under the same assumptions of Theorem 4.3, it holds with probability ε

$$\limsup_{k\to\infty}\|\varphi_{ik}-\varphi_k^c\| \leq \frac{\zeta' B^J + N\zeta\delta B^{J\prime}}{1-\rho} \tag{100}$$

for all $i \in \mathbb{I}$. □

Now we prove that the distributed estimates of the temporal differences approach the centralized ones

Lemma 1.12: Under the same assumptions of Theorem 4.3, it holds with probability ε

$$\limsup_{k\to\infty}|d_{ik}-d_k| \leq L^a + (\alpha+1)L^b, \tag{101}$$

for all $i \in \mathbb{I}$, where

$$\begin{aligned}
L_a &= N\left(B^h + \frac{\zeta\delta}{N(1-\rho)}\sum_{j=1}^N B_j^{y'}\right), \\
L_b &= N\left(\frac{\zeta' B^J + N\zeta\delta B^{J\prime}}{1-\rho}\right).
\end{aligned}$$

Proof: Let us compute

$$\begin{aligned}
&|d_{j,k+1}-d_{k+1}| = \Bigg|\sum_{i=1}^N (h_{i,k+1}+\alpha\psi_{i,k+1}-\varphi_{i,k+1} \\
&-g_i(x_{i,k},u_{i,k},\sigma(x_k)) - \alpha\psi_{k+1}^c + \varphi_{k+1}^c)\Bigg| \\
&\leq \sum_{i=1}^N \Big(|h_{i,k+1}-g_i(x_{i,k},u_{i,k},\sigma(x_k))| \\
&+\alpha|\psi_{i,k+1}-\psi_{k+1}^c| + |\varphi_{i,k+1}-\varphi_{k+1}^c|\Big)
\end{aligned}$$

Taking the $\limsup$ as $k\to\infty$ and using the previous results we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\sum_{i=1}^N |h_{i,k+1}-g_i(x_{i,k},u_{i,k},\sigma(x_k))| \\
&\leq N\left(B^h + \frac{\zeta\delta}{N(1-\rho)}\sum_{j=1}^N B_j^{y'}\right) := L^a
\end{aligned}$$

$$\limsup_{k\to\infty}\sum_{i=1}^N |\psi_{i,k+1}-\psi_{k+1}^c| \leq N\left(\frac{\zeta' B^J + N\zeta\delta B^{J\prime}}{1-\rho}\right) := L^b$$

$$\limsup_{k\to\infty}\sum_{i=1}^N |\varphi_{i,k+1}-\varphi_{k+1}^c| \leq L^b$$

It thus follows that

$$\limsup_{k\to\infty}|d_{j,k+1}-d_{k+1}| = L^a + (\alpha+1)L^b. \tag{102}$$

■

Now we are ready to prove Theorem 4.3.

Proof: Point (i) follows by Lemma 1.12. Point (ii) follows by Lemma 1.4. Point (iv) follows trivially by point (i). As for point (iii), it holds

$$\limsup_{k\to\infty}\|\mu_{i,k}-\mu_k^c\| = \limsup_{k\to\infty}\|d_{ik}z_{ik} - d_k^c z_k^c\|,$$

and the proof follows by combining points (i) and (ii). ■