Operations Research Primer

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1 Introduction

OR Primer is a collection of resources to teach fundamentals of computational optimization regarding Linear Programming and Mixed Integer (Linear) Programming.

This collection targets those who want to get a jumpstart without any technical and theoretical details, except the absolutely required fundamental terms. Therefore, there will be lots of examples.

Primary focus of OR Primer is to give the reader the ability to discern if a "business" problem can be converted to an OR (LP or MILP) problem. In essence "yes" or "no" depends on the answers of the following questions: "Is there a decision to be made? (*i.e. Decision Variables*)", "Are there limitations and requirements? (*i.e. Constraints*)", "What is the ultimate aim? (*i.e. Objective Function*)" and "Can we describe the problem in linear terms? (*i.e. Linearity*)".

At the second stage, we will focus on transferring the "business" problem from verbal to mathematical form. Reader is expected to get a sense of how to write a mathematical model in a standard way.

Finally, we will learn how to describe the model in code. We will learn about Algebraic Modelling Languages (AML) and solvers. Our focus will be on scripting languages such as R, Python and Julia.

Optionally, we will discuss theory. Topics such as simplex, duality, interior points etc. will be briefly explained but it is up to the reader to really dive in the theory or if she is just happy with the coding.

There will be lots of external resources. Feel free to add them on Discussions.

Part I

Linear Programming

Linear programming (LP) is the fundamental modeling method of Operations Research. But briefly an LP should adhere to the following rules.

- **Neither** constraints **nor** the objective function may contain non-linear terms.
- Decision variables are all **continuous**. They **may not** be *binary* or *integer*.
- Decision variables can be either non-negative ($x \geq 0$) or unrestricted (**urs**).

2 Mathematical Model

A mathematical representation of LP model is provided below.

$$\min c^{\top} x \tag{2.1}$$
$$Ax = b \tag{2.2}$$
$$x \geq 0 \tag{2.3}$$
$$\tag{2.4}$$

2.1 Parts of a Model

An LP model requires the following object types to be complete.

- **Decision Variables (DV)**: Decision variables are the objects which the algorithm (i.e. solver) decides its value. A combination of a decision variable value set is a **solution**. In LP it is not possible to define a DV in non-linear (e.g., x^2) terms or in interaction with other DVs (e.g., xy). In the model, elements of x are decision variables. x is an n-sized vector.
- **Coefficients and constants**: It is possible to add pre-defined constants as coefficients to decision variables or by themselves. In the model, elements of A, b and c are constants and coefficients. A is an m x n matrix, b is an m-sized vector and c is an n-sized vector.
- **Constraints**: Constraints are the rules which the decision variable values should satisfy in order to be a valid (i.e. **feasible**) solution. In the model, $Ax = b$ system of equations and non-negativity terms ($x \geq 0$) are the constraints.
- **Objective Function**: Objective function defines the direction (either minimization or maximization) and the evaluation formula of the solution quality. In the model, the term $\min c^{\top} x$ is the objective function and the solver will try to minimize $c^{\top} x$.

2.2 Indices

There are also **indices** which we will use to define elements in decision variables, coefficients, constants and constraints. For instance x_i is the i th element of the decision variable vector and $A_{i,j}$ is the (i,j)th element of the coefficient matrix. Let's rewrite the model.

$$\min \sum_{j=1}^{n} c_j x_j \tag{1}$$

$$\sum_{j=1}^{n} A_{i,j} x_j = b_i \ \forall_{i \in 1..m} \tag{2}$$

$$x_j \geq 0 \ \forall_{j \in 1..n} \tag{3}$$

(2.5)

These parts may also be multi-dimensional. For instance $x_{i,j,k,t}$ is possible.

3 Giapetto Example

Giapetto is the introductory example of Linear Programming. This example is directly taken from Winston's Operations Research (4th Edition).

3.1 Problem

"Giapetto's Woodcarving, Inc., manufactures two types of wooden toys: **soldiers** and **trains**.

A soldier sells for **$27** and uses **$10** worth of raw materials. Each soldier that is manufactured increases Giapetto's variable labor and overhead costs by **$14**. A train sells for **$21** and uses **$9** worth of raw materials. Each train built increases Giapetto's variable labor and overhead costs by **$10**.

The manufacture of wooden soldiers and trains requires two types of skilled labor: **carpentry** and **finishing**. A soldier requires **2 hours** of finishing labor and **1 hour** of carpentry labor. A train requires **1 hour** of finishing labor and **1 hour** of carpentry labor.

Each week, Giapetto can acquire all of the needed raw material, but he is only allotted **100 finishing hours** and **80 carpentry hours**. There is an unlimited demand for trains. However, **at most, 40 soldiers are sold** each week.

Giapetto wants to **maximize his weekly profit** (Revenues - Costs). Formulate a mathematical model for Giapetto's situation that can be used to maximize Giapetto's weekly profit."

Let's convert problem statement into a number of model building steps in the next section.

3.2 Model Building Steps

1. Let's calculate the net profit of a soldier and a train, respectively. Sale price of a soldier is **$27**, raw material cost is **$10** and labor/overhead costs are **$14**. So producing a soldier toy yields **$3** of net profit. With the same process a train's net profit is **$2**.
2. Our aim is to maximize our total net profit. Let's denote x_1 as the number of soldiers produced and x_2 as the number of trains produced. Values x_1 and x_2 will be determined by the solver. Therefore they are **decision variables**.

3. So, our total net profit can be defined as $z = 3x_1 + 2x_2$. This is also our **objective function**.
4. For finishing tasks, a soldier requires **2 hours** and a train requires **1 hour** of labor. Finishing labor capacity is **100 hours**. So, its mathematical expression is $2x_1 + x_2 \leq 100$.
5. For carpentry tasks, a soldier requires **1 hour** and a train requires **1 hour** of labor. Carpentry labor capacity is **80 hours**. So, its mathematical expression is $x_1 + x_2 \leq 80$.
6. Demand for soldiers is limited with 40. So, its mathematical expression is $x_1 \leq 40$.
7. Also, it is not possible to sell negative amounts of soldiers or toys (no returns). Therefore both x_1 and x_2 should be greater than zero (non-negativity constraints).

Let's gather all the steps in a single model in the next section.

3.3 Mathematical Model

3.3.1 Decision Variables

- x_1: Number of soldiers to be manufactured.
- x_2: Number of trains to be manufactured.

3.3.2 Model

$$\max z = 3x_1 + 2x_2 \tag{3.1}$$

$$s.t.$$

$$2x_1 + x_2 \leq 100 \tag{3.2}$$

$$x_1 + x_2 \leq 80 \tag{3.3}$$

$$x_1 \leq 40 \tag{3.4}$$

$$x_1, x_2 \geq 0 \tag{3.5}$$

$$\tag{3.6}$$

3.3.3 Constraints

- (3.1) is the objective function to maximize total profits. Each soldier yields **$3** profit and each train **$2**.
- (3.2) is the finishing task capacity constraint. Each soldier requires **2 hours** of labor and each train requires **1 hour**. Total capacity for finishing task is **100 hours**.
- (3.3) is the carpentry task capacity constraint. Each soldier requires **1 hour** of labor and each train requires **1 hour**. Total capacity for finishing task is **80 hours**.
- (3.4) is the maximum demand constraint for soldiers. Maximum available demand for soldiers is **40**.
- (3.5) Non-negativity constraint. It is not possible to sell negative amount of each toys (i.e. no backorders, no returns etc. in this case).