Agenda

- Review of Simplex method
- Optimality condition question
- Degeneracy and cycling
- Example

Logistics

- HW 5 due Friday March 25 9pm
- Midterm 2 Thursday April 12
- Midterm 1 grades out

Standard form polyhedra

Standard form LP?

$$\min_x \ c^T x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$

What is P?

$$P = \{x \mid Ax = b, \ x \geq 0\}$$

Feasible direction

a vector d is a feasible direction at point $x \in P$

$\Longleftrightarrow$

$\exists \theta > 0$ s.t. $x + \theta d \in P$

Claim: $Ad = 0$

since
$$A(x + \theta d) = b$$
$$Ax + \theta Ad = b$$
$$\theta Ad = 0$$
$$Ad = 0$$

$x + \theta d \geq 0$ since $x + \theta d \in P$

for small enough θ

Basic feasible solution

- A basis: e.g. $B = \{1, 4, 5\}$
- A basis matrix $A_B = [A_{B(1)}, \ldots, A_{B(m)}]$
- What does x satisfy?

$$\left.\begin{array}{l} \cdot\ A_B x_B = b \\ \cdot\ x_i = 0 \quad \forall i \notin B \end{array}\right\}$$

- When is x a basic feasible solution?
  - $\rightarrow$ when $x_B \geq 0$

- basic direction d (also feasible)
  - jth basic direction means we are considering adding j to the basis
  - Remember, we need $Ad = 0$ and we don't want to add other non-basic variables to the basis

Set $d_j = 1$

Set $d_k = 0$ for all $k \neq j$, $k \notin B$

Then d_B will be set by solving the linear system

$$Ad = 0$$

$$Ad = \sum_{i=1}^{n} A_i d_i = A_B d_B + A_j = 0 \implies A_B d_B = -A_j$$

$$d_B = -A_B^{-1} A_j$$

Cost Improvement

$$\underbrace{c^T(x+\theta d)}_{\text{new cost}} - \underbrace{c^T x}_{\text{old cost}} = \overbrace{\theta\, c^T d}^{\text{cost improvement}}$$

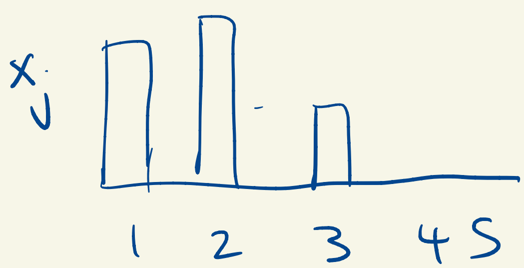
$c^T d$ is the reduced cost associated with this d

$$\bar{c}_j = c^T d = \sum_{i=1}^{n} c_i d_i = c_j + c_B^T d_B = c_j - c_B^T A_B^{-1} A_j$$

Example: $m=3,\ n=5$

$$B = \{1, 2, 3\}$$

want to add 4 to the basis



$$A_B x_B = b$$

$$(A_1\ A_2\ A_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = b$$

- if $\bar{c} \geq 0$ then? x is optimal
- if some $\bar{c}_j < 0$ then? we can decrease the cost by bringing x_j into the basis
- How far can we step after finding d?

$$\theta^* = \max\{\theta \mid \theta \geq 0,\ x + \theta d \geq 0\}$$

- if $d \geq 0$ in this case, then? $\theta^* = \infty$, LP is unbdd

Optimality conditions

(a) a feasible solution x is optimal $\Leftrightarrow c^Td \geq 0$ for any feasible direction at x

$\Rightarrow$

Suppose $\exists$ a feas. dir. d s.t. $c^Td < 0$

Then $\exists\ \theta > 0$ s.t. $x + \theta d$ is a feas. soln

Thus $c^Tx > c^T(x + \theta d)$ contradiction bc x not optimal

$\Leftarrow$

Suppose $c^Td \geq 0$ for every feas. dir. d at x

Let $y \in P$

Then let $d = y - x$

Thus $c^Td \geq 0 \Rightarrow c^T(y-x) \geq 0$

$c^Ty \geq c^Tx$ so x is optimal

(b) a feasible solution x is the unique optimal solution if and only if $c^T d > 0$ for every nonzero feasible direction d at x