Agenda

- Converting LPs
- Steel company operations
- The moment problem
- Chebyshev center

Logistics

- HW4 out, due Friday March 5 at 9pm
- Midterm 1 Thurs March 11
- Spring break Mon March 15 – Tues March 16
  - no precepts this week
  - no lecture on Tuesday
- A bit of a break until HW5 due
  - Friday March 26

Converting LPs

we'd like to convert $\min_x \|Ax-b\|_1$ s.t. $\|x\|_\infty \le k$

$A \in \mathbb{R}^{m,n}$

$b \in \mathbb{R}^m$

$x \in \mathbb{R}^n$ $\quad k \in \mathbb{R}$

into 2 forms

form (1)

$$\min_x c^T x \quad \text{s.t. } Ax \le b, \quad Cx = d$$

form (2)

$$\min_x c^T x \quad \text{s.t. } Ax = b, \quad x \ge 0$$

form (1)

- $\|x\|_\infty \le k \rightarrow x_i \le k, -x_i \le k \quad i = 1, \dots, n$
- Let s be a proxy for $|Ax-b|$

$s \ge Ax - b$, $s \ge -Ax + b$ since it's a minimization problem

$$\left[\begin{array}{l} \min_{x,s} \mathbb{1}^T s \\ \text{s.t. } x \le \mathbb{1}k \\ \quad -x \le \mathbb{1}k \\ \quad s \ge Ax - b \\ \quad s \ge -Ax + b \end{array}\right]$$

form (1)

$$\left[\begin{array}{l} \min_{x,s} \mathbb{1}^T s \\ \text{s.t. } x \le \mathbb{1}k \\ \quad -x \le \mathbb{1}k \\ \quad Ax - s \le b \\ \quad -Ax - s \le -b \end{array}\right]$$

Can write it in block form

$$\begin{pmatrix} I & 0 \\ -I & 0 \\ A & -I \\ -A & -I \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} \le \begin{pmatrix} \mathbb{1}k \\ \mathbb{1}k \\ b \\ -b \end{pmatrix}$$

$$\tilde{A} \tilde{x} \le \tilde{b}$$

form (1)

$$\begin{aligned} \min_{x,s} \quad & \mathbb{1}^T s \\ \text{s.t.} \quad & x \le \mathbb{1}k \\ & -x \le \mathbb{1}k \\ & Ax - s \le b \\ & -Ax - s \le -b \end{aligned}$$

→ add in y_1, y_2, y_3, y_4 slack vars

$$\begin{aligned} \min_{x, s, y_1, \dots, y_4} \quad & \mathbb{1}^T s \\ \text{s.t.} \quad & x + y_1 = \mathbb{1}k \\ & -x + y_2 = \mathbb{1}k \\ & Ax - s + y_3 = b \\ & -Ax - s + y_4 = -b \\ & y_1, \dots, y_4 \ge 0 \end{aligned}$$

We just need to have all variables with a non-negativity constraint

- Let $x = x^+ - x^-$, $x^+ \ge 0$, $x^- \ge 0$
- $s \ge 0$ already, so we can simply add this redundant constraint

$$\begin{aligned} \min_{\substack{x^+, x^-, s \\ y_1, y_2, y_3, y_4}} \quad & \mathbb{1}^T s \\ \text{s.t.} \quad & x + y_1 = \mathbb{1}k \\ & -x + y_2 = \mathbb{1}k \\ & Ax - s + y_3 = b \\ & -Ax - s + y_4 = -b \\ & x^+, x^-, s, y_1, y_2, y_3, y_4 \ge 0 \end{aligned}$$

form (2)

Steel Company Operations

- Steel company can produce bands and coils
- Goal is to maximize revenue

	Production rate (tons/hr)	revenue ($/ton)	upper bounds (tons) ← from orders
Bands	200	25	6000
Coils	140	30	4000

- There are 40 hours of production time this week
- Decide how many tons of bands and coils should be produced to maximize revenue

- decision variables b, c represent # of hours allocated to produce bands and coils respectively

$$\max_{b,c} \ \overbrace{200(25)}^{5000} b + \overbrace{140(30)}^{4200} c$$

$$\text{s.t. } b + c \le 40$$

$$b, c \ge 0$$

$$200\, b \le 6000 \quad \rightarrow (b \le 30)$$

$$140\, c \le 4000 \quad \rightarrow (c \le 28.6)$$

- we see that it's better to produce as many bands as possible since $5000 > 4200$
- Set $b^* = 30$, $c^* = 10$

The moment problem

- Suppose that Z is a random variable taking values in the set $\{0, 1, \dots, K\}$ with probabilities $p_0, \dots, p_K$
- We are given $\mathbb{E}[Z] = \sum_{i=0}^{K} p_i \, i$

$$\mathbb{E}[Z^2] = \sum_{i=0}^{K} p_i \, i^2$$

- We would like to obtain upper and lower bounds on the 4th moment: $\mathbb{E}[Z^4] = \sum_{i=0}^{K} i^4 p_i$
- Show how LPs can be used to approach this problem

- Decision variables are $p_0, \dots, p_K$
- constraints: $\mathbb{E}[Z]$, $\mathbb{E}[Z^2]$, $p \geq 0$, $\mathbb{1}^T p = 1$

- Use 1 LP to obtain a lower bound and another LP to obtain an upper bound

$$LB = \min_{p} \sum_{i=1}^{K} i^4 p_i$$
$$\text{s.t. } p \geq 0$$
$$\mathbb{1}^T p = 1$$
$$\mathbb{E}[Z] = \sum_{i=1}^{K} i \, p_i$$
$$\mathbb{E}[Z^2] = \sum_{i=1}^{K} i^2 p_i$$

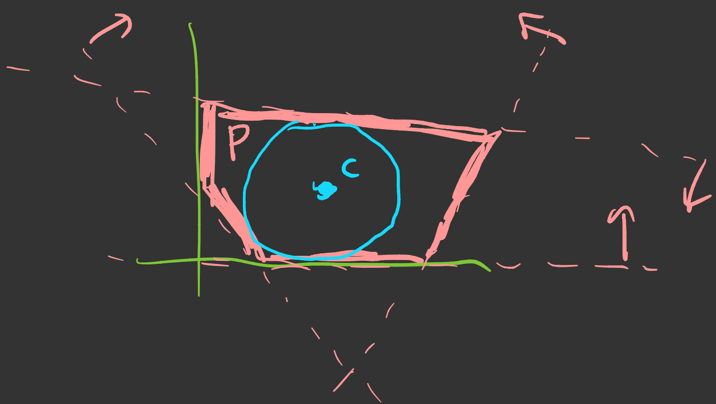
$$UB = \max_{p} \sum_{i=1}^{K} i^4 p_i$$
$$\text{s.t. } p \geq 0$$
$$\mathbb{1}^T p = 1$$
$$\mathbb{E}[Z] = \sum_{i=1}^{K} i \, p_i$$
$$\mathbb{E}[Z^2] = \sum_{i=1}^{K} i^2 p_i$$

Chebyshev Center

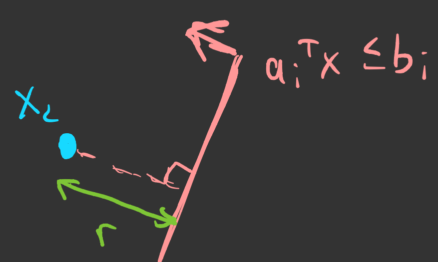
- Consider a set P described by linear inequality constraints

$$P = \{ x \in \mathbb{R}^n \mid a_i^T x \le b_i \quad i=1,\dots,m \}$$ ← called a polyhedron

- Goal: find a ball with the largest possible radius which is entirely contained within the set P
- Provide an LP formulation of this problem



Look at a particular constraint



we want to keep a_i but change b_i

- $a_i^T(x_c + u) \le b_i \quad$ for all $\{u \mid \|u\|_2 \le r\}$
- only need to look at the worst case

$$\max_u \; a_i^T(x_c + u) \le b_i$$

- The worst-case u is parallel to a_i

$$u^* = \frac{a_i}{\|a_i\|_2} r$$

$$a_i^T x_c + \frac{a_i^T a_i}{\|a_i\|_2} r \le b_i$$

$$a_i^T x_c + \|a_i\|_2 r \le b_i$$

Now write it in the form of an LP

$$\max_{x_c, r} \quad r$$
$$\text{s.t.} \quad a_i^T x_c + \|a_i\|_2 r \leq b_i \quad i=1,\dots,m$$