ORF307_HW2

February 7, 2023

ORF307 Homework 2

Due: Friday, Feburary 17, 2023 9:00 pm ET

- The jupyter file is available at https://github.com/ORF307/companion
- Please export your code with output as `pdf`.
- If there is any additional answers, please combine them as **ONE** pdf file before submitting to the Gradescope.

Q1 Least angle property and orthogonality principle of least squares

Suppose the $m \times n$ matrix A has linearly independent columns, and b is an m-vector. Let $x^\star$ denote the least squares approximate solution of $Ax = b$, i.e., $A^T A x^\star = A^T b$.

(a) Show that for any n-vector x, $(Ax)^T b = (Ax)^T (Ax^\star)$, i.e., the inner product of Ax and b is the same as the inner product of Ax and $Ax^\star$.

Hint. Use $(Ax)^T b = x^T (A^T b)$.

(b) Show that when $Ax^\star$ and b are both nonzero, we have

$$\frac{(Ax^\star)^T b}{\|Ax^\star\|_2 \|b\|_2} = \frac{\|Ax^\star\|_2}{\|b\|_2}.$$

The left-hand side is the cosine of the angle between $Ax^\star$ and b.

Hint. Apply part (a) with $x = x^\star$.

(c) *Least angle property of least squares.* Show that $x = x^\star$ also minimizes the angle between Ax and b. (You can assume that Ax and b are nonzero.)

Remark. For any positive scalar α, $x = \alpha x^\star$ also minimizes the angle between Ax and b.

(d) *Orthogonality principle.* The optimal residual r is defined as $r = Ax^\star - b$. Show that r is orthogonal to the columns of A, and therefore, it is orthogonal to any linear combination of the columns of A. In other words, for any n-vector z, we have $(Az)^T r = 0$.

Remark. This is one of the most important properties of least squares solutions: $Ax^\star$ is the linear combination of the columns of A that is closest to b and, it is orthogonal to the residual vector $r = Ax^\star - b$. In other words, $Ax^\star$ is the "projection" of b on the space of linear combinations of the columns of A.

Q2 Matrix inversion lemma

(a) It is sometimes very cheap to solve subsequent linear systems when the coefficient matrix gets slightly modified. Suppose we have a $n \times n$ matrix M. Given n-vectors u and v, prove that, if $v^T M^{-1} u \neq -1$, the following update, called *matrix inversion lemma* holds:

$$(M + uv^T)^{-1} = M^{-1} - \frac{1}{1 + v^T M^{-1} u} M^{-1} u v^T M^{-1}.$$

Hint. Directly apply the matrix inverse definition, i.e., $(M + uv^T)^{-1}(M + uv^T) = I$.

Remark. The assumption $v^T M^{-1} u \neq -1$ can be shown to correspond to $M + uv^T$ being invertible.

(b) Suppose now that M is symmetric positive definite and that you have already computed a Cholesky decomposition LL^T (therefore, we have a convenient way to solve $Mx = b$), but that we want to solve the linear system

$$(M + uv^T)x = b.$$

Explain how we can use the recursion in point (a) to do so by solving a few prefactored linear systems.

Hint. Remember, we never compute inverses explicitly! Every time you have $M^{-1}q$ for some vector n-vector q, it means solving the linear system $Mz = q$ with n-vector z.

(c) Work out the total flop count for this method keeping only the dominant terms and compare it to factoring matrix $M + uv^T$ from scratch.

Q3 Moore's Law

[2]:
```
# Import required packages for coding exercises
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from numpy.linalg import cholesky as llt
seed = 1
np.random.seed(seed) # We use np.random.seed for reproducible results
```

For convenience we have included the following function `lstsq` that solves the least squares problem as we have seen in class returning `x` and factorization `L`.

[3]:
```
def forward_substitution(L, b):
    n = L.shape[0]
    x = np.zeros(n)
    for i in range(n):
        x[i] = (b[i] - L[i,:i] @ x[:i])/L[i, i]
    return x

def backward_substitution(U, b):
    n = U.shape[0]
    x = np.zeros(n)
```

```python
    for i in reversed(range(n)):
        x[i] = (b[i] - U[i,i+1:] @ x[i+1:])/U[i, i]
    return x

def lstsq(A, b):
    M = A.T.dot(A)  # Form Gram matrix
    q = A.T.dot(b)  # Form right hand side
    L = llt(M)      # Factor
    x = forward_substitution(L, q)
    x = backward_substitution(L.T, x)
    return x, L
```

The figure and table below show the number of transistors N in 13 microprocessors, and the year of their introduction.

[1]:

```python
# Data
t = np.array([1971, 1972, 1974, 1978, 1982, 1985, 1989, 1993, 1997, 1999, 2000,␣
 ↪2002, 2003])
N = np.array([2250, 2500, 5000, 29000, 120000, 275000, 1180000, 3100000,␣
 ↪7500000, 24000000, 42000000, \
              220000000, 410000000])

# Show data as dataframe
# NB. Jupyter makes a nice table out of a pandas dataframe
pd.DataFrame({'t': t,
              'N': N})
```

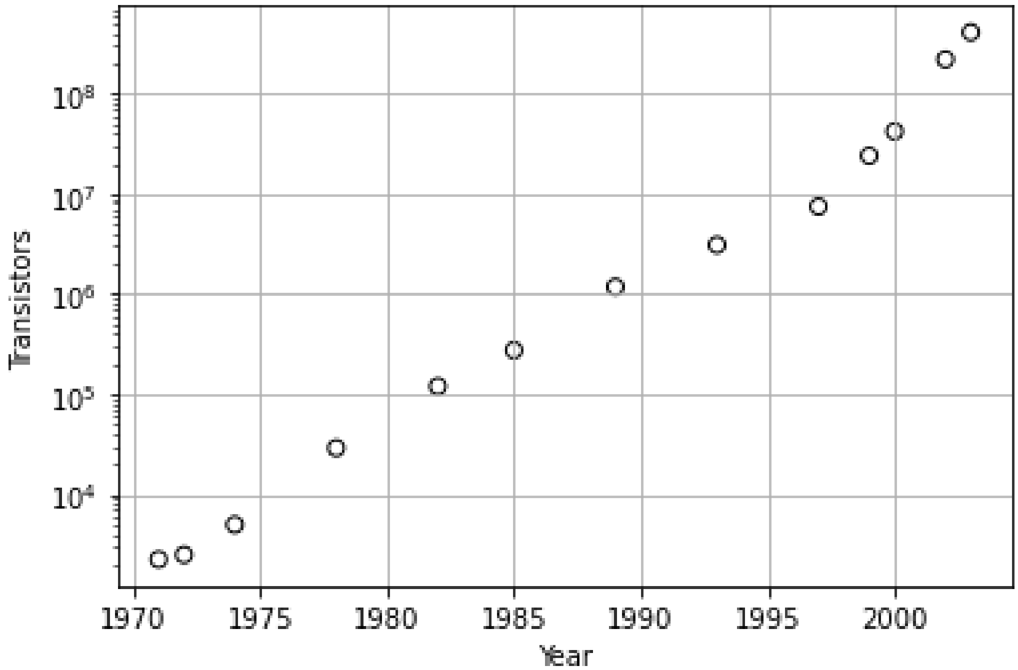
[1]:

	t	N
0	1971	2250
1	1972	2500
2	1974	5000
3	1978	29000
4	1982	120000
5	1985	275000
6	1989	1180000
7	1993	3100000
8	1997	7500000
9	1999	24000000
10	2000	42000000
11	2002	220000000
12	2003	410000000

[2]:

```python
fig, ax = plt.subplots(1,1)
ax.scatter(t, N, marker='o', color='k', facecolors='none')
ax.set_xlabel(r'Year')
ax.set_ylabel(r'Transistors')
ax.set_yscale('log')
```

```
ax.grid('on')
```



The plot gives the number of transistors on a logarithmic scale. Find the least squares straight-line fit of the data using the model

$$\log_{10} N \approx \theta_1 + \theta_2(t - 1970),$$

where t is the year and N is the number of transistors. Note that θ_1 is the model's prediction of the log of the number of transistors in 1970, and 10^{θ_2} gives the model's prediction of the fractional increase in number of transistors per year.

(a) Find the coefficients θ_1 and θ_2 that minimize the mean square error on the data using the `lstsq` function above. Give the mean square error on the data. Plot the model you find along with the data points.

(b) The first point-contact transistor was invented in 1947. Now let $N = 1$, where $\log_{10} N = 0$, and then find t. See if your model fits the historical data well.

(c) Use your model to predict the number of transistors in a microprocessor introduced in 2015. Compare the prediction to the IBM Z13 microprocessor, released in 2015, which has around 4×10^9 transistors.

(d) Compare your result with Moore's law, which states that the number of transistors per integrated circuit roughly doubles every one and a half to two years.