Agenda

- Review of Simplex method
- Optimality condition question
- Degeneracy and cycling
- Example

Logistics

- HW 5 due Friday March 26 9pm
- HW 6 out Thursday due Friday April 2 9pm
- Midterm 2 Thursday April 15
- Midterm 1 grades out

standard form polyhedra

$$\min_x \; c^T x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$

Feasible direction

$$P = \{ x \mid Ax = b, \; x \geq 0 \}$$

a vector d is a feasible direction at point $x \in P$

$\Longleftrightarrow$

$\exists \, \theta > 0$ s.t. $x + \theta d \in P$

$Ad = 0$

since $A(x + \theta d) = b$

$Ax + \theta Ad = b$

$\theta Ad = 0$

$Ad = 0$

$x + \theta d \geq 0$ since $x + \theta d \in P$

Basic feasible solution

- A basis: e.g. $B = \{1, 4, 5\}$
- A basis matrix $A_B = [A_{B(1)}, \dots, A_{B(m)}]$
- x satisfies
  - $A_B x_B = b$
  - $x_i = 0 \quad \forall i \notin B$

$\Big\}$ satisfies $Ax = b$, $x \geq 0$

if $x_B \geq 0$

then we have a basic feasible solution

- basic direction d (also feasible)
  - jth basic direction means we are considering adding j to the basis
  - Remember, we need $Ad = 0$ and we don't want to add other non-basic variables to the basis

Set $d_j = 1$

Set $d_k = 0$ for all $k \neq j$, $k \notin B$

Then d_B will be set by solving the linear system

$$Ad = 0$$

$$Ad = \sum_{i=1}^{n} A_i d_i = A_B d_B + A_j = 0 \implies A_B d_B = -A_j$$

$$d_B = -A_B^{-1} A_j$$

Cost Improvement

$$\underbrace{c^T(x+\theta d)}_{\text{new cost}} - \underbrace{c^T x}_{\text{old cost}} = \overbrace{\theta\, c^T d}^{\text{cost improvement}}$$

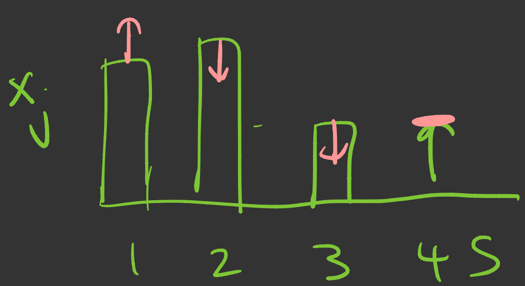
$c^T d$ is the reduced cost associated with this d

$$\bar{c}_j = c^T d = \sum_{i=1}^{n} c_i d_i = c_j + c_B^T d_B = c_j - c_B^T A_B^{-1} A_j$$

Example: $m=3,\ n=5$ $\quad A = \begin{pmatrix} & & \end{pmatrix}$

$B = \{1, 2, 3\}$

want to add 4 to the basis



$A_B x_B = b$

$$(A_1\ A_2\ A_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = b$$

- if $\bar{c} \geq 0$ then x is optimal
- if some $\bar{c}_j < 0$ then we can decrease the cost by bringing x_j into the basis

  $\theta^* = \max\{\theta \mid \theta \geq 0,\ x + \theta d \geq 0\}$

  - if $d \geq 0$ in this case, then $\theta^* = \infty$, LP is unbdd

Optimality conditions

(a) a feasible solution x is optimal $\Leftrightarrow c^T d \geq 0$ for any feasible direction at x

$\Rightarrow$ proof by contradiction

Suppose $\exists$ a feas. dir. d s.t. $c^T d < 0$

Then $\exists\ \theta > 0$ s.t. $x + \theta d$ is a feas. soln

Thus $c^T x > c^T(x + \theta d)$ contradiction bc x not optimal

$\Leftarrow$ Proof by contradiction

Suppose $c^T d \geq 0$ for every feas. dir. d at x

Let $y \in P$

Then let $d = y - x$

Thus $c^T d \geq 0 \Rightarrow c^T(y - x) \geq 0$

$c^T y \geq c^T x$ so x is optimal

(b) a feasible solution x is the unique optimal solution if and only if $c^T d > 0$ for every nonzero feasible direction d at x