There is one more precept this semester (excluding this one)

No precept next week (April 20th, April 21st)

The last precept will be on (April 27th, April 28th)

# Midterm - linear optimization.

- Piecewise linear optimization.
- Geometry and the simplex method
- Duality and Sensitivity analysis.
- Network optimization
- Interior-point method

Midterm will not have coding exercises, but you can use python to solve linear system

first part: brief review.

second part: P6, $\boxed{\text{P7}}$ of midterm exercises.

## Piecewise linear optimization.

$\|x\|_1 = \sum_i |x_i|$ $\|x\|_\infty = \max_i |x_i|$.

Turning vector norm problems in LPs.

for example, formulate

minimizing $\|Ax-b\|_1$ or $\|Ax-b\|_\infty$

in LPs.

$\triangle$

# Geometry and simplex method.

## Standard form.

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0. \end{aligned}$$

- Extreme points
- Basic feasible solution
- Standard form
- Feasible directions
- Reduce costs
- Optimality conditions
- An iteration of the simplex method
- Phase I / Phase II
- Degeneracy / cycling

# Duality and sensitivity analysis.

## Inequality form LP.

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \end{aligned}$$

$$\boxed{\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{s.t.} \quad & A^T y + c = 0 \\ & y \geq 0. \end{aligned}}$$

## Standard form LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & -b^T y \\ \text{s.t.} \quad & A^T y + c \geq 0 \end{aligned}$$

- Weak and strong duality
- Relationship between $p^*$ and $d^*$.
- Complementary slackness

- Optimality condition
- Farkas lemma.

duality gap

$c^T x + b^T y$

# Sensitivity Analysis.

Goal: extract information from $x^*, y^*$ about their sensitivity with respect to changes in problem data.

- Adding variables
- Adding constraints
- Global sensitivity
- Local sensitivity

# Network flow optimization.

- Arc-node incidence matrix
- Minimum cost network flow problem
- Integrety theorem.

# Interior-point method.

- Newton's method
- LP as a root finding problem
- The central path.
- Mehrotra predictor-corrector algorithm

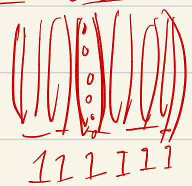
# Ex 6.

$\boxed{P6}$, P7 midterm exercise.

column-stochastic matrix $P \in \mathbb{R}^{n\times n}$

$$\begin{cases} p_{ij} \ge 0 & i,j=1,\dots,n. \\ \sum_{i=1}^{n} p_{ij} = 1 & j=1,\dots,n. \quad \boxed{①} \end{cases}$$

P



① $\Longleftrightarrow$ $P^T \cdot \mathbb{1} = \mathbb{1}$

(a) Show $(P^T x)_i \le \max_i x_i = x_{max}$

$$(P^T x)_i = \sum_{j=1}^{n} p_{ji} \cdot \boxed{x_j} \le \sum_{j=1}^{n} p_{ji} \boxed{x_{max}}$$

$$= x_{max} \cdot \left(\sum_{j=1}^{n} p_{ji}\right) = x_{max}$$

$x_{max} = \max_i x_i$ $(P^T x)_i$ ith entry of $P^T x$

$P \in \mathbb{R}^{n\times n}$ $x \in \mathbb{R}^{n\times 1}$ $P^T x \in \mathbb{R}^{n\times 1}$

(b) Using LP duality show that there exists a $y \in \mathbb{R}^n$ such that (a)

$$Py = y, \quad y \geq 0, \quad \mathbb{1}^T y = 1.$$

Consider the following LP.

$$\begin{aligned} \text{minimize} \quad & -\mathbb{1}^T x = -1 \to -\infty \quad 0 \\ \text{s.t.} \quad & (P-I)x = 0 \\ & x \geq 0. \end{aligned}$$

$-\mathbb{1}^T x = -1$

$Px - x = 0$

$\Leftrightarrow Px = x$

This LP is feasible as ($x=0$ is a feasible solution). Its dual is written as.

$A^T y + c$

$$\begin{aligned} \text{maximize} \quad & 0 \\ \text{s.t.} \quad & (P-I)^T y - \mathbb{1} \geq 0. \end{aligned}$$

$\boxed{y=0}$

dual infeasible + primal feasible $\Rightarrow$ primal unbounded

For any given $y$, we show.

$(P-I)^T y + \mathbb{1} \geq 0$ does not hold.

Let $j = \arg\max_{i=1}^{n} y_i$. $j$ index of largest entry of $y$.

$\left((P-I)^T y - \mathbb{1}\right)_j = (P^T y)_j - y_j - 1$ $(P^T y - y - \mathbb{1})_j$.

$\leq \underbrace{y_{max} - y_j}_{0} + 1 = -1$ $y_j = y_{max}$

(a)

Therefore, the dual LP is infeasible + primal feasible

$\Rightarrow$ primal is unbounded.

Thus, there exists $x$ such that

$(P-I)x = 0$, $x \geq 0$, $-\mathbb{1}^T x = -1$.

$$\Rightarrow Px = x,\ x \geq 0,\ \mathbb{1}^T x = 1.$$

$$Py = y,\ y \geq 0,\ \mathbb{1}^T y = 1$$

$\mathbb{P}$ b

$\Delta$

Ex 7 Consider the problem

$$\text{minimize } -2x_1 - x_2$$
$$\text{s.t. } x_1 - x_2 \overset{+s_1}{\leq} 2$$
$$x_1 + x_2 \overset{+s_2}{\leq} 6.$$
$$x_1, x_2 \geq 0.$$

$$\boxed{\begin{array}{c}\text{minimize } c^T x \\ Ax = b \\ x \geq 0.\end{array}}$$

(a) Convert the problem into standard form and construct a basic feasible solution at $(x_1, x_2) = (0, 0)$.

Standard form

$$\text{minimize} \quad -2x_1 - x_2$$

$$\text{s.t.} \quad x_1 - x_2 + s_1 = 2$$

$$x_1 + x_2 + s_2 = 6$$

$$x_1, x_2, s_1, s_2 \geq 0.$$

basic feasible solution $(0, 0, 2, 6)$

$= (\bar{x}_1, \bar{x}_2, s_1, s_2)$

(b) Carry out one iteration of the simplex method. Report the values for $\bar{c}$, $\theta$, $d$, and $x$ (including any additional variables you added to make it into standard form)

$x = (x_1, x_2, s_1, s_2)$

$p = (A_B^T)^{-1} c_B$

$\bar{c} = c - A^T p$ where $A_B^T p = c_B$

$B = (3,4)$ $c_B = (0,0)$ $A_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ $p = (0,0)$ and $\bar{c} = (-2,-1,0,0)$

So we could let either $x_1$ or $x_2$ enter the basis; we take $x_1$ by Bland's rule. The search direction $d$ solves the system $A_B d_B = -A_j$

Here $A_j = (1,1)$ so $d = (1,0,-1,-1)$

We see $\theta = 2$ to force $S_1$ to leave the basis. The new $x$ is $(2,0,0,4)$

$$
\begin{array}{rl}
(0,0,2,6) & \leftarrow x \\
+\, d\,(1,0,-1,-1) \times 2 & \theta \\
\hline
(2,0,0,4) & \leftarrow x_{new}
\end{array}
$$

(1,2,0,0,4)

(c) Compute the next $\bar{c}$, what would you do if you wanted to complete the simplex method?

Now $B=(1,4)$ $\quad c_B=(-2,0)$

$(A_B^T)^{-1}$ $\quad A_B=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ so $p=(-2,0)$ and

$\bar{c}=(0,-3,2,0)$ Since $\bar{c}_2<0$, we know that we need at least one more simplex iteration.

codes are available on Canvas

02_midterm_exam_exercises_sol.pdf

P7.