- Formulate problems as LPs.
- Equivalence theorem
- Minimum cost trading to achieve target sector exposures.

Midterm 25 points x 4.

- Least squares
- Multi-obj / constrained LS.
- Linear optimization modeling
- Linear optimization geometry. (equivalence theorem)

no coding / no simplex

you are allowed to use all course materials (lecture slides, precept, homework, books)

no internet no communications

no precept next week. !

Q1 Formulate problems as LPs

(a) $A \in \mathbb{R}^{m \times n}$ $b \in \mathbb{R}^m$ find the vector $x \in \mathbb{R}^n$ that minimizes.

$$\sum_{i=1}^{m} \max\{0, a_i^T x + b_i\}. \quad (1)$$

$$c_i = \max\{0, a_i^T x + b_i\}.$$

$$\begin{cases} c_i \geq 0 \\ c_i \geq a_i^T x + b_i \end{cases}$$

minimize $\mathbb{1}^T \cdot c$. $\quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$

$$
\begin{aligned}
\underset{(c,x)}{\text{minimize}} \quad & (\mathbb{1}^T, 0)\begin{pmatrix} c \\ x \end{pmatrix} \qquad \boxed{\mathbb{1}^T c} \qquad ② \\
\text{s.t.} \quad & \begin{cases} c_i \geq 0 \\ c_i - a_i^T x \geq b_i \end{cases}
\end{aligned}
$$

Let x_1 minimize ①

(c^*, x_2) minimize ② $\qquad (c, x)$

By definition of ①.

$$\sum \max\{0, a_i^T x_1 + b\} \leq \sum \max\{0, a_i^T x_2 + b\}$$

for any c satisfies $\begin{cases} c_i \geq 0 \\ c_i - a_i^T x_1 \geq b_i \end{cases}$

$\boxed{(c, x_1)}$ is a feasible solution of ②

$$\Rightarrow \quad \mathbb{1}^T c \geq \mathbb{1}^T c^*$$

$$(\mathbb{1}^T, 0)\begin{pmatrix} c \\ x_1 \end{pmatrix} \geq (\mathbb{1}^T, 0)\begin{pmatrix} c^* \\ x_2 \end{pmatrix}$$

$$\underset{(c,x)}{\text{minimize}} \quad (\mathbb{1}^T, 0)\begin{pmatrix} c \\ x \end{pmatrix}$$

$\boxed{2}$

$$\text{s.t.} \quad \begin{cases} c_i \geq 0 \\ c_i - a_i^T x \geq b_i \end{cases}$$

$$c_i \geq \max\{0, a_i^T x + b_i\}.$$

$$\Rightarrow \quad \mathbb{1}^T c = \sum \max\{0, a_i^T x_1 + b\}.$$

$$\mathbb{1}^T c^* = \sum \max\{0, a_i^T x_2 + b\}.$$

$$\underline{\sum \max\{0, a_i^T x_1 + b\} \geq \sum \max\{0, a_i^T x_2 + b\}}$$

$$\leq \quad \Delta$$

$$\Rightarrow \underline{\sum \max\{0, a_i^T x_1 + b\}}$$

$$\underline{= \sum \max\{0, a_i^T x_2 + b\}.}$$

Q2 equivalence theorem.

$P = \{x \mid Ax \le b\}$.

① Show that every vertex is an extreme point.

vertex: if $\exists c$ such that. x is. the unique optimum of

$$\begin{aligned} \text{minimize} \quad & c^T y \\ \text{s.t.} \quad & y \in P. \end{aligned}$$

extreme point: x not on a straight line between any other points in the set. P

vertex $\Rightarrow$ extreme point.

(proof by contradiction)

Assume x is vertex, but not extreme point.

$\exists\, y, z \neq x \quad y, z \in P \quad x = \lambda y + (1-\lambda) z$

vertex $\exists\, c$. such that

$c^T x < c^T w \quad \forall\, w \in P$

$c^T y,\ c^T z > c^T x \qquad c^T x < c^T y,\ c^T z.$

$\Rightarrow \quad c^T x = \lambda c^T y + (1-\lambda) c^T z > c^T x$

Contradiction!

$I(X) = \{1, 2, 4, 6\}$
$\{a_1, a_2, a_4, a_6\}$

$P = \{x \mid Ax \leq b\}$
$a_i^T x \leq b_i$

② extreme point $\Rightarrow$ feasible solution

$\{a_i \mid i \in I(X)\}$ has n linear independent vectors.

$\boxed{I(X)} = \{ i \in \{1, \cdots, m\} \mid a_i^T x = b_i \}.$

(proof by contradiction)

Assume x is extreme point but not basic feasible solution.

Then $\{a_i \mid i \in I(x)\}$ does not span $\mathbb{R}^n$ $\Rightarrow$ $\exists d \in \mathbb{R}^n$ $a_i^T d = 0$ $i \in I(x)$

Let $y = x + \varepsilon d$ $z = x - \varepsilon d$. $a_i^T d = 0$.

$$\begin{cases} \forall i \in I(x) & a_i^T x = b_i \Rightarrow \begin{cases} a_i^T y = b_i \\ a_i^T z = b_i \end{cases} \\ \forall i \notin I(x) & \boxed{a_i^T x < b_i}. \quad \varepsilon \text{ small enough} \\ \Rightarrow a_i^T y < b_i \quad a_i^T z < b_i \end{cases}$$

$x = 0.5y + 0.5z$. and $y, z \in P$.

$\Rightarrow$ x not extreme point

contradiction!

$a_i^T y = a_i^T x + (\varepsilon - a_i^T d) < b_i$

b_i

εn $a_i^T x$ $a_i^T x$ b_i

Q3. $h^{curr} \to$ n-vector (portfolio) with entries giving the dollar value invested in the n assets.

total value $= \mathbb{1}^T h^{curr}$ Δ

we seek a new portfolio h.

s.t. $\mathbb{1}^T h = \mathbb{1}^T h^{curr}$

same total ~~vector~~ value.

m - industry sectors.

(Tech / bio).

s m-vector denotes the sector exposures (dollar value) to the m sectors. $s = Sh$.

S is $m \times n$ matrix

$$\begin{cases} S_{ij} = 1 & \text{asset } j \text{ in sector } i \\ S_{ij} = 0 & \text{asset } j \text{ not in sector } i. \end{cases}$$

The new portfolio must have a given sector exposure s^{des}.

Among all portfolios that have the same value as our current portfolio h^{curr} and achieve the desire exposures we wish to minimize the trading cost.

$$\sum_{i=1}^{n} k_i \left(h_i - \boxed{h_i^{curr}}\right)^2 .$$

$k_i > 0.$

Explain how to find h. LP.

give KKT equations.

$$h^{curr} = (v_1, v_2, \ldots, v_n) \cdot n.$$

$$h = (h_1, h_2, \ldots, h_n)\ n$$

constrains: $\begin{cases} Sh = s^{des} \\ \mathbb{1}^T h = \mathbb{1}^T h^{curr} \end{cases}$

minimize $\sum_{i=1}^{n} k_i (h_i - h_i^{curr})^2$

Let $K = \begin{pmatrix} k_1 & & \\ & \ddots & \\ & & k_n \end{pmatrix}$

$$\mathbb{1}^T h^{curr} = V$$

LP:

$$\underset{h}{\text{minimize}} \; \| Kh - Kh^{curr} \|_2 :$$

$$\begin{bmatrix} S \\ \mathbb{1}^T \end{bmatrix} h = \begin{bmatrix} s^{des} \\ v \end{bmatrix}$$

KKT equation:

$\|Ax-b\|^2$
s.t. $Cx=d$

$$A = \underline{\underline{K}} \quad . \quad \underline{b} = \underline{Kh^{curr}}$$

$$C = \begin{bmatrix} S \\ \mathbb{1}^T \end{bmatrix} \quad d = \begin{bmatrix} s^{des} \\ v \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2A^TA & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} x^* \\ z \end{bmatrix} = \begin{bmatrix} 2A^Tb \\ d \end{bmatrix}$$

$\underline{x^*, t}$