


Agenda

- Converting LPs
- Steel company operations
- The moment problem
- Chebyshev center

Logistics

- HW4 out, due Friday Feb 25 at 9pm
- Midterm 1 Thurs March 3 in class
- Spring break the following week
- A bit of a break until HW5 due
 - Friday March 25

Converting LPs

we'd like to convert $\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_1$
 s.t. $\|\mathbf{x}\|_\infty \leq k$

$$\begin{aligned} A &\in \mathbb{R}^{m,n} \\ b &\in \mathbb{R}^m \\ x &\in \mathbb{R}^n \\ k &\in \mathbb{R} \end{aligned}$$

into 2 forms

form (1)

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{Cx} = \mathbf{d} \end{array}$$

form (2)

$$\begin{array}{ll} \min_{\mathbf{x}} & \mathbf{c}^T \mathbf{x} \\ \text{s.t.} & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

form (1)

$\|\mathbf{x}\|_\infty \leq k \rightarrow x_i \leq k, -x_i \leq k \quad i=1, \dots, n$

Let s be a proxy for $\|\mathbf{Ax} - \mathbf{b}\|_1$

$s \geq \mathbf{Ax} - \mathbf{b}, s \geq -\mathbf{Ax} + \mathbf{b}$ since it's a minimization problem

$$\left\{ \begin{array}{l} \min_{\mathbf{x}, s} \mathbf{1}^T s \\ \text{s.t. } \mathbf{x} \leq \mathbf{1}k \\ -\mathbf{x} \leq \mathbf{1}k \\ s \geq \mathbf{Ax} - \mathbf{b} \\ s \geq -\mathbf{Ax} + \mathbf{b} \end{array} \right.$$

can write it in block form

form (1)

$$\left\{ \begin{array}{l} \min_{\mathbf{x}, s} \mathbf{1}^T s \\ \text{s.t. } \mathbf{x} \leq \mathbf{1}k \\ -\mathbf{x} \leq \mathbf{1}k \\ \mathbf{Ax} - \mathbf{s} \leq \mathbf{b} \\ -\mathbf{Ax} - \mathbf{s} \leq -\mathbf{b} \end{array} \right.$$

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{I} & \mathbf{0} \\ \mathbf{A} & -\mathbf{I} \\ -\mathbf{A} & -\mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{s} \end{pmatrix} \leq \begin{pmatrix} \mathbf{1}k \\ \mathbf{1}k \\ \mathbf{b} \\ -\mathbf{b} \end{pmatrix}$$

$$\tilde{\mathbf{A}} \tilde{\mathbf{x}} \leq \tilde{\mathbf{b}}$$

Steel Company Operations

- Steel company can produce bands and coils

- Goal is To maximize revenue

	Production rate (tons/hr)	revenue (\$/ton)	upper bounds (tons)
Bands	200	25	6 000
Coils	140	30	4000

- There are 40 hours of production time this week
- Decide how many tons of bands and coils should be produced to maximize revenue
- decision variables b, c represent # of hours allocated to produce bands and coils respectively

$$\max_{b,c} \underbrace{200(25)}_{5000} b + \underbrace{140(30)}_{4200} c$$

$$\text{s.t. } b + c \leq 40$$

$$b, c \geq 0$$

$$200 b \leq 6000 \rightarrow (b \leq 30)$$

$$140 c \leq 4000 \rightarrow (c \leq 28.6)$$

- we see that it's better to produce as many bands as possible since $5000 > 4200$

- Set $b^* = 30, c^* = 10$

The moment problem

- Suppose that Z is a random variable taking values in the set $\{0, 1, \dots, K\}$ with probabilities p_0, \dots, p_K
- We are given $E[Z] = \sum_{i=0}^K p_i i$

$$E[Z^2] = \sum_{i=0}^K p_i i^2$$

- We would like to obtain upper and lower bounds on the 4th moment: $E[Z^4] = \sum_{i=0}^K i^4 p_i$
- Show how LPs can be used to approach this problem

- Decision variables are p_0, \dots, p_K

- constraints: $E[Z], E[Z^2], p \geq 0, \mathbf{1}^T p = 1$

- Use 1 LP to obtain a lower bound and another LP to obtain an upper bound

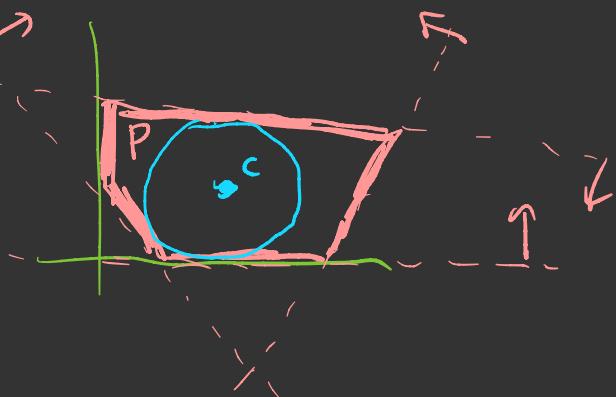
$$\begin{aligned} LB &= \min_p \sum_{i=1}^K i^4 p_i \\ \text{s.t. } & p \geq 0 \\ & \mathbf{1}^T p = 1 \\ & E[Z] = \sum_{i=1}^K i p_i \\ & E[Z^2] = \sum_{i=1}^K i^2 p_i \end{aligned}$$

$$\begin{aligned} UB &= \max_p \sum_{i=1}^K i^4 p_i \\ \text{s.t. } & p \geq 0 \\ & \mathbf{1}^T p = 1 \\ & E[Z] = \sum_{i=1}^K i p_i \\ & E[Z^2] = \sum_{i=1}^K i^2 p_i \end{aligned}$$

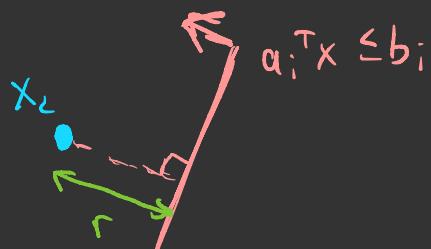
Chebychev Center

- Consider a set P described by linear inequality constraints

$$P = \{x \in \mathbb{R}^n \mid a_i^T x \leq b_i, i=1, \dots, m\}$$
 ← called a polyhedron
- Goal: find a ball with the largest possible radius which is entirely contained within the set P
- Provide an LP formulation of this problem



Look at a particular constraint



- $a_i^T(x_c + v) \leq b_i$ for all $\{v \mid \|v\|_2 \leq r\}$
 - only need to look at the worst case
- $$\max_v a_i^T(x_c + v) \leq b_i$$

The worst-case v is parallel to a_i

$$v^* = \frac{a_i}{\|a_i\|_2} r$$

$$a_i^T x_c + \frac{a_i^T a_i}{\|a_i\|_2} r \leq b_i$$

$$a_i^T x_c + \|a_i\|_2 r \leq b_i$$

Now write it in the form of an LP

$$\begin{array}{ll}\max_{x_c, r} & r \\ \text{s.t. } & a_i^T x_c + \|a_i\|_2 r \leq b_i \quad i=1, \dots, m\end{array}$$

form
(1)

$$\left\{ \begin{array}{l} \min_{x,s} \mathbf{1}^T s \\ \text{s.t. } x \leq \mathbf{1}k \\ -x \leq \mathbf{1}k \\ Ax - s \leq b \\ -Ax - s \leq -b \end{array} \right\} \xrightarrow{\text{add in } y_1, y_2, y_3, y_4} \text{Slack vars}$$

$$\begin{array}{ll} \min_{x,s,y_1,\dots,y_4} & \mathbf{1}^T s \\ \text{s.t. } & x + y_1 = \mathbf{1}k \\ & -x + y_2 = \mathbf{1}k \\ & Ax - s + y_3 = b \\ & -Ax - s + y_4 = -b \\ & y_1, \dots, y_4 \geq 0 \end{array}$$

We just need to have all variables with a non-negativity constraint

- Let $x = x^+ - x^-$, $x^+ \geq 0, x^- \geq 0$
- $s \geq 0$ already, so we can simply add this redundant constraint

$$\left\{ \begin{array}{l} \min_{x^+, x^-, s, y_1, y_2, y_3, y_4} \mathbf{1}^T s \\ \text{s.t. } x + y_1 = \mathbf{1}k \\ -x + y_2 = \mathbf{1}k \\ Ax - s + y_3 = b \\ -Ax - s + y_4 = -b \\ x^+, x^-, s, y_1, y_2, y_3, y_4 \geq 0 \end{array} \right\} \text{form (2)}$$