Agenda

- Duality Review
  - Motivation example
  - Weak/Strong duality, duality for LPs
  - Farkas Lemma
- Taking the dual example
- Dual of 1 norm example

Logistics

- HW 5 out, due Friday March 25 at 9pm
- HW 6 due Friday April 1 at 9pm
- Midterm 2 on Tuesday April 12

Duality motivation / obtaining lower bounds

- We want to get a lower bound for

$$\min_x \; x_1 + 3x_2$$
$$\text{s.t. } x_1 + x_2 \geq 2$$
$$x_2 \geq 1$$
$$x_1 - x_2 \geq 3$$

- One idea: add the constraints, match cost

$$\begin{aligned} & y_1(x_1 + x_2 \geq 2) \\ + & y_2(\quad\;\; x_2 \geq 1) \\ + & y_3(x_1 - x_2 \geq 3) \\ \hline & (y_1 + y_3)x_1 + (y_1 + y_2 - y_3)x_2 \geq \underbrace{2y_1 + y_2 + 3y_3}_{\text{think: dual objective}} \end{aligned}$$

match these → cost: $x_1 + 3x_2$

$$\Rightarrow \underbrace{y_1 + y_3 = 1, \quad y_1 + y_2 - y_3 = 3, \; y \geq 0}_{\text{think: dual constraints}}$$

$y = (1,2,0) \rightarrow$ bound $= 4$

$y = (0,4,1) \rightarrow$ bound $= 7$

Farkas Lemma

Given A and b, one of the following 2 statements true

1. $\exists x$ s.t. $Ax=b$, $x\geq 0$
2. $\exists y$ s.t. $A^Ty\geq 0$, $b^Ty<0$

- 1 and 2 can't both be true : simple

  Assume $x\geq 0$, $Ax=b$, $y^TA\geq 0$

  $$\Rightarrow y^Tb = \underbrace{y^TA}_{\geq 0}\underbrace{x}_{\geq 0} \geq 0 \quad \Rightarrow y^Tb\geq 0$$

- 1 and 2 can't both be false : harder, need duality

Primal	Dual
$\min\ 0$	$\max\ -b^Ty$
s.t. $Ax=b$, $x\geq 0$	s.t. $A^Ty\geq 0$

Recall

Dual \ Primal	$p^*=\infty$	p^* finite	$p^*=-\infty$
$d^*=\infty$	primal inf dual unb		
d^* finite		opt. vals equal	
$d^*=-\infty$	exception		primal unb. dual inf.

dual: $y=0$ always feas.

$\Rightarrow d^*=0$ or $d^*=\infty$

$d^*=0$: $\Rightarrow$ 1. is true

$d^*=\infty$ alt 2:

$\exists y$ s.t. $A^Ty\geq 0$, $b^Ty<0$

2. holds

Dual of std LP

$$\min_x \; c^T x \quad \text{s.t. } Ax = b, \; x \geq 0 \qquad\qquad \max_y \; -b^T y \quad \text{s.t. } A^T y + c \geq 0$$

Inequality form

$$\min_x \; c^T x \quad \text{s.t. } Ax \leq b \qquad\qquad \max_y \; -b^T y \quad \text{s.t. } A^T y + c = 0, \; y \geq 0$$

Example

$$\min_x \; 3x_1 + 4x_2 \quad \text{s.t. } x_1 + x_2 \geq 5, \; 2x_1 + x_2 \geq 6, \; x \geq 0$$

Pattern match!

$$A = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix} \qquad A^T = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$\min_x \; 3x_1 + 4x_2 \quad \text{s.t. } -x_1 - x_2 \leq -5, \; -2x_1 - x_2 \leq -6, \; x \geq 0 \;\; \xrightarrow{\text{dual}} \;\; \max_y \; 5y_1 + 6y_2 \quad \text{s.t. } -y_1 - 2y_2 + 3 = 0, \; -y_1 - y_2 + 4 = 0, \; y \geq 0$$

Other way: derive the dual

$$\min_x \; c^T x \quad \text{s.t. } Ax \geq b$$

formulate Lagrangian: $L(x, y) = c^T x + y^T(Ax - b)$ s.t. $y \geq 0$

dual fn: $g(y) = \min_x c^T x + y^T(Ax - b)$ $\{\mathrm{dom}(g) : y \geq 0\}$

$$= \min_x (c + A^T y)^T x - b^T y$$

$$= \begin{cases} -\infty & \text{if } c + A^T y \neq 0 \\ -b^T y & \text{if } c + A^T y = 0 \end{cases}$$

Dual prob: $\max_y g(y)$ s.t. $y \geq 0$ $\rightarrow$ $\max_y -b^T y$ s.t. $y \geq 0$, $c + A^T y = 0$

Dual of 1 norm problem

$$\min_x \; c^T x \quad \text{s.t. } \|Ax+b\|_1 \le 1$$

a. formulate this LP inequality form

$$\min_x \; c^T x$$
$$\text{s.t. } s_1 + \dots + s_m \le 1$$
$$s \ge Ax+b$$
$$s \ge -Ax-b$$

$$\min_x \; c^T x$$
$$\text{s.t. } s_1 + \dots + s_m \le 1$$
$$Ax - s \le -b$$
$$-Ax - s \le b$$

b. derive the dual LP and show that it's equivalent to

$$\max_z \; b^T z - \|z\|_\infty$$
$$\text{s.t. } A^T z + c = 0$$

Recall

$$\min_x \; c^T x \quad Ax \le b \quad \overset{\text{dual}}{\longleftrightarrow} \quad \max_y \; -b^T y \quad \text{s.t. } A^T y + c = 0, \; y \ge 0$$

$$\begin{pmatrix} x \\ s \end{pmatrix} = \tilde{x}$$

$$\min_{\tilde{x}} \; \tilde{c}^T \tilde{x}$$
$$\text{s.t. } \tilde{A}\tilde{x} \le \tilde{b}$$

$$\tilde{A} = \begin{pmatrix} A & -I \\ -A & -I \\ 0 & \mathbb{1}^T \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} -b \\ b \\ 1 \end{pmatrix}$$

$$\tilde{c} = \begin{pmatrix} c \\ 0 \end{pmatrix}$$

dual:

$$\max_{\tilde{y}} \; -\tilde{b}^T \tilde{y}$$
$$\text{s.t. } \tilde{y} \ge 0$$
$$\tilde{A}^T y + \tilde{c} = 0$$

$$\tilde{y} = \begin{pmatrix} y \\ z \\ w \end{pmatrix}$$

$$\max_y \; b^T y - b^T z - w$$
$$\text{s.t. } y, z, w \ge 0$$

$$\begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} A^T & -A^T & 0 \\ -I & -I & \mathbb{1} \end{pmatrix} \begin{pmatrix} y \\ z \\ w \end{pmatrix} = 0$$

$$\max_{y,z,w} \quad b^T y - b^T z - w$$

s.t. $y, z, w \geq 0$

$$\begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} A^T & -A^T & 0 \\ -I & -I & \mathbb{1} \end{pmatrix} \begin{pmatrix} y \\ z \\ w \end{pmatrix} = 0 \quad \Big\} \begin{array}{l} A^T y - A^T z + c = 0 \\ -y - z + \mathbb{1} w = 0 \end{array}$$

$$\max_{y,z,w} \quad b^T (y - z) - w$$

Let $\tilde{z} = y - z$

s.t. $y, z, w \geq 0$

$A^T(y-z) + c = 0$

$y + z = \mathbb{1} w$

$\rightarrow$ $\boxed{\|\tilde{z}\|_\infty \leq w}$ because the max absolute difference between 2 non-neg. vectors can't be larger than w

$\rightarrow$ also uses the fact that $\underbrace{y_i}_{\geq 0} + \underbrace{z_i}_{\geq 0} = w$ $\Rightarrow y_i \leq w,\ z_i \leq w \quad \forall i$

$\Rightarrow |y_i - z_i| \leq w \quad \forall i$

$\Rightarrow \|\tilde{z}\|_\infty \leq w$

$\rightarrow$ In this case, we will get $\boxed{\|\tilde{z}\|_\infty = w}$ since at least one element of y or z is zero. If this weren't true, then y, z, w wouldn't be optimal because we could decrease y and z by the same amount to improve the objective

$$\max_{\tilde{z}} \quad b^T \tilde{z} - \|\tilde{z}\|_\infty$$

s.t. $A^T \tilde{z} + c = 0$

✓

c. give a direct argument that whenever x is primal feasible and z is dual feasible, $c^T x \geq b^T z - \|z\|_\infty$

2 arguments

1. weak duality
2. Recall primal and dual

$$\min_x \; c^T x \quad \text{s.t. } \|Ax+b\|_1 \leq 1 \qquad\qquad \max_z \; b^T z - \|z\|_\infty \quad \text{s.t. } A^T z + c = 0$$

Use Cauchy-Schwarz $u^T v \leq \|u\|_1 \|v\|_\infty$

$$(Ax+b)^T z \leq \underbrace{\|Ax+b\|_1}_{\leq 1} \|z\|_\infty$$

$$\Rightarrow (Ax+b)^T z \leq \|z\|_\infty$$

$$b^T z - \|z\|_\infty \leq -x^T A^T z = c^T x$$