Agenda

- Duality Review
  - Motivation example
  - Weak/Strong duality, duality for LPs
  - Farkas Lemma
- Taking the dual example
- Dual of 1 norm example

Logistics

- HW 6 out, due Friday April 2 at 9pm
- HW 7 due Friday April 9 at 9pm
- Midterm 2 on Thursday April 15

Duality motivation / obtaining lower bounds

- We want to get a lower bound for

$$\min_x \; x_1 + 3x_2$$
$$\text{s.t. } x_1 + x_2 \geq 2$$
$$x_2 \geq 1$$
$$x_1 - x_2 \geq 3$$

- One idea: add the constraints, match cost

$$
\begin{aligned}
& y_1(x_1 + x_2 \geq 2) \\
+ \; & y_2(\qquad x_2 \geq 1) \\
+ \; & y_3(x_1 - x_2 \geq 3) \\
\hline
& (y_1 + y_3)x_1 + (y_1 + y_2 - y_3)x_2 = \underbrace{2y_1 + y_2 + 3y_3}_{\text{think: dual objective}}
\end{aligned}
$$

match these → cost: $x_1 + 3x_2$

$$\Rightarrow \underbrace{y_1 + y_3 = 1, \quad y_1 + y_2 - y_3 = 3, \; y \geq 0}_{\text{think: dual constraints}}$$

$y = (1, 2, 0) \rightarrow$ bound $= 4$

$y = (0, 4, 1) \rightarrow$ bound $= 7$

Farkas Lemma

Given A and b, one of the following 2 statements true

1. $\exists x$ s.t. $Ax = b$, $x \geq 0$
2. $\exists y$ s.t. $A^T y \geq 0$, $b^T y < 0$

- 1 and 2 can't both be true : simple

  Assume $x \geq 0$, $Ax = b$, $y^T A \geq 0$

  $$\Rightarrow y^T b = \underbrace{y^T A}_{\geq 0} \underbrace{x}_{\geq 0} \geq 0 \quad \Rightarrow y^T b \geq 0$$

- 1 and 2 can't both be false : harder, need duality

Primal	Dual
$\min 0$	$\max -b^T y$
s.t. $Ax = b$	s.t. $A^T y \geq 0$
$x \geq 0$	

Recall

		Primal		
		$p^* = \infty$	p^* finite	$p^* = -\infty$
Dual	$d^* = \infty$	primal inf, dual unb		
	d^* finite		opt. vals equal	
	$d^* = -\infty$	exception		primal unb., dual inf.

dual: $y = 0$ always feas.

$\Rightarrow d^* = 0$ or $d^* = \infty$

$d^* = 0$: alt 1: $A^T y \geq 0 \Rightarrow b^T y \geq 0$

sys. 2 can't hold

$d^* = \infty$ alt 2:

$\exists y$ s.t. $A^T y \geq 0$, $b^T y < 0$

then $p^* = \infty$ primal infeas.

sys. 1 can't hold

Dual of std LP

$$\min_x \; c^T x \quad \text{s.t. } Ax = b, \; x \geq 0 \qquad\qquad \max_y \; -b^T y \quad \text{s.t. } A^T y + c \geq 0$$

Inequality form

$$\min_x \; c^T x \quad \text{s.t. } Ax \leq b \qquad\qquad \max_y \; -b^T y \quad \text{s.t. } A^T y + c = 0, \; y \geq 0$$

Example

$$\min_x \; 3x_1 + 4x_2 \quad \text{s.t. } x_1 + x_2 \geq 5, \; 2x_1 + x_2 \geq 6, \; x \geq 0$$

Pattern match!

$$A = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix} \qquad A^T = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$$

$$\min_x \; 3x_1 + 4x_2 \quad \text{s.t. } -x_1 - x_2 \leq -5, \; -2x_1 - x_2 \leq -6, \; x \geq 0 \quad \xrightarrow{\text{dual}} \quad \max_y \; 5y_1 + 6y_2 \quad \text{s.t. } -y_1 - 2y_2 + 3 = 0, \; -y_1 - y_2 + 4 = 0, \; y \geq 0$$

Other way: derive the dual

$$\min_x \; c^T x \quad \text{s.t. } Ax \geq b$$

formulate Lagrangian: $L(x,y) = c^T x + y^T(Ax - b)$ s.t. $y \geq 0$

dual fn: $g(y) = \min_x c^T x + y^T(Ax - b)$ $\{\text{dom}(g) : y \geq 0\}$

$$= \min_x (c + A^T y)^T x - b^T y$$

$$= \begin{cases} -\infty & \text{if } c + A^T y \neq 0 \\ -b^T y & \text{if } c + A^T y = 0 \end{cases}$$

Dual prob: $\max_y g(y)$ s.t. $y \geq 0$ $\rightarrow$ $\max_y -b^T y$ s.t. $y \geq 0$, $c + A^T y = 0$

Dual of 1 norm problem

$$\min_x \; c^T x \quad \text{s.t. } \|Ax+b\|_1 \le 1$$

a. formulate this LP inequality form

$$\min_x \; c^T x$$
$$\text{s.t. } s_1 + \dots + s_m \le 1$$
$$s \ge Ax + b$$
$$s \ge -Ax - b$$

$$\min_x \; c^T x$$
$$\text{s.t. } s_1 + \dots + s_m \le 1$$
$$Ax - s \le -b$$
$$-Ax - s \le b$$

b. derive the dual LP and show that it's equivalent to

$$\max_z \; b^T z - \|z\|_\infty$$
$$\text{s.t. } A^T z + c = 0$$

Recall

$$\min_x \; c^T x \quad Ax \le b \quad \xleftrightarrow{\text{dual}} \quad \max_y \; -b^T y \quad \text{s.t. } A^T y + c = 0, \; y \ge 0$$

$$\begin{pmatrix} x \\ s \end{pmatrix} = \tilde{x}$$

$$\min_{\tilde{x}} \; \tilde{c}^T \tilde{x}$$
$$\text{s.t. } \tilde{A}\tilde{x} \le \tilde{b}$$

$$\tilde{A} = \begin{pmatrix} A & -I \\ -A & -I \\ 0 & \mathbb{1}^T \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} -b \\ b \\ 1 \end{pmatrix}$$

$$\tilde{c} = \begin{pmatrix} c \\ 0 \end{pmatrix}$$

dual:

$$\max_{\tilde{y}} \; -\tilde{b}^T \tilde{y}$$
$$\text{s.t. } \tilde{y} \ge 0$$
$$\tilde{A}^T y + \tilde{c} = 0$$

$$\tilde{y} = \begin{pmatrix} y \\ z \\ w \end{pmatrix}$$

$$\max_y \; b^T y - b^T z - w$$
$$\text{s.t. } y, z, w \ge 0$$

$$\begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} A^T & -A^T & 0 \\ -I & -I & \mathbb{1} \end{pmatrix} \begin{pmatrix} y \\ z \\ w \end{pmatrix} = 0$$

$$\max_{y,z,w} \quad b^Ty - b^Tz - w$$
$$\text{s.t.} \quad y,z,w \geq 0$$

$$\begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} A^T & -A^T & 0 \\ -I & -I & \mathbb{1} \end{pmatrix} \begin{pmatrix} y \\ z \\ w \end{pmatrix} = 0 \quad \Bigg\} \begin{matrix} A^Ty - A^Tz + c = 0 \\ -y - z + \mathbb{1}w = 0 \end{matrix}$$

$$\max_{y,z,w} \quad b^T(y - z) - w$$
$$\text{s.t.} \quad y,z,w \geq 0$$
$$A^T(y-z) + c = 0$$
$$y + z = \mathbb{1}w$$

Let $\tilde{z} = y - z$

$y + z = \tilde{z} + 2z$

$$\max_{\tilde{z},w} \quad b^T\tilde{z} - w$$
$$\text{s.t.} \quad A^T\tilde{z} + c = 0$$
$$\tilde{z} \leq \mathbb{1}w \longrightarrow \|\tilde{z}\|_\infty \leq w$$
$$w \geq 0$$

since $w > 0$, no reason to make $w > \|\tilde{z}\|_\infty$ since it will hurt obj.

So set $w = \|\tilde{z}\|_\infty$

$$\max_{\tilde{z}} \quad b^T\tilde{z} - \|\tilde{z}\|_\infty$$
$$\text{s.t.} \quad A^T\tilde{z} + c = 0$$

c. give a direct argument that whenever x is primal feasible and z is dual feasible, $c^Tx \geq b^Tz - \|z\|_\infty$

2 arguments

1. weak duality
2. $c^Tx + z^TAx = 0$

$-\|z\|_\infty \leq z^T(Ax-b) \leq \|z\|_\infty$ since $\|Ax-b\|_1 \leq 1$

$$c^Tx = z^TAx$$
$$= z^T(Ax + b - b) = b^Tz + \underbrace{z^T(Ax-b)}_{} \geq b^Tz - \|z\|_\infty$$

$$\Rightarrow c^Tx \geq b^Tz - \|z\|_\infty$$