# Agenda

- Optimality Conditions
- Sensitivity Analysis
  - new variable
  - new constraint
  - global / local
  - example
- Farkas Lemma

# Logistics

- HW 6 due Fri 4/1
- HW 7 due Fri 4/8
- Midterm 2 4/12

# Optimality Conditions

Primal and dual solns are optimal

$$\Updownarrow$$

1. $x$ primal feasible
2. $y$ dual feasible
3. Duality gap is zero $\quad (c^T x + b^T y = 0)$

What is maintained for all iterations?

| | Primal feas. | Dual feas. | Duality gap |
| --- | --- | --- | --- |
| Primal Simplex | ✓ | | ✓ |
| Dual Simplex | | ✓ | ✓ |

Proof for duality gap $=0$ for primal simplex

$$y = -A_B^{-T} c_B$$

$$b^T y = -b^T A_B^{-T} c_B$$

$$c^T x = c_B^T x_B$$

$$b^T y + c^T x = c_B^T (x_B - A_B^{-1} b) = 0 \quad ✓$$

# Sensitivity Analysis: new variable

$$\min_x \; c^T x \quad \text{s.t. } Ax = b,\; x \geq 0$$

$(x^*, y^*)$ opt

Add new var: $$\min_{x, x_{n+1}} \; c^T x + c_{n+1} x_{n+1} \quad \text{s.t. } Ax + A_{n+1} x_{n+1} = b,\; x, x_{n+1} \geq 0$$

$$\bar{A} = \left( A \mid A_{n+1} \right)$$

$(x^*, 0)$ is primal feasible

$$\bar{A}^T = \begin{pmatrix} A^T \\ \hline A_{n+1}^T \end{pmatrix}$$

Dual: $$\max_y \; -b^T y \quad \text{s.t. } A^T y + c \geq 0,\; A_{n+1}^T y + c_{n+1} \geq 0$$

- If $A_{n+1}^T y^* + c_{n+1} \geq 0$ then $y^*$ dual feasible and $(x^*, 0)$ optimal
- Else, run primal simplex

# Sensitivity Analysis: new constraint

$$\begin{aligned} \min_x \ & c^T x \\ \text{s.t. } & Ax = b \\ & a_{m+1}^T x = b_{m+1} \\ & x \geq 0 \end{aligned}$$

Dual: $$\begin{aligned} \max_y \ & -b^T y \\ \text{s.t. } & A^T y + a_{m+1} y_{m+1} + c \geq 0 \end{aligned}$$

$(y^*, 0)$ is dual feasible

- If $a_{m+1}^T x^* = b_{m+1}$ then $x^*$ is opt.
- Else: run dual simplex

# Global / Local Sensitivity Analysis

Consider the problem

$$\min_x \; -5x_1 - x_2 + 12x_3$$
$$\text{s.t. } 3x_1 + 2x_2 + x_3 = 10$$
$$5x_1 + 3x_2 + x_4 = 16$$
$$x \geq 0$$

Opt soln : $\bar{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

- Suppose we change $a_{11}$ from 3 to $3+\delta$
- Keep $x_1, x_2$ as basic variables, let $B(\delta)$ be the corresponding basis matrix

a. Compute $B(\delta)^{-1}b$. For which values of $\delta$ is $B(\delta)$ a feasible basis?

$$B(\delta) = \begin{pmatrix} 3+\delta & 2 \\ 5 & 3 \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$B(\delta)^{-1} = \frac{1}{3\delta - 1}\begin{pmatrix} 3 & -2 \\ -5 & 3+\delta \end{pmatrix}$$

$$B(\delta)^{-1}b = \frac{1}{3\delta-1}\begin{pmatrix} 3 & -2 \\ -5 & 3+\delta \end{pmatrix}\begin{pmatrix} 10 \\ 16 \end{pmatrix}$$

$$= \frac{1}{3\delta-1}\begin{pmatrix} -2 \\ -2+16\delta \end{pmatrix}$$

Require $\delta \leq \frac{1}{3}$ and $-2+16\delta \leq 0$

$$\boxed{\delta \leq \frac{1}{8}}$$

b. Compute $c_B^T B(\delta)^{-1}$. For which values of $\delta$ is $B(\delta)$ an optimal basis?

$$c_B^T B(\delta)^{-1} = \frac{1}{3\delta-1}(-5,-1)\begin{pmatrix} 3 & -2 \\ -5 & 3+\delta \end{pmatrix}$$

$$= \left(\frac{10}{1-3\delta}, \frac{\delta-7}{1-3\delta}\right)$$

$$y = -(A_B^T)^{-1} c_B = \begin{pmatrix} \frac{-10}{1-3\delta} \\ \frac{7-\delta}{1-3\delta} \end{pmatrix}$$

- Dual feasibility: $A^T y + c \geq 0$

$$A^T y + c = \begin{pmatrix} 3+\delta & 5 \\ 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{-10}{1-3\delta} \\ \frac{7-\delta}{1-3\delta} \end{pmatrix} + \begin{pmatrix} -5 \\ -1 \\ 12 \\ 0 \end{pmatrix} \geq 0$$

$$\frac{1}{1-3\delta}\begin{pmatrix} 5-15\delta \\ 1-3\delta \\ -10 \\ 7-\delta \end{pmatrix} \geq \begin{pmatrix} 5 \\ 1 \\ -12 \\ 0 \end{pmatrix}$$

$\delta \leq 18$

$\delta \leq \frac{1}{3}$

Notice: The 1st 2 constraints tight (near $\delta = 0$)

$$10 \leq 12(1-3\delta)$$

$$\frac{5}{6} \leq 1-3\delta$$

$$3\delta \leq \frac{1}{6} \quad \boxed{\delta \leq \frac{1}{18}}$$

# Another Farkas Lemma

Prove that exactly 1 of the following 2 statements holds

(1) $\exists x$ s.t. $Rx > 0$

(2) $\exists y$ s.t. $R^T y = 0$, $y \geq 0$, $y \neq 0$

- Claim: Both can't be true

Assume $\exists x, y$ s.t. $Rx > 0$, $R^T y = 0$, $y \geq 0$

$0 = x^T R^T y = \underbrace{(Rx)^T}_{>0} \underbrace{y}_{\geq 0}$ This can only hold true if $y = 0$

- Claim: Both can't be false

$$\min_y \; -\mathbb{1}^T y \quad \text{s.t. } R^T y = 0, \; y \geq 0 \quad \xrightarrow{\text{dual}} \quad \max_x \; 0 \quad \text{s.t. } Rx - \mathbb{1} \geq 0$$

Case 1: $\exists x$ s.t. $Rx \geq 1 \Rightarrow$ (1) holds

Case 2: $\nexists x$ s.t. $Rx \geq 1 \Rightarrow$ dual infeasible

Primal always feasible, so primal unbounded

$\Rightarrow \exists y$ s.t. $R^T y = 0$, $y \geq 0$, $y \neq 0$