Agenda

- LP reformulation
- Portfolio least squares
- Questions

Logistics Friend 006

- Midterm Thursday 1:30 - 2:50 (ODS will start at 12:30)
- 1 page cheat sheet
- Up to lecture 9 on
  - Not including equivalence theorem or anything on extreme points/vertices/basic feasible solutions

Formulate the following problem as an LP

$$\min_x \sum_{i=1}^{m} \max(0, a_i^T x + b_i)$$

goal: $y_i = \max(0, a_i^T x + b_i)$

$$\min_{x,y} \sum_{i=1}^{m} y_i$$

$$\text{s.t. } \underbrace{y_i = \max(0, a_i^T x + b_i)}_{\Downarrow\ \not\Uparrow\ \overbrace{y_i \ge 0,\ y_i \ge a_i^T x + b_i}} \quad i=1,\dots,m$$

$$\min_{x,y} \mathbb{1}^T y$$

$$\text{s.t. } y_j \ge 0 \qquad j=1,\dots,m$$

$$y_j \ge a_j^T x + b_j \qquad j=1,\dots,m$$

Things to remove

- max
- $|\cdot| \rightarrow \|\cdot\|_1$
- $\|\cdot\|_\infty$

Portfolio Problem

- Current portfolio: h^{curr} ← amount invested in n assets

  $h^{curr} \in \mathbb{R}^n$

- Current total value is $\mathbb{1}^T h^{curr}$
- $h - h^{curr}$ is the trade vector
- n assets divided into m industry sectors   tech, pharma
- $s \in \mathbb{R}^m$ denotes the dollar value sector exposure

$$\underset{(m)}{s} = \underset{(m,n)}{S}\underset{(n)}{h} \qquad S \in \mathbb{R}^{m,n} \qquad S_{ij} = \begin{cases} 1 & \text{if asset } j \text{ in sector } i \\ 0 & \text{else} \end{cases}$$

- New portfolio must have a given sector exposure $s^{des} \in \mathbb{R}^m$
- Minimize the trading cost, given by

$$\sum_{i=1}^{n} K_i (h_i - h_i^{curr})^2 \qquad (K_i > 0)$$

- Explain how to find h using constrained least squares

$$S = \begin{array}{c|ccc} & \text{FB} & \text{Google} & \text{Kellogg} \\ \hline \text{tech} & 1 & 1 & 0 \\ & & & \\ & & & \end{array} \quad \Big\updownarrow m \ (\#\text{sectors})$$

$$\xleftrightarrow[(\#\text{assets})]{n}$$

HW: Illumination

$$\min_{I,p} \| I - I^{des} \|_2^2 + \mu \| p - \tfrac{1}{2} \|_2^2$$
$$\text{s.t. } 0 \le p \le 1$$
$$I = Ap$$

$$\sum_{i=1}^{n} \kappa_i (h_i - h_i^{curr})^2 \quad , \quad s^{des} = Sh$$

current total value $= \mathbb{1}^T h^{curr}$

min this

$$\kappa \in \mathbb{R}^n$$

$$\|Ax-b\|_2^2 = (Ax-b)^T(Ax-b)$$

Constrained LS

$$\min_x \|Ax-b\|_2^2 \quad \text{s.t. } Cx = d$$

$$\min_h \sum_{i=1}^{n} \kappa_i (h_i - h_i^{curr})^2$$

$$\text{s.t. } s^{des} = Sh \quad (m)$$

$$\mathbb{1}^T h = \mathbb{1}^T h^{curr} \quad (1)$$

$$\min_h \|Kh - Kh^{curr}\|_2^2$$

$$\text{s.t.} \begin{array}{c} m \\ 1 \end{array} \underbrace{\begin{pmatrix} S \\ \mathbb{1}^T \end{pmatrix}}_{C} h = \underbrace{\begin{pmatrix} s^{des} \\ \mathbb{1}^T h^{curr} \end{pmatrix}}_{d} \begin{array}{c} m \\ 1 \end{array}$$

$$y \geq -x$$

$$y \geq x$$

$$y \geq 0, \ y_i \geq a_i^T x + b_i$$

$$\|h - h^{curr}\|_2^2 = \sum_{i=1}^{n} (h_i - h_i^{curr})^2$$

$$\sum_{i=1}^{n} \kappa_i (h_i - h_i^{curr})^2 = \sum_{i=1}^{n} (\sqrt{\kappa_i}\, h_i - \sqrt{\kappa_i}\, h_i^{curr})^2$$

$$\sqrt{\kappa}^T h - \sqrt{\kappa}^T h^{curr}$$

$$K = \begin{pmatrix} \sqrt{\kappa_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\kappa_n} \end{pmatrix}$$

$$K^T K = \begin{pmatrix} \kappa_1 & & 0 \\ & \ddots & \\ 0 & & \kappa_n \end{pmatrix}$$

$$h^T K^T K h = \|Kh\|_2^2 = \sum_{i=1}^{n} (\sqrt{\kappa_i}\, h_i)^2$$

$$\sum_{i=1}^{n} \kappa_i (h_i - h_i^{curr})^2 = \|\underset{A}{K}h - \underbrace{Kh^{curr}}_{b}\|_2^2$$

$$\kappa \in \mathbb{R}^n$$

$$\|Ax-b\|_2^2 = (Ax-b)^T(Ax-b)$$

$$= x^T A^T A x - 2b^T A x + b^T b$$

Constrained LS

$$\min_x \|Ax-b\|_2^2 \quad \text{s.t. } Cx = d$$

- Suppose you want to include a penalty on shorting assets. How might you include this as a linear constraint?

  - $h_i < 0$
  - penalty ν_i $\quad (\nu_i > 0)$

$$\left.\begin{array}{l}\text{If } h_i < 0 \text{ then you must pay} \\ \qquad \nu_i |h_i| \ \$ \\ \text{If } h_i \geq 0 \text{ then no penalty}\end{array}\right\} \quad \max(-\nu_i h_i, 0)$$

$$\min_{h} \sum_{i=1}^{n} \kappa_i (h_i - h_i^{curr})^2 \cancel{+ \sum_{i=1}^{n} \max(-\nu_i h_i, 0)}$$

$$\text{s.t. } s^{des} = Sh \qquad (m)$$

$$\cancel{\mathbb{1}^T h = \mathbb{1}^T h^{curr} \quad (1)}$$

self-financing

$$\mathbb{1}^T h + \sum_{i=1}^{n} \max(-\nu_i h_i, 0) \leq \mathbb{1}^T h^{curr}$$

KKT to solve $\min_x \|Ax - b\|_2^2$ s.t. $Cx = d$

$$\begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$$