Agenda

- 3, 6, 7 of the additional exercises

Logistics

- Midterm 2 Tues April 12
  - same terms as midterm 1

- No precept next week
- Last precept on integer optimization after

3) Let $A = A^T$

Consider $\min_x \ c^T x$ s.t. $Ax \geq c$, $x \geq 0$

$\min_x \ c^T x$ s.t. $Ax = b$, $x \geq 0$

$\max_y \ -b^T y$ s.t. $A^T y + c \geq 0$

Show that if x^* satisfies $Ax^* = c$, $x^* \geq 0$ then x^* is optimal

$$\min_x \ c^T x \quad \text{s.t. } -Ax \leq -c, \quad x \geq 0$$

$$\min_{x,s} \ c^T x \quad \text{s.t. } -Ax + s = -c, \quad x, s \geq 0$$

std form

$$\min_{\tilde{x}} \ \tilde{c}^T \tilde{x} \quad \text{s.t. } \tilde{A}\tilde{x} = \tilde{b}, \quad \tilde{x} \geq 0$$

$$\tilde{x} = \begin{pmatrix} x \\ s \end{pmatrix} \quad \tilde{c} = \begin{pmatrix} c \\ 0 \end{pmatrix}$$

$$\tilde{b} = -c \quad \tilde{A} = \begin{pmatrix} -A & I \end{pmatrix}$$

take the dual

$$\max_y \ -\tilde{b}^T y \quad \text{s.t. } \tilde{A}^T y + \tilde{c} \geq 0$$

$$\max_y \ c^T y \quad \text{s.t. } \begin{pmatrix} -A^T \\ I \end{pmatrix} y + \begin{pmatrix} c \\ 0 \end{pmatrix} \geq 0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\max_y \ c^T y \quad \text{s.t. } -A^T y + c \geq 0, \quad y \geq 0$$

Use $A = A^T$

dual

$$\max_y \ c^T y \quad \text{s.t. } Ay \leq c, \quad y \geq 0$$

primal

$$\min_x \ c^T x \quad \text{s.t. } Ax \geq c, \quad x \geq 0$$

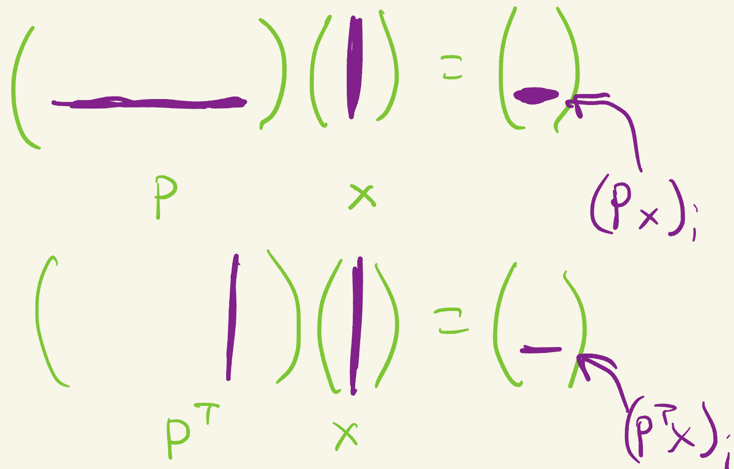
(x^*, y^*) is primal/dual opt

- Primal feas ✓
- Dual feas ✓
- Zero duality gap ✓

6) P is column-stochastic if

- $P \geq 0$
- $P^T \mathbb{1} = \mathbb{1}$ (i.e. $\sum_{i=1}^{n} P_{ij} = 1 \quad j = 1, \dots, n$)

6.1) Show that if P is column-stochastic then $(P^T x)_i \leq x_{max}$



$$(P^T x)_i = \sum_{j=1}^{n} P_{ji} x_j \overset{P \geq 0}{\leq} \sum_{j=1}^{n} P_{ji} x_{max} = x_{max} \left(\sum_{j=1}^{n} P_{ji} \right) \underset{P^T \mathbb{1} = \mathbb{1}}{=} x_{max}$$

6.2) Assume P is column-stochastic. Show using LP duality that $\exists\, y \in \mathbb{R}^n$ s.t. $Py = y,\ y \geq 0,\ \mathbb{1}^T y = 1$

there exists

		dual infeas.	dual finite opt	dual unbdd
primal	infeas.	✗		✗
primal	finite opt		✗	
primal	unbdd	✓		

dual unbdd $\Rightarrow$ p inf.

$\cancel{\Leftarrow}$

p unbdd $\Rightarrow$ d inf.

$\cancel{\Leftarrow}$

Consider this Primal/Dual pair

$$\min_x\ -\mathbb{1}^T x \qquad\qquad \max_y\ 0$$

$$\text{s.t. } (P-I)x = 0 \qquad\qquad \text{s.t. } (P-I)^T y - \mathbb{1} \geq 0$$

$$x \geq 0$$

$x=0$ always feas

$(P^T x)_i \leq x_{max}$

idea: show that the dual is infeas. ; use property from 6.1

Let $j = \operatorname{argmax} y$

$$((P-I)^T y)_j - 1 = (P^T y)_j - y_j - 1 \leq y_{max} - y_j - 1 = -1$$

this violates the constraint

$\Rightarrow$ dual infeas. $\Rightarrow$ primal unbdd

$\exists\, x$ s.t. $x \geq 0,\ Px = x,\ \mathbb{1}^T x > 0$

Scale x to make $\mathbb{1}^T x = 1$

$\Rightarrow \exists\, x$ s.t. $x \geq 0,\ Px = x,\ \mathbb{1}^T x = 1$

7) Consider $\min_x \; -2x_1 - x_2$

$$\text{s.t. } x_1 - x_2 \le 2$$
$$x_1 + x_2 \le 6$$
$$x_1, x_2 \ge 0$$

7.1) Convert the problem into std form, construct a basic feasible solution where $(x_1, x_2) = (0,0)$

$$\min_{x,s} \; -2x_1 - x_2 + 0s_1 + 0s_2$$
$$\text{s.t. } x_1 - x_2 + s_1 = 2$$
$$x_1 + x_2 + s_2 = 6$$
$$x, s \ge 0$$

BFS

$(0, 0, 2, 6)$

(x_1, x_2, s_1, s_2)

$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

7.2) Carry out 1 iteration of simplex

1. $\bar{c} = c + A^T y$, $-A_B^T y = c_B$

$B = (3,4)$, $A_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $c_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\bar{c} = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 0 \end{pmatrix}$ ⟵ add x_1 to basis

$x \to x + \theta d$

$x_2 \to x_2 + \theta d_2$

$\underset{0}{x_2} \to x_2 + \underset{>0}{\theta}\, d_2$

2. figure out what d is

non-basic $\begin{cases} 1. \; d_1 = 1 \\ 2. \; d_2 = 0 \end{cases}$

$d = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}$

basic 3. $A_B d_B = -A_1 = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$A_B d_B + A_1 = 0$

need $A(x + \theta d) = b$ for feas.

$\cancel{Ax} + \theta A d = \cancel{b}$, $\Rightarrow Ad = 0 \Rightarrow A_B d_B + A_N d_N = 0$

$d = \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \quad x = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \end{pmatrix}$

$Ax = b, \; x \geq 0$

$\underbrace{A(x + \theta d) = b}_{\text{always satisfied for any } \theta}, \quad \underbrace{x + \theta d \geq 0}_{\text{might get violated at some } \theta}$

$\theta^* = 2$

$\Rightarrow$ new $x = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 4 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 4 \end{pmatrix}$

$\theta = 2.1$

$x = \begin{pmatrix} 2.1 \\ 0 \\ -.1 \\ 3.9 \end{pmatrix}$

violate $x \geq 0$