- Review of Simplex Method.
- Optimality Conditions
- Degeneracy and cycling
- Example.

Standard form polyhedra

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 . \end{aligned}$$

Feasible direction.

$P = \{ x \mid Ax = b, x \geq 0 \}$.

$x \in P$, a vector d is a feasible direction $\Longleftrightarrow$

$\exists\ \theta > 0$ s.t. $x + \theta d \in P$.

- $Ad = 0$
- $x + \theta d \geq 0$

$x + \theta d \in P$:

$A(x+\theta d) = b \Rightarrow Ad = 0$

$x + \theta d \geq 0$.

Basic feasible solution

a basis matrix $A_B = [A_{B(1)}, \cdots, A_{B(m)}]$

a basic feasible solution

x satisfies

- $A_B x_B = b$
- $x_i = 0 \quad \forall i \notin B(1), \cdots, B(m)$

Here $A \in \mathbb{R}^{m \times n}$.

Example: when $m=1$

if A_B is invertible

$$x_B = A_B^{-1} b.$$

Reduced costs

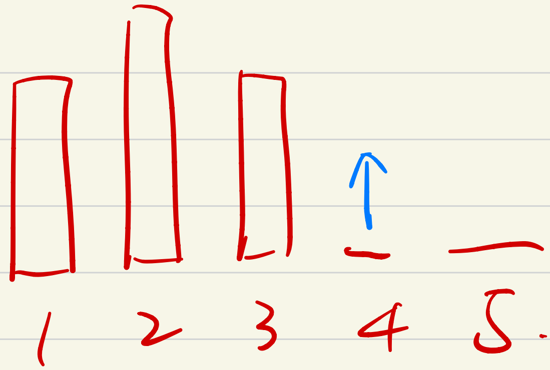
Change in objective costs of adding x_j to the basis.

$$\bar{c}_j = c_j - c_B^T A_B^{-1} A_j .$$

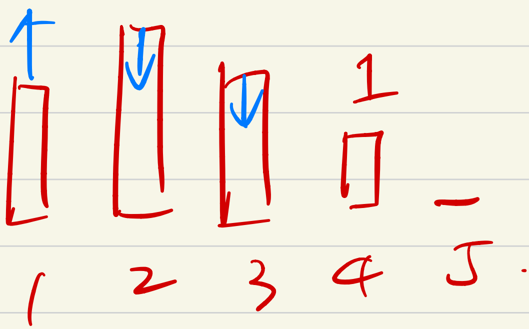
Example: $m=3$, $n=5$.

$B = (B(1), B(2), B(3)) = (1, 2, 3)$

Now we want to add $j=4$



if we want to let $x_j = 1$.



$$A_B x_B = b.$$

$$[A_1, A_2, A_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$$

$$[A_1, A_2, A_3, A_4] \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ 1 \end{bmatrix} = b.$$

$$[A_1, A_2, A_3] \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} + A_4 = b$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = A_B^{-1} b - A_B^{-1} A_4.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A_B^{-1} b$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = A_B^{-1} A_4$$

$$= A_B^{-1} A_j .$$

(in our example $j = 4$)

Now we compare objective cost. $c^T x$.

- $c(c_1, c_2, c_3) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ ①

- $c(c_1, c_2, c_3, c_4) \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ 1 \end{pmatrix}$ ②

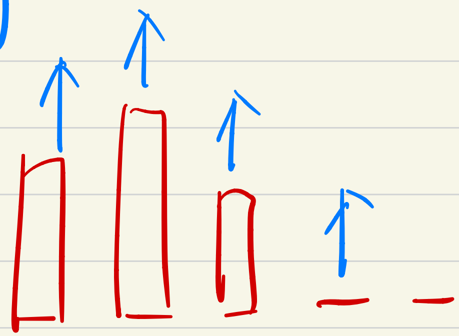
$$② - ① = C_q + C_B^T \left[\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right]$$

$$\bar{C_j} = C_j - C_B^T A_B^{-1} A_j$$

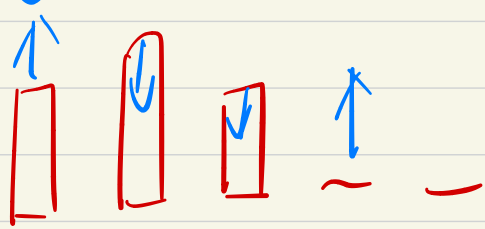
$\bar{C_j} > 0$ increase obj

$\bar{C_j} < 0$ decrease obj

Moving to new basis

① 

$\bar{c}_j < 0$ unbounded.

② 

$\bar{c}_j < 0$ there always exist some i first ~~to~~ touch 0, then we eliminate index i.

③ $\bar{c}_j > 0$ we don't choose j.

Optimality conditions.

(a) a feasible solution x is optimal $\Leftrightarrow$ $c^T d \geq 0$ for any feasible direction at x.

Proof: proof by contradiction

$\Rightarrow$ if $\exists$ a feasibe direction d such that $c^T d < 0$.

by the definition of d.

$\exists$ $\theta > 0$ such that $x + \theta d$ is a feasible solution

Then $c^T x > c^T(x + \theta d)$,

which means x is not optimal contradiction.

$\Leftarrow$ if x is not optimal

Then $\exists\, x^*$ such that.

$c^T x^* < c^T x$ we can

proof that. $d = x^* - x$ is

a feasibe direction at x

and $c^T d < 0$

contradiction

(b) A feasible solution x is the unique optimal solution if and only if $c^T d > 0$ for every nonzero feasible direction d at x

Proof: same idea as. (a)

Two-phased simplex method

① $\min \quad 1^T y$

$\text{s.t.} \quad Ax + y = b$

$x, y \geq 0.$

Phase 1

- construct auxiliary problem such that $b \geq 0$
- solving auxiliary problem using simplex from $(0, b)$

- If the optimal is greater than 0, problem is infeasible, break.

Phase II.

- Solve the original LP using simplex

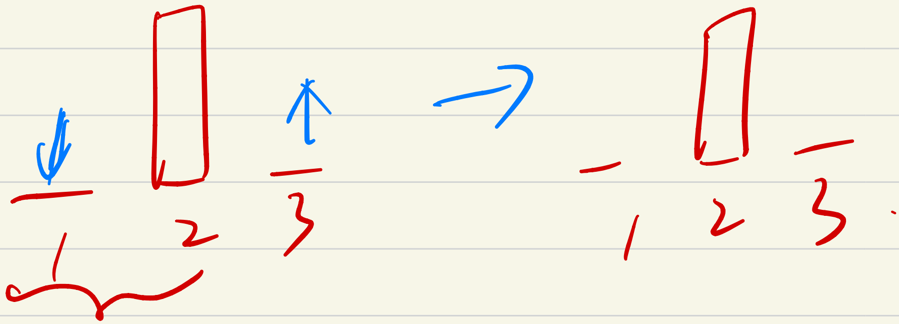
Degeneracy and cycling

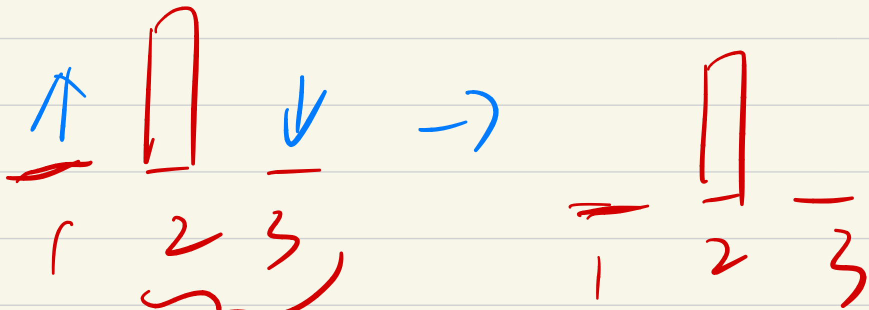
$$\min \quad -2x_1 - x_2$$
$$\text{s.t.} \quad x_1 + x_2 + x_3 = 1$$
$$-x_1 + x_2 - x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$

- Basis $B = \{1,2\} \rightarrow x = (0,1,0)$
- Basis $B = \{2,3\} \rightarrow x = (0,1,0)$

$\rightarrow$ ∘ Basis $B = \{1,2\} \rightarrow x = (0,1)$

cycling !





nothing changed!

Bland's rule to avoid cycles:

If we use the smallest index rule for choosing both the j entering the basis and i leaving the basis, then no cycling will occur.

Simplex method example.

$$\min \quad -2x_1 - x_2$$

$$\text{s.t.} \quad x_1 - x_2 \le 2$$

$$x_1 + x_2 \le 6$$

$$x_1, x_2 \ge 0$$