# Precept 09

- Optimality Conditions $\boxed{\text{(primal}}$ + dual)
- Sensitivity Analysis
  - Adding a new variable
  - Adding a new constraint
  - Global / Local

  - Example

$\boxed{LP}$

$\boxed{x^*}$

$\Delta$

- Network Flows
  - Total unimodularity

  - $\boxed{\text{Example}}$

# Optimality Condition. ← $x, y$ optimal ?

| Primal | Dual |
|---|---|
| $\min\ c^T x$ | $\max\ -b^T y$ |
| s.t. $Ax = b$ | s.t. $A^T y + c \geq 0$ |
| $x \geq 0$ | |

$x$ is a primal feasible point

$y$ is a dual feasible point

$x, y$ are optimal solutions to the primal and dual problems

$\Longleftrightarrow$ duality gap is zero ($c^T x + b^T y = 0$)

$X_B$

$y \quad c^T x + b^T y = 0$

$x$ + zero gap $\longrightarrow$ $\boxed{y \text{ feasible?}}$

$\delta < \frac{1}{8}$

$x$

$y$ + zero gap $\longrightarrow$ $\boxed{x \text{ feasible}}$

# Primal and Dual Simplex

| Primal | Dual |
|---|---|
| $\min\ c^T x$ | $\max\ -b^T y$ |
| s.t. $Ax = b$ | s.t. $A^T y + c \geq 0$ |
| $x \geq 0$ | |

Suppose we have a primal feasible $x$ with base $B$

$A_B x_B = b$, $x \geq 0$ $\Rightarrow$ $x_B = A_B^{-1} b \geq 0$

$x_i = 0\ i \notin B$

Set $y = -(A_B^T)^{-1} c_B$

duality gap $= c^T x + b^T y = c_B^T x_B - b^T (A_B^T)^{-1} c_B$

$= c_B^T x_B - c_B^T A_B^{-1} b = c_B^T (x_B - A_B^{-1} b) = 0$

$y = -(A_B^T)^{-1} \cdot c_B$ $\quad z_2^T \quad z_1$

$= 0$ $\quad z = z_1^T \cdot z_2 = z_2^T \cdot z_1$

$z^T = z \quad z^T = z_2^T \cdot z_1$

$A^T y + c \geq 0$ ? ✓

$\boxed{A^T y + c < 0}$ $\quad$ $\boxed{\text{simplex method}}$

# Sensitivity Analysis (add new variables)

$$\min \ c^T x \quad \text{s.t. } Ax = b \quad x \geq 0.$$

$x^*$ optimal ①

$$\Rightarrow \min \ c^T x + c_{n+1} x_{n+1}$$
$$\text{s.t. } Ax + A_{n+1} x_{n+1} = b$$
$$x, x_{n+1} \geq 0$$

$x_{n+1} \rightarrow$

$c^T x + c_{n+1} \cdot x_{n+1}$

$\leftarrow Ax^* = b$

① $(x^*, 0)$ is still primal feasible

Dual

$$\max \ -b^T y$$
$$\text{s.t. } A^T y + A_{n+1}^T y + c + c_{n+1} \geq 0$$

set $y$ such that zero gap.

$\boxed{y^*}$ is the optimal of dual of ①.

$y^*$ optimal $(A^T y^* + c) \geq 0$

if $(A_{n+1}^T y^* + c_{n+1}) \geq 0$ then $y^*$ is dual feasible $(x^*, 0)$ optimal

Otherwise, run primal simplex

(adding new constraints)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & a_{m+1}^T x = b_{m+1} \\ & x \geq 0. \end{aligned}$$

$x^*$

$\cancel{c^T x^* + b^T y^* = 0}$

Dual

$$\begin{aligned} \max \quad & -b^T y \\ \text{s.t.} \quad & A^T y + \cancel{a_{m+1} y_{m+1}} + c \geq 0 \end{aligned}$$

0

Dual

max $-b$

$A^T y + c$

$(y^*, 0)$ is still dual feasible

$a_{m+1}^T x^* = b_{m+1}$ ?

$y \to$ construct $x^*$

If yes. $x^*$ is still optimal

otherwise, run dual simplex

# Global / Local Sensitivity Analysis.

$$\boxed{\begin{aligned} \min \quad & -5x_1 - x_2 + 12x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + x_3 = 10. \\ & 5x_1 + 3x_2 + x_4 = 16 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}}$$

$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{pmatrix}$

$b = \begin{pmatrix} 10 \\ 16 \end{pmatrix}$

an optimal solution $x^* = (2, 2, 0, 0)$

Suppose we change $a_{11}$ from 3 to $3+\delta$.

Keeping $x_1$ and $x_2$ as the basic variables, and let $B(\delta)$ be the corresponding basis matrix

$A_B = \begin{pmatrix} 3+\delta & 2 \\ 5 & 3 \end{pmatrix}$ $\quad B = \{1, 2\}$

(a) Compute $B(\delta)^{-1} b$, for which values of $\delta$ is $B(\delta)$ a feasible basis?

$B = \{1, 2\}$

$x_B = A_B^{-1} b \Rightarrow x_B \geq 0$

$\boxed{A_B^{-1} b \geq 0} \Rightarrow \delta$

$$B(\delta) = \begin{pmatrix} 3+\delta & 2 \\ 5 & 3 \end{pmatrix}$$

$B(\delta) = A_B$

$$B(\delta)^{-1} = \frac{1}{3\delta - 1} \begin{pmatrix} 3 & -2 \\ -5 & 3+\delta \end{pmatrix}.$$

$B = \{1, 2\}$

$x_B.$

$$B(\delta)^{-1} b = \begin{pmatrix} \frac{2}{1-3\delta} \\ \frac{2-16\delta}{1-3\delta} \end{pmatrix} \geq 0 \Rightarrow \delta \leq \frac{1}{8}.$$

(b) Compute $c_B^T B(\delta)^{-1}$, For which values of $\delta$ is $B(\delta)$ an optimal basis. ?

$B = \{1, 2\}$. $\boxed{x_B}$

$$c_B^T B(\delta)^{-1} = \left( \frac{5}{1-3\delta}, \frac{2\delta - 4}{1-3\delta} \right)$$

$$y = -(A_B^T)^{-1} c_B = \begin{pmatrix} -\frac{5}{1-3\delta} \\ \frac{4-2\delta}{1-3\delta} \end{pmatrix}$$

$$\boxed{A^T y + c \geq 0 \Rightarrow}$$

$$-A^T (A_B^T)^{-1} c_B + c \geq 0.$$

$$\begin{pmatrix} 3 & 5 \\ 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{5}{1-3\delta} \\ \frac{4-2\delta}{1-3\delta} \end{pmatrix} + \begin{pmatrix} -5 \\ -1 \\ 12 \\ 0 \end{pmatrix} \geq 0$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} \frac{5\delta}{1-3\delta} \geq 0. \\ \frac{1-3\delta}{1-3\delta} \geq 0 \\ 12 - \frac{5}{1-3\delta} \geq 0 \\ \frac{4-2\delta}{1-3\delta} \geq 0 \end{cases} \Rightarrow$$

$$\left(\delta \leq \frac{1}{18}\right)$$

$$\boxed{\delta \leq \frac{1}{18}.}$$

# Network Flows

## Total Unimodularity

- a matrix $A$ is totally unimodular if all its minors are $-1, 0, 1$
- the inverse of an nonsigular square submatrix of $A$ has entries $-1, 1, 0$.

## Integrality theorem

polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$

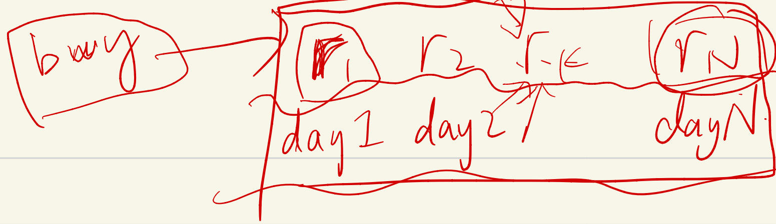
where $A$ totally unimodular

$b$ integer vector

$\Rightarrow$ all extreme points of $P$ are integer vectors.

node-arc incidence matrix of a directed graph is totally unimodular

# Example.




A catering company must provide to a client $r_i$ tablecloths on each of $N$ consecutive days.

buy new table cloth $p$ dollars

launder
- fast: $f$ dollars unavaible for the next $n$ days
- slow: $g$ dollars unavaible for the next $m$ days

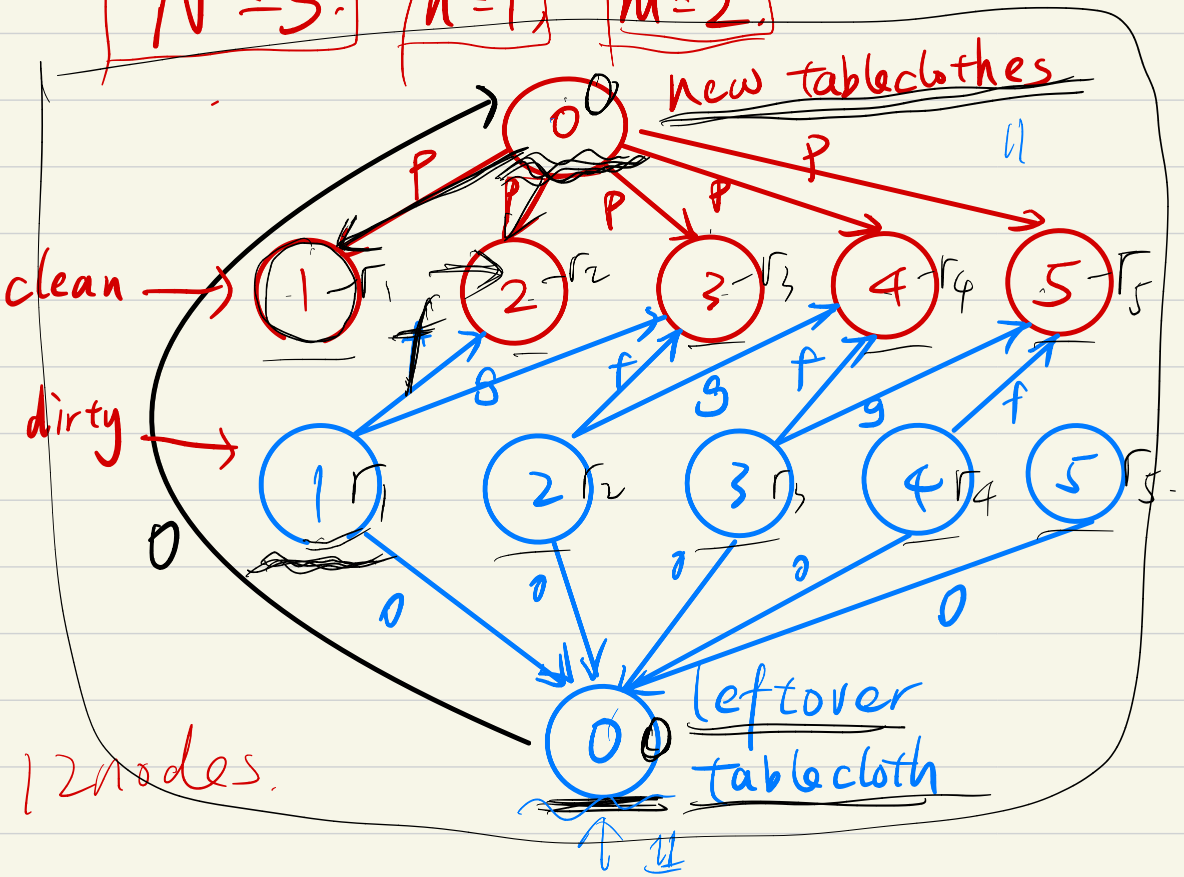
$f > g$

Goal: meet the client's demand at minimum cost, starting with no tableclothes, and any leftover tablecloths have no value

Formulate this problem as network flow.

$N=5$. $n=1$, $m=2$.

①




12 nodes.