Precept 09

- Optimality Conditions (primal + dual)
- Sensitivity Analysis
  - Adding a new variable
  - Adding a new constraint
  - Global / Local
  - Example
- Network Flows
  - Total unimodularity
  - Example

Optimality Condition.

Primal

$\min \; c^T x$

s.t. $Ax = b$

$x \geq 0$

Dual

$\max \; -b^T y$

s.t. $A^T y + c \geq 0$

x is a primal feasible point x^*

y is a dual feasible point.

x, y are optimal solutions to the primal and dual problems

$\Longleftrightarrow$ duality gap is zero ($c^T x + b^T y = 0$)

x + zero gap $\longrightarrow$ $\boxed{y \text{ feasible?}}$

y + zero gap $\longrightarrow$ x feasible?

Primal and Dual Simplex

Primal

$\min \; c^T x$

s.t. $Ax = b$

$x \geq 0$

Dual

$\max \; -b^T y$

s.t. $A^T y + c \geq 0$

Suppose we have a primal feasible $\boxed{x}$ with base $\boxed{B}$

$A_B x_B = b$, $x \geq 0$ $\Rightarrow$ $x_B = A_B^{-1} b \geq 0$

Set $y = -(A_B^T)^{-1} c_B$

$z^T = z$

$z = x^T y = y^T x$

duality gap $= c^T x + b^T y = c_B^T x_B - b^T (A_B^T)^{-1} c_B$

$= c_B^T x_B - c_B^T A_B^{-1} b = c_B^T (x_B - A_B^{-1} b) = 0$

$\boxed{A^T y + c \geq 0 \; ?}$ ✓ $\Rightarrow$ $\boxed{x \text{ optimal}}$

$A^T y + c < 0$ simplex method

Sensitivity Analysis (add new variables)

① $\min\ c^T x$
s.t. $Ax = b \quad x \geq 0$

x^* optimal.

$\Rightarrow \min\ c^T x + c_{n+1} x_{n+1}$
s.t. $Ax + A_{n+1} x_{n+1} = b$
$x, x_{n+1} \geq 0$

$\leftarrow x^*_{new}$

① $(x^*, 0)$ is still primal feasible

$(x^*, 0) \quad \triangle \quad y^*$

Dual

$A^T y^* + c \geq 0.$

$\max\ -b^T y$
s.t. $A^T y + A_{n+1}^T y + c + c_{n+1} \geq 0$

y^* optimal of Dual of ①

if $A_{n+1}^T y^* + c_{n+1} \geq 0$ then

y^* is dual feasible $(x^*, 0)$ optimal

$A^T y^* + A_{n+1}^T y^* + c + c_{n+1} \geq 0.$

Otherwise, run primal simplex

(adding new constraints)

$$\min \; c^T x$$
$$\text{s.t.} \quad Ax = b$$
$$\boxed{a_{m+1}^T x = b_{m+1}}$$
$$x \geq 0.$$

$(c, c_{n+1})^T (x^*, 0) + b^T y^*$
$= c^T x^* + b^T y^* = 0.$

$\leftarrow$ Dual

Dual

$$\max \; -b^T y$$
$$\text{s.t.} \quad A^T y + a_{m+1} y_{m+1} + c \geq 0$$

$-(b, b_m)^T (y^*, 0)$

primal

$(y^*, 0)$ is still dual feasible

$\boxed{a_{m+1}^T x^* = b_{m+1}}$? $\rightarrow$ $c^T x^* = (b, b_m)^T y^*$

If yes. x^* is still optimal

otherwise, run dual simplex

Global / Local Sensitivity Analysis.

$$\begin{aligned} \min \quad & -5x_1 - x_2 + 12x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + x_3 = 10 \\ & 5x_1 + 3x_2 + x_4 = 16 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

$+u$

A

an optimal solution $x^* = (2, 2, 0, 0)$

Suppose we change a_{11} from 3 to $3+\delta$.

Keeping x_1 and x_2 as the basic variables, and let $B(\delta)$ be the corresponding basis matrix

$$A_B = \begin{pmatrix} 3+\delta & 2 \\ 5 & 3 \end{pmatrix}$$

(a) Compute $B(\delta)^{-1} b$, for which values of δ is $B(\delta)$ a feasible basis?

$B = (1, 2)$ (x_1, x_2)

$$B(\delta) = \begin{pmatrix} 3+\delta & 2 \\ 5 & 3 \end{pmatrix}$$

$X = B(\delta)^{-1} b$

$A_B^{-1} b$

$$B(\delta)^{-1} = \frac{1}{3\delta - 1} \begin{pmatrix} 3 & -2 \\ -5 & 3+\delta \end{pmatrix}.$$

$\delta \leq \frac{1}{3}.$

$$X = B(\delta)^{-1} b = \begin{pmatrix} \frac{2}{1-3\delta} \\ \frac{2-16\delta}{1-3\delta} \end{pmatrix} \geq 0 \Rightarrow \boxed{\delta \leq \frac{1}{8}}$$

(X_1, X_2)

(b) Compute $C_B^T B(\delta)^{-1}$. For which values of δ is $B(\delta)$ an optimal basis?

$$C_B^T B(\delta)^{-1} = \left(\frac{10}{1-3\delta}, \frac{\delta - 7}{1-3\delta} \right)$$

$$y = -(A_B^T)^{-1} C_B = \begin{pmatrix} -\frac{10}{1-3\delta} \\ \frac{7-\delta}{1-3\delta} \end{pmatrix}$$

$\boxed{A^T y + c \geq 0} \Rightarrow$ (X_1, X_2)

$$\begin{pmatrix} 3+\delta & 5 \\ 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{10}{1-3\delta} \\ \frac{7-\delta}{1-3\delta} \end{pmatrix} + \begin{pmatrix} -5 \\ -1 \\ 12 \\ 0 \end{pmatrix} \geq 0$$

$$\Rightarrow \begin{cases} \frac{0}{1-3\delta} \geq 0 \\ \frac{0}{1-3\delta} \geq 0 \\ \frac{2-36\delta}{1-3\delta} \geq 0 \\ \frac{7-\delta}{1-3\delta} \geq 0 \end{cases} \Rightarrow \delta \leq \frac{1}{18}$$

Network Flows

Total Unimodularity

- a matrix A is totally unimodular if all its minors are $-1, 0, 1$
- the inverse of an nonsigular square submatrix of A has entries $-1, 1, 0$.

Integrality theorem

polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$

where A totally unimodular
b integer vector

$\Rightarrow$ all extreme points of P are integer vectors.

node-arc incidence matrix of a directed graph is totally unimodular

$0 \quad r_1 \quad r_2 \quad \dots \quad r_N$

day 1 day 2 day N

Example.

A catering company must provide to a client r_i tablecloths on each of N consecutive days.

buy new table cloth p dollars

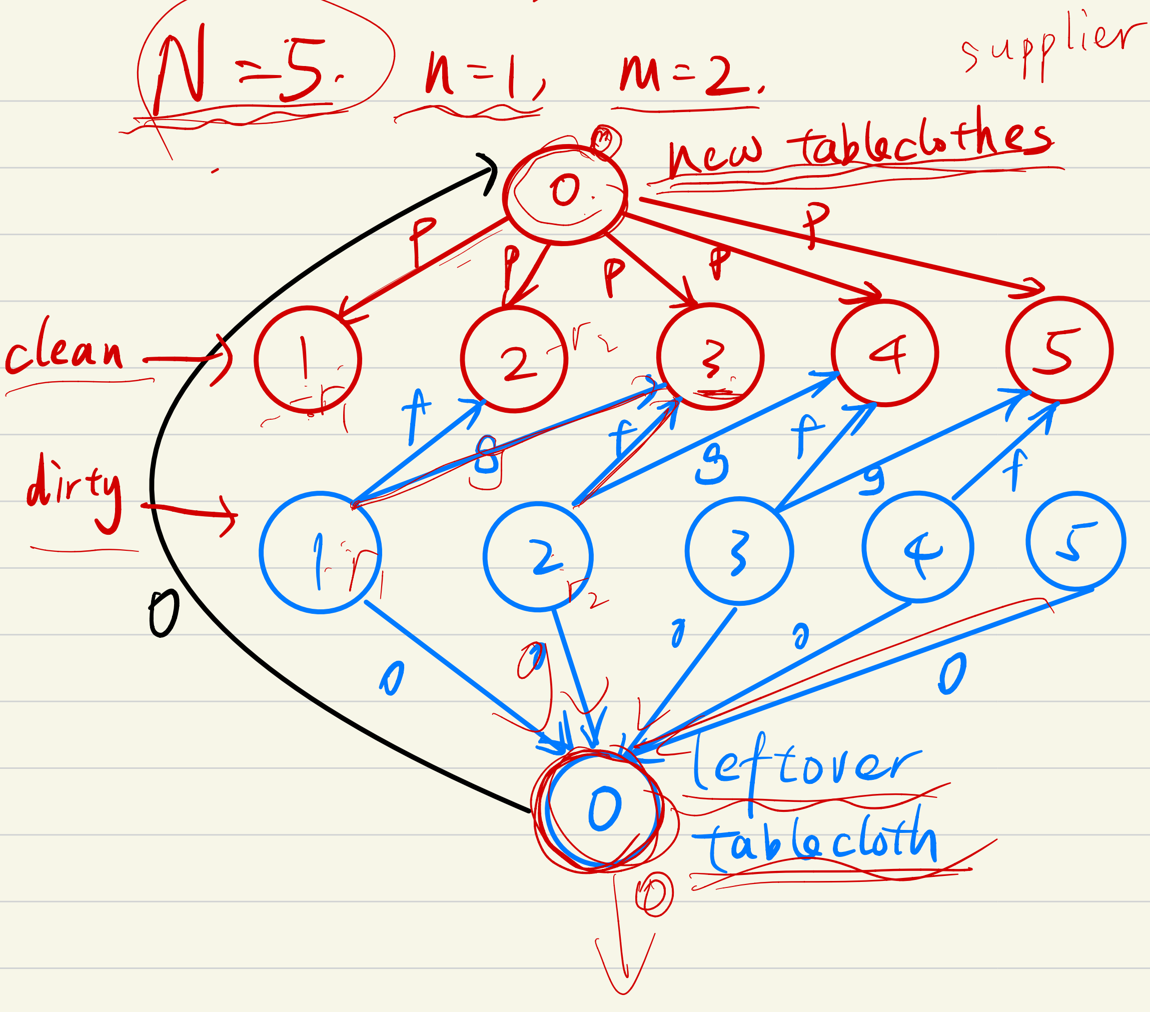
launder fast f dollars unavaible for the next n days

slow g dollars unavaible for the next m days

Goal: meet the client's demand at minimum cost, starting with no tableclothes, and any leftover tablecloths have no value

Formulate this problem as network flow.

N = 5. n = 1, m = 2.


supplier
0
new tablecloths
p
p
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clean
1
2
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4
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r1
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dirty
1
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leftover tablecloth
0