


Agenda

- Optimality Conditions
- Sensitivity Analysis
 - new variable
 - new constraint
 - global/local
 - example
- Farkas Lemma

Logistics

- HW 6 due Fri 4/1
- HW 7 due Fri 4/8
- Midterm 2 4/12

Optimality Conditions

Primal and dual solns are optimal



1. x primal feasible
2. y dual feasible
3. Duality gap is zero ($c^T x + b^T y = 0$)

What is maintained for all iterations?

	Primal feas.	Dual feas.	Duality gap
Primal Simplex	✓		✓
Dual Simplex		✓	✓

Proof for duality gap = 0 for primal simplex

$$y = -A_B^{-T} C_B$$

$$b^T y = -b^T A_B^{-T} C_B$$

$$c^T x = C_B^T X_B$$

$$b^T y + c^T x = C_B^T (X_B - A_B^{-1} b) = 0 \quad \checkmark$$

Sensitivity Analysis: new variable

$$\begin{array}{ll} \min_x & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \quad (x^*, y^*) \text{ opt}$$

Add new var:

$$\begin{array}{ll} \min_{x, x_{n+1}} & c^T x + c_{n+1} x_{n+1} \\ \text{s.t.} & Ax + A_{n+1} x_{n+1} = b \\ & x, x_{n+1} \geq 0 \end{array}$$

$$\bar{A} = \left(\begin{array}{c|c} A & A_{n+1} \end{array} \right)$$

$(x^*, 0)$ is primal feasible

$$\bar{A}^T = \left(\begin{array}{c|c} A^T & A_{n+1}^T \end{array} \right)$$

Dual:

$$\begin{array}{ll} \max_y & -b^T y \\ \text{s.t.} & A^T y + c \geq 0 \\ & A_{n+1}^T y + c_{n+1} \geq 0 \end{array}$$

- If $A_{n+1}^T y^* + c_{n+1} \geq 0$ then y^* dual feasible
and $(x^*, 0)$ optimal
- Else, run primal simplex

Sensitivity Analysis: new constraint

$$\begin{array}{ll}\min_x & c^T x \\ \text{s.t.} & Ax = b \\ & a_{m+1}^T x = b_{m+1} \\ & x \geq 0\end{array}$$

Dual: $\begin{array}{ll}\max_y & -b^T y \\ \text{s.t.} & A^T y + a_{m+1} y_{m+1} + c \geq 0\end{array}$

$(y^*, 0)$ is dual feasible

- If $a_{m+1}^T x^* = b_{m+1}$ then x^* is opt.

'Else': run dual simplex

Global/Local Sensitivity Analysis

Consider the problem

$$\begin{aligned} \min_x \quad & -5x_1 - x_2 + 12x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + x_3 = 10 \\ & 5x_1 + 3x_2 + x_4 = 16 \\ & x \geq 0 \end{aligned}$$

$$\text{Opt soln: } \bar{x} = \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

Suppose we change a_{11} from 3 to $3 + \delta$

Keep x_1, x_2 as basic variables, let $B(\delta)$ be the corresponding basis matrix

a. Compute $B(\delta)^{-1}b$. For which values of δ is $B(\delta)$ a feasible basis?

$$B(\delta) = \begin{pmatrix} 3+\delta & 2 \\ 5 & 3 \end{pmatrix} \quad (a \ b)^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$B(\delta)^{-1} = \frac{1}{3\delta-1} \begin{pmatrix} 3 & -2 \\ -5 & 3+\delta \end{pmatrix}$$

$$\begin{aligned} B(\delta)^{-1}b &= \frac{1}{3\delta-1} \begin{pmatrix} 3 & -2 \\ -5 & 3+\delta \end{pmatrix} \begin{pmatrix} 10 \\ 16 \end{pmatrix} \\ &= \frac{1}{3\delta-1} \begin{pmatrix} -2 \\ -2+16\delta \end{pmatrix} \end{aligned}$$

Require $\delta \leq \frac{1}{3}$ and $-2+16\delta \leq 0$

$$\boxed{\delta \leq \frac{1}{8}}$$

b. Compute $C_B^T B(\delta)^{-1}$. For which values of δ is $B(\delta)$ an optimal basis?

$$C_B^T B(\delta)^{-1} = \frac{1}{3\delta-1} (-5, -1) \begin{pmatrix} 3 & -2 \\ -5 & 3+\delta \end{pmatrix}$$

$$= \left(\frac{10}{1-3\delta}, \frac{\delta-7}{1-3\delta} \right)$$

$$y = -(A_B^T)^{-1} C_B = \begin{pmatrix} \frac{-10}{1-3\delta} \\ \frac{7-\delta}{1-3\delta} \end{pmatrix}$$

• Dual feasibility: $A^T y + c \geq 0$

$$A^T y + c = \begin{pmatrix} 3+\delta & 5 \\ 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{-10}{1-3\delta} \\ \frac{7-\delta}{1-3\delta} \end{pmatrix} + \begin{pmatrix} -5 \\ -1 \\ 12 \\ 0 \end{pmatrix} \geq 0$$

$$\frac{1}{1-3\delta} \begin{pmatrix} 5-15\delta \\ 1-3\delta \\ -10 \\ 7-\delta \end{pmatrix} \geq \begin{pmatrix} 5 \\ 1 \\ -12 \\ 0 \end{pmatrix}$$

$$\begin{aligned} 5-15\delta &\geq 5 \\ 1-3\delta &\geq 1 \\ -10 &\geq -12 \\ 7-\delta &\geq 0 \end{aligned}$$

$$\begin{aligned} \delta &\leq 1/3 \\ \delta &\leq 1/18 \end{aligned}$$

Notice: The 1st 2 constraints tight (near $\delta=0$)

$$10 \leq 12(1-3\delta)$$

$$\frac{5}{6} \leq 1-3\delta$$

$$3\delta \leq \frac{1}{6}$$

$\delta \leq \frac{1}{18}$

Another Farkas Lemma

Prove that exactly 1 of the following 2 statements holds

(1) $\exists x \text{ s.t. } Ax \leq b$

(2) $\exists y \text{ s.t. } A^T y = 0, b^T y < 0, y \geq 0$

• Claim: Both can't be true

Assume $\exists x \text{ s.t. } Ax \leq b, \exists y \text{ s.t. } A^T y = 0, y \geq 0$

$$(Ax - b)^T y = -b^T y + x^T A^T y = \underbrace{-b^T y}_{\leq 0} + \underbrace{x^T A^T y}_{\geq 0} \Rightarrow b^T y \geq 0 \text{ contradiction}$$

• Claim: Both can't be false

$$\begin{array}{ll} \min_x & 0 \\ \text{s.t. } & Ax \leq b \end{array} \xrightarrow{\text{dual}} \begin{array}{ll} \max_y & -b^T y \\ \text{s.t. } & A^T y = 0 \\ & y \geq 0 \end{array}$$

Case 1: $\exists x \text{ s.t. } Ax \leq b \rightarrow (1) \text{ holds}$

Case 2: $\nexists x \text{ s.t. } Ax \leq b$

- Primal is feasible
- Dual also always feasible ($y \geq 0$)
- By strong duality, dual is unbounded

$$\exists y \text{ s.t. } A^T y = 0, y \geq 0, -b^T y \geq 0$$