Agenda

- Branch & Bound Overview
- Disjunctive constraints problem
- Production/Distribution problem

Logistics

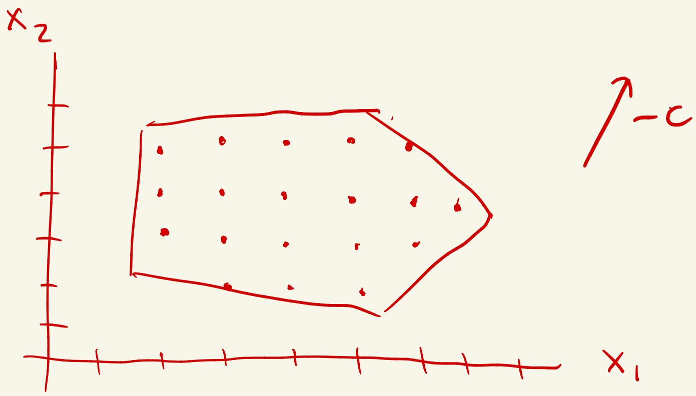
- HW8 out due Friday April 30 9pm
- This is the last precept
- Final project
  - more details to be announced
  - coding project
  - 24 hours to complete
  - Window open May 5
  - Window closed May 7
- Midterm 2 - 9 point bonus given

Graduate School

- Reach out to me if you're interested/ have questions

Mixed integer program

$$p^{IP} = \begin{array}{ll} \min_x & c^T x \\ \text{s.t.} & Ax \le b \\ & x_i \in \mathbb{Z}, \ i \in \mathcal{I} \end{array}$$



Relaxation

$$p^{rel} = \begin{array}{ll} \min_x & c^T x \\ \text{s.t.} & Ax \le b \end{array}$$

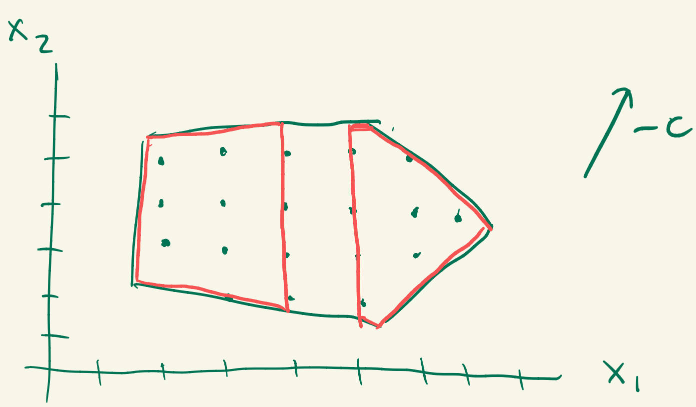
$p^{rel} \le p^{IP}$ way to achieve LBs

If x^{*rel} is integer then $p^{rel} = p^{IP}$,

– this happens if $\text{conv}\, P = \{Ax \le b\}$

Branch & Bound

- main idea: divide & conquer



Partition into smaller sets

Solve subproblems

$$\Phi(S^j) = \begin{array}{ll} \min_x & c^T x \\ \text{s.t.} & x \in S^j \end{array}$$

$$\Phi_{LB}(S^j) \le \Phi(S^j) \le \Phi_{UB}(S^j)$$

relaxation

any feas. point

Branch & Bound Algorithm

$$\min_x \; c^T x \quad \text{s.t. } x \in P_{IP} \qquad P_{IP} = \{ x \mid Ax \le b,\; x_i \in \mathbb{Z} \;\; i \in \mathcal{I} \}$$

Iterations

1. Branch: create/refine the partition of P_{IP} and get S^j
2. Bound:
   - Compute lower and upper bounds

     $$L_j = \Phi_{LB}(S^j) \qquad U_j = \Phi_{UB}(S^j) \quad \forall j$$

   - Update global bounds on $c^T x^*$

     $$L = \min_j L_j \qquad U = \min_j U_j$$

3. If $U - L \le \epsilon$ break

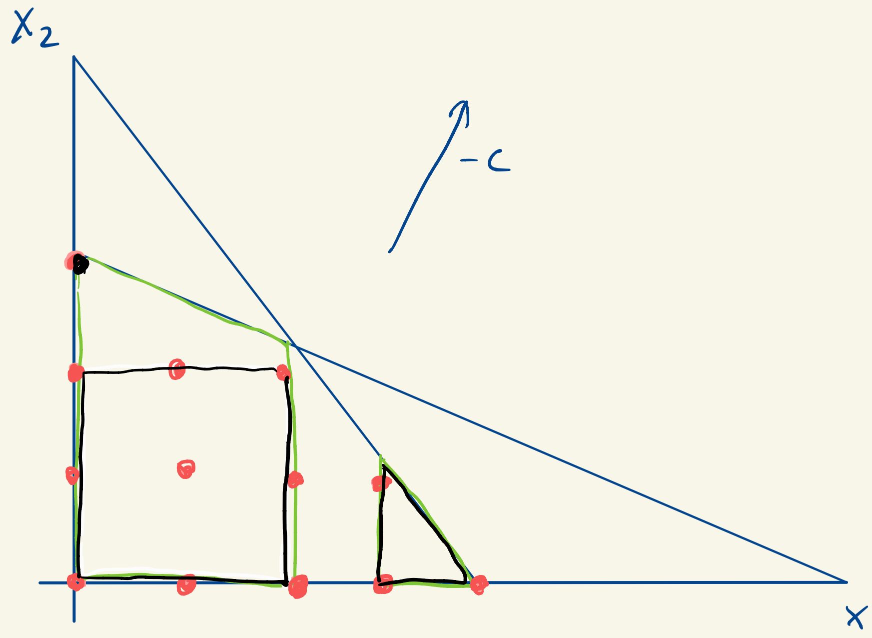
- Assume U is nonincreasing, L nondecreasing
- Pruning: S^j is active if $L_j \le \min_j U_j$

  else: it is inactive and we can prune it

Branch & Bound example

$$\min_{x} \; -2x_1 - 3x_2$$
$$\text{s.t. } \tfrac{2}{9}x_1 + \tfrac{1}{4}x_2 \le 1$$
$$\tfrac{1}{7}x_1 + \tfrac{1}{3}x_2 \le 1$$
$$x \ge 0$$
$$x \in \mathbb{Z}$$

$$x^{*IP} = (2,2)$$



$\bar{x} = (2.17, 2.06)$
$(-10.55, -10)$

$x_1 \le 2$ | $x_1 \ge 3$

$\bar{x} = (2, 2.14)$
$(-10.43, -10)$

$\bar{x} = (3, 1.33)$
$(-10, -9)$ } prune

$x_2 \le 2$ | $x_2 \ge 3$

$\bar{x} = (2,2)$
$(-10, -10)$

$\bar{x} = (0,3)$
$(-9, -9)$

(L, U)

relaxation

feasible point

Disjunctive constraints

Suppose we are given m constraints $a_i^T x \geq b_i \quad i=1,\dots,m$
Model the requirement that at least k of them are satisfied

Recall from lecture

goal: either $a^T x \leq b$ or $d^T x \leq f$ is valid

$$a^T x \leq b + yM$$
$$d^T x \leq f + (1-y)M$$
$$y \in \{0,1\}$$
$$x \in \mathbb{R}$$

$y \in \{0,1\}^m$ idea: $y_i = \begin{cases} 1 & \text{if constraint } i \text{ satisfied} \\ 0 & \text{else} \end{cases}$

$$\sum_{i=1}^{m} y_i \geq k$$

$$a_i^T x \geq b_i - (1-y_k)M \quad i=1,\dots,m$$

Production/ Distribution problem

A company produces a set of K products at I plants and ships these products to J market zones $(k=1,\dots,K)$ $(i=1,\dots,I)$ $(j=1,\dots,J)$

v_{ik}: cost of producing 1 unit of product k at plant i

c_{ijk}: cost of shipping 1 unit of product k from plant i to zone j

f_{ik}: fixed cost of producing product k at plant i

M_{ik}: maximal quantity of product k produced at plant i

m_{ik}: minimal quantity of product k that can be produced at plant i if plant i introduces a nonzero quantity

q_{ik}: capacity of plant i to produce 1 unit of product k

Q_i: capacity of plant i

d_{jk}: demand for product k at market zone j

(a) formulate minimizing the total cost as an integer program

vars: $x_{ik} \in \mathbb{Z}^{I,K}$: units of product k produced at plant i

$y_{ijk} \in \mathbb{Z}^{I,J,K}$: units of product k shipped from plant i to market zone j

$z_{ik} \in \{0,1\}^{I,K}$: binary variable indicating if product k produced at plant i

$$\min_{x,y,z} \sum_{i=1}^{I}\sum_{k=1}^{K} v_{ik}x_{ik} + \sum_{i=1}^{I}\sum_{k=1}^{K}\sum_{j=1}^{J} c_{ijk}y_{ijk} + \sum_{i=1}^{I}\sum_{k=1}^{K} f_{ik}z_{ik}$$

$$\begin{aligned}
\text{s.t. } & x_{ik} \le M_{ik}z_{ik} && \forall i,k \\
& x_{ik} \ge m_{ik}z_{ik} && \forall i,k \\
& \sum_{k=1}^{K} q_{ik}x_{ik} \le Q_i && \forall i \\
& d_{jk} = \sum_{i=1}^{I} y_{ijk} && \forall j,k \\
& \sum_{j=1}^{J} y_{ijk} = x_{ik}
\end{aligned}$$

$$x \in \mathbb{Z}^{I,K}$$
$$y \in \mathbb{Z}^{I,J,K}$$
$$z \in \{0,1\}^{I,K}$$

(b) No plant can produce more than K_1 products

$$\sum_{k=1}^{K} z_{ik} \le K_1 \qquad \forall i$$

(c) Every product can be produced in at most I_1 plants

$$\sum_{i=1}^{I} z_{ik} \le I_1 \qquad \forall k$$

(d) For a particular product k_0, plant 3 must produce it if neither plant 1 or plant 2 produces it

$$z_{1k_0} + z_{2k_0} + z_{3k_0} \ge 1$$

(e) Each market zone must be sourced by exactly one plant for all products

introduce new variables $w_{ijk} \in \{0,1\}^{I,J,K}$

$$w_{ijk} = \begin{cases} 1 & \text{if product } k \text{ shipped to zone } j \text{ from plant } i \\ 0 & \text{else} \end{cases}$$

$$w_{ijk} = 0 \Rightarrow y_{ijk} = 0$$

$$0 \le y_{ijk} \le w_{ijk} M$$

$$\sum_{i=1}^{I} w_{ijk} \le 1 \qquad \forall j, k$$