MATH 665 PROBLEM SET 1

FALL 2024

Due Monday, September 19. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

Problem 1.

(1) Compute the sizes of the conjugacy classes of S_4, S_5, S_6.
(2) Use (1) to show that A_4 is not simple, but A_5 and A_6 are.

Problem 2.

(1) Show that $|\mathrm{SL}_2(\mathbf{F}_7)| = 2 \cdot 3 \cdot 7 \cdot 8$.
(2) Find a reference listing the eleven conjugacy classes of $\mathrm{SL}_2(\mathbf{F}_7)$.
(3) Use (2) to compute the six conjugacy classes of
$$\mathrm{PSL}_2(\mathbf{F}_7) = \mathrm{SL}_2(\mathbf{F}_7)/\{\pm 1\}$$
and their sizes.
(4) Use (1) and (3) to show that $\mathrm{PSL}_2(\mathbf{F}_7)$ is simple.

Problem 3. Over $\bar{\mathbf{F}}_q$, for q odd, let $G = \mathrm{SL}_2$. Let $B = TU$ be its upper-triangular subgroup, where T is the diagonal torus and U the unipotent radical of B. Let $F : G \to G$ correspond to the split $\mathbf{F}_q$-form (**Sep 3**), so that B, T, U are F-stable. For any character χ of T^F, viewed as a character of B^F, let $I_\chi = \mathrm{Ind}_{B^F}^{G^F}(\chi)$.

(1) Taking $q = 3$:
   (a) Use Bruhat to find the number of double cosets of U^F in G^F.
   (b) For all χ, use Mackey (**Sep 5**) to decompose I_χ into its irreducible summands as a representation of G^F. The total number of summands, as we run over all χ, should match your answer to (a).
(2) Repeat (2), now taking $q = 5$.

Problem 4. Keep the setup of the previous problem. Recall the Deligne–Lusztig variety (**Sep 10**)
$$\tilde{X}_s = \{gU \in G/U \mid g^{-1}F(g) \in U\dot{s}U\}, \quad \text{where } \dot{s} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$
The G-action on $\mathbf{A}^2$ induces an isomorphism $G/U \xrightarrow{\sim} \mathbf{A}^2 \setminus \{0\}$. Show that at the level of $\bar{\mathbf{F}}_q$-points, this isomorphism identifies $\tilde{X}_s$ with the plane curve $xy^q - x^q y = 1$, where x, y are the standard coordinates on $\mathbf{A}^2$.

Problem 5. Let q be any prime power. Over $\bar{\mathbf{F}}_q$, let X be a smooth algebraic variety with an action of an algebraic group H. Suppose that there are Frobenius maps F on X and H such that $F(h \cdot x) = F(h) \cdot F(x)$. Show that:

(1) If H is *connected*, then every F-stable $H(\bar{\mathbf{F}}_q)$-orbit on $X(\bar{\mathbf{F}}_q)$ has an F-fixed point. *Hint:* Pick a point and apply Lang's theorem (**Sep 5**).
(2) In the setting of (1), deduce that there is a bijection $(X/H)^F \simeq X^F/H^F$.

(3) If H is not connected, then the conclusions to (1)–(2) fail, even when $X = \mathbf{A}^1$.

Problem 6. Over any algebraically closed field k, let $Z \subseteq \mathrm{GL}_2$ be the subgroup of scalar matrices, acting on the larger group by multiplication.

(1) Compute the subring $k[\mathrm{GL}_2]^Z \subseteq k[\mathrm{GL}_2]$.
(2) Deduce that the embedding $\mathrm{SL}_2 \to \mathrm{GL}_2$ descends to an isomorphism

$$\mathrm{GL}_2 /\!\!/ Z \xrightarrow{\sim} \mathrm{SL}_2 /\!\!/ \{\pm 1\}.$$

Above, $k[X /\!\!/ H] := k[X]^H$ for any algebraic variety X over k with an action of an algebraic group H.

This problem suggests why we prefer not to define an algebraic group PSL_2 distinct from PGL_2.[1]

[1] See https://mathoverflow.net/a/16150 for further context.