# Lusztig's classification of representations of finite reductive groups

We give an overview of Lusztig's classification of irreducible representations of reductive groups over finite fields and provide explicit examples. The parametrization involves a semisimple conjugacy class of the dual group and a unipotent representation of its centralizer, in a spirit similar to Langlands parameters for irreducible admissible representations of reductive groups over local fields.

This is a note prepared for an Alcove Seminar talk at Harvard, Spring 2015. Our main reference is [1].

[-] **Contents**





## Dual groups and the Jordan decomposition

Let $G$ be a connected reductive group over $\overline{\mathbb{F}}_q$ together with a Frobenius map $F: G \to G$ defining a $\mathbb{F}_q$-structure on $G$. Recall that the main theorem of Deligne-Lusztig says that every irreducible representation (over $\overline{\mathbb{Q}_\ell}$) of the finite group $G^F = G(\mathbb{F}_q)$ appears in the Deligne-Lusztig induction $R_T^G(\theta)$ for some $F$-stable (i.e., defined over $\mathbb{F}_q$) maximal torus $T \subseteq G$ and some character $\theta: T^F \to \overline{\mathbb{Q}_\ell}^\times$. Moreover, when $\theta$ is in general position (i.e., ${}^w\theta \neq \theta$ for any nontrivial element $w \in W^F$, where $W = N_G(T)/T$ is the Weyl group), $R_T^G(\theta)$ is the character (up to sign) of an irreducible representation of $G^F$.

To construct the remaining irreducible representations of $G^F$, one needs to further *decompose* reducible Deligne-Lusztig characters $R_T^G(\theta)$, which can be done using Howlett-Lehrer theory nowadays. After obtaining all irreducible representations, one also hopes to *parametrize* them in a structural manner so that one can actually use the classification in practice.

The first step of Lusztig's parametrization deals with the pair $(T, \theta)$ appeared in the Deligne-Lusztig characters. Recall that if two pairs $(T, \theta)$, $(T', \theta')$ are not $G^F$-conjugate, then $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ are orthogonal to each other. However, it may happen that $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ share the same irreducible factors (since they are *virtual* characters). To ensure that they don't share any irreducible factors, one needs the stronger notion of *geometric conjugacy classes*.

**Definition 1** Two pairs $(T, \theta)$, $(T', \theta')$ are *geometrically conjugate* if there exists some $F$ such that $(T, \theta \circ \mathbb{N}_{\mathbb{F}_{q^n}/\mathbb{F}_q})$ and $(T', \theta' \circ \mathbb{N}_{\mathbb{F}_{q^n}/\mathbb{F}_q})$ are $G^{F^n}$-conjugate.

By [2, Cor 6.3], if two pairs $(T, \theta)$ and $(T', \theta')$ are not geometrically conjugate, then $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ are *disjoint*. So the first step is to classify geometric conjugacy classes of the pair $(T, \theta)$. This is neatly done using the *dual group* $G^*$ of $G$. The dual group $G^*$ is a reductive group over $\overline{\mathbb{F}}_q$ with the dual root datum and the dual Frobenius $F: G^* \to G^*$ (defining an $\mathbb{F}_q$-structure on $G^*$). Notice this is *not* the Langlands dual group, which is defined over $\mathbb{C}$. By the very definition, the geometric conjugacy classes of $(T, \theta)$ then correspond bijectively to geometric conjugacy classes of semisimple elements of $(G^*)^F = G^*(\mathbb{F}_q)$.

**Remark 1** If $G$ further has connected center, then by a theorem of Steinberg, (the derived group of) $G^*$ is simply-connected (up to inseparable coverings) and the centralizer of a semisimple element of $G^*$ is *connected*. Hence by Lang's theorem, there is no distinction between conjugacy classes and geometric conjugacy classes, hence the geometric conjugacy classes of $(T, \theta)$ correspond bijectively to conjugacy classes of semisimple elements of $(G^*)^F$. We will assume $G$ has connected center for simplicity from now on.

**Remark 2** The set of semisimple conjugacy classes of $(G^*)^F$ has size $|Z^0(G)^F| \cdot q^l$, where $l$ is the semisimple rank of $G$ ([2, 5.6 (ii)]).

**Definition 2** Let $(s)$ be a semisimple conjugacy class of the dual group $(G^*)^F$, the *Lusztig series* $\mathcal{E}(G^F, (s))$ is the set of irreducible representations of $G^F$ appearing in $R_T^G(\theta)$ for any pair $(T, \theta)$ corresponding to $(s)$.

In particular, the Lusztig series $\mathcal{E}(G^F, (s))$ form a disjoint partition of $\mathrm{Irr}(G^F)$. The second step is to parametrize each Lusztig series.

**Definition 3** The representations in the Lusztig series $\mathcal{E}(G^F, (1))$ are called *unipotent representations*.

**Remark 3** These are simply the irreducible factors of $R_T^G(\mathbf{1})$ where $T$ runs over all $F$-stable maximal torus. As we will see very soon, these unipotent representations are closely related to unipotent conjugacy classes.

**Remark 4** The study of unipotent representations can be reduced to the adjoint case: $Z(G)^F$ acts trivially on the Deligne-Lusztig varieties and thus lies in the kernel of every unipotent representation of $G^F$.

When $(s)$ is regular semisimple, $(T,\theta)$ is in general position and hence the Lusztig series is a singleton. When $(s)=(1)$ is trivial, the Lusztig series consists of the unipotent representations of $G^F$. When $(s)$ lies between the two extremal cases, it turns out that the Lusztig series $\mathcal{E}(G^F,(s))$ can be parametrized by the unipotent representations of a *smaller* connected reductive group $H$, where $H$ is the dual of the centralizer $Z_{G^*}(s)$ ([3, Theorem 4.23]).

**Theorem 1** Let $(s)$ be a semisimple conjugacy class of $(G^*)^F$ and $H$ be the dual group of the centralizer $Z_{G^*}(s)$. Then there is a bijection
$$\mathcal{E}(G^F,(s))\to\mathcal{E}(H^F,(1)),\quad \chi\mapsto\chi_u$$
such that for any pair $(T,\theta)$ corresponding to $s\in(G^*)^F$,
$$\langle\chi,\varepsilon_G\varepsilon_T\cdot R_T^G(\theta)\rangle=\langle\chi_u,\varepsilon_H\varepsilon_S\cdot R_S^H(\psi)\rangle,$$
where $(S,\psi)$ is any pair corresponding to $s\in(H^*)^F$ and $\varepsilon_G=(-1)^{\mathbb{F}_q\text{-}\mathrm{rk}(G)}$.

**Remark 5** The requirement in the theorem does not characterize the bijection completely. However, it implies that the degree $\chi(1)$ is (up to sign) the product of $\chi_u(1)$ and the prime-to-$q$ part of $|G^F|/|H^F|$. The latter is exactly the degree of the unique semisimple character $\chi_s\in\mathcal{E}(G^F,(s))$ associated to $(s)$ ([1, 8.4.8]). So this theorem can be thought of as a *Jordan decomposition* for irreducible representations. We remark that the construction of the bijection is done case by case *after* decomposing all the Deligne-Lusztig characters $R_T^G(\theta)$.

By induction, the problem remaining is to parametrize the unipotent representations for any connected reductive group $G$. This is done case by case in Lusztig's several papers and books ([4],[5],[6], [7],[8], [9]). We aim to summarize his results.

## Unipotent representations

Classifying all unipotent representations again uses an inductive strategy in terms of unipotent *cuspidal* representations and parabolic induction. For each classical group, there is either no or only one unipotent cuspidal representation. The cases for which there exists one unipotent cuspidal representation are listed as follows.

| $B_l$ | $C_l$ | $D_l$ | ${}^2A_l$ | ${}^2D_l$ |
|---|---|---|---|---|
| $l=s^2+s$ | $l=s^2+s$ | $l=s^2, 2\mid s$ | $l=\frac{1}{2}(s^2+s)-1$ | $l=s^2, 2\nmid s$ |

For each exceptional group, there are at least two unipotent cuspidal representations. The number of unipotent cuspidal representations and the total number of unipotent representations are listed as follows.

| $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ | ${}^3D_4$ | ${}^2E_6$ | ${}^2B_2$ | ${}^2G_2$ | ${}^2F_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 13 | 7 | 4 | 2 | 3 | 2 | 6 | 10 |
| 30 | 76 | 166 | 37 | 10 | 8 | 30 | 4 | 8 | 21 |

**Remark 6** A systematical supply of unipotent cuspidal representations comes from the $F$*-eigenspaces* of the middle cohomology $H_c^l(X_w,\overline{\mathbb{Q}_\ell})$ of the Deligne-Lusztig varieties associated to the Coxeter element $w\in W$ ([10]). In each case Lusztig was able to find most of the unipotent cuspidal representations. He then counted the number unipotent representations thus obtained. Together with a formula for the sum of the squares of the dimension of unipotent representations, he concluded that a very small number of unipotent cuspidal representation is missing: 3 for $F_4$, 7 for $E_8$ and at most *one* in the other cases (which also explains the extra cases for classical groups). The remaining cuspidal ones are then constructed case by case using brutal force.

After all the unipotent representations are obtained, Lusztig observed that they naturally form families in a remarkable way, parametrized by *special unipotent conjugacy classes* (justifying the name).

**Example 1** To see how this might be possible, let us look at the case $G=\mathrm{PGL}_n(\mathbb{F}_q)$. It was known from Green's work that each irreducible unipotent character of $G$ is of the form
$$R_\phi=\frac{1}{|W|}\sum_{w\in W}\phi(w)\cdot R_w(\mathbf{1}),\qquad(1)$$
a linear combination of the Deligne-Lusztig characters $R_w(\mathbf{1})$ weighted by an irreducible character $\phi\in\mathrm{Irr}(W)$ of the Weyl group $W\cong S_n$. This gives a bijection between unipotent representations of $G$ and $\mathrm{Irr}(S_n)$. On the other hand, the irreducible representations of $S_n$ is in bijection with the partitions of $F$, hence is in bijection with unipotent conjugacy classes of $G$ (= shape of Jordan blocks)! In particular, the unipotent representations should be parametrized in a way that is independent of $q$!

**Definition 4** We say two irreducible unipotent character $\chi$ and $\chi'$ are *in the same family* if there exists a chain of characters $\chi=\chi_1,\chi_2,\cdots,\chi_k=\chi'$ and $\phi_1,\ldots,\phi_{k-1}\in\mathrm{Irr}(W)$ such that $\langle\chi_i,R_{\phi_i}\rangle\neq 0$ and $\langle\chi_{i+1},R_{\phi_i}\rangle\neq 0$.

It turns out that something stronger is true: for a family $\mathcal{F}$, there exists a *unique* $\phi\in\mathrm{Irr}(W)$ such that $\langle\chi,\phi\rangle>0$ for any $\chi\in\mathcal{F}$. This gives a bijection between families and a subset of $\mathrm{Irr}(W)$, called *special characters* of $W$. All characters are special for type $A_l$, but not for other types of groups. By the Springer correspondence, there is a bijection between $\mathrm{Irr}(W)$ and certain pairs $(u,\sigma)$, where $u$ is a unipotent conjugacy class of $G^{\mathrm{ad}}(\mathbb{C})$ and $\sigma\in\mathrm{Irr}(A(u))$ (here $A(u):=Z_G(u)/Z_G(u)^0$). Moreover, even though $\sigma$ may be nontrivial, it turns out that the set of unipotent classes of $G^{\mathrm{ad}}(\mathbb{C})$ injects into $\mathrm{Irr}(W)$. Those unipotent classes corresponding to the special characters of $W$ are naturally called *special unipotent*.

Now Lusztig's parameter for an irreducible representation of $G(\mathbb{F}_q)$ consists of a semisimple conjugacy class $(s)$ of $G^*(\mathbb{F}_q)$ and a special unipotent conjugacy class $(u)$ of $H^{\mathrm{ad}}(\mathbb{C})$, where $H$ is the Langlands dual group of $C_{G^*}(s)$. The shape of Lusztig's parameters is a reminisce of local Langlands parameters: the Weil group of $\mathbb{F}_q$ is simply $\mathbb{Z}$ and a homomorphism from the Weil group to $G^*(\mathbb{F}_q)$ corresponds to a semisimple class $(s)$. However, $G^*$ is not the Langlands dual group and one needs to add the adjective *special* to the complex unipotent class $(u)$.

**Remark 7** The finite component group $A(u)$ is an elementary 2-group $(\mathbb{Z}/2\mathbb{Z})^e$ when $G$ is of classical type and is a symmetric group on $\leq 5$ letters when $G$ is of exceptional type. Miraculously, the members of a family $\mathcal{F}$ can be parameterized by pairs $(x,\psi)$, where $x\in A(u)$ and $\psi\in\mathrm{Irr}(Z_{A(u)}(x))$. Families thus can be thought as analogues of $L$-packets. Notice that $\mathcal{F}$ has size 1, $4^e$, 8, 21 or 39 according to $A(u)=\{1\},(\mathbb{Z}/2\mathbb{Z})^e,S_3,S_4,S_5$. (Added on 04/18/2016: as Cheng-Chiang pointed out to me, this is not quite true. The correct statement is to replace $A(u)$ appearing in this remark by a certain quotient of $A(u)$ known as *Lusztig quotient group*.)

**Example 2** For $G=E_8$, there are 46 families of unipotent characters corresponding to the 46 special representations of the Weyl group $W(E_8)$. There are 23 families with 1 character, 18 families with 4 characters, 4 families with 8 characters, and one family with 39 characters, which gives a total number of $23+18\cdot4+4\cdot8+39=166$ unipotent representations.

## Examples

For classical groups, Lusztig parametrized the unipotent representations using *symbols*, a combinatorial gadget (independent of $q$) corresponding to special characters of $W$.

**Example 3** For $G=\mathrm{PGL}_n(\mathbb{F}_q)$. As in Example 1, the symbols are simply partitions of $F$. The partition $n=n$ corresponds to the trivial representation of $W$ and the trivial representation of $G$. The partition $n=1+1+\cdots+1$ corresponds to the sign representation of $W$ and the Steinberg representation of $G$, which has dimension $q^{\dim U}=q^{n(n-1)/2}$ ($U$ is the maximal unipotent subgroup of $G$). For a general partition $n=a_1+a_2+\cdots+a_m$, $a_1\leq a_2\leq\cdots\leq a_m$. Let $b_1=a_1$, $b_2=a_2+1$, $b_3=a_3+2$ and so on. Then the corresponding unipotent representation has dimension

$$\frac{(q-1)(q^2-1)\cdots(q^n-1)\prod_{i<j}(q^{b_j}-q^{b_i})}{q^{\binom{m-1}{2}+\binom{m-2}{2}+\cdots}\prod_{i=1}^m\prod_{k=1}^{b_i}(q^k-1)}.$$

For example,

- $3=1+2$ corresponds to the standard representation of $S_3$ of dimension 2. The corresponding unipotent representation has dimension
$$\frac{(q-1)(q^2-1)(q^3-1)(q^3-q^1)}{(q-1)(q-1)(q^2-1)(q^3-1)}=q(q+1)=q^2+q.$$
- $4=1+3$ corresponds to the standard representation of $S_4$ of dimension 3. The corresponding unipotent representation has dimension
$$\frac{(q-1)(q^2-1)(q^3-1)(q^4-1)(q^4-q)}{(q-1)(q-1)(q^2-1)(q^3-1)(q^4-1)}=q(q^2+q+1)=q^3+q^2+q.$$
- $4=2+2$ corresponds to the 2-dimensional representation of $S_4$. The corresponding unipotent representation has dimension
$$\frac{(q-1)(q^2-1)(q^3-1)(q^4-1)(q^3-q^2)}{(q-1)(q^2-1)(q-1)(q^2-1)(q^3-1)}=q^4+q^2.$$
- $4=1+1+2$ corresponds to the twisted (by the sign character) standard representation of $S_4$. The corresponding unipotent representation has dimension
$$\frac{(q-1)(q^2-1)(q^3-1)(q^4-1)(q^4-q)(q^4-q^2)(q^2-q)}{q(q-1)(q-1)(q^2-1)(q-1)(q^2-1)(q^3-1)(q^4-1)}=q^5+q^4+q^3.$$

We conclude that

- $\mathrm{PGL}_2(\mathbb{F}_q)$ has 2 unipotent representations: they have dimension 1 and $q$.
- $\mathrm{PGL}_3(\mathbb{F}_q)$ has 3 unipotent representations: they have dimension 1, $q^2+q$, and $q^3$.

- $\mathrm{PGL}_4(\mathbb{F}_q)$ has 5 unipotent representations: they have dimension 1, $q^3+q^2+q$, $q^4+q^2$, $q^5+q^4+q^3$, $q^6$.

It could be a lot of fun to check the identity computing the number of $\mathbb{F}_q$-points of the flag variety

$$G(\mathbb{F}_q)/B(\mathbb{F}_q) = \dim R_{\{1\}}(\mathbf{1}) = \sum_{\chi\in\mathrm{Irr}(W)} \chi(1)\chi_u(1),$$

where $\chi_u$ is the unipotent character corresponding to the special character $\chi\in\mathrm{Irr}(W)$. For example, when $n=4$ this says

$$\frac{(q^4-1)(q^3-1)(q^2-1)}{(q-1)^3} = 1+3(q^3+q^2+q)+2(q^4+q^2)+3(q^5+q^4+q^3)+q^6.$$

**Example 4** Consider $G=\mathrm{PGL}_2(\mathbb{F}_q)$. We list Lusztig's parametrization as follows.

| $(s)$ | $H$ | $\lvert\chi_s(1)\rvert=\frac{\lvert G^F\rvert}{\lvert H^F\rvert}_{q'}$ | $\lvert\chi_u(1)\cdot\chi_s(1)\rvert$ |
|---|---|---|---|
| $\mathrm{diag}\{1,1\}$ | $\mathrm{PGL}_2$ | $1$ | $1(\text{trivial}), q(\text{Steinberg})$ |
| $\mathrm{diag}\{\zeta_{q-1},1\}$ | $\mu_{q-1}$ | $q+1$ | $q+1(\text{principal series})$ |
| $\mathrm{diag}\{\zeta_{q+1},\zeta_{q+1}^q\}$ | $\mu_{q+1}$ | $q-1$ | $q-1(\text{cuspidal})$ |

**Remark 8** Using the orthogonal relation, the formula (1) is also equivalent to

$$R_w(\mathbf{1}) = \sum_{\phi\in\mathrm{Irr}(W)} \phi(w)\cdot R_\phi.$$

We see that the parabolic induction is $R_1(\mathbf{1})=\mathbf{1}+\mathrm{St}$ and the Deligne-Lusztig induction (realized on the Drinfeld curve) is $R_w(\mathbf{1})=\mathbf{1}-\mathrm{St}$. See Drinfeld curves for an elementary introduction.

**Remark 9** When $q=2$, $G=\mathrm{PGL}_2(\mathbb{F}_2)\cong S_3$. The two unipotent representations are the trivial representations and the 2-dimensional representation. There is a unique cuspidal representation, the sign representation of $S_3$. A cool consequence: the local representation at 2 of any elliptic curve over $\mathbb{Q}$ with conductor $4D$ is the *unique* depth zero supercuspidal representation of $\mathrm{PGL}_2(\mathbb{Q}_2)$, i.e., the compact induction of the $\mathrm{PGL}_2(\mathbb{Z}_2)\to\mathrm{PGL}_2(\mathbb{F}_2)\cong S_3\xrightarrow{\text{sign}}\pm1$.

**Example 5** Consider $G=\mathrm{PGL}_4(\mathbb{F}_2)\cong A_8$. Using the hook-length formula we find that it has the irreducible representations of the following dimensions: 1, 7, 14, 20 ,21, 21, 21, 28, 35, 45, 45, 56, 64, 70. We check this using Lusztig's classification (plug in $q=2$ in the previous example).

| $(s)$ | $H$ | $\lvert\chi_s(1)\rvert=\frac{\lvert G^F\rvert}{\lvert H^F\rvert}_{q'}$ | $\lvert\chi_u(1)\cdot\chi_s(1)\rvert$ |
|---|---|---|---|
| $\mathrm{diag}\{1,1,1,1\}$ | $\mathrm{PGL}_4$ | $1$ | $1,14,20,56,64$ |
| $\mathrm{diag}\{\zeta_3,\zeta_3^2,1,1\}$ | $\mathrm{PGL}_2\times\mu_3$ | $\frac{15\cdot7\cdot3}{3\cdot3}=35$ | $35,70$ |
| $\mathrm{diag}\{\zeta_3,\zeta_3^2,\zeta_3,\zeta_3^2\}$ | $\mathrm{PGL}_{2/\mathbb{F}_4}$ | $\frac{15\cdot7\cdot3}{15\cdot3}=7$ | $7,28$ |
| $\mathrm{diag}\{\zeta_7,\zeta_7^2,\zeta_7^4,1\}$ | $\mu_7$ | $\frac{15\cdot7\cdot3}{7}=45$ | $45,45$ |
| $\mathrm{diag}\{\zeta_{15},\zeta_{15}^2,\zeta_{15}^4,\zeta_{15}^8\}$ | $\mu_{15}$ | $\frac{15\cdot7\cdot3}{15}=21$ | $21,21,21$ |

In particular, we see that there are three cuspidal representations, all of them have dimension 21.

**Example 6** Define a *symbol* for type $BC$ to be a tuple $(\lambda_1,\ldots,\lambda_a;\mu_1,\ldots,\mu_b)$, where $\lambda_i$, $\mu_j$ are non-negative and strictly increasing, $b-a$ is a positive odd integer, and $\lambda_1,\mu_1$ are not both zero. Define the *rank* of this symbol by $\sum\lambda_i+\sum\mu_j-((a+b-1)/2)^2$. Then there is a bijection between unipotent characters of type $B_l$ and $C_l$ and symbols of rank $l$. There is a similar formula for the dimension of the corresponding unipotent character as that for type $A$. A unipotent character with $b-a=2s+1$ appears in the parabolic induction of the unique unipotent cuspidal representation of $B_{s^2+s}$ or $C_{s^2+s}$. The unipotent characters fall into families according to the multi-set $\{\lambda_1,\ldots,\lambda_a,\mu_1,\ldots,\mu_b\}$.

For example, consider $G=\mathrm{PGSp}_4(\mathbb{F}_q)$. There are 6 symbols of rank 2, corresponding to 6 unipotent representations with the following dimensions:

- $(2;)$, 1 (trivial);
- $(1,2;0)$, $\frac{1}{2}q(q^2+1)$;
- $(0,2;1)$, $\frac{1}{2}q(q+1)^2$;
- $(0,1;2)$, $\frac{1}{2}q(q^2+1)$;
- $(0,1,2;)$, $\frac{1}{2}q(q-1)^2$ (unipotent cuspidal);
- $(0,1,2;1,2)$, $q^4$ (Steinberg).

They form families of size 1, 4 and 1.

**Example 7** $G=\mathrm{PGSp}_4(\mathbb{F}_2)\cong S_6$. Using the hook-length formula we find that it has irreducible representations of the following dimensions: 1, 1, 5, 5, 5, 5, 9, 9, 10, 10, 16. We check this using Lusztig's classification (plug in $q=2$ in the previous example).

| $(s)$ | $H$ | $\lvert\chi_s(1)\rvert = \frac{\lvert G^F\rvert}{\lvert H^F\rvert_{q'}}$ | $\lvert\chi_u(1)\cdot\chi_s(1)\rvert$ |
|---|---|---|---|
| $\mathrm{diag}\{1,1,1,1\}$ | $\mathrm{PGSp}_4$ | $1$ | $1,5,9,5,1,16$ |
| $\mathrm{diag}\{\zeta_3,1,\zeta_3^{-1},1\}$ | $\mathrm{PGL}_2\times\mu_3$ | $\frac{5\cdot3\cdot3}{3\cdot3}=5$ | $5,10$ |
| $\mathrm{diag}\{\zeta_3,\zeta_3^2,\zeta_3^{-1},\zeta_3^{-2}\}$ | $\mathrm{U}_2$ | $\frac{5\cdot3\cdot3}{3\cdot3}=5$ | $5,10$ |
| $\mathrm{diag}\{\zeta_5,\zeta_5^2,\zeta_5^{-1},\zeta_5^{-2}\}$ | $\mu_5$ | $\frac{5\cdot3\cdot3}{5}=9$ | $9$ |

In particular, there are two cuspidal representations: they have dimension 1 (unipotent, the sign character of $S_6$ ) and 9.

## References


[1] Carter, Roger W., *Finite groups of Lie type,* John Wiley \& Sons, Inc., New York, 1985.

[2] Deligne, P. and Lusztig, G., *Representations of reductive groups over finite fields,* Ann. of Math. (2) **103** (1976), no.1, 103--161.

[3] Lusztig, George, *Characters of reductive groups over a finite field,* Princeton University Press, Princeton, NJ, 1984.

[4] Lusztig, G., *Irreducible representations of finite classical groups,* Invent. Math. **43** (1977), no.2, 125--175.

[5] Lusztig, George, *Representations of finite Chevalley groups,* American Mathematical Society, Providence, R.I., 1978.

[6] Lusztig, George, *Unipotent representations of a finite Chevalley group of type $E_8$,* Quart. J. Math. Oxford Ser. (2) **30** (1979), no.119, 315--338.

[7] Lusztig, George, *On the unipotent characters of the exceptional groups over finite fields,* Invent. Math. **60** (1980), no.2, 173--192.

[8] Lusztig, George, *Unipotent characters of the symplectic and odd orthogonal groups over a finite field,* Invent. Math. **64** (1981), no.2, 263--296.

[9] Lusztig, George, *Unipotent characters of the even orthogonal groups over a finite field,* Trans. Amer. Math. Soc. **272** (1982), no.2, 733--751.

[10] Lusztig, G., *Coxeter orbits and eigenspaces of Frobenius,* Invent. Math. **38** (1976/77), no.2, 101--159.