CHAPTER 1. TOPOLOGY OF ALGEBRAIC VARIETIES, HODGE DECOMPOSITION, AND APPLICATIONS

Contents

In this chapter we will review a number of fundamental facts on the topology of smooth complex projective varieties, and the Hodge decomposition of their singular cohomology with complex coefficients. We will then see them in action by proving the Kodaira Vanishing theorem, the invariance of Hodge numbers under deformations, and the birational invariance of certain Hodge numbers. Some basic references for this material are [GH] Chapter 0 and 1, [La] §3.1 and §4.2, and [Vo].

1. The Lefschetz hyperplane theorem

Theorem 1.1 (**Lefschetz hyperplane theorem**). *Let X be a smooth complex projective variety of dimension n, and let D be an effective ample divisor on X. Then the restriction map*

$$r_i : H^i(X, \mathbf{Z}) \longrightarrow H^i(D, \mathbf{Z})$$

is an isomorphism for $i \le n-2$, and injective for $i = n-1$.

Proof. A conceptual approach is via the following theorem essentially saying that complex affine manifolds have only half as much topology as one might expect:[1]

Theorem 1.2 (**Andreotti-Frankel**). *Let $Y \subset \mathbf{C}^r$ be a closed n-dimensional complex submanifold. Then Y has the homotopy type of a CW complex of real dimension $\le n$. As a consequence*

$$H^i(Y, \mathbf{Z}) = 0 \text{ and } H_i(Y, \mathbf{Z}) = 0 \text{ for } i > n.$$

[1] Note that every $\mathcal{C}^\infty$ manifold of real dimension $2n$ has the homotopy type of a CW complex of real dimension $\le 2n$.

Assuming this for now, let's continue with the proof of the theorem on hyperplane sections. Since D is ample, for some $m \gg 0$ we have that mD is very ample, and therefore there exists an embedding $X \subset \mathbf{P}^N$ and a hyperplane H in $\mathbf{P}^N$ such that $mD = X \cap H$. This implies that $Y = X - D = X - mD$ is a smooth affine complex variety of dimension n. The Andreotti-Frankel theorem implies then that $H_j(Y, \mathbf{Z}) = 0$ for $j > n$. On the other hand, for all j one has by Alexander-Lefschetz duality[2]

$$H_j(Y, \mathbf{Z}) \simeq H^{2n-j}(X, D; \mathbf{Z})$$

and therefore $H^i(X, D; \mathbf{Z}) = 0$ for $i < n$. This is equivalent to the desired conclusion, by the long exact sequence of (relative) cohomology

$$\ldots \longrightarrow H^i(X, D; \mathbf{Z}) \longrightarrow H^i(X, \mathbf{Z}) \longrightarrow H^i(D, \mathbf{Z}) \longrightarrow H^{i+1}(X, D; \mathbf{Z}) \longrightarrow \ldots$$

□

The proof of Theorem 1.2 is a very nice application of basic Morse theory, as in [Mi]. We start by recalling some of its fundamental facts. Let M be a $\mathcal{C}^\infty$ manifold of real dimension n, and let

$$f : M \longrightarrow \mathbf{R}$$

be a $\mathcal{C}^\infty$ function on M. Recall that $p \in M$ is a *critical point* of f if $df_p = 0$, in which case $q = f(p)$ is the corresponding *critical value*. If p is a critical point, then there exists a symmetric bilinear form called the *Hessian* of f at p,

$$\mathrm{Hess}(f)_p = d^2 f_p : T_pX \times T_pX \longrightarrow \mathbf{R},$$

given in local coordinates by $\mathrm{Hess}(f)_p = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$. We say that the p is a *nondegenerate* critical point if $\mathrm{Hess}(f)_p$ is nondegenerate, in which case we define

$$\lambda_p = \mathrm{index}_p(f) = \text{ number of negative eigenvalues of } \mathrm{Hess}(f)_p.$$

The Morse Lemma [Mi] Lemma 2.2 states that in suitable local coordinates around a nondegenerate critical point, f can be written as the quadratic function

$$f(x_1, \ldots, x_n) = -x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2.$$

Theorem 1.3 (**Basic theorem of Morse Theory,** [Mi] **Theorem 3.5**)**.** *With the notation above, assume that f has the property that $f^{-1}((-\infty, a])$ is compact for every $a \in \mathbf{R}$. Assume in addition that f has only nondegenerate critical points. Then M has the homotopy type of a CW complex with one cell of dimension λ for each critical point of index λ.*

Example 1.4. The height function on the standard two-dimensional torus is a typical example of a function as in Theorem 1.3. The index at each of the critical points can easily be computed to be as in Figure 1. We recover the well known fact that the two-dimensional torus has the homotopy type of a *CW* complex with 1 cell of dimension 0, 2 cells of dimension 1, and 1 cell of dimension 2.

[2] See e.g. [Hat] §3.3; note that Poincaré duality is the special case $D = 0$.

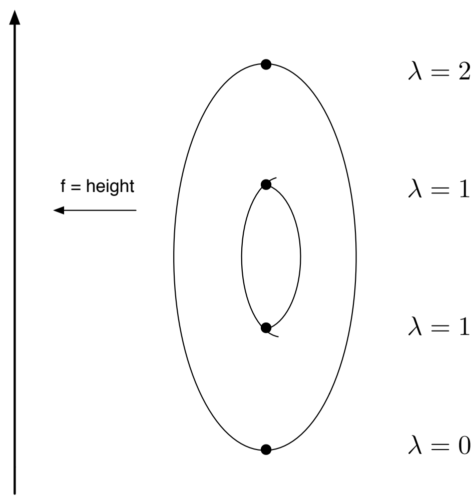

FIGURE 1. Morse theory for the standard torus.

For us, the key example of a function with only nondegenerate critical points (a *Morse function*) is given by the following construction. Assume that $M \subset \mathbf{R}^m$ is a closed submanifold of dimension n. Given $c \in \mathbf{R}^m$, define

$$\varphi_c : M \longrightarrow \mathbf{R}, \ \ \varphi_c(p) = ||p - c||^2,$$

i.e. the usual Euclidian distance in $\mathbf{R}^m$.

Lemma 1.5 ([Mi] Theorem 6.6)**.** *For almost all $c \in \mathbf{R}^m$, φ_c has only nondegenerate critical points.*

Note that φ_c obviously satisfies the other condition in the statement of Theorem 1.3.

Proof. (of Theorem 1.2.) According to the discussion above, the Theorem reduces to giving an upper bound for the index of a critical point for the distance function on a closed submanifold of $\mathbf{C}^r$. More precisely, it is a consequence of the following result, combined with Theorem 1.3. $\square$

Lemma 1.6. *Let M be an n-dimensional closed complex submanifold of $\mathbf{C}^r = \mathbf{R}^{2r}$, and let $c \in \mathbf{C}^r$. If p is a nondegenerate critical point of φ_c, then $\lambda_p \leq n$.*

Proof. Here is a sketch of an elementary proof, as in the original work of Andreotti-Frankel. (A more conceptual proof can be found for instance in [Mi] §7.) Write $k = r - n$. Choose coordinates on $\mathbf{C}^r$ in such a way that $p = 0$, $c = (0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in the $(n+1)$-st position, and M is the graph of a holomorphic function $f : \mathbf{C}^n \to \mathbf{C}^k$ with $df_0 = 0$. (Exercise: check that you can do this.) In other words, denoting the coordinates by $z_1, \ldots, z_n$, we have

$$M = \{\big(z_1, \ldots, z_n, f_1(z), \ldots, f_k(z)\big) \mid f_1, \ldots, f_k \text{ holomorphic with } \mathrm{ord}_0(f_i) \geq 2\}.$$

The distance function φ_c is then given by the formula

$$\varphi_c(z) = (1 - 2 \cdot \mathrm{Re}\ f_1(z)) + \sum_{i=1}^{n} |z_i|^2 + \sum_{i=2}^{k} |f_i(z)|^2.$$

Since $\mathrm{ord}_0(f_i) \geq 2$ for all i, the last sum in the formula does not contribute to $\mathrm{Hess}(\varphi_c)_0$. Now write

$$f_1(z) = Q(z) + \text{ terms of order } \geq 3,$$

where $Q(z)$ is a homogeneous quadratic polynomial in $z_1, \ldots, z_n$. Putting all of this together we get

$$\mathrm{Hess}(\varphi_c)_0 = -2 \cdot \mathrm{Hess}(\mathrm{Re}\ Q(z))_0 + 2 \cdot \mathrm{Id}.$$

As the second term is positive definite, the result follows from the following standard

Lemma 1.7. *If Q is a complex homogeneous quadratic polynomial in $z_1, \ldots, z_n$, then* $\mathrm{Hess}(\mathrm{Re}\ Q(z))_0$ *has at most n positive and at most n negative eigenvalues.*

Proof. After a complex change of coordinates $z \to w$ one can write

$$Q(w) = w_1^2 + \ldots + w_s^2$$

with $s \leq n$. Writing $w_j = x_j + i \cdot y_j$, we have

$$\mathrm{Re}\ Q(w) = (x_1^2 - y_1^2) + \ldots + (x_s^2 - y_s^2),$$

and for this the statement is clear. □

□

2. The Hodge decomposition

For a smooth complex projective variety, or more generally a compact Kähler manifold, a fundamental result is the Hodge decomposition of its singular cohomology with complex coefficients.

Theorem 2.1 (Hodge decomposition). *If X is a compact Kähler manifold, then there is a decomposition*

$$H^i(X, \mathbf{C}) \simeq \bigoplus_{p+q=i} H^{p,q}(X)$$

with:

- $H^{p,q}(X) = \overline{H^{q,p}(X)}$
- $H^{p,q}(X) \simeq H^q(X, \Omega_X^p)$.

As $H^{p,q}$ usually denotes the cohomology group obtained from (p,q)-forms via the $\overline{\partial}$ operator, the second bullet is in fact the independent *Dolbeaut Theorem*. In particular, the $H^{q,0}(X)$ spaces can be seen as the global sections of the various bundles of holomorphic forms on X, or after conjugation as the cohomology groups of the structure sheaf. As such, we will see for instance that they are birational invariants.

We use the standard notation

$$b_i(X) = \dim_{\mathbf{C}} H^i(X, \mathbf{C}),$$

the i-th *Betti number* of X, and

$$h^{p,q}(X) = \dim_{\mathbf{C}} H^{p,q}(X) = \dim_{\mathbf{C}} H^q(X, \Omega_X^p) = \dim_{\mathbf{C}} H^p(X, \Omega_X^q),$$

the (p,q)-*Hodge number* of X. By the Hodge decomposition theorem we have

$$b_i(X) = \sum_{p+q=i} h^{p,q}(X) \text{ and } h^{p,q}(X) = h^{q,p}(X).$$

As a first example of application, this gives an immediate obstruction to a complex manifold being Kähler:

Corollary 2.2. *If X is compact Kähler and k is an odd integer, then $b_k(X)$ is even.*

Given the Hodge decomposition, there is a holomorphic version of the Lefschetz Hyperplane Theorem 1.1:

Corollary 2.3. *Let X be a smooth complex projective variety of dimension n, and let D be a smooth effective ample divisor on X. Then the restriction maps*

$$r_{p,q} : H^q(X, \Omega_X^p) \longrightarrow H^q(D, \Omega_D^p)$$

are isomorphisms for $p+q \leq n-2$, and injective for $p+q = n-1$.

Proof. The restriction maps naturally commute with the functorial decomposition given by the Hodge theorem, so that $r_i = \oplus_{p+q=i} r_{p,q}$. The statement follows then immediately from Theorem 1.1. $\square$

Definition 2.4 (Hodge diamond)**.** We usually collect the Hodge numbers of a compact Kähler manifold X in a *Hodge diamond*, as represented below.

$$\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & \vdots & & \ddots & \\
h^{n,0} & h^{n-1,1} & & \cdots & & h^{1,n-1} & h^{0,n} \\
 & \ddots & & \vdots & & & \\
 & & h^{n,n-1} & & h^{n-1,n} & & \\
 & & & h^{n,n} & & &
\end{array}$$

This diamond has a few symmetries. Conjugation $H^{p,q} = \overline{H^{q,p}}$ implies that it is invariant under reflection across the middle column. Serre duality gives the isomorphism $H^{p,q} \simeq H^{n-p,n-q*}$, which gives diagonal symmetry (in other words the Hodge diamond is left invariant under rotation by $180°$). This properties imply invariance under reflection across the middle row as well; this isomorphism can also be seen directly via the Hodge $*$-operator.

Exercise 2.5. Find the Hodge diamonds of the following varieties:

(i) $\mathbf{P}^n$.

(ii) a smooth degree 3 hypersurface in $\mathbf{P}^3$ (cubic surface).

(iii) a smooth degree 3 hypersurface in $\mathbf{P}^4$ (cubic threefold).

One Hodge space that has a special interpretation is $H^{1,1}(X)$. Every class here is known to be analytic, in other words it comes from a divisor on X. For a proof of the theorem below see e.g. [GH] p.163.

Theorem 2.6 (**Lefschetz theorem on** $(1,1)$**-classes**). *Let X be a smooth projective complex variety of dimension n. Then every class in $H^{1,1}(X) \cap H^2(X, \mathbf{Z})$ (i.e. integral $(1,1)$-class) is the first Chern class of a line bundle on X, or equivalently the Poincaré dual of $[D] \in H_{2n-2}(X, \mathbf{Z})$ for some divisor D on X.*

3. Hodge numbers in smooth families

Definition 3.1 (**Families**). A holomorphic map $\pi : \mathcal{X} \to B$ between complex manifolds is called a *family of complex manifolds* if π is a proper submersion. If $\mathcal{X}$ and B are smooth varieties, we can express this by saying that π is a smooth proper morphism (in the algebraic sense). For $b \in B$, we denote by $X_b = \pi^{-1}(b)$, seen as a complex manifold (or smooth algebraic variety).

From the topological, or even differential, point of view, the fibers of a family as above over a contractible base (for instance a disk) are all the same. For a proof of the following theorem, and for further extensions, see e.g. [Vo] §9.1.

Theorem 3.2 (**Ehresmann**). *Let $\pi : \mathcal{X} \to B$ be a proper submersion between smooth manifolds, with B contractible. Consider a base point $b_0 \in B$. Then there exists a diffeomorphism*

$$\Phi : \mathcal{X} \longrightarrow X_{b_0} \times B$$

relative to B, i.e. such that $\pi = p_2 \circ \Phi$.

Remark 3.3. The statement of the theorem is far from being true if one requires the trivialization Φ to be holomorphic (or algebraic) rather than just $\mathcal{C}^\infty$, hence the theory of moduli in these more restrictive categories. On the other hand, the diffeomorphisms $X_b \to X_{b_0}$ induced by Φ for any $b \in B$ enable us to see the family as given by different complex structures varying with b, on the fixed differentiable manifold X_{b_0}.

As any manifold has by definition an open cover with contractible submanifolds, for which the previous theorem applies, we obtain that the topology of the fibers of a family is the same. In particular:

Corollary 3.4. *If $\pi : \mathcal{X} \to B$ is a proper submersion of smooth manifolds, then*

$$H^i(X_{b_1}, \mathbf{Z}) \simeq H^i(X_{b_2}, \mathbf{Z})$$

for any integer i and any $b_1, b_2 \in B$.

Theorem 3.5. *Let $\pi : \mathcal{X} \to B$ be a smooth family of complex projective varieties (or compact Kähler manifolds). Then the Hodge numbers $h^{p,q}(X_b)$ are constant for $b \in B$.*

Proof. Note that by the Dolbeaut isomorphism, we can see the Hodge numbers as representing the dimension of various cohomology groups of holomorphic vector bundles:

$$h^{p,q}(X) = \dim_{\mathbf{C}} H^q(X, \Omega_X^p).$$

But by the Semicontinuity theorem, these vary upper-semicontinuously when we move b. (Indeed, one can apply the Semicontinuity theorem to the bundle of relative differentials $\Omega^p_{\mathcal{X}/B}$ on $\mathcal{X}$, whose restriction to each X_b is $\Omega^p_{X_b}$.) In other words, if we fix any point $b_0 \in B$, we have for all p and q that

$$h^{p,q}(X_b) \le h^{p,q}(X_{b_0})$$

for b in a neighborhood of b_0. This implies for each i, by the Hodge decomposition, that

$$b_i(X_b) = \sum_{p+q=i} h^{p,q}(X_b) \le \sum_{p+q=i} h^{p,q}(X_{b_0}) = b_i(X_{b_0})$$

and since the Betti numbers on the two extremes are equal according to Corollary 3.4, we obtain that all Hodge numbers are constant in a neighborhood of b_0. Now cover the base B with sufficiently small open subsets on which the above applies. □

Remark 3.6. A more general statement holds in the setting of compact complex manifolds (see e.g. [Vo] Proposition 9.20): let $\pi : \mathcal{X} \to B$ be a family (i.e. a proper submersion) of compact complex manifolds such that X_{b_0} is Kähler for some $b_0 \in B$. Then for b in a neighborhood of b_0 we have

$$h^{p,q}(X_b) = h^{p,q}(X_{b_0}) \text{ for all } p \text{ and } q,$$

and moreover the Hodge to de Rham spectral sequence degenerates at E_1. In fact one can also show that X_b must be Kähler for b sufficiently close to b_0 (see [Vo] Theorem 9.23).

4. Birationally invariant Hodge numbers

Let X be a smooth projective variety of dimension n over an algebraically closed field. It is an elementary result that the Hodge numbers

$$h^{p,0}(X) = h^0(X, \Omega_X^p)$$

are birational invariants for all q.

Proposition 4.1. *If X and Y are birational smooth projective varieties over an algebraically closed field, then*

$$h^{p,0}(X) = h^{p,0}(Y) \text{ for all } p.$$

Proof. By symmetry, it is enough to show that $h^{p,0}(Y) \le h^{p,0}(X)$. Let f be a (bi)rational map from X to Y, $V \subset X$ the maximal open set on which f is defined, and $U \subset V$ an open subset on which the induced $f : U \to f(U)$ is an isomorphism.

By pulling back p-forms via the morphism $f : V \to Y$, we get an induced map

$$f^* : H^0(Y, \Omega_Y^p) \longrightarrow H^0(V, \Omega_V^p).$$

The first claim is that this map is injective; indeed via f we get an isomorphism $\Omega^p_{V|U} \simeq \Omega^p_{Y|f(U)}$. If f^* were not injective, it would mean that a nonzero section in $H^0(Y, \Omega^p_Y)$ would vanish on a nonempty open set, which is a contradiction.

The second claim is that the restriction map

$$H^0(X, \Omega^p_X) \longrightarrow H^0(V, \Omega^p_V)$$

is an isomorphism, which combined with the above finishes the proof. But it is well-known that any rational map on a smooth (or more generally normal) variety is defined in codimension 1, i.e. $\operatorname{codim}_X(X - V) \geq 2$. Now by Hartogs' theorem (or its algebraic analogue) any regular function, hence any regular q-form, extends over a codimension 2 subset. This implies the surjectivity of the restriction map, while injectivity follows as in the paragraph above. $\square$

Exercise 4.2. Check the assertion above, namely that $\operatorname{codim}_X(X - V) \geq 2$, by using the valuative criterion of properness. (Cf. also [Har] II Ex. 3.20 and III Ex. 3.5 for the case of rational functions, referred to as *Hartogs' theorem* in the proof of the Proposition.)

Remark 4.3. If X is defined over (a subfield of) the complex numbers, then by Hodge duality we have $h^{p,0} = h^{0,p}$, hence Proposition 4.1 automatically implies that if X and Y are birational then

$$h^p(X, \mathcal{O}_X) = h^p(Y, \mathcal{O}_Y).$$

This statement is still true in arbitrary characteristic, but the argument above does not work any more. Instead, one can use the following fundamental result on birational morphisms:[3]

Theorem 4.4. *Let $f : X \to Y$ be a birational morphism between smooth varieties. Then*

$$f_*\mathcal{O}_X \simeq \mathcal{O}_Y \text{ and } R^i f_*\mathcal{O}_X = 0 \text{ for } i > 0.$$

This is well-known (but nontrivial) in characteristic 0, using fundamental facts on resolution of singularities. If resolution were known in characteristic $p > 0$, the argument would go through; at the moment this is not the case. However, the statement above was recently proved, with different methods, by Chatzistamatiou-Rülling [CR].

Going back to the proof of our invariance, since X and Y are birational, by the general resolution of singularities machinery they are dominated by a common model, i.e. there exists a smooth projective Z and surjective birational morphisms

$$f : Z \longrightarrow X \quad \text{and} \quad g : Z \longrightarrow Y.$$

We can therefore assume that there is a birational *morphism* $f : X \to Y$. But by Theorem 4.4, the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_*\mathcal{O}_X) \Rightarrow H^{p+q}(X, \mathcal{O}_X)$$

degenerates at E_2, giving isomorphisms

$$H^p(Y, \mathcal{O}_Y) \simeq H^p(X, \mathcal{O}_X).$$

[3] This result could be phrased as saying that smooth varieties have rational singularities.

Example 4.5. It is not true in general that all Hodge numbers are birational invariants. The simplest example is that of blow-ups. Let X be a smooth projective variety (or more generally complex manifold) of dimension n, and let $\tilde{X} = \mathrm{Bl}_x(X)$ be the blow-up of X at an arbitrary point x. Denote by $E \simeq \mathbf{P}^{n-1}$ the exceptional divisor. This divisor introduces its own new cycles (including E itself) to $H_i(\tilde{X}, \mathbf{Z})$. More precisely, for each $i > 0$ one has

$$H_i(\tilde{X}, \mathbf{Z}) \simeq H_i(X, \mathbf{Z}) \oplus H_i(E, \mathbf{Z}).$$

This reflects in the Hodge decomposition. Since all analytic cycles are in $H^{p,p}$ for various p, we obtain

$$H^{p,p}(\tilde{X}) \simeq H^{p,p}(X) \oplus \mathbf{C}$$

for all $0 < p < n$, so the non-extremal Hodge numbers on the middle column go up by 1.

Exercise 4.6. Verify carefully all the statements made in the Example above.

5. The topological approach to the Kodaira vanishing theorem

My emphasis in these notes is on results directly focused on the singular cohomology and Hodge numbers of smooth projective varieties. However, here I take a moment to exemplify how such results can be applied in order to derive statements of fundamental importance in birational geometry.

Concretely, we will use the previous topological and Hodge theoretic results in this chapter, together with a covering construction, in order to derive the celebrated Kodaira Vanishing theorem. That this is possible was first observed by Ramanujam; Kodaira's original proof was of a more differential geometric nature (cf. [GH] Ch.I §2). We follow the treatment in [La] §4.2. This approach has led to numerous important generalizations of the Kodaira Vanishing theorem (see for instance [EV]).

Theorem 5.1 (Kodaira Vanishing Theorem). *Let X be a smooth complex projective variety of dimension n, and let D be an ample divisor on X. Then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0 \text{ for all } i > 0.$$

Equivalently,

$$H^i(X, \mathcal{O}_X(-D)) = 0 \text{ for all } i < n.$$

Proof. **Step 1.** We first prove the theorem in the case when D is a *smooth effective* divisor. In this case we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

which induces the long exact sequence on cohomology

$$\cdots \longrightarrow H^{i-1}(X, \mathcal{O}_X) \longrightarrow H^{i-1}(D, \mathcal{O}_D) \longrightarrow H^i(X, \mathcal{O}_X(-D)) \longrightarrow$$
$$\longrightarrow H^i(X, \mathcal{O}_X) \longrightarrow H^i(D, \mathcal{O}_D) \longrightarrow \cdots$$

But the case $p = 0$ and $q = i$ in Corollary 2.3 says that the restriction maps

$$H^j(X, \mathcal{O}_X) \longrightarrow H^j(D, \mathcal{O}_D)$$

are isomorphisms for $j \le n-2$ and injective for $j = n-1$, which using the sequence above implies $H^i(X, \mathcal{O}_X(-D)) = 0$ for $i < n$.

Step 2. We now reduce to the case treated in Step 1 by means of a standard *cyclic covering* construction. Since D is ample, for $m \gg 0$ there is a smooth irreducible divisor $B \in |mD|$. Proposition 5.7 below implies that we can take an m-th root of this divisor, at the expense of working on a finite cover of X. Concretely, consider $f : Y \to X$ to be the m-fold cyclic cover branched along B. Denote $D' = f^*D$. Proposition 5.7 says that Y is smooth and that there exists a smooth effective divisor $B' \in |D'|$, which is obviously ample. We now claim that in order to conclude it suffices to have

$$H^i(Y, \mathcal{O}_Y(-B')) = 0 \text{ for all } i < n,$$

which holds by the previous step. Indeed, we have by definition

$$\mathcal{O}_Y(-B') = \mathcal{O}_Y(-D') \simeq f^*\mathcal{O}(-D)$$

and so the claim follows from Lemma 5.8 below. □

One can in fact use a similar argument to prove the more general Nakano Vanishing, a statement about arbitrary bundles of holomorphic forms (Kodaira Vanishing is the special case $p = n$).

Theorem 5.2 (**Nakano Vanishing Theorem**)**.** *Let X be a smooth complex projective variety, and L an ample line bundle on X. Then*

$$H^q(X, \Omega_X^p \otimes L) = 0 \ \text{ for } p + q > n,$$

or equivalently

$$H^q(X, \Omega_X^p \otimes L^{-1}) = 0 \ \text{ for } p + q < n.$$

Before proving the Theorem, we need to introduce one more construction:

Definition 5.3 (Forms with log-poles)**.** Let X be a smooth variety, and D a smooth effective divisor on X. The sheaf of 1-forms on X with log-poles along D is

$$\Omega_X^1(\log\, D) = \Omega_X^1 < \frac{df}{f} >, \ f \text{ local equation for } D.$$

Concretely, if $z_1, \ldots, z_n$ are local coordinates on X, chosen such that $D = (z_n = 0)$, then $\Omega_X^1(\log\, D)$ is locally generated by $dz_1, \ldots, dz_{n-1}, \frac{dz_n}{z_n}$. This is a free system of generators, so $\Omega_X^1(\log\, D)$ is locally free of rank n. For any integer p, we define

$$\Omega_X^p(\log\, D) := \bigwedge^p \big(\Omega_X^1(\log\, D)\big).$$

Lemma 5.4. *There are short exact sequences:*

(i) $0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^p(\log\, D) \longrightarrow \Omega_D^{p-1} \longrightarrow 0.$

(ii) $0 \longrightarrow \Omega_X^p(\log\, D)(-D) \longrightarrow \Omega_X^p \longrightarrow \Omega_D^p \longrightarrow 0.$

Proof. I will sketch the proof for $p = 1$; in general it is only notationally more complicated. The comprehensive source for this is [EV] §2.

Choose local analytic coordinates $z_1, \ldots, z_n$ so that $D = (z_n = 0)$. For (i), the map on the right is the *residue map* along D

$$\mathrm{res_D} : \Omega^1_X(\log\, D) \longrightarrow \mathcal{O}_D$$

given by

$$f_1 dz_1 + \ldots + f_{n-1} dz_{n-1} + f_n \frac{dz_n}{z_n} \mapsto f_{n|D},$$

where $f_1, \ldots, f_n$ are local functions on X. The right hand side is 0 if one can write $f = z_n \cdot g$ for an arbitrary regular function g. Therefore we can see the kernel as being locally generated by $dz_1, \ldots, dz_n$, hence isomorphic to Ω^1_X.

For (ii), the map on the right is given by restriction of forms. Since locally $D = (z_n = 0)$, the kernel of the restriction map $\Omega^1_X \to \Omega^1_D$ is locally generated by $z_n dz_1, \ldots z_n dz_{n-1}, dz_n$. But these obviously generate the subsheaf $\Omega^p_X(\log\, D)(-D) \subset \Omega^p_X(\log\, D)$. □

Lemma 5.5. *Let $f : Y \to X$ be the m-fold cyclic cover branched along D, as in Proposition 5.7. Let D' be the divisor in Y such that $f^*D = mD'$, mapping isomorphically onto D. Then*

$$f^*\Omega^p_X(\log\, D) \simeq \Omega^p_Y(\log\, D').$$

Proof. It is enough to prove this for $p = 1$. As usual, choose local coordinates $z_1, \ldots, z_n$ so that $D = (z_n = 0)$. We then have local coordinates $z_1, \ldots, z_{n-1}, w$ on Y such that $D' = (w = 0)$ and f is given by $z_n = w^m$. Then we see that

$$f^*\Big(\frac{dz_n}{z_n}\Big) = m \cdot \frac{dw}{w},$$

which implies what we want. □

Proof of Theorem 5.2. We will show the second version of the statement of the Theorem. For $m \gg 0$, let $D \in |mL|$ be a smooth divisor. Let $f : Y \to X$ be the m-fold cyclic cover branched along D as in Proposition 5.7, with $f^*D = mD'$ and $L' = \mathcal{O}_Y(D')$.

We can assume by induction on $n = \dim X$ that we already know Nakano vanishing on D, so that

$$H^q(D, \Omega^{p-1}_D \otimes L^{-1}_{|D}) = 0 \text{ for } p + q < n.$$

Using this and passing to cohomology in the sequence in Lemma 5.4(i), it suffices then to prove that

$$H^q(X, \Omega^p_X(\log\, D) \otimes L^{-1}) = 0 \text{ for } p + q < n.$$

Now using Lemma 5.5 together with Lemma 5.8 below, this is equivalent with proving

$$H^q(Y, \Omega^p_Y(\log\, D') \otimes \mathcal{O}_Y(-D')) = 0 \text{ for } p + q < n.$$

Finally, we appeal to the exact sequence in Lemma 5.4(ii). Using this, our desired statement is equivalent to the fact that the restriction maps

$$r_{p,q} : H^q(Y, \Omega^p_Y) \longrightarrow H^q(D', \Omega^p_{D'})$$

are isomorphisms for $p + q \le n - 2$, and injective for $p + q = n - 1$. But this is precisely the statement of the holomorphic Lefschetz hyperplane theorem, Corollary 2.3. □

Remark 5.6. Note that one can completely reverse the argument above, and deduce the Lefschetz hyperplane theorem from Nakano vanishing (which in turn can be proved by various other methods); many references, especially of a more differential geometric nature, follow that approach.

Appendix: Covering Lemmas. I will state here without proof a useful technical result needed in order to "take m-th roots" of divisors $B \in |mD|$ as in the previous section. This is only the tip of the iceberg, as more complicated constructions are needed for deeper applications. For a thorough survey and clean proofs see [La] §4.1.B.

Proposition 5.7. *Let X be a variety over an algebraically closed field k, and let L be a line bundle on X. Let $0 \neq s \in H^0(X, L^{\otimes m})$ for some $m \geq 1$, with $D = Z(s) \in |mL|$. Then there exists a finite flat morphism $f : Y \to X$, where Y is a scheme over k such that if $L' = f^*L$, there is a section*

$$s' \in H^0(Y, L') \text{ satisfying } (s')^m = f^*s.$$

Moreover:

- *if X and D are smooth, then so are Y and $D' = Z(s')$.*
- *the divisor D' maps isomorphically onto D.*
- *there is a canonical isomorphism $f_*\mathcal{O}_Y \simeq \mathcal{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-(m-1)}$.*

The scheme Y in Proposition 5.7 is called the *m-fold cyclic cover of X branched along D*. We also need to compare cohomology via finite covers.

Lemma 5.8. *Let $f : Y \to X$ be a finite surjective morphism of normal complex varieties, and let $\mathcal{E}$ be a locally free sheaf on X. If for some $i \geq 0$ one has $H^i(Y, f^*\mathcal{E}) = 0$, then $H^i(X, \mathcal{E}) = 0$.*

Exercise 5.9. Prove Lemma 5.8.

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CHAPTER 2. POINTS OVER FINITE FIELDS AND THE WEIL CONJECTURES

Contents

In this chapter we will relate the topology of smooth projective varieties over the complex numbers with counting points over finite fields, via the Weil conjectures. If X is a variety defined over a finite field $\mathbf{F}_q$, one can count its points over the various finite extensions of $\mathbf{F}_q$; denote $N_m = |X(\mathbf{F}_q^m)|$ (for instance, if $X \subset \mathbf{A}^n_{\mathbf{F}_q}$ is affine, given by equations $f_1, \ldots, f_k$, then $N_m = |\{x \in \mathbf{F}_q^m \mid f_i(x) = 0, \ \forall i\}|$). The local Weil zeta function of X,

$$Z(X;t) := \exp\left(\sum_{m\geq 1} \frac{N_m}{m} \cdot t^m\right) \in \mathbf{Q}[[t]],$$

satisfies a number of fundamental properties, known as the Weil conjectures, which are known to be true mainly by work of Deligne. Some of these are its rationality, a functional equation, and an analogue of the Riemann Hypothesis. Most importantly for this course, for varieties specializing to smooth projective varieties over $\mathbf{C}$, it is related via its rational representation to the Betti numbers of the latter. My main sources of inspiration for this chapter are [Ha] Appendix C, [Mi], and especially [Mu].

1. Varieties over finite fields

Basics on finite fields. I start by recalling a few facts on finite fields; one standard reference is [La]. Let k be a finite field of characteristic $p > 0$. Then $|k| = p^r$ for some integer $r \geq 1$; for each p and r there exists a unique (up to isomorphism) finite field with this cardinality, which can be described as the splitting field of the polynomial $X^{p^r} - X$ in an algebraic closure $\overline{\mathbf{F}}_p$ (its elements are the roots of this polynomial). This field will be denoted $\mathbf{F}_q$, with $q = p^r$.

Fix such a $k = \mathbf{F}_q$. If $k \subset K$ is a finite field extension and $[K : k] = m$, then $|K| = q^m$. On the other hand, for any $m \geq 1$, there exists a finite extension $k \subset K$ of degree m. In a fixed algebraic closure there exists a unique such extension, namely $\mathbf{F}_{q^m}$. If $[K : k] = m$ and $[K' : k] = n$, there exists a morphism of k-algebras $K' \to K$ if and only if $m|n$.

Fix an algebraic closure $k \subset \overline{k}$. The *Frobenius mapping*

$$\sigma : \mathbf{F}_q \longrightarrow \mathbf{F}_q, \ x \mapsto x^q$$

can be extended to an element in $\mathrm{Gal}(\overline{k}/k)$, sometimes called the *arithmetic Frobenius* (with the inverse in $\mathrm{Gal}(\overline{k}/k)$ called the *geometric Frobenius*). We can see the unique finite extension of k in $\overline{k}$ of degree m as the field fixed by the m-th power of σ, i.e.

$$\mathbf{F}_{q^m} = \{x \in \overline{k} \mid \sigma^m(x) = x\}.$$

The Galois group $\mathrm{Gal}(\overline{k}/k)$ can be described as follows. First, one can see that the Galois group of a finite extension is cyclic, namely $\mathrm{Gal}(\overline{\mathbf{F}}_{q^m}/\mathbf{F}_q) \simeq \mathbf{Z}/m\mathbf{Z}$. Then one has isomorphisms

$$\mathrm{Gal}(\overline{k}/k) \ \simeq \ \varprojlim_m \mathrm{Gal}(\overline{\mathbf{F}}_{q^m}/\mathbf{F}_q) \ \simeq \ \varprojlim_m \mathbf{Z}/m\mathbf{Z} \ = \ \widehat{\mathbf{Z}},$$

the profinite completion of $\mathbf{Z}$.

Varieties over $\mathbf{F}_q$. Let X be a reduced scheme of finite type over a field k. We consider and relate various notions of points of X.

Definition 1.1 (Degree of a closed point)**.** Let $x \in X$ be a closed point, with local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$. According to Nullstellensatz, the residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is a finite extension of k. The *degree* of x is

$$\deg(x) := [k(x) : k].$$

Definition 1.2. A K*-valued point* of X, with $k \subset K$ a field extension, is an element of the set

$$X(K) := \mathrm{Hom}_{\mathrm{Spec}\ k}(\mathrm{Spec}\ K, X) = \bigcup_{x \in X} \mathrm{Hom}_k(k(x), K).$$

We relate these notions when $k = \mathbf{F}_q$ is a finite field.

Lemma 1.3. *If X is defined over the finite field $k = \boldsymbol{F}_q$, and K is an extension of k of degree m, then*

$$|X(K)| = \sum_{d|m} d \cdot |\{x \in X \mid x \text{ closed with } \deg(x) = d\}|.$$

Proof. Let $x \in X$ be the image in X of a K-valued point $\mathrm{Spec}\ K \to X$. Then x is a closed point; indeed since the extension $k \subset K$ is algebraic, so is $k \subset k(x)$, hence

$$\dim \overline{\{x\}} = \mathrm{trdeg}_k(k(x)) = 0.$$

Assuming that $[K:k]=m$, we then get

$$X(K) \; = \bigcup_{\deg(x)|m} \mathrm{Hom}_k(k(x),K).$$

But one can see that if $\deg(x)=d$ with $d|m$, then

$$|\mathrm{Hom}_k(k(x),K)| = d,$$

which finishes the proof. To this end, note that the Galois group $\mathrm{Gal}(\mathbf{F}_{q^m}/\mathbf{F}_q) \simeq \mathbf{Z}/m\mathbf{Z}$ acts transitively on $\mathrm{Hom}_k(k(x),K)$, with the stabilizer of any element isomorphic to $\mathrm{Gal}(\mathbf{F}_{q^m}/\mathbf{F}_{q^d})$, which implies the claim. $\square$

Note that if $k \subset K$ is a finite extension, there are only finitely many points in $X(K)$. (By taking a finite affine open cover of X, it is enough to see this in the affine case, where things are very explicit: if $X \subset \mathbf{A}^n_k$ is defined by the equations $f_1,\ldots,f_k$, then $X(K)$ is the set of common solutions of these equations in K^n.) By Lemma 1.3, we deduce that for each e there are only finitely many closed points $x \in X$ with $\deg(x)=d$.

A key tool is the interpretation of points over finite extensions of $\mathbf{F}_q$ as being those points fixed by various powers of the Frobenius.

Definition 1.4. The *Frobenius morphism* of X over $\mathbf{F}_q$ is the morphism of ringed spaces

$$\mathrm{Frob}_{X,q} : X \longrightarrow X$$

defined as the identity on the topological space X, and the Frobenius map $a \mapsto a^q$ on the sheaf of rings $\mathcal{O}_X$. This is a morphism of schemes over $\mathbf{F}_q$, since $a^q = a$ for any $a \in \mathbf{F}_q$.

Consider now an algebraic closure $\overline{\mathbf{F}}_q$, and let

$$\overline{X} := X \times_{\mathrm{Spec}\ \mathbf{F}_q} \mathrm{Spec}\ \overline{\mathbf{F}}_q.$$

Note that $\overline{X}$ is a variety[1] over $\overline{\mathbf{F}}_q$ and $\overline{X}(\overline{\mathbf{F}}_q) = X(\overline{\mathbf{F}}_q)$. There is an induced morphism of schemes over $\overline{\mathbf{F}}_q$:

$$\mathrm{Frob}_{\overline{X},q} = \mathrm{Frob}_{X,q} \times_{\mathbf{F}_q} \mathrm{id}_{\overline{\mathbf{F}}_q} : \overline{X} \longrightarrow \overline{X}.$$

For any $m \geq 1$ this can be composed with itself m times to obtain $\mathrm{Frob}^m_{\overline{X},q}$.

Lemma 1.5. *For any $m \geq 1$, the points in $X(\boldsymbol{F}_{q^m})$ can be identified with the points of $\overline{X}(\overline{\boldsymbol{F}}_q)$ fixed by* $\mathrm{Frob}^m_{\overline{X},q}$.

Proof. Since each such points lives in an open set of an affine open cover of X, it is enough to look at the case when $X \subset \mathbf{A}^n_{\mathbf{F}_q}$ is a closed subset. In this case, $\mathrm{Frob}_{\overline{X},q}$ is the restriction of $\mathrm{Frob}_{\mathbf{A}^n_{\overline{\mathbf{F}}_q},q}$, which on $\overline{\mathbf{F}}_q$-points is given by

$$(x_1,\ldots,x_n) \mapsto (x_1^q,\ldots,x_n^q).$$

According to the description of $\mathbf{F}^m_q$ in the previous section, it is clear then that the $\mathbf{F}_{q^m}$-points are precisely those fixed by the m-th power of this map. $\square$

[1] This is true since $\mathbf{F}_q$ is a perfect field; cf. [Ha] Ch.II, Exercise 3.15. Do this exercise!

Corollary 1.6. *Denoting by Δ and Γ_m the diagonal and the graph of* $\mathrm{Frob}^m_{\overline{X},q}$ *in* $\overline{X}\times\overline{X}$*, there is a one-to-one correspondence between* $X(\boldsymbol{F}_{q^m})$ *and the closed points of* $\Delta\cap\Gamma_m$*.*

Proposition 1.7. *If X is smooth over $\boldsymbol{F}_q$, then the intersection $\Delta\cap\Gamma_m$ is transverse at every point, so that $\Delta\cap\Gamma_m$ consists of a reduced set of points.*

Proof. We first show this when $X = \mathbf{A}^n_{\mathbf{F}_q}$. Write the affine coordinate ring of $\overline{X}\times\overline{X}$ as $\overline{\mathbf{F}}_q[X_1,\ldots,X_n,Y_1,\ldots,Y_n]$. The diagonal is defined by the ideal $(X_1-Y_1,\ldots,X_n-Y_n)$, while by the discussion above Γ_m is defined by the ideal $(Y_1-X_1^{q^m},\ldots,Y_n-X_n^{q^m})$. It follows that

$$\Delta\cap\Gamma_m \simeq \prod_{i=1}^n \mathrm{Spec}\ \mathbf{F}_q[X_i]/(X_i-X_i^{q^m}),$$

which is reduced since the polynomial $X^{q^m}-X$ has no multiple roots.

Consider now the case of an arbitrary smooth X. Let $x\in X$ be a closed point in X corresponding to a point in $X(\mathbf{F}_{q^m})$ as above. Pick a regular system of parameters $t_1,\ldots,t_n$ for the regular local ring $\mathcal{O}_{X,x}$, which define an étale map $U\to\mathbf{A}^n_{\mathbf{F}_q}$ for some Zariski open neighborhood U of X. The restrictions of Δ and Γ_m to $\overline{U}\times\overline{U}$ are precisely the preimages of the analogous sets in $\mathbf{A}^n_{\overline{\mathbf{F}}_q}\times\mathbf{A}^n_{\overline{\mathbf{F}}_q}$ via the induced morphism $\overline{U}\times\overline{U}\to\mathbf{A}^n_{\overline{\mathbf{F}}_q}\times\mathbf{A}^n_{\overline{\mathbf{F}}_q}$. The statement follows from the case of $\mathbf{A}^n$, since the preimage of a reduced set via an étale morphism is reduced. $\square$

2. The local Weil zeta function

Let X be a variety defined over the finite field $k=\mathbf{F}_q$. For every integer $m\geq 1$, we define

$$N_m (= N_m(X)) := |X(\mathbf{F}_{q^m})|.$$

Definition 2.1. The *local Weil zeta function* of X is the formal power series

$$Z(X;t) := \exp\left(\sum_{m\geq 1}\frac{N_m}{m}\cdot t^m\right)\in\mathbf{Q}[[t]].$$

Example 2.2 (Affine space)**.** Let $X=\mathbf{A}^n_{\mathbf{F}_q}$. For each $m\geq 1$, we clearly have $X(\mathbf{F}_{q^m}) = (\mathbf{F}_{q^m})^n$, which is of cardinality q^{mn}. Therefore[2]

$$Z(\mathbf{A}^n;t) = \exp\left(\sum_{m\geq 1}\frac{q^{nm}}{m}\cdot t^m\right) = \exp(-\log(1-q^nt)) = \frac{1}{1-q^nt}.$$

Exercise 2.3. Show that if X is a variety over $\mathbf{F}_q$, then

$$Z(X\times\mathbf{A}^n_{\mathbf{F}_q};t) = Z(X;q^nt).$$

[2] Recall that $\log(1+t)=\sum_{m\geq 1}\frac{(-1)^{m+1}t^m}{m}$.

Example 2.4 ("Motivic" behavior of the zeta function)**.** This refers to the behavior of the zeta function with respect to the decomposition of varieties into disjoint unions. Consider a variety X over $\mathbf{F}_q$, with $Y \subset X$ a closed subvariety and $U = X - Y$. Then

$$Z(X;t) = Z(Y;t) \cdot Z(U;t).$$

Indeed, note that by definition $X(\mathbf{F}_{q^m}) = Y(\mathbf{F}_{q^m}) \cup U(\mathbf{F}_{q^m})$, so one can use the multiplicative behavior of the exponential with respect to sums.

Example 2.5 (Projective space)**.** Let $X = \mathbf{P}^n_{\mathbf{F}_q}$. We can write $X = \mathbf{A}^n_{\mathbf{F}_q} \cup \mathbf{P}^{n-1}_{\mathbf{F}_q}$, and continue this decomposition inductively with respect to n. Using Example 2.2 and Example 2.4, we obtain

$$Z(\mathbf{P}^n;t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^nt)}.$$

Equivalently, for all $m \geq 1$,

$$N_m(\mathbf{P}^n_{\mathbf{F}_q}) = 1 + q^m + q^{2m} + \ldots + q^{nm}.$$

Exercise 2.6 (Grassmannians)**.** For any $1 \leq k \leq n-1$, let $G(k,n)$ be the Grassmannian defined over Spec $\mathbf{Z}$; for each field K, its K-valued points are the k-dimensional linear subspaces in K^n.

(1) Show that $GL_n(\mathbf{F}_q)$ acts transitively on $G(k,n)(\mathbf{F}_q)$, and the stabilizer of each point is isomorphic to $GL_k(\mathbf{F}_q) \times GL_{n-k}(\mathbf{F}_q) \times M_{k,n-k}(\mathbf{F}_q)$.

(2) Show that for each $k \geq 1$ one has

$$|GL_k(\mathbf{F}_q)| = q^{\frac{k(k-1)}{2}}(q^k-1)(q^{k-1}-1)\cdots(q-1).$$

(3) Use the previous parts to show that

$$|G(k,n)(\mathbf{F}_q)| = \frac{(q^n-1)\cdots(q^{n-k+1}-1)}{(q^k-1)\cdots(q-1)} =: \binom{n}{k}_q,$$

the Gaussian binomial coefficient.

(4) Show that

$$\binom{n}{k}_q = q^k\binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

and use this to deduce that

$$\binom{n}{k}_q = \sum_{i=0}^{k(n-k)} \lambda_{n,k}(i)q^i,$$

where $\lambda_{n,k}(i)$ can be interpreted as the number of partitions of i into at most $n-k$ parts, each of size at most k.

(5) With the notation in (4), deduce that

$$Z(G(k,n);t) = \prod_{i=0}^{k(n-k)} \frac{1}{(1-q^it)^{\lambda_{n,k}(i)}}.$$

Analogy with the Riemann zeta function. This serves to motivate some of the Weil conjectures in the next sections. Recall that, for $s \in \mathbf{C}$, the *Riemann zeta function* is defined as

$$\zeta(s) := \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Note that the primes numbers correspond precisely to the closed points of Spec $\mathbf{Z}$. We can then define analogously, for any scheme X of finite type over $\mathbf{Z}$, the following zeta function:

$$\zeta_X(s) := \prod \frac{1}{1 - N(x)^{-s}},$$

where the product is taken over all closed points $x \in X$, and $N(x)$ denotes the number of elements of the residue field $k(x)$. Now if $k = \mathbf{F}_q$, using the notation above we can rewrite this as

$$\zeta_X(s) := \prod \frac{1}{1 - \left(q^{\deg(\mathrm{x})}\right)^{-s}}.$$

On the other hand, we have the following product formula:

Proposition 2.7. *If X is a variety over $\boldsymbol{F}_q$, then*

$$Z(X;t) = \prod \frac{1}{1 - t^{\deg(x)}},$$

where the product is taken over all closed points $x \in X$. In particular $Z(X;t) \in 1 + t\mathbf{Z}[[t]]$.

Proof. For $d \geq 1$, write $a_d = |\{x \in X \mid x \text{ closed with } \deg(x) = d\}|$. Recall that Lemma 1.3 says that $N_m = \sum_{d|m} d \cdot a_d$. Note also that the right hand side of the formula in the statement is equal to $\prod_{d\geq 1}(1 - t^d)^{-a_d}$. We then have

$$\log Z(X;t) = \sum_{m\geq 1} \frac{N_m}{m} t^m = \sum_{m\geq 1} \sum_{d|m} \frac{d \cdot a_d}{m} t^m = \sum_{d\geq 1} a_d \cdot \sum_{e\geq 1} \frac{t^{de}}{e} =$$

$$\sum_{d\geq 1} a_d \cdot \log\bigl((1-t^d)^{-1}\bigr) = \sum_{d\geq 1} \log\bigl((1-t^d)^{-a_d}\bigr) = \log \left(\prod_{d\geq 1}(1-t^d)^{-a_d}\right).$$

□

Corollary 2.8. *We have the identification $Z(X; q^{-s}) = \zeta_X(s)$.*

3. Statement of the Weil conjectures

Let X be a smooth projective variety of dimension n over $\mathbf{F}_q$. The following four theorems are known as the *Weil conjectures*. The first was proved by Dwork [Dwo] (without the assumption that X is smooth and projective) and by Grothendieck [Gro], while the other three were proved by Deligne [De1] (a new proof of the analogue of the Riemann hypothesis was also given by Laumon [Lau]).

Theorem 3.1 (Rationality). *$Z(X;t)$ is a rational function, i.e. $Z(X;t) = \frac{P(t)}{Q(t)}$ with $P(t), Q(t) \in \mathbf{Q}[t]$.*

Theorem 3.2 (**Functional equation**)**.** *Denote by $E = \Delta^2$ the self-intersection number of the diagonal in $X \times X$. Then*

$$Z(X; \frac{1}{q^n t}) = \pm\, q^{nE/2} t^E Z(X; t).$$

Theorem 3.3 (**Analogue of the Riemann hypothesis**)**.** *One can write*

$$Z(X;t) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2n}(t)}$$

with $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - q^n t$, and for $1 \le i \le 2n-1$, $P_i(t) \in \mathbf{Z}[t]$ with

$$P_i(t) = \prod_j (1 - \alpha_{i,j} t)$$

where $\alpha_{i,j}$ are algebraic integers such that $|\alpha_{i,j}| = q^{i/2}$.

Theorem 3.4 (**Betti numbers**)**.** *Using the notation in Theorem 3.3, define the i-th* Betti number of X *as $b_i(X) = \deg P_i(t)$. Then*

$$E = \sum_{i=0}^{2n} (-1)^i b_i(X).$$

If in addition there exists a finitely generated $\mathbf{Z}$-algebra R, $\mathcal{X}$ a smooth projective variety over Spec R, *and $\mathfrak{p} \subset R$ a maximal ideal such that $R/\mathfrak{p} \simeq \boldsymbol{F}_q$ and $X = \mathcal{X} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ R/\mathfrak{p}$, then*

$$b_i(X) = b_i\big((\mathcal{X} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C})^{\mathrm{an}}\big).$$

In plain English, this last property says that if X is the reduction mod p of a smooth projective complex variety as in §3, then its Betti numbers coincide with the usual Betti numbers for the singular cohomology of that variety. As formulated here, Theorem 3.4 depends on Theorem 3.3; we will see later that there is a way of formulating it independently as well.

Exercise 3.5. Let X be a smooth projective variety X of dimension n over a field, and let $\Delta \in X \times X$ be the diagonal. Show that

$$\Delta^2 = c_n(T_X) \in A^n(X),$$

i.e. the number E above equals the top Chern class of the tangent bundle of X.

Exercise 3.6. Verify the Weil conjectures for $\mathbf{P}^n$.

Exercise 3.7. Using Exercise 2.6, verify the Weil conjectures for $G(k,n)$. Use them to deduce that the Betti numbers of the complex Grassmannian are

$$b_{2i+1}(G(k,n)) = 0 \text{ for all } i, \text{ and } b_{2i} = \lambda_{n,k}(i) \text{ for } 1 \le i \le k(n-k).$$

Remark 3.8 (Number of points from the rational representation)**.** Given Theorem 3.1, write $Z(X;t) = \frac{P(t)}{Q(t)}$ with $P, Q \in \mathbf{Q}[t]$ normalized (after possibly dividing by powers of t) such that $P(0) = Q(0) = 1$. Write

$$P(t) = \prod_{i=1}^{r} (1 - \alpha_i t) \text{ and } Q(t) = \prod_{j=1}^{s} (1 - \beta_j t).$$

Then a simple calculation shows that, for every $m \geq 1$,

$$N_m = \sum_{j=1}^{s} \beta_j^m - \sum_{i=1}^{r} \alpha_i^m.$$

Exercise 3.9 (**$K3$ surfaces**)**.** A smooth projective complex surface S is called a $K3$ *surface* if $\omega_S \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$. Show the following:

(1) The Hodge diamond of a $K3$ surface is

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

(2) A smooth quartic hypersurface in $\mathbf{P}^3$ is a $K3$ surface.

(3) If X is a surface over $\mathbf{F}_q$ which is the reduction mod p of a complex $K3$ surface (as in Theorem 3.4), then

$$\big||X(\mathbf{F}_q)| - |\mathbf{P}^2(\mathbf{F}_q)|\big| \leq 23q.$$

[Hint: use (1), the Weil conjectures, and Remark 3.8.] Deduce that every such surface has points over a field with more than 22 elements.

(4) For completeness: there are examples of $K3$ surfaces with no points whatsoever over a particular finite field. For instance, take the Fermat quartic $X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0$. Show that this has no points over $\mathbf{F}_5$. Show on the other hand that it has points over every other finite field (use (3) if necessary).

4. Some proofs via Weil cohomology theories

Here I will sketch the approach to the proofs of some of the Weil conjectures. Since I will not treat the analogue of the Riemann hypothesis in general, I will start by proving it in the case of curves, where only elementary tools are needed, and then look at the other conjectures in general via the notion of Weil cohomology theory.

The case of curves. Let X be a smooth projective curve over $\mathbf{F}_q$. Fix an algebraic closure $\mathbf{F}_q \subset \overline{\mathbf{F}}_q$. Recall that we denote $\overline{X} = X \times_{\mathrm{Spec}\ \mathbf{F}_q} \mathrm{Spec}\ \overline{\mathbf{F}}_q$. We assume that $\overline{X}$ is a smooth projective irreducible curve over $\overline{\mathbf{F}}_q$, of genus

$$g = h^1(X, \mathcal{O}_X) = h^1(\overline{X}, \mathcal{O}_{\overline{X}}).[3]$$

Note first that the rationality and the Betti numbers theorems imply that we have

$$Z(X; t) = \frac{P(t)}{(1-t)(1-qt)},$$

[3]Note that via the natural projection $p : \overline{X} \to X$, for any quasi-coherent sheaf $\mathcal{F}$ on X we have

$$H^i(X, \mathcal{F}) \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q \simeq H^i(\overline{X}, p^*\mathcal{F}).$$

with $P(t) \in \mathbf{Z}[t]$ of degree $2g$. Write $P(t) = \prod_{i=1}^{2g}(1-\alpha_i t)$, with $\alpha_i \in \mathbf{C}$. The analogue of the Riemann hypothesis amounts to the following

Theorem 4.1. *Using the notation above, the α_i are algebraic integers satisfying $|\alpha_i| = q^{1/2}$ for all i.*

It is clear to begin with that the α_i are algebraic integers: since $P(t)$ has integer coefficients, all symmetric functions $s_j(\alpha_1, \ldots, \alpha_{2g})$ are integers, and the α_i are roots of the polynomial $\sum_{j=0}^{2g}(-1)^j s_j(\alpha_1, \ldots, \alpha_{2g}) t^{2g-j}$. For the absolute value statement, one can first go through a few simple reduction steps.

First, if $\Delta \subset \overline{X} \times \overline{X}$ is the diagonal, then the self-intersection Δ^2 can be computed using the genus fomula for curves on surfaces, i.e.

$$2g-2 = \Delta^2 + K_{\overline{X}\times\overline{X}} \cdot \Delta.$$

If p_1 and p_2 are the projections to the two factors of $\overline{X} \times \overline{X}$, we have

$$K_{\overline{X}\times\overline{X}} = p_1^* K_{\overline{X}} + p_2^* K_{\overline{X}} \equiv (2g-2)\cdot p_1^*\{\mathrm{pt}\} + (2g-2)\cdot p_1^*\{\mathrm{pt}\} \equiv (2g-2)F_1 + (2g-2)F_2,$$

where F_1 and F_2 are fibers of the two projections. Since clearly $\Delta \cdot F_1 = \Delta \cdot F_2 = 1$, we get $K_{\overline{X}\times\overline{X}} \cdot \Delta = 4g-4$, so finally

$$\Delta^2 = 2-2g.$$

The functional equation then says in this case

$$Z(X; \frac{1}{qt}) = q^{1-g} t^{2-2g} Z(X;t).$$

A simple calculation shows that this implies

$$\prod_{i=1}^{2g}(1-\alpha_i t) = q^g \prod_{i=1}^{2g}(t - \frac{\alpha_i}{q}) = \frac{\prod_{i=1}^{2g}\alpha_i}{q^g} \cdot \prod_{i=1}^{2g}(1 - \frac{q}{\alpha_i} t).$$

This has the following consequences:

- $\prod_{i=1}^{2g}\alpha_i = q^g$ (in particular $\alpha_i \neq 0$ for all i).
- the set $\{\alpha_1, \ldots, \alpha_{2g}\}$ is invariant under the mapping $\alpha \mapsto q/\alpha$.

The last property shows that it is enough to prove the inequality

$$|\alpha_i| \le q^{1/2} \quad \text{for all } i. \tag{1}$$

Another standard piece of notation when referring to zeta functions of curves is the following: recalling that $N_m = |X(\mathbf{F}_{q^m})|$, denote

$$a_m = 1 - N_m + q^m.$$

(Note that $|a_m|$ is the difference between the number of points of $\mathbf{P}^1$ and of X over $\mathbf{F}_q$.)

Exercise 4.2. Show that if X is an elliptic curve, then using the notation above

$$Z(X;t) = \frac{1 - a_1 t + qt^2}{(1-t)(1-qt)}.$$

Lemma 4.3. *The set of inequalities (1) above is equivalent to the condition*

$$|a_m| \le 2gq^{m/2} \quad \text{for all } m \ge 1. \tag{2}$$

Proof. The first claim is that for all $m \ge 1$ we have

$$a_m = \sum_{i=1}^{2g} \alpha_i^m.$$

Indeed, note that from the formula for the zeta function we have

$$\sum_{m\ge 1} N_m \frac{t^m}{m} = \sum_{i=1}^{2g} \log(1-\alpha_i t) - \log(1-t) - \log(1-qt) =$$

$$= \sum_{m\ge 1} \Big(-\sum_{i=1}^{2g} \alpha_i^m + 1 + q^m \Big) \frac{t^m}{m},$$

which implies what we want. Now if we assume that $|\alpha_i| \le q^{1/2}$ for all i, then the triangle inequality immediately implies (2).

For the reverse implication, observe that

$$\sum_{m\ge 1} a_m t^m = \sum_{i=1}^{2g} \sum_{m\ge 1} \alpha_i^m t^m = \sum_{i=1}^{2g} \frac{\alpha_i t}{1-\alpha_i t}.$$

This rational function has a pole at each $t = 1/\alpha_i$. Assuming that we have $|a_m| \le 2gq^{m/2}$ for all $m \ge 1$, we get

$$|\sum_{m\ge 1} a_m t^m| \le 2g \sum_{m\ge 1} (q^{1/2}|t|)^m = \frac{2gq^{1/2}|t|}{1-q^{1/2}|t|}.$$

This cannot have poles with $|t| < q^{-1/2}$, and therefore we must have $|\alpha_i| \le q^{1/2}$ as claimed. $\square$

We finally come to the main content of the proof of Theorem 4.1:[4] Weil's interpretation of counting rational points by looking at the intersection of the diagonal with the graph of the Frobenius can be combined with the Hodge index theorem for surfaces to give a quick proof of the estimate in (2). Since the same argument works for all $\mathbf{F}_{q^m}$, it is enough to do this for $m = 1$.

Denote by $\Gamma \subset \overline{X} \times \overline{X}$ the graph of $\mathrm{Frob}_{\overline{X},q}$. We have seen in Corollary 1.6 and Proposition 1.7 that the intersection $\Delta \cap \Gamma$ is reduced, and that

$$N := |X(\mathbf{F}_q)| = |\Delta \cap \Gamma| = \Delta \cdot \Gamma.$$

We use the following standard application of the Hodge index theorem:

[4] This is essentially [Ha] Ch. V, Exercise 1.10.

Exercise 4.4 ([Ha], Ch. V, Exercise 1.9)**.** Let C_1 and C_2 be two smooth projective curves over $k = \overline{k}$. Denote by F_1 and F_2 any fibers of the projections of $C_1 \times C_2$ onto the respective factors. Then for every divisor D on $C_1 \times C_2$ we have

$$D^2 \leq 2 \cdot (D \cdot F_1) \cdot (D \cdot F_2).$$

We apply this with $C_1 = C_2 = \overline{X}$ and $D = a\Delta + b\Gamma$, for arbitrary $a, b \in \mathbf{Z}$. Recall that $\Delta \cdot F_1 = \Delta \cdot F_2 = 1$ and $\Delta^2 = 2 - 2g$. Note also that clearly $\Gamma \cdot F_1 = 1$, while $\Gamma \cdot F_2 = q$ (the cardinality of the set $\{(x, y) \mid y = x^q\}$). We need to compute Γ^2 as well; for this we again use the genus formula

$$2g - 2 = \Gamma^2 + K_{\overline{X} \times \overline{X}} \cdot \Gamma.$$

We have seen before that $K_{\overline{X} \times \overline{X}} \equiv (2g - 2)(F_1 + F_2)$, and so using the information above we get $\Gamma^2 = q(2 - 2g)$. For $D = a\Delta + b\Gamma$, the inequality in the Exercise can be written (after a small calculation) as

$$ga^2 - (q + 1 - N)ab + gqb^2 \geq 0.$$

Since this holds for all a and b, we must have

$$(q + 1 - N)^2 - 4g^2q \leq 0$$

which after taking square roots gives precisely the inequality we're after.

Weil cohomology theories. Here I am closely following [Mu], which in turn is closely following [deJ] and [Mi]. The main point is the following: Weil realized that the rationality and the functional equation for the zeta function would follow formally from the existence of a cohomology theory in characteristic $p > 0$ with axioms closely resembling those of singular cohomology over $\mathbf{C}$. More precisely:

Definition 4.5. A *Weil cohomology theory* for varieties over a field k, with coefficients in a field K with char $K = 0$, is given by the **data**

(D1) A contravariant functor $X \mapsto H^*(X) = \oplus_i H^i(X)$, mapping smooth projective varieties over k to graded commutative[5] K-algebras. We use the cup-product notation $\alpha \cup \beta$ for the product in $H^*(X)$.

(D2) For every such X, a linear trace map $\mathrm{Tr}_X : H^{2\dim X}(X) \to K$.

(D3) For every such X, and for every closed subvariety $Z \subset X$ of codimension c, a cohomology class $\mathrm{cl}(Z) \in H^{2c}(X)$.

satisfying the following **axioms**

(A1) (Finite dimensionality and vanishing) For every X, all $H^i(X)$ are finite dimensional vector spaces over K, and in addition

$$H^i(X) = 0 \text{ for } i < 0 \text{ and } i > 2\dim X.$$

[5]meaning $ab = (-1)^{\deg(a)\deg(b)}ba$ for every a and b.

(A2) (Künneth formula) For every X and Y, if p_X, p_Y are the projections of $X \times Y$ onto the two factors, then the K-algebra homomorphism

$$H^*(X) \otimes_K H^*(Y) \to H^*(X \times Y),\ \alpha \otimes \beta \mapsto p_X^*\alpha \cup p_Y^*\beta$$

is an isomorphism.

(A3) (Poincaré duality) For every X, the trace map Tr_X is an isomorphism, and for every $0 \leq i \leq 2\mathrm{dim}X$, the K-bilinear map

$$H^i(X) \otimes_K H^{2\mathrm{dim}X-i}(X) \to K,\ \alpha \otimes \beta \mapsto \mathrm{Tr}_X(\alpha \cup \beta)$$

is a perfect pairing.

(A4) (Trace maps and products) For every X and Y and every $\alpha \in H^{2\mathrm{dim}X}(X)$ and $\beta \in H^{2\mathrm{dim}Y}(Y)$, one has

$$\mathrm{Tr}_{X\times Y}(p_X^*\alpha \cup p_Y^*\beta) = \mathrm{Tr}_X(\alpha) \cdot \mathrm{Tr}_Y(\beta).$$

(A5) (Exterior product of cohomology classes) For every X and Y, and every closed subvarieties $Z \subset X$ and $W \subset Y$, one has

$$\mathrm{cl}(Z \times W) = p_X^*\mathrm{cl}(Z) \cup p_Y^*\mathrm{cl}(W).$$

(A6) (Push-forward of cohomology classes) For every morphism $f : X \to Y$ and every closed subvariety $Z \subset X$, for every class $\alpha \in H^{2\mathrm{dim}Z}(Y)$ one has

$$\mathrm{Tr}_X(\mathrm{cl}(Z) \cup f^*\alpha) = \deg(Z/f(Z)) \cdot \mathrm{Tr}_Y(\mathrm{cl}(f(Z)) \cup \alpha).$$

(A7) (Pull-back of cohomology classes) For every morphism $f : X \to Y$ and every closed subvariety $Z \subset Y$ satisfying the conditions

• all irreducible components $W_1, \ldots, W_r$ of $f^{-1}(Z)$ have dimension $\dim Z + \dim X - \dim Y$;

• either f is flat in a neighborhood of Z, or Z is generically transverse to f, i.e. $f^{-1}(Z)$ is generically smooth,

assuming that $[f^{-1}(Z)] = \sum_{i=1}^r m_i W_i$ as a cycle ($m_i = 1$ for all i in the generically transverse case), then

$$f^*\mathrm{cl}(Z) = \sum_{i=1}^r m_i \mathrm{cl}(W_i).$$

(A8) (Case of a point) If $x = \mathrm{Spec}\ k$, then $\mathrm{cl}(x) = 1$ and $\mathrm{Tr}_x(1) = 1$.

As mentioned above, when $k = \mathbf{C}$ and $K = \mathbf{Q}$, singular cohomology provides a Weil cohomology theory. When char $k = p > 0$, the Weil cohomology theory we will discuss below is ℓ-adic cohomology, with $\ell \neq p$ and $K = \mathbf{Q}_\ell$.

Fix a Weil cohomology theory over the field k. We will need a few extra properties that follow from the axioms.

Lemma 4.6. *Let X be a smooth projective variety over k, of dimension n. Then:*

(1) The morphism $K \to H^0(X)$ given by the K-algebra structure is an isomorphism.

(2) In $H^0(X)$, one has $\mathrm{cl}(X) = 1$.

(3) If $x \in X$ is a closed point, then $\mathrm{Tr}_X(\mathrm{cl}(x)) = 1$.

(4) Suppose $f : X \to Y$ is a generically finite surjective morphism of degree d to a smooth projective variety Y. Then

$$\mathrm{Tr}_X(f^*\alpha) = d \cdot \mathrm{Tr}_Y(\alpha),\ \forall\ \alpha \in H^{2n}(Y).$$

Consequently, if $Y = X$, f^* acts as multiplication by d on $H^{2n}(X)$.

Proof. Using Poincaré duality (A3) with $i = 0$ we obtain that $\dim_K H^0(X) = 1$, which immediately gives (1). Part (2) follows by applying (A7) to the natural morphism $X \to \mathrm{Spec}\, k$, combined with (A8). For $x \in X$ a closed point, we can apply (A6) to $X \to \mathrm{Spec}\, k$, taking $Z = \{x\}$ and $\alpha = 1$. We get that $\mathrm{Tr}_X(\mathrm{cl}(x)) = \mathrm{Tr}_{\mathrm{Spec}\ k}(1)$, to which we apply (A8) to get (3).

Consider now $f : X \to Y$ as in (4), and take a general point $Q \in Y$. We have that as a cycle $[f^{-1}(Q)] = \sum_{i=1}^r m_i P_i$, where P_i are the reduced points of the fiber over Q, and $\sum_{i=1}^r m_i = d$. By generic flatness, since Q is general we have that f is flat over a neighborhood of Q. Therefore by (A7) and (A6) we have

$$\mathrm{Tr}_X(f^*\mathrm{cl}(Q)) = \mathrm{Tr}_X(\sum_{i=1}^r m_i \mathrm{cl}(P_i)) = d \cdot \mathrm{Tr}_Y(\mathrm{cl}(Q)).$$

This proves (4), since by (3) $\mathrm{cl}(Q)$ generates $H^{2n}(Y)$. □

Definition 4.7 (Push-forward)**.** The data of a contravariant functor guarantees only the existence of a pull-back map in cohomology, given a morphism $f : X \to Y$. However, the Poincaré duality axiom allows one to define a push-forward in cohomology as well. If $\alpha \in H^i(X)$, then the *push-forward* of α via f is the unique class $f_*\alpha \in H^{2\dim Y - 2\dim X + i}(Y)$ such that

$$\mathrm{Tr}_Y(f_*\alpha \cup \beta) = \mathrm{Tr}_X(\alpha \cup f^*\beta)$$

for every $\beta \in H^{2\dim X - i}(Y)$. This is clearly K-linear, and further properties are collected in the following:

Lemma 4.8. *With the notation above, one has:*

(1) (Projection formula) $f_*(\alpha \cup f^*\gamma) = f_*\alpha \cup \gamma$, for any $\alpha \in H^*(X)$ and any $\gamma \in H^*(Y)$.

(2) If $g : Y \to Z$ is another morphism, then $(g \circ f)_* = g_* \circ f_*$.

(3) If $Z \subset X$ is a closed subvariety, then

$$f_*\mathrm{cl}(Z) = \deg(Z/f(Z)) \cdot \mathrm{cl}(f(Z)).$$

Proof. Exercise. □

Lemma 4.9. *If X and Y are smooth projective varieties and $\alpha \in H^i(Y)$, then $p_{X*}(p_Y^*\alpha) = \mathrm{Tr}_Y(\alpha)$ if $i = 2\dim Y$, and $p_{X*}(p_Y^*\alpha) = 0$ otherwise.*

Proof. By definition $p_{X*}(p_Y^*\alpha) \in H^{i-2\dim Y}(X)$, so it follows automatically by (A1) that $p_{X*}(p_Y^*\alpha) = 0$ when $i \neq 2\dim Y$. Now if $\alpha \in H^{2\dim Y}(Y)$ and $\beta \in H^{2\dim X}(X)$, then

$$\mathrm{Tr}_X(p_{X*}(p_Y^*\alpha) \cup \beta) = \mathrm{Tr}_{X\times Y}(p_Y^*\alpha \cup p_X^*\beta) = \mathrm{Tr}_Y(\alpha)\cdot \mathrm{Tr}_X(\beta).$$

(The first equality follows by the definition of the trace, and the second by axiom (A4).) This implies that $p_{X*}(p_Y^*\alpha) = \mathrm{Tr}_Y(\alpha)$. □

Finally, completely analogously to the well-known case of singular cohomology (see [Fu] Ch.19), one can define for each i a *cycle class map*

$$\mathrm{cc} : A^i(X) \longrightarrow H^{2i}(X)$$

from the Chow group of codimension i cycles, such that when putting these together we obtain a ring homomorphism

$$\mathrm{cc} : A^*(X) \longrightarrow H^{2*}(X)$$

compatible with f^* and f_*. This gives in particular the following:

Lemma 4.10. *Let X be a smooth projective variety, and let $\alpha_i \in A^{m_i}(X)$, with $i = 1,\ldots,r$, such that $\sum_{i=1}^r m_i = \dim X$. Then*

$$\alpha_1\cdot\ldots\cdot\alpha_r = \mathrm{Tr}_X\big(\mathrm{cc}(\alpha_1)\cup\ldots\cup\mathrm{cc}(\alpha_r)\big).$$

Proof. Since the cycle class map is a ring homomorphism, it is enough to prove that for $\beta \in Z_0(X)$ one has $\deg(\beta) = \mathrm{Tr}_X(\beta)$. Furthermore, one can assume that β is just a point by additivity, in which case the assertion follows from Lemma 4.6 (3). □

Trace formula. The key formal result related to Weil cohomology theories that we will use below is the following (for the singular cohomology of complex projective varieties with coefficients in $\mathbf{Q}$, this is a special case of the famous *Lefschetz fixed point theorem*):

Theorem 4.11 (Trace formula). *Let $\varphi : X \to X$ be an endomorphism of a smooth projective variety X of dimension n. If $\Delta, \Gamma_\varphi \subset X\times X$ denote the diagonal and the graph of φ respectively, then*

$$\Delta\cdot\Gamma_\varphi = \sum_{i=0}^{2n}(-1)^i\ \mathrm{Tr}(\varphi^*|H^i(X)).$$

If Δ and Γ_φ intersect transversely, this number is precisely the cardinality of the fixed point set $\{x\in X \mid \varphi(x) = x\}$.

Applying the Theorem to $\varphi = \mathrm{id}_X$, we obtain a familiar formula:

Corollary 4.12. *If X is a smooth projective variety and $\Delta \subset X\times X$ is the diagonal, then*

$$\Delta^2 = \sum_{i=0}^{2n}(-1)^i\ h^i(X) =: \chi(X).$$

To prove Theorem 4.11, we need some preparatory lemmas. We use the same notation as in the Theorem, and in addition we denote by p_1, p_2 the projections of $X\times X$ onto the two factors.

Lemma 4.13. *For any $\alpha \in H^*(X)$, we have*

$$p_{1*}(\mathrm{cl}(\Gamma_\varphi) \cup {p_2}^*\alpha) = \varphi^*\alpha.$$

Proof. Denote by $j : X \hookrightarrow X \times X$ the embedding of the graph of φ. We clearly have $p_1 \circ j = \mathrm{Id}_X$ and $p_2 \circ j = \varphi$. Now $j_*\mathrm{cl}(X) = \mathrm{cl}(\Gamma_\varphi)$ by Lemma 4.8 (3). Therefore

$$p_{1*}(\mathrm{cl}(\Gamma_\varphi) \cup {p_2}^*\alpha) = p_{1*}(j_*\mathrm{cl}(X) \cup {p_2}^*\alpha) = p_{1*}j_*(\mathrm{cl}(X) \cup j^*{p_2}^*\alpha) = p_{1*}j_*(\varphi^*\alpha) = \varphi^*\alpha,$$

where the second identity follows from the projection formula. □

Lemma 4.14. *Let $\{e_i^r\}_{i=1,\dots,k_r}$ be a basis for $H^r(X)$, for each r. Let $\{f_i^{2n-r}\}_{i=0,\dots,k_r}$ be the dual basis for $H^{2n-r}(X)$ via the Poincaré duality pairing, so that $\mathrm{Tr}_X(e_i^r \cup f_j^{2n-r}) = \delta_{i,j}$. Then*

$$\mathrm{cl}(\Gamma_\varphi) = \sum_{r,i} p_1^*\varphi^*e_i^r \cup p_2^*f_i^{2n-r} \in H^{2n}(X \times X).$$

Proof. The Künneth property (A2) implies that we can write

$$\mathrm{cl}(\Gamma_\varphi) = \sum_{s,j} p_1^*a_j^s \cup p_2^*f_j^{2n-s}$$

for some unique classes $a_j^s \in H^s(X)$ for each s and j. Using Lemma 4.13 and the projection formula we obtain

$$\varphi^*e_i^r = \sum_{s,j} p_{1*}\big(p_1^*a_j^s \cup p_2^*f_j^{2n-s} \cup p_2^*e_i^r\big) = \sum_{s,j} a_j^s \cup p_{1*}\big(p_2^*(f_j^{2n-s} \cup e_i^r)\big).$$

Now by Lemma 4.9 we have that $p_{1*}\big(p_2^*(f_j^{2n-s} \cup e_i^r)\big) = 0$ when $r \neq s$, and $p_{1*}\big(p_2^*(f_j^{2n-r} \cup e_i^r)\big) = \mathrm{Tr}_X(f_j^{2n-r} \cup e_i^r)$. But by definition this is zero unless $i = j$, when it is equal to 1. This implies that $a_i^r = \varphi^*e_i^r$. □

Proof. (*of Theorem 4.11*) Applying the formula in Lemma 4.14 with $\varphi = \mathrm{Id}$, but with the dual bases $\{f_j^s\}$ and $\{(-1)^s e_j^{2n-s}\}$, we get

$$\mathrm{cl}(\Delta) = \sum_{s,j} (-1)^s p_1^*f_j^s \cup p_2^*e_j^{2n-s}.$$

By Lemma 4.10 we have

$$\Delta \cdot \Gamma_\varphi = \mathrm{Tr}_{X\times X}(\mathrm{cl}(\Delta) \cup \mathrm{cl}(\Gamma_\varphi)) = \mathrm{Tr}_{X\times X}\left(\sum_{r,s,i,j} p_1^*(f_j^s \cup \varphi^*e_i^r) \cup p_2^*(e_j^{2n-s} \cup f_i^{2n-r})\right) =$$

$$= \sum_{r,i} \mathrm{Tr}_X(f_i^{2n-r} \cup \varphi^*e_i^r) \cdot \mathrm{Tr}_X(e_i^r \cup f_i^{2n-r}) = \sum_r (-1)^r \mathrm{Tr}(\varphi^*|H^r(X)).$$

□

For later use, let's also note the following formula for the characteristic polynomial of a linear transformation, in terms of traces of iterates.

Lemma 4.15. *Let V be a vector space over the field K, and $\varphi : V \to V$ a K-linear transformation. Then*

$$\det(\mathrm{Id} - t\varphi) = \exp\left(-\sum_{m\geq 1} \mathrm{Tr}(\varphi^m)\cdot \frac{t^m}{m}\right).$$

Proof. By extending everything to $\overline{K}$, we can assume that K is algebraically closed. Then there exists a basis for V in which the matrix of φ is upper triangular. If the entries on the diagonal are $a_1, \ldots, a_n$, then

$$\det(\mathrm{Id} - t\varphi) = \prod_{i=1}^{n} (1 - a_i t).$$

On the other hand

$$\exp\left(-\sum_{m\geq 1} \mathrm{Tr}(\varphi^m)\cdot \frac{t^m}{m}\right) = \exp\left(-\sum_{m\geq 1}\sum_{i=1}^{n} \frac{a_i^m t^m}{m}\right) =$$

$$= \exp\left(\sum_{i=1}^{n} \log(1 - a_i t)\right) = \prod_{i=1}^{n}(1 - a_i t).$$

$\square$

Rationality. Assume that there exists a Weil cohomology theory for varieties over $\overline{\mathbf{F}}_p$, with p a prime. We will use this to deduce the rationality Theorem 3.1 for varieties over $\mathbf{F}_q$, with $q = p^r$. Denote $F := \mathrm{Frob}_{\overline{X},q} : \overline{X} \to \overline{X}$. More precisely, we have

Theorem 4.16. *Let X be a smooth projective geometrically connected variety over $\boldsymbol{F}_q$, of dimension n. Then*

$$Z(X;t) = \frac{P_1(t)\cdot P_3(t)\cdots P_{2n-1}(t)}{P_0(t)\cdot P_2(t)\cdots P_{2n}(t)}$$

where $P_i(t) = \det\big(\mathrm{Id} - tF^*|H^i(\overline{X})\big)$ for all $0 \leq i \leq 2n$. In particular, $Z(X;t) \in \mathbf{Q}(t)$.

Proof. We have seen in Proposition 1.7 that Δ and Γ_m are transverse in $\overline{X}\times\overline{X}$, and $N_m = \Delta\cdot\Gamma_m$. Theorem 4.11 implies then

$$N_m = \sum_{i=0}^{2n} (-1)^i\ \mathrm{Tr}\big((F^m)^*|H^i(\overline{X})\big).$$

Using Lemma 4.15, we get

$$Z(X;t) = \exp\left(\sum_{m\geq 1}\sum_{i=0}^{2n} (-1)^i\ \mathrm{Tr}\big((F^m)^*|H^i(\overline{X})\big)\frac{t^m}{m}\right) = \prod_{i=0}^{2n} \det\big(\mathrm{Id} - tF^*|H^i(\overline{X})\big)^{(-1)^{i+1}},$$

which gives the first part. Since the cohomology theory has coefficients in K, this gives $Z(X;t) \in K(t)\cap \mathbf{Q}[[t]]$, which in turn implies $Z(X;t) \in \mathbf{Q}(t)$ by the Exercise below. $\square$

Exercise 4.17. Let L be a field, and $f = \sum_{m\geq 1} a_m t^m \in L[[t]]$. Then $f \in L(t)$ if and only if there exist natural numbers m, n such that the linear span of the vectors

$$\{(a_i, a_{i+1}, \ldots, a_{i+n}) \in L^{n+1} \mid i \geq m\}$$

is a proper subspace of L^{n+1}. In particular, for any field extension $L \subset L'$, $f \in L(t)$ if and only if $f \in L'(t)$.

Functional equation. We will need another linear algebra lemma (cf. [Ha] Appendix C, Lemma 4.3).

Lemma 4.18. *Let $\varphi : V \times W \to K$ be a perfect pairing of vector spaces of dimension r over a field K. Let $f \in \mathrm{End}_K(V)$, $g \in \mathrm{End}_K(W)$ and $\lambda \in K^*$ be such that $\varphi(f(v), g(w)) = \lambda\varphi(v, w)$ for all $v \in V$ and $w \in W$. Then*

$$\det(\mathrm{Id} - tg) = \frac{(-1)^r \lambda^r t^r}{\det(f)} \cdot \det(\mathrm{Id} - \lambda^{-1} t^{-1} f)$$

and

$$\det(g) = \frac{\lambda^r}{\det(f)}.$$

Proof. Again we may assume, by extending scalars, that K is algebraically closed. We can then put the matrix of f in upper triangular form; in other words, there exists a basis $e_1, \ldots, e_r$ of V such that $f(e_i) = \sum_{j=1}^r a_{i,j} e_j$, and $a_{i,j} = 0$ for $i > j$. By the perfect pairing property, there exists a basis $e'_1, \ldots, e'_r$ of W such that $\varphi(e_i, e'_j) = \delta_{i,j}$.

The hypothesis implies that f and g are invertible. Let's check this for g: if $g(w) = 0$, then $\varphi(f(v), g(w)) = \lambda\varphi(v, w) = 0$ for all $v \in V$, so $w = 0$. Write now $g^{-1}(e'_j) = \sum_{l=1}^r b_{j,l} e'_l$. Then $b_{j,i} = 0$ for $i > j$; indeed, note that since $\varphi(f(e_i), e'_j) = 0$ for $j < i$, we also have $\varphi(e_i, g^{-1}(e'_j)) = 0$. We can relate the diagonal entries as well:

$$a_{i,i} = \varphi(f(e_i), e'_i) = \lambda\varphi(e_i, g^{-1}(e'_i)) = \lambda b_{i,i}.$$

The second identity follows by noting that $\det(f) = \prod_{i=1}^r a_{i,i}$ and $\det(g) = \prod_{i=1}^r b_{i,i}^{-1} = \lambda^r / \prod_{i=1}^r a_{i,i}$. For the first, note that

$$\det(\mathrm{Id} - tg) = \det(g) \cdot \det(g^{-1} - t\mathrm{Id}) = \frac{\lambda^r}{\det(f)} \cdot \prod_{i=1}^r (a_{i,i}\lambda^{-1} - t) =$$

$$= \frac{(-1)^r \lambda^r t^r}{\det(f)} \cdot \prod_{i=1}^r (1 - a_{i,i}\lambda^{-1} t^{-1}) = \frac{(-1)^r \lambda^r t^r}{\det(f)} \cdot \det(\mathrm{Id} - \lambda^{-1} t^{-1} f).$$

□

We can now deduce the functional equation for the zeta function.

Proof. (*of Theorem 3.2.*) We will apply Lemma 4.18 to the perfect pairing given by Poincaré duality (axiom (A3)):

$$\varphi_i : H^i(\overline{X}) \otimes H^{2n-i}(\overline{X}) \to K,\ \varphi_i(\alpha \otimes \beta) = \mathrm{Tr}_{\overline{X}}(\alpha \cup \beta),$$

with f, g taken to be F^* acting on the respective cohomology groups. Note that by Lemma 4.6, F^* acts on $H^{2n}(\overline{X})$ as multiplication by q^n, since $F : \overline{X} \to \overline{X}$ is a finite morphism of degree q^n (check!). For every $\alpha \in H^i(\overline{X})$ and $\beta \in H^{2n-i}(\overline{X})$ we have

$$\varphi_i(F^*\alpha, F^*\beta) = \mathrm{Tr}_{\overline{X}}(F^*(\alpha \cup \beta)) = \mathrm{Tr}_{\overline{X}}(q^n(\alpha \cup \beta)) = q^n \varphi_i(\alpha, \beta).$$

Denote $b_i = \dim_K H^i(\overline{X})$ and $P_i(t) = \det\big(\mathrm{Id} - tF^* | H^i(\overline{X})\big)$. Lemma 4.18 implies then that

$$\det(F^*|H^{2n-i}(\overline{X})) = \frac{q^{nb_i}}{\det(F^*|H^i(\overline{X}))}$$

and

$$P_{2n-i}(t) = \frac{(-1)^{b_i} q^{nb_i} t^{b_i}}{\det(F^*|H^i(\overline{X}))} \cdot P_i(1/q^n t).$$

Finally, recall that by Corollary 4.12 that $E = \sum_{i=0}^{2n} (-1)^i b_i$. Using the two identities above and Theorem 4.16, we obtain

$$Z(X; \frac{1}{q^n t}) = \prod_{i=0}^{2n} P_i(1/q^n t)^{(-1)^{i+1}} = \prod_{i=0}^{2n} P_{2n-i}(t)^{(-1)^{i+1}} \cdot \frac{(-1)^E q^{nE} t^E}{\prod_{i=0}^{2n} \det(F^*|H^i(\overline{X}))^{(-1)^i}} =$$

$$= \pm Z(X;t) \cdot \frac{q^{nE} t^E}{q^{nE/2}} = \pm\, q^{nE/2} t^E Z(X;t).$$

$\square$

It follows from the proof that one can make the sign in the formula more precise: it is $(-1)^E$ if $\det(F^*|H^n(\overline{X})) = q^{nb_n/2}$ and $(-1)^{E+1}$ if $\det(F^*|H^n(\overline{X})) = -q^{nb_n/2}$. Note also that our discussion for P_1 in the case of curves can be generalized: if $P_n(t) = \prod_{i=1}^{b_n}(1 - \alpha_i t)$, then the second identity above (for $i = n$) implies that the set $\{\alpha_1, \ldots, \alpha_{b_n}\}$ is invariant under the operation $\alpha \mapsto q^n/\alpha$, and $\prod_{i=1}^{b_n} \alpha_i = \det(F^*|H^n(\overline{X}))$, computed as above.

A brief introduction to ℓ-adic cohomology. This is just a very quick review, necessary to at least define and mention some properties of ℓ-adic cohomology; details can be found for instance in Milne's book [Mi]. Let X be a Noetherian scheme. We consider the category $\mathrm{\acute{E}t}(X)$ of all étale morphisms $f : Y \to X$ from a scheme to X. We think of this as the analogue of the category of open subsets of a topological space: an object can be thought of as an *étale open subset* of X.[6] For instance, inclusions of open subsets correspond here to morphisms in $\mathrm{\acute{E}t}(X)$: for any étale schemes Y and Z over X, any morphism $Y \to Z$ over X is étale. Intersections of open sets correspond to fiber products, which exist in $\mathrm{\acute{E}t}(X)$. An *étale open cover* in $\mathrm{\acute{E}t}(X)$ is a family of étale morphisms $f_i : U_i \to U$ in $\mathrm{\acute{E}t}(X)$ such that $U = \cup_i f(U_i)$. The set of all such étale covers of U is denoted $\mathrm{Cov}(U)$.

This data defines the *étale topology* on X. It is an example of a *Grothendieck topology*, in the sense that it satisfies the following properties:

(0) (Fiber products) Fiber products exist in $\mathrm{\acute{E}t}(X)$.

[6]Recall for instance that an étale morphism of complex algebraic varieties is the same thing as a local analytic isomorphism in the classical topology.

(1) (Isomorphims) If $f : U \to V$ is an isomorphism in $\text{Ét}(X)$, then $(f) \in \text{Cov}(V)$.

(2) (Local character) If $(U_i \to U) \in \text{Cov}(U)$ and for every i we have $(U_{i,j} \to U_i) \in \text{Cov}(U_i)$, then $(U_{i,j} \to U)_{i,j} \in \text{Cov}(U)$.

(3) (Stability under base change) If $(U_i \to U) \in \text{Cov}(U)$ and $V \to U$ is an morphism in $\text{Ét}(X)$, then we have $(U_i \times_U V \to V) \in \text{Cov}(V)$.

Sometimes $\text{Ét}(X)$ together with the above Grothendieck topology is called the *étale site* of X. The main point is that in order to develop a sheaf theory and a cohomology theory for sheaves, using for instance injective resolutions according to Grothendieck's recipe, one need only have families of coverings for the objects of a category, satisfying the properties above, i.e. a Grothendieck topology.

We consider the category of sheaves on the étale site, or sheaves in the étale topology, denoted $\text{Sh}(X_{\text{ét}})$. Concretely, an étale presheaf, say of abelian groups, on X is a contravariant functor $\text{Ét}(X) \to \text{AbGps}$. It is a sheaf if in addition it satisfies the gluing property: for every $U \in \text{Ét}(X)$ and every $(U_i \to U) \in \text{Cov}(U)$, the complex of abelian groups

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is exact. Note that every such $\mathcal{F}$ defines a sheaf $\mathcal{F}_U$ in the usual sense on the domain U of each object in $\text{Ét}(X)$, but there is more information contained in the definition of an étale sheaf. It can be shown that $\text{Sh}(X_{\text{ét}})$ is an abelian category with enough injectives, and therefore we can consider the right derived functors of the left exact functor $\mathcal{F} \mapsto \mathcal{F}(X)$. These are called the *étale cohomology groups* of $\mathcal{F}$ on X, denoted by $H^i_{\text{ét}}(X, \mathcal{F})$ for $i \geq 0$.

Constant sheaves. Let G be any abelian group. The étale constant sheaf on X, denoted by G as well, is the functor

$$\text{Ét}(X) \to \text{AbGps}, \ (U \to X) \mapsto G^{\pi_0(U)},$$

where $\pi_0(X)$ is the number of connected components of U. We will be especially interested in the cohomology groups $H^i_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z})$, for $n \geq 1$. For instance, when $n = 1$ we will see a rather elementary interpretation of these groups below.

$\mathcal{O}_X$-modules. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules. One defines the associated $W(\mathcal{F}) \in \text{Sh}(X_{\text{ét}})$ as follows: for every $U \to X$ in $\text{Ét}(X)$, $W(\mathcal{F})(U) := f^*\mathcal{F}(U)$. The fact that this is an étale sheaf follows from the theory of faithfully flat descent.[7] One can show that for all i there are canonical isomorphisms

$$H^i_{\text{ét}}(X, W(\mathcal{F})) \simeq H^i(X, \mathcal{F}).$$

[7] First, one easily reduces to checking the sheaf condition only for Zariski open covers, and for étale covers consisting of a single map $V \to U$ with V and U both affine. This then becomes equivalent to checking the exactness of the natural sequence

$$M \to B \otimes_A M \to B \otimes_A B \otimes_A M$$

(with $B \to B \otimes_A B$ given by $b \mapsto 1 \otimes b - b \otimes 1$) for every ring homomorphism $A \to B$ corresponding to an étale surjective morphism $\text{Spec}(A) \to \text{Spec}(B)$, and every A-module M. For this it is in fact enough to assume that $A \to B$ is *faithfully flat*, a weaker condition. Cf. [Mi] p.51.

Group schemes. Let $\mathbf{G}$ be an abelian group scheme over X. We can consider $\mathbf{G}$ as an étale sheaf on X, via the functor

$$(U \to X) \mapsto \mathbf{G}(U) := \mathrm{Hom}_X(U, \mathbf{G}).$$

Once more, this presheaf is a sheaf due to faithfully flat descent. As an important example, consider the multiplicative group scheme over X,

$$\mathbf{G}_m := X \times_{\mathrm{Spec}\ \mathbf{Z}} \mathrm{Spec}\ \mathbf{Z}[t, t^{-1}].$$

For every $U \to X$ in $\acute{\mathrm{E}}\mathrm{t}(X)$, we have that $\mathbf{G}_m(U) = \mathcal{O}_U(U)^*$. We can also consider the closed sub-group scheme corresponding to n-th roots of unity

$$\mu_n := X \times_{\mathrm{Spec}\ \mathbf{Z}} \mathrm{Spec}\ \mathbf{Z}[t]/(t^n - 1) \subset \mathbf{G}_m,$$

where $\mu_n(U) = \{u \in \mathcal{O}_U(U) \mid u^n = 1\}$. If X is a defined over a field k which is separably closed, then for every integral k-algebra A one has $\{u \in A \mid u^n = 1\} \subset k$. Hence if X is integral, any choice of an n-th root of unity in k determines as isomorphism of étale sheaves $\mu_n \simeq \mathbf{Z}/n\mathbf{Z}$.

Assume now X is defined over a field k such that $\mathrm{char}(k)$ does not divide n. Then there is an exact sequence of étale sheaves called the *Kummer sequence*:

$$0 \longrightarrow \mu_n \longrightarrow \mathbf{G}_m \longrightarrow \mathbf{G}_m \longrightarrow 0,$$

where the morphism $\mathbf{G}_m \to \mathbf{G}_m$ is given by $u \mapsto u^n$.[8] Note that there is an isomorphism similar to the isomorphism $H^1(X, \mathcal{O}_X^*) \simeq \mathrm{Pic}(X)$ in the Zariski topology, namely

$$H^1_{\acute{e}t}(X, \mathbf{G}_m) \simeq \mathrm{Pic}(X).$$

(This is sometimes called *Hilbert's Theorem 90*; see [Mi] Proposition 4.9.) Making all the assumptions above, namely that X is an integral scheme over a separably closed field k with $\mathrm{char}(k)$ not dividing n, by passing to cohomology in the Kummer sequence we obtain an exact sequence

$$\Gamma(X, \mathcal{O}_X)^* \xrightarrow{(\cdot)^n} \Gamma(X, \mathcal{O}_X)^* \to H^1_{\acute{e}t}(X, \mathbf{Z}/n\mathbf{Z}) \to \mathrm{Pic}(X) \xrightarrow{(\cdot)^n} \mathrm{Pic}(X) \to H^2_{\acute{e}t}(X, \mathbf{Z}/n\mathbf{Z}). \tag{3}$$

The case of curves. The étale cohomology of constant sheaves associated with finite abelian groups can often be described by reducing to the case of curves. In that case, one has the following general result:

Theorem 4.19. *Let X be a smooth projective curve of genus g over an algebraically closed field k. Let n be a natural number not divisible by* $\mathrm{char}(k)$. *Then one has:*

- $H^0_{\acute{e}t}(X, \mathbf{Z}/n\mathbf{Z}) \simeq \mathbf{Z}/n\mathbf{Z}$.
- $H^1_{\acute{e}t}(X, \mathbf{Z}/n\mathbf{Z}) \simeq \{L \in \mathrm{Pic}^0(X) \mid L^n \simeq \mathcal{O}_X\} \simeq (\mathbf{Z}/n\mathbf{Z})^{2g}$.
- $H^2_{\acute{e}t}(X, \mathbf{Z}/n\mathbf{Z}) \simeq \mathbf{Z}/n\mathbf{Z}$.
- $H^i_{\acute{e}t}(X, \mathbf{Z}/n\mathbf{Z}) = 0$ *for* $i > 2$.

[8] This is surjective due to the fact that for any k-algebra A and every $a \in A$, the natural morphism $A \to A[t]/(t^n - a)$ is étale and surjective, and the image of a via this morphism is an n-th power.

The main technical point for proving the result above is to show that

$$H^i_{\text{ét}}(X, \mathbf{G}_m) = 0 \text{ for all } i \geq 2.$$

(This is a consequence of Tsen's theorem saying that a non-constant homogeneous polynomial of degree $< n$ in $k(X)[X_1, \ldots, X_n]$ must have a non-trivial zero; cf. [Mi] III.2.22(d).) Using the long exact sequence in cohomology associated to the Kummer sequence, one immediately obtains the first and last statement, together with the fact that the sequence (3) is exact on the right. Note now that multiplication by n is surjective on $\mathrm{Pic}^0(X)$, which means that $H^2_{\text{ét}}(X, \mathbf{Z}/n\mathbf{Z})$ is the same as the cokernel of multiplication by n on the Neron-Severi group of X, hence isomorphic to $\mathbf{Z}/n\mathbf{Z}$. Note also that if $L \in \mathrm{Pic}^0(X)$ satisfies $L^n \simeq \mathcal{O}_X$, then we must have $L \in \mathrm{Pic}^0(X)$. Therefore we are looking at the subgroup of n-torsion points of the abelian variety $\mathrm{Pic}^0(X)$ defined over k, and given the assumption on the characteristic it is well-known that this is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{2g}$.

Exercise 4.20. Revisit the Weil conjectures for curves in view of Theorem 4.19 and the section below (cf. also [Mi] V.2).

ℓ-adic cohomology. Let now k be an algebraically closed field. If $\mathrm{char}(k) = p > 0$, consider a prime $\ell \neq p$. As above, given any $m \geq 1$, the étale cohomology $H^i_{\text{ét}}(X, \mathbf{Z}/\ell^m\mathbf{Z})$ is a $\mathbf{Z}/\ell^m\mathbf{Z}$-module, and there are natural maps

$$H^i_{\text{ét}}(X, \mathbf{Z}/\ell^{m+1}\mathbf{Z}) \longrightarrow H^i_{\text{ét}}(X, \mathbf{Z}/\ell^m\mathbf{Z})$$

forming an inductive system.

Definition 4.21. The i-th ℓ-adic cohomology of X is

$$H^i_{\text{ét}}(X, \mathbf{Z}_\ell) := \varprojlim_m H^i_{\text{ét}}(X, \mathbf{Z}/\ell^m\mathbf{Z}).$$

This has a natural structure of $\mathbf{Z}_\ell$-module, where $\mathbf{Z}_\ell$ is the ring of ℓ-adic integers. We also consider

$$H^i_{\text{ét}}(X, \mathbf{Q}_\ell) := H^i_{\text{ét}}(X, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

One of the main facts on ℓ-adic cohomology is that, when restricted to smooth projective varieties over k, it forms a Weil cohomology theory with coefficients in $\mathbf{Q}_\ell$ (see [Mi] Ch. VI). Theorem 4.16 applied in this setting implies that if X is smooth, projective, geometrically connected over $\mathbf{F}_q$, $q = p^r$, and $\ell \neq p$, then

$$Z(X; t) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2n}(t)}$$

where $P_i(t) = \det\big(\mathrm{Id} - tF^* | H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell)\big)$ for all $0 \leq i \leq 2n$.

Along the same lines, one can give a proof of rationality for arbitrary varieties (i.e. not necessarily smooth or projective) over $\mathbf{F}_q$. The only difference is that one needs to consider instead ℓ-adic cohomology with compact supports, denoted $H^i_c(\overline{X}, \mathbf{Q}_\ell)$. With this modification, the exact same formula as above holds.

Going back to the case of smooth projective varieties, as we saw above the functional equation for the zeta function follows from Poincaré duality for ℓ-adic cohomology. Finally, the hardest conjecture, the analogue of the Riemann Hypothesis, is verified in [De1], where it is shown that

$$P_i(t) = \det\big(\mathrm{Id} - tF^*|H^i_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell)\big) = \prod_i (1-\alpha_i t) \in \mathbf{Z}[t],$$

and $|\alpha_i| = q^{i/2}$ for all i, for any choice of isomorphism $\mathbf{Q}_\ell \simeq \mathbf{C}$. This is substantially more complicated than the proofs of the other conjectures explained above.

Betti numbers. The proof of Theorem 3.4 now follows from a general theorem comparing étale and singular cohomology in characteristic 0: if X is a smooth complex variety, then the étale and singular cohomology of X with coefficients in a finite abelian group[9] are isomorphic (see e.g [Mi] Theorem III.3.12).

Assume now that there exists a finitely generated $\mathbf{Z}$-algebra R, $\mathcal{X}$ a scheme which is smooth and projective over Spec R, and $\mathfrak{p} \subset R$ a maximal ideal such that $R/\mathfrak{p} \simeq \mathbf{F}_q$ and

$$X = \mathcal{X} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ R/\mathfrak{p}.$$

By the comparison theorem mentioned above, one gets for every m:

$$H^i_{\text{ét}}(\mathcal{X} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C}, \mathbf{Z}/\ell^m\mathbf{Z}) \simeq H^i((\mathcal{X} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C})^{\mathrm{an}}, \mathbf{Z}/\ell^m\mathbf{Z}).$$

Passing to the limit and tensoring with $\mathbf{Q}_\ell$, we obtain

$$H^i_{\text{ét}}(\mathcal{X} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C}, \mathbf{Q}_\ell) \simeq H^i((\mathcal{X} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C})^{\mathrm{an}}, \mathbf{Q}_\ell).$$

On the other hand, the smooth base change theorem for étale cohomology and the proper smooth base change theorem for locally constant sheaves (see [Mi] VI. §4) imply that

$$H^i_{\text{ét}}(\mathcal{X} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C}, \mathbf{Q}_\ell) \simeq H^i_{\text{ét}}(X, \mathbf{Q}_\ell).$$

Putting these two facts together, we obtain a comparison result between the ℓ-adic cohomology of X and the singular cohomology of the complex points of its lifiting,

$$H^i_{\text{ét}}(X, \mathbf{Q}_\ell) \simeq H^i((\mathcal{X} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C})^{\mathrm{an}}, \mathbf{Q}_\ell),$$

and consequently

$$b_i(X) = \deg\ P_i(t) = b_i\big((\mathcal{X} \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C})^{\mathrm{an}}\big).$$

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[9] The result is not necessarily true otherwise, for instance for $\mathbf{Z}$-coefficients.

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CHAPTER 3. p-ADIC INTEGRATION

Contents

The aim of this chapter is to set up the theory of p-adic integration to an extent which is sufficient for proving Igusa's theorem [Ig] on the rationality of the p-adic zeta function, and Weil's interpretation [We1] of the number of points of a variety over a finite field as a p-adic volume, in case this variety is defined over the ring of integers of a p-adic field. After recalling a few facts on p-adic fields, I will introduce p-adic integrals, both in the local setting and with respect to a global top differential form on a K-analytic manifold. I will explain Igusa's proof of the rationality of the zeta function using Hironaka's resolution of singularities in the K-analytic case, and the change of variables formula. Weil's theorem on counting points over finite fields via p-adic integration will essentially come as a byproduct; it will be used later in the course to compare the number of rational points of K-equivalent varieties.

1. Basics on p-adic fields

We will first look at different approaches to constructing the p-adic integers $\mathbf{Z}_p$ and the p-adic numbers $\mathbf{Q}_p$, as well as more general rings of integers in p-adic fields, and recall some of their basic properties. I highly recommend [Ne] Chapter II for a detailed discussion of this topic.

p-adic numbers. Let p be a prime, and $x \in \mathbf{Q}$. One can write uniquely $x = p^m \cdot \frac{a}{b}$ with $m \in \mathbf{Z}$ and a and b integers not divisible by p. We define the *order* and the *norm* of x with respect to p as

$$\mathrm{ord}_p(x) := m \text{ and } |x|_p := \frac{1}{p^m}.$$

This norm on $\mathbf{Q}$ is an example of the following general concept:

Definition 1.1. Let K be a field. A *non-archimedean absolute value* on K is a map $|\cdot| : K \to \mathbf{R}_{\geq 0}$ satisfying, for all $x, y \in K$, the following properties:

(i) $|x| \geq 0$ for all x, and $|x| = 0$ if and only if $x = 0$.

(ii) $|xy| = |x| \cdot |y|$.

(iii) $|x + y| \leq \max\{|x|, |y|\}$.

If we consider the mapping $d(x, y) := |x - y|$, then this is a (non-archimedean) distance function (or metric), which in turn induces a topology on K.

Definition 1.2. The field of *p-adic numbers* $\mathbf{Q}_p$ is the completion of the topological space $\mathbf{Q}$ in the norm $|\cdot|_p$, i.e. the set of equivalence classes of all Cauchy sequences with respect to this norm. [1] Note that $\mathbf{Q}_p$ is a field of characteristic 0.

It is standard to see that every $x \in \mathbf{Q}_p$ has a unique "Laurent series (base p) expansion", namely a representation of the form

$$x = a_m p^m + a_{m+1} p^{m+1} + \dots \tag{1}$$

where $m = \mathrm{ord}_p(x) \in \mathbf{Z}$ and $a_i \in \{0, 1, \dots, p-1\}$ for all i.

Exercise 1.3. Check that in $\mathbf{Q}_p$ one has:

(i) $\frac{1}{1-p} = 1 + p + p^2 + \dots$

(ii) $-1 = (p-1) + (p-1)p + (p-1)p^2 + \dots$

Exercise 1.4. Show that $\mathbf{Q}_p$ is a totally disconnected, locally compact topological space.

Definition 1.5. The *ring of p-adic integers* $\mathbf{Z}_p$ is the unit disk in the space $\mathbf{Q}_p$ with the norm $|\cdot|_p$, namely

$$\mathbf{Z}_p = \{x \in \mathbf{Q}_p \mid |x|_p \leq 1\}.$$

This is precisely the set of x with no Laurent part in the expression in (1), i.e. such that $a_i = 0$ for $i < 0$, easily checked to be a ring via the properties of the norm.

The following exercise collects some of the most important properties of $\mathbf{Z}_p$.

Exercise 1.6. (i) $\mathbf{Z}_p$ is open and closed in $\mathbf{Q}_p$.

(ii) $\mathbf{Z}_p$ is compact.

(iii) $\mathbf{Z}_p$ is a local ring, with maximal ideal $p\mathbf{Z}_p = \{x \in \mathbf{Z}_p \mid |x|_p < 1\}$, and

$$\mathbf{Z}_p / p\mathbf{Z}_p \simeq \mathbf{Z}/p\mathbf{Z}.$$

Completions of DVR's. More generally, one can approach and extend the constructions above is via completions in the $\mathfrak{m}$-adic topology. Consider a DVR $(R, \mathfrak{m})$, with field of fractions $K = Q(R)$ and associated discrete valuation $v : K \to \mathbf{Z}$. On R, or K, one can consider the $\mathfrak{m}$-adic topology, which is the unique translation invariant topology with a

[1] Thus in $\mathbf{Q}_p$ every Cauchy sequence is convergent, and $\mathbf{Q}$ can be identified with its subfield consisting of classes of constant sequences.

basis of neighborhoods of 0 consisting of $\{\mathfrak{m}^i\}_{i\geq 1}$. (See for instance [Ma] §8 for the general setting, and for a detailed treatment of the properties discussed below.)

Definition 1.7. The *completion* of R with respect to the $\mathfrak{m}$-adic topology is

$$\widehat{R} := \varprojlim_i R/\mathfrak{m}^i.$$

This is a Noetherian local ring with a canonical embedding $R \hookrightarrow \widehat{R}$. Its maximal ideal is $\mathfrak{m}\cdot\widehat{R}$, and we have

$$\widehat{R}/(\mathfrak{m}\cdot\widehat{R})^i \simeq R/\mathfrak{m}^i \quad \text{for all } i \geq 1.$$

This implies in particular that $\dim \widehat{R} = \dim R = 1$, and that the maximal ideal of $\widehat{R}$ is generated by the image in $\widehat{R}$ of a uniformizing parameter π of R, so that $\widehat{R}$ is in fact a DVR as well. Note that

$$\widehat{K} := Q(\widehat{R}) \simeq \widehat{R}_{(\pi)} \simeq K \otimes_R \widehat{R}.$$

Recall now that if $v : K \to \mathbf{Z}$ is the discrete valuation corresponding to R, one has for every $r \in R$, $v(r) = \max\{i \mid r \in \mathfrak{m}^i\}$.

Definition 1.8. Let $0 < \alpha < 1$. For every $x \in K$, define

$$|x| \ (= |x|_v) := \alpha^{v(x)} \quad \text{for } x \neq 0$$

and $|0| = 0$. This can be easily seen to be a non-archimedean norm, as in the special case of $|\cdot|_p$ above. Its corresponding distance function is $d(x,y) = |x-y|$, and one can check that the associated topology is the $\mathfrak{m}$-adic topology described above (and hence independent of the choice of α).

Exercise 1.9. Check that $\widehat{K}$ has a valuation and a non-archimedean norm extending those on K, and that as such it is the completion of K with respect to the topology induced by $|\cdot|$.

Example 1.10. The main example of course is that of p-adic integers discussed in the previous subsection. Concretely, fix a prime p, and take $R = \mathbf{Z}_{(p)}$, the localization of $\mathbf{Z}$ in the prime ideal generated by p. One has $K = Q(\mathbf{Z}_{(p)}) \simeq \mathbf{Q}$, and

$$\widehat{R} = \varprojlim_i \mathbf{Z}_{(p)}/p^i\mathbf{Z}_{(p)} \simeq \varprojlim_i \mathbf{Z}/p^i\mathbf{Z} = \mathbf{Z}_p \quad \text{and } \widehat{K} = \mathbf{Q}_p.$$

By taking $\alpha = 1/p$, we obtain the p-adic absolute value $|\cdot|_p$ defined before.

p-adic fields and rings of integers. We collect only a few properties necessary later on for working with K-analytic manifolds.

Definition 1.11. A *p-adic field* K is a finite extension of $\mathbf{Q}_p$. The *ring of integers* $\mathcal{O}_K \subset K$ is the integral closure of $\mathbf{Z}_p$ in K.

Lemma 1.12. *We have the following:*

(i) $K = Q(\mathcal{O}_K)$.

(ii) $K \simeq \mathcal{O}_K \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

(iii) $\mathcal{O}_K$ *is a DVR.*

Proof. Take any $x \in K$. Since K is algebraic over $\mathbf{Q}_p$, one can easily check that there is some $a \in \mathbf{Z}_p$ such that $ax \in \mathcal{O}_K$. This implies that $K = Q(\mathcal{O}_K)$, and in fact $K = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathcal{O}_K$, which proves (i) and (ii). For (iii), note first that clearly $\mathcal{O}_K$ is normal, while since $\mathbf{Z}_p \subset \mathcal{O}_K$ is an integral extension and $\dim \mathbf{Z}_p = 1$, we also have $\dim \mathcal{O}_K = 1$. To conclude that $\mathcal{O}_K$ is a DVR, it remains to show that it is local. Now every integral extension of a DVR is a finite algebra over it, and therefore in our case $\mathcal{O}_K$ is a finite $\mathbf{Z}_p$-algebra. The assertion then follows from the following general statement: if $(R, \mathfrak{m})$ is a complete local ring, and S is a finite R-algebra, then S is a local ring as well. This is a well-known consequence of Hensel's Lemma (explain). $\square$

Given the Lemma, let $v_K : K \to \mathbf{Z}$ be the discrete valuation of K correponding to $\mathcal{O}_K$. The *ramification index* of K over $\mathbf{Q}_p$ is $e_K := v_K(p)$; K is called *unramified* if $e_K = 1$, and otherwise *ramified*. We can define a non-archimedean norm on K extending $|\cdot|_p$ on $\mathbf{Q}_p$ by

$$|\cdot| = |\cdot|_p : K \to \mathbf{Q},\ |x|_p := \frac{1}{p^{v_K(x)/e_K}} \text{ for } x \neq 0$$

and $|0| = 0$. We clearly have

$$\mathcal{O}_K = \{x \in K \mid |x|_p \leq 1\},$$

while the maximal ideal $\mathfrak{m}_K \subset \mathcal{O}_K$ is given by the condition $|x|_p < 1$. As before, we consider on K and $\mathcal{O}_K$ the topology corresponding to this norm.

Proposition 1.13. *Let K be a p-adic field, with the topology associated to $|\cdot|_p$. Then:*

(i) As a $\mathbf{Z}_p$-module, $\mathcal{O}_K$ is isomorphic to the free module $\mathbf{Z}_p^d$, where $d := [K : \mathbf{Q}_p]$.

(ii) There exists a basis of open neighborhoods of 0 *in $\mathcal{O}_K$ given by $\{p^i\mathcal{O}_K\}_{i\geq 1}$.*

(iii) Fixing an isomorphism $\mathcal{O}_K \simeq \mathbf{Z}_p^d$, the topology on $\mathcal{O}_K$ corresponds to the product topology on $\mathbf{Z}_p^d$.

(iv) $\mathcal{O}_K$ and K are complete topological spaces.

Proof. (i) The ring $\mathbf{Z}_p$ is a PID (every ideal is generated by a power of p) and $\mathcal{O}_K$ is a torsion-free $\mathbf{Z}_p$-module. Since $\mathcal{O}_K$ is finite over $\mathbf{Z}_p$, by the structure theorem for modules over PID's we get that $\mathcal{O}_K$ is a free $\mathbf{Z}_p$-module, of finite rank equal to $d = [K : \mathbf{Q}_p]$.

(ii) The topology given by $|\cdot|_p$ coincides with the $\mathfrak{m}_K$-adic topology, and so the family $\{\mathfrak{m}_K^i\}_{i\geq 1}$ gives a basis of open neighborhoods of the origin. Now the statement follows by observing that by the definition of ramification it follows that $p\mathcal{O}_K = \mathfrak{m}_K^{e_K}$, so the $\mathfrak{m}_K$-adic topology and the $p\mathcal{O}_K$-adic topology on $\mathcal{O}_K$ coincide.

(iii) Via such an isomorphism, the ideal $p\mathcal{O}_K$ corresponds to the product of the ideals $p\mathbf{Z}_p$. Now by (ii) and the basic properties of $\mathbf{Z}_p$, the powers of these ideals on the two sides give bases for the respective topologies.

(iv) This follows immediately from (iii): since the topology on $\mathbf{Z}_p$ is complete, so is the product topology on $\mathbf{Z}_p^d$ and hence that on $\mathcal{O}_K$. As every point in K has a neighborhood homeomorphic to $\mathcal{O}_K$, this implies that K is complete as well. □

Remark 1.14. The reasoning in (i) and (ii) above can be made a bit more precise. On one hand the quotient $\mathcal{O}_K/p\mathcal{O}_K$ is free of rank d over $\mathbf{F}_p$. On the other hand, it has a filtration with successive quotients isomorphic to $\mathcal{O}_K/\mathfrak{m}_K$, namely

$$(0) \subset \mathfrak{m}_K^{e_K-1}/\mathfrak{m}_K^{e_K} \subset \cdots \subset \mathfrak{m}_K/\mathfrak{m}_K^{e_K} \subset \mathcal{O}_K/\mathfrak{m}_K^{e_K}.$$

This implies that the residue field $\mathcal{O}_K/\mathfrak{m}_K$ is a finite extension of $\mathbf{F}_p$ of degree $[K:\mathbf{Q}_p]/e_K$.

Proposition 1.15. *A p-adic field K is locally compact, and its ring of integers $\mathcal{O}_K$ is compact.*

Proof. Since $\mathcal{O}_K$ is complete with respect to the topology given by $|\cdot|_p$, which is the same as the $\mathfrak{m}_K$-adic topology, we have

$$\mathcal{O}_K \simeq \varprojlim_i \mathcal{O}_K/\mathfrak{m}_K^i.$$

Now, as in any discrete valuation ring, we have

$$m_K^n/m_K^{n+1} \simeq \mathcal{O}_K/\mathfrak{m}_K.$$

(If $\mathfrak{m}_K = (\pi_K)$, then the mapping is given by $a\pi_K^n \mapsto a(\mathrm{mod}\ m_K)$.) Since $\mathcal{O}_K/\mathfrak{m}_K$ is a finite field, the exact sequences

$$0 \longrightarrow \mathfrak{m}_K^{i-1}/\mathfrak{m}_K^i \longrightarrow \mathcal{O}_K/\mathfrak{m}_K^i \longrightarrow \mathcal{O}_K/\mathfrak{m}_K^{i-1} \longrightarrow 0.$$

imply inductively that all the rings $\mathcal{O}_K/\mathfrak{m}_K^i$ are finite, and hence compact. The product $\prod_{i=1}^\infty \mathcal{O}_K/\mathfrak{m}_K^i$ is then compact, and so the closed subset $\varprojlim_i \mathcal{O}_K/\mathfrak{m}_K^i$ is compact as well. Now $\mathcal{O}_K$ is open in K, so for every $x \in K$ the set $x + \mathcal{O}_K$ is a neighborhood of x which is compact. □

Finally, let's note the following fact, in analogy with the p-adic expansion of an element in $\mathbf{Q}_p$. Let K be a p-adic field, and let π_k be a uniformizing parameter for $\mathcal{O}_K$. Recall that $\mathcal{O}_K/\mathfrak{m}_K \simeq \mathbf{F}_q$ with $q = p^{[K:\mathbf{Q}_p]/e_K}$, and choose a system of representatives $S \subset \mathcal{O}_K$ for $\mathcal{O}_K/\mathfrak{m}_K$ (so a finite set of cardinality q, including 0). Then every element $x \in K$ admits a unique representation as a convergent Laurent series

$$x = a_m\pi_K^m + a_{m+1}\pi_K^{m+1} + \dots$$

with $a_i \in S$, $a_m \neq 0$, and $m \in \mathbf{Z}$.

Remark 1.16. A p-adic field is a (not quite so) special example of the more general notion of a *local field* (see [Ne] Ch.II §5). Most of the general aspects of the discussion above can be extended to the setting of local fields.

2. p-adic integration

Let G be a topological group, i.e. endowed with a topology which makes the group operation $G \times G \to G$, $(g, h) \mapsto gh$, and the inverse map $G \to G$, $g \mapsto g^{-1}$, continuous. If G is abelian[2] and locally compact, it is well-known that it has a non-zero translation invariant[3] measure μ with a mild regularity property, which is unique up to scalars. This is called the *Haar measure*. More precisely, a Haar measure is characterized by the following integration properties:

- Any continuous function $f : G \to \mathbf{C}$ with compact support is μ-integrable.
- For any μ-integrable function f and any $g \in G$, one has

$$\int_G f(x) d\mu = \int_G f(gx) d\mu.$$

Other important properties of the Haar measure are as follows: every Borel subset of G is μ-measurable, $\mu(A) > 0$ for every nonempty open subset A of G , while A is compact subset if and only if $\mu(A)$ is finite. For a thorough treatment, including a proof of existence and uniqueness, see [RV] §1.2.

We use this in the p-adic setting (see for instance [Ig] §7.4 or [We1]). Let's start with the more down-to-earth case of $\mathbf{Q}_p$. Since $\mathbf{Q}_p$ is locally compact, it has Haar measure μ, which according to the discussion above can be normalized so that for the compact subring $\mathbf{Z}_p$ it satisfies

$$\mu(\mathbf{Z}_p) = 1.$$

For any measurable function $f : \mathbf{Q}_p \to \mathbf{C}$, one can consider the integrals

$$\int_{\mathbf{Q}_p} f d\mu \quad \text{and especially} \quad \int_{\mathbf{Z}_p} f d\mu.$$

Here are some first examples of calculations.

Example 2.1. $\mu(p\mathbf{Z}_p) = \frac{1}{p}$.

Proof. Since $\mathbf{Z}_p/p\mathbf{Z}_p \simeq \mathbf{F}_p$, with a set of representatives $0, 1, \ldots, p-1$, we have a disjoint union decomposition

$$\mathbf{Z}_p = p\mathbf{Z}_p \cup (p\mathbf{Z}_P + 1) \cup \ldots \cup (p\mathbf{Z}_P + p - 1).$$

By translation invariance, all of the sets on the right have the same measure, and since $\mu(\mathbf{Z}_p) = 1$, this immediately gives the result. □

Exercise 2.2. Show more generally that for every $m \geq 1$ one has $\mu(p^m\mathbf{Z}_p) = \frac{1}{p^m}$.

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[2]This is not necessary, but otherwise we would have to speak separately of left and right invariant measures.

[3]This means that for every measurable set $A \subset G$ and any $g \in G$, one has $\mu(A) = \mu(gA)$.

A useful observation for calculating integrals is the following: the functions $f : \mathbf{Z}_p \to \mathbf{C}$ we will be dealing with have their image equal to a countable subset, say $C \subset \mathbf{C}$. Suppose we want to calculate the integral $\int_A f(x)d\mu$, for some measurable set A. If

$$A_f(c) := \{x \in A \mid f(x) = c\}$$

are the level sets of f in A, then

$$\int_A f(x)d\mu = \sum_{c\in C} \int_{A_f(c)} f(x)d\mu = \sum_{c\in C} c \cdot \mu(A_f(c)).$$

For arbitrary functions $f : \mathbf{Z}_p \to \mathbf{C}$, the definition and calculation of the integral is of course much more complicated.

Example 2.3. Let $s \geq 0$ be a real number, and $d \geq 0$ an integer. Then

$$\int_{\mathbf{Z}_p} |x^d|^s d\mu = \frac{p-1}{p - p^{-ds}}.$$

Proof. We take advantage of the fact that in this context the function we are integrating is the analogue of a step function, as in the comment above. We clearly have:

- $|x^d|^s = 1$ for $x \in \mathbf{Z}_p - p\mathbf{Z}_p$.
- $|x^d|^s = \frac{1}{p^{ds}}$ for $x \in p\mathbf{Z}_p - p^2\mathbf{Z}_p$.
- $|x^d|^s = \frac{1}{p^{2ds}}$ for $x \in p^2\mathbf{Z}_p - p^3\mathbf{Z}_p$.

and so on. Since these sets partition $\mathbf{Z}_p$ we get

$$\int_{\mathbf{Z}_p} |x^d|^s d\mu = 1 \cdot \mu(\mathbf{Z}_p - p\mathbf{Z}_p) + \frac{1}{p^{ds}} \cdot \mu(p\mathbf{Z}_p - p^2\mathbf{Z}_p) + \frac{1}{p^{2ds}} \cdot \mu(p^2\mathbf{Z}_p - p^3\mathbf{Z}_p) + \ldots$$

Using the exercise above, this sum is equal to

$$1 \cdot \left(1 - \frac{1}{p}\right) + \frac{1}{p^{ds}} \cdot \left(\frac{1}{p} - \frac{1}{p^2}\right) + \frac{1}{p^{2ds}} \cdot \left(\frac{1}{p^2} - \frac{1}{p^3}\right) + \ldots =$$
$$= \left(1 + \frac{1}{p^{ds+1}} + \frac{1}{p^{2ds+2}} + \ldots\right) - \frac{1}{p} \cdot \left(1 + \frac{1}{p^{ds+1}} + \frac{1}{p^{2ds+2}} + \ldots\right) =$$
$$= \left(1 - \frac{1}{p}\right) \cdot \frac{1}{1 - p^{-ds-1}} = \frac{p-1}{p - p^{-ds}}.$$

□

Let now K be more generally any p-adic field, with ring of integers $\mathcal{O}_K$. Recall that if $\mathfrak{m}_K = (\pi_K)$ is the maximal ideal, then $\mathcal{O}_K/\mathfrak{m}_K \simeq \mathbf{F}_q$, where $q = p^r$ for some $r > 0$. Take on K the topology dicussed in the previous section, namely that induced by the norm $|\cdot|_p$ extending the p-adic norm on $\mathbf{Q}_p$. We have seen that K is a (totally disconnected) locally compact abelian topological group, hence we can consider a Haar measure μ on K. According to the discussion above, since $\mathcal{O}_K$ is compact we can normalize this measure so that it satisfies

$$\mu(\mathcal{O}_K) = 1.$$

For any $r \geq 1$, we can also consider the Haar measure on K^r with the product topology, normalized such that $\mu(\mathcal{O}_K^r) = 1$. We can integrate any measurable function f defined on K^r, for instance $f \in \mathcal{O}_K[X_1, \ldots, X_r]$.

Lemma 2.4. *For every $k \geq 1$, $\mu(\mathfrak{m}_K^k) = \frac{1}{q^k}$.*

Proof. First note that we have a disjoint union

$$\mathcal{O}_K = \bigcup_{s \in S} (\mathfrak{m}_K + s),$$

where S is a set of representatives in $\mathcal{O}_K$ for $\mathcal{O}_K/\mathfrak{m}_K \simeq \mathbf{F}_q$. By translation invariance, this immediately implies that $\mu(\mathfrak{m}_k) = 1/q$. By a completely similar argument we see that

$$\mu(\mathfrak{m}^k) = \frac{1}{|\mathcal{O}_K/\mathfrak{m}_K^k|}.$$

Now a simple filtration argument as in the previous subsection shows that

$$|\mathcal{O}_K/\mathfrak{m}_K^k| = |\mathfrak{m}_K^{k-1}/\mathfrak{m}_K^k| \cdot \ldots \cdot |\mathfrak{m}_K/\mathfrak{m}_K^2| \cdot |\mathcal{O}_K/\mathfrak{m}_K| = q^k.$$

The last equality follows since for each i we have $\mathfrak{m}_K^{i-1}/\mathfrak{m}_K^i \simeq \mathcal{O}_K/\mathfrak{m}_K$. □

Exercise 2.5. Show that $\mu(\mathcal{O}_K^*) = 1 - \frac{1}{q}$.

Exercise 2.6. Show that for any non-negative integers $k_1, \ldots, k_r$, one has

$$\mu(\mathfrak{m}_K^{k_1} \times \ldots \times \mathfrak{m}_K^{k_r}) = \frac{1}{q^{k_1 + \ldots + k_r}}.$$

Exercise 2.7. Let $s \geq 0$ be a real number, and $d \geq 0$ an integer. Then

$$\int_{\mathcal{O}_K} |x^d|^s d\mu = \frac{q-1}{q - q^{-ds}}.$$

More generally, for any non-negative integers $k_1, \ldots, k_r$,

$$\int_{\mathcal{O}_K^r} |x_1^{k_1} \cdot \ldots \cdot x_r^{k_r}|^s d\mu = \prod_{i=1}^r \frac{q-1}{q - q^{-k_i s}}.$$

(Hint: Fubini's formula holds for p-adic integrals.)

Remark 2.8. (1) The computations above still work if one takes $s \in \mathbf{C}$, imposing the condition $\mathrm{Re}(s) > -1$.

(2) What makes an integral as above easy to compute is the fact that the integrand is a monomial in the variables, i.e. it involves only multiplication (for which we have the norm formula $|xy| = |x||y|$). As soon as addition appears in the integrand, things become a lot more complicated, partly due to the absence of a formula for $|x + y|$. Here is a typical example:

Exercise 2.9 (Challenge; cf. [duS])**.** For $s \geq 0$, compute the integral

$$\int_{\mathbf{Z}_p^2} |xy(x+y)|^s d\mu.$$

Definition 2.10. Let $f \in \mathcal{O}_K[X_1, \ldots, X_r]$ (or more generally any K-analytic function as defined below), and let $s \in \mathbf{C}$. The *(local) zeta function* of f is

$$Z(f, s) := \int_{\mathcal{O}_K^r} |f(x)|^s d\mu.$$

It is a holomorphic function of s for $\mathrm{Re}(s) > 0$ (exercise). (This is a special example of a more general type of zeta functions introduced in [We1].)

3. Integration on K-analytic manifolds

K-analytic functions and manifolds. In this section, besides Igusa's book [Ig], I am benefitting from lecture notes of Lazarsfeld [La]. Let K be a p-adic field, and $r > 0$ an integer. For any open set $U \subset K^r$, a *K-analytic function* $f : U \to K$ is a function which is locally around any point in U given by a convergent power series. Such a function can be seen in a standard fashion to be differentiable, with all partial derivatives again K-analytic functions (see [Ig] Ch.2). We call $f = (f_1, \ldots, f_m) : U \to K^m$ a *K-analytic map* if all f_i are K-analytic functions.

Definition 3.1 (K-analytic manifold)**.** Let X be a Haussdorff topological space, and $n \geq 0$ an integer. A *chart* of X is a pair (U, φ_U) consisting of an open subset of X together with a homeomorphism $\varphi_U : U \to V$ onto an open set $V \subset K^n$. An *atlas* is a family of charts $\{(U, \varphi_U)\}$ such that for every U_1, U_2 with $U_1 \cap U_2 \neq \emptyset$ the composition

$$\varphi_{U_2} \circ \varphi_{U_1}^{-1} : \varphi_{U_1}(U_1 \cap U_2) \to \varphi_{U_2}(U_1 \cap U_2)$$

is bi-analytic. Two atlases are equivalent if their union is also an atlas. Finally, X together with an equivalence class of atlases as above is called a *K-analytic manifold* of dimension n. If we vary x around a point $x_0 \in U$, where U is an open set underlying a chart, then $\varphi_U(x) = (x_1, \ldots, x_n)$ is called a *system of coordinates* around x_0. K-analytic maps between manifolds X and Y are defined in the obvious way.

From the similar properties of K (and hence K^r) we get:

Lemma 3.2. *A K-analytic manifold is a locally compact, totally disconnected topological space.*

Example 3.3. (1) Every open set $U \subset K^n$ is a K-analytic manifold.

(2) $X = \mathcal{O}_K^n \subset K^n$ is a compact K-analytic manifold; note that it is a manifold since it is an open subset of K^n.

(3) Consider the projective line $\mathbf{P}^1$ over K, with homogeneous coordinates $(x : y)$. This is covered by two disjoint compact open sets (sic!), namely

$$U := \{(x : y) \mid |x/y| \leq 1\} \quad \text{and} \quad V := \{(x : y) \mid |y/x| < 1\}.$$

We have bi-analytic maps

$$U \to \mathcal{O}_K, \ (x : y) \mapsto x/y \quad \text{and} \quad V \to \mathfrak{m}_K \simeq_{\text{homeo}} \mathcal{O}_K, \ (x : y) \mapsto y/x.$$

(4) Let $\pi : \mathrm{Bl}_0(K^2) \to K^2$ be the blow-up of the origin in the affine plane over K, naturally defined inside $K^2 \times \mathbf{P}^1$. Let

$$X = \mathcal{O}_K^2 \subset K^2 \quad \text{and} \quad Y = \pi^{-1}(X),$$

both compact K-analytic manifolds. Recall that $\mathrm{Bl}_0(K^2)$ is covered by two copies of K^2, mapping to the base K^2 via the rules

$$K^2 \to K^2 \ (s,t) \mapsto (s, st) \quad \text{and} \quad K^2 \to K^2 \ (u,v) \mapsto (uv, u).$$

We can then express Y as the disjoint union of the compact open sets

$$U = \{(s,t) \mid |s| \leq 1, |t| \leq 1\} \quad \text{and} \quad V = \{(u,v) \mid |u| < 1, |v| \leq 1\},$$

by noting that

$$\pi(U) = \{(x,y) \mid |y| \leq |x| \leq 1\} \quad \text{and} \quad \pi(V) = \{(x,y) \mid |x| < |y| \leq 1\} \cup \{(0,0)\}.$$

What we saw in the examples above is a general fact:

Exercise 3.4. Every compact K-analytic manifold of dimension n is bi-analytic to a finite disjoint union of copies of $\mathcal{O}_K^n$.

Differential forms and measure. If X is an n-dimensional K-analytic manifold, we can define differential forms in the usual way. Locally on an open set U with coordinates $x_1, \ldots, x_n$, a form of degree k can be written as

$$\nu = \sum_{i_1 < \ldots < i_k} f_{i_1, \ldots, i_k} \, dx_{i_1} \wedge \ldots \wedge dx_{i_k}, \tag{2}$$

with $f_{i_1,\ldots,i_k}$ K-valued functions on U. If these functions are K-analytic, then ν is a K-*analytic differential form.*

We will be particularly interested in forms of top degree n, and the measure they define. Let ω be such a K-analytic form on X. The associated measure $\mu_\omega = |\omega|$ on X is defined as follows. Let's assume first that X is a local $\pi_K^{p_1}\mathcal{O}_K \times \ldots \times \pi_K^{p_n}\mathcal{O}_K$, on which (2) holds globally. For any compact-open polycylinder

$$A \simeq (x_1 + \pi_K^{k_1}\mathcal{O}_K) \times \ldots \times (x_r + \pi_K^{k_n}\mathcal{O}_K) \subset X$$

we set

$$\mu_\omega(A) := \int_A |f(x)| d\mu,$$

where μ is the usual normalized Haar measure. This is easily checked to define a Borel measure on X.

To define the measure μ_ω on a global X, we need to check that it transforms precisely like differential forms when changing coordinates.

Theorem 3.5 (Change of variables formula, I)**.** *Let $\varphi = (\varphi_1, \ldots, \varphi_n) : K^n \to K^n$ be a K-analytic map. Suppose $x \in K^n$ is a point where* $\det\left(\frac{\partial \varphi_i}{\partial x_j}(x)\right) \neq 0$. *Then φ restricts to a bi-analytic isomorphism*

$$\varphi : U \subset K^n \xrightarrow{\simeq} V \subset K^n$$

with U a neighborhood of x and V a neighborhood of $\varphi(x)$, and $\mu^V_{\mathrm{Haar}} = \left|\det\left(\frac{\partial\varphi_i}{\partial x_j}(x)\right)\right|_K \cdot \mu^U_{\mathrm{Haar}}$*, which means that for every measurable set $A \subset U$ one has*

$$\int_{\varphi(A)} d\mu^V_{\mathrm{Haar}} = \int_A \left|\det\left(\frac{\partial\varphi_i}{\partial x_j}(x)\right)\right|_K \cdot d\mu^U_{\mathrm{Haar}}.$$

Here is just a brief sketch of the proof (for more details see [Ig] §7.4). The essential point is to treat the case when φ is given by multiplication by an invertible matrix M. For this in turn the essential case is that of a diagonal matrix $M = \mathrm{diag}(\pi_K^{k_1}, \ldots, \pi_K^{k_n})$. This maps the polydisk $\mathcal{O}_K^n$, of measure 1, to $\pi_K^{k_1}\mathcal{O}_K \times \ldots \times \pi_K^{k_n}\mathcal{O}_K$, of measure $\frac{1}{q^{k_1+\ldots+k_n}}$; on the other hand $|\det(M)| = \frac{1}{q^{k_1+\ldots+k_n}}$, since $|\pi_K| = \frac{1}{q}$.

Remark 3.6. Let's change notation slightly in order to make this look more familiar: denote $|dx| = |dx_1 \wedge \ldots \wedge dx_n|$ and $|dy| = |dy_1 \wedge \ldots \wedge dy_n|$ the Haar measure on U and V respectively. Then the Theorem says that for $A \subset U$ measurable,

$$\int_{\varphi(A)} |dy| = \int_A |\varphi^* dy| = \int_A |\det(\mathrm{Jac}(\varphi))| \cdot |dx|.$$

Putting together all of the above, we obtain

Corollary 3.7. *Let X be a compact K-analytic manifold, and ω a K-analytic n-form on X. Then there exists a globally defined measure μ_ω on X. In particular, for any continuous function $f : X \to \mathbf{C}$, the integral $\int_X f(x) d\mu_\omega$ is well-defined.*

Proof. By Exercise 3.4, we can cover X by finitely many disjoint compact open subsets U on which $\omega = f(x)dx_1 \wedge \ldots \wedge dx_n$. But by Theorem 3.5, μ_ω is independent of the particular choice of coordinates, hence it gives a globally defined measure. ☐

Since removing sets of measure 0 does not affect integrals, Theorem 3.5 immediately implies the following slightly more general statement.

Theorem 3.8 (Change of variables formula, II)**.** *Let $\varphi : Y \to X$ be a K-analytic map of compact K-analytic manifolds. Assume that φ is bi-analytic away from closed subsets $Z \subset Y$ and $\varphi(Z) \subset X$ of measure 0. If ω is a K-analytic n-form on X and f is a K-analytic function on X, then*

$$\int_X |f|^s d\mu_\omega = \int_Y |f \circ \varphi|^s d\mu_{\varphi^*\omega}.$$

Example 3.9. Let $\pi : Y = \mathrm{Bl}_0(\mathcal{O}_K^2) \to X = \mathcal{O}_K^2$ be the blow-up of the origin. Recall that in Exercise 2.7 we've computed

$$\int_X |x^a y^b|^w |dx \wedge dy| = \frac{1}{1-q^{-wa-1}} \cdot \frac{1}{1-q^{-wb-1}} \cdot \left(\frac{q-1}{q}\right)^2. \tag{3}$$

We verify the change of variables formula on the blow-up Y. We know from Example 3.3 (4) that Y is covered by two disjoint polydisks $Y = U \cup V$ with

$$U = \{(s,t) \mid |s| \leq 1, |t| \leq 1\} \text{ and } V = \{(u,v) \mid |u| < 1, |v| \leq 1\}$$

with maps to X given by

$$(s,t) \mapsto (s, st) \quad \text{and} \quad (u,v) \mapsto (uv, u).$$

This means that

$$\pi^*(dx \wedge dy) = sds \wedge dt \text{ on } U \text{ and } \pi^*(dx \wedge dy) = vdu \wedge dv \text{ on } V.$$

while on the other hand

$$|\pi^* f|^s = |s^{a+b}|^w |t^b|^w \text{ on } U \text{ and } |\pi^* f|^s = |u^a|^w |v^{a+b}|^w \text{ on } V.$$

This gives

$$\int_Y |\pi^* f|^w \cdot |\pi^*(dx \wedge dy)| = \int_{|s|\le 1, |t| \le 1} |s|^{w(a+b)+1} |t|^{wb} ds \wedge dt + \int_{|u| \le \frac{1}{q}, |v| \le 1} |u|^{wa} |t|^{w(a+b)+1} du \wedge dv.$$

We now use result similar to Exercise 2.7, namely

Exercise 3.10. For every non-negative integer m and any c,

$$\int_{|x| \le \frac{1}{q^m}} |x|^c |dx| = \frac{q^{-m(c+1)}}{1 - q^{-(c+1)}} \cdot \frac{q-1}{q}.$$

Given this formula, we can write the integral above as

$$\left(\frac{q-1}{q}\right)^2 \left(\frac{1}{(1 - q^{-w(a+b)-2})(1 - q^{-wb-1})} + \frac{1}{(1 - q^{-wa-1})(1 - q^{-w(a+b)-2})} \right)$$

and a simple calculation leads to the same formula as (3).

Resolution of singularities. We start with an example. Generalizing a previous exercise, given integers a, b, c let's consider the integral

$$I := \int_{X = \mathcal{O}_K^2} |f(x,y)|^w |dx \wedge dy|, \quad \text{with} \quad f(x,y) = x^a y^b (x-y)^c.$$

Given the change of variables formula, and the fact that integrals of monomial functions are much easier to compute, a natural idea is to pass to a birational model of X on which the function f can be brought to a monomial form. In this particular case, fortunately one needs to consider only the blow-up $\pi : Y \to X$ at the origin. Recall yet again that Y is covered by two disjoint polydisks $Y = U \cup V$ with

$$U = \{(s,t) \mid |s| \le 1, |t| \le 1\} \quad \text{and} \quad V = \{(u,v) \mid |u| < 1, |v| \le 1\}$$

with maps to X given by

$$(s,t) \mapsto (s, st) \quad \text{and} \quad (u,v) \mapsto (uv, u).$$

Consider now

$$I_U := \int_U |\pi^* f| |\pi^*(dx \wedge dy)|.$$

A simple calculation shows that on U we have

$$(\pi^* f)(s,t) = s^{a+b+c} t^b (1-t)^c \quad \text{and} \quad \pi^*(dx \wedge dy) = sds \wedge dt,$$

which gives

$$I_U = \int_U |s^{a+b+c+1}t^b(1-t)^c|^s|ds \wedge dt| =$$

$$= \left(\int_{|s|\le 1} |s^{a+b+c+1}|^w|ds|\right) \cdot \left(\int_{|t|\le 1} |t^b|^w|(1-t)^c|^w|dt|\right).$$

We have already seen the calculation of the first integral in the product a few times. For the second integral, let's choose a set $S = \{\alpha_1 = 0, \alpha_2 = 1, \alpha_3, \ldots, \alpha_q\} \subset \mathcal{O}_K$ of representatives for $\mathcal{O}_K/\mathfrak{m}_K \simeq \mathbf{F}_q$. We can then split the region $\{|t| \le 1\}$ as a disjoint union

$$\{|t| \le 1\} = T_1 \cup \ldots \cup T_q, \;\; T_i := \{|t - \alpha_i| \le \frac{1}{q}\},$$

so that if we denote $g(t) = t^b(1-t)^c$, we have

$$\int_{|t|\le 1} |g(t)|^w|dt| = \int_{T_1} |g(t)|^w|dt| + \ldots + \int_{T_q} |g(t)|^w|dt|.$$

The point is that each of the integrals in the sum on the right can now be computed as a "monomial" integral. For instance, note that the condition $|t| \le \frac{1}{q}$ defining T_1 implies that $|t-1| = 1$, so that

$$\int_{T_1} |g(t)|^w|dt| = \int_{T_1} |t^b|^w|dt|,$$

which we know how to compute. Similarly, the condition defining T_2 implies $|t| = 1$, so that

$$\int_{T_2} |g(t)|^w|dt| = \int_{|t-1|\le \frac{1}{q}} |(t-1)^c|^w|dt|,$$

which again we know how to compute after making the change of variables $t' = t - 1$. The same thing can be done for all the other T_i's, which means that we can complete the calculation of I_U via only monomial computations. One can similarly define I_V and deal with it in an analogous fashion, while finally by the change of variables formula and the decomposition of Y we have $I = I_U + I_V$.

Exercise 3.11. Complete all the details of the calculation above to find a formula for I.

What happened here? Geometrically, by blowing up the origin in $\mathcal{O}_K^2$, we "resolved the singularities" of the curve $f(x,y) = 0$. (Draw the picture.) The essential point is that on $\mathrm{Bl}_0(\mathcal{O}_K^2)$, the function $\pi^* f$ is locally monomial. Hironaka's famous theorem on resolution of singularities says that we can always do this. Let's recall first the better known version over $\mathbf{C}$, restricting only to the case of hypersurfaces in affine space.

Theorem 3.12 (Resolution of singularities over $\mathbf{C}$). *Let $X = \mathbf{C}^n$, and let $f \in \mathbf{C}[X_1, \ldots, X_n]$ be a non-constant polynomial. Then there exists a complex manifold Y and a proper surjective map $\pi : Y \to X$ such that the divisor*

$$\mathrm{div}(\pi^* f) + \mathrm{div}(\pi^*(dx_1 \wedge \ldots \wedge dx_n))$$

has simple normal crossings support.

Remark 3.13. Let's recall and expand a bit the terminology in the Theorem. A *simple normal crossings divisor* on X is an effective divisor $D = \sum_{i=1}^{k} F_i$ such that each F_i is a non-singular codimension 1 subvariety of X, and in the neighborhood of each point, D is defined in local coordinates $x_1, \ldots, x_n$ by the equation $x_1 \cdot \ldots \cdot x_k = 0$. The conclusion of the Theorem is that one can write

$$\mathrm{div}(\pi^* f) = \sum_{i=1}^{k} a_i F_i \quad \text{and} \quad \mathrm{div}(\pi^*(dx_1 \wedge \ldots \wedge dx_n)) = \sum_{i=1}^{k} b_i F_i$$

for some integers a_i, b_i, $i \in \{1, \ldots, k\}$, with $\sum_{i=1}^{k} F_i$ having simple normal crossings support. A mapping π as in the statement is called an *embedded resolution of singularities* of the hypersurface $(f = 0)$. The integers a_i, b_i are important invariants of the resolution, called *discrepancies*.

Example 3.14. Let $f(x, y) = y^2 - x^3$, i.e. the equation defining a cusp in $\mathbf{C}^2$. Its embedded resolution is one of the best known examples of this procedure. In order to get to a simple normal crossings divisor, we have to blow-up the origin three successive times, keeping track of multiplicities; for the geometric picture, see [Ha] Example 3.9.1. If $Y \to \mathbf{C}^2$ is the composition of the three blow-ups, denoting by C the proper transform of $(f = 0)$, and by E_1, E_2, E_3 the three exceptional divisors in Y (coming from the succesive blow-ups, in this order), we have

$$\mathrm{div}(\pi^* f) = C + 2E_1 + 3E_2 + 6E_3$$

and

$$K_{Y/\mathbf{C}^2} = \mathrm{div}(\pi^*(dx \wedge dy)) = E_1 + 2E_2 + 4E_3.$$

Exercise 3.15. Complete the details of the calculation.

Since we're at this, let's record a few more things about general birational maps between smooth complex varieties. Let $\pi : Y \to X$ be a birational map (not necessarily proper) between two such varieties of dimension n. A key point to note is that while we cannot talk about canonically defined divisors K_X and K_Y, there is a canonically defined *relative* canonical divisor $K_{Y/X}$, namely the zero locus of the Jacobian of the map π. This supported on the exceptional locus of π: if E_i are the exceptional divisors of π, with $1 \le i \le k$, then there exist positive integers a_{E_i} such that

$$K_{Y/X} := Z(\mathrm{Jac}(\pi)) = \sum_{i=1}^{k} a_{E_i} \cdot E_i. \tag{4}$$

Let's make this explicit. Fix an exceptional divisor E for π, and consider $Z = f(E) \subset X$, with $\mathrm{codim}_X Z = c \ge 2$. Let $y \in E$ be a general point, and $x = f(y) \in Z$. Since these are smooth points, we can then choose systems of coordinates $y_1, \ldots, y_n$ around y, and $x_1, \ldots, x_n$ around x, such that $E = (y_1 = 0)$ and $Z = (x_1 = \ldots = x_c = 0)$. Then for each $1 \le i \le c$ there exist integers $k_i \ge 1$ such that

$$\pi^* x_i = y_i^{k_i} \cdot \psi_i,$$

with ψ_i an analytic function which is invertible at y. Noting that

$$\pi^* dx_i = k_i \psi_i \cdot y_i^{k_i - 1} \cdot dy_i + d\psi_i \cdot y_i^{k_i},$$

this gives

$$\pi^*(dx_1 \wedge \ldots \wedge dx_n) = y_1^{a_E} \cdot (\text{unit}) \cdot (dy_1 \wedge \ldots \wedge dy_n),$$

with $a_E = (\sum_{i=1}^c k_i) - 1$. In other words, locally around a general point of E we have $K_{Y/X} = a_E \cdot E$. Globalizing this to include all exceptional divisors of π we get (4). Note also that for every effective divisor $D \subset X$ we have

$$\pi^* D = \tilde{D} + \sum_{i=1}^k b_i \cdot E_i,$$

where $\tilde{D}$ is the proper transform of D, and b_i are non-negative integers.

Going back to the K-analytic picture, for our present purposes the important fact is that a statement completely analogous to Theorem 3.12 holds for K-analytic manifolds.

Theorem 3.16 (K-analytic resolution of singularities)**.** *Let K be a p-adic field, $X = K^n$, and $f \in K[X_1, \ldots, X_n]$ a non-constant polynomial. Then there exists an n-dimensional K-analytic manifold Y, a proper surjective K-analytic map $\pi : Y \to X$ which is an isomorphism outside a set of measure 0, and finitely many submanifolds $F_1, \ldots, F_k$ of Y of codimension 1, such that the following hold:*

- *the divisor $\sum_{i=1}^k F_i$ has simple normal crossings support.*
- $\operatorname{div}(\pi^* f) = \sum_{i=1}^k a_i F_i$ *for some non-negative integers $a_1, \ldots, a_k$.*
- $\operatorname{div}(\pi^*(dx_1 \wedge \ldots \wedge dx_n)) = \sum_{i=1}^k b_i F_i$ *for some non-negative integers $b_1, \ldots, b_k$.*

In terms of equations, this means that in suitable coordinates $y = (y_1, \ldots, y_n)$ around any point $y \in Y$ we have

$$\pi^* f = \mu(y) \cdot y_1^{a_1} \cdot \ldots \cdot y_n^{a_n}$$

and

$$\pi^*(dx_1 \wedge \ldots \wedge dx_n) = \nu(y) \cdot y_1^{b_1} \cdot \ldots \cdot y_n^{b_n} \cdot (dy_1 \wedge \ldots \wedge dy_n)$$

with $\mu(0) \neq 0$ and $\nu(0) \neq 0$.

Roughly speaking, the Theorem follows from the usual version: by Theorem 3.12, there exists a smooth algebraic variety Z defined over K a morphism $\pi : Z \to \mathbb{A}^n_K$ which is an embedded resolution of $(f = 0)$. One takes $Y = Z(K)$.

4. Igusa's theorem on the rationality of the zeta function

In this section we prove a theorem of Igusa which was one of the first important applications of the theory of p-adic integration. The number theoretic set-up is as follows: fix a prime p, and let $f \in \mathbf{Z}_p[X_1, \ldots, X_n]$ (for instance $f \in \mathbf{Z}[X_1 \ldots, X_n]$). For any integer $m \geq 0$, define

$$N_m := |\{x \in (\mathbf{Z}/p^m\mathbf{Z})^n \mid f(\text{mod } p^m)(x) = 0\}|,$$

with the convention $N_0 = 1$. Consider the *Poincaré series*

$$Q(f, t) := \sum_{m=0}^{\infty} N_m \cdot t^m.$$

The following result was conjectured by Borevich and Shafarevich [BS] and proved by Igusa (see e.g. §8.2).

Theorem 4.1. *$Q(f,t)$ is a rational function.*

A more general statement appears in Theorem 4.4, and a more precise statement appears in Theorem 4.8 below. Let's start by looking at some examples.

Example 4.2. (1) Take $f(x) = x$. Then $x \equiv 0 \pmod{p^m}$ has precisely one solution, so $N_m = 1$ for all m. Then

$$Q(f,t) = 1 + t + t^2 + \ldots = \frac{1}{1-t}.$$

(2) Take $f(x) = x^2$. Then N_m is the number of solutions of $x^2 \equiv 0 \pmod{p^m}$:

- For $m = 1$, we have $p|x^2$ iff $p|x$, so $N_1 = 1$.
- For $m = 2$, we have $p^2|x^2$ iff $p|x$, so $N_2 = p$.
- For $m = 3$, we have $p^3|x^2$ iff $p^2|x$, so $N_3 = p$.
- For $m = 4$, we have $p^4|x^2$ iff $p^2|x$, so $N_4 = p^2$.
- For $m = 5$, we have $p^5|x^2$ iff $p^3|x$, so $N_5 = p^3$.

The pattern is now clear. We have

$$Q(f,t) = 1 + t + pt^2 + pt^3 + p^2t^4 + p^2t^5 + \ldots =$$
$$= (1+t)(1 + pt^2 + p^2t^4 + \ldots) = \frac{1+t}{1-pt^2}.$$

(3) Take $f(x,y) = y - x^2$. Fixing an arbitrary x, the congruence $y \equiv x^2 \pmod{p^m}$ determines y, so we easily get $N_m = p^m$ for each m. We have

$$Q(f,t) = 1 + pt + p^2t^2 + p^3t^3 + \ldots = \frac{1}{1-pt}.$$

Exercise 4.3. (1) Compute $Q(f,t)$ for $f(x) = x^d$ with $d \ge 3$.

(2) Challenge: compute $Q(f,t)$ for $f(x,y) = y^2 - x^3$.

(2) Challenge: compute $Q(f,t)$ for $f(x_1, \ldots, x_n) = x_1^{d_1} \cdot \ldots \cdot x_n^{d_n}$ with $n \ge 1$ and $d_1, \ldots, d_n$ arbitrary positive integers. Start with some small cases, like $x \cdot y$, etc.

Theorem 4.1 can be proved in the more general context of arbitrary p-adic fields. Consider such a field K, with ring of integers $\mathcal{O}_K$, such that $\mathcal{O}_K/\mathfrak{m}_K \simeq \mathbf{F}_q$, $q = p^r$. Let $f \in \mathcal{O}_K[X_1, \ldots, X_n]$, and for any $m \ge 0$ define

$$N_m := |\{x \in (\mathcal{O}_K/\mathfrak{m}_K^m)^n \mid f(\text{mod } \mathfrak{m}_K^m)(x) = 0\}|,$$

with the convention $N_0 = 1$. Define as before the Poincaré series of f to be

$$Q(f,t) := \sum_{m=0}^{\infty} N_m \cdot t^m.$$

Theorem 4.4 ([Ig] §8.2). *$Q(f,t)$ is a rational function.*

The key idea in Igusa's approach to Theorem 4.4 is to relate $Q(f,t)$ to a p-adic integral via the following:

Proposition 4.5. *With the notation above, we have*

$$Z(f,s) = Q\left(f, \frac{1}{q^{n+s}}\right)(1-q^s) + q^s.$$

Proof. For every $m \geq 0$, consider the subset of $\mathcal{O}_K^n$ given by

$$V_m := \{x \in \mathcal{O}_K^n \mid |f(x)| \leq \frac{1}{q^m}\},$$

so that $V_m - V_{m+1}$ are the level sets of f. In Lemma 4.6 below we will show that

$$\mu(V_m) = N_m \cdot \frac{1}{q^{nm}}.$$

Assuming this, and decomposing the domain into the disjoint union of these level sets, we have

$$Z(f,s) = \int_{\mathcal{O}_K^n} |f(x)|^s d\mu =$$

$$= 1 \cdot (\mu(V_0) - \mu(V_1)) + \frac{1}{q^s} \cdot (\mu(V_1) - \mu(V_2)) + \frac{1}{q^{2s}} \cdot (\mu(V_2) - \mu(V_3)) + \ldots =$$

$$= 1 \cdot \left(1 - N_1 \cdot \frac{1}{q^n}\right) + \frac{1}{q^s} \cdot \left(N_1 \cdot \frac{1}{q^n} - N_2 \cdot \frac{1}{q^{2n}}\right) + \frac{1}{q^{2s}} \cdot \left(N_2 \cdot \frac{1}{q^{2n}} - N_3 \cdot \frac{1}{q^{3n}}\right) + \ldots =$$

$$= \left(1 + N_1 \cdot \frac{1}{q^{n+s}} + N_2 \cdot \frac{1}{q^{2(n+s)}} + \ldots\right) - q^s \cdot \left(N_1 \cdot \frac{1}{q^{n+s}} + N_2 \cdot \frac{1}{q^{2(n+s)}} + \ldots\right) =$$

$$= Q\left(f, \frac{1}{q^{n+s}}\right) - q^s \cdot \left(Q\left(f, \frac{1}{q^{n+s}}\right) - 1\right).$$

∎

Lemma 4.6. *With the notation in the proof of Proposition 4.5, we have $\mu(V_m) = N_m \cdot \frac{1}{q^{nm}}$.*

Proof. Note that V_m is the preimage in $\mathcal{O}_K^n$ of the subset

$$\{x \mid f(\text{mod } \mathfrak{m}_K^m)(x) = 0\} \subset (\mathcal{O}_K/\mathfrak{m}_K^m)^n.$$

But this is a disjoint union of N_m translates of the kernel of the natural mapping $\mathcal{O}_K^n \to (\mathcal{O}_K/\mathfrak{m}_K^m)^n$, i.e. a disjoint union of translates of $(\mathfrak{m}_K^m)^n$, from which the result follows using Lemma 2.4. ∎

Example 4.7. Let's verify the statement of Proposition 4.5 in the case of $f(x) = x^d$ for some positive integer d. We have seen before that

$$Z(f,s) = \frac{q-1}{q-q^{-ds}}.$$

Rewriting this in terms of $t = q^{-s}$, we have

$$Z(f,t) = \frac{q-1}{q-t^d}.$$

By the theorem we then have

$$Q(f,t) = \left(Z(f,qt) - \frac{1}{qt}\right) \cdot \left(\frac{qt}{qt-1}\right) = \frac{qt - t - 1 + q^{d-1}t^d}{(1-q^{d-1}t^d)(qt-1)}.$$

This agrees with the examples above for $d = 1, 2$ (and also tells you what the answer should be in Exercise 4.3(1)).

Using the substitution $t = q^{-s}$, the result in Proposition 4.5 can be rewritten as

$$Q(f, q^{-n}t) = \frac{tZ(f,s) - 1}{t-1}.$$

A more precise version of Theorem 4.4 is then given by the following:

Theorem 4.8. *Let K be a p-adic field, and $f \in \mathcal{O}_K[X_1, \ldots, X_n]$. Then the zeta function*

$$Z(f,s) = \int_{\mathcal{O}_K} |f|^s d\mu$$

is a rational function of q^{-s}. Moreover, let $(a_1, b_1), \ldots, (a_k, b_k)$ be the discrepancies associated to an embedded resolution of singularities of $(f = 0)$ (as in Remark 3.13). Then

$$Z(f,s) = \frac{P(q^{-s})}{(1-q^{-a_1 s - b_1 - 1}) \cdot \ldots \cdot (1 - q^{-a_k s - b_k - 1})},$$

where $P \in \mathbf{Z}[1/q][X]$. Consequently, the poles of $Z(f,s)$ occur among the values $-\frac{b_1+1}{a_1}, \ldots, -\frac{b_k+1}{a_k}$.

Proof. Let $\pi : V \to K^n$ be an embedded resolution of singularities of $(f = 0)$, and let $X = \mathcal{O}_K^n \subset K^n$ and $Y = \pi^{-1}(X)$, so that we get a restriction $\pi : Y \to X$. We've seen that Y is a compact K-analytic manifold which is covered by disjoint compact open charts U_i on which in coordinates we have

$$\pi^* f = \mu(y) \cdot y_1^{a_1} \cdot \ldots \cdot y_n^{a_n}$$

and

$$\pi^*(dx_1 \wedge \ldots \wedge dx_n) = \nu(y) \cdot y_1^{b_1} \cdot \ldots \cdot y_n^{b_n} \cdot (dy_1 \wedge \ldots \wedge dy_n)$$

with $\mu(y) \neq 0$ and $\nu(y) \neq 0$ for all $y \in U_i$. By the change of variables formula, we have

$$Z(f,s) = \sum_i \int_{U_i} |\mu(y)|^s |\nu(y)| |y_1|^{a_1 s + b_1} \cdot \ldots \cdot |y_n|^{a_n s + b_n} |dy_1 \wedge \ldots \wedge dy_n|.$$

Note now that the functions $|\mu(y)|$ and $|\nu(y)|$ are locally constant on each U_i. By shrinking the U_i, we can then assume that they are in fact constant, say

$$|\mu| = q^{-a} \ \text{ and } \ |\nu| = q^{-b}.$$

Since U_i can be identified with a polydisk P_i given by $|y_i| \le q^{-k_i}$ for all i, we obtain that $Z(f,s)$ is a sum of terms of the form

$$q^{-as-b} \cdot \int_{P_i} |y_1|^{a_1 s + b_1} \cdot \ldots \cdot |y_n|^{a_n s + b_n} |dy_1 \wedge \ldots \wedge dy_n| =$$

$$= q^{-as-b} \cdot \int_{|y_1|\leq q^{-k_1}} |y_1|^{a_1 s+b_1} |dy_1| \cdot \ldots \cdot \int_{|y_n|\leq q^{-k_n}} |y_n|^{a_n s+b_n} |dy_n| =$$

$$= q^{-as-b} \cdot \left(\frac{q-1}{q}\right)^n \cdot \frac{q^{-k_1(a_1 s+b_1+1)}}{1-q^{-(a_1 s+b_1+1)}} \cdot \ldots \cdot \frac{q^{-k_n(a_n s+b_n+1)}}{1-q^{-(a_n s+b_n+1)}},$$

where we use Exercise 3.10 for the last line. So the statement follows if we check that no extra poles can arise from the term involving q^{-as} (note that a and b can be negative). To this end, note that $(f \circ \pi)(P_i) \subset \mathcal{O}_K$, so that $|\pi^* f| \leq 1$ on P_i. This means that for every $y \in P_i$ we have

$$|\mu(y)| \cdot |y_1|^{a_1} \cdot \ldots \cdot |y_n|^{a_n} \leq 1.$$

This is equivalent to

$$q^{-a-k_1 a_1-\ldots-k_n a_n} \leq 1, \text{ i.e. } \; a + k_1 a_1 + \ldots + k_n a_n \geq 0,$$

so indeed q^{-s} appears with a non-negative power in the numerator of the expression above. $\square$

Remark 4.9 (Monodromy conjecture)**.** Going back to the statement of Theorem 4.8, most of the time the poles of the local zeta function do not account for all the values $-\frac{b_i+1}{a_i}$ (at an even more basic level, some of these values are not invariant with respect to the choice of embedded resolution). A deeper conjecture due to Igusa, called the *monodromy conjecture*, aims to identify these poles more precisely.

Here is a brief explanation. Consider f as a mapping $f : \mathbf{C}^n \to \mathbf{C}$, and fix a point $x \in f^{-1}(0)$. The *Milnor fiber* of f at x is

$$M_{f,x} := f^{-1}(t) \cap B_\varepsilon(x),$$

where $B_\varepsilon(x)$ is the ball of radius ε around x, and $0 < t \ll \varepsilon \ll 1$. It was shown by Milnor that as a C^∞ manifold $M_{f,x}$ does not depend on t and ε. Each lifting of a path in a small disk of radius t around $0 \in \mathbf{C}$ induces a diffeomorphism $M_{f,x} \to M_{f,x}$, whose action on the cohomology $H^i(M_{f,x}, \mathbf{C})$ for each i is called the *monodromy action*.

Conjecture 4.10 (Igusa's Monodromy Conjecture)**.** *Let s be a pole of $Z(f, s)$. Then $e^{2\pi i s}$ is an eigenvalue of the monodromy action on some $H^i(M_{f,x}, \mathbf{C})$ at some point of $x \in f^{-1}(0)$.*

This truly remarkable conjecture, known in only a few cases, relates number theoretic invariants of $f \in \mathbf{Z}[X_1, \ldots, X_n]$ to differential topological invariants of the corresponding function $f : \mathbf{C}^n \to \mathbf{C}$. An even stronger conjecture relates the poles of $Z(f, s)$ to the roots of the so-called *Bernstein-Sato polynomial* of f.

5. Weil's measure and the relationship with rational points over finite fields

Let K be a p-adic field, with ring of integers $\mathcal{O}_K$, and residue field $\mathcal{O}_K/\mathfrak{m}_K \simeq \mathbf{F}_q$. In the next chapter we will need a result of Weil, roughly speaking relating the p-adic volume of a K-analytic manifold to the number of points of the manifold over $\mathbf{F}_q$.

Let $\mathcal{X}$ be a scheme over $S = \text{Spec } \mathcal{O}_K$, flat of relative dimension n. Recall that the set of $\mathcal{O}_K$-points of $\mathcal{X}$ is the set $\mathcal{X}(\mathcal{O}_K)$ of sections of the morphism $\mathcal{X} \to S$. We can also consider the set $\mathcal{X}(K)$ of K-points of $\mathcal{X}$, i.e. sections of the induced $X_K := \mathcal{X} \times_S \text{Spec } K \to \text{Spec } K$.

Exercise 5.1. (1) If $\mathcal{X}$ is an affine S-scheme, then

$$\mathcal{X}(\mathcal{O}_K) = \{x \in \mathcal{X}(K) \mid f(x) \in \mathcal{O}_K \text{ for all } f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})\} \subset \mathcal{X}(K).$$

(2) If $\mathcal{X}$ is proper over S, then $\mathcal{X}(\mathcal{O}_K) = \mathcal{X}(K)$. (Hint: use the valuative criterion for properness.)

Definition 5.2. Assume that $\mathcal{X}$ is smooth over S. A *gauge form* on $\mathcal{X}$ is a global section $\omega \in \Gamma(\mathcal{X}, \Omega^n_{\mathcal{X}/S})$ which does not vanish anywhere on $\mathcal{X}$. Note that such a form exists if and only if $\Omega^n_{\mathcal{X}/S}$ is trivial; more precisely, we have an isomorphism

$$\mathcal{O}_{\mathcal{X}} \to \Omega^n_{\mathcal{X}/S}, \ 1 \mapsto \omega.$$

(Therefore gauge forms always exist locally on $\mathcal{X}$.)

Weil's p-adic measure. We saw in §3 that if ω is a K-analytic n-form on $\mathcal{X}(K)$, one can associate to it a measure μ_ω. In a completely similar way, one can associate a measure μ_ω on $\mathcal{X}(\mathcal{O}_K)$ to any n-form $\omega \in \Gamma(\mathcal{X}, \Omega^n_{\mathcal{X}/S})$. (Note that $\mathcal{X}(\mathcal{O}_K)$ is a compact space in the p-adic topology.) Although not important for our discussion here, following the arguments below one can see that when ω is a gauge form, the measure μ_ω can in fact be defined over the entire $\mathcal{X}(K)$. This is called the *Weil p-adic measure* associated to ω.

Theorem 5.3 (Weil, [We2] 2.25)**.** *Let $\mathcal{X}$ be a smooth scheme over S of relative dimension n, and let ω be a gauge form on $\mathcal{X}$, with associated Weil p-adic measure μ_ω. Then*

$$\int_{\mathcal{X}(\mathcal{O}_K)} d\mu_\omega = \frac{|\mathcal{X}(\mathbf{F}_q)|}{q^n}.$$

Proof. This is really just a globalization of the argument in Lemma 4.6. Consider the reduction modulo $\mathfrak{m}_K$ map

$$\varphi : \mathcal{X}(\mathcal{O}_K) \longrightarrow \mathcal{X}(\mathbf{F}_q), \quad x \mapsto \bar{x}.$$

It is enough to show that for every $\bar{x} \in \mathcal{X}(\mathbf{F}_q)$, one has

$$\int_{\varphi^{-1}(\bar{x})} d\mu_\omega = \frac{1}{q^n}.$$

This follows from the following two observations. On one hand, we have a K-analytic isomorphism $\varphi^{-1}(\bar{x}) \simeq \mathfrak{m}_K^n$, since locally the mapping φ looks like $\mathcal{O}_K^n \to (\mathcal{O}_K/\mathfrak{m}_K)^n$. On the other hand, if we write in local coordinates $\omega = f(x) \cdot dx_1 \wedge \ldots \wedge dx_n$, by virtue of the fact that ω is a gauge form we have that $f(x)$ is a p-adic unit for every x, so that $|f(x)| = 1$. This facts combined give

$$\int_{\varphi^{-1}(\bar{x})} d\mu_\omega = \int_{\mathfrak{m}_K^n} |dx_1 \wedge \ldots \wedge dx_n| = \mu(\mathfrak{m}_K^n) = \frac{1}{q^n}.$$

□

Canonical measure. Let again $\mathcal{X}$ be a smooth scheme over S, this time not necessarily endowed with a gauge form. One can nevertheless naturally produce a measure on $\mathcal{X}(\mathcal{O}_K)$ (but not necessarily on $\mathcal{X}(K)$ if $\mathcal{X}$ is not proper over S) by gluing local measures given by gauge forms.

Consider a finite cover $U_1, \ldots, U_k$ of $\mathcal{X}$ by Zariski open S-schemes such that for each i the line bundle $\Omega^n_{\mathcal{X}/S}$ is trivial over U_i, i.e. $\Omega^n_{\mathcal{X}/S_{|U_i}} \simeq \mathcal{O}_{U_i}$. This means that we can pick a gauge form ω_i on each U_i, with associated Weil measure μ_{ω_i} defined on $\mathcal{X}(K)$ as above. Consider its restriction to $\mathcal{X}(\mathcal{O}_K)$. Now any two gauge forms clearly differ by an invertible function, i.e. by a section $s_i \in \Gamma(U_i, \mathcal{O}^*_{U_i})$, so that for any $\mathcal{O}_K$-point $x \in U_i$ we have $|s_i(x)| = 1$. Recalling how the measure associated to an n-form is defined in §3, this has the following consequences:

- The Weil measure μ_{ω_i} on $U_i(\mathcal{O}_K)$ does not depend on the choice of gauge form.
- These measures glue together to a global measure μ_{can} on the compact $\mathcal{X}(\mathcal{O}_K)$, called the *canonical measure*.

Weil's result Theorem 5.3 continues to hold in this setting.

Corollary 5.4. *Let $\mathcal{X}$ be a smooth scheme over S of relative dimension n. Then*

$$\int_{\mathcal{X}(\mathcal{O}_K)} d\mu_{\text{can}} = \frac{|\mathcal{X}(\mathbf{F}_q)|}{q^n}.$$

Proof. Consider a Zariski-open covering $U_1, \ldots, U_k$ of $\mathcal{X}$ as above, such that one has gauge forms on each U_i. Since μ_{can} is obtained by gluing the local Weil measures, we have

$$\int_{\mathcal{X}(\mathcal{O}_K)} d\mu_{\text{can}} = \sum_i \int_{U_i(\mathcal{O}_K)} d\mu_{\text{can}} - \sum_{i<j} \int_{(U_i\cap U_j)(\mathcal{O}_K)} d\mu_{\text{can}} + \ldots + (-1)^k \int_{(U_1\cap\ldots\cap U_k)(\mathcal{O}_K)} d\mu_{\text{can}}.$$

Now for each of these integrals one can apply Theorem 5.3. The result follows by noting that by the inclusion-exclusion principle one has

$$|\mathcal{X}(\mathbf{F}_q)| = \sum_i |U_i(\mathbf{F}_q)| - \sum_{i<j} |(U_i \cap U_j)(\mathbf{F}_q)| + \ldots + (-1)^k |(U_1 \cap \ldots \cap U_k)(\mathbf{F}_q)|.$$

$\square$

Let's conclude by noting a useful technical result which essentially says that proper Zariski closed subsets are irrelevant for the calculation of integrals with respect to the canonical measure.

Proposition 5.5. *Let $\mathcal{X}$ be a smooth scheme over S, and let $\mathcal{Y}$ be a reduced closed S-subscheme of codimension ≥ 1. Then $\mathcal{Y}(\mathcal{O}_K)$ has measure zero in $\mathcal{X}(\mathcal{O}_K)$ with respect to the canonical measure μ_{can}.*

Proof. Using an affine open cover of $\mathcal{X}$, we can immediately reduce to the case when $\mathcal{X}$ is a smooth affine S-scheme. Considering some hypersurface containing $\mathcal{Y}$, we can also reduce to the case of a principal divisor, i.e. $\mathcal{Y} = (f = 0)$ with $f \in \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$

irreducible. By the Noether normalization theorem, we can then further assume that $\mathcal{X} = \mathbf{A}^n_{\mathcal{O}_K} = \mathrm{Spec}\ \mathcal{O}_K[X_1, \ldots, X_n]$ and $f = X_1$.[4]

To show that $\mu_{\mathrm{can}}(\mathcal{Y}(\mathcal{O}_K)) = 0$, we will use a limit argument. Define for every integer $m \geq 1$ the subsets of $\mathbf{A}^n(\mathcal{O}_K)$

$$\mathcal{Y}_m(\mathcal{O}_K) := \{(x_1, \ldots, x_n) \in \mathcal{O}_K^n \mid x_1 \in \mathfrak{m}_K^m\}.$$

Noting that $\cap_{m=1}^{\infty} \mathfrak{m}_K^m = 0$, we have

$$\mathcal{Y}(\mathcal{O}_K) = \bigcap_{m=1}^{\infty} \mathcal{Y}_m(\mathcal{O}_K).$$

It suffices then to show

$$\int_{\mathcal{Y}(\mathcal{O}_K)} d\mu_{\mathrm{can}} = \lim_{m \to \infty} \int_{\mathcal{Y}_m(\mathcal{O}_K)} d\mu_{\mathrm{can}} = 0.$$

But each one of the terms in the limit can be easily computed using Fubini:

$$\int_{\mathcal{Y}_m(\mathcal{O}_K)} d\mu_{\mathrm{can}} = \int_{\mathfrak{m}_K^m} |dx_1| \cdot \prod_{i=2}^{n} \int_{\mathcal{O}_K} |dx_i| = \frac{1}{q^m} \ ,$$

hence the limit is indeed equal to zero. □

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[4]Exercise: do this carefully, as we are not directly applying the usual Noether normalization for a finitely generated algebra over a field.

CHAPTER 4. K-EQUIVALENT VARIETIES AND BETTI NUMBERS

Contents

In this chapter we will prove one of the main results in this course, namely Batyrev theorem [Ba] on the equality of Betti numbers for birational Calabi-Yau varieties, and more generally for K-equivalent varieties (cf. also [It], [Wa]). This is done by means of a reduction mod p and lifting to a p-adic field procedure, which allows one to use the p-adic integration and Weil conjecture methods studied in the previous chapters. I will take the opportunity to recall some important notions from birational geometry. Later in the course we will encounter the motivic integration methods introduced by Kontsevich, which allow one to improve the theorem mentioned above and obtain equality for all Hodge numbers.

1. Generalities on K-equivalent varieties

In the minimal model program, starting with a smooth variety X (of non-negative Kodaira dimension), one tries to produce a minimal model of X. Roughly speaking this is a variety Y with mild singularities, birational to X, but whose canonical class K_Y is in some technical sense "smallest" among all such varieties birational to X (the correct notion is that K_X is *nef*). Minimal models are not necessarily unique; this and other considerations lead to trying to look for ways of putting some sort of ordering on the elements of the birational equivalence class of X by means of comparing canonical bundles. The discussion below is a first step towards this in the case of smooth varieties, which is the relevant case for our questions here; a full discussion from the point of view of birational geometry would necessarily have to involve singularities.

Definition 1.1. Let X and Y be smooth projective varieties over the complex numbers. They are called K-*equivalent* if there exsists a smooth projective variety Z and birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that $f^*\omega_X \simeq g^*\omega_Y$.

An important thing to keep in mind: due to the general resolution of singularities package, any two birational X and Y are dominated by common smooth birational models

Z as in the definition above. However, in general the relationship between $f^*\omega_X$ and $g^*\omega_Y$ is more complicated. Here is though a simple but important example where any common model Z does the job.

Definition 1.2. Let X be a smooth projective (or compact complex) manifold. Then X is called (weakly[1]) *Calabi-Yau* if $\omega_X \simeq \mathcal{O}_X$.

Example 1.3. Any two birational Calabi-Yau varieties are K-equivalent.

Note that any two birational smooth projective curves are isomorphic. The K-equivalence condition does not produce anything interesting in the case of surfaces either.

Proposition 1.4. *If X and Y are K-equivalent surfaces, then $X \simeq Y$.*

Proof. Consider a smooth projective birational model Z as in Definition 1.1, so that $f^*\omega_X \simeq g^*\omega_Y$. The key point is that in the case of surfaces, any birational morphism can be factored into a finite sequence of standard one-point blow-ups (see e.g. [Ha] Ch.V Corollary 5.4). Consider therefore a blow-up $\pi : Z = \mathrm{Bl}_p(Z') \to Z'$ at a point p of a smooth projective surface Z', such that say g is obtained as a composition

$$Z \xrightarrow{\pi} Z' \xrightarrow{g'} Y.$$

Denote by E the exceptional divisor of π. Then clearly $(g^*\omega_Y)_{|E} \simeq \mathcal{O}_E$, and so also $(f^*\omega_X)_{|E} \simeq \mathcal{O}_E$. This implies that E is contracted by f, so that f factors through π as well. Indeed, denoting by $E_i, \ldots, E_k$ the components of the exceptional locus of f, we have

$$K_Z = f^*K_X + E_1 + \ldots + E_k.$$

Since E is a (-1)-curve, by the genus formula we have $K_Z \cdot E = -1$. Combining this with the triviality above, we obtain

$$E \cdot (E_i + \ldots + E_k) = -1,$$

from which it follows that E must be one of the E_i. Consequently, one can replace Z by Z', with birational maps to f' to X and g' to Y, such that the K-equivalence condition still holds. But then one can inductively continue this argument along the finite sequence of blow-ups which factor g, until one necessarily reaches $X \simeq Y$. □

In the rest of the section I will explain a few geometric consequences of K-equivalence.

Exercise 1.5. Let Z be a common smooth birational model for X and Y as in Definition 1.1. Then

$$f^*\omega_X \simeq g^*\omega_Y \iff K_{Z/X} = K_{Z/Y}.$$

(Hint: use the fact that for any birational morphism $f : Z \to X$ with Z and X smooth, and any exceptional divisor $E \subset Z$, one has $f_*\mathcal{O}_Z(E) \simeq f_*\mathcal{O}_Z \simeq \mathcal{O}_X$; the second isomorphism is standard, and the first is a well-known result of Fujita.)

[1] For the full Calabi-Yau condition one usually also requires that X be simply connected and $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$.

Lemma 1.6. *Let X and Y be K-equivalent varieties. Then there exist Zariski open subsets $U \subset X$ and $V \in Y$ such that*

$$U \simeq V, \quad \operatorname{codim}_X(X - U) \geq 2 \quad \text{and} \quad \operatorname{codim}_Y(Y - V) \geq 2.$$

In other words, X and Y are isomorphic in codimension 1.

Proof. By what we have seen in Ch.3 §3, the exceptional locus of f is precisely the support of the relative canonical divisor $K_{Z/X}$. Exercise 1.5 says that this is the same as the exceptional locus of g. Denoting this locus by E, this immediately implies that X and Y are isomorphic outside of the images $f(E)$ and $g(E)$, which obviously have codimension at least 2 in X and Y respectively. □

We can put this discussion in a somewhat broader context, starting with an explanation of the terminology *minimal model* from the point of view of the "size" of the canonical bundle.

Definition 1.7. (1) A line bundle L on a projective variety X is called *nef* if $L \cdot C \geq 0$ for every irreducible curve $C \subset X$.

(2) A smooth projective variety X is called *minimal* if ω_X is nef.

(3) Let X and Y be smooth projective birational varieties. We say that $K_X \leq K_Y$ if there exist a smooth projective variety Z and birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that $g^*K_Y - f^*K_X$ is linearly equivalent to an effective divisor (in other words $H^0(Z, g^*\omega_Y \otimes f^*\omega_X^{-1}) \neq 0$).

Example 1.8. If $f : Y \to X$ is the blow-up of X along a smooth subvariety, then $K_X \leq K_Y$.

Proposition 1.9. *Let X and Y be smooth projective birational varieties, with K_X nef. Then $K_X \leq K_Y$. In particular, any two birational minimal varieties are K-equivalent.*[2]

Proof. Let Z be a common birational model for X and Y, with birational morphisms $f : Z \to X$ and $g : Y \to Z$. We can write

$$K_{Z/X} - A = K_{Z/Y} - B,$$

where A and B are effective divisors with no common components, and $\operatorname{codim}_Y(\operatorname{Supp}(B)) \geq 2$. The result follows if we show that $B = 0$, so let's assume that $B \neq 0$ and derive a contradiction.

Let's denote $n = \dim X = \dim Y = \dim Z$ and $d = \dim g(\operatorname{Supp}(B))$. Let H and M be very ample divisors on Y and Z respectively. Given general divisors $H_1, \ldots, H_d \in |H|$ and $M_1, \ldots, M_{n-d-2} \in |M|$, we can consider the generic surface section

$$S := g^*H_1 \cap \ldots \cap g^*H_d \cap M_1 \cap \ldots \cap M_{n-d-2}$$

[2] Let me repeat that in general one needs to work with singular varieties; besides special situations like the Calabi-Yau case, it is rare that one can reach a minimal model which is non-singular, let alone more than one.

of Z, which is smooth by Bertini's theorem. Note now that on Z we have

$$g^*H^d \cdot M^{n-d-2} \cdot B^2 \geq 0.$$

Indeed, note that $B \equiv f^*K_X + A - g^*K_Y$, so this follows from the following three facts:

- $g^*H^d \cdot M^{n-d-2} \cdot B \cdot f^*K_X \geq 0$ since K_X is nef.
- $g^*H^d \cdot M^{n-d-2} \cdot B \cdot A \geq 0$ since A and B have no common components.
- $g^*H^d \cdot M^{n-d-2} \cdot B \cdot g^*K_Y = 0$ since B is contracted by g.

Consequently, denoting $D = B_{|S}$, we have $D^2 \geq 0$ on S. But note that by construction D consists of a union of exceptional curves on X (with respect to the restriction of g to S), and so by a standard consequence of the Hodge index theorem[3] $D^2 < 0$, a contradiction. □

Remark 1.10. A slight refinement of this argument shows that if $f : X \cdots \to Y$ is a birational map between smooth varieties such that K_X is nef along the exceptional locus Z of f, then $K_X \leq K_Y$, and Z has codimension at least 2.

We are aiming towards relating topological and holomorphic invariants of K-equivalent varieties. While the full results are quite deep, part of this follows from the structural results covered in this section.

Proposition 1.11. *Let $f : X \cdots \to Y$ be a birational (rational) map between smooth projective complex varieties, which is an isomorphism outside of codimension at least 2 closed subsets. Then there are natural isomorphisms*

$$H^i(X, \mathbf{Z}) \simeq H^i(Y, \mathbf{Z}) \quad \text{for } i \leq 2,$$

compatible (after complexification) with the Hodge decompositions on the two sides.

Proof. Poincaré duality implies that equivalently we can aim for natural isomorphisms

$$H_{2n-i}(X, \mathbf{Z}) \simeq H_{2n-i}(Y, \mathbf{Z}),$$

where n is the dimension of X, and $i \leq 2$. Now X and Y are diffeomorphic as real manifolds outside closed subsets of real codimension at least 4, and therefore this diffeomorphism sees all $(2n-i)$-cycles on X and Y with $i \leq 2$, inducing the desired natural isomorphism.

On the other hand, we have seen in Chapter 1 that an isomorphism of open sets that are complements of closed analytic subvarietes induces via Hartogs' theorem isomorphisms

$$H^{0,i}(X) \simeq H^{0,i}(Y) \quad \text{and} \quad H^{i,0}(X) \simeq H^{i,0}(Y) \quad \text{for all } i.$$

Since for $i \leq 2$, $H^i(X, \mathbf{C})$ and $H^i(Y, \mathbf{C})$ have at most one component in their Hodge decomposition which is not of this form, it is clear that the complexification of the isomorphisms above respects the Hodge structures. □

Remark 1.12. One can similarly show the same result at the level of homotopy groups:

$$\pi_i(X) \simeq \pi_i(Y) \quad \text{for } i \leq 2.$$

[3] See for example [Ha] Ch.V, Exercise 5.4(a).

2. Reduction mod p and lifting to the p-adics

Let's briefly go through two standard arithmetic reduction procedures:

Reduction mod p. Let X be a scheme of finite type over $\mathbf{C}$. Then there exists a finitely generated $\mathbf{Z}$-algebra R and X_R a scheme over Spec R such that

$$X \simeq X_R \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C}.$$

This is obtained in general by gluing the affine case. In the affine case, X is given in some $\mathbf{A}^n$ as the zero locus of a finite number of polynomials. We can take $R = \mathbf{Z}[a_1, \ldots, a_N]$, where the a_i are all the coefficients of these polynomials; X can clearly be lifted to a scheme X_R over R by considering the exact same equations. Note that X_R does not depend only on R, but also on the choice of defining equations.

Now let $\mathfrak{p} \subset R$ be a maximal ideal, lying over $(p) \subset \mathbf{Z}$ with p a prime number. Then we have that $R/\mathfrak{p}$ is a finite extension of $\mathbf{Z}/p\mathbf{Z}$, so

$$R/\mathfrak{p} \simeq \mathbf{F}_q, \text{ with } q = p^r, r > 0.$$

The fiber of X_R over the corresponding point in Spec R,

$$X_{\mathfrak{p}} := X_R \times_{\mathrm{Spec} R} \mathrm{Spec}\ R/\mathfrak{p},$$

is a scheme of finite type over $\mathbf{F}_q$, called the *reduction mod p of X*.

Assume now that X is a smooth variety over $\mathbf{C}$. Then there exists a nonempty open set $U \subset \mathrm{Spec}\ R$, containing (0), such that the structural map $\pi : X_R \to \mathrm{Spec}\ R$ is smooth over U. This implies that all the reductions mod p corresponding to points in U are smooth varieties over the corresponding $\mathbf{F}_q$. (In such a situation we say that X has *good reduction* at those primes.)

Completely analogously, reduction mod p can be done simultaneously for a finite collection of schemes of finite type, coherent sheaves on them, and morphisms between them. Indeed, thinking again locally, the schemes are given by a finite number of equations, the morphisms are determined by their graphs, which are again schemes of finite type, while the coherent sheaves are locally finitely presented modules, so again given by a finite number of equations.

We will in fact only use the following:

Proposition 2.1. *Let $f : X \to Y$ be a morphism between varieties over $\mathbf{C}$. Then there exists a finitely generated $\mathbf{Z}$-algebra R and schemes X_R and Y_R over* Spec R*, together with a morphism $f_R : X_R \to Y_R$ over* Spec R*, such that $X_R \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C} \simeq X$, $X_R \times_{\mathrm{Spec}\ R} \mathrm{Spec}\ \mathbf{C} \simeq Y$ and $f_R \times_{\mathrm{Spec} R} \mathrm{Spec}\ \mathbf{C} = f$. In addition:*

- *if X and Y are smooth (respectively projective), then there exists a Zariski open set $U \subset$* Spec R *such that $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ are smooth (respectively projective) for every $\mathfrak{p} \in U$.*

- *if f is birational (respectively proper), then there exists a Zariski open set $U \subset$* Spec R *such that the induced $f_{\mathfrak{p}} : X_{\mathfrak{p}} \to Y_{\mathfrak{p}}$ is birational (respectively proper) for every $\mathfrak{p} \in U$.*

Exercise 2.2. Prove the assertions in the Proposition that do not immediately follow from the discussion above.

Lifting to the p-adics. Let R be a finitely generated $\mathbf{Z}$-algebra as above, and X_R a scheme of finite type over Spec R. Let $\mathfrak{p} \subset R$ be a maximal ideal, with $X_{\mathfrak{p}}$ the fiber of X_R over $\mathfrak{p}$; as noted above, this is a scheme of finite type over $\mathbf{F}_q$, the residue field of R at $\mathfrak{p}$.

Proposition 2.3. *With the notation above, there exists a p-adic field K with ring of integers $(\mathcal{O}_K, \mathfrak{m}_K)$ such that $\mathcal{O}_K/\mathfrak{m}_K \simeq \boldsymbol{F}_q$, and a scheme of finite type $\mathcal{X}$ over* Spec $\mathcal{O}_K$ *which is a* lifting *of $X_{\mathfrak{p}}$ over $\mathcal{O}_K$, i.e. such that*

$$X_{\mathfrak{p}} \simeq \mathcal{X} \times_{\mathrm{Spec}\ \mathcal{O}_K} \mathrm{Spec}\ \boldsymbol{F}_q.$$

If $X_{\mathfrak{p}}$ is smooth over $\boldsymbol{F}_q$, then $\mathcal{X}$ can be taken to be smooth over $\mathcal{O}_K$.

Proof. Let $L = Q(R)$ be the quotient field of R. Since R is a finitely generated algebra over $\mathbf{Z}$, we have a finite field extension $\mathbf{Q} \subset L$ (L is a *number field*). By restricting to a general complete intersection curve through the origin in Spec R, we can assume that $\dim R = 1$, and that for a general non-zero prime ideal $\mathfrak{p} \subset R$ the localization $R_{\mathfrak{p}}$ is regular, i.e. a DVR (in fact by further restriction to a suitable affine open subset, we can assume that R is a Dedekind domain).

Now the completion $\widehat{R_{\mathfrak{p}}}$ with respect to the $\mathfrak{p}R_{\mathfrak{p}}$-adic topology is a complete DVR, and hence the ring of integers $\mathcal{O}_K$ of a p-adic field K. It has the same residue field, since

$$\widehat{R_{\mathfrak{p}}}/\mathfrak{p}\widehat{R_{\mathfrak{p}}} \simeq R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \simeq \mathbf{F}_q.$$

Note that we could have gone to the reduction mod p by passing first to a scheme over the DVR $R_{\mathfrak{p}}$, namely $\bar{X}_{\mathfrak{p}} := X_R \times_{\mathrm{Spec} R} \mathrm{Spec}\ R_{\mathfrak{p}}$, so that

$$X_{\mathfrak{p}} \simeq \bar{X}_{\mathfrak{p}} \times_{\mathrm{Spec} R_{\mathfrak{p}}} \mathrm{Spec}\ \mathbf{F}_q.$$

But we have a natural injective homomorphism $R_{\mathfrak{p}} \hookrightarrow \widehat{R_{\mathfrak{p}}} = \mathcal{O}_K$, and so we can define

$$\mathcal{X} := \bar{X}_{\mathfrak{p}} \times_{\mathrm{Spec} R_{\mathfrak{p}}} \mathrm{Spec}\ \mathcal{O}_K,$$

which is a scheme of finite type over Spec $\mathcal{O}_K$. This is a lifting of $X_{\mathfrak{p}}$, since

$$\mathcal{X} \times_{\mathrm{Spec}\ \mathcal{O}_K} \mathrm{Spec}\ \mathbf{F}_q \simeq (\bar{X}_{\mathfrak{p}} \times_{\mathrm{Spec} R_{\mathfrak{p}}} \mathrm{Spec}\ \mathcal{O}_K) \times_{\mathrm{Spec}\ \mathcal{O}_K} \mathrm{Spec}\ \mathbf{F}_q \simeq X_{\mathfrak{p}}.$$

Finally, it is well known that passing to the completion preserves smoothness (over the completed ring), so if $X_{\mathfrak{p}}$ is smooth then so it $\mathcal{X}$ over Spec $\mathcal{O}_K$.

□

Remark 2.4. One can also approach this type of result using the following theorem of independent interest (see e.g. [Ca] Ch.V):

Theorem 2.5 ("Embedding theorem")**.** *Let F be a number field, and $C \subset F$ a finite subset of non-zero elements. Then there exist infinitely many primes p for which there is an embedding $\alpha : F \hookrightarrow \mathbf{Q}_p$, which can be chosen such that $|\alpha(c)|_p = 1$ for all $c \in C$ (in other words C is mapped into the invertible elements of $\mathbf{Z}_p$).*

Let now $F = Q(R)$, the quotient field of R. Since R is a finitely generated $\mathbf{Z}$-algebra, F is a number field. Note that $\mathcal{X}$ is defined over F. According to the Embedding Theorem, for an infinite number of primes p there exist embeddings $\alpha : F \hookrightarrow \mathbf{Q}_p$, which can be chosen such that any finite set in R can be mapped to the p-adic units. In particular we can apply this to the generators of R as an algebra over $\mathbf{Z}$ (i.e. basically the coefficients of the polynomials defining X), so that these have images in $\mathbf{Z}_p$.

Using a further finite extension $\mathbf{Q}_p \subset K$, so that we have a local homomorphism $R \subset \mathcal{O}_K$, the defining equations of X can be lifted to equations with coefficients in $\mathcal{O}_K$, and this gives the desired lifting $\mathcal{X}$ over Spec $\mathcal{O}_K$.

3. Batyrev's theorem on the invariance of Betti numbers

The main goal of this section, and in fact of most of the course up to this point, is to prove the following result due to Batyrev [Ba].

Theorem 3.1. *Let X and Y be birational complex Calabi-Yau varieties. Then*

$$b_i(X) = b_i(Y) \text{ for all } i.$$

More generally, the same statement holds for any two complex smooth projective K-equivalent varieties.

Proof. Assuming that X and Y are K-equivalent varieties of dimension n, we consider a smooth Z together with birational maps $f : Z \to X$ and $g : Z \to Y$ as in Definition 1.1. We first use reduction mod p and lifting to the p-adics to obtain models for X, Y, Z, f and g over the ring of integers of a p-adic field.

More precisely, by the reduction mod p procedure (especially Proposition 2.1), there exist a finitely generated $\mathbf{Z}$-algebra R and liftings X_R, Y_R, Z_R, $f_R : Z_R \to X_R$ and $g_R : Z_R \to Y_R$ over Spec R. We can assume (by passing to an open subset if necessary), that X_R, Y_R and Z_R are smooth and projective over Spec R, f_R and g_R are birational, and

$$f_R^* \, \Omega^n_{X_R/R} \simeq g_R^* \, \Omega^n_{Y_R/R}$$

on Z_R.

Next, by the lifting to the p-adics procedure (see Proposition 2.3), there exists a p-adic field K, with ring of integers $\mathcal{O}_K$ whose residue field is $\mathbf{F}_q$, and smooth schemes $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ over $S = \text{Spec } \mathcal{O}_K$ with special fibers $X_{\mathfrak{p}}$, $Y_{\mathfrak{p}}$, $Z_{\mathfrak{p}}$ over the closed point corresponding to $\mathfrak{m}_K$ (i.e. $\bar{X} = \mathcal{X} \times_S \text{ Spec } \mathbf{F}_q$, etc.) which are isomorphic to the fibers of X_R, Y_R, Z_R over the prime $\mathfrak{p}$. In addition, there are proper birational morphisms over S

$$\tilde{f} : \mathcal{Z} \to \mathcal{X}, \;\; \tilde{g} : \mathcal{Z} \to \mathcal{Y}$$

extending f_R and g_R, such that on $\mathcal{Z}$ we have

$$\tilde{f}^* \, \Omega^n_{\mathcal{X}/S} \simeq \tilde{g}^* \, \Omega^n_{\mathcal{Y}/S}. \tag{1}$$

Under this assumption, the claim is that

$$|\mathcal{X}(\mathbf{F}_q)| = |\mathcal{Y}(\mathbf{F}_q)|.$$

Using Corollary 5.4 in Chapter 3, i.e. Weil's result on the interpretation of the number of points over $\mathbf{F}_q$ as a normalized volume with respect to the canonical p-adic measure, it is enough then to show that

$$\int_{\mathcal{X}(\mathcal{O}_K)} d\mu_{\mathrm{can}} = \int_{\mathcal{Y}(\mathcal{O}_K)} d\mu_{\mathrm{can}}.$$

Recall now that the canonical measure is defined by gluing the p-adic measures defined by any local generators ω of $\Omega^n_{\mathcal{X}/S}$ on open sets where this line bundle is trivial. By pull-pack by f and g, these induce local generators for $\tilde{f}^*\,\Omega^n_{\mathcal{X}/S}$ and $\tilde{g}^*\,\Omega^n_{\mathcal{Y}/S}$ respectively. Using the change of variable formula, Theorem 3.8 in Ch.III, on both sides, we obtain our equality via

$$\int_{\mathcal{X}(\mathcal{O}_K)} d\mu_{\mathrm{can}} = \int_{\mathcal{Z}(\mathcal{O}_K)} |\tilde{f}^*\,\Omega^n_{\mathcal{X}/S}| d\mu = \int_{\mathcal{Z}(\mathcal{O}_K)} |\tilde{g}^*\,\Omega^n_{\mathcal{Y}/S}| d\mu = \int_{\mathcal{Y}(\mathcal{O}_K)} d\mu_{\mathrm{can}}.$$

Note that in the middle terms we have used the notation $|\tilde{f}^*\,\Omega^n_{\mathcal{X}/S}|d\mu$ for the measure on $\mathcal{Z}$ coming from gluing the local measures defined by the generators $f^*\omega$ (and similarly for $\mathcal{Y}$). By (1), these coincide.

Finally, for any $m \geq 2$ we can replace the finitely generated $\mathbf{Z}$-algebra R by a cyclotomic extension $R_m \subset \mathbf{C}$, which is obtained by adjoining to R all complex $(q^m - 1)$-roots of unity. It is easy to check that R_m has a maximal ideal $\mathfrak{p}_m$ lying over $\mathfrak{p} \subset R$ such that $R_m/\mathfrak{p}_m \simeq \mathbf{F}_{q^m}$, the degree m extension of $\mathbf{F}_q \simeq R/\mathfrak{p}$. Over $\mathbf{C}$, this still reduces to the initial X. We can repeat the exact same arguments as above, the conclusion being that

$$|\mathcal{X}(\mathbf{F}_{q^m})| = |\mathcal{Y}(\mathbf{F}_{q^m})| \text{ for all } m \geq 1.$$

This implies that the local Weil zeta functions of $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ studied in Ch.II satisfy

$$Z(X_{\mathfrak{p}}; t) = Z(Y_{\mathfrak{p}}; t).$$

Since $X_{\mathfrak{p}}$ and $Y_{\mathfrak{p}}$ are reductions mod p of the complex smooth projective varieties X and Y, the Betti numbers component of the Weil conjectures, Theorem 3.4 in Ch.II, implies that $b_i(X) = b_i(Y)$ for all i. □

Remark 3.2. (1) Although with what we have covered until now we are not yet able to show this, a stronger statement holds, namely and two K-equivalent varieties have the same *Hodge numbers*. To show this, inspired by p-adic integration Kontsevich introduced the technique of *motivic integration*; we will see this, together with the proof of the more general statement, in the next chapter. Note however that it is now known how to recover the Hodge numbers using p-adic techniques as well, using the above methods plus results from p-adic Hodge theory (see [It]).

(2) The proof above shows in fact slightly more than what we have stated (see [Wa] Theorem 3.1): if X and Y are not necessarily K-equivalent, but satisfy $K_X \leq K_Y$, then

$$\int_{\mathcal{X}(\mathcal{O}_K)} d\mu_{\mathrm{can}} \;\leq\; \int_{\mathcal{Y}(\mathcal{O}_K)} d\mu_{\mathrm{can}}.$$

This implies, in the notation above, that for each $m \geq 1$

$$|X_{\mathfrak{p}}(\mathbf{F}_{q^m})| \leq |Y_{\mathfrak{p}}(\mathbf{F}_{q^m})|.$$

(3) It is interesting to note that, unlike in the motivic integration approach, here it is not crucial to use the change of variables formula (except somewhat trivially via isomorphisms). Let's see this in the case of Calabi-Yau manifolds; the argument can be quickly adapted to the general case. Assume therefore that there are gauge forms $\omega_{\mathcal{X}}$ and $\omega_{\mathcal{Y}}$ on $\mathcal{X}$ and $\mathcal{Y}$ respectively. We know that there are isomorphic open subsets

$$\alpha : \mathcal{U} \subset \mathcal{X} \xrightarrow{\simeq} \mathcal{V} \subset \mathcal{Y}$$

over S, with complements of codimension at least 2. The restriction $\omega_{\mathcal{U}}$ of $\omega_{\mathcal{X}}$ is a gauge form on $\mathcal{U}$, and via the isomorphism α so is $\alpha^*\omega_{\mathcal{V}}$. This means that there exists a nowhere vanishing regular function $f \in \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}^*)$ such that

$$\alpha^*\omega_{\mathcal{V}} = f \cdot \omega_{\mathcal{U}}.$$

But since $\operatorname{codim}_{\mathcal{X}}(\mathcal{X} - \mathcal{U}) \geq 2$, it follows that f extends to an invertible function on $\mathcal{X}$, i.e. an element in $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}^*)$, so that $|f(x)| = 1$ for all $x \in \mathcal{X}(K)$. Consequently, the Weil p-adic measures on $\mathcal{U}(K)$ given by $\alpha^*\omega_{\mathcal{V}}$ and $\omega_{\mathcal{U}}$ are the same. This implies the equality of integrals

$$\int_{\mathcal{U}(K)} d\mu_{\omega_{\mathcal{X}}} = \int_{\mathcal{V}(K)} d\mu_{\omega_{\mathcal{Y}}}.$$

But we have seen in Proposition 5.5 in Ch. III that the complements of $\mathcal{U}$ and $\mathcal{V}$ are sets of measure zero with respect to the p-adic measure, so this is the same as

$$\int_{\mathcal{X}(K)} d\mu_{\omega_{\mathcal{X}}} = \int_{\mathcal{Y}(K)} d\mu_{\omega_{\mathcal{Y}}}.$$

Finally, since $\mathcal{X}$ and $\mathcal{Y}$ are projective over S, we have that $\mathcal{X}(K) = \mathcal{X}(\mathcal{O}_K)$ and $\mathcal{Y}(K) = \mathcal{Y}(\mathcal{O}_K)$ (see Exercise 5.1 in Ch.II).

(4) It is an open problem whether the analogue of Theorem 3.1 is true for compact Kähler manifolds.

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CHAPTER 5. JET SPACES AND ARC SPACES

Contents

In this chapter we will discuss the basic theory of jet and arc spaces for schemes over $\mathbf{C}$ (or over any field of characteristic zero). There are by now many good surveys of the topic, especially from the point of view of motivic integration (see for instance [Bl], [Ve]) which will be our main focus in this course. The main sources I will use are [Mu] and [La].

Before getting into details, I would like to recall the p-adic inspiration for this theory. Here is a list of things we looked at in Chapter III.

(1) Polynomials $f \in \mathbf{Z}[X_1, \ldots, X_n]$ and their solutions over $\mathbf{Z}/p^{m+1}\mathbf{Z} \simeq \mathbf{Z}_p/p^{m+1}\mathbf{Z}_p$, where p is a fixed prime and $m \geq 0$. Such a solution can be written as

$$x_1 = a_{10} + a_{11}p + \ldots + a_{1m}p^m \, , \, \ldots \, , \, x_n = a_{n0} + a_{n1}p + \ldots + a_{nm}p^m$$

with $a_i \in \{0, \ldots, p-1\}$.

(2) Solutions of f over $\mathbf{Z}_p = \varprojlim_m \mathbf{Z}/p^{m+1}\mathbf{Z}$.

(3) The p-adic norm $|f(x)|_p = \frac{1}{p^{\mathrm{ord}_p f(x)}}$.

(4) p-adic integrals on $\mathbf{Z}_p^n$, with respect to the Haar measure $d\mu$.

We would now like to look at polynomials $f \in \mathbf{C}[X_1, \ldots, X_n]$. The point is to replace by analogy formal power series in the p-adic setting with the usual formal power series over $\mathbf{C}$. More precisely, we will look at:

(1') Polynomials $f \in \mathbf{C}[X_1, \ldots, X_n]$ and their solutions over $\mathbf{C}[t]/(t^{m+1}) \simeq \mathbf{C}[[t]]/(t^{m+1})$, with $m \geq 0$. Such a solution can be written as

$$x_1 = a_{10} + a_{11}t + \ldots + a_{1m}t^m \, , \, \ldots \, , \, x_n = a_{n0} + a_{n1}t + \ldots + a_{nm}t^m$$

with $a_i \in \mathbf{C}$, i.e. an *m-th jet* of $f = 0$.

(2') Solutions of f over $\mathbf{C}[[t]] = \varprojlim_{m} \mathbf{C}[[t]]/(t^{m+1})$. Such a solution is a collection of formal power series $\gamma = (\gamma_1, \ldots, \gamma_n)$, called an arc of $f = 0$.

(3') A norm $|f(\gamma)|_{\mathbf{C}((t))} = \frac{1}{q^{\mathrm{ord}_\gamma f}}$, where $q > 1$ is any fixed real numeber.

(4') Integrals over the arc space $\mathbf{A}^n_\infty = \mathrm{Spec}(\mathbf{C}[\![t]\!])^n$. This last step should be useful, like in the p-adic case, but it is a priori unclear how to do it since there are no known good $\mathbf{R}$-valued measures on $\mathbf{A}^n_\infty$. Following Kontsevich, we will need to introduce a measure with values, roughly speaking, in the Grothendieck ring of varieties.

1. Jet spaces

Given a non-negative integer m, we use the notation

$$\Delta_m = \mathrm{Spec}\ \mathbf{C}[t]/(t^{m+1}) \quad \text{and} \quad \Delta = \mathrm{Spec}\ \mathbf{C}[\![t]\!].$$[1]

There are obvious inclusion maps

$$\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_m \subset \cdots \subset \Delta. \tag{1}$$

Definition 1.1. Let X be a scheme over $\mathbf{C}$. Set theoretically, the *space of m-th order jets* on X consists of the morphisms of schemes over $\mathbf{C}$

$$X_m := \mathrm{Hom}(\Delta_m, X),$$

while the *space of arcs* is

$$X_\infty := \mathrm{Hom}(\Delta, X).$$

The inclusions in (1) give rise to *truncation maps*

$$\pi_m : X_\infty \longrightarrow X_m$$

and

$$\pi^k_m : X_k \longrightarrow X_m, \quad k \geq m.$$

These spaces have natural scheme structures. To see this in a rather elementary way, it is important to understand first the situation for affine space and then affine schemes.

Lemma 1.2. *We have $(\mathbf{A}^n)_m = \mathbf{A}^{n(m+1)}$ and for each $k \geq m$, the truncation map*

$$\pi^k_m : (\mathbf{A}^n)_k \longrightarrow (\mathbf{A}^n)_m$$

is given (up to permutation) by projection onto the first $n(m+1)$ coordinates.

Proof. An element in $(\mathbf{A}^n)_m$ is given by a $\mathbf{C}$-algebra homomorphism

$$\varphi : \mathbf{C}[X_1, \ldots, X_n] \longrightarrow \mathbf{C}[t]/(t^{m+1}).$$

This is described by the images of the variables $u_i = \varphi(X_i)$. Each u_i can be written as

$$u_i = u_{i0} + u_{i1}t + \ldots + u_{im}t^m,$$

[1]This is meant to symbolize that Δ is playing the role of a disk in the usual topology; it is the "formal disk".

with no conditions on the u_{ij}. Thinking of $(u_{i0}, \ldots, u_{im})$, $i = 1, \ldots, n$, as coordinate vectors, this gives the stated isomorphism. The second statement is clear, since for the truncation map we would be factoring φ through a homomorphism to $\mathbf{C}[t]/(t^{k+1})$. $\square$

Let now $X \subset \mathbf{A}^n$ be an arbitrary affine scheme, given by an ideal $I \subset \mathbf{C}[X_1, \ldots, X_n]$. Assume that I is generated by $f_1, \ldots, f_r$. An m-th jet on X is given by a homomorphism

$$A(X) = \mathbf{C}[X_1, \ldots, X_n]/I \longrightarrow \mathbf{C}[t]/(t^{m+1}).$$

If we write a jet as a vector $(u_1, \ldots, u_n)$ as above, we can then see X_m as being the affine subscheme of $(\mathbf{A}^n)_m$ given by the equations

$$f_i(u_1, \ldots, u_n) = 0, \quad i = 1, \ldots, r.$$

Exercise 1.3. Check that the scheme structure on X_m defined above is independent of the generators f_i and on the affine embedding $X \subset \mathbf{A}^n$. Conclude that for an arbitrary scheme X we obtain a well-defined scheme structure on X_m. Check that for every open subset $U \subset X$ we have $U_m = (\pi_0^m)^{-1}(U)$ (in fact start the whole thing by doing this for X affine).

There is a more functorial way of recognizing the natural scheme structure on X_m (and in particular get the conclusion of the Exercise above), by means of the fact that X_m represents a functor. For the general theory of the functor of points of a scheme, and how it determines it, see e.g. [Mu] Ch.II §6. Concretely, consider the functor

$$J_m : \{\text{Schemes over } \mathbf{C}\} \longrightarrow \{\text{Sets}\}, \quad J_m(S) = \mathrm{Hom}(S \times \Delta_m, X).$$

Proposition 1.4. *The functor J_m is represented by X_m, i.e. for each $\mathbf{C}$-scheme S we have functorial bijections*

$$\mathrm{Hom}(S \times \Delta_m, X) \simeq \mathrm{Hom}(S, X_m).$$

Proof. It suffices to show this for affine schemes, in other words to show that for all $\mathbf{C}$-algebras A we have functorial bijections

$$\mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{m+1}), X) \simeq \mathrm{Hom}(\mathrm{Spec}\ A, X_m).$$

If X is affine, this follows precisely as in the argument above. If X is arbitrary, we cover it by open affines $U_1, \ldots, U_r$, so that we do have the conclusion we want for $(U_i)_m$ for all i. As in the Exercise above, both $(\pi_0^{m,i})^{-1}(U_i \cap U_j)$ and $(\pi_0^{m,j})^{-1}(U_i \cap U_j)$ are isomorphic over $U_i \cap U_j$ with $(U_i \cap U_j)_m$. So the scheme X_m can be constructed by gluing $(U_i)_m$ over these overlaps, and it clearly satisfies the property we want. $\square$

Example 1.5. Note that by definition, when X is a smooth variety, the $\mathbf{C}$-valued points of the space of first-order jets form the total space of the tangent bundle T_X. More generally, given a scheme X, the first jet-scheme X_1 is the total space of the tangent sheaf, i.e. $X_1 = \mathbf{Spec}(\mathrm{Sym}\ \Omega_X^1)$. It is enough to see this for affine schemes $X = \mathrm{Spec}\ R$. We use the interpretation in Proposition 1.4: given any $\mathbf{C}$-algebra A, note that giving a morphism

$$\mathrm{Spec}\ A \longrightarrow \mathbf{Spec}(\mathrm{Sym}\ \Omega_X^1)$$

is equivalent to giving a morphism of $\mathbf{C}$-algebras $f : R \to A$ and a $\mathbf{C}$-derivation $D : R \to A$. This in turn is the same as specifying a homomorphism

$$g : R \longrightarrow A[t]/(t^2), \quad \text{with } g(r) = f(r) + tD(r).$$

Example 1.6. Let's describe all X_m for $X = (xy = 0) \subset \mathbf{A}^2$. We are looking for solutions of the form

$$x = x_0 + x_1 t + \ldots + x_m t^m \quad \text{and} \quad y = y_0 + y_1 t + \ldots + y_m t^m$$

such that $xy = 0 \pmod{t^{m+1}}$. (Recall that one such gives a point $(x_0, \ldots, x_m, y_0, \ldots, y_m) \in \mathbf{A}^{2(m+1)} = (\mathbf{A}^2)_m$.) They can be produced as follows: consider for any integers $k, l \geq 0$ such that $k + l = m + 1$, the subsets

$$V_{k,l} = \{x_0 = \ldots = x_{k-1} = y_0 = \ldots = y_{l-1} = 0\} \subset \mathbf{A}^{2(m+1)}.$$

Clearly $V_{k,l} \simeq \mathbf{A}^{m+1}$. We see that $V_{k,l}$ parametrizes jets of the form

$$(x, y) = (x_k t^k + x^{k+1} t^{k+1} + \ldots,\ y_l t^l + y^{l+1} t^{l+1} + \ldots)$$

and all solutions must belong to one of them. The $V_{k,l}$ are in fact precisely the irreducible components of X_m. Note that under the truncation map

$$\pi_0^m : X_m \longrightarrow X$$

$V_{m+1,0}$ and $V_{0,m+1}$ map to the x and y axes respectively, while all the other $V_{k,l}$ map to the origin (which is the singularity of X).

Example 1.7. Let $X = (xy - z^3 = 0) \subset \mathbf{C}^3$. We will show that for each $m \geq 1$, X_m is irreducible of dimension $2(m+1)$. For a fixed m, we are looking for

$$x(t) = a_0 + a_1 t + \ldots + a_m t^m,\ y(t) = b_0 + b_1 t + \ldots + b_m t^m,\ z(t) = c_0 + c_1 t + \ldots + c_m t^m$$

such that

$$x(t) \cdot y(t) = z(t)^3 \bmod t^{m+1}. \tag{2}$$

Note that the identity (2) consists of $m + 1$ equations in the $a's$, $b's$ and $c's$, i.e. in the affine space $(\mathbf{A}^3)_m = \mathbf{A}^{3(m+1)}$. This implies that each irreducible component of X_m has dimension at least $2(m+1)$. We now have to distinguish between those irreducible components of X_m that do not lie over the origin, and those that do. We will show that there is precisely one of the former, and none of the latter.

Let $Y \subset X_m$ be the union of components that do not lie over the origin. It is easy to see that for the general $(x(t), y(t), z(t)) \in Y$ we have $a_0, b_0, c_0 \neq 0$. Now if $a_0 \neq 0$, x can be inverted, so that we can solve for y in terms of x and z. In other words, if the a's and c's are general, one can uniquely solve for the b's. This implies that Y is irreducible, of dimension precisely $2(m+1)$.

On the other hand, I claim that

$$\dim\{(x(t), y(t), z(t)) \mid a_0 = b_0 = c_0 = 0\} < 2(m+1),$$

which immediately implies that there are no extra components lying over the origin. Indeed, if all the initial terms are 0, after factoring out (at least) one power of t in x, y and z, we can repeat the same process and obtain loci of dimension at most $2m$.

Exercise 1.8. (1) Let $X = (x^3+y^3+z^3=0) \subset \mathbf{C}^3$. Show that for every $m \geq 1$, X_m has a unique irreducible component which dominates X via the truncation map $\pi_0^m : X_m \to X$. This component has dimension $2(m+1)$. Show that for $3|m+1$, X_m has extra irreducible components lying over the origin in X, which are of dimension $2(m+1)$ as well.

(2) Let $X = (x^d + y^d + z^d = 0) \subset \mathbf{C}^3$, with $d \geq 4$. Show that for every $m \geq 1$, X_m has a unique irreducible component which dominates X via the truncation map $\pi_0^m : X_m \to X$. This component has dimension $2(m+1)$. Show that for $d|m+1$, X_m has extra irreducible components lying over the origin in X, which this time have dimension $> 2(m+1)$.

Definition 1.9. We say that a morphism $f : X \to Y$ is *locally trivial with fiber F* if there exists a Zariski open cover $Y = U_1 \cup \ldots \cup U_k$ such that $f^{-1}(U_i) \simeq U_i \times F$ for all i, with the restriction of f to $f^{-1}(U_i)$ being the projection onto the first component. If this condition holds for a cover where U_i are only locally closed, we call f *piecewise trivial with fiber F*.

Theorem 1.10. *Let X be a smooth complex variety of dimension n. Then, for each $m \geq 1$, X_m is smooth of dimension $n(m+1)$. Moreover, the truncation maps*

$$\pi_m^{m+e} : X_{m+e} \longrightarrow X_m, \quad \text{with } e > 0,$$

are locally trivial in the Zariski topology, with fiber $\mathbf{A}^{ne}$.

Proof. It is clear that the local-triviality statement implies the smoothness and dimension conclusions as well. For this statement, it is enough to assume that $e = 1$, since then we can proceed inductively.

To this end, since we know that for an open set $U \subset X$ we have $(\pi_0^m)^{-1}(U) = U_m$, it suffices to prove the following: for any open set $U \subset X$ having an algebraic coordinate system $x_1, \ldots, x_n$, we have an isomorphism $U_m \simeq U \times \mathbf{A}^{mn}$ such that π_m^{m+1} is the projection onto the first mn components. The condition of being an algebraic coordinate system means that $x_1, \ldots, x_n \in \mathcal{O}_X(U)$ are such that the differentials $dx_1, \ldots, dx_n$ trivialize Ω_X^1 over U; such a system exists around every point since X is smooth.

On the other hand, an algebraic coordinate system is equivalent to the data of an étale morphism $U \to \mathbf{A}^n$. By Lemma 1.11 below and Lemma 1.2, we conclude that

$$U_m \simeq (\mathbf{A}^n)_m \times_{\mathbf{A}^n} U = \mathbf{A}^{n(m+1)} \times_{\mathbf{A}^n} U \simeq \mathbf{A}^{nm} \times U,$$

and the assertion about the projection map is clear. $\square$

Note that by functoriality each morphism of schemes $f : X \to Y$ induces morphisms $f_m : X_m \to Y_m$, compatible with the truncation maps.

Lemma 1.11. *Let $f : X \to Y$ be an étale morphism of schemes. Then, for every $m \geq 1$, the induced commutative diagram*

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \downarrow{\scriptstyle \pi_0^m} & & \downarrow{\scriptstyle \pi_0^m} \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian (i.e a fiber product).

Proof. Using the interpretation of X_m and Y_m as representing the respective functors J_m as in Proposition 1.4, it is enough to show that for every $\mathbf{C}$-algebra A and every commutative diagram

$$\begin{tikzcd} \text{Spec } A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \text{Spec } A[t]/(t^{m+1}) \arrow[r] & Y \end{tikzcd}$$

there is a unique morphism Spec $A[t]/(t^{m+1}) \to X$ of schemes over $\mathbf{C}$ making the two induced triangles commutative. This is a consequence of the commutative algebra fact that étale morpisms are formally étale (see e.g. [Ma] §28). □

Exercise 1.12. Show that for an arbitrary scheme X, the truncation maps π_m^{m+1} : $X_{m+1} \to X_m$ are affine morphisms.

Exercise 1.13. Show that if C is a singular curve, then C_1 is not irreducible.

2. Arc spaces

Generalities. Given a scheme X, we have constructed jet-schemes X_m for each $m \geq 0$, together with truncation maps $\pi_m^{m+1} : X_{m+1} \to X_m$, which are affine morphisms; in other words, for every open affine subset $U \subset X_m$, $(\pi_m^{m+1})^{-1}(U)$ is affine, and the corresponding restriction of the truncation map corresponds to a ring homomorphism with arrows reversed. Since inductive limits exist in the category of $\mathbf{C}$-algebras, this implies that we can pass to the projective limit in the category of schemes over $\mathbf{C}$ to obtain:

Definition 2.1. The (scheme-theoretic) *arc space* of X is

$$X_\infty := \varprojlim_m X_m.$$

Even if X is a scheme of finite type over $\mathbf{C}$, in general X_∞ will not be of finite type. Note that by definition there exist *projection morphisms*

$$\psi_m : X_\infty \longrightarrow X_m \text{ for all } m \geq 0,$$

and by the definition of the projective limit scheme structure on X_∞, for every open set $U \subset X$ we have

$$\mathcal{O}_{X_\infty}(\psi_0^{-1}(U)) \simeq \varinjlim_m \mathcal{O}_{X_m}((\pi_0^m)^{-1}(U)).$$

The Zariski topology on X_∞ is the projective limit topology; in other words, the closed subsets of X_∞ are limits of compatible closed subsets in $Z_m \subset X_m$, i.e. satisfying

$$\pi_m^{m+1}(Z_{m+1}) = Z_m \text{ for } m \gg 0.$$

For example, given any closed subscheme $Z \subset X$, Z_∞ is a closed subset of X_∞.

For every $\mathbf{C}$-algebra A, using Proposition 1.4 we have

$$\text{Hom}(\text{Spec } A, X_\infty) \simeq \varprojlim_m \text{Hom}(\text{Spec } A, X_m) \simeq$$

$$\simeq \varprojlim_m \operatorname{Hom}(\operatorname{Spec} A[t]/(t^{m+1}), X) \simeq \operatorname{Hom}(\operatorname{Spec} A[\![t]\!], X).$$

In particular, the $\mathbf{C}$-valued points of X_∞ are precisely the set-theoretic space of arcs $\operatorname{Hom}(\Delta, X)$ defined in the previous section.

Exercise 2.2. (1) Show that if $f : X \to Y$ is a morphism of schemes, there is an induced natural morphism $f_\infty : X_\infty \to Y_\infty$, commuting with all the projection morphisms ψ_m. If f is a closed immersion, show that f_∞ is a closed immersion.

(2) If f is étale, show that the commutative diagram

$$\begin{tikzcd} X_\infty \arrow[r, "f_\infty"] \arrow[d, "\psi_0"'] & Y_\infty \arrow[d, "\psi_0"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a fiber product.

Equations for jet and arc spaces. Here I will note that there is an algorithmic procedure for producing equations for X_m and X_∞ starting with equations for an affine $X \subset \mathbf{A}^n$. Starting with the polynomial ring $S = \mathbf{C}[X_1, \dots X_n]$, we formally introduce new variables $X_i^{(m)}$ for each $m \geq 1$, with $X_i^{(0)} = X_i$. Define

$$S_\infty = \mathbf{C}[\ X_i^{(m)} \mid i = 1, \dots, n,\ m \geq 0\ ].$$

We have $(\mathbf{A}^n)_\infty = \operatorname{Spec} S_\infty$. We can define a $\mathbf{C}$-derivation on S_∞ by the rule

$$D : S_\infty \longrightarrow S_\infty, \ \ D(X_i^{(m)}) = X_i^{(m+1)}.$$ [2]

Given a polynomial $f \in S$, letting $f^{(0)} = f$, we can define recursively $f^{(m)} = D(f^{(m-1)})$.

Let now $X \subset \mathbf{A}^n$ be given by the ideal $I = (f_1, \dots, f_r) \subset S$, and let $R = A(X) = S/I$. Define

$$R_\infty := S_\infty / I_\infty, \ \ I_\infty = (f_i, f_i^{(1)}, \dots, f_i^{(m)}, \dots), \ i = 1, \dots, r.$$

Lemma 2.3. $X_\infty \simeq \operatorname{Spec} R_\infty$.

Proof. For any $\mathbf{C}$-algebra A, a $\mathbf{C}$-homomorphism

$$\varphi : \mathbf{C}[X_1, \dots, X_n] \longrightarrow A[\![t]\!]$$

is defined by the images of the variables

$$\varphi(X_i) = \sum_{m \geq 0} \frac{a_i^{(m)}}{m!} \cdot t^m.$$

This gives after a small calculation that for every $f \in \mathbf{C}[X_1, \dots, X_n]$ we have

$$\varphi(f) = \sum_{m \geq 0} \frac{f^{(m)}(a, a^{(1)}, \dots, a^{(m)})}{m!} \cdot t^m,$$

[2] The basic thinking is that we are introducing a new variable $X_i^{(m)}$ for each differential operator given by the partial derivative $\frac{\partial^m}{\partial X_i^m}$.

where $a = (a_1, \ldots, a_r)$ and similarly for $a^{(m)}$. To say that φ induces a homomorphism $R \to A[\![t]\!]$ is then equivalent to saying that

$$f_i^{(m)}(a, a^{(1)}, \ldots, a^{(m)}) = 0 \quad \text{for all } m \geq 0 \quad \text{and all } 1 \leq i \leq r.$$

□

By truncating the above discussion at level m, we obtain of course equations for the jet-scheme X_m. In other words, we can consider

$$S_m = \mathbf{C}[\ X_i^{(j)} \mid i = 1, \ldots, n,\ j = 0, \ldots, m\ ]$$

and

$$R_m := S_m/(f_i, f_i^{(1)}, \ldots, f_i^{(m)} \mid i = 1, \ldots, r).$$

The conclusion is that $X_m \simeq \operatorname{Spec} R_m$, and the truncation maps are obtained from the natural morphisms $R_m \to R_{m+1}$.

Example 2.4. Let X be the cusp $y^2 - x^3 = 0$ in $\mathbf{A}^2$. Then X_1 is defined in Spec $\mathbf{C}[X, Y, X', Y']$ by the ideal

$$(Y^2 - X^3, 2YY' - 3X^2X')$$

while X_2 is defined in Spec $\mathbf{C}[X, Y, X', Y', X'', Y'']$ by the ideal

$$(Y^2 - X^3, 2YY' - 3X^2X', 2Y'^2 + 2YY' - 6XX'^2 - 3X^2X'').$$

Kolchin's theorem. Before proving the next result, let's observe the following. An arc $\gamma : \operatorname{Spec} \mathbf{C}[\![t]\!] \to X$ determines two points on X, corresponding to the two points of Spec $\mathbf{C}[\![t]\!] \to X$; one is the image $\gamma\big((t)\big)$ of its closed point, while the other is the image $\gamma\big((0)\big)$ of its generic point. The latter corresponds to the induced morphism

$$\bar{\gamma} : \operatorname{Spec} \mathbf{C}((t)) \longrightarrow X.$$

Proposition 2.5. *Let $f : Y \to X$ be a proper birational morphism of varieties. If $Z \subset X$ is a proper closed subset such that f is an isomorphism over $X - Z$, then the induced map*

$$Y_\infty - f^{-1}(Z)_\infty \longrightarrow X_\infty - Z_\infty$$

is a bijection.

Proof. Let $\gamma \in X_\infty$. Then $\gamma \notin Z_\infty$ if and only if the induced $\bar{\gamma} : \operatorname{Spec} \mathbf{C}((t)) \longrightarrow X$ has its image in $U = X - Z$. In this case $\bar{\gamma}$ has a lifting to a morphism

$$\operatorname{Spec} \mathbf{C}((t)) \longrightarrow Y - f^{-1}(Z),$$

since f is an isomorphism over U. We now apply the Valuative Criterion of Properness (see [Ha], Ch.II Theorem 4.7) for such an arc. Note that (identifying $\bar{\gamma}$ with its lifting) we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} \mathbf{C}((t)) \arrow[r, "\bar{\gamma}"] \arrow[d] & Y \arrow[d, "f"] \\ \operatorname{Spec} \mathbf{C}[\![t]\!] \arrow[r, "\gamma"] & X \end{tikzcd}$$

The valuative criterion says then that there exists a unique lifting of γ to an arc Spec $\mathbf{C}[\![t]\!] \to Y$, which necessarily has to be in $Y_\infty - f^{-1}(Z)_\infty$. $\square$

Theorem 2.6 (Kolchin). *If X is a complex variety, then X_∞ is irreducible.*

Proof. We prove this only for the complex points of X_∞ (but the general statement holds as well). First, when X is smooth, we have seen that all X_m are smooth varieties. Therefore passing to the limit $X_\infty = \varprojlim_m X_m$ we obtain an irreducible scheme as well.

Let's assume now that X is singular, and prove the result by induction on $n = \dim X$. Consider a resolution of singularities $f : Y \to X$, and let $Z \subset X$ be a proper closed subset such that f is an isomorphism over $X - Z$. Proposition 2.5 implies that

$$X_\infty = Z_\infty \cup \mathrm{Im}(f_\infty).$$

Now since Y is smooth, by the above we know that Y_∞ is irreducible, and therefore $\mathrm{Im}(f_\infty)$ is irreducible as well. To conclude, it suffices then to show that $Z_\infty \subset \overline{\mathrm{Im}(f_\infty)}$.

Consider now the decomposition of Z into its irreducible components, $Z = Z_1 \cup \ldots \cup Z_k$. This gives

$$Z_\infty = Z_{1\infty} \cup \ldots Z_{k\infty}.$$

For every $i = 1, \ldots, k$, pick an irreducible component Y_i of $f^{-1}(Z_i)$ such that the induced $Y_i \to Z_i$ is surjective. By generic smoothness (see [Ha], Ch.III Corollary 10.7), there exist open subsets $U_i \subset Y_i$ and $V_i \subset Z_i$ such that f restricts to a surjective smooth morphism $g_i : U_i \to V_i$. This implies that

$$V_{i\infty} = \mathrm{Im}(g_{i\infty}) \subset \mathrm{Im}(f_\infty).$$

Now by induction on dimension we know that all $Z_{i\infty}$ are irreducible. They contain $V_{i\infty}$ as non-empty open sets, so passing to closures we conclude that $Z_{i\infty} \subset \overline{\mathrm{Im}(f_\infty)}$. $\square$

3. Cylinders and the birational transformation rule

Let X be a smooth complex variety of dimension n. While X_∞ is infinite dimensional, it often suffices to restrict one's attention to subsets which come from a finite level, i.e. from nice enough subsets of some X_m.

Definition 3.1. (1) Let Y be a variety. A *constructible subset* of Y is a finite union of locally closed subsets of Y.

(2) A *cylinder* in X_∞ is a set of the form $\psi_m^{-1}(C_m)$ for some m, where $C_m \subset X_m$ is a constructible set. A cylinder C is open, closed, locally closed or irreducible if it can be written as above, with C_m having the respective property. Cylinders are closed under finite unions, intersections, and complements, and therefore they form an algebra of sets.

Example 3.2. The image of the morphism $f : \mathbf{A}^2 \to \mathbf{A}^2$, $f(x,y) = (x, xy)$ (i.e. the restriction of the blow-up of $\mathbf{A}^2$ at the origin to one of its charts) is constructible, but not locally closed around the origin. The image of any morphism of finite type is constructible, according to Chevalley's theorem.

Example 3.3. (1) The most interesting cylinders we will discuss arise from looking at the vanishing orders of arcs. Fix a proper closed subscheme $Z \subset X$. One defines a function

$$\mathrm{ord}_Z : X_\infty \longrightarrow \mathbf{N} \cup \infty, \quad \gamma \mapsto \mathrm{ord}_Z(\gamma)$$

given by the vanishing order of γ along Z: explicitly, if $\gamma : \mathrm{Spec}\ \mathbf{C}[\![t]\!] \to X$, then the scheme theoretic preimage of Z is defined by an ideal in $\mathbf{C}[\![t]\!]$, generated by $t^{\mathrm{ord}_Z(\gamma)}$ (note that every ideal in $\mathbf{C}[\![t]\!]$ is generated by some power of t). We define the *m-th contact locus* of Z to be

$$\mathrm{Cont}^m(Z) := \mathrm{ord}_Z^{-1}(m) \subset X_\infty.$$

Similarly, we can define $\mathrm{Cont}^{\geq m}(Z) := \mathrm{ord}_Z^{-1}(\geq m)$, and we note that

$$\mathrm{Cont}^{\geq m}(Z) = \psi_{m-1}^{-1}(Z_{m-1}).$$

(Indeed, it is enough to assume that $X = \mathrm{Spec}\ R$, with Z being given by an ideal $I \subset R$. An arc $\gamma \in \mathrm{Cont}^{\geq m}(Z)$ is given by a homomorphism $\varphi : R \to \mathbf{C}[\![t]\!]$ such that $\varphi(I) = (t^k)$ with $k \geq m$. This implies that I is mapped to 0 by composition with the truncation map $\mathbf{C}[\![t]\!] \to \mathbf{C}[\![t]\!]/(t^m)$, so that finally this data is equivalent to a homomorphism $R/I \to \mathbf{C}[\![t]\!]/(t^m)$, or in other words to an arc in Z_{m-1}.) This implies that $\mathrm{Cont}^{\geq m}(Z)$ is a closed cylinder, and consequently $\mathrm{Cont}^m(Z)$ is a locally closed cylinder.

(2) The basic non-example is the following: if $Z \subset X$ is a proper subscheme, then $Z_\infty \subset X_\infty$ is not a cylinder. This amounts to the fact that one cannot test membership in Z_∞ by looking at jets of some fixed order m.

Exercise 3.4. Let $f : X \to Y$ be a proper birational morphism of smooth varieties. Show that:

(1) $f_m : X_m \to Y_m$ is surjective for every $m \geq 0$.

(2) If $F \subset X_m$ is a union of fibers of f_m and $C = \psi_m^{-1}(F)$, then

$$f_\infty(C) = \psi_m^{-1}(f_m(F)).$$

In particular, $f_\infty(C)$ is a cylinder.

(3) $f_\infty : X_\infty \to Y_\infty$ is surjective.

Let now $f : X \to Y$ be a proper birational morphism of smooth varieties of dimension n. We are particularly interested in the contact loci defined by the relative canonical divisor

$$C^p := \mathrm{Cont}^p(K_{X/Y}).$$

The behavior of these loci is explained by the following celebrated Birational Transformation Theorem. This will be the key ingredient for the Change of Variables Theorem for motivic integration, discussed in the next chapter.

Theorem 3.5 (Kontsevich)**.** *With the notation above, let m and e be integers such that $m \geq 2e$. Then:*

(i) For every $\gamma, \gamma' \in X_m$ such that $\gamma \in \psi_m(C^e)$ and $f_m(\gamma) = f_m(\gamma') \in Y_m$, we have

$$\pi^m_{m-e}(\gamma) = \pi^m_{m-e}(\gamma').$$

(In other words, the fiber of f_m through a point in $\psi_m(C^e)$ is contained in a fiber of π^m_{m-e}.)

(ii) $\psi_m(C^e)$ is a union of fibers of f_m, each isomorphic to $\mathbf{A}^e$.

(iii) The induced map

$$\psi_m(C^e) \longrightarrow f_m\big(\psi_m(C^e)\big)$$

is piecewise trivial with fiber $\mathbf{A}^e$.

Corollary 3.6. *In the notation of the Theorem, $f_\infty(C^e)$ is a cylinder.*

Proof. This follows from (ii) in the Theorem and Exercise 3.4. □

Example 3.7. Let

$$f : \mathbf{A}^2 \to \mathbf{A}^2, \quad f(x,y) = (x, xy)$$

be the restriction of the blow-up of the origin of $\mathbf{A}^2$ to one of its two charts. The Jacobian of f is x, and therefore the relative canonical divisor is $K = (x = 0)$. Consider now a jet

$$\gamma = (x_0 + x_1 t + \ldots + x_m t^m, y_0 + y_1 t + \ldots + y_m t^m)$$

in $(\mathbf{A}^2)_m$. The condition $\gamma = \psi_m(C^e)$ (i.e. of γ having order of contact with K precisely e) translates into $x_0 = x_1 = \ldots = x_{e-1} = 0$ and $x_e \neq 0$. We have

$$f_m(\gamma) = \big(x_e t^e + \ldots + x_m t^m, (x_e t^e + \ldots + x_m t^m)(y_0 + y_1 t + \ldots + y_m t^m) \bmod t^{m+1}\big)\,.$$

As a consequence

$$f_m^{-1}(f_m(\gamma)) = \{\gamma + (0, z_{m-e+1} t^{m-e+1} + \ldots + z_m t^m)\}$$

with z_j arbitrary, since nothing beyond order $m-e$ in the y's can influence the expression for $f_m(\gamma)$. This clearly implies that $f_m^{-1}(f_m(\gamma)) \simeq \mathbf{A}^e$, and for every $\gamma' \in f_m^{-1}(f_m(\gamma))$ we have $\pi^m_{m-e}(\gamma) = \pi^m_{m-e}(\gamma')$.

Remark 3.8. Let $f : X \to Y$ be a proper birational morphism of smooth varieties, and let $Z \subset Y$ be the image of the exceptional locus of f. Since Z_∞ consists precisely of those arcs having infinite contact order with Z, and same for $f^{-1}(Z)$, we have

$$X_\infty - f^{-1}(Z)_\infty = \coprod_{e \in \mathbf{N}} C^e \quad \text{and} \quad Y_\infty - Z_\infty = \coprod_{e \in \mathbf{N}} f_\infty(C^e).$$

Note that f_∞ induces a bijection on these loci, and in particular the mapping $C^e \to f_\infty(C^e)$ is bijective for each e. However, Theorem 3.5 says that when truncating to finite level $m \geq 2e$, the induced mapping in piecewise trivial with fiber $\mathbf{A}^e$.

The rest of this section is concerned with the proof of Theorem 3.5. Let's first note that the statement in part (iii) follows from the previous statements and the following general

Lemma 3.9. *Let $f : V \to W$ be a morphism of schemes of finite type such that the fibers over all (not necessarily closed) points of W are isomorphic as schemes to $\mathbf{A}^e$. Then f is a piecewise trivial fibration with fiber $\mathbf{A}^e$.*

Proof. We can assume that W is irreducible. The fiber of f over the generic point $\eta \in W$ is isomorphic to $\mathbf{A}^e$, which means that there exists an open set $U \subset W$ such that $f^{-1}(U) \simeq U \times \mathbf{A}^e$. But now the induced morphism $V - f^{-1}(U) \to W - U$ is a morphism between schemes of finite type of strictly lower dimension than before, and we can conclude by induction. $\square$

On the other hand, the fact that $\psi_m(C^e)$ is a union of fibers of f_m follows from (i) due to the fact that C^e is stable at level e in the sense of the following

Definition 3.10. A subset $S \subset X_\infty$ is *stable at level* k if for any $m \geq k$ we have that $\psi_m(S) \subset X_m$ is constructible, $\psi_m^{-1}(\psi_m(S)) = S$, and the induced truncation map

$$\pi_m^{m+1} : \psi_{m+1}(S) \longrightarrow \psi_m(S)$$

is a locally trivial fibration with fiber $\mathbf{A}^n$. S is called *stable* if it is stable at some level.

Indeed, it is clear that C^e is stable at level e, since the condition of an arc being in C^e depends only on its truncation to level e. Since $m \geq 2e$, we then have that $\psi_m(C^e) \to \psi_{m-e}(C^e)$ is locally trivial with fiber $\mathbf{A}^{ne}$, which is the entire fiber of π_{m-e}^m. This, combined with (i), immediately implies the statement in (ii).

In conclusion, for the Theorem one only needs to check that each fiber $f_m^{-1}(f_m(\gamma))$ with $\gamma \in \psi_m(C^e)$ is contained in a fiber of π_{m-e}^m, and is isomorphic to $\mathbf{A}^e$.

The case of affine space. Since the ideas are more transparent in this case, for intuition let's first treat the case $X = Y = \mathbf{A}^n$. Let

$$f = (f_1, \ldots, f_n) : \mathbf{A}^n \longrightarrow \mathbf{A}^n, \quad f_i \in \mathbf{C}[X_1, \ldots, X_n].$$

Let

$$\gamma' \in X_\infty = \mathbf{A}^n_\infty,$$

i.e. a collection of n formal power series in the variable t. Assume that $\gamma' \in C^e$. Write

$$\psi_m(\gamma') = \gamma = (\gamma_1, \ldots, \gamma_n),$$

so that γ_i are polynomials of degree m in t, with $\mathrm{ord}_{K_{X/Y}}(\gamma) = e$. In order to describe the fiber

$$f_m^{-1}(f_m(\gamma))$$

we need to describe all n-tuples $\beta = (\beta_1, \ldots, \beta_n)$ of polynomials of degree m in t satisfying

$$f(\gamma + \beta) \equiv f(\gamma) \bmod t^{m+1}.$$

Since f is given by polynomials, we can expand using Taylor's formula, to get

$$f(\gamma + \beta) = f(\gamma) + Df(\gamma) \cdot \beta + \text{ higher order terms in } \beta, \tag{3}$$

where we have

$$Df(\gamma) = \left(\frac{\partial f_i}{\partial X_j} (\gamma_1(t), \ldots, \gamma_n(t)) \right)_{1 \leq i,j \leq n}$$

which we treat as a matrix in $M_{n,n}(\mathbf{C}[\![t]\!])$. Our hypothesis implies that $\mathrm{ord}_t \, |Df(\gamma)| = e$.

The next claim is that it is enough to assume that $Df(\gamma)$ is diagonal, of the form

$$Df(\gamma) = \mathrm{Diag}(t^{e_1}, \ldots, t^{e_n}), \quad \sum_{i=1}^{n} e_i = e.$$

Let's assume for now that this is the case, and finish the proof. First, in this case having $Df(\gamma) \cdot \beta \equiv 0 \bmod t^{m+1}$ is equivalent to saying that the i-th component of β looks like

$$\beta_i = t^{m+1-e_i} \cdot P_i,$$

where P_i is an arbitrary polynomial in t of degree at most $e_i - 1$. This immediately gives

$$\{\beta \mid Df(\gamma) \cdot \beta \equiv 0 \bmod t^{m+1}\} \simeq \mathbf{A}^e = \mathbf{A}^{e_1 + \ldots + e_n}.$$

On the other hand, since $m \geq 2e$, the quadratic terms in (3) all have degree in t higher than $m+1$. Combined with the above, this gives

$$\{\beta \mid f_m(\gamma + \beta) = f_m(\gamma)\} \simeq \mathbf{A}^e.$$

It is also clear that the set $\{\gamma + \beta\}$ is contained in a fiber of π^m_{m-e}, as the description of β_i above implies that $\gamma + \beta \equiv \gamma \bmod t^{m-e+1}$.

It remains to see that it is enough to assume that $Df(\gamma)$ is diagonal. As a matrix over $\mathbf{C}[\![t]\!]$, it can be diagonalized; choose $P, Q \in \mathrm{GL}_n\big(\mathbf{C}[\![t]\!]\big)$ such that

$$P \cdot Df(\gamma) \cdot Q = \mathrm{Diag}\ (t^{e_1}, \ldots, t^{e_n}).$$

Write

$$P_m, Q_m \in \mathrm{GL}_n\big(\mathbf{C}[t]/(t^{m+1})\big)$$

for the truncations modulo t^{m+1}. Note that we still have

$$P_m \cdot Df(\gamma) \cdot Q_m = \mathrm{Diag}\ (t^{e_1}, \ldots, t^{e_n}),$$

while multiplying by P_m and Q_m produces automorphisms of $(\mathbf{A}^n)_m$. In particular we have the identification

$$\{\beta \in (\mathbf{A}^n)_m \mid f(\gamma + \beta) \equiv f(\gamma) \bmod t^{m+1}\} \simeq$$

$$\simeq \{\delta \in (\mathbf{A}^n)_m \mid P_m \cdot f(\gamma + Q_m \cdot \delta) \equiv P_m \cdot f(\gamma) \bmod t^{m+1}\}$$

But now note that

$$P_m \cdot f(\gamma + Q_m \cdot \beta) = P_m \cdot f(\gamma) + P_m \cdot Df(\gamma) \cdot Q_m \cdot \beta + \ldots,$$

which shows that we can reduce to the diagonal setting.

The general case. I will now give the proof in the general case, following [Bl] Appendix A and [Mu] §2.3, which in turn follow [Lo]. For the entire proof, the convenient language is that of derivations, for which a very brief reminder is provided in the Appendix below.

Let $\gamma \in C^e$, with $\gamma_m = \psi_m(\gamma) \in X_m$ and $\psi_0(\gamma) = x \in X$. We first record a crucial interpretation of this information. Note that the birational morphism $f : X \to Y$ induces a short exact sequence

$$0 \longrightarrow f^*\Omega^1_Y \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/Y} \longrightarrow 0.$$

The pull-back of this sequence to Spec $\mathbf{C}[\![t]\!]$ via the arc γ is equivalent to the exact sequence

$$0 \longrightarrow \Omega^1_{Y,f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathbf{C}[\![t]\!] \xrightarrow{\gamma^* df} \Omega^1_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathbf{C}[\![t]\!] \longrightarrow \Omega^1_{X/Y,x} \otimes_{\mathcal{O}_{X,x}} \mathbf{C}[\![t]\!] \longrightarrow 0. \tag{4}$$

Now by the definition of the contact order, we have that $\det(\gamma^* df) = (t^e)$. Since $\mathbf{C}[\![t]\!]$ is a PID, we can choose bases of the rank n free modules $\Omega^1_{Y,f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathbf{C}[\![t]\!]$ and $\Omega^1_{X,x} \otimes_{\mathcal{O}_{Y,f(x)}} \mathbf{C}[\![t]\!]$ such that $\gamma^*(df)$ is given by a diagonal matrix, and in fact the sequence above becomes

$$0 \longrightarrow (\mathbf{C}[\![t]\!])^n \xrightarrow{\mathrm{Diag}(t^{e_1},\ldots,t^{e_n})} (\mathbf{C}[\![t]\!])^n \longrightarrow \bigoplus_{i=1}^{n} \mathbf{C}[\![t]\!]/(t^{e_i}) \longrightarrow 0 \tag{5}$$

with $e_1 + \ldots + e_n = e$.

To prove the statement, let's first assume that we know that $f_m^{-1}(f_m(\gamma_m))$ is contained in a fiber of π^m_{m-e}, and deduce that it is isomorphic to $\mathbf{A}^e$.

Step 1. The first claim is that we have an identification

$$(\pi^m_{m-e})^{-1}(\pi^m_{m-e}(\gamma_m)) \simeq \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1})\right).$$

Indeed, given any γ'_m with $\pi^m_{m-e}(\gamma_m) = \pi^m_{m-e}(\gamma'_m)$, we have that the two jets correspond to morphisms $\mathcal{O}_{X,x} \to \mathbf{C}[t]/(t^{m+1})$ such that the compositions with the natural projection

$$\mathcal{O}_{X,x} \longrightarrow \mathbf{C}[t]/(t^{m+1}) \longrightarrow \mathbf{C}[t]/(t^{m-e+1})$$

are equal. This implies that their difference defines a map to the kernel of the projection,

$$\gamma_m - \gamma'_m : \mathcal{O}_{X,x} \longrightarrow (t^{m-e+1})/(t^{m+1})$$

which is easily seen to be a derivation (note that since $m \geq 2e$ the square of the ideal $(t^{m-e+1})/(t^{m+1})$ is zero, which makes the Leibniz rule work).

Step 2. We now show that we have an identification

$$f_m^{-1}(f_m(\gamma_m)) \simeq \mathrm{Der}_{\mathcal{O}_{Y,f(x)}}\left(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1})\right).$$

Using the sequence (8) below, we have that $\mathrm{Der}_{\mathcal{O}_{Y,f(x)}}(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1}))$ is the kernel of the natural morphism

$$\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1})\right) \longrightarrow \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)}, (t^{m-e+1})/(t^{m+1})\right),$$

and therefore using Step 1. and the current assumption that

$$f_m^{-1}(f_m(\gamma_m)) \subset (\pi^m_{m-e})^{-1}(\pi^m_{m-e}(\gamma_m)),$$

the identification will be made inside the space $\mathrm{Der}_{\mathbf{C}}(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1}))$. Consider $\gamma'_m \in (\pi^m_{m-e})^{-1}(\pi^m_{m-e}(\gamma_m))$, so that by Step 1.

$$\gamma'_m - \gamma_m \in \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x}, (t^{m-e+1})/(t^{m+1})\right).$$

Then clearly $\gamma'_m - \gamma_m$ is mapped to 0 in $\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)}, (t^{m-e+1})/(t^{m+1})\right)$ if and only if $f \circ (\gamma'_m - \gamma_m) = 0$, i.e. if and only if $f_m(\gamma'_m) = f_m(\gamma_m)$.

Step 3. Finally, let's note that

$$\mathrm{Der}_{\mathcal{O}_{Y,f(x)}}\left(\mathcal{O}_{X,x},(t^{m-e+1})/(t^{m+1})\right)\simeq \mathbf{A}^e$$

as claimed. We identify again the space on the left with the kernel of the natural morphism

$$\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x},(t^{m-e+1})/(t^{m+1})\right)\longrightarrow \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{m-e+1})/(t^{m+1})\right),$$

or equivalently, according to the isomorphism in (7), that of the morphism

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}\left(\Omega^1_{X,x},(t^{m-e+1})/(t^{m+1})\right)\longrightarrow \mathrm{Hom}_{\mathcal{O}_{Y,f(x)}}\left(\Omega^1_{Y,f(x)},(t^{m-e+1})/(t^{m+1})\right).$$

This is the same as the natural morphism

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}\left(\Omega^1_{X,x},\mathbf{C}[\![t]\!]/(t^{m+1})\right)\longrightarrow \mathrm{Hom}_{\mathcal{O}_{Y,f(x)}}\left(\Omega^1_{Y,f(x)},\mathbf{C}[\![t]\!]/(t^{m+1})\right)$$

since the jet γ_m has contact of order e with the Jacobian. (As before, the structure of $\mathcal{O}_{X,x}$-module on $\mathbf{C}[\![t]\!]/(t^{m+1})$ is that induced by the jet $\gamma_m:\mathcal{O}_{X,x}\to\mathbf{C}[\![t]\!]/(t^{m+1})$.) Given the sequences (4) and (5), the kernel of this last morphism is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}\left(\Omega^1_{X/Y,x},\mathbf{C}[\![t]\!]/(t^{m+1})\right)\simeq \mathbf{A}^e.$$

The last isomorphism as a $\mathbf{C}$-space follows via the $\mathbf{C}[\![t]\!]/(t^{m+1})$-module structure, since $m>e$ and we have seen that $\Omega^1_{X/Y,x}\otimes_{\mathcal{O}_{X,x}}\mathbf{C}[\![t]\!]$ is torsion of length e.

We are now left with proving the statement that $f_m^{-1}(f_m(\gamma_m))$ is contained in a fiber of π^m_{m-e}. We again do this in several steps.

Step 4. Let $\gamma\in X_\infty$ such that $\psi_m(\gamma)=\gamma_m$. Consider now $\gamma'_m\in f_m^{-1}(f_m(\gamma_m))$ and $\gamma'\in X_\infty$ such that $\psi_m(\gamma')=\gamma'_m$. To prove what we want, it is enough to construct an element $\tilde\gamma\in X_\infty$ such that

$$f_\infty(\tilde\gamma)=f_\infty(\gamma')\quad\text{and}\quad \psi_{m-e}(\tilde\gamma)=\psi_{m-e}(\gamma). \tag{6}$$

(Note that this is sufficient since the stability of C^e puts $\tilde\gamma$ in C^e, on which f_∞ is injective, and therefore $\tilde\gamma=\gamma'$.) To this end, starting with $\gamma^m:=\gamma_m$, we construct by induction on $k\geq m$ a sequence of jets $\gamma^k\in X_k$ satisfying the properties:

(i) $f_k(\gamma^k)=f_k(\psi_k(\gamma'))$.

(ii) $\pi^{k+1}_{k-e}(\gamma^{k+1})=\pi^k_{k-e}(\gamma^k)$.

By the definition of a projective limit, this sequence of jets produces an element $\tilde\gamma\in X_\infty$, which then clearly satisfies the conditions (6).

Step 5. Let's first see what are the options for satisfying condition (ii) above. Assume that for some $k\geq m$ we have constructed γ^k. Fix an arbitrary lifting $\beta^{k+1}\in X_{k+1}$ of γ^k. We prove that there is a bijection

$$\{\alpha^{k+1}\in X_{k+1}\mid \pi^{k+1}_{k-e}(\alpha^{k+1})=\pi^k_{k-e}(\gamma^k)\}\longrightarrow \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x},(t^{k-e+1})/(t^{k+2})\right),$$

where we consider $\mathbf{C}[\![t]\!]/(t^{k+2})$ as an $\mathcal{O}_{X,x}$-module via the jet β^{k+1}. Indeed, we can interpret the differences $\alpha^{k+1}-\beta^{k+1}$ as derivations, and the proof of the statement is completely identical to that in Step 1.

Step 6. We now focus on condition (i). We use as before $f_{k+1}(\beta_{k+1})$ as a fixed "anchor" lifting of $f_k(\gamma^k)$. Having done this, as in the previous step all other elements in Y_{k+1} having the same truncation as $f_k(\gamma^k)$ in Y_{k-e} are parametrized by

$$\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{k-e+1})/(t^{k+2})\right).$$

In particular, the other lifting $f_{k+1}(\psi_{k+1}(\gamma'))$ of $f_k(\gamma^k)$ corresponds to such a derivation D. To conclude the inductive step, we need to find an element α^{k+1} as in Step 5. such that $f_{k+1}(\alpha^{k+1}) = f_{k+1}(\psi_{k+1}(\gamma'))$ (then we set this element to be γ^{k+1}). We conclude that this is equivalent to showing that D is in the image of the natural map

$$u : \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x},(t^{k-e+1})/(t^{k+2})\right) \longrightarrow \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{k-e+1})/(t^{k+2})\right).$$

Step 7. The natural surjection

$$t^{k-e+1}/t^{k+2} \longrightarrow t^{k-e+1}/t^{k+1}$$

induces a commutative diagram

$$\begin{array}{ccc}
\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x},(t^{k-e+1})/(t^{k+2})\right) & \xrightarrow{u} & \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{k-e+1})/(t^{k+2})\right) \\
\downarrow & & \downarrow \\
\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{X,x},(t^{k-e+1})/(t^{k+1})\right) & \xrightarrow{v} & \mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{k-e+1})/(t^{k+1})\right)
\end{array}$$

To show that $D \in \mathrm{Im}(u)$, let's first observe that $D \in \mathrm{Im}(v)$. Indeed, this follows from the inductive step: the image of D in $\mathrm{Der}_{\mathbf{C}}\left(\mathcal{O}_{Y,f(x)},(t^{k-e+1})/(t^{k+1})\right)$ corresponds to the element $f_k(\psi_k(\gamma'))$, which is the same as $f_k(\gamma^k)$ and hence puts D in the image of v.

Now all that's left to note is that u and v have the same cokernel, so that D also lies in the image of u as desired. But as in the discussion before Step 1. (as well as that in Step 3.), we can identify v, for instance, with the linear transformation of $\mathbf{C}[\![t]\!]/(t^{k+1})$-vector spaces

$$\left(\mathbf{C}[\![t]\!]/(t^{k+1})\right)^n \overset{\mathrm{Diag}(t^{e_1},\ldots,t^{e_n})}{\longrightarrow} \left(\mathbf{C}[\![t]\!]/(t^{k+1})\right)^n,$$

so that

$$\mathrm{Coker}(v) \simeq \bigoplus_{i=1}^{n} \mathbf{C}[\![t]\!]/(t^{e_i}).$$

The exact same thing can be said about $\mathrm{Coker}(u)$, replacing $k+1$ with $k+2$.

Appendix: Derivations (see e.g. [Ma] §25). Let k be a ring, A a k-algebra, and M an A-module. A k-*derivation* from A to M is a map $D : A \to M$, satisfying the following two properties:

- $D(\lambda(a+b)) = \lambda D(a) + \lambda D(b)$ for all $\lambda \in k$ and $a, b \in B$.
- $D(ab) = aD(b) + bD(a)$ for all $a, b \in A$.

The set of all such derivations is denoted $\mathrm{Der}_{\mathbf{C}}(A, M)$. A fundamental property is the isomorphism

$$\mathrm{Der}_{\mathbf{C}}(A,M) \simeq \mathrm{Hom}_A(\Omega_{A/k}, M), \tag{7}$$

where $\Omega_{A/k}$ is the A-module of differentials of A over k.

Given an algebra homomorphism $A \to B$ (which in particular turns B into a k-algebra as well) and a B-module N, there is an induced exact sequence

$$0 \longrightarrow \mathrm{Der}_A(B, N) \longrightarrow \mathrm{Der}_{\mathbf{C}}(B, N) \longrightarrow \mathrm{Der}_{\mathbf{C}}(A, N) \tag{8}$$

which is induced via the isomorphism above from the standard exact sequence on differentials

$$\Omega_{A/k} \otimes_A B \longrightarrow \Omega_{B/k} \longrightarrow \Omega_{A/k} \longrightarrow 0.$$

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CHAPTER 6. MOTIVIC INTEGRATION

Contents

In this chapter we introduce the motivic measure and motivic integrals on smooth varieties, following an introductory section describing the Grothendieck ring of varieties and universal Euler characteristics. We prove the Change of Variables Formula, and use it to deduce the fact that K-equivalent varieties have the same Hodge numbers (all of these results are due to Kontsevich). The main references I will use are the lecture notes [Bl] and [La]. Other useful sources that I will use at times are [Cr], [Lo], [Mu] and [Ve].

1. Grothendieck ring and generalized Euler characteristics

Kontsevich's idea was to replace the Haar measure from the case of p-adic integration with a measure taking values in the Grothendieck ring of all varieties over the complex numbers.

Definition 1.1. The *Grothendieck group* $K_0(\mathrm{Var}_{\mathbf{C}})$ of complex algebraic varieties is the group generated by classes $[X]$ associated to each such variety, subject to the relations

- $[X] = [Y]$ if X and Y are isomorphic
- $[X] = [Z] + [X - Z]$ if $Z \subset X$ is a closed subset.

It is also called the *Grothendieck ring* if in addition one introduces the multiplication operation

$$[X] \cdot [Y] = [X \times Y].$$

The unit element for addition is $0 = [\emptyset]$, and for multiplication $1 = [\mathrm{pt}]$. It is also convenient to formally introduce the symbol

$$\mathbb{L} := [\mathbf{A}^1].$$

Example 1.2. By inductively using the usual affine cover of $\mathbf{P}^n$, we get

$$[\mathbf{P}^n] = 1 + [\mathbf{A}^1] + \ldots + [\mathbf{A}^n] = 1 + \mathbb{L} + \ldots + \mathbb{L}^n.$$

(Note that this is analogous to the formula $|\mathbf{P}^n(\mathbf{F}_q)| = 1 + q + \ldots + q^n$, analogy which will be developed further.)

Example 1.3 (Constructible sets)**.** Every constructible subset C of an algebraic variety X defines a class in $K_0(\mathrm{Var}_{\mathbf{C}})$. Indeed, write

$$C = \coprod_{i=1}^{k} S_i$$

where S_i are disjoint locally closed subsets of X. We can the write each S_i as $S_i = Y_i - Z_i$ with $Y_i, Z_i \subset X$ closed. This means we can define

$$[S_i] = [Y_i] - [Z_i], \;\; i = 1, \ldots, k$$

and then

$$[C] = \sum_{i=1}^{k} [S_i] = \sum_{i=1}^{k} [Y_i] - \sum_{i=1}^{k} [Z_i].$$

Exercise 1.4. Show that $K_0(\mathrm{Var}_{\mathbf{C}})$ is generated by the classes of smooth varieties. Even better, show that it is generated by the classes of smooth quasi-projective varieties. (See Theorem 1.18 below for a stronger statement.)

Exercise 1.5. Let $f : X \to Y$ be a piecewise trivial fibration with fiber F. Show that $[X] = [Y] \cdot [F]$.

The ring $K_0(\mathrm{Var}_{\mathbf{C}})$ is quite hard to understand and use directly; for instance, it is very difficult to know when two classes are equal. The main way it is used is via homomorphisms

$$K_0(\mathrm{Var}_{\mathbf{C}}) \longrightarrow R$$

where R is some more familiar ring. In the rest of this section we will study a few important such homomorphisms, which come from the topology or Hodge theory of algebraic varieties and are sometimes called *generalized Euler characteristics*. The main difficulty is that by virtue of the very definition of the Grothendieck ring, one needs to deal with the behavior of natural invariants on varieties which are not necessarily compact.

Euler characteristics. Let X be an algebraic variety over $\mathbf{C}$. We denote by $H^*_c(X, \mathbf{Q})$ the rational singular cohomology of X with compact support. Some properties of this cohomology theory are as follows (see [Ha] Ch.3 or [Fu] §22.c and §24.c):

- If X is compact, then $H^*_c(X, \mathbf{Q}) = H^*(X, \mathbf{Q})$.
- For each i one has an isomorphism

$$H^i_c(X, \mathbf{Q}) \simeq \varinjlim_{K} H^i(X, X - K; \mathbf{Q}),$$

where the limit is taken over all compact subsets $K \subset X$.

• If X is smooth, than the real version $H_c^*(X, \mathbf{R})$ can be computed as the de Rham cohomology defined by smooth forms with compact support.

• If $Y \subset X$ is a Zariski closed subset and $U = X - Y$, then there exists a long exact sequence

$$\ldots \longrightarrow H_c^i(U, \mathbf{Q}) \longrightarrow H_c^i(X, \mathbf{Q}) \longrightarrow H_c^i(Y, \mathbf{Q}) \longrightarrow H_c^{i+1}(U, \mathbf{Q}) \longrightarrow \ldots \tag{1}$$

• (Künneth formula.) For any two varieties X, Y and any i one has a natural isomorphism

$$H_c^i(X \times Y, \mathbf{Q}) \simeq \bigoplus_{k+l=i} H_c^k(X, \mathbf{Q}) \otimes H_c^l(Y, \mathbf{Q}). \tag{2}$$

Definition 1.6. The *compactly supported Euler characteristic* of X is

$$\chi_c(X) := \sum_{i \geq 0} (-1)^i \dim H_c^i(X, \mathbf{Q}).$$

This Euler characteristic provides the first example of a ring homomorphism of the type mentioned above.

Lemma 1.7. *There is a ring homomorphism*

$$\chi_c : K_0(\mathrm{Var}_{\mathbf{C}}) \longrightarrow \mathbf{Z}, \ \ [X] \mapsto \chi_c(X).$$

Proof. The fact that χ_c is well defined on $K_0(\mathrm{Var}_{\mathbf{C}})$, i.e. that

$$\chi_c(X) = \chi_c(Y) + \chi_c(U)$$

whenever one has a decomposition $X = Y \cup U$ as above, follows from (1). The fact that it is a ring homomorphism follows from the Künneth formula (2). $\square$

We are however mostly interested in the usual Euler characteristic

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim H^i(X, \mathbf{Q}) = \sum_{i \geq 0} (-1)^i b_i(X),$$

even in the non-compact case. It turns out though that this is the same as the compactly supported one; this is a slightly deeper result.

Theorem 1.8. *If X is a complex algebraic variety, then $\chi_c(X) = \chi(X)$.*

Proof. (Sketch.) In general, if X is an oriented real manifold of dimension m, Poincaré duality in the non-compact case states that

$$H_c^i(X, \mathbf{Q}) \simeq H_{m-i}(X, \mathbf{Q}).$$

In particular, this quickly takes care of the case of smooth complex varieties.

Let now X be an arbitrary complex algebraic variety, and denote by Y its singular locus. We can take a resolution of singularities $\pi : X' \to X$ such that π is an isomorphism over $U = X - Y$, and let $E = \pi^{-1}(Y)$. From the smooth case we know

$$\chi(X') = \chi_c(X') \ \text{ and } \ \chi(U) = \chi_c(U).$$

By induction on dimension we can also assume $\chi(Y) = \chi_c(Y)$ and $\chi(E) = \chi_c(E)$. Therefore we have

$$\chi_c(X) = \chi_c(U) + \chi_c(Y) = \chi(U) + \chi(Y),$$

so it is enough to show that this last sum is equal to $\chi(X)$. Note also that the above implies that we know the identity

$$\chi(X') = \chi(U) + \chi(E). \tag{3}$$

To show the required identity, one uses the existence of a small classical smooth neighborhood of $Y \subset N$ such that

- N deformation retracts onto Y
- $\pi^{-1}(N)$ deformation retracts onto E.[1]

Now an application of the Mayer-Vietoris sequence on X gives

$$\chi(X) = \chi(U) + \chi(N) - \chi(N - Y),$$

and $\chi(N) = \chi(Y)$, so it is enough to prove that $\chi(N - Y) = 0$. This can be checked on X': note that $N - Y \simeq \pi^{-1}(N) - E$, while the Mayer-Vietoris sequence on X' gives

$$\chi(X') = \chi(U) + \chi(\pi^{-1}(N)) - \chi(\pi^{-1}(N) - E).$$

Using (3) and the fact that $\pi^{-1}(N)$ retracts to E, one obtains the desired conclusion. $\square$

Corollary 1.9. *There is a ring homomorphism*

$$\chi : K_0(\mathrm{Var}_{\mathbf{C}}) \longrightarrow \mathbf{Z}, \ \ [X] \mapsto \chi(X).$$

Example 1.10. This result is not true if we leave the world of complex varieties. For instance, take $X = S^1$, so that $X = \mathbf{R} \cup \mathrm{pt}$. We have $\chi(S^1) = 0$, while $\chi(\mathbf{R}) = \chi(\mathrm{pt}) = 1$.

Virtual Poincaré polynomial. Using work of Deligne on Hodge theory, one can go one step further with respect to the previous example, and take into account the behavior of the individual Betti numbers with respect to the Grothendieck ring.

Theorem 1.11 (Deligne)**.** *To every quasi-projective complex algebraic variety X, one can associate a* virtual Poincaré polynomial

$$P_X(t) \in \mathbf{Z}[t]$$

satisfying the following properties:

(i) If X is smooth and projective, then

$$P_X(t) = \sum_{i \geq 0} b_i(X) \cdot t^i.$$

(ii) If $Y \subset X$ is a closed subset, and $U = X - Y$, then

$$P_X(t) = P_Y(t) + P_U(t).$$

(iii) $P_{X \times Y}(t) = P_X(t) \cdot P_Y(t)$.

[1] The first condition follows since one can triangulate X using Y as a subcomplex. The second follows from the first and the properness of π.

Corollary 1.12. *There is a ring homomorphism*

$$P : K_0(\mathrm{Var}_{\mathbf{C}}) \longrightarrow \mathbf{Z}[t], \;\; [X] \mapsto P_X(t).$$

Proof. We use the fact that by Exercise 1.4 it is enough to define P on quasi-projective varieties, in which case we can use (ii) and (iii) in the Theorem above. $\square$

Remark 1.13. (1) Properties (i) and (ii) in the theorem uniquely determine the polynomial $P_X(t)$ for each X. Indeed, by induction on dimension, it is enough to characterize $P_U(t)$ with U smooth. General resolution of singularities shows that we can embed U as an open set in a smooth projective variety X, such that

$$X - U = \sum E_i \text{ is an SNC divisor.}$$

Using (ii) have

$$P_U(t) = P_X(t) - \sum P_{E_i}(t) + \sum P_{E_i \cap E_j}(t) - \ldots$$

which determines P_U by (i).

(2) As with the Euler characteristic, no such polynomial can exist for real varieties. Again consider the case of $S^1 = \mathbf{R} \cup \mathrm{pt}$. If P existed, we'd have

$$P_{S^1}(t) = 1 + t \;\text{ and }\; P_{\mathrm{pt}}(t) = 1,$$

which would then force $P_{\mathbf{R}}(t) = t$. But we can also consider two points $p_1, p_2 \in S^1$ (think the north and south pole), and the new decomposition $S^1 = \mathbf{R} \cup \mathbf{R} \cup p_1 \cup p_2$ then leads to a contradiction.

Remark 1.14. The proof of Theorem 1.11 uses Deligne's construction of a *weight filtration* on $H^*(X, \mathbf{Q})$, a first step towards mixed Hodge theory. I am planning to include this at some point.

Virtual Hodge polynomial. Further work of Deligne shows that we can refine the picture above to take into account the Hodge numbers as well. His result is the following:

Theorem 1.15 (Deligne)**.** *To every quasi-projective complex algebraic variety X, one can associate a* virtual Hodge polynomial

$$H_X(u, v) \in \mathbf{Z}[u, v]$$

satisfying the following properties:

(i) If X is smooth and projective, then

$$H_X(u, v) = \sum_{p,q \geq 0} (-1)^{p+q} \cdot h^{p,q}(X) \cdot u^p v^q.$$

(ii) If $Y \subset X$ is a closed subset, and $U = X - Y$, then

$$H_X(u, v) = H_Y(u, v) + H_U(u, v).$$

(iii) $H_{X \times Y}(u, v) = H_X(u, v) \cdot H_Y(u, v)$.

Exactly as in the case of the virtual Poincaré polynomial, we obtain:

Corollary 1.16. *There is a ring homomorphism*

$$H : K_0(\mathrm{Var}_{\mathbf{C}}) \longrightarrow \mathbf{Z}[u,v], \ \ [X] \mapsto H_X(u,v).$$

Remark 1.17. Continuing Remark 1.14, the theorem above follows Deligne's result saying that one can endow the complex cohomology with compact support of every complex quasi-projective variety with a natural mixed Hodge structure (extending the usual Hodge theory for smooth projective varieties): the associated graded pieces of the weight filtration on $H^*(X,\mathbf{Q})$ carry after complexification natural Hodge filtrations. Again, I am planning to include a discussion at some point.

There is an alternative approach to the existence of the virtual Poincaré and Hodge polynomials, which avoids mixed Hodge theory, but is based on another deep result, namely the Weak Factorization Theorem for birational maps [AKMW]. Using this result, Bittner [Bi] was able to give the following simpler description of the Grothendieck ring of varieties.

Theorem 1.18 (Bittner)**.** *Let $K_0'(\mathrm{Var}_{\mathbf{C}})$ be the ring generated by isomorphism classes of smooth projective varieties, subject to the following relations: if $Y \subset X$ is a smooth projective subvariety, then*

$$[X] - [Y] = [\mathrm{Bl}_Y(X)] - [E],$$

where $\mathrm{Bl}_Y(X)$ is the blow-up of X along Y and E is its exceptional divisor. Then the natural ring homomorphism

$$K_0'(\mathrm{Var}_{\mathbf{C}}) \longrightarrow K_0(\mathrm{Var}_{\mathbf{C}}), \ \ [X] \mapsto [X]$$

is an isomorphism.

Exercise 1.19. Using the theorem above, define P_X and H_X by reducing the arbitrary case to the smooth projective case, where they are well understood.

2. Motivic measure and integrals on smooth varieties

To understand the origins of the motivic measure, let's continue the parallel with the p-adic situation started at the beginning of Ch.V.

(1) We have $\mathbf{A}^n(\mathbf{Z}_p)(=\mathbf{Z}_p^n)$, together with a reduction mod p^m map

$$\psi_m : \mathbf{A}^n(\mathbf{Z}_p) \longrightarrow \mathbf{A}^n(\mathbf{Z}/p^m\mathbf{Z}).$$

On $\mathbf{A}^n(\mathbf{Z}_p)$ we have the Haar measure μ, normalized such that

$$\mu(\mathbf{A}^n(\mathbf{Z}_p)) = 1 = \frac{p^n}{|\mathbf{A}^n(\mathbf{Z}/p\mathbf{Z})|}.$$

(2) For any set $C_m \subset \mathbf{A}^n(\mathbf{Z}/p^{m+1}\mathbf{Z})$ we have

$$\mu(\psi_m^{-1}(C_m)) = \frac{|C_m|}{p^{n(m+1)}} = \frac{|C_m|}{|\mathbf{A}^n(\mathbf{Z}/p^{m+1}\mathbf{Z})|}. \tag{4}$$

In practice we used only the measure of subsets $C \subset \mathbf{A}^n(\mathbf{Z}_p)$ of the form $C = \psi_m^{-1}(C_m)$ as above. (This, assuming we know things are well defined, means that we could have even forgotten about the Haar measure and just used formula (4) directly.)

Extrapolating from the case of affine space, in the arc space situation we have

(1') For any $\mathbf{C}$-scheme X, the arc space X_∞ and jet schemes X_m, with truncation maps

$$\psi_m : X_\infty \longrightarrow X_m.$$

We look for a measure μ on X_∞, normalized such that $\mu(\mathbf{A}^n_\infty) = 1$.

(2') The role of sets C_m as above is played by constructible subsets $C_m \subset X_m$, while that of C by the cylinders $C = \psi_m^{-1}(C_m)$. By analogy we look for a formula of the type

$$\mu(C) = \frac{|C_m|}{|(\mathbf{A}^n)_{m+1}|},$$

only this time the "number of points" does not make sense any more, and the symbol $|\cdot|$ will have to stand for something else (namely the class in the Grothendieck ring). This in particular leads to considering the following:

Definition 2.1. The ring $\mathcal{M}_\mathbf{C}$ is the localization of $K_0(\mathrm{Var}_\mathbf{C})$ in the class $\mathbb{L}$, i.e.

$$\mathcal{M}_\mathbf{C} := K_0(\mathrm{Var}_\mathbf{C})[\mathbb{L}^{-1}].$$

Let now X be a smooth complex variety of dimension n. As a preliminary step, we define the measure of a cylinder set in X_∞ as an element in $\mathcal{M}_\mathbf{C}$.[2]

Definition 2.2. Let $C \subset X_\infty$ be a cylinder, written as $C = \psi_m^{-1}(C_m)$, with $C_m \subset X_m$ a constructible set. The *motivic measure* of C is

$$\mu(C) := \frac{[C_m]}{\mathbb{L}^{n(m+1)}} \in \mathcal{M}_\mathbf{C}.$$

This is well defined: indeed, say $C = \psi_k^{-1}(C_k)$ as well, with $k > m$, and $C_k = (\pi^k_m)^{-1}(C_m)$. Then via π^k_m, C_k is an $\mathbf{A}^{n(k-m)}$-bundle over C_m, which in view of Exercise 1.5 gives

$$[C_k] = [C_m] \cdot \mathbb{L}^{n(k-m)}.$$

We also define the motivic measure of X as

$$\mu(X) := \mu(X_\infty).$$

Since $X_\infty = \psi_0^{-1}(X)$, we have

$$\mu(X) = \frac{[X]}{\mathbb{L}^n}.$$

In particular, we have $\mu(\mathbf{A}^n) = 1$.

The ring $\mathcal{M}_\mathbf{C}$ is a first step, but it is not yet the suitable answer for fully defining a measure. The problem is that one cannot take limits in it. To understand the convergence problem a bit better, and motivate the definition of motivic integrals at the same time,

[2] so something that typically looks like $[V]/\mathbb{L}^k$, where V is a variety.

let's go through a calculation analogous to that for p-adic integrals. Given a polynomial $f \in \mathbf{Z}_p[X_1, \ldots, X_n]$, we have computed

$$\int_{\mathbf{Z}_p^n} |f| d\mu = \sum_{m=0}^{\infty} \mu(\{x \mid |f(x)|_p = p^{-m}\}) \cdot p^{-m} = \sum_{m=0}^{\infty} \mu(\mathrm{ord}_f^{-1}(m)) \cdot \frac{1}{|\mathbf{A}^m(\mathbf{Z}/p\mathbf{Z})|}.$$ [3]

Now given $f \in \mathbf{C}[X_1, \ldots, X_n]$, or the divisor $D = (f = 0) \subset \mathbf{A}^n$, we can try to define completely analogously

$$\int_{X_\infty} \mathbb{L}^{-\mathrm{ord}_f} := \sum_{m=0}^{\infty} \mu(\mathrm{ord}_f^{-1}(m)) \cdot \frac{1}{\mathbb{L}^m}.$$

However the sum on the right cannot usually make sense in the ring $\mathcal{M}_{\mathbf{C}}$. We need to complete this ring in order to be able to talk about the (possible) convergence of such an infinite sum. Let's be even more concrete: we have seen that essentially by definition

$$\mathrm{Cont}^m(D) = \mathrm{ord}_f^{-1}(m) = \psi_{m-1}^{-1}(D_{m-1}) - \psi_m^{-1}(D_m).$$

Using the definition above, we then must have

$$\mu\left(\mathrm{ord}_f^{-1}(m)\right) = \frac{[D_{m-1}]}{\mathbb{L}^{nm}} - \frac{[D_m]}{\mathbb{L}^{n(m+1)}}.$$

In particular, we need to deal with the convergence of infinite sums whose general term is of the form $\frac{[D_{m-1}]}{\mathbb{L}^{(n+1)m}}$. Consider the following notion: the *virtual dimension* of a class $\frac{[V]}{\mathbb{L}^k} \in \mathcal{M}_{\mathbf{C}}$ is $\dim V - i$. We would like to arrange that classes in $\mathcal{M}_{\mathbf{C}}$ become "small" when their virtual dimension becomes very negative. This is motivated in part by the following general fact, which we assume for now:

Theorem 2.3. *Let $Z \subset X$ be any closed subscheme. Then there exists a constant $C = C_Z > 0$ such that*

$$\mathrm{codim}_{X_{m-1}} Z_{m-1} \geq C \cdot m \quad \text{for} \quad m \gg 0.$$

Using this fact, note that the virtual dimension of $\frac{[D_{m-1}]}{\mathbb{L}^{(n+1)m}}$ is at most $-C \cdot m$, tending to $-\infty$ as $m \to \infty$.

The solution to this convergence problem is to pass to a completion of $\mathcal{M}_{\mathbf{C}}$ with respect to the following filtration induced by the virtual dimension. Define for each k the subgroup $F^k\mathcal{M}_{\mathbf{C}}$ spanned by the classes $\frac{[V]}{\mathbb{L}^i}$ with $\dim V - i \leq -k$. We have

$$\cdots \subset F^k\mathcal{M}_{\mathbf{C}} \subset F^{k-1}\mathcal{M}_{\mathbf{C}} \subset \cdots \quad \text{and} \quad F^k\mathcal{M}_{\mathbf{C}} \cdot F^l\mathcal{M}_{\mathbf{C}} \subset F^{k+l}\mathcal{M}_{\mathbf{C}}$$

so as usual this filtration defines a topology on $\mathcal{M}_{\mathbf{C}}$.

Definition 2.4. We define

$$\widehat{\mathcal{M}_{\mathbf{C}}} = \lim_{\substack{\longleftarrow \\ m}} \mathcal{M}_{\mathbf{C}}/F^k\mathcal{M}_{\mathbf{C}}$$

i.e. the completion of $\mathcal{M}_{\mathbf{C}}$ with respect to the filtration $F^\bullet\mathcal{M}_{\mathbf{C}}$.

[3] Recall that $|f(x)|_p = p^{-m}$ is equivalent to $\mathrm{ord}_p(f(x)) = m$. I denoted by ord_f the function $\mathrm{ord}_f(x) := \mathrm{ord}_p(f(x))$, to make it look like the order function in the arc case.

In concrete terms, given a sequence of elements

$$\alpha_p = \frac{[V_p]}{\mathbb{L}^{i_p}} \in \mathcal{M}_{\mathbf{C}},$$

in $\widehat{\mathcal{M}_{\mathbf{C}}}$ we have

$$\alpha_p \to 0 \iff \dim V_p - i_p \to -\infty.$$

Exercise 2.5. Show that $\sum_n \alpha_n$ converges in $\widehat{\mathcal{M}_{\mathbf{C}}}$ if and only if $\alpha_n \to 0$ when $n \to \infty$.

For instance, by what we said earlier, the sum with terms $\frac{[D_{m-1}]}{\mathbb{L}^{(n+1)m}}$ considered above makes sense in $\widehat{\mathcal{M}_{\mathbf{C}}}$.

Remark 2.6. It is not known at the moment whether the natural map $\mathcal{M}_{\mathbf{C}} \to \widehat{\mathcal{M}_{\mathbf{C}}}$ is injective. Note that

$$\operatorname{Ker}\left(\mathcal{M}_{\mathbf{C}} \longrightarrow \widehat{\mathcal{M}_{\mathbf{C}}}\right) = \bigcap_{m \geq 0} F^m \mathcal{M}_{\mathbf{C}}.$$

Exercise 2.7. Show that in $\widehat{\mathcal{M}_{\mathbf{C}}}$ one has for each p:

$$\sum_{i=1}^{\infty} \mathbb{L}^{-pi} = \frac{1}{1 - \mathbb{L}^{-p}}.$$

We are now ready to give the definition of a measurable function with respect to the motivic measure, and of the associated motivic integral.

Definition 2.8. Let X be a smooth complex variety. Let

$$F : X_\infty \longrightarrow \mathbf{N} \cup \{\infty\}$$

be a function such that

- $F^{-1}(m)$ is a cylinder for every $m \in \mathbf{N}$.
- $F^{-1}(\infty)$ has measure 0 (i.e. it is the intersection of a decreasing family of cylinder sets whose measure tends to 0 in $\widehat{\mathcal{M}_{\mathbf{C}}}$).

Such a function is called *measurable*. The *motivic integral* of a measurable function F is defined as

$$\int_{X_\infty} \mathbb{L}^{-F} = \sum_{m=0}^{\infty} \mu\left(F^{-1}(m)\right) \cdot \frac{1}{\mathbb{L}^m},$$

provided the right-hand-side converges in $\widehat{\mathcal{M}_{\mathbf{C}}}$. In particular, if $Z \subset X$ is a closed subscheme we can consider $F = \operatorname{ord}_Z$, and

$$\int_{X_\infty} \mathbb{L}^{-\operatorname{ord}_Z} = \sum_{m=0}^{\infty} \mu\left(\operatorname{Cont}^m(Z)\right) \cdot \frac{1}{\mathbb{L}^m}.$$

Exercise 2.9. Let $Z \subset X$ be a locally closed subvariety of a smooth variety. Show that Z_∞ has measure 0 with respect to the motivic measure on X_∞. This implies in particular that the function $F = \operatorname{ord}_Z$ defined above is indeed measurable.

Example 2.10. If $F \equiv 0$, then

$$\int_{X_\infty} \mathbb{L}^{-0} = \mu(X_\infty) = \frac{[X]}{\mathbb{L}^n}.$$

Example 2.11. Let $Z = \{0\} \subset X = \mathbf{A}^1$. We have

$$Z_m = \{0\} \subset X_m = \mathbf{A}^{m+1}.$$

Recall that we have

$$\mathrm{Cont}^m(Z) = \psi_{m-1}^{-1}(Z_{m-1}) - \psi_m^{-1}(Z_m) = \psi_{m-1}^{-1}(0) - \psi_m^{-1}(0),$$

which gives

$$\mu\left(\mathrm{Cont}^m(Z)\right) = \frac{1}{\mathbb{L}^m} - \frac{1}{\mathbb{L}^{m+1}}.$$

We obtain

$$\int_{\mathbf{A}^1_\infty} \mathbb{L}^{-\mathrm{ord}_{\{0\}}} = \sum_{m\geq 0}\left(\frac{1}{\mathbb{L}^m} - \frac{1}{\mathbb{L}^{m+1}}\right)\cdot\frac{1}{\mathbb{L}^m} = \left(1-\frac{1}{\mathbb{L}}\right)\cdot\sum_{m\geq 0}\frac{1}{\mathbb{L}^{2m}} =$$

$$= \frac{\mathbb{L}-1}{\mathbb{L}}\cdot\frac{1}{1-\mathbb{L}^{-2}} = \frac{\mathbb{L}}{\mathbb{L}+1}.$$ [4]

Example 2.12. Generalizing the example above, let's consider a smooth divisor D in a smooth variety X, and compute $\int_{X_\infty}\mathbb{L}^{-\mathrm{ord}_D}$. We have

$$\mathrm{Cont}^m(D) = \psi_{m-1}^{-1}(D_{m-1}) - \psi_m^{-1}(D_m)$$

and therefore, for $m \geq 1$,

$$\mu(\mathrm{Cont}^m(D)) = \frac{[D_{m-1}]}{\mathbb{L}^{nm}} - \frac{[D_m]}{\mathbb{L}^{n(m+1)}} =$$

$$= [D]\cdot\mathbb{L}^{(n-1)(m-1)}\cdot\frac{1}{\mathbb{L}^{nm}} - [D]\cdot\mathbb{L}^{(n-1)m}\cdot\frac{1}{\mathbb{L}^{n(m+1)}} = [D]\cdot\left(\frac{1}{\mathbb{L}^{n+m-1}} - \frac{1}{\mathbb{L}^{n+m}}\right),$$

where the next to last identity uses the fact that D_k is a locally trivial $\mathbf{A}^{(n-1)k}$-bundle over D for each k. This finally gives

$$\int_{X_\infty}\mathbb{L}^{-\mathrm{ord}_D} = \sum_{m=0}^{\infty}\mu\left(\mathrm{Cont}^m(D)\right)\cdot\frac{1}{\mathbb{L}^m} = \frac{[X-D]}{\mathbb{L}^n} + [D]\cdot\sum_{m=1}^{\infty}\left(\frac{1}{\mathbb{L}^{n+m-1}} - \frac{1}{\mathbb{L}^{n+m}}\right)\cdot\frac{1}{\mathbb{L}^m} =$$

$$= \frac{[X-D]}{\mathbb{L}^n} + \frac{[D]}{\mathbb{L}^{n+1}}\cdot\sum_{m=1}^{\infty}\left(\frac{1}{\mathbb{L}^{m-1}} - \frac{1}{\mathbb{L}^m}\right)\cdot\frac{1}{\mathbb{L}^{m-1}} = \frac{[X-D]}{\mathbb{L}^n} + \frac{[D]}{\mathbb{L}^n\cdot(\mathbb{L}+1)}.$$

This example is generalized further by the following two exercises.

Exercise 2.13. Compute $\int_{X_\infty}\mathbb{L}^{-\mathrm{ord}_Z}$, where $Z \subset X$ is a smooth subvariety of codimension c.

[4] Compare this with the p-adic integral calculation $\int_{\mathbf{Z}_p}|x|d\mu = \frac{p}{p+1}$ in Ch.III.

Exercise 2.14. Let X be a smooth complex variety, and $D = \sum_{i=1}^k a_i D_i$ a divisor with simple normal crossings support on X. Show that $F = \mathrm{ord}_D$ is integrable, and

$$\int_{X_\infty} \mathbb{L}^{-\mathrm{ord}_D} = \sum_{J \subset \{1,\ldots,k\}} [D_J^0] \cdot \prod_{j \in J} \frac{\mathbb{L}-1}{\mathbb{L}^{a_i}-1},$$

where $D_J^0 := \cap_{j\in J} D_j - \cap_{j \notin J} D_j$. (This is combinatorially quite intricate – see e.g. [Cr] Theorem 1.17 or the exercises at the end of §2 in [Bl] – so start for instance with the cases $k = 1, 2$.)

3. Change of variables formula

The main result making the theory work is the following *Change of Variables Formula* for motivic integrals due to Kontsevich, and based on the Birational Transformation Theorem in the previous chapter.

Theorem 3.1. *Let $f : X \to Y$ be a proper birational morphism between smooth complex varieties, and let $F : Y_\infty \to \mathbf{N} \cup \{\infty\}$ be an integrable function with respect to the motivic measure. Then*

$$\int_{Y_\infty} \mathbb{L}^{-F} = \int_{X_\infty} \mathbb{L}^{-(F \circ f_\infty + \mathrm{ord}_{K_{X/Y}})}.$$

In particular, if $D \subset X$ is an effective divisor, then

$$\int_{Y_\infty} \mathbb{L}^{-\mathrm{ord}_D} = \int_{X_\infty} \mathbb{L}^{-\mathrm{ord}_{f^*D + K_{X/Y}}}.$$

Proof. Recall from Exercise 3.4 in Ch.IV that $f_\infty : X_\infty \to Y_\infty$ is surjective. For every $p \geq 0$, consider the cylinders

- $C_p = F^{-1}(p) \subset Y_\infty$.
- $D_p = (F \circ f_\infty)^{-1}(p) \subset X_\infty$.
- $D_{p,e} = D_p \cap \mathrm{Cont}^e(K_{X/Y}) \subset X_\infty$.

We have

$$\int_{Y_\infty} \mathbb{L}^{-F} = \sum_{p=0}^{\infty} \mu(C_p) \cdot \frac{1}{\mathbb{L}^p}$$

while

$$\int_{X_\infty} \mathbb{L}^{-(F \circ f_\infty + \mathrm{ord}_{K_{X/Y}})} = \sum_{l=0}^{\infty} \mu\left((F \circ f_\infty + \mathrm{ord}_{K_{X/Y}})^{-1}(l)\right) \cdot \frac{1}{\mathbb{L}^l} =$$

$$= \sum_{p=0}^{\infty} \sum_{e=0}^{\infty} \mu\left((F \circ f_\infty)^{-1}(p) \cap \mathrm{ord}_{K_{X/Y}}^{-1}(e)\right) \cdot \frac{1}{\mathbb{L}^{p+e}} = \sum_{p=0}^{\infty} \sum_{e=0}^{\infty} \mu(D_{p,e}) \cdot \frac{1}{\mathbb{L}^{p+e}}.$$

Claim: the set

$$C_{p,e} := f_\infty(D_{p,e}) \subset Y_\infty$$

is a cylinder set, and the induced map $D_{p,e} \to C_{p,e}$ is an $\mathbf{A}^e$-piecewise trivial fibration in the following sense: there exists some integer m, and locally closed subsets $C^m_{p,e} \subset Y_m$ and $D^m_{p,e} \subset X_m$ such that

$$C_{p,e} = \psi_p^{-1}(C^m_{p,e}) \text{ and } D_{p,e} = \psi_p^{-1}(D^m_{p,e}),$$

and the restriction

$$f_m : D^m_{p,e} \to C^m_{p,e}$$

is an $\mathbf{A}^e$-piecewise trivial fibration.

Assuming the Claim for the moment, let's conclude the proof of the Theorem. First, the $\mathbf{A}^e$-fibration assertion implies

$$\mu(D_{p,e}) = \mu(C_{p,e}) \cdot \mathbb{L}^e.$$

Going back to the formulas above, note then that

$$\sum_{e=0}^{\infty} \mu(D_{p,e}) \cdot \frac{1}{\mathbb{L}^{p+e}} = \sum_{e=0}^{\infty} \mu(C_{p,e}) \cdot \frac{1}{\mathbb{L}^p}.$$

To finish the proof, it remains to note that

$$\mu(C_p) = \sum_{e=0}^{\infty} \mu(C_{p,e}).$$

But this is clear, since we have

$$C_p = \left(\coprod_{e \geq 0} C_{p,e} \right) \cup C_{p,\infty}$$

where $C_{p,\infty}$ is a subset of E_∞ (where $E \subset X$ is the exceptional locus), hence a set of measure 0.

We are left with proving the Claim. By the definition of a measurable function, each C_p is a cylinder, so for say $m \gg e$ there exists constructible

$$C^m_p \subset X_m \quad \text{such that } C_p = \psi_m^{-1}(C^m_p).$$

By definition this implies also that

$$D_p = \psi_m^{-1}(D^m_p) \text{ with } D^m_p = f_m^{-1}(C^m_p)$$

and

$$D_{p,e} = \psi_m^{-1}(D^m_{p,e}) \text{ with } D^m_{p,e} = D^m_p \cap \psi_m \left(\mathrm{Cont}^e(K_{X/Y})\right).$$

Assume in particular that $m \geq 2e$. Then by the Birational Transformation Rule, Theorem 3.5 in Ch.V, we know that $\psi_m \left(\mathrm{Cont}^e(K_{X/Y})\right)$ is a union of fibers of f_m, as is D^m_p. This implies that

$$D^m_{p,e} = f_m^{-1}(C^m_{p,e}) \text{ with } C^m_{p,e} = f_m(D^m_{p,e}).$$

Now as $C_{p,e} = \psi_m^{-1}(C^m_{p,e})$, we see that $C_{p,e}$ is a cylinder. Moreover, by the same Birational Transformation Rule, the induced map

$$f_m : D^m_{p,e} \to C^m_{p,e}$$

is an $\mathbf{A}^e$-piecewise trivial fibration. $\square$

Note in particular that after a log-resolution, in the order function case the right-hand-side of the Change of Variables formula can be computed explicitly as in Exercise 2.14.

4. Kontsevich's theorem on K-equivalent varieties

Recall that in §1 we defined a few "motivic" invariants on $K_0(\mathrm{Var}_{\mathbf{C}})$, namely the Euler characteristic χ, the virtual Poincaré polynomial P, and the virtual Hodge polynomial H. Note that:

- $\chi(\mathbb{L}) = \chi(\mathbf{A}^1) = 1$.
- $P_{\mathbb{L}} = P_{\mathbf{A}^1} = P_{\mathbf{P}^1} - P_{\mathrm{pt}} = t^2$.
- $H_{\mathbb{L}} = H_{\mathbf{A}^1} = H_{\mathbf{P}^1} - H_{\mathrm{pt}} = uv$.

As a consequence, passing to the localization in $\mathcal{M}_{\mathbf{C}} = K_0(\mathrm{Var}_{\mathbf{C}})[\mathbb{L}^{-1}]$, we obtain ring homomorphisms

- $\chi : \mathcal{M}_{\mathbf{C}} \longrightarrow \mathbf{Z}$.
- $P : \mathcal{M}_{\mathbf{C}} \longrightarrow \mathbf{Z}[t, \frac{1}{t}]$.
- $H : \mathcal{M}_{\mathbf{C}} \longrightarrow \mathbf{Z}[u, v, \frac{1}{uv}]$.

Lemma 4.1. *Let $\overline{\mathcal{M}}_{\mathbf{C}} := \mathrm{Im}(\mathcal{M}_{\mathbf{C}} \to \widehat{\mathcal{M}_{\mathbf{C}}})$. Then χ, P and H factor through $\overline{\mathcal{M}}_{\mathbf{C}}$, i.e. we have an induced*

$$H : \overline{\mathcal{M}}_{\mathbf{C}} \longrightarrow \mathbf{Z}[u, v, \frac{1}{uv}],$$

and analogous statements for χ and P.

Proof. We show the statement for H; the others are analogous. We need to show that if

$$\alpha \in \mathrm{Ker}(\mathcal{M}_{\mathbf{C}} \longrightarrow \widehat{\mathcal{M}_{\mathbf{C}}}) = \bigcap_{m \geq 0} F^m \mathcal{M}_{\mathbf{C}},$$

then $H(\alpha) = 0$. Now by definition $F^m \mathcal{M}_{\mathbf{C}}$ is generated by classes $[V]/\mathbb{L}^i$, with $\dim V - i \leq -m$. For each such class, the associated virtual Hodge polynomial has degree $2 \dim V - 2i \leq -2m$, and so we obtain that

$$\deg\, H(\alpha) \leq -2m \quad \text{for all } m,$$

which implies that $H(\alpha) = 0$. $\square$

Remark 4.2. One can extend H to

$$H : \widehat{\mathcal{M}_{\mathbf{C}}} \longrightarrow \mathbf{Z}[\![u, v, \frac{1}{uv}]\!]$$

as well, and similarly for the other invariants.

We are finally able to prove the main result of the notes, improving Batyrev's theorem on Betti numbers discussed in Ch.IV.

Theorem 4.3 (Kontsevich)**.** *Let X and Y be smooth projective complex K-equivalent varieties. Then*

$$h^{p,q}(X) = h^{p,q}(Y) \text{ for all } p, q.$$

Proof. By Lemma 4.1, it is enough to show that

$$[X] = [Y] \in \widehat{\mathcal{M}_{\mathbf{C}}}$$

since then they have the same Hodge polynomial. We will in fact show $\mu(X) = \mu(Y)$. Recall that the fact that X and Y are K-equivalent means that there exists a smooth projective complex variety Z, and a diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

with f and g birational and $K_{Z/X} = K_{Z/Y}$. We apply the Change of Variables Formula, Theorem 3.1, to the function $F \equiv 0$ with respect to both f and g. Denoting $\dim X = \dim Y = n$, this gives

$$\frac{[X]}{\mathbb{L}^n} = \mu(X) = \int_{X_\infty} \mathbb{L}^{-0} = \int_{Z_\infty} \mathbb{L}^{-\mathrm{ord}_{K_{Z/X}}} = \int_{Z_\infty} \mathbb{L}^{-\mathrm{ord}_{K_{Z/Y}}} = \int_{Y_\infty} \mathbb{L}^{-0} = \mu(Y) = \frac{[Y]}{\mathbb{L}^n}.$$

□

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