Langlands correspondence for reductive groups over function fields

These are my live-TeXed notes for the course *Math G6659: Langlands correspondence for general reductive groups over function fields* taught by Michael Harris at Columbia, Spring 2016.

Any mistakes are the fault of the notetaker. Let me know if you notice any mistakes or have any comments!

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Introduction

This course will discuss one of the most exciting recent development in automorphic forms and number theory: the paper (still under revision) *Chtoucas pour les groupes reductifs et parametrisation de Langlands globale* of Vincent Lafforgue on the global Langlands correspondence over function fields. I have been thinking about his paper for a few years and more intensively in the past several months. The more I think about it the more I realize that my original intention was unrealistic due to the huge amount of technical material. I was trying the write up the notes for each lecture during the break, but every single day (at least 80 percent of the time) I realised the topic I was tring to discuss could well be a semester course in itself. To be more realistic, I will not try to explain the entire proof but rather to explain the background, present the framework of the Langlands correspondence and explain some highlights of the proof in details. We will also have five extra sessions on background of Langlands correspondence: including L-groups, the Satake isomorphism and representations of adelic groups and so on (the notetaker will not attend these due to scheduling conflicts but refers to his other notes for the background.)

Let us begin with the standard function field set-up. Let $q=p^r$ be a prime power and $k=\mathbb{F}_q$. Let X/k be a smooth irreducible projective curve and $F=k(X)$ its field of rational functions. Recall that the valuations on F corresponds bijectively to closed points of X over finite extensions of k. Let v a valuation of F and F_v be its completion. If $k(v)=\mathbb{F}_{q^{f_v}}$ is the residue field, then $F_v\cong k(v)((T))$, with its ring of integers $\mathcal{O}_v\cong k(v)[[T]]$. Notice there is only one kind of localization in the function field case, which simplifies things a bit (but not too much) compared to the number field case.

Let G be a connected reductive group over F. For simplicity we assume G is coming from base extension from a split group over k. We almost always assume G is semisimple (e.g., SL_n, PGL_n, Sp_{2n}, SO_n, E_6, E_7, E_8, F_4, G_2), or otherwise $G=GL_n$. Notice V. Lafforgue works in much more generality.

Recall that $G(\mathbb{A})$ is the restricted product $\prod' G(F_v)$ with respect to $\prod G(\mathcal{O}_v)$. Let $N=\sum n_v v$ be an effective divisor on X. Let $K_N=\{k\in\prod G(\mathcal{O}_v): k\equiv 1\pmod{m_v^{n_v}}\}$ be the open compact subgroup of level N. Let A be a coefficient ring (usually $A=\overline{\mathbb{Q}_\ell}$, $\ell\ne p$, sometimes $A=\mathbb{C}$). Recall:

Definition 1 We have the space of *automorphic forms of level* N *with coefficient in* A
$$\mathcal{A}(G,K_N,A)=C(G(F)\backslash G(\mathbb{A})/K_N,A).$$
Notice these automorphic forms are automatically locally constant. The space $\mathcal{A}(G,K_N,A)$ is the fixed space $\mathcal{A}(G,A)^{K_N}$ of the space of all automorphic forms

$$\mathcal{A}(G,A)=C_{u.l.c}(G(F)\backslash G(\mathbb{A}),A)=\bigcup_N \mathcal{A}(G,K_N,A).$$

Here the subscript means uniformly locally constant functions.

Remark 1 When $G=GL_n$ (or G is not semisimple), we choose a discrete subgroup $\Xi\subseteq Z(G)$ in the center so that $Z(F)\Xi\backslash Z(G)$ is compact. All I am about to say will be true after this modification.

The adelic group $G(\mathbb{A})$ acts on $\mathcal{A}(G,A)$ by right translation. The central question in the theory of automorphic forms is to decompose this space as $G(\mathbb{A})$-representations.

Let $\mathcal{A}_0(G,A)\subseteq\mathcal{A}(G,A)$ be the space of cusp forms. If A is algebraically closed, then we have a decomposition

$$\mathcal{A}_0(G,A)\cong\bigoplus_\pi n(\pi)\pi, \tag{1}$$

where π runs through (a countable set of) admissible irreducible representations of $G(\mathbb{A})$ and $n(\pi)$ is a non-negative integer. The central question is to determine the multiplicity $n(\pi)$ for any π. In particular, to determine which abstract representations π are automorphic (i.e. $n(\pi)>0$). Fix N, the question becomes to determine π with $n(\pi)>0$ and $\pi^{K_N}\neq 0$. The conjectural answer is provided by Langlands parameters:

Definition 2 Let $\Gamma=\pi_1(X-\mathrm{supp}(N))$ be the Galois group of the maximal separable extension of F unramified outside the support of N. A *Langlands parameter* (unramified outside N) is a homomorphism $\sigma:\Gamma\to\hat G(A)$, up to conjugacy by $\hat G(A)$, where $\hat G$ is the Langlands dual group of G.

Remark 2 When $G=GL(1)$, the conjecture boils down to class field theory. When $G=GL(n)$, Laurent Lafforgue proved the conjecture and in this case the multiplicity of automorphic representations is always 1.

Remark 3 To be more rigorous, the source of a Langlands parameter should be the conjectural *Langlands group*. In the function field case, there is no issue of archimedean places and the conjectural Langlands group should be the motivic Galois group of the category of motives realizing the space of automorphic forms of level N. This is the Galois group Γ, at least after taking the ℓ-adic realization: the independence of ℓ for the Langlands parameters is still an issue.

The beginning of V. Lafforgue's theorem is:

Theorem 1 There is a canonical decomposition

$$\mathcal{A}_0(G,K_N,A)\cong\bigoplus_\sigma\mathcal{A}_{0,\sigma},$$

where σ runs over Langlands parameters.

Remark 4 The theorem as stated is meaningless so far. The main point is that this decomposition is compatible with the the local Langlands correspondence (Satake parameters). In particular, each $\mathcal{A}_{0,\sigma}$ is a sum of π^{K_N}, i.e. this decomposition is an refinement of the decomposition (1).

Recall that the decomposition of $\mathcal{A}_0(G,A)$ is given by the action of the Hecke algebra

$$\mathcal{H}_N=C_c(K_N\backslash G(\mathbb{A})/K_N,A),$$

which is an algebra under convolution with identity $\mathbf{1}_{K_N}$. The Hecke algebra $\mathcal{H}_N$ acts on $\mathcal{A}_0(G,K_n,A)$ by convolution operators: for $f\in\mathcal{A}_0(G,K_N,A)$, $T\in\mathcal{H}_N$,

$$T(f)(g)=\int_{G(\mathbb{A})}f(gh)T(h)dh.$$

The Hecke algebra decomposes as restricted product of local Hecke algebras $\prod'\mathcal{H}_v$. The basic fact is that

Proposition 1 (Satake isomorphism) If $v\nmid N$, then

a. $\mathcal{H}_v$ is a commutative algebra over A, isomorphic to a polynomial algebra.
b. There is a canonical bijection between the characters $\mathrm{Hom}(\mathcal{H}_v,A)$ with semisimple conjugacy classes in $\hat G(A)$.

It follows that $\mathbb{T}_N=\prod'_{v\nmid N}\mathcal{H}_v$ is a commutative algebra and acts on $\mathcal{A}_0(G,K_N,A)$ and decomposes it into a direct sum over $\lambda\in\mathrm{Hom}(\mathbb{T}_N,A)$. Each λ gives a collection of semisimple conjugacy classes $\lambda_v\in\hat G(A)$ indexed by $v\nmid N$.

V. Lafforgue defines a new commutative algebra $\mathcal{B}_N$ of *excursion operators* acting on $\mathcal{A}_0(G,K_N,A)$. It contains the image of $\mathbb{T}_N$ and moreover connects to Galois representations: any Langlands parameter σ is in fact a character of $\mathcal{B}_N$!

More precisely, for any $h \in \mathrm{Hom}(\mathcal{B}_N, A)$, V. Lafforgue associates a Langlands parameter $\sigma_h : \Gamma \to \hat{G}(A)$. When $v \nmid N$, σ_h is unramified at v and hence defines a conjugacy class $\sigma_h(\mathrm{Frob}_v) \in \hat{G}(A)$. The crucial property of the decomposition in Theorem 1 is

Theorem 2 $\sigma_h(\mathrm{Frob}_v)$ corresponds to $h|_{\mathcal{H}_v}$ under the Satake isomorphism.

Remark 6 For π a globally generic automorphic representation, let $X(\pi)$ be the subset of $B(F)$-orbits of generic characters that π is generic with respect to. If π and π' are locally isomorphic everywhere but not globally, then the conjecture is that $X(\pi) \cap X(\pi') = \varnothing$. The different way to extend from the Hecke algebra to the excursion algebra should be indexed by these subsets of generic characters. When G is adjoint, one can compute that the set of generic character is a singleton and therefore the image of the Hecke algebra and the (semi-simplification of) the excursion algebra should be the same.

Blasius (for $SL(n)$) and Larsen (for F_4) have constructed examples of generic automorphic representations (over number fields) which are locally isomorphic but not globally. For $SL(n)$ there are n orbits of generic characters and these automorphic representations can be distinguished by these different orbits. But for F_4 (adjoint), there is only one orbit of generic characters and such examples of generic automorphic representations are unramified everywhere. The Whittaker functional $\pi_1^K \oplus \pi_2^K \to \mathbb{C}$ is nonzero on both factors and has 1-dimensional kernel. They can be detected by the global parameter (the character of the Hecke algebra extends to the excursion algebra in different ways). Question: can they be distinguished in a purely automorphic way (e.g., by non-Whittaker type of Fourier coefficients?)

Today we will introduce the objects which V. Lafforgue uses to construct the global Langlands parameter. One novelty of his work is that he does not construct the global parameter directly but instead he constructs some combinatorial invariant involving the Galois group and $\hat{G}$. He then uses geometric invariant theory to show that the combinatorial invariant is equivalent to a global parameter.

Definition 3 For $n \geq 0$ and an algebraic dimensional representation $\hat{G}^n \to \mathrm{Aut}(W)$, where $W = \otimes_{i=1}^n W_i$, V. Lafforgue, following Varshavsky, associates a moduli stack $\mathrm{Cht}^n_{N,W}$, the moduli stack of G*-shtukas of level* N *and paws (or legs) bounded by* W. It is a Deligne-Mumford (ind-)stack, hence can be thought of as a finite quotient of a scheme locally. For a test scheme S/k, $\mathrm{Cht}^n_{N,W}(S)$ classifies the following data:

a. An n-tuple $x = (x_1, \ldots, x_n)$ of S-points on $X - N$.
b. A G-torsor $\mathcal{G}$ over $X \times S$.
c. An isomorphism $\phi : \mathcal{G} \cong {}^{\tau}\mathcal{G}$ away from the union of the graphs of $x_i : S \to X$, where $\tau = (\mathrm{Id}_X \times \mathrm{Frob}_S)^*$, such that the relative position of ϕ at each x_i is bounded by the dominant weight of W_i (a coweight of G).
d. A trivialization of $(\mathcal{G}, \phi)$ along $N \times S$, i.e., an isomorphism $\mathcal{G}|_{N\times S} \cong G|_{N \times S}$.

Remark 7 One can view a G-torsor as a functor $\mathrm{Rep}(G) \to \mathrm{Bun}_X$. When $G = GL_n$, this is the same as a vector bundle of rank n. For general G, this may be thought of as a vector bundle with extra structures.

Example 1 When $G = GL_2$, $n = 2$, $N = v_1 + v_2$ consists of two points, $W_1 = \mathrm{Std}$ and $W_2 = \mathrm{Std}^\vee$, a point of $\mathrm{Cht}^2_{N,W}$ is exactly a Drinfeld shtukas with two legs with minuscule modification (a simple pole at v_1 and a simple zero at v_2).

Remark 8 Fixing a bound μ (dominant coweight of G) on the Harder-Narasimhan polygon gives an open substack $\mathrm{Cht}^{n,\leq\mu}_{N,W}$ of *finite type*, which is in fact represented by a scheme when N is sufficiently large.

Definition 4 Sending each G-shtukas to its paws defines a *paw morphism* $\mathfrak{p} : \mathrm{Cht}^n_{N,W} \to (X - N)^n$. We define a sheaf on $(X - N)^n$,

$$\mathcal{H}^{\leq\mu}_{N,n,W} = R^0\mathfrak{p}_!(\mathcal{IC}^{\cdot}_{\mathrm{Cht}^{n,\leq\mu}_{N,W}}),$$

whose stalks should be thought of as the middle intersection cohomology with compact support.

Theorem 3

a. When $W = \mathbf{1}$ is the trivial representation, $\mathrm{Cht}^n_{N,\mathbf{1}}$ is the constant discrete stack $G(F)\backslash G(\mathbb{A})/K_N$ over $(X - N)^n$. It follows that

$$\varinjlim_{\mu} \mathcal{H}_{N,n,\mathbf{Id}}^{\leq\mu} \cong C_c(G(F)\backslash G(\mathbb{A})/K_N, \overline{\mathbb{Q}_\ell}).$$

b. The map $W \mapsto \mathcal{H}_{N,n,W}^{\leq\mu}$ extends to an additive functor to $\mathrm{Rep}(\hat{G}^n)$.

c. The stalk of $\varinjlim_\mu \mathcal{H}_{N,n,W}^{\leq\mu}$ at a (good) geometric point contains the subspace $H_{n,W}$ of Hecke finite elements.

d. Each $H_{n,W}$ carries a monodromy action of Γ^n.

e. Given any morphism $\zeta : [m] \to [n]$ of finite sets of m and n elements. There is a natural projection $[\zeta] : \hat{G}^n \to \hat{G}^m$ and similarly $[\zeta] : \Gamma^n \to \Gamma^m$. For any $W \in \mathrm{Rep}(\hat{G}^m)$, composing with $[\zeta]$ defines $W^\zeta \in \mathrm{Rep}(\hat{G}^n)$. There is a canonical isomorphism

$$\chi_\zeta : H_{m,W} \cong H_{n,W^\zeta}$$

equivariant for the action of Γ^n (which acts on the left hand side via $[\zeta]$). Moreover,

$$\chi_{\zeta_1\circ\zeta_2} = \chi_{\zeta_1} \circ \chi_{\zeta_2}.$$

In fancier language, these data can be thought of as a morphism from the classifying stack of Γ to the classifying stack of $\hat{G}$.

f. There is a canonical action of the Hecke algebra $\mathcal{H}(G(\mathbb{A}), K_N)$ on each $H_{n,W}$ and all morphisms in (e) commute with this action.

Remark 9 All one really needs is the special case $[m] \hookrightarrow [m+1]$.

Remark 10 When $n = 0$, the space $H_{0,\mathbf{1}}$ is known as the *vacuum space*, if one thinks of the paws as moving particles. The vacuum space has no Galois action but for larger n, the space $H_{n,W}$ admits more and more Galois action. Drinfeld and L. Lafforgue computed $H_{2,\mathrm{Std}\otimes\mathrm{Std}^\vee}$ by comparing the Grothendieck-Lefschetz trace formula and Arthur-Selberg trace formula. V. Lafforgue, however, does not compute these spaces at all (except checking some commutative diagrams).

Remark 11 Over function fields, the space of Hecke finite compactly supported functions of level N (i.e., which spans a finite dimensional space under the unramified Hecke operators) is exactly the space of cups forms of level N. On the one hand, cusp forms are compactly supported by a theorem of Harder (see Borel-Jacquet 5.2). The space of cups forms of level N is finite dimensional, hence Hecke finite. On the other hand, let P be a rational maximal parabolic subgroup of G with Levi subgroup M, we would like to show that

$$f_P(g) := \int_{N_P(F)\backslash N_P(\mathbb{A})} f(ng)dn = 0.$$

We may assume P is standard, so $P \supseteq B$ and $M \supseteq T$. We may assume that $M(F_v) \cap K_v = M(\mathcal{O}_v)$ after conjugation. Since f is compactly supported, one can check that f_P is also compactly supported. On the other hand, the Satake transform $\mathcal{H}(G_v, K_v) \to \mathcal{H}(T_v, T(\mathcal{O}_v))$ is given by integration on the unipotent radical of B, hence factors through $\mathcal{H}(M_v, M(\mathcal{O}_v))$. Because f is Hecke finite over $\mathcal{H}(G_v, K_v)$ and $\mathcal{H}(M_v, M(\mathcal{O}_v))$ is a finite $\mathcal{H}(G_v, K_v)$-algebra, it follows that f_P is Hecke finite over $\mathcal{H}(M_v, M(\mathcal{O}_v))$. The center $Z = Z_M(F_v)/Z_M(\mathcal{O}_v)$ is infinite and preserves the space $\mathcal{H}(M_v, M(\mathcal{O}_v)) \cdot f_P$. Since f_P is compactly supported, this contradicts the Hecke finiteness unless $f_P = 0$.

When $G = GL_n$, any Galois representation $\nu : \Gamma \to \hat{G} = GL_n$ is uniquely determined by its trace function $\mathrm{tr}(\nu) \in \mathcal{O}(\hat{G})^{\hat{G}}$. For general G, a global parameter $v : \Gamma \to \hat{G}$ is not determined by any single invariant function. Instead, consider for any $n \geq 2$,

$$f \in \mathcal{O}(\hat{G}^{n-1} /\!\!/ \hat{G}) = (\mathcal{O}(\hat{G})^{n-1})^{\hat{G}} = \mathcal{O}(\hat{G}\backslash\hat{G}^n/\hat{G}).$$

When $n = 2$, f can be identified as a function on the maximal torus invariant under the Weyl group. In general, f can be thought of as a generalized matrix coefficient, namely there exists a triple (W, x, ξ) where

a. $W \in \mathrm{Rep}(\hat{G}^n)$,

b. $x \in W^{\hat{G}}$ an invariant vector of W,

c. $\xi \in (W^\vee)^{\hat{G}}$ an invariant covector of W, such that

$$f(g_1, \ldots, g_n) = \xi((g_1, \ldots, g_n)x).$$

It turns out any $\nu : \Gamma \to \hat{G}$ can be uniquely determined as a function on the space of triples (n, f, γ) given by $\xi(\nu(\gamma_1, \ldots, \gamma_n)x)$ where $\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma^n$. Our final goal is then to construct such functions on (n, f, γ) using the geometry of the moduli space of shtukas.

Definition 5 The invariant vector and covector can be thought of as a *creation operator* $x : \mathbf{1} \to W$ and an *annihilation operator* $\xi : W \to \mathbf{1}$. The diagonal map $\zeta : \hat{G} \to \hat{G}^n$ then induces $x : \mathbf{1} \to W^\zeta$ and $\xi : W^\zeta \to \mathbf{1}$. We define the *excursion operator*

$$S_{n,f,\gamma} \in \mathrm{End}(\mathcal{A}_0(G, K_N, \overline{\mathbb{Q}_\ell})) = \mathrm{End}(H_{0,\mathbf{1}}) \cong \mathrm{End}(H_{1,\mathbf{1}})$$

to be the composition:

$$H_{1,\mathbf{1}} \xrightarrow{[x]} H_{1,W^\zeta} \cong H_{n,W} \xrightarrow{(\gamma_1,\dots,\gamma_n)} H_{n,W} \cong H_{1,W^\zeta} \xrightarrow{[\xi]} H_{1,\mathbf{1}}.$$

It does not depend on the choice of the triple (W, x, ξ) representing f. Let $\mathcal{B}_N$ be the *excursion algebra* generated by the excursion operators.

Theorem 4

a. The excursion algebra $\mathcal{B}_N$ is commutative. For fixed (n, γ), $f \mapsto S_{n,f,\gamma}$ is an algebra morphism.

b. (Face relations) $S_{f,n,\gamma}$ satisfies natural relations subject to morphisms $\hat{G}^n \to \hat{G}^m$ and $\Gamma^n \to \Gamma^m$.

c. (Degeneracy relations) $S_{f,n,\gamma}$ satisfies natural relations subject to multiplications on $\hat{G}^n$ and Γ^n.

d. For fixed n, f, the homomorphism $\Gamma^n \to \mathrm{End}(\mathcal{A}_0(G, K_n, \overline{\mathbb{Q}_\ell}))$ is continuous under the the ℓ-adic topology.

e. The unramified Hecke algebra $\mathbb{T}_N \subseteq \mathcal{B}_N$. In fact, for $V \in \mathrm{Rep}(\hat{G})$, let $f_V(g_1, g_2) = \mathrm{Tr}_V(g_1 g_2^{-1})$, then $S_{2,f_V,(\mathrm{Frob}_v,1)} = h_{V,v} \in \mathcal{H}_v$, for $v \nmid N$.

Let $\nu : \mathcal{B}_N \to \overline{\mathbb{Q}_\ell}$ be a character. It turns out (see next section) from this theorem that $(n, f, \gamma) \mapsto \nu(S_{n,f,\gamma})$ is a $\overline{\mathbb{Q}_\ell}$-valued $\hat{G}$*-pseudo-representation* of Γ. Our next goal is to make the following theorem precise, hence reduce V. Lafforgue's construction of global Langlands parameters to Theorem 4.

Theorem 5 Any $\hat{G}$-pseudo-representation comes from a unique semisimple global Langlands parameter $\nu : \Gamma \to \hat{G}(\overline{\mathbb{Q}_\ell})$.

Remark 12 For GL_n, this is a theorem of R. Taylor. For general groups, it can be deduced from a theorem of Richardson, based on geometric invariant theory.

Pseudo-representations

Definition 6 Let R be a topological ring. Let G be a topological group with unit e. An R-valued *pseudo-representation* of G of dimension d is a continuous function $T : G \to R$ such that

a. $T(e) = d$ (sometimes also requiring that $d!$ is not a zero divisor in R, not needed for V. Lafforgue's work).

b. $T(g_1 g_2) = T(g_2 g_1)$ for any $g_1, g_2 \in G$.

c. $d \geq 0$ is the smallest integer with the following property: let S_{d+1} be the symmetric group on $d+1$ letters and $\mathrm{sgn} : S_{d+1} \to \{\pm 1\}$ be the sign character, then for all $g_1, \dots, g_{d+1} \in G^{d+1}$, the following holds:

$$\sum_{\sigma \in S_{d+1}} \mathrm{sgn}(\sigma) T_\sigma(g_1, \dots, g_{d+1}) = 0, \qquad (2)$$

Here if $\sigma = \sigma_1 \sigma_2 \cdots \sigma_s$ is the is the cycle decomposition, $\sigma_j = (i_1^{(j)} \cdots i_{r_j}^{(j)})$ has length r_j and

$$T_\sigma(g_1, \dots, g_{d+1}) = \prod_{j=1}^{s} T(g_{i_1^{(j)}} g_{i_2^{(j)}} \cdots g_{i_{r_j}^{(j)}}).$$

Remark 13 As we will see, the identity in (c) comes from the fact $(d+1)$-st exterior power of a d-dimensional vector space is zero.

Theorem 6 (Taylor, Rouquier)

a. Suppose $\rho : G \to GL(d, R)$ is a continuous representation, then its character $\mathrm{Tr}\,\rho$ is a pseudo-representation of dimension d.

b. Conversely, if R is an algebraically closed field of characteristic zero or $p > d$, then any d-dimensional pseudo-representation of G is the trace of a semisimple representation of dimension

d, unique up to homeomorphism.

Remark 14 Recall that the Brauer-Nesbitt theorem implies that the semi-simplification of a representation is determined by its character. This is the best possible one can do because one can only pin down a representation up to semi-simplification from its character.

Remark 15 Here is how Theorem 6 is used in deformation theory of Galois representations. Suppose $\mathcal{O}$ is a p-adic ring with maximal ideal $\mathfrak{m}$ and fraction field L. Suppose for any r, we have a torsion representation $\rho_r : G \to GL(d, \mathcal{O}/\mathfrak{m}^r)$ with the compatibility $\mathrm{Tr}(\rho_{r+1}) \equiv \mathrm{Tr}(\rho_r) \pmod{\mathfrak{m}^r}$. Then $T = \varprojlim \mathrm{Tr}(\rho_r)$ is a pseudo-representation. Hence Theorem 6 implies that T actually comes from a genuine representation $\rho : G \to GL(d, \overline{L})$ deforming all ρ_r's.

Proof Let us show that the character of a representation $\rho : G \to GL(d, R)$ satisfies the identity (2) in (c). Write $\Theta(g_1, \dots, g_{d+1}) : G^{d+1} \to R$ to be the left hand side of (2). Writing R as a quotient of two integral domains of characteristic zero, we reduce to the case that R is an algebraically closed field of characteristic 0. Let $V = R^d$ and $W = \mathrm{End}(V)$. We may reduce to the universal case $G = GL(d, R)$ and Θ can be viewed as a function on W^{d+1}.

We observe that Θ is invariant under S_{d+1} since the the cycle decomposition is invariant under conjugation and $T_\sigma(g_1, \dots, g_{d+1})$ is replaced by $T_{\xi\sigma\xi^{-1}}(g_{\xi(1)}, \dots, g_{\xi(d+1)})$ under the action of $\xi \in S_{d+1}$. We can extend Θ to a multi-linear map on $W^{\otimes d+1}$, then Θ is determined by its values on the subspace of symmetric tensors $\mathrm{Sym}^{d+1} W \subseteq W^{\otimes d+1}$ by the invariance under S_{d+1}.

Since $\mathrm{char}\, R = 0$, $\mathrm{Sym}^{d+1} W$ is spanned by symmetric tensors of the form $\Delta(w) = w \otimes w \cdots \otimes w$. It remains to show that $\Theta(\Delta(w)) = 0$ for all $w \in W$. It suffices to check semisimple w since these semisimple elements are Zariski dense in W. Define

$$\Xi = \Delta(w) \circ \sum_{\sigma \in S_{d+1}} \mathrm{sgn}(\sigma)\sigma \in \mathrm{End}(V^{\otimes d+1}).$$

We claim that $\mathrm{Tr}(\Xi) = \Theta(\Delta(w))$. Since the skew-symmetrization $A = \sum \mathrm{sgn}(\sigma)\sigma$ maps $V^{\otimes d+1}$ into $\bigwedge^{d+1} V = 0$, we know $\Xi = 0$. The claim then implies that $\Theta(\Delta(w)) = 0$ as desired.

It remains to prove the claim. Choose a basis $\{e_i\}$ of V so that w is diagonalized to be $\mathrm{diag}(\lambda_1, \dots, \lambda_d)$ under this basis. Then $V^{\otimes d+1}$ has a basis

$$e_{\mathbf{i}} = e_{\mathbf{i}(1)} \otimes \cdots \otimes e_{\mathbf{i}(d+1)},$$

where $\mathbf{i}$ runs all maps $[d+1] \to [d]$ and

$$\Delta(w) e_{\mathbf{i}} = \lambda_{\mathbf{i}} e_{\mathbf{i}}, \quad \lambda_{\mathbf{i}} = \prod_{j=1}^{d+1} \lambda_{\mathbf{i}(j)}.$$

Therefore

$$\mathrm{Tr}(\Xi) = \sum_{\mathbf{i}} \sum_{\sigma(e_{\mathbf{i}}) = e_{\mathbf{i}}} \mathrm{sgn}(\sigma)\lambda_{\mathbf{i}}.$$

Notice that $\sigma(e_{\mathbf{i}}) = e_{\mathbf{i}}$ if and only if $\mathbf{i}$ is *constant* on each cycle in the cycle decomposition of σ. It follows that

$$\mathrm{Tr}(\Xi) = \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{j=1}^{s} \mathrm{Tr}(w^{r_j}),$$

which is equal to $\Theta(\Sigma(w))$ by definition. □

For a character $\nu : \mathcal{B}_N \to A = \overline{\mathbb{Q}_\ell}$, Theorem 4 gives a continuous algebra homomorphism

$$\Theta_n^\nu : \mathcal{O}[\hat{G}^n /\!\!/ \hat{G}] \to C(\Gamma^n, A), \quad f \mapsto \nu(n, f, \gamma).$$

By the face relation, for a map $\zeta : [m] \to [n]$, we have

$$\Theta_n^\nu(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Theta_m^\nu(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)}).$$

By the degeneracy relation we have

$$\Theta_{n+1}^\nu(f^{+1})(\gamma_1, \dots, \gamma_{n+1}) = \Theta_n^\nu(f)(\gamma_1, \dots, \gamma_n\gamma_{n+1}),$$

where $f^{+1} \in \mathcal{O}[\hat{G}^{n+1} /\!\!/ \hat{G}]$ is given by

$$f^{+1}(g_1, \dots, g_{n+1}) = f(g_1, \dots, g_n g_{n+1}).$$

It turns out the collection $\{\Theta_n^\nu\}$ gives a global Langlands parameter $\sigma : \Gamma \to \hat{G}$.

Remark 16 When $G = GL(d)$, the global Langlands parameter can be constructed using Theorem 6. Notice that $\tau = \Theta_1(\mathrm{Tr\,Std})$ determines all $\Theta_n(f)$ since all representations of $GL(d)$ can be constructed from the standard representation using tensor powers. It remains to check that τ satisfies the identity (2). In fact, for $\sigma \in S_n$, one can compute that

$$\mathrm{Tr}(\sigma(g_1,\dots,g_n)|_{\mathrm{Std}^{\otimes n}}) = \prod_{j=1}^{s} \mathrm{Tr}(g_{i_1^{(j)}}\cdots g_{i_{r_j}^{(j)}}|_{\mathrm{Std}}).$$

Taking $f = \mathrm{Tr}(\sigma(g_1,\dots,g_n))$, the degeneracy relation (and Θ_n^ν is an algebra homomorphism) implies that

$$\Theta_n(f)(\gamma_1,\dots,\gamma_n) = \tau_\sigma(\gamma_1,\dots,\gamma_n).$$

Now take $n = d+1$ and take skew-symmetrization, we obtain the identity (2) for τ by using the fact $\bigwedge^{d+1}\mathrm{Std} = 0$.

Now let us consider general G. The construction of the global Langlands parameter does not follow directly from Theorem 6. We need more inputs from geometric invariant theory.

Definition 7 We say an n-tuple $(g_1,\dots,g_n) \in \hat{G}^n$ is *semisimple* if $\overline{\langle g_1,\dots,g_n\rangle}$, the Zariski closure of the subgroup generated by $g_1,\dots,g_n$ in $\hat{G}$, is semisimple.

Theorem 7 (Richardson) $\hat{G}$-orbit of $(g_1,\dots,g_n)$ (under conjugation) is Zariski closed in $\hat{G}^n$ if and only if $(g_1,\dots,g_n)$ is semisimple.

Remark 17 Notice by geometric invariant theory, the geometric points of GIT quotient $X /\!\!/ G$ is in bijection with closed orbits of the reductive G on the affine variety X. Thus $(\hat{G}^n /\!\!/ \hat{G})(A)$ is in bijection with $\hat{G}(A)$-conjugacy classes of semisimple n-tuples in $\hat{G}^n(A)$.

Definition 8 A homomorphism $\rho : \Gamma \to \hat{G}(A)$ is *semisimple* if whenever the image of ρ is contained in a parabolic $P \subseteq \hat{G}$, the $\mathrm{Im}(\rho)$ also contained in the Levi of P. In characteristic 0, this is the same as saying that the Zariski closure of $\mathrm{Im}(\rho)$ is reductive.

The following theorem constructs a global Langlands parameter σ from the collection Θ_n^ν, which makes Theorem 5 more precise.

Theorem 8 Let $\nu : \mathcal{B}_N \to A = \overline{\mathbb{Q}_\ell}$ be a homomorphism. Then there is a homomorphism $\sigma : \Gamma \to \hat{G}(\overline{\mathbb{Q}_\ell})$, unique up to conjugacy, such that

a. σ is continuous and takes values in $\hat{G}(E)$, for a finite extension $E/\mathbb{Q}_\ell$.
b. σ is semisimple.
c. σ corresponds to ν at unramified places under the Satake isomorphism.

Proof For any n-tuple γ, $f \mapsto \Theta_n^\nu(f)(\gamma)$ gives a homomorphism $\mathcal{O}[\hat{G}^n /\!\!/ \hat{G}] \to A$, i.e. a point in $(\hat{G}^n /\!\!/ \hat{G})(A)$. Let $\xi_n^{\mathrm{ss}}(\gamma)$ be the corresponding semisimple conjugacy class in $\hat{G}^n$ given by Theorem 7.

Choose n, $\gamma_1,\dots,\gamma_n$ and a representative $(g_1,\dots,g_n) \in \xi_n^{\mathrm{ss}}(\gamma_1,\dots,\gamma_n)$ so that

- (H1) $\overline{\langle g_1,\dots,g_n\rangle}$ is of maximal dimension.
- (H2) The centralizer $C(g_1,\dots,g_n)$ of $\langle g_1,\dots,g_n\rangle$ (which is also the centralizer of the Zariski closure) is of the smallest dimension with smallest number of connected components.

Let $\gamma \in \Gamma$, we define $\sigma(\gamma)$ to be $g \in \hat{G}(A)$ such that

$$(g_1,\dots,g_n,g) \in \xi_{n+1}^{\mathrm{ss}}(\gamma_1,\dots,\gamma_n,\gamma).$$

We need to verify the following:

- (A) Such g exists.
- (B) Such g is unique.
- (C) The map $\gamma \mapsto \sigma(\gamma)$ is a homomorphism.
- (D) The map $\gamma \mapsto \sigma(\gamma)$ is continuous.

Let $(h_1,\dots,h_n,h) \in \xi_{n+1}^{\mathrm{ss}}(\gamma_1,\dots,\gamma_n,\gamma)$. We will show that the n-tuple $(h_1,\dots,h_n)$ is semisimple. In fact, the *face relation* implies that $(h_1,\dots,h_n)$ lies over $\xi_n^{\mathrm{ss}}(\gamma_1,\dots\gamma_n)$. Theorem 5.2 of Richardson then implies that $\overline{\langle h_1,\dots,h_n\rangle}$ has a Levi isomorphic to $\overline{\langle g_1,\dots,g_n\rangle}$. By (H1), they must have the same dimension. Hence

$(h_1, \ldots, h_n)$ is semisimple and thus equal to $t^{-1}(g_1, \ldots, g_n)t$ for some t. Therefore $t(h_1, \ldots, h_n, h)t^{-1} \in \xi_{n+1}^{\mathrm{ss}}(\gamma_1, \ldots, \gamma_n, \gamma)$. We can then define $g = tht^{-1}$, which proves (A).

Since $C(g_1, \ldots, g_n, g) \subseteq C(g_1, \ldots, g_n)$, by (H2) we know that they must be equal. The uniqueness of g (B) then follows from the fact that g lies in the center of $C(g_1, \ldots, g_n)$. The *degeneracy relation* implies that $(g_1, \ldots, g_n, gg') \in \xi_{n+1}^{\mathrm{ss}}(\gamma_1, \ldots, \gamma_n, \gamma\gamma')$. Hence (C) follows from the uniqueness.

Notice σ takes value in a reductive group D, the center of $C(g_1, \ldots, g_n)$. To show (D), it suffices to show that for any $\phi \in \mathcal{O}[D]$, the composition $\phi \circ \sigma : \Gamma \to A$ is continuous. It follows from geometric invariant theory that the map

$$q : \mathcal{O}[\hat{G}^{n+1} /\!\!/ \hat{G}] \to \mathcal{O}[D], \quad f \mapsto (g \mapsto f(g_1, \ldots, g_n, g))$$

is *surjective*. If we lift ϕ to f, then by the construction of $\sigma(\gamma)$ we know that $\phi \circ \sigma$ is equal to the map

$$\Gamma \to A, \quad \gamma \mapsto \Theta_{n+1}^{\nu}(f)(\gamma_1, \ldots, \gamma_n, \gamma),$$

which is *continuous*. □

The rest of the course will focus on proving Theorem 4, using the geometry of moduli spaces of shtukas.

Moduli of G-bundles

Our next goal is to explain that $\mathrm{Cht}_{\varnothing, W, N}$, the moduli space of shtukas of level N with no paws, is the discrete stack $\mathrm{Bun}_G(k) \cong G(F) \backslash G(\mathbb{A}) / K_N$. Since the IC sheaf on the discrete stack is simply the constant sheaf $\overline{\mathbb{Q}_\ell}$, it follows that

$$R^0(\mathfrak{p}_{\varnothing, W, N})_! \overline{\mathbb{Q}_\ell} \cong C_c(G(F) \backslash G(\mathbb{A}) / K_N),$$

whose Hecke finite is exactly the space of cusp forms (Remark 11).

Proposition 2 Let X be a scheme over k. Let G be an affine group scheme over k. The following data (called a G-*bundle*) are equivalent:

a. A sheaf $\mathcal{P}$ on the fpqc site of X with a left action of G such that

$$G \times \mathcal{P} \cong \mathcal{P} \times_X \mathcal{P}, \quad (g, s) \mapsto (gs, s)$$

is an isomorphism and there exists an fpqc cover $Y \to X$ with $\mathcal{P}(Y) \neq \varnothing$ (local triviality in fpqc topology).

b. A scheme $\tilde{X} \to X$ with a left action of G and an fpqc cover $Y \to X$ such that $Y \times_X \tilde{X} \cong Y \times G$ in a G-equivariant way.

c. An fpqc scheme $\tilde{X} \to X$ with a left action of G such that

$$G \times \tilde{X} \cong \tilde{X} \times_X \tilde{X}, \quad (g, x) \mapsto (gx, x).$$

Proof (a) to (b): Take Y as in (a). Then $\mathcal{P}(Y) \neq \varnothing$ implies that $\mathcal{P}_Y \cong G_Y$ is trivial. The existence of the scheme $\tilde{X}$ then follows from fpqc descent for affine morphisms since $G_Y \to Y$ is affine.

(b) to (a): take $\mathcal{P} : T \mapsto \mathrm{Hom}_X(T, \tilde{X})$.

(b) to (c): it follows from fpqc descent for isomorphisms.

(c) to (b): take $Y = \tilde{X}$. □

Remark 18 If G is smooth, then any G-bundle is automatically locally trivial in the etale topology.

Remark 19 If X is a curve, then any G-bundle is automatically locally trivial in the Zariski topology.

Example 2 Let $G = GL(V) \cong GL(n)$. Let $\tilde{X}$ be a G-bundle over X. Then the associated bunlde $\tilde{V} = \tilde{X} \times^G V$ is a rank n vector bundle. This gives a functor from G-bundle on X to rank n vector bundle on X. It is an equivalence of groupoids. The inverse functor sends a vector bundle $\mathcal{V}$ on X to the fpqc sheaf $\mathcal{P}_{\mathcal{V}} : T \mapsto \mathrm{Iso}(V_T, \mathcal{V}_T)$.

More generally, let $\tilde{X}$ be a G-bundle over X. If Z is affine (or quasi-projective with a G-equivariant ample line bundle), then then quotient $\tilde{Z} := \tilde{X} \times^G Z$ always exists. This defines a functor

$$\mathrm{Rep}(G) \to \mathrm{Vect}(X), \quad V \mapsto \tilde{V}.$$

It is exact, commutes with direct sum, tensor product and sends the trivial representation to the trivial bundle.

Theorem 9 The category of G -bundles on X is equivalent to the category of exact tensor functors $\mathrm{Rep}(G) \to \mathrm{Vect}(X)$.

Example 3 The trival G -bundle corresponds to the functor $V \mapsto \tilde{V} \cong V \otimes \mathcal{O}_X$.

Proof Let us describe the inverse construction. Given $F : \mathrm{Rep}(G) \to \mathrm{Vect}(X)$, we would like to define a G -bundle $\tilde{X} \to X$. We can extend F to direct limits of finite dimensional represetations by

$$F(\varinjlim V_i) := \varinjlim F(V_i).$$

Since each $F(V_i)$ is a vector bundle on X and flatness is preserved under direct limit, we know that $F(\varinjlim V_i)$ is a flat $\mathcal{O}_X$ -module. Take $V = k[G]$ to be the regular representation, then $\mathcal{A} = F(V)$ is a $\mathcal{O}_X$ -algebra, flat as an $\mathcal{O}_X$ -module. Define the scheme $\tilde{X} = \mathrm{Spec}_X \mathcal{A}$. Then $\tilde{X}$ is flat over X and admits a G -action over X. By the exactness of F, the short exact sequence

$$0 \to k \to k[G] \to k[G]/k \to 0$$

induces an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{A} \to F(k[G]/k) \to 0.$$

Because $F(k[G]/k)$ is a flat $\mathcal{O}_X$ -module, we know that $\mathcal{O}_X \to \mathcal{A}$ remains injective when reducing any maximal ideal. In particular, $\mathcal{A}_x$ is not zero for all $x \in X$. Hence $\tilde{X} \to X$ is surjective. Since for any representation V, the G -representation $k[G] \otimes V$ is isomorphic to $k[G] \otimes \mathbf{V}$, where $\mathbf{V}$ is the vector space underlying V with trivial G -action. It follows from F is a tensor functor that

$$\tilde{X} \times_X \tilde{X} = \mathrm{Spec}_X(F(k[G] \otimes k[G])) \cong \mathrm{Spec}_X(F(k[G]) \otimes \mathcal{O}_X[G])) = \tilde{X} \times G.$$

is a G -equivariant isomorphism. □

Theorem 10 Let X/k be a projective scheme. Then the functor $\mathcal{F}_0 \to \mathcal{F}_{0,\bar{k}}$ induces an equivalence between the category of coherent sheaves on X and the category of pairs $(\mathcal{F}, \alpha)$, where $\mathcal{F}$ is a coherent sheaf on $X_{\bar{k}}$ and $\alpha : \mathcal{F} \cong {}^{\tau}\mathcal{F}$. Here if S/k is a scheme, then $\tau = \mathrm{Id} \times \mathrm{Frob}_q : S \times \bar{k} \to S \times \bar{k}$.

Proof Let $\mathcal{O}(1)$ be a very ample line bundle on X. The functor $\mathcal{F} \mapsto \bigoplus_{n \geq 0} H^0(X, \mathcal{F}(n))$ gives an equivalence between coherent sheaves on X and the quotient category of graded modules of finite type over $\bigoplus_{n \geq 0} H^0(X, \mathcal{O}(n))$ modulo the subcategory of eventually zero modules. We have a similar functor for $X_{\bar{k}}$, which also commutes with τ. Using this equivalence we are reduced to the case when X is a point, which follows from the following lemma (Galois descent over finite fields). □

Lemma 1 Let U, V be vector spaces over $\bar{k}$. Let $\lambda : U \to V$ be a $\bar{k}$ -linear map and let $\phi : U \to V$ be a Frob_q -semi-linear map. Let $U_0 = \ker(\lambda - \phi)$. Then

a. $U_0 \otimes_k \bar{k} \to V$ is injective if λ is injective;

b. $U_0 \otimes_k \bar{k} \to V$ is bijective if λ and ϕ are bijective.

Proof

a. Suppose $\{e_i\}$ is a k -basis of U_0, we need to show that $\{e_i\}$ are linearly independent over $\bar{k}$. Suppose there is a relation $\sum a_i e_i = 0$. Apply ϕ and use the definition of U_0 and the injectivity of λ, we obtain that $\sum a_i^q e_i = 0$. It follows that (a_i) is a scalar multiple of (a_i^q). Hence all $a_i \in k$, by Hilbert 90, thus the relation $\sum a_i e_i = 0$ must be trivial.

b. It suffices to show that $|U_0| = q^n$, if $\dim_{\bar{k}} U = n$. Let $Z \subseteq \mathbb{A}^n_{GL(n) \times GL(n)}$ be a closed subscheme defined by

$$g_1(x_1, \dots, x_n)^t = g_2(x_1^q, \dots, x_n^q)^t,$$

where $g_1, g_2 \in GL(n)$ are the matrices corresponding to λ, ϕ. Using the addition on $\mathbb{A}^n$, this becomes an affine group scheme over $GL(n) \times GL(n)$. One can check that it is also etale by computing the relative differential. Let $Z' \subseteq \mathbb{P}^n_{GL(n) \times GL(n)}$ be its Zariski closure. In homogeneous coordinates we have

$$x_0^{q-1} g_1(x_1, \dots, x_n)^t = g_2(x_1^q, \dots, x_n^q)^t.$$

So $x_0 =$ implies that all $x_i = 0$, i.e., $Z' = Z$. It follows that Z is in fact *finite* over $GL(n) \times GL(n)$. But the fiber over $(\mathrm{Id}, \mathrm{Id})$ is nothing but k^n, hence the fibers have constant cardinality q^n. □

Now let us come back to the situation that X is a smooth projective curve over $k = \mathbb{F}_q$. Let $\mathrm{Bun}_{G,N}(X_{\bar{k}})$ (resp. $\mathrm{Bun}_{G,N}(X)$) be the moduli stack of G bundles on $X_{\bar{k}}$ (resp. X) with a trivialization along N. We obtain the following corollary.

Corollary 1 $\mathrm{Bun}_{G,N}(X) = \{(\mathcal{G}, \alpha), \mathcal{G} \in \mathrm{Bun}_{G,N}(X_{\bar{k}}), \alpha : \mathcal{G} \cong {}^{\tau}\mathcal{G}\}$.

Remark 20 This nothing but $\mathrm{Cht}_{\varnothing,N}$, the moduli of shtukas with no paws.

Let us consider the case $G = GL(n)$. In this case $\mathrm{Bun}_{n,N} := \mathrm{Bun}_{G,N}(X)$ is the moduli stack of vector bundle of rank n with trivialization along N. Consider the tuples $(\mathcal{E}, \alpha_x, \gamma)$, where

- $\mathcal{E} \in \mathrm{Bun}_{n,N}(k)$,
- $\alpha_x : \mathcal{E}|_{\mathrm{Spec}\,\mathcal{O}_x} \cong \mathcal{O}_x^n$, where $x \in |X|$,
- $\gamma : X_\eta \cong F^n$, where η is the generic point of X.

From this tuple we can define $g_x : F_x^n \to F_x^n$ given by $\gamma \circ \alpha_x^{-1}$. Then $g_x \in GL(n, F_x)$. Since γ is integral at x for almost all P, we know that $(g_x)_{x \in |X|} \in GL(n, \mathbb{A})$. Forgetting γ and replacing α_x by the given trivialization $\alpha_N : \mathcal{E}|_{\mathrm{Spec}\,\hat{\mathcal{O}}/N} \cong (\hat{\mathcal{O}}/N)^n$ amounts to taking quotient by $GL(n, F)$ on one side and by K_N on the other side. Conversely, an element $g_x \in GL(n, \mathbb{A})$ gives a projective module of rank n over each affine open of X (since projective modules over a Dedekind domain are equivalent to its local data) and they glue together to a rank n vector bundle on X. So we obtain *Weil's uniformization*

$$\mathrm{Bun}_{n,N}(k) \cong GL(n, F) \backslash GL(n, \mathbb{A}) / K_N.$$

For general G, we need the following theorem (Hasse principle):

Theorem 11 (Steinberg, Borel-Springer, Kottwitz) Let G be a split reductive connected group. Then

$$\ker^1(F, G) = \{1\}.$$

Here

$$\ker^1(F, G) = \ker\left(H^1(F, G) \to \prod H^1(F_v, G)\right).$$

Notice any G-bundle over X becomes locally trivial by Lang's theorem and Hensel's lemma. This theorem implies that the generic fiber of any G-bundle over X is trivial as well. Hence we obtain the same uniformization for general G using Tannakian formalism (Theorem 9),

$$\mathrm{Bun}_{G,N}(k) \cong G(F) \backslash G(\mathbb{A}) / K_N.$$

Notice $\mathrm{Bun}_{n,N}^d$ is not of finite type as one can easily see from the example of $X = \mathbb{P}^1$. The Harder-Narasimhan (i.e. slope) filtration of vector bundle on curves naturally gives a stratification on $\mathrm{Bun}_{n,N}^d$ such that the strata with bounded slopes become finite type. For general G, we choose $T \subseteq B \subseteq G$ a maximal torus and a Borel. Let T^{ad} be the image of T in G^{ad}. Let $\mu \in X_*(T^{\mathrm{ad}}) \otimes \mathbb{Q}$, which one can think of as the "slopes" for a G-bundle.

Remark 21 For $G = GL(n)$, if a vector bundle $\mathcal{E}$ has slope corresponding to μ. Then one can see that the $GL(n)$-structure of $\mathcal{E}$ can be reduced to the parabolic subgroup P_μ associated to μ. Intuitively smaller parabolics correspond to more degenerate "cusps" of Bun_G. In fact, one can use this correspondence to give a geometric proof of Remark 11.

Definition 9 Define $\mathrm{Bun}_G^{\le \mu}$ to be the substack of Bun_G such that for any dominant weight $\lambda \in X_+^*(T^{\mathrm{ad}})$, any geometric point $s \in S$ and any B structure $\mathcal{B}$ on $\mathcal{G}_s$, we have $\deg \mathcal{B}_\lambda \le \langle \mu, \lambda \rangle$. Here a B*-structure* on $\mathcal{G}_s$ means a B-bundle $\mathcal{B}$ such that $\mathcal{B} \times^B G \cong \mathcal{G}_s$ and $\mathcal{B}_\lambda$ is the line bundle associated to $\mathcal{B}$ using the 1-dimensional representation λ of B.

Remark 22 The Harder-Narasimhan stratification on Bun_G parallels Arthur's truncation process in the trace formula. In fact a comparison between them already appeared in the work of L. Lafforgue.

Theorem 12

a. Bun_n^d is represented by an algebraic stack, locally of finite type.

b. $\mathrm{Bun}_n^{d, \le \mu}$ is an open substack.

c. If $\deg N$ is sufficiently large (relative to μ), then $\mathrm{Bun}_{n,N}^{d, \le \mu}$ is a smooth quasi-projective scheme.

Remark 23 All the statements are valid for general G. In fact, the $GL(n)$ case implies the general case: one can embed G into some $GL(n)$ to view G-bundles as vector bundles with G-structure and deduce the relative representativity of $\mathrm{Bun}_G \to \mathrm{Bun}_{GL(n)}$ from that the quotient $GL(n)/G$ is affine (since G is reductive). See Behrend's thesis, Sec 4.

Remark 24 The finite group $G(\hat{\mathcal{O}}/N\hat{\mathcal{O}})$ acts on the level structure and hence acts on the stack $\mathrm{Bun}_{n,N}^{\le\mu}$. Hence (c) implies that the quotient stack $\mathrm{Bun}_G^{d,\le\mu}$ is Deligne-Mumford and of finite type. This together with (b) implies (a).

Let us briefly sketch the proof of the algebraicity statement (c). Fix an ample line bundle $\mathcal{O}(1)$ on X. For fixed μ, there exists an integer m such that for any S/k, and any $\mathcal{E} \in \mathrm{Bun}_n^{\le\mu}$, the following (relative version of Serre's theorem, uniform in S) holds:

a. The direct image $p_*\mathcal{E}(m)$ is a vector bundle, where $p : X \times S \to S$ is the projection.
b. $R^1p_*\mathcal{E}(m) = 0$.
c. $\mathcal{E}$ is generated by $p^*(p_*(\mathcal{E}(m)))(-m)$.

Moreover, for fixed μ and m, when N with sufficiently larger degree, the vector bundle $p_*\mathcal{E}(m)$ is a subbundle of $p_*\mathcal{E}(m)|_{N\times S}$. Using the level N structure, we can then embed $\mathrm{Bun}_N^{d,\le\mu}$ into the moduli space classifying pairs (M, Q), where M is a subbundle of $p_*\mathcal{O}_{N\times S}^n$ of fixed rank and Q is a locally free quotient of $p^*M(m)$ of rank n and degree d. The latter moduli space is a generalized Grassmannian represented by a quasi-projective scheme (using Grothendieck's Quot scheme construction). The smoothness follows from the vanishing of H^2 for curves.

Affine Grassmannians and Beilinson-Drinfeld affine Grassmannians ▲

Let $\mathcal{O} = k[[t]]$ and $K = k((t))$. Let G be a split group. Let LG (resp. L^+G) be the loop (resp. positive loop) group. Let $\mathrm{Gr}_G = LG/L^+G$ be the affine Grassmannian. All these are ind-schemes over k. We have

$$LG(k) = G(K), \quad L^+G(k) = G(\mathcal{O}), \quad \mathrm{Gr}_G(k) = G(K)/G(\mathcal{O}).$$

I was notified by X. Zhu during the weekend that there are some gaps in the literature on the foundation of affine Grassmannians. His recent PCMI notes *Introduction to Affine Grassmannians* filled the gaps.

Definition 10 Recall we have the Cartan decomposition

$$G(K) = \coprod_{\mu \in X_*(T)^+} G(\mathcal{O})\mu(t)G(\mathcal{O}).$$

It defines an invariant map

$$\mathrm{inv} : G(\mathcal{O})\backslash G(K)/G(\mathcal{O}) \to X_*(T)^+.$$

For $\mu \in X_*(T)^+$, we define $\mathrm{Gr}_{\le\mu} \subseteq \mathrm{Gr}_G$ to classify pairs $(\mathcal{E}, \beta)$, where $\mathcal{E}$ is a G-bundle on the disk D_R and $\beta : \mathcal{E}|_{D_R^*} \cong G \times D_R^*$ is a trivialization on the puncture disk D_R^* such that the invariant $\mathrm{inv}(\beta) \le \mu$. We endow it with the closed reduced subscheme structure. We further define

$$\mathrm{Gr}_\mu = \mathrm{Gr}_{\le\mu} - \bigcup_{\lambda<\mu} \mathrm{Gr}_{\le\lambda}.$$

Proposition 3 The subscheme Gr_μ is a single L^+G-orbit. It is a smooth quasi-projective variety of dimension $\langle 2\rho, \mu\rangle$. Moreover, $\overline{\mathrm{Gr}_\mu} = \mathrm{Gr}_{\le\mu}$ is a (possibly singular) projective variety.

Proof The first claim follows from Cartan decomposition. We have

$$\mathrm{Gr}_\mu = L^+G/\mathrm{stab}(\mu(t)) = L^+G/L^+G \cap \mu(t) \cdot L^+G \cdot \mu(t)^{-1}.$$

It follows that

$$\dim \mathrm{Gr}_\mu = \dim L^+\mathfrak{g}/L^+\mathfrak{g} \cap \mathrm{Ad}(\mu(t))L^+\mathfrak{g} = \langle 2\rho, \mu\rangle.$$

The smoothness follows since it is an orbit of the group. The claim $\overline{\mathrm{Gr}_\mu} = \mathrm{Gr}_{\le\mu}$ can be seen by doing long combinatorics or reducing to the case of $GL(2)$ inductively. □

Remark 25 The crucial idea is that the $\overline{\mathbb{Q}_\ell}$-IC sheaves on $\mathrm{Gr}_{\le\mu}$ form a basis (when varying μ) of the abelian category of L^+G-equivariant perverse sheaves $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$ on Gr_G. Our next goal is to give a multiplicative structure (convolution product) on perverse sheaves and prove this product is commutative. This is the analogue of the commutativity of the spherical Hecke algebra. Under the sheaf-function dictionary this indeed recovers the convolution on the classical Hecke algebra. Moreover, taking (appropriate signed summation of) the cohomology provides a fiber functor. This gives $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$ a neutral Tannakian structure and hence we can abstractly identify $\mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$ with the category of representations of an affine group G'. The content of the *geometric Satake* *correspondence* is that G' is exactly the Langlands dual group $\hat{G}$. Under the sheaf-function dictionary, this indeed recovers the classical Satake isomorphism.

The proof (we will follow Richarz's proof) of geometric Satake requires global input: the Beilinson-Drinfeld affine Grassmannian (a global analogue of the affine Grassmannian).

Definition 11 We define the *Beilinson-Drinfeld affine Grassmannian* $\mathrm{Gr}_{G,I}^{I_1,\ldots,I_k}$ to be the (ind-)stack classifying the same data as the Hecke stack (see Definition 14) $\mathrm{Hecke}_{1,I}^{I_1,\ldots,I_k}$ (with no level structure) plus a trivialization for the last G-bundle $\mathcal{G}_k \cong G_{X\times S}$ along $\hat{D}$. Notice we recover the usual affine Grassmannian when there is only one point of modification ($k=1$, $\#I_k=1$).

The similar proof as in the case of affine Grassmannians gives:

Theorem 13 $\mathrm{Gr}_{G,I}$ is represented by an ind-scheme, ind-projective and of finite type over X^I.

Definition 12 Let Σ be the moduli space of relative effective Cartier divisors on X. Then Σ is represented by $\coprod_{n\geq 1} X^n/S_n$. Let $D=\sum \Gamma_{x_i}$ be such a divisor. We write $\hat{D}$ be the formal completion of X along D and $\hat{D}^*=\hat{D}-D$. We define the *Beilinson-Drinfeld Grassmannian*

$$\mathrm{Gr}_{G,X}=\cup_I \mathrm{Gr}_{G,I}.$$

It is an ind-scheme, ind-proper over Σ.

Definition 13 We define $\mathcal{L}G$ (resp. $\mathcal{L}^+G$) to be the *global loop group* (resp. the *global positive loop group*), classifying pairs (s,D), where $D\in\Sigma(R)$ and $s\in G(\hat{D}^*)$ (resp. $s\in G(\hat{D})$).

Theorem 14

a. $\mathcal{L}G$ is represented by an ind-group scheme over Σ. It classifies tuples $(D,\mathcal{G},\beta,\alpha)$, where $D\in\Sigma(R)$, $\mathcal{G}$ a G-torsor on X, β a trivialization of $\mathcal{G}$ away from D, α a trivialization along $\hat{D}$.

b. $\mathcal{L}G$ is represented by an ind-group scheme over Σ.

c. The map $\mathcal{L}G\to \mathrm{Gr}_{G,X}$ given by forgetting α is a right $\mathcal{L}^+G$-torsor and induces an isomorphism of fpqc sheaves over Σ,

$$\mathcal{L}G/\mathcal{L}^+G\cong \mathrm{Gr}_{G,X}.$$

Remark 26 The proof of geometric Satake is discussed in the course. Since we will only need its statement (Theorem 15), we refer to Richarz's paper for the proof and do not reproduce the lectures here.

Moduli of shtukas

Definition 14 Let I be a finite set and $I=I_1\coprod\cdots\coprod I_k$ be a partition. We define the *Hecke stack* $\mathrm{Hecke}_{N,I}^{I_1,\ldots,I_k}$ to be the moduli stack whose S-points are given by tuples consisting of:

a. $x_i\in(X-N)(S)$, $i\in I$.

b. $(\mathcal{G}_j,\alpha_j)\in \mathrm{Bun}_{G,N}(S)$, $j=0,\ldots,k$.

c. Isomorphisms

$$\phi_j:\mathcal{G}_{j-1}|_{X\times S-\cup_{i\in I_j}\Gamma_{x_i}}\cong \mathcal{G}_j|_{X\times S-\cup_{i\in I_j}\Gamma_{x_i}},\quad j=1,\ldots,k,$$

compatible with the level structure (α_i).

In other words, a point in the Hecke stack is a length k sequence of modifications of G-bundles and the j-th modification has prescribed location $(x_i)_{i\in I_j}$. We can further restrict the order of poles for these modifications.

Definition 15 Let $\mathbf{w}=(w_i)_{i\in I}$ be an I-tuple of dominant coweights of G. We define $\mathrm{Hecke}_{N,I,\leq\mathbf{w}}^{I_1,\ldots,I_k}$ to be the closed substack of $\mathrm{Hecke}_{N,I}^{I_1,\ldots,I_k}$ such that for all dominant weights λ of G,

$$\phi_j(\mathcal{G}_{j-1})_\lambda\subseteq(\mathcal{G}_j)_\lambda\left(\sum_{i\in I_j}\langle\lambda,w_i\rangle\Gamma_{x_i}\right).$$

Here $\mathcal{G}_\lambda$ is the vector bundle associated to $\mathcal{G}$ using the finite dimensional representation of G of highest weight λ.

Definition 16 We define the *moduli stack of shtukas* $\mathrm{Cht}_{N,I}^{I_1,\ldots,I_k}$ to be the stack over $(X-N)^I$ classifying the data (a-c) together with an isomorphism $\sigma : {}^\tau\mathcal{G}_0 \cong \mathcal{G}_k$ preserving the level structure. Thus we have a Cartesian diagram

$$\begin{array}{ccc} \mathrm{Cht}_{N,I}^{I_1,\ldots,I_k} & \longrightarrow & \mathrm{Hecke}_{N,I}^{I_1,\ldots,I_k} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{G,N} & \xrightarrow{(\mathrm{Id},\mathrm{Frob}_q)} & \mathrm{Bun}_{G,N}\times \mathrm{Bun}_{G,N}\ . \end{array}$$

Similarly define $\mathrm{Cht}_{N,I,\mathbf{w}}^{I_1,\ldots,I_k,\leq\mu}$.

Proposition 4 There is a canonical isomorphism

$$\beta_{I,\mathbf{w}}^{I_1,\ldots,I_k} : \mathrm{Hecke}_{I,w}^{I_1,\ldots,I_k} \to \mathrm{Gr}_{I,\mathbf{w}}^{I_1,\ldots,I_k} \times_{X^I} \mathrm{Bun}_{G,\Sigma n_i x_i} / G_{\sum n_i x_i}.$$

Here $\mathrm{Bun}_{G,\sum n_i x_i}$ parametrizes a G bundle G together with a trivialization along $\hat{D}$ modulo the equivalence mod $\mathfrak{m}_{x_i}^{n_i}$.

Remark 27 For Z an effective divisor on X , $G_Z = G_{\hat{Z}}/G(Z)$ (a finite dimensional quotient of two infinite dimensional objects). The quotient by $G_{\sum n_i x_i}$ concretely means forgetting the trivialization α along $\sum n_i x_i$. The morphism β is the same as a $G_{\sum n_i x_i}$ -torsor $\mathcal{A}$ over $\mathrm{Gr}_{I,\mathbf{w}} \times_{X^I} \mathrm{Bun}_{G,\sum n_i x_i}$. In terms of moduli interpretation, the above isomorphism is simply reorganizing the same data.

Remark 28 If n_i is sufficiently large relative to w_i , then the action of $G_{\sum \infty x_i}$ on $\mathrm{Gr}_{I,\mathbf{w}}$ factors through a finite dimensional quotient.

Definition 17 We define $\mathrm{Gr}_{I,\mathbf{w}} \subseteq \mathrm{Gr}_{I,\leq\mathbf{w}}$ be the reduced closed subscheme given by the Zariski closure of $(\prod \mathrm{Gr}_{i,w_i})|_{U^I}$, where $U^I \subseteq X^I$ is the complement of all diagonals. Define $\mathrm{Hecke}_{I,\mathbf{w}}$ be the inverse image of $\mathrm{Gr}_{I,\mathbf{w}}$.

Definition 18 More generally for any W a finite dimensional representation of $\hat{G}^I$, we define $\mathrm{Gr}_{I,W}^{I_1,\ldots,I_k} \subseteq \mathrm{Gr}_I^{I_1,\ldots,I_k}$ to be the union of $\mathrm{Gr}_{I,\mathbf{w}}^{I_1,\ldots,I_k}$ where the representation $V_{\mathbf{w}}$ appears in W . Define $\mathrm{Cht}_{N,I,W}^{I_1,\ldots,I_k} \subseteq \mathrm{Cht}_{N,I}^{I_1,\ldots,I_k}$ using $\mathrm{Hecke}_{I,W}^{I_1,\ldots,I_k}$ similarly.

We have a map analogous to the map β ,

$$\varepsilon_{N,I,W,(n_i)}^{I_1,\ldots,I_k} : \mathrm{Cht}_{N,I}^{I_1,\ldots,I_k,\leq\nu} \to \mathrm{Gr}_{I,W}^{I_1,\ldots,I_k} / G_{\sum n_i x_i}^{\mathrm{ad}}.$$

Proposition 5 This map ε is smooth of dimension equal to $\sum n_i \cdot \dim G$.

Proof Use the fact that Bun_G is smooth and the derivative of Frobenius is zero. □

Now we can restate the geometric Satake correspondence for all possible parameters (Theorem 1.17 in V. Lafforgue's paper).

Theorem 15 There is a canonical functor of tensor categories $W \mapsto \mathcal{S}_{I,W}^{I_1,\ldots,I_k}$ from the representations of $\hat{G}$ to $\mathrm{Perv}_{(\mathcal{L}+G)^I}(\mathrm{Gr}_I^{I_1,\ldots,I_k})^{\mathrm{ULA}}$ (universally locally acyclic equivariant perverse sheaves). The support of $\mathcal{S}_{I,W}$ lies in $\mathrm{Gr}_{I,W}$.

a. If W is irreducible, then $\mathcal{S}_{I,W}$ is the IC sheaf of $\mathrm{Gr}_{I,W}$.

b. Suppose $I_1,\ldots,I_k$ is a refinement of $I'_1,\ldots,I'_{k'}$, this induces natural maps $\pi_{I'_1,\ldots,I'_k}^{I_1,} : \mathrm{Gr}_I^{I_1,\ldots,I_k} \to \mathrm{Gr}_I^{I'_1,\ldots,I'_{k'}}$ (and similarly for Cht). These maps turns out to be *small* and hence maps IC sheaves to IC sheaves and $\mathcal{S}_{I,W}^{I_1,\ldots,I_k}$ to $\mathcal{S}_{I,W}^{I'_1,\ldots,I'_{k'}}$.

c. Suppose $W = \boxtimes W_j$, where $W_j \in \mathrm{Rep}(\hat{G}^{I_j})$. Then $\mathcal{S}_{I,W} \cong \tilde{\kappa}^*(\boxtimes \mathcal{S}_{I_j,W_j})$, where $\kappa : \mathrm{Gr}_{I,W} \to \prod \mathrm{Gr}_{I_j,W_j}/G_\Sigma$.

d. Suppose $\zeta : I \to J$ and $\Delta_\zeta : X^J \to X^I$. Let $W \in \mathrm{Rep}(\hat{G}^I)$ and $W^\zeta \in \mathrm{Rep}(\hat{G}^J)$ via pullback. Let $J = \coprod J_j$ and $I_j = \zeta^{-1}(J_j)$. Then we have isomorphisms, functorial in W ,

$$\Delta_\zeta^*(\mathcal{S}_{I,W}^{I_1,\ldots,I_j}) \cong \mathcal{S}_{J,W^\zeta}^{J_1,\ldots,J_j}.$$

Here

$$\Delta_\zeta : \mathrm{Gr}_J \to \mathrm{Gr}_I \times_{X^I} X^J \to \mathrm{Gr}_I\ .$$

Definition 19 If W is irreducible. We set $\mathcal{F}_{N,I,W}^{I_1,\ldots,I_k}$ over $(X-N)^I$ to be the IC sheaf of $\mathrm{Cht}_{N,I,W}^{I_1,\ldots,I_k}$. More generally, we define $\mathcal{F}_{N,I,W} = \varepsilon^* \mathcal{S}_{I,W}$.

Definition 20 Let $\mathfrak{p}_{N,I,W}^{I_1,\ldots,I_k} : \mathrm{Cht}_{N,I,W}^{I_1,\ldots,I_k} \to (X-N)^I$ be the paw morphism. We define
$$\mathcal{H}_{N,I,W}^{\le\nu} = R^0\mathfrak{p}_{N,I,W,!}^{I_1,\ldots,I_k,\le\nu}(\mathcal{F}_{N,I,W}^{I_1,\ldots,I_k}|_{\mathrm{Cht}_{N,I,W}^{I_1,\ldots,I_k,\le\nu}}).$$
Notice the result does not depend on the partition of I, which is a consequence of the smallness in b).

Excursion operators

The local system $\mathcal{H}_{N,I,W}^{\le\mu}$ corresponds to a $\overline{\mathbb{Q}_\ell}$-local system on the arithmetic etale fundamental group $\pi_1((X-N)^I)$, which is an extension of G_k by the geometric etale fundamental group. Drinfeld's lemma (see the next section) allows us to extend the action of $\pi_1((X-N)^I)$ to $\pi_1(X-N)^I$ (the latter has $|I|$ copies of G_k).

Proposition 6 Let $\zeta : I \to J$. Let (ρ, W) be a finite dimensional representation of $\hat{G}^I$. Then there is a canonical *coalescence isomorphism* $\kappa_\zeta : \Delta_\zeta^*(\mathcal{H}_{N,I,W}^{\le\mu}) \cong \mathcal{H}_{N,J,W^\zeta}^{\le\mu}$.

Proof It is enough to treat the case when ζ is injective and surjective.

a. Suppose ζ is injective. For simplicity let us assume W is irreducible. The shtukas in question involves no modification along $J-I$ (since W^ζ is a trivial representation of $\hat{G}^{J-I}$). So we have a canonical isomorphism
$$\mathrm{Cht}_{N,I,W} \times (X-N)^{J-I} \cong \mathrm{Cht}_{N,J,W^\zeta}.$$
(In particular, the $\mathrm{Cht}_{N,I,\mathbf{1}}$ is nothing but $\mathrm{Bun}_G(\mathbb{F}_q) \times (X-N)^I$.)

b. Suppose ζ is surjective. Then
$$\mathrm{Cht}_{N,I,W} \times_{(X-N)^I} (X-N)^J \cong \mathrm{Cht}_{N,J,W^\zeta}.$$
It follows from d) of Theorem 15 that $\mathrm{pr}_1^*(\mathcal{F}_{N,I,W}) \cong \kappa^*\mathcal{F}_{N,J,W^\zeta}$. The result then follows from proper base change. □

Taking $\zeta = \zeta_\varnothing^I : I \to I \sqcup \{0\}$ and $E \subseteq \overline{\mathbb{Q}_\ell}$ be the trivial representation, we obtain the isomorphism
$$\mathcal{H}_{N,I,W} \boxtimes E_{X-N} \cong \mathcal{H}_{N,J,W\otimes\mathbf{1}}.$$
Taking $\zeta = \zeta_J^I : I \sqcup J \to I \sqcup \{0\}$, we obtain the isomorphism
$$\mathcal{H}_{N,I\sqcup J,W\boxtimes U}|_{(X-N)^I\times\Delta(X-N)} \cong \mathcal{H}_{N,I\sqcup\{0\},W\otimes U^\zeta}.$$
Now combining these two isomorphisms we can define the creation/annihilation operators.

Definition 21 Let $x : \mathbf{1} \to U^{\zeta_J}$ be an invariant vector. Let $\mathcal{H}(x)$ be the corresponding morphism on cohomology. We define the *creation operator* to be $e_x^\#$ to be the composition
$$\mathcal{H}_{N,I,W} \boxtimes E \cong \mathcal{H}_{N,I\sqcup\{0\},W\boxtimes\mathbf{1}} \xrightarrow{\mathcal{H}(\mathrm{Id}\otimes x)} \mathcal{H}_{N,I\sqcup\{0\},W\boxtimes U^{\zeta_J}} \cong \mathcal{H}_{N,I\sqcup J,W\boxtimes U}.$$
Similarly define the *annihilation operator* $c_\xi^\flat$ for $\xi : U^{\zeta_J} \to \mathbf{1}$ an invariant covector.

Proposition 7

a. The creation and annihilation operators commute when the paws are disjoint.

b. The composition of two creation/annihilation operators correspond to disjoint set of paws is the creation/annihilation operators for their union.

Partial Frobenius operators

Reference: thesis of Eike Lau and L. Lafforgue.

Theorem 10 has the following consequence:

Corollary 2 Let X_0/k be smooth of finite type. Let L/k be algebraically closed. Let $F : \mathrm{Frob}_q \times \mathrm{Id} : X = X \times_k L \to X \times_k L$. Then the functor $Y_0 \to Y = Y_0 \times_k L$ gives an equivalence between finite etale covers $Y_0 \to X_0$ and finite etale covers $Y \to X$ together with an isomorphism $\beta : Y \cong F^*Y$.

Proof View Y as relative spectrum $\mathrm{Spec}_{\mathcal{O}_X}\mathcal{O}_Y$. For any $Y \to X$ finite etale, we have $\mathrm{Frob}_q^* Y \cong Y$ (see Stacks project 50.79). So the isomorphism β is equivalent to the isomorphism $\alpha : Y \cong {}^\tau Y$ since $\mathrm{Frob}_q = \tau \circ F$. The same argument as in Theorem 10 gives that $Y_0 \mapsto (Y, \alpha)$ is fully faithful.

The hard part is to show the essential surjectivity. By fully faithfulness it suffices to deal with the case X is affine. Let $\tilde{X}$ be a compactification, let $\tilde{Y}$ be the normalization of $\tilde{X}$ in the function field $E = L(Y)$. Because X is smooth, we know that $\tilde{Y}_X = X$. So we are in a situation of a normal morphism between projective schemes, which on an open part becomes etale. Since τ does not change the scheme (but only change the L-structure), ${}^\tau\tilde{Y}$ is the normalization of ${}^\tau X$ in ${}^\tau E$. Since normalization is canonical, it follows that ${}^\tau\tilde{Y} \cong \tilde{Y}$. By Theorem ##VLgaloisdescent applying to the projective scheme $\tilde{Y}$, we get $Y_0 = \tilde{Y}_0|_{X_0} \to X_0$. Since $Y \to X$ is etale, we know that $Y_0 \to X_0$ is also etale (by base change). □

Definition 22 Let $\pi(X)$ be the *fundamental groupoid* of X, whose objects are geometric points of X and morphisms are the isomorphisms $\mathrm{Hom}(x, x') = \mathrm{Iso}(\phi_x, \phi_{x'})$. Here ϕ_x is the fiber functor at P from the category of finite etale coverings of X to sets.

Definition 23 A *representation of* $\pi(X)$ is a functor from $\pi(X)$ to sets, namely, a collection of sets S_x together with $\gamma: S_x \cong S_{x'}$ for any $\gamma \in \mathrm{Hom}(x, x')$. Notice the category of representations of $\pi(X)$ is equivalent to the category of finite etale coverings of X.

Theorem 16 Let $X_1, \ldots, X_n$ be smooth schemes of finite type over a finite field k. Let $\overline{X_i} = X_i \times \bar{k}$. Let $\bar{X} = \prod \overline{X_i}$. Let $F_i : \bar{X} \to \bar{X}$ be the i-th partial Frobenius morphism $\mathrm{Id} \times \cdots \times F \times \cdots \mathrm{Id}$. Then

a. let Λ be a local system over $\Omega \subseteq X^I$ open dense, equipped with isomorphism $\phi_i : F_i^*\Lambda \to \Lambda$ over $F_i^{-1}(\Omega) \cap \Omega$ such that ϕ_i commute and $\prod \phi_i : \mathrm{Frob}_q^* \Lambda \cong \Lambda$ is the canonical isomorphism. Then there exists an $U \subseteq X$ open dense such that Λ extends a local system to U^I.

b. we have an equivalence $\pi(\bar{X})/(\prod F_i^{\mathbb{Z}}) \cong \prod \pi(X_i)$: namely, a representation of $\prod \pi(X_i)$ is the same as a finite etale covering $\bar{Y} \to \bar{X}$ plus compatible isomorphisms $\beta_i : \bar{Y} \cong F_i^* \bar{Y}$.

Proof

a. Assume Ω is the largest subset of X^I to which Λ extends. Let $Z = X^I - \Omega$. Then $F_i^{-1}(Z) = Z$ for all A. Say $Z = \cup Z_j$ is a union of irreducible components. We will show that each $Z_j \subseteq X^J \times \text{ finite sets}$ for some $J \subseteq I$, $|J| = |I| - 1$.

   a. First assume that $|I| = 2$. Let $\zeta \subseteq X^2$ be an irreducible divisor. Let $p_1, p_2 : \zeta \to X$ be the two projections. Suppose otherwise both projections are surjective. Let $d(\zeta) = \deg p_1 / \deg p_2 \in \mathbb{Q}_{>0}$. Then $d(F_1^*(\zeta)) = q \cdot d(\zeta)$. But $(F_1^*)^n(\zeta)$ is finite (only finitely many component), a contradiction. So every component of Z is either horizontal or vertical.

   b. For general I, let $q_i : X^I \to X^{I-i}$. Let $Z_i \subseteq Z$ be the maximal subset such that $q_i^{-1} q_i(Z_i) \subseteq Z$. Then $Z = Z_1 \cup Z_2$ by the previous case. We may then replace Z_i by their subsets of pure codimension 1. Hence $q_i(Z_i) \subseteq X^{I-i}$ is closed of pure codimension 1 and invariant under the remaining partial Frobenius. Then we induct on I to prove the claim.

b. For simplicity let us assume all X_i's in the theorem are the same. Let $\eta_n \in X^n$ be the geometric generic point. Now we apply Corollary 2 to $L_i = L(U^{I-i})^{\mathrm{alg}}$, we know for each i the finite etale cover $Y|_{U \times \mathrm{Spec}\, L_i} \cong Y_{0, L_i}$ for some Y_0. Thus Y factors through $\pi_1(U, \eta) \times \mathrm{Gal}_{I-i}$ for each i, hence factors through their intersection $\prod \pi_1(U, \eta)$ by part a). □

Remark 29 Notice though each partial Frobenius does not act on $\mathrm{Cht}_{N,I,W}$ directly, the action of $\pi_1((X-N)^I)$ on $\mathcal{H}_{N,I,W}$ does extend to an action of $\pi_1(X-N)^I$. In fact, the first partial Frobenius

$$\mathrm{Frob}_1 : \mathrm{Cht}_{N,I,W}^{1,\ldots,n} \to \mathrm{Cht}_{N,I,W}^{2,\ldots,n,1},$$

which lies over the map $\mathrm{Frob}_1(x_1, x_2, \ldots, x_n) = ({}^\tau x_1, x_2, \ldots, x_n)$. Because the cohomology $\mathcal{H}_{N,I,W}$ does not depend on the partition of I and the partial Frobenius induces an isomorphism of etale sites, we know that the Frob_1 induces an automorphism

$$F_i \in \mathrm{Aut}(\varinjlim_{\mu} \mathcal{H}_{N,I,W}^{\le \mu}).$$

(it increases μ but does not effect the Hecke finite classes). The composition of the n permutation

$$\mathrm{Frob}_n \circ \cdots \mathrm{Frob}_2 \circ \mathrm{Frob}_1$$

maps $\mathrm{Cht}_{N,I,W}^{1,\ldots,n}$ to itself, and is exactly the absolute Frobenius. Since F_i covers Frob_i on $(X-N)^I$, commutes with each other and $\prod F_i$ is the absolute Frobenius, by Theorem 16 we know that the action of $\pi_1((X-N)^I)$

extends to an action of $\pi_1(X-N)^I$.

Compatibility of excursion operators and Hecke operators

We now come to the last key point of this course, i.e., item e) in Theorem 4), which is Lemma 10.2/Prop. 6.2 in V. Lafforgue's paper:

Lemma 2 Let $I=\{1,2\}$, $W=V\boxtimes V^\vee$ an irreducible representation of $\hat{G}^I$. Let $v\in|X-N|$. Let $\gamma\in\Gamma_v$ have degree d under $\Gamma_v\to\mathrm{Gal}(\bar{k}/k)\cong\phi_v^{\hat{\mathbb{Z}}}$. Then $\mathcal{S}_{I,W,\mathrm{tr}_V,(\gamma,1)}$ depends only on d. In particular, the inertia I_v acts trivially. Moreover, if $d=1$, then $\mathcal{S}_{I,W,\mathrm{tr}\,V,(\gamma,1)}=T(h_{V,v})$.

Here is a rough strategy. Consider the Deligne-Mumford stack $\mathcal{Z}^{\{1,2\}}=\mathrm{Cht}_{N,\{1,2\},V\boxtimes V^\vee}^{\{1\},\{2\}}$ over $\Delta(v)$. Then one constructs two closed substack $\mathcal{Y}_i\hookrightarrow\mathcal{Z}^{\{1,2\}}$ ($i=1,2$) and together with morphisms $\alpha_i:\mathcal{Y}_i\to\mathcal{Z}=\mathrm{Cht}_{N,\varnothing,\mathbf{1}}$ such that

a. The first two stages of the excursion operator (creation and Frob_v) is realized by a cohomological correspondence supported on $\mathcal{Y}_1$ (after normalizing by a half-integral power of q).
b. The last stage (annihilation) is realized by a cohomological correspondence supported on $\mathcal{Y}_2$. Thus $\mathcal{S}_{V,v}$ is a cohomological correspondence supported on $\mathcal{Y}_1\times_{\mathcal{Z}^{\{1,2\}}}\mathcal{Y}_2$. Now:
c. The fiber product $\mathcal{Y}_1\times_{\mathcal{Z}^{\{1,2\}}}\mathcal{Y}_2$ *is* the usual etale Hecke correspondence $T(h_{V,v})$,
d. $S_{V,v}$ equals to this fiber product.