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Induction and Restriction of Character Sheaves

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To I. M. Gelfand on his 80th birthday

Introduction

This paper may be viewed as a continuation of [Gi]. In [Gi], we gave various equivalent definitions of character sheaves that, we believe, are more transparent than the original definitions of Lusztig [Lu1, Lu2]. The purpose of the present paper is quite similar: we provide simple conceptual proofs to a number of results concerning restriction and induction, proved in [Lu2] by rather long arguments involving case by case analysis. We do not claim however to be able to obtain all of the results of [Lu2] in an elementary way. The part of [Lu2] concerning classification of character sheaves remains as difficult (and as mysterious) as it was. There are actually certain reasons preventing that part being simplified. We will try to explain those reasons in the rest of the introduction below.

Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$. Let N^* denote the nilpotent cone in $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. The character sheaves on G can be defined as irreducible $\operatorname{Ad} G$-equivariant perverse sheaves on G with nilpotent characteristic variety (see [Gi]). There are analogous objects on $\mathfrak{g}$ (see [Lu3] and §9 below), the irreducible $\operatorname{Ad} G$-equivariant perverse sheaves on $\mathfrak{g}$ whose characteristic variety is contained in $\mathfrak{g} \times N^*$. Hence, a perverse sheaf M on $\mathfrak{g}$ is a character sheaf if and only if the Fourier transform of M is the intersection cohomology complex associated to an irreducible G-equivariant local system on a G-orbit in N^*. Thus, character sheaves on $\mathfrak{g}$ are parametrized by all irreducible G-equivariant local systems on nilpotent G-orbits in $\mathfrak{g}^*$.

Now, let M be a character sheaf on G. Pulling M back to $\mathfrak{g}$ via the exponential map gives a character sheaf on $\mathfrak{g}$. Let $u(M)$ denote the irreducible local system on a nilpotent G-orbit in $\mathfrak{g}^*$ attached to the latter sheaf via the above parametrization. The assignment $M \rightsquigarrow u(M)$ does not lead, however, to a parametrization of character sheaves on G, because there is

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an additional invariant of character sheaves on G, called a *central character* (see [Gi, §1] for an algebraic definition of a central character), that does not exist for character sheaves on $\mathfrak{g}$. The central character of a character sheaf M is a semisimple conjugacy class, $h(M) \subset G^\vee$, where $G^\vee$ stands for the complex semisimple Lie group dual to G in the sense of Langlands. To define $h(M)$ geometrically, choose a Levi subgroup $L \subset G$ of minimal dimension such that $M_{|L} \neq 0$ (cf. §7). Let C be the connected center of L. It turns out that the complex $M_{|L}$ is locally-constant along C-orbits (of the multiplication action) in L. The monodromy of M along the C-orbits gives rise to a finite set of points in the dual torus $C^\vee \subset G^\vee$ that belong, in fact, to the same conjugacy class in $G^\vee$. This conjugacy class is $h(M)$, by definition. Clearly, there is no Lie algebra analogue of such a definition, for all orbits of the additive group $\mathrm{Lie}(C)$ are simply connected.

One would like to view the invariants $h(M)$ and $u(M)$ attached to a character sheaf M on G as the "semisimple" and the "unipotent" parts of a single invariant, in analogy with the Jordan decomposition of an element of an algebraic group. There is a difference, spoiling that analogy, however. In the case of the Jordan decomposition of an element a, the semisimple part $h(a)$ dominates over the unipotent part $u(a)$ in the sense that, usually, the semisimple part is found first, and the unipotent part can then be looked for in the centralizer of $h(a)$, a smaller group. In the character sheaves case, the invariants $h(M)$ and $u(M)$ have been defined in a totally different way so that $h(M)$ by no means plays a dominant role. In particular, there is no counterpart to the key property of the Jordan decomposition, saying that the semisimple and the unipotent parts commute. For that (and other) reasons Lusztig used another invariant $v(M)$ instead of $u(M)$, which fits better into the Jordan decomposition philosophy and which is dominated, in the above sense, by $h(M)$. The replacement of $u(M)$ by $v(M)$ was made however at the expense of simplicity; there is no conceptual definition of the invariant $v(M)$ as yet. Lusztig defined the assignment $M \rightsquigarrow v(M)$ by hand, on a case by case basis (see [Lu2]). We will not reproduce his definition, just saying instead that $v(M)$ is an Ad-equivariant local system on a special unipotent orbit in the Langlands dual group $G^\vee$ (rather than on G). Thus, character sheaves on G are parametrized by the set of all pairs $(h(M), v(M))$, which is similar to the Jordan decomposition of conjugacy classes in $G^\vee$, in conjunction with Langlands's philosophy. The sophisticated nature of the invariant $v(M)$ is due to the absence of a procedure relating objects on G to those on $G^\vee$ (see, however, [Gi4]).

We indicate an approach to the definition of the invariant $v(M)$ in the framework of [Gi] (due to Ian Grojnowski; see his MIT Thesis, 1992), which is different from the original definition of Lusztig. Let $D(G)$ denote the algebra of regular algebraic differential operators on G, $U(\mathfrak{g})$ the enveloping algebra of the Lie algebra $\mathfrak{g}$, and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. The action of G on itself by left and right translations gives rise to an injective algebra homomorphism: $U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g}) \hookrightarrow D(G)$. Recall [Gi] that a finitely generated $D(G)$-module M is said to be admissible if the action on M of the subalgebra $Z(\mathfrak{g}) \subset D(G)$ and the action of the Lie subalgebra $\mathfrak{g}_{\mathrm{ad}} = \{x \otimes 1 - 1 \otimes x \in U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g}) \subset D(G),\, x \in \mathfrak{g}\}$ are both locally finite. By a result of [Gi], character sheaves are precisely the perverse sheaves corresponding to simple admissible $D(G)$-modules via the Riemann-Hilbert correspondence. Given such a simple admissible module M, choose a simple $(U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{g}))$-submodule M_0. Then M_0 is an irreducible Harish-Chandra $(\mathfrak{g} \times \mathfrak{g}, G_\Delta)$-module. Hence, $SS(M_0)$, the characteristic variety of M_0, is the closure of a single special nilpotent orbit in $\mathfrak{g}^*$. The orbit does not depend on the choice of submodule $M_0 \subset M$ due to [Gi, Theorem 4.3.3] and the equality $M = D(G) \cdot M_0$. Let $\mathscr{O}$ be the unipotent orbit in G corresponding to that nilpotent orbit in $\mathfrak{g}^* \simeq \mathfrak{g}$ via the exponential map. It is known further that there is a (one-to-one) correspondence between special unipotent orbits in G and those in $G^\vee$. The special orbit in $G^\vee$ corresponding to $\mathscr{O}$ turns out to be the support of the local system $v(M)$.

Admissible modules form an abelian category $\mathrm{Admiss}(G)$. One would like to have not only a parametrization of the simple objects of $\mathrm{Admiss}(G)$, i.e., of character sheaves, but also an explicit description of the category itself. This goes as follows. Let L be a Levi subgroup of G and M a cuspidal character sheaf on L (see [Lu2] and §7 below). To each G-conjugacy class of pairs (L, M) one associates a full abelian subcategory $\mathrm{Admiss}_L^G(M) \subset \mathrm{Admiss}(G)$ (see §8), called a "block". It turns out that the category $\mathrm{Admiss}(G)$ breaks up into a direct sum of subcategories:

$$\mathrm{Admiss}(G) = \bigoplus_{\text{all blocks}} \mathrm{Admiss}_L^G(M).$$

Thus, it suffices to describe each of the blocks separately. We prove that the category $\mathrm{Admiss}_L^G(M)$ is equivalent to $A(L, M)$-Mod, the category of finite-dimensional modules over an explicitly constructed algebra $A(L, M)$ (see §8). The algebra $A(L, M)$ has a natural grading by nonnegative integers and its zero-degree component is a group algebra introduced by Lusztig in [Lu1, §9.2]. Moreover, $A(L, M)$ turns out to be a Koszul algebra (see [BGS]).

The algebra $A(L, M)$ also has a "dual" geometric interpretation in terms of the group $G^\vee$. To that end, recall that the character sheaf M is the intersection cohomology complex associated with a cuspidal local system $\mathscr{M}$ on L. Write the "Jordan decomposition" of $\mathscr{M}$ (see [Lu2]) as $\mathscr{M} = \mathscr{M}_C \boxtimes \mathscr{M}_{\mathscr{O}}$, where $\mathscr{M}_C$ is a one-dimensional local system on a connected component of C, the center of L, and $\mathscr{M}_{\mathscr{O}}$ is (essentially) a cuspidal local system on a special unipotent orbit $\mathscr{O} \subset L$. Let $L^\vee$ be the Levi subgroup in the Langlands dual of G corresponding to L. Let $C^\vee$ be the center of $L^\vee$. Thus, $C^\vee$ is the Langlands dual of C, that is, the group of all one-dimensional local systems on C. Further, let $\mathscr{O}^\vee$ be the special unipotent orbit in $L^\vee$

corresponding to the orbit $\mathscr{O}$ via the bijection between special orbits in L and $L^\vee$. Let $c \in C^\vee$ denote the point corresponding to the local system $\mathscr{M}_C$. Form the set $c \cdot \mathscr{O}^\vee \subset L^\vee$, and let $\operatorname{Ad} G^\vee(c \cdot \mathscr{O}^\vee)$ denote the set of elements of $G^\vee$ that are conjugate to $c \cdot \mathscr{O}^\vee$.

Now, Lusztig has defined a bijective correspondence from the set of cuspidal local systems on $\mathscr{O}$ to a set of $L^\vee$-equivariant local systems on $\mathscr{O}^\vee$. Let $\mathscr{M}_{\mathscr{O}}^\vee$ denote the local system on $\mathscr{O}^\vee$ corresponding to $\mathscr{M}_{\mathscr{O}}$ and let $IC(\mathscr{M}_{\mathscr{O}}^\vee)$ be the intersection cohomology complex on $L^\vee$ associated with $\mathscr{M}_{\mathscr{O}}^\vee$. The element c is central in $L^\vee$ and it acts on $L^\vee$ by multiplication. Let $c_* IC(\mathscr{M}_{\mathscr{O}}^\vee)$ denote the push-forward of $IC(\mathscr{M}_{\mathscr{O}}^\vee)$ with respect to that action on $L^\vee$. Form the induced sheaf $M^\vee = \operatorname{Ind}_{L^\vee}^{G^\vee}(c_* IC(\mathscr{M}_{\mathscr{O}}^\vee))$ (see §5). The complex $M^\vee$ is supported on the closure of the set $\operatorname{Ad} G^\vee(c \cdot \mathscr{O}^\vee)$ and, moreover, it is an object of the $\operatorname{Ad} G^\vee$-equivariant derived category of constructible complexes on $G^\vee$ (see, e.g., [Gi4]). Let $\operatorname{Ext}^{\bullet}_{G^\vee}$ stand for the Ext-groups in that category.

THEOREM 0.1. *There is a natural graded algebra isomorphism*

$$\operatorname{Ext}^{\bullet}_{G^\vee}(M^\vee, M^\vee) \cong A(L, M).$$

In the case when L is a maximal torus in G the theorem was first proved in [Gi3] using the results of [Gi5] (for $q = 1$). In the case of arbitrary Levi subgroup L and $\mathscr{M}_C =$ trivial local system, the theorem follows from [Lu4, §5] (for $q = \exp(c_i) = 1$). The arguments of [Lu4] can be extended to the general case.

The results of the last paragraphs fit into the framework of a general Koszul-Langlands philosophy, an extension of the Langlands philosophy advocated for some time by Wolfgang Soergel and the author. The Langlands philosophy says that, generally, simple objects of a representation-theoretic category associated with a reductive group are parametrized by certain objects associated with the Langlands dual group. The Koszul-Langlands philosophy says that, moreover, an appropriate mixed version (see [BGS, Chapter 2]) of the representation-theoretic category in question is, under favorable circumstances, a Koszul category [BGS, Chapter 2], and that the Koszul dual category [BGS] has a natural description in terms of the Langlands dual group. In particular, the simple objects of the two categories correspond to each other (whence the ordinary Langlands philosophy); however, it is more appropriate to regard simple objects of one of the categories as corresponding to the indecomposable projectives in the dual category. Now, in the special case studied in this paper, one can define a category $\operatorname{Admiss}_{\mathrm{mix}}(G)$, a mixed version of the category $\operatorname{Admiss}(G)$ formed by suitable mixed Hodge modules on G (in the sense of M. Saito). One then shows, modifying the proof of Theorem 8.1, that the category $\operatorname{Admiss}_{\mathrm{mix}}(G)$ is equivalent to the category of graded modules over a Koszul algebra, hence, is a Koszul category.

Furthermore, the Koszul dual category is equivalent, due to Theorem 0.1, to an appropriate category of $\operatorname{Ad} G^\vee$-equivariant perverse sheaves on $G^\vee$.

A detailed treatment of the above ideas, including a complete proof of Theorem 0.1, will be given elsewhere.

§1. Naive restriction

Let G be a complex connected reductive group. Lusztig has defined in [Lu1, §3] a decomposition of G into smooth locally-closed algebraic subvarieties. His construction was extended in [Gi, §3.4] to the more general case of a complex symmetric variety G/K. However, the definition given in [Gi] is incorrect. Below, we give a corrected definition of decomposition of a symmetric variety, although it will be used in this paper in the group case only.

Let θ be an involutive automorphism of G. Let $K = G^\theta$ be the fixed point subgroup of G, and let P be the identity component of the set $\{x \in G, \theta(x) = x^{-1}\}$. A torus (resp., a semisimple element of G) contained in P is said to be *split*. The centralizer of a split torus is called a split Levi subgroup of G. This is a θ-stable reductive subgroup of G (in [Gi] the term "relevant" instead of "split" was used). Following [Lu1, Definition 2.6], call a split element $h \in P$ *isolated* if its connected centralizer $Z_G^0(h)$ is not contained in a proper split Levi subgroup of G. Observe further that the set P is stable under the adjoint K-action. A K-orbit in P is said to be isolated if any element of the orbit has an isolated semisimple part. In particular, any unipotent K-orbit is isolated.

Given a split Levi subgroup L, let L^{reg} denote the Zariski open part of L defined by

$$L^{\mathrm{reg}} = \{x = s_x \cdot u_x \in L \mid Z_G^0(s_x) \subset L\}, \tag{1.1}$$

where $x = s_x \cdot u_x$ stands for the Jordan decomposition of x. Let $Z^0(L)$ denote the connected center of L and let $\overline{L} = L/Z^0(L)$. The involution θ on G induces an involution on $\overline{L}$ and we let $K(\overline{L})$ and $P(\overline{L})$ denote the "$\overline{L}$-counterparts" of the objects K and P (for G).

We now define, imitating [Lu1, §3], a decomposition $P = \bigsqcup P_{L,\mathscr{O}}$ indexed by all pairs $(L, \mathscr{O})$, where L is a split Levi subgroup of G and $\mathscr{O}$ is an isolated $K(\overline{L})$-orbit in $P(\overline{L})$. Let $\widetilde{\mathscr{O}}$ be the inverse image of $\mathscr{O}$ under the natural projection $P(L) \to P(\overline{L})$. The piece $P_{L,\mathscr{O}}$ attached to a pair $(L, \mathscr{O})$ is defined by

$$P_{L,\mathscr{O}} = \textit{union of } K(G)\textit{-orbits in } P(G) \textit{ that meet } (\widetilde{\mathscr{O}} \cap L^{\mathrm{reg}}). \tag{1.2}$$

With this definition of the partition of P, all the arguments of [Gi, §3] go through.

From now on, we restrict our attention to the "group case", i.e., assume that $\theta = \mathrm{id}$. Then we have $G = K = P$, and we write $G_{L,\mathscr{O}}$ for the piece of the decomposition of G given by (1.2).

LEMMA 1.3. *Let L be a Levi subgroup of G. Then for any piece $G_{M,\mathcal{O}}$ the intersection $L^{\mathrm{reg}} \cap G_{M,\mathcal{O}}$ is transverse.*

PROOF. Let $x = s \cdot u$ be the Jordan decomposition of $x \in L^{\mathrm{reg}}$. Let M be the centralizer of the connected center of $Z_G^0(s)$. Then $s, u \in M$ and, moreover, $x \in G_{M,\mathcal{O}}$, where $\mathcal{O}$ is the $\overline{M}$-conjugacy class of the image of x in $\overline{M}$. Let $\mathfrak{g}$, $\mathfrak{m}$, $\mathfrak{l}$, $\mathfrak{r}$ denote the Lie algebras of the groups G, M, L, $Z_G^0(s)$ respectively. Observe that $z(\mathfrak{l}) \subset \mathfrak{r}$, for $s \in L$. Moreover, $s \in L^{\mathrm{reg}}$ yields $\mathfrak{r} \subset \mathfrak{l}$, hence, $z(\mathfrak{l}) \subset z(\mathfrak{r})$. It follows that $\mathfrak{l} = z(z(\mathfrak{l})) \supset z(z(\mathfrak{r})) = \mathfrak{m}$. Whence, $\mathfrak{m} \cap \mathfrak{l}^{\perp} = 0$, where $\mathfrak{l}^{\perp}$ stands for the annihilator of $\mathfrak{l}$ in $\mathfrak{g}$ with respect to an invariant form.

Now let N_1 and N_2 denote the fibers at x of conormal bundles to the subvarieties $L^{\mathrm{reg}} \subset G$ and $G_{M,\mathcal{O}} \subset G$ respectively. We have:

$$N_1 = \mathfrak{l}^{\perp} \quad \text{and} \quad N_2 = [\mathfrak{m}, \mathfrak{m}] \cap z_{\mathfrak{m}}(u) \tag{1.4}$$

(see [Gi, (3.6.6)]). Observe that $[\mathfrak{m}, \mathfrak{m}] \cap z_{\mathfrak{m}}(u) \subset \mathfrak{m}$. Thus, $N_1 \cap N_2 \subset \mathfrak{l}^{\perp} \cap \mathfrak{m} = 0$, and the lemma follows. □

Let $\mathrm{Perv}_{\mathrm{strat}}(G)$ denote the abelian category of $\mathrm{Ad}\, G$-equivariant perverse sheaves on G which are locally constant along the pieces $G_{M,\mathcal{O}}$. By Lemma 1.3, any object of $\mathrm{Perv}_{\mathrm{strat}}(G)$ is noncharacteristic with respect to the subvariety $L^{\mathrm{reg}} \subset G$. Hence, the restriction $V \mapsto V_{|L^{\mathrm{reg}}}$ gives rise to an exact functor from $\mathrm{Perv}_{\mathrm{strat}}(G)$ to $\mathrm{Ad}\, L$-equivariant perverse sheaves on L^{reg}.

Now, let V be an admissible module on G. Then, $V \in \mathrm{Perv}_{\mathrm{strat}}(G)$ and $SSV \subset \mu^{-1}(N_{\mathfrak{g}}^*)$ ([Gi, Theorem 1.4.2]). The restriction to L^{reg} being noncharacteristic, it follows easily that $SS(V_{|L^{\mathrm{reg}}}) \subset N_{\mathfrak{l}}$. In particular, $V_{|L^{\mathrm{reg}}}$ is smooth in the direction of orbits of the group $Z(L)$, the center of L.

In §4 we will prove the following

PROPOSITION 1.5. *For any admissible module V on G the restriction $V_{|L^{\mathrm{reg}}}$ can be uniquely extended to a $Z(L)$-monodromic module on L.*

The uniqueness of a $Z(L)$-monodromic extension of $V_{|L^{\mathrm{reg}}}$ trivially follows from the equality $L = Z(L) \cdot L^{\mathrm{reg}}$, which shows that any $Z(L)$-monodromic module on L is completely determined by its restriction to L^{reg}.

The existence of a $Z(L)$-monodromic extension implies, in particular, that for any $Z(L)$-orbit $C \subset L$, the locally-constant complex $V_{|C \cap L^{\mathrm{reg}}}$ has no monodromy about $C \cap (L \setminus L^{\mathrm{reg}})$. We will not attempt to prove this monodromy vanishes. Instead, we will give an alternative construction in §3 of a module on L that extends $V_{|L^{\mathrm{reg}}}$.

DEFINITION 1.6. For an admissible module V let $\mathrm{Res}_L^G V$ denote the $Z(L)$-monodromic extension of $V_{|L^{\mathrm{reg}}}$.

The uniqueness of the extension yields

COROLLARY 1.7. *The assignment $V \mapsto \mathrm{Res}_L^G V$ gives rise to an exact functor commuting with the Verdier duality.*

REMARK 1.8. Let $i\colon L \hookrightarrow G$ be the inclusion. One might expect naively that $\mathrm{Res}_L^G V = i^* V[\dim G - \dim L]$, for the equality, obviously holds on L^{reg}. However, in general, one has two functors i^* and $i^!$ and there is no way to make a choice between them compatible with the Verdier duality. The correct answer given in §3 below involves, in a sense, a combination of both i^* and $i^!$.

§2. Digression to the Harish-Chandra homomorphism

Let P be a parabolic subgroup of G, U_P the unipotent radical of P, and L a Levi subgroup of P so that $P = L \cdot U_P$. Let $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{l}$, $\mathfrak{u}$ denote the respective Lie algebras and let $\overline{\mathfrak{u}}$ be the nilpotent Lie subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{u}$, so that $\mathfrak{g} = \overline{\mathfrak{u}} \oplus \mathfrak{l} \oplus \mathfrak{u}$.

Let $Z(\mathfrak{g})$ and $Z(\mathfrak{l})$ be the centers of the enveloping algebras $U(\mathfrak{g})$ and $U(\mathfrak{l})$ respectively. First, we recall the definition of the Harish-Chandra homomorphism $a\colon Z(\mathfrak{g}) \to Z(\mathfrak{l})$ and after that explain its geometric meaning.

The decomposition $\mathfrak{g} = (\overline{\mathfrak{u}} \oplus \mathfrak{l}) \oplus \mathfrak{u}$ gives rise to a vector space decomposition:

$$U(\mathfrak{g}) = U(\overline{\mathfrak{u}} \oplus \mathfrak{l}) \oplus U(\mathfrak{g}) \cdot \mathfrak{u}. \tag{2.1}$$

Let a denote the first projection.

PROPOSITION 2.2. (i) *If $x \in U(\mathfrak{g})^{\mathfrak{l}}$ (= the ring of $\mathrm{ad}\,\mathfrak{l}$-invariants), then $a(x) \in Z(\mathfrak{l})$ ($\subset U(\mathfrak{l}) \subset U(\overline{\mathfrak{u}} \oplus \mathfrak{l})$).*

(ii) *The restriction of a to $Z(\mathfrak{g})$ gives rise to an algebra homomorphism $Z(\mathfrak{g}) \to Z(\mathfrak{l})$.*

PROOF. The direct sum decomposition (2.1) is, clearly, $\mathrm{ad}\, L$-stable. Hence for $x \in U(\mathfrak{g})^{\mathfrak{l}}$ we have $a(x) \in U(\overline{\mathfrak{u}} \oplus \mathfrak{l})^{\mathfrak{l}}$. Let h be an element in the center of $\mathfrak{l}$ such that all the eigenvalues of $\mathrm{ad}\, h$ on $\overline{\mathfrak{u}}$ (resp. $\mathfrak{l}$) are > 0 (resp. $= 0$). We see that any $\mathrm{ad}\, h$-invariant element of $U(\overline{\mathfrak{u}} + \mathfrak{l})$ belongs to $U(\mathfrak{l})$. Hence, $U(\overline{\mathfrak{u}} + \mathfrak{l})^{\mathfrak{l}} = U(\mathfrak{l})^{\mathfrak{l}} = Z(\mathfrak{l})$ and (i) follows.

Verification of (ii) is left to the reader. □

Set $Y_P = G/U_P$. The group L normalizes U_P, so that there is a right L-action on Y_P commuting with the natural left G-action.

Let $D(Y_P)$ denote the ring of globally defined differential operators on Y_P. The left G-action and the right L-action on Y_P give rise to algebra homomorphisms

$$e_L\colon U(\mathfrak{g}) \to D(Y_P) \quad \text{and} \quad e_R\colon U(\mathfrak{l}) \to D(Y_P)$$

(the subscript L stands for "left" and R for "right") with commuting images. The intersection $\mathrm{im}\, e_L \cap \mathrm{im}\, e_R$ turns out to be equal to $e_L(Z(\mathfrak{g}))$ and we have

Proposition 2.3. $e_L(z) = e_R(a(z))$, $z \in Z(\mathfrak{g})$.

To prove the proposition one regards $e_L(z)$ and $e_R(a(z))$ as sections of the vector bundle $\underline{\mathfrak{u}} \cdot U(\underline{\mathfrak{g}})$ (see [Gi, Proposition 5.6.2]). The equality is then verified fiber by fiber, using Proposition 2.2 (i). The details are left to the reader. □

§3

Given a parabolic subgroup $P = L \cdot U_P$, let $i_P: P \hookrightarrow G$ denote the inclusion and $q_P: P \to P/U_P = L$ the natural projection. Following Lusztig, for a D-module V on G, set

$$\mathrm{res}_P V = (q_P)_*(i_P^! V)[\dim U_P].$$

This is a complex of D-modules on L.

Lemma 3.1. *If V is an admissible module on G, then $\mathrm{res}_P V$ is an admissible complex on L, i.e., a complex with admissible cohomology.*

Proof. Let $q: G \to G/U_P = Y_P$ be the natural projection and $i: L = P/U_P \hookrightarrow G/U_P = Y_P$ the inclusion. The base change theorem yields $\mathrm{res}_P V = (q_P)_*(i_P^! V)[\dim U_P] = i^!(q_* V)[\dim U_P]$. We show that the $Z(\mathfrak{l})$-action on cohomology of the complex $i^!(q_* V)$ is locally-finite.

Consider the direct image $q_* V$ first. We construct a Zariski-open affine covering of Y_P by right L-stable subsets W, so that the action of $e_R(Z(\mathfrak{l}))$ (see §2) on $\Gamma(W, H^i q_* V)$ turns out to be locally-finite. To that end, note that the map q is smooth and affine. Hence the complex $q_* V$ can be computed by means of the relative de Rham complex, which effectively reduces to the Koszul complex:

$$0 \to q.V \to q.V \otimes \mathfrak{u}^* \to \cdots \to q.V \otimes \Lambda^{\dim \mathfrak{u}} \mathfrak{u}^* \to 0 \tag{3.2}$$

for the Lie algebra cohomology $H^*(\mathfrak{u}, q.V)$, where $\mathfrak{u}$ is viewed as the Lie algebra of vector fields on G generated by the right U_P-action.

Now, let W be a Zariski-open affine part of Y_P which is stable under the right L-action (i.e., W is the inverse image of a subset in G/P via the natural projection $G/U_P \to G/P$). We choose W in such a way that $q^{-1}(W)$ is affinely embedded into G and such that the hypersurface $G \setminus q^{-1}(W)$ is defined by an equation $f = 0$, where the function f is an eigenfunction with respect to the right P-action on G. A subset W of that type may be obtained, for instance, as the inverse image of the open B-orbit in G/P, where B is an appropriate Borel subgroup of G, so that $q^{-1}(W) = B \cdot P$.

Let $(\cdot)_{(f)}$ denote the localization with respect to the function f on G, associated to W. The localization functor being exact and the variety G being affine, we get (using 3.2):

$$\Gamma(W, H^j q_* V) = H^j(\mathfrak{u}, \Gamma(G, V))_{(f)}. \tag{3.3}$$

We now examine the $Z(\mathfrak{g})$-action on the right-hand side of (3.3). The $Z(\mathfrak{g})$-action on $\Gamma(G, V)$ is locally-finite since V is admissible. The $Z(\mathfrak{g})$-action on $\Gamma(G, V) \otimes \Lambda^j \mathfrak{u}^*$ is also locally-finite, for the Lie algebra $\mathfrak{g}$ acts trivially on the factor $\Lambda^i \mathfrak{u}^*$. Thus, the $Z(\mathfrak{g})$-action on $H^j(\mathfrak{u}, \Gamma(G, V))$, and on its image in $\Gamma(W, H^j q_* V)$, is locally-finite. But, the left-hand side of (3.3) may be viewed as a D-module on W. So, the $Z(\mathfrak{g})$-action on the left-hand side is induced by the action of $e_R(Z(\mathfrak{l}))$ via the Harish-Chandra homomorphism 2.3. Hence, the $e_R(Z(\mathfrak{l}))$-action on the image of $H^j(\mathfrak{u}, \Gamma(G, V))$ is locally-finite (we have used here that $Z(\mathfrak{l})$ is a finitely-generated module over $a(Z(\mathfrak{g}))$, the image of the Harish-Chandra homomorphism $a: Z(\mathfrak{g}) \to Z(\mathfrak{l})$).

Further, the equality (3.3) shows that $\Gamma(W, H^j q_* V)$ is a quotient of

$$H^j(\mathfrak{u}, \Gamma(G, V)) \otimes_{\mathbb{C}} \mathbb{C}[f^{-1}]. \tag{3.4}$$

We have just established that the first factor here is a locally-finite $Z(\mathfrak{l})$-module. The factor $\mathbb{C}[f^{-1}]$, on the other hand, is a direct sum of one-dimensional $\mathfrak{l}$-submodules spanned by f^{-k}, for f was chosen to be an L-eigenfunction. Hence, (3.4) is a locally-finite $Z(\mathfrak{l})$-module by [Gi, Proposition 2.2] applied to the Lie algebra $\mathfrak{l}$. Therefore, the same holds for $\Gamma(W, H^j q_* V)$.

Next, we study the restriction of $q_* V$ to the submanifold $L = P/U_P \subset Y_P$. Let W be an open neighborhood of L in Y_P of the type considered above, so that its image in G/P is a neighborhood of the base point $l \in G/P$. Choose a local coordinate system on G/P with the origin at the base point l and pull the coordinate functions back on G/U_P. The resulting functions $t_1, \ldots, t_r$ are right L-invariant functions defined on a neighborhood of L in Y_P, so that $L = \{y \in Y_P \mid t_i(y) = 0,\ i = 1, \ldots, r\}$. Hence, we may compute the restriction functor $i^!$ by means of the standard Koszul complex $K(t_1, \ldots, t_r)$ associated to the functions $t_1, \ldots, t_r$. Thus, the object $i^!(q_* V)$ is represented by the simple complex associated with the double complex (cf. (3.2)–(3.3))

$$\begin{aligned} \cdots &\to K(t_1, \ldots, t_r)\Gamma(G, V)_{(f)} \otimes \Lambda^j \mathfrak{u}^* \\ &\to K(t_1, \ldots, t_r)\Gamma(G, V)_{(f)} \otimes \Lambda^{j+1} \mathfrak{u}^* \to \cdots. \end{aligned}$$

Let

$$H^n i^!(H^m q_* V) = E_2^{n,m} \Rightarrow \mathrm{Gr}\, H^{n+m}(i^! q_* V)$$

be the standard spectral sequence associated with a double complex. As we know, the $e_R(Z(\mathfrak{l}))$-action on $H^m q_* V$ is locally finite. Furthermore, the multiplication by t_j commutes with the $e_R(Z(\mathfrak{l}))$-action, for t_j is a right-invariant function on Y_P. Hence, the induced $e_R(Z(\mathfrak{l}))$-action on a Koszul-complex cohomology $H^n i^!(H^m q_* V)$ is locally-finite. The spectral sequence yields that the same holds for the $Z(\mathfrak{l})$-action on $H^{n+m}(i^! q_* V)$. This completes the proof. □

We now indicate the proof of [Gi, Proposition 8.7.1]. Consider the following commutative diagram:

$$\begin{tikzcd} G \times (Y/T) \arrow[rr, "q"] \arrow[dr, "\mathrm{pr}_2"'] & & (Y \times Y)/T \arrow[dl, "\mathrm{pr}"] \\ & Y/T & \end{tikzcd}$$

where both pr_2 and pr denote the second projection (see [Gi] for notation). Given $y \in Y/T$, let $U_y = y \cdot U \cdot y^{-1}$ be the unipotent radical of a Borel subgroup of G. The fiber of the map q over a point of $pr^{-1}(y)$ is naturally isomorphic to U_y. We see that the map q may be viewed as a relative version (with respect to the base Y/T) of the projection $G \to G/U$. Hence, the direct image functor q_* can be computed via the "relative" Koszul complex with respect to the family of Lie algebras $\{\mathrm{Lie}\, U_y,\, y \in Y/T\}$. Repeating the argument of the first part of the proof of Lemma 3.1 (applied to $P = T \cdot U =$ Borel subgroup), one shows that the right $U(\mathfrak{t})$-action on ${}^pH^i(q_* p^! V)$ is locally-finite, provided V is an admissible $D(G)$-module.

§4

Let $\mathrm{res}_P^i\, V$ denote the ith cohomology D-module of the complex $\mathrm{res}_P\, V$, introduced at the beginning of §3.

Observe that any admissible module on L is a $Z(L)$-monodromic module, since the Lie algebra of the group $Z(L)$ is obviously contained in $Z(\mathfrak{l})$, the center of the enveloping algebra. With that understood we can state

Theorem 4.1. *Let V be an admissible module on G. Then:*

(i) *The complex $\mathrm{res}_P\, V$ is concentrated in degree 0, i.e., $\mathrm{res}_P^i\, V = 0$ for $i \neq 0$.*

(ii) *$\mathrm{res}_P^0\, V$ is the $Z(L)$-monodromic extension of $V_{|L^{\mathrm{reg}}}$ so that*

$$\mathrm{res}_P^0\, V = \mathrm{Res}_L^G\, V$$

(see §1).

(iii) *The assignment $V \mapsto \mathrm{Res}_L^G\, V$ gives rise to an exact functor from the category of admissible $D(G)$-modules to the category of admissible $D(L)$-modules.*

(iv) *The functor Res_L^G commutes with the Verdier duality and preserves the weights (of mixed modules).*

Corollary 4.2. *The functor res_P does not depend on the choice of parabolic subgroup P that has L as a Levi factor.*

Corollary 4.3. *If V is a semisimiple admissible module, then so is $\mathrm{Res}_L^G\, V$.*

Corollary 4.2 follows from parts (i), (ii) of the theorem and Corollary 4.3 from part (iv), since Res_L^G maps pure objects into pure ones.

Proof of Theorem 4.1. Choose a one-dimensional torus C ($\cong \mathbb{C}^*$) in the center of L in such a way that all the weights of the adjoint C-action on the Lie algebra $\mathfrak{u}$ are positive integers. Since the group U_P is unipotent, it is algebraically and C-equivariantly isomorphic to $\mathfrak{u}$ via the exponential map. So, the projection $q_P \colon P \to P/U_P = L$ may (and will) be viewed as a vector bundle on L with C-action along the fibers.

Now, $i_P^! V$ is obviously an $\mathrm{Ad}\, L$-equivariant, hence C-equivariant, complex on P. But for a C-equivariant complex on a vector bundle, one knows that $(q_P)_* = i_L^*$ where $i_L \colon L \hookrightarrow P$ denotes the zero-section embedding (see, e.g., [Gi2]; it is used here that all the weights of the C-action on the vector bundle are of the same sign). Hence,

$$\mathrm{res}_P\, V = i_L^* i_P^! V.$$

We see that the complex $\mathrm{res}_P\, V$ depends only on the restriction of V on a small open neighborhood of L.

Let $x \in L^{\mathrm{reg}}$. It follows from Lemma 1.3 that both the restriction from G to P and the restriction from P to L are noncharacteristic at x. Hence, on L^{reg} we have $i_L^* = i_L^![\dim P - \dim L]$, so that

$$(\mathrm{res}_P\, V)_{|L^{\mathrm{reg}}} = (i_L \cdot i_P)^! V[\dim P - \dim L]_{|L^{\mathrm{reg}}} = V_{|L^{\mathrm{reg}}}.$$

This shows that $(\mathrm{res}_P^i\, V)_{|L^{\mathrm{reg}}} = 0$ for $i \neq 0$ and $(\mathrm{res}_P^0\, V)_{|L^{\mathrm{reg}}} = V_{|L^{\mathrm{reg}}}$. But since the complex $\mathrm{res}_P\, V = 0$ is $Z(L)$-monodromic (Lemma 3.1), we conclude that $\mathrm{res}_P^i = 0$ (for $i \neq 0$) everywhere on L and that $\mathrm{res}_P^0\, V = \mathrm{Res}_L^G\, V$. This proves (i) and (ii). Part (iii) follows from Lemma 3.1 and from the exactness of Res_L^G.

Finally, it is well known that the functors $i_P^!$ and $(q_P)_*$ increase the weights of mixed complexes. Hence, so does res_P. But since res_P commutes with the Verdier duality (Corollary 1.7), it must also decrease the weights. Part (iv) follows. □

§5. Induction

(Cf. [Lu2, Chapter 4].) Given a parabolic subgroup P of G, set

$$G_P = \{(h, y) \in G \times (G/P) \mid h \in y \cdot P \cdot y^{-1}\}. \tag{5.1}$$

Clearly, G_P is a G-equivariant fiber bundle on G/P with fiber P. The restriction to the fiber P over the base point $l \in G/P$ sets up a bijective correspondence between G-equivariant complexes on G_P and $\mathrm{Ad}\, P$-equivariant complexes on P. Let $\widetilde{V}_P$ denote the complex on G_P corresponding to an equivariant complex V_P on P.

Let U_P be the unipotent radical of P and $L = P/U_P$, the Levi factor. Given an $\mathrm{Ad}\, L$-equivariant complex V on L, let V_P be its pull-back on P via the natural projection $P \to P/U_P = L$. It is clear that V_P is an $\mathrm{Ad}\, P$-equivariant complex on P. We set $\mathrm{ind}_P\, V = f_* \widetilde{V}_P$ where $f \colon G_P \to G$

is defined by $f(h, y) = h$ (see (5.1)). The complex $\operatorname{ind}_P V$ is said to be *induced* from V.

Viewing L as a Levi subgroup of G, we have the following simple

Lemma 5.2 [Lu2, (4.6.2)]. *The functor* ind_P *is the right adjoint of* res_P.

Next, we shall prove the following

Proposition 5.3. *Let V be an L-admissible $D(L)$-module. Then, any cohomology group of* $\operatorname{ind}_P V$ *is a G-admissible $D(G)$-module.*

Proof. Let $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{u}$, $\mathfrak{l}$ denote the Lie algebras of the groups G, P, U_P, L. The algebra $\mathfrak{l}$ may be viewed as a Levi subalgebra of $\mathfrak{p}$, so that $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$.

First, it follows from the construction that $\operatorname{ind}_P V$ is an $\operatorname{Ad} G$-equivariant complex on G. By [Gi, Theorem 1.4.2], it suffices to show that the characteristic variety $SS(\operatorname{ind}_P V)$ is contained in $\mu^{-1}(N^*_{\mathfrak{g}})$, where $N^*_{\mathfrak{g}}$ is the nilpotent variety of $\mathfrak{g}^*$ and $\mu\colon T^*G \to N^*_{\mathfrak{g}}$ is the moment map.

We identify T^*G with the trivial bundle $\mathfrak{g}^* \times G$, T^*L with $\mathfrak{l}^* \times L$, etc., and write

$$T^*(G \times G/P) = \{(\lambda, x, \xi_y) \in \mathfrak{g}^* \times G \times T^*(G/P)\}$$

where ξ_y denotes a covector at a point $y \in G/P$. Now, let the module $\widetilde{V}_P$ be viewed as a D-module on $G \times (G/P)$ supported on G_P. The module being G-equivariant, one obtains

$$SS\widetilde{V}_P = \{(\lambda, h, \xi_y) \in g^* \times G \times T^*(G/P) \mid \\ (y^{-1} \cdot \lambda \cdot y_{|\mathfrak{p}}, \ \ y^{-1} \cdot h \cdot y) \in SSV_P\}. \tag{5.4}$$

Let $\bar{\mathfrak{u}}$ denote the nilpotent subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{u}$, so that $\mathfrak{g} = \bar{\mathfrak{u}} + \mathfrak{l} + \mathfrak{u}$. We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ and $\mathfrak{p}^*$ with $\bar{\mathfrak{u}} + \mathfrak{l}$ via the Killing form. Then we can write (notation of (5.4)) $y^{-1} \cdot \lambda \cdot y = \bar{v} + x + v$, $\bar{v} \in \bar{\mathfrak{u}}$, $x \in \mathfrak{l}$, $v \in \mathfrak{u}$. But we have $\bar{v} = 0$ and $x \in N^*_{\mathfrak{l}}$, since $\widetilde{V}_P$ is the pull-back to P of an admissible $D(L)$-module. Thus

$$SS\widetilde{V}_P \in \{(\lambda, h, \xi_y) \mid y^{-1} \cdot \lambda \cdot y = x + v \in N_{\mathfrak{l}} + \mathfrak{u}\}. \tag{5.5}$$

Now, $\operatorname{ind}_P V$ is the direct image of $\widetilde{V}_P$ with respect to the first projection $G \times (G/P) \to G$. The projection being proper, one knows (see [Ka]) that $SS(\operatorname{Ind}_P V) \subset \operatorname{pr}(SS\widetilde{V}_P)$, where pr denotes the projection $T^*(G \times G/P) = T^*G \times T^*(G/P) \to T^*G$. Hence, the estimate (5.5) yields

$$SS(\operatorname{ind}_P V) \subset \{(\lambda, h) \in \mathfrak{g} \times G \mid y^{-1} \cdot \lambda \cdot y \in N^*_{\mathfrak{l}} + \mathfrak{u}\} \subset N^*_{\mathfrak{g}} \times G.$$

An equality $N^*_{\mathfrak{g}} \times G = \mu^{-1}(N^*_{\mathfrak{g}})$ completes the proof.

§6

Recall the category $\operatorname{Perv}_{\text{strat}}(G)$ introduced in §1.

Proposition 6.1. (i) *The functor* ind_P *yields an exact functor* $\operatorname{Perv}_{\text{strat}}(L) \to \operatorname{Perv}_{\text{strat}}(G)$.

(ii) *If V_L is the intersection cohomology complex associated to an* $\operatorname{Ad} L$-*equivariant local system on a stratum $L_{M,\mathscr{O}}$, then* $\operatorname{ind}_P(V_L)$ *is the intersection cohomology complex associated to an* $\operatorname{Ad} G$-*equivariant local system on the stratum $G_{M,\mathscr{O}}$.*

The proof of part (ii) is essentially due to Lusztig. More specifically, given a stratum $L_{M,\mathscr{O}}$, let

$$\widetilde{G}_{M,\mathscr{O}} = G \times_{\operatorname{Ad} P} (L_{M,\mathscr{O}} \cdot U_P)$$

be the corresponding "stratum" of G_P (see §5). Clearly, $f(\widetilde{G}_{M,\mathscr{O}}) = \overline{G}_{M,\mathscr{O}}$, where the bar stands for the closure. One can derive from dimension estimates of [Lu1, §1] that the map $f\colon \widetilde{G}_{M,\mathscr{O}} \to \overline{G}_{M,\mathscr{O}}$ is small (cf. [BM]). Statement (ii) now follows from the decomposition theorem [BBD] (this was observed in [Lu1, Proposition 4.5]).

To prove (i), observe that for any simple object $V \in \operatorname{Perv}_{\text{strat}}(L)$ we have just obtained that

$${}^pH^i(\operatorname{ind}_P V) = 0 \quad \text{for } i \neq 0.$$

Hence, ind_P is an exact functor from $\operatorname{Perv}_{\text{strat}}(L)$ to $\operatorname{Perv}_{\text{strat}}(G)$. □

Given an admissible module V on L, set $\operatorname{Ind}_L^G V = \operatorname{ind}_P V$.

Theorem 6.2. (i) *The functor* Ind_L^G *is an exact functor from* $\operatorname{Admiss}(L)$ *to* $\operatorname{Admiss}(G)$; *it is the right adjoint of* Res_L^G.

(ii) *The functor* Ind_L^G *does not depend on the choice of parabolic subgroup P containing L as a Levi factor.*

(iii) Ind_L^G *commutes with the Verdier duality and takes semisimple* (*pure*) *modules on L into similar ones on G.*

Proof. (i) follows from Proposition 5.3, Proposition 6.1 (i), and Lemma 5.2. Part (ii) follows from the adjointness of Ind_L^G and Res_L^G and the analogue of (ii) for Res (see Theorem 4.1 (ii)). Part (iii) is clear since the projection $P \to P/U_P$ is a smooth morphism and the map $G_P \to G$ is proper. □

§7. Cuspidal modules

Definition 7.1 [Lu1, Lu2]. An irreducible module $V \in \operatorname{Perv}_{\text{strat}}(G)$ is said to be cuspidal if, for any proper parabolic subgroup $P \subset G$, we have $\operatorname{res}^0_P V = 0$.

Cuspidal modules are the most basic ones in the following sense.

PROPOSITION 7.2 [Lu2, Theorem 4.4 (a)]. *Let* $V \in \mathrm{Admiss}(G)$ *be an irreducible module. If* V *is not cuspidal, then there exists a Levi subgroup* $L \neq G$ *and an irreducible cuspidal module* $V_L \in \mathrm{Admiss}(L)$ *such that* V *is a direct summand of* $\mathrm{Ind}_L^G(V_L)$.

PROOF. If V is an admissible module, then by Theorem 4.1 (ii) we have $\mathrm{res}_P^0 V = \mathrm{Res}_L^G V$.

Let L be a Levi subgroup of minimal dimension such that $\mathrm{Res}_L^G V \neq 0$. Then $\mathrm{Res}_L^G V$ is a sum of cuspidal modules on L by the transitivity of restriction and Theorem 4.1 (iv). Observe next that $L \neq G$, for V is not cuspidal. So, if V_L is a simple constituent of $\mathrm{Res}_L^G V$ then:

$$\mathrm{Hom}(V, \mathrm{Ind}_L^G(V_L)) = \mathrm{Hom}(\mathrm{Res}_L^G V, V_L) \neq 0.$$

Hence, V is a direct summand of $\mathrm{Ind}_L^G(V_L)$ (Theorem 6.2 (iii)). □

We have the following result that explains the relation between cuspidal and admissible modules.

THEOREM 7.3. *Let* $V \in \mathrm{Perv}_{\mathrm{strat}}(G)$ *be an irreducible perverse sheaf with support* $\overline{G}_{M,\mathscr{O}}$. *Then the following conditions are equivalent:*

(i) V *is an admissible module and* $M = G$;
(ii) V *is cuspidal.*

PROOF OF (i) $\Longrightarrow$ (ii). Suppose V is admissible but not cuspidal. Then V is a direct summand of $\mathrm{Ind}_L^G(V_L)$ for a certain cuspidal module V_L on a Levi subgroup $L \neq G$ (Proposition 7.2). Since V_L is irreducible, we have $\mathrm{supp}\, V_L = L_{M',\mathscr{O}'}$, where $M' \subset L$. Hence, $\mathrm{supp}\, V = \overline{G}_{M',\mathscr{O}'}$ by Proposition 4.1 (ii). But the assumption of the theorem implies $\mathrm{supp}\, V = G_{G,\mathscr{O}} \neq G_{M',\mathscr{O}'}$, since $M' \subset L \neq G$. The contradiction completes the proof.

(ii) $\Longrightarrow$ (i). Let V be a cuspidal module with $\mathrm{supp}\, V = G_{M,\mathscr{O}}$. Choose a parabolic subgroup $P = M \cdot U_P$ with Levi component M. Obviously, $V_{|M^{\mathrm{reg}}} \neq 0$. Hence, $(\mathrm{res}_P^0 V)_{|M^{\mathrm{reg}}} \neq 0$. It follows that $M = G$. Admissibility of V will be proved later in §9. □

COROLLARY 7.4. *The support of a cuspidal module is the closure of a distinguished stratum* (*see* [Gi, §3]).

PROOF. Let V be a cuspidal module and let $S = G_{G,\mathscr{O}}$ be the open stratum in $\mathrm{supp}\, V$. Obviously, we have $T_S^*G \subset SSV$. On the other hand, [Gi, Theorem 1.4.2] and 7.3 yield $SSV \subset \mu^{-1}(N_{\mathfrak{g}}^*)$, so that $T_S^*G \subset \mu^{-1}(N_{\mathfrak{g}}^*)$. [Gi, Proposition 3.5.1] completes the proof. □

The following result was verified by Lusztig using case by case analysis.

COROLLARY 7.5. *Any cuspidal sheaf is a characteristic sheaf.*

The proof follows from Theorem 7.3 and [Gi, Theorem 1.6.1]. □

COROLLARY 7.6. *Any cuspidal sheaf* V *is strongly cuspidal in the sense of* [Lu2, *Chapter* 7], *that is,* $\mathrm{res}_P V = 0$ *for any proper parabolic subgroup* $P \subsetneq G$.

The proof follows from Theorem 7.3 and Theorem 4.1 (i), (ii). □

§8

We now fix a Levi subgroup $L \subset G$ and a cuspidal module M on L. As we know, $\mathrm{Ind}_L^G M$ is a semisimple admissible module on G (Theorem 7.3 and Theorem 6.2). Let $\mathrm{Admiss}_L^G(M)$ be the full subcategory of $\mathrm{Admiss}(G)$ consisting of all those modules whose irreducible subquotients are direct summands of $\mathrm{Ind}_L^G M$.

To study the category $\mathrm{Admiss}_L^G(M)$ more closely we introduce some notation. Let $\mathrm{supp}\, M = \overline{L_{L,\mathscr{O}}}$, so that M is the intersection cohomology complex associated to an equivariant irreducible local system on $Z^{\mathrm{reg}} \cdot \mathscr{O}$, where Z^{reg} is the regular part of a component of $Z(L)$. The local system can be written as $\mathscr{M} = \mathscr{M}_Z \times \mathscr{M}_{\mathscr{O}}$ where $\mathscr{M}_Z$ and $\mathscr{M}_{\mathscr{O}}$ are local systems on Z^{reg} and $\mathscr{O}$ respectively. Observe that $\mathscr{M}_Z$ extends to a local system on the whole of Z, since M is $Z(L)$-monodromic. Hence, $\mathscr{M}_Z$ is a one-dimensional local system associated with a character $\lambda \in \mathrm{Hom}(\pi_1(Z(L)), \mathbb{C}^*) = Z(L)^\vee$, where $Z(L)^\vee$ denotes the dual torus.

Let $N(L)$ denote the normalizer of L in G. Following [Lu1, §3.4], form the group $N(L, M)$ of all $n \in N(L)$ such that $n \cdot Z \cdot n^{-1} = Z$, $n \cdot \mathscr{O} \cdot n^{-1} = \mathscr{O}$, and such that the automorphism $x \mapsto n \cdot x \cdot n^{-1}$ of $Z^{\mathrm{reg}} \cdot \mathscr{O}$ can be lifted to a morphism $\mathscr{M} \to (\mathrm{Ad}\, n)^*\mathscr{M}$ of local systems. Then $L \subset N(L, M)$ and we set $W(L, M) = N(L, M)/L$ (this was shown by Lusztig to be a Coxeter group).

For $n \in N(L, M)$, the space $\mathrm{Hom}(\mathscr{M}, (\mathrm{Ad}\, n)^*\mathscr{M})$ is one-dimensional ($\mathscr{M}$ irreducible) and can be canonically identified with a similar space for $n' \in N(L, M)$, provided $n' \equiv n \bmod L$. So, if w is the image of n in $W(L, M)$, we write $A_w := \mathrm{Hom}(\mathscr{M}, (\mathrm{Ad}\, n)^*\mathscr{M})$. The space $A := \bigoplus_{w \in W(L,M)} A_w$ has a natural algebra structure (see [Lu1, §3.4]); the algebra A is isomorphic to the group algebra of the group $W(L, M)$. In particular, A is a semisimple algebra.

Let $\mathfrak{z} = \mathrm{Lie}\, Z(L)$, and let $S(\mathfrak{z})$ be the symmetric algebra on $\mathfrak{z}$. The adjoint action of $N(L, M)$ on L preserves $Z(L)$ and induces an action of the group $W(L, M)$ on $\mathfrak{z}$, hence, on $S(\mathfrak{z})$. Let $A(L, M) := A \# S(\mathfrak{z})$ be the smash-product of algebras arising from this action.

THEOREM 8.1. *The category* $\mathrm{Admiss}_L^G(M)$ *is equivalent to the category of finite-dimensional* $A(L, M)$*-modules.*

This theorem strengthens the following result of Lusztig.

THEOREM 8.2 [Lu1]. $\mathrm{Hom}(\mathrm{Ind}_L^G M, \mathrm{Ind}_L^G M) \cong A$.

Sketch of proof of Theorem 8.1 (cf. [Gi4]). Let S_+ denote the augmentation ideal of $S(\mathfrak{z})$ and $S_n := S(\mathfrak{z})/S_+^n$, $n = 1, 2, \ldots$. We view S_n as a $\mathfrak{z}$-module by restricting the natural $S(\mathfrak{z})$-action on S_n to $\mathfrak{z}$. Clearly, the $\mathfrak{z}$-action is nilpotent.

Next, we define a unipotent representation e_n of the group $\pi_1(Z(L))$ in the space S_n, by identifying $\pi_1(Z(L))$ with a lattice in $\mathfrak{z}$ and setting:

$$e_n(z)\cdot s = (\exp z)\cdot s, \qquad s \in S_n, \quad z \in \mathfrak{z}.$$

Let $\mathscr{E}_n$ denote the local system on $Z^0(L)$ corresponding to the representation e_n of $\pi_1(Z(L))$. The local systems $\mathscr{E}_n$ form a projective system:

$$\mathscr{E}_1 \leftarrow \mathscr{E}_2 \leftarrow \cdots.$$

Now, let $\mathscr{M} = \mathscr{M}_Z \times \mathscr{M}_{\mathscr{O}}$ be the local system on $Z^{\mathrm{reg}}\cdot\mathscr{O}$ associated with the cuspidal module M. We pull the systems $\mathscr{E}_n$ on Z via a $Z^0(L)$-equivariant isomorphism $Z^0(L) \xrightarrow{\sim} Z$ and form the tensor product $\mathscr{E}_n \otimes \mathscr{M}_Z$. So, the collection $\{\mathscr{E}_n \otimes \mathscr{M}\}$ may be regarded as a projective system of $\operatorname{Ad} L$-equivariant local systems on $Z^{\mathrm{reg}}\cdot\mathscr{O}$. We have $\mathscr{E}_1 \otimes \mathscr{M} \cong \mathscr{M}$ and, moreover, all irreducible subquotients of $\mathscr{E}_n \otimes \mathscr{M}$ are isomorphic to $\mathscr{M}$.

Let M_n be the intersection cohomology complex on L associated with the local system $\mathscr{E}_n \otimes \mathscr{M}$, $E_n = \operatorname{Ind}_L^G M_n$, and let $E = \varprojlim E_n$. By Proposition 4.1 (ii), we have $E_n = \overline{G_{L,\mathscr{O}}}$. To prove the theorem it suffices to show the following statements:

(A) E_n is an object of $\operatorname{Admiss}_L^G(M)$, for each $n = 1, 2, \ldots$;

(B) E is a projective generator of the category $\operatorname{Admiss}_L^G(M)$;

(C) $\operatorname{Hom}(E_{|G_{L,\mathscr{O}}}, E_{|G_{L,\mathscr{O}}}) \cong A(L, M)$;

(D) the natural restriction morphism $\operatorname{Hom}(E, E) \to \operatorname{Hom}(E_{|G_{L,\mathscr{O}}}, E_{|G_{L,\mathscr{O}}})$ is an isomorphism.

We will comment on the proofs of (A)–(D). One first notes that all the simple subquotients of M_n are isomorphic to M because of a similar property of the local systems $\mathscr{E}_n \otimes \mathscr{M}$. Hence, the exactness of the induction functor yields (A).

Next, observe (cf. [Lu1]) that the restriction of $\operatorname{Ind}_L^G M$ to $G_{L,\mathscr{O}}$ is an equivariant local system. Hence, the same is true for E. The proof of isomorphism (C) is straightforward. This is similar to an isomorphism

$$\operatorname{Hom}(\operatorname{Ind}_L^G M_{G_{L,\mathscr{O}}}, \operatorname{Ind}_L^G M_{G_{L,\mathscr{O}}}) = A$$

established in [Lu1].

Let $j_{!*}$ denote the DGM-extension functor of a local system on $G_{L,\mathscr{O}}$. We have $E = j_{!*}(E_{G_{L,\mathscr{O}}})$, by Proposition 6.1 (ii). This yields (D). Finally, proving that E is projective amounts to showing that $\operatorname{Ext}^1(E, \operatorname{Ind}_L^G M) = 0$. Let

$$0 \to \operatorname{Ind}_L^G M \to N \to E \to 0$$

be a nontrivial extension. Since both $\operatorname{Ind}_L^G M$ and E are obtained by applying $j_{!*}$ they have neither quotients nor submodules supported on $\overline{G_{L,\mathscr{O}}} \setminus G_{L,\mathscr{O}}$. Hence, the same holds for N, so that N is also obtained by applying $j_{!*}$. Thus, it suffices to show the sequence

$$0 \to \operatorname{Ind}_L^G M_{|G_{L,\mathscr{O}}} \to N_{|G_{L,\mathscr{O}}} \to E_{|G_{L,\mathscr{O}}} \to 0$$

is split. This is easy. □

§9. Admissible modules on a semisimple Lie algebra

Let G be a connected complex semisimple Lie group with Lie algebra $\mathfrak{g}$, $D(\mathfrak{g})$ the ring of differential operators on $\mathfrak{g}$ with polynomial coefficients, and $Z(\mathfrak{g}) \subset D(G)$ the subring of $\operatorname{Ad} G$-invariant differential operators with constant coefficients. We have $T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g}^*$. Recall that N denotes the nilpotent variety of $\mathfrak{g}$ and N^* the "nilpotent" variety of $\mathfrak{g}^*$, i.e., the variety arising from N via the Killing form isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$.

Recall that given a $D(\mathfrak{g})$-module V, one can define a $D(\mathfrak{g}^*)$-module FV, the Fourier transform of V. It is defined via the natural algebra isomorphism $D(\mathfrak{g}) = D(\mathfrak{g}^*)$ which is an algebraic counterpart of the isomorphism

$$T^*(\mathfrak{g}) \cong \mathfrak{g} \times \mathfrak{g}^* \cong T^*(\mathfrak{g}^*). \tag{9.1}$$

It is known [Br] that $SS(FV) = SSV$, provided V is a $\mathbb{C}^*$-monodromic module, with respect to the natural $\mathbb{C}^*$-action on $\mathfrak{g}$ by multiplication. In that case FV is regular holonomic iff so is V.

Proposition 9.2. *Let V be an $\operatorname{Ad} G$-equivariant $D(\mathfrak{g})$-module. Then the following conditions are equivalent:*

(i) *V is a locally-finite $Z(\mathfrak{g})$-module;*

(ii) *V is regular holonomic such that $SSV \subset \mathfrak{g} \times N^*$;*

(iii) *FV is supported on N^*.*

Proof. The equivalence (i) $\Longleftrightarrow$ (iii) is straightforward. Let us prove that (iii) $\Longrightarrow$ (ii). Recall that any equivariant D-module on a G-variety that has finitely-many G-orbits is regular holonomic. The module FV is, clearly, $\operatorname{Ad} G$-equivariant. Hence, it is regular holonomic. Furthermore FV is a $\mathbb{C}^*$-monodromic module, for it is known that any $\operatorname{Ad} G$-orbit in N^* is a $\mathbb{C}^*$-stable variety. It follows that $F(FV)$ is a regular holonomic module such that $SSF(FV) \subset \mathfrak{g} \times N^*$. But the module V is obtained from $F(FV)$ by the sign-involution of the space $\mathfrak{g}$. Thus, (ii) is proved.

To prove that (ii) $\Longrightarrow$ (iii), we introduce a stratification $\mathfrak{g} = \bigsqcup \mathfrak{g}_{\mathfrak{l},\mathscr{O}}$ similar to the stratification of G, considered in §1. The strata are indexed by pairs $(\mathfrak{l}, \mathscr{O})$ where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathscr{O}$ is a nilpotent conjugacy class in $\mathfrak{l}$. The stratum $\mathfrak{g}_{\mathfrak{l},\mathscr{O}}$ is defined by

$$\mathfrak{g}_{\mathfrak{l},\mathscr{O}} = \operatorname{Ad} G\text{-saturation of } \mathfrak{z}^{\mathrm{reg}} + \mathscr{O} \tag{9.3}$$

where $\mathfrak{z}^{\mathrm{reg}} = \{x \in Z(\mathfrak{l}) \mid Z_{\mathfrak{g}}(x) = 1\}$.

Now let V be a regular holonomic module such that $SSV \subset \mathfrak{g} \times N^*$. One can verify, as in [Gi, §3], that V is smooth along the strata $\mathfrak{g}_{\mathfrak{l},\mathcal{O}}$. Further, it follows from (9.3) that each stratum of the stratification is a $\mathbb{C}^*$-stable variety. Hence, V is a $\mathbb{C}^*$-monodromic module, and the statement (iii) follows. □

REMARK 9.4. Proposition 9.2 is a Lie algebra counterpart of [Gi, Theorem 1.4.2]. Strangely enough, the Lie algebra case turns out to be much easier to prove.

DEFINITION 9.5. An $\operatorname{Ad} G$-equivariant $D(\mathfrak{g})$-module V is called admissible if it satisfies the equivalent conditions (i)–(iii) of Proposition 9.2.

Given a parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ let $\mathfrak{u}_{\mathfrak{q}}$ denote the nilradical of $\mathfrak{q}$, $i_{\mathfrak{q}}: \mathfrak{q} \hookrightarrow \mathfrak{g}$ the inclusion, and $p_{\mathfrak{q}}: \mathfrak{q} \to \mathfrak{q}/\mathfrak{u}_{\mathfrak{q}}$ the natural projection. Let V be an irreducible regular holonomic $\operatorname{Ad} G$-equivariant $D(\mathfrak{g})$-module supported on N. Set $\operatorname{res}_{\mathfrak{q}} V = (p_{\mathfrak{q}})_*(i_{\mathfrak{q}}^! V)[\dim \mathfrak{u}_{\mathfrak{p}}]$. The module V is said to be *cuspidal* (cf. [Lu3]) if, for any proper parabolic subalgebra $\mathfrak{l} \subset \mathfrak{g}$, we have

$$^pH^0(\operatorname{res}_{\mathfrak{q}} V) = 0.$$

THEOREM 9.6 (cf. [Lu3]). *Let V be an irreducible* $\operatorname{Ad} G$*-equivariant $D(G)$-module. The following properties are equivalent*:

(i) *V is cuspidal*;
(ii) *V is an admissible $D(\mathfrak{g})$-module supported on N*;
(iii) *Both V and FV are supported on nilpotent varieties* (*of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively*).

PROOF. The equivalence of (ii) and (iii) follows from Proposition 9.2.

(i) $\Longrightarrow$ (iii). We identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form. Since V is supported on N, we see that FV is an admissible irreducible $D(\mathfrak{g})$-module. Let $\mathfrak{g}_{\mathfrak{l},\mathcal{O}}$ be the open stratum in $\operatorname{supp} FV$ and $\mathfrak{l} \neq \mathfrak{g}$. The restriction of FV to $\mathfrak{g}_{\mathfrak{l},\mathcal{O}}$ is a smooth local system. Let $\mathfrak{q} = \mathfrak{l}+\mathfrak{u}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi component $\mathfrak{l}$. For any $s \in \mathfrak{z}^{\text{reg}}$ and $n \in \mathcal{O}$ the affine space $s+n+\mathfrak{u}$ is contained in $\mathfrak{g}_{\mathfrak{l},\mathcal{O}}$. Hence, the restriction to $s+n+\mathfrak{u}$ of the complex $i_{\mathfrak{q}}^* FV$ is the constant sheaf. It follows that $^pH^m(p_{\mathfrak{q}})_!(i_{\mathfrak{q}}^* FV) \neq 0$. The identity $F((p_{\mathfrak{q}})_*(i_{\mathfrak{q}}^! V)) = (p_{\mathfrak{q}})_!(i_{\mathfrak{q}}^* FV)$ yields $^pH^m(p_{\mathfrak{q}})_*(i_{\mathfrak{q}}^* FV) \neq 0$. This contradicts the assumption that V is cuspidal. Hence, $\mathfrak{l} = \mathfrak{g}$ so that $\operatorname{supp} FV \subset N$ and (iii) is proved.

The proof of (ii) $\Longrightarrow$ (i) is similar to the proof of the implication (i) $\Longrightarrow$ (ii) of Theorem 7.3. We omit the details. □

We are now able to complete the proof of Theorem 7.3 by proving implication (ii) $\Longrightarrow$ (i) of that theorem. Let V be a cuspidal module on G. An elementary reduction argument shows that we may assume that G is semisimple and that $\operatorname{supp} V$ is the closure of a unipotent orbit. Let $\exp: \mathfrak{g} \to G$ be the exponential map and $\exp^* V$ the pull-back of V on $\mathfrak{g}$. It is not hard

to verify that $\exp^* V$ is a cuspidal module on $\mathfrak{g}$. So, it is admissible by Theorem 9.6. Hence, the characteristic variety of $\exp^* V$ is contained in $\mathfrak{g} \times N^*$. It follows that $SSV \subset G \times N^* \subset T^*G$. Thus, V is admissible by [Gi, Theorem 1.4.2].

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