6.

On geometrizing the relation between HOMFLY-PT and $\mathfrak{sl}(n)$ link homology.

6.1.

6.1. One of Webster's results in "Khovanov–Rozansky homology via a canopolis formalism" is to rewrite the matrix-factorization (MF) construction of $\mathfrak{sl}_n$ link homology explicitly in terms of Soergel bimodules. This gives an $\mathfrak{sl}_n$ analogue of Khovanov's theorem matching the MF and SBim constructions of HOMFLY-PT homology. It also clarifies, at least for me, the origin of Rasmussen's spectral sequence from HOMFLY-PT to $\mathfrak{sl}_n$.

6.2. Webster works with $\mathbf{Z}$-graded matrix factorizations. Suppose that S is a ring and $N = S/I$ for some ideal I. For any regular sequence $\vec{x} = (x_1, \dots, x_d) \in S^d$ such that $I = \langle x_1, \dots, x_d \rangle$, there is a well-known free resolution of N called the Koszul complex

$$Z_{\vec{x}} = \bigotimes_i (S \xrightarrow{x_i} S).$$

Fix a potential $\varphi \in I$: say, $\varphi = \sum_i x_i y_i$ for some $\vec{y} = (y_1, \dots, y_d) \in S^d$. Then replacing the complex $S \xrightarrow{x_i} S$ with the matrix factorization $S \xrightarrow{x_i} S \xrightarrow{y_i} S$, we can form what Webster calls the Koszul matrix factorization

$$Z_{\vec{x},\vec{y}} = \bigotimes_i (S \xrightarrow{x_i} S \xrightarrow{y_i} S).$$

By construction, it is a matrix factorization with potential φ.

Let $S = R \otimes R^{\mathrm{op}}$, where $R = \mathbf{C}[t_1, \dots, t_d]$. Let $p(x) \in \mathbf{C}[x]$ be a polynomial such that $(p(x) - p(y))/(x - y)$ is polynomial in x and y, and let $\vec{x}, \vec{y} \in S^d$ be defined by

$$x_i = t_i \otimes 1 - 1 \otimes t_i, \quad y_i = \frac{p(t_i \otimes 1) - p(1 \otimes t_i)}{t_i \otimes 1 - 1 \otimes t_i}.$$

In this case, we set $Z = Z_{\vec{x}}$ and $Z(p) = Z_{\vec{x},\vec{y}}$.

6.3. Let $\beta \in Br_n$ be a braid on n strands, and let $F(\beta)$ be its Rouquier complex. Webster's Theorem 2.7 implies that the cohomology of $F(\beta) \otimes Z(p)$ is, up to grading shift, the $\mathfrak{sl}_{\lambda(p)}$ homology of the link closure of β, where $\lambda(p)$ is the partition of multiplicities of the roots of p. By comparison, the cohomology of $F(\beta) \otimes Z$ is, up to grading shift, the HOMFLY-PT homology of the link.

Given a $\mathbf{Z}$-graded matrix factorization $(M, d = d_+ + d_-)$ in which the forward differential d_+ satisfies $d_+^2 = 0$, we write M_+ to denote the underlying complex formed by M under d_+. Thus $Z(p)_+ = Z$. It seems to be a general fact that for any complex F and matrix factorization M, we have a spectral sequence from the cohomology of $F \otimes M_+$ to that of $F \otimes M$.

6.2.

6.4. Let H_d be the Hecke algebra of S_d over $\mathbf{Z}[v^{\pm 1}]$. I would like to understand how the above viewpoint on $\mathfrak{sl}_n$ link homology is related to the categorification of the tensor-product representation

$$V_n^{\otimes d} \curvearrowleft H_d, \quad \text{where } V_n = \mathbf{Z}[v^{\pm 1}]^n.$$

Of course there are many categorifications, going back to work of Frenkel–Khovanov and others (that I should know better). I will explain the one related to Soergel bimodules that I understand best.

6.5. This paragraph dispenses with technicalities. We work over either $k = \bar{\mathbf{F}}_q$ or $k = \mathbf{C}$. Let $G = \mathrm{GL}_d$, equipped with the split $\mathbf{F}_q$-structure in the first case. Over $\bar{\mathbf{F}}_q$, "mixed" will mean "ℓ-adic sheaves with the Weil structure coming from the split $\mathbf{F}_q$-structure on G". Over $\mathbf{C}$, "mixed" will mean "mixed Hodge modules". The "shift-twist" will be the functor $[1](\frac{1}{2})$, where $[1]$ is the usual degree shift in the constructible derived category and $(\frac{1}{2})$ is a choice of half-Tate twist.

6.6. Let $\mathcal{B}$ be the flag variety of G, and let $\mathcal{C}_n = \coprod_{\vec{e} \in \mathbf{Z}^n_{\geq 0}} \mathcal{C}_{\vec{e}}$, where $\mathcal{C}_{\vec{e}}$ is defined by

$$\mathcal{C}_{\vec{e}}(k) = \{0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = k^d \mid \dim(V_i/V_{i-1}) = e_i\}.$$

Thus, we require $\sum_i e_i = d$, but we allow the entries of $\vec{e}$ to be arbitrary nonnegative integers. Note that G acts on $\mathcal{C}_{\vec{e}}$ and hence $\mathcal{C}_n$, just as it acts on $\mathcal{B}$.

The Hecke algebra H_n, *resp.* the module $V_n^{\otimes d}$, is the split Grothendieck group of an additive category of G-equivariant mixed perverse sheaves on $\mathcal{B} \times \mathcal{B}$, *resp.* $\mathcal{C}_n \times \mathcal{B}$, together with their shift-twists. The shift-twist categorifies v. The action of H_n on $V_n^{\otimes d}$ is geometrized by a convolution action of the sheaves over $\mathcal{B} \times \mathcal{B}$ on the sheaves over $\mathcal{C}_n \times \mathcal{B}$, which restricts to the additive categories in question.

To give more detail about the categorification of H_n: The G-orbits on $\mathcal{B} \times \mathcal{B}$ are indexed by the Weyl group S_d. The wth orbit defines a mixed equivariant perverse sheaf IC_w supported on its closure, called its intersection cohomology complex. The cohomology functor $\mathrm{H}^*_G(\mathcal{B} \times \mathcal{B}, -)$ sends IC_w to the indecomposable Soergel bimodule $\mathbf{B}_w$ indexed by w.

6.7. Webster–Williamson gave a geometric model for the Hochschild homology of $\mathbf{B}_w$:

$$\mathrm{HH}^*(\mathbf{B}_w) \simeq \mathrm{gr}^{\mathsf{W}}_* \mathrm{H}^*_{\mathrm{Ad}(G)}(G, pr_! act^* IC_w),$$

where $pr_! act^*$ is a pullback-pushforward functor taking sheaves on $\mathcal{B} \times \mathcal{B}$ to sheaves on G, and W refers to the weight filtation on cohomology. This isomorphism is functorial with respect to Soergel bimodules on the left and sheaves on the right. In this model, the Hochschild (*i.e.*, Tor) grading corresponds to the difference between the cohomological and weight degrees in a precise sense.

I myself gave a different model for the Hochschild homology, involving restricting $pr_! act^* IC_w$ to the unipotent locus of G and pulling it back along the Springer resolution. However, my work passes through Webster–Williamson's, and their model may be sufficient for what follows.

I hope to determine explicitly how the Koszul resolution of R, and hence, the complex $F(\beta) \otimes Z$, enters in their work. Then I would like to find a similar story with $\mathcal{C}_n \times \mathcal{B}$ in place of $\mathcal{B} \times \mathcal{B}$, and with $F(\beta) \otimes Z(p)$ in place of $F(\beta) \otimes Z$, where $p(x) = x^n$. This would be an algebro-geometric explanation of the Khovanov–Rozansky differential d_- depending on p.