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QUANTUM GROUPS AND FLAG VARIETIES.

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1. Introduction

By classical Schur-Weyl duality there is a one-to-one correspondence between irreducible representations of the Symmetric group S_d and certain finite dimensional irreducible representations of the group $GL_n(\mathbb{C})$. The correspondence is obtained by decomposing the tensor space $(\mathbb{C}^n)^{\otimes d}$ with respect to the natural (commuting) actions of the two groups. The main goal of the present paper is to extend this construction to the *quantum affine* setup and to provide its geometric interpretation.

By "quantizing" we mean replacing objects by their q-analogues. The natural q-analogue of the Symmetric group S_d, or rather of its group algebra, is the Hecke algebra of type A_d (cf. e.g. [KL 1]). The natural q-analogue of $GL_n(\mathbb{C})$, or rather of the enveloping algebra of its Lie algebra, is the quantum group $\mathbf{U}_q(\mathfrak{gl}_n)$ introduced by Drinfeld and Jimbo. A q-analogue of the Weyl correspondence was found by Jimbo [J] about the same time as the definition of the quantum group itself. A very interesting geometric construction of that correspondence was given later by Grojnowski-Lusztig [GL] following the earlier work [BLM] where a "Hecke-like" interpretation of $\mathbf{U}_q(\mathfrak{gl}_n)$ was discovered.

The next step in that direction consists of replacing 'finite dimensional' quantum objects by their 'affine' counterparts, which amounts heuristically to replacing the complex field $\mathbb{C}$ by the p-adic field. Representation theory of *affine Hecke algebras* was studied in [Gi] and [KL 2] (see [CG for details) and turned out to be quite an intersting problem in itself. A similar finite dimensional Representaion theory of $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$ was worked out in [GV] (some special cases were treated earlier in [CP] and [Ch 1]). In the present paper we 'match' the geometric approach to affine Hecke algebras used in [Gi] and [KL 2] with the corresponding geometric approach to $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$

used in [GV], thus providing a geometric approach to the 'quantum affine' Weyl reciprocity and 'explaining' the results of [Dr 3] for Yangians and [Ch 2] for quantum algebras.

The role of the tensor representation $(\mathbb{C}^n)^{\otimes d}$ above is played, in the quantum affine setup, by a *Polynomial Tensor Representation* introduced in section 4 below. In the subsequent sections we produce three different geometric constructions of the Polynomial Tensor Representation. The first one is based on equivariant K-theory, and the other two, which are "dual" to the first in the sense of Langlands, involve affine flag varieties over a finite field. Thus, this paper may be viewed as a continuation of [GV] with a new geometric ingredient being borrowed from [GL].

There are several reasons to suggest that it is the interplay between the three constructions of the paper that seems to be of most importance. We would like to mention some of them. First, the Polynomial Tensor Representation was constructed originally via an *R-matrix realization* of the algebra $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$ discovered in [FRT] and [RS 1]. On the other hand, the formulas for the representation of $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$ given in [GV] can be modified to produce the Polynomial Tensor Representation in Drinfeld's *loop-like realization* [Dr 2]. Comparison of those two approaches yields an independent proof of the main result of [DF]. Second, the Polynomial Tensor Representation arises naturally in Physics in connection with quantum integrable systems and Bethe Ansatz (see [KR]). It is a somewhat miraculous coincidence that the $d \to \infty$ limit of the construction, which comes out quite naturally from the point of view of [BLM] and [GV], is also of great importance in Physics, where it is known as the 'Thermodinamics Limit'. This subject is closely related to the so-called "Zamolodchikov conjecture" on deformations of conformal field theories. It was approached quite differently in [KKM] via the techniques of the crystal bases. Finally, the last two of geometric constructions presented below provide distinguished *Intersection homology bases* in the Polynomial Tensor Representation. Those bases are likely to enjoy various nice properties similar (and possibly related) to Lusztig-Kashiwara canonical bases.

2. The matrix loop realization of $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$

Let $\mathfrak{g} = \mathrm{End}\,(\mathbb{C}^n)$ be the endomorphism (associative) algebra of the vector space $\mathbb{C}^n$ with the standard basis formed by the matrix units E_{ij} , the $n\times n$-matrices with the only non-zero entry 1 at the $i\times j$-th place. Let $q\in\mathbb{C}^*$ be a complex number and z a formal variable. The Drinfeld-Jimbo *R-matrix* in the fundamental representation $\mathbb{C}^n$ of the affine algebra $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$ has the following form:

$$R(z)=\frac{z-1}{qz-q^{-1}}\sum_{1\le i\neq j\le n}E_{ii}\otimes E_{jj}+\sum_{1\le i\le n}E_{ii}\otimes E_{ii}+ \tag{2.1}$$

$$+\frac{z(q-q^{-1})}{qz-q^{-1}}\sum_{i<j}E_{ij}\otimes E_{ji}+\frac{q-q^{-1}}{qz-q^{-1}}\sum_{i>j}E_{ij}\otimes E_{ji}$$

Thus, $R(z)$ is a $\mathfrak{g}\otimes\mathfrak{g}$-valued rational function in z.

A somewhat mysterious role throughout the theory of affine quantum groups is played by a function θ that depends on a complex parameter $q\in\mathbb{C}^*$ and an integral parameter m, and which is given by the formula:

$$\theta_m(z):=\frac{q^m\,z-1}{z-q^m} \tag{2.2}$$

The R-matrix (2.1) can be written in terms of the function θ_{-1} as follows

$$R(z)=\sum_{1\le i,j\le n}E_{ij}\otimes E_{ji}+\theta_{-1}(qz)\sum_{1\le i\neq j\le n}(E_{ii}\otimes E_{jj}-q^{\mathrm{sgn}(i-j)}E_{ij}\otimes E_{ji})$$

where $sgn(k)$ equals $+1$ if $k>0$ and -1 if $k<0$.

It can be verified that the R-matrix satisfies the following *Yang-Baxter identity* in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$:

$$R_{12}(\tfrac{z_1}{z_2})\cdot R_{13}(\tfrac{z_1}{z_3})\cdot R_{23}(\tfrac{z_2}{z_3})=R_{23}(\tfrac{z_2}{z_3})\cdot R_{13}(\tfrac{z_1}{z_3})\cdot R_{12}(\tfrac{z_1}{z_2}) \tag{2.3}$$

along with the *unitarity condition*: $\quad R_{21}(z)^{-1}=R(z^{-1})$.

Write $\mathfrak{g}=\mathfrak{n}^+\oplus\mathfrak{h}\oplus\mathfrak{n}^-$ for the standard triangular decomposition, where $\mathfrak{h}$ and $\mathfrak{n}^\pm$ stand for the subalgebras of diagonal and strictly upper (resp. lower) triangular matrices. Let $\mathfrak{g}[t,t^{-1}]$ be the algebra of $\mathfrak{g}$-valued Laurent polynomials; it contains $\mathfrak{g}$ as the subalgebra of constant polynomials. We introduce the following subalgebras:

$$\mathbf{L}^\pm\mathfrak{g}=\mathfrak{n}^\pm\oplus t^{\pm1}\,\mathfrak{g}[t^{\pm1}]\quad\subset\quad\mathfrak{g}[t,t^{-1}]$$

Let $\mathbf{K}=\mathbb{C}[E_{11},E_{11}^{-1},\dots,E_{nn},E_{nn}^{-1}]$ be the polynomial algebra. Form the vector space:

$$\mathbf{L}\mathfrak{g}=\mathbf{L}^+\mathfrak{g}\ \oplus\ \mathbf{K}\ \oplus\ \mathbf{L}^-\mathfrak{g}$$

There are two imbeddings of the space $\mathfrak{h}$ of diagonal matrices into $\mathbf{L}\mathfrak{g}$ that arise from two different imbeddings $\mathfrak{h} \hookrightarrow \mathbf{K}$ given by the formulas:

$$\mathfrak{h} \ni \sum a_i\, E_{ii} = A \mapsto A^{\pm} = \sum a_i\, E_{ii}^{\pm 1} \in \mathbf{K}$$

For $A \in \mathfrak{h}$, introduce two generating functions in the formal variable z:

$$A^{\pm}(z) = A^{\pm} \oplus \sum_{i=1}^{\infty} (A\, t^{\pm i})\, z^{\mp i} \quad \in \; \mathbf{K} \oplus \mathbf{L}^{\pm}\mathfrak{g}[[z^{\mp 1}]]$$

Similarly, for any $A \in \mathfrak{n}^{\pm}$ form the generating function:

$$A(z) = \sum_{i=-\infty}^{\infty} (A\, t^{i})\, z^{-i} \quad \in \; (\mathfrak{g}[t, t^{-1}])[[z, z^{-1}]]$$

Observe that this power series has a unique decomposition:

$$A(z) = A^{+}(z) + A^{-}(z) \quad \text{so that} \quad A^{\pm}(z) \in \; \mathbf{L}^{\pm}\mathfrak{g}[[z^{\mp 1}]]$$

Thus, to each element A of $\mathfrak{h}$, $\mathfrak{n}^{+}$ or $\mathfrak{n}^{-}$, we assigned two generating functions $A^{+}(z)$ and $A^{-}(z)$.

Let $T(\mathbf{L}\mathfrak{g})[C, C^{-1}]$ be the Laurent polynomial ring in the variables $C^{\pm 1}$ with coefficients in the tensor algebra of the $\mathbb{C}$-vector space $\mathbf{L}\mathfrak{g}$. It is convenient to introduce the following 'universal' generating functions, usually referred to as 'L-operators'

$$L^{\pm}(z) := \sum_{\alpha,\beta=1}^{n} E_{\alpha\beta} \otimes E^{\pm}_{\alpha\beta}(z) \in \mathfrak{g} \otimes \mathbf{L}^{\pm}\mathfrak{g}[[z^{\mp 1}]] \tag{2.4}$$

and set

$$L_1^{\pm}(z) = \sum_{\alpha,\beta=1}^{n} E_{\alpha\beta} \otimes Id \otimes E^{\pm}_{\alpha\beta}(z) \in \mathfrak{g} \otimes \mathfrak{g} \otimes T(\mathbf{L}\mathfrak{g})[[z^{\mp 1}]]$$

$$L_2^{\pm}(z) = \sum_{\alpha,\beta=1}^{n} Id \otimes E_{\alpha\beta} \otimes E^{\pm}_{\alpha\beta}(z) \in \mathfrak{g} \otimes \mathfrak{g} \otimes T(\mathbf{L}\mathfrak{g})[[z^{\mp 1}]]$$

Following [FRT] and [RS 1], define an associative $\mathbb{C}$-algebra $\mathbf{U}(R)$ to be the quotient of $T(\mathbf{L}\mathfrak{g})[C, C^{-1}]$ modulo the following relations involving the R-matrix (2.1) and the generating functions introduced above:

$$A \cdot B = A\, B, \qquad \forall\, A, B \in \mathbf{K}$$

$$R(\tfrac{z}{w}) \cdot L_1^{\pm}(z) \cdot L_2^{\pm}(w) \;=\; L_2^{\pm}(w) \cdot L_1^{\pm}(z) \cdot R(\tfrac{z}{w})$$

$$R(C^2\tfrac{z}{w}) \cdot L_1^{+}(z) \cdot L_2^{-}(w) \;=\; L_2^{-}(w) \cdot L_1^{+}(z) \cdot R(C^{-2}\tfrac{z}{w})$$

Thus, for all $A, B \in \mathfrak{h}$, $\mathfrak{n}^{\pm}$, one gets an infinite number of quadratic relations on generators from $\mathbf{L}\mathfrak{g}$. The dot-product denotes either the product in $T(\mathbf{L}\mathfrak{g})[C, C^{-1}]$ or the product in $\mathfrak{g}$ and in the above formulas we identify $R(\frac{z}{w})$ with $R(\frac{z}{w}) \otimes 1 \in \mathfrak{g} \otimes \mathfrak{g} \otimes T(\mathbf{L}\mathfrak{g})[C, C^{-1}]$.

We record the following result that will be used later.

Theorem 2.5 [DF]. *There exists a factorization*

$$L^{\pm}(z) = \left(Id \otimes 1 + \sum_{\alpha>\beta} E_{\alpha\beta} \otimes A^{\pm}_{\alpha\beta}(z)\right) \cdot$$
$$\cdot \left(\sum_{\alpha} E_{\alpha\alpha} \otimes A^{\pm}_{\alpha\alpha}(z)\right) \cdot \left(Id \otimes 1 + \sum_{\alpha<\beta} E_{\alpha\beta} \otimes A^{\pm}_{\alpha\beta}(z)\right)$$

where $A^{\pm}_{\alpha\beta}(z) \in \mathbf{U}(R)[[z^{\mp 1}]]$, $1 \leq \alpha, \beta \leq n$ are certain uniquely determined elements. □

Next recall Drinfeld's "loop-like" presentation [Dr 2] of $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$ as the associative $\mathbb{C}$-algebra with unit 1 and generators

$$E_{\alpha k}\,,\; F_{\alpha k}\,,\; K^{\pm}_{\beta l} \quad,\quad \alpha \in [1, n-1]\,,\; \beta \in [1, n]\,,\; k \in \mathbb{Z}\,,\; l \in \pm\mathbb{N} \tag{2.6}$$

The relations among the generators are conveniently expressed in terms of the following generating functions in the formal variable z :

$$E_\alpha(z) = \sum_{k=-\infty}^{\infty} E_{\alpha k} z^{-k} \quad F_\alpha(z) = \sum_{k=-\infty}^{\infty} F_{\alpha k} z^{-k} \quad K^{\pm}_{\beta}(z) = \sum_{k\in\pm\mathbb{N}} K^{\pm}_{\beta k} z^{-k}.$$

Write also

$$\delta(z) := \sum_{n\in\mathbb{Z}} z^n \quad,\quad \epsilon_{\alpha\beta} := \delta_{\alpha+1,\beta} - \delta_{\alpha,\beta}. \tag{2.7}$$

Using the Cartan matrix $\mathbf{m}_{\alpha\beta}$ of the Lie algebra $\mathfrak{sl}_n$ the relations among the generating functions are written as follows

$$K^+_{\beta 0}K^-_{\beta 0} = K^-_{\beta 0}K^+_{\beta 0} = 1 \tag{2.8.1}$$

$$[K^\pm_\alpha(z), K^\pm_\beta(w)] = 0 = [K^\pm_\alpha(z), K^\mp_\alpha(w)] \tag{2.8.2}$$

$$\theta_1(q^{-1}C^{\pm 2}\tfrac{z}{w})K^\pm_\alpha(z)K^\mp_\beta(w) = \theta_1(q^{-1}C^{\mp 2}\tfrac{z}{w})K^\mp_\beta(w)K^\pm_\alpha(z) \quad \text{if } \alpha < \beta \tag{2.8.3}$$

$$K^\pm_\beta(z)E_\alpha(w) = \theta_{\epsilon_{\alpha\beta}}(q^{\epsilon_{\alpha\beta}}C^{\pm 1}\tfrac{z}{w})\, E_\alpha(w)K^\pm_\beta(z) \tag{2.8.4}$$

$$K^\pm_\beta(z)F_\alpha(w) = \theta_{-\epsilon_{\alpha\beta}}(q^{\epsilon_{\alpha\beta}}C^{\mp 1}\tfrac{z}{w})\, F_\alpha(w)K^\pm_\beta(z) \tag{2.8.5}$$

$$E_\alpha(z)E_\beta(w) = \theta_{\mathbf{m}_{\alpha\beta}}(q^{\alpha-\beta}\tfrac{z}{w})\, E_\beta(w)E_\alpha(z) \tag{2.8.6}$$

$$F_\alpha(z)F_\beta(w) = \theta_{-\mathbf{m}_{\alpha\beta}}(q^{\alpha-\beta}\tfrac{z}{w})\, F_\beta(w)F_\alpha(z) \tag{2.8.7}$$

$$[E_\alpha(z), F_\beta(w)] = (q - q^{-1})\cdot \delta_{\alpha\beta}\cdot \tag{2.8.8}$$

$$\Big(\delta(C^{-2}\tfrac{z}{w})K^+_{\alpha+1}(Cw)/K^+_\alpha(Cw) - \delta(C^2\tfrac{z}{w})K^-_{\alpha+1}(Cz)/K^-_\alpha(Cz)\Big)$$

$$\{E_\alpha(z_1)E_\alpha(z_2)E_\beta(w) - (q+q^{-1})E_\alpha(z_1)E_\beta(w)E_\alpha(z_2) + \tag{2.8.9}$$

$$+ E_\beta(w)E_\alpha(z_1)E_\alpha(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0 \text{ if } |\alpha-\beta| = 1$$

$$\{F_\alpha(z_1)F_\alpha(z_2)F_\beta(w) - (q+q^{-1})F_\alpha(z_1)F_\beta(w)F_\alpha(z_2) + \tag{2.8.10}$$

$$+ F_\beta(w)F_\alpha(z_1)F_\alpha(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0 \text{ if } |\alpha-\beta| = 1.$$

Remark 2.9. Drinfeld's realization of the subalgebra $U_q(\widehat{\mathfrak{sl}_n}) \subset \mathbf{U}_q(\widehat{\mathfrak{gl}_n})$ is given, in the notation of [Dr 2], by the formulas (see [DF], [Dr 2]):

$$\xi^+_\alpha(z) = (q-q^{-1})^{-1}E_\alpha(zq^{-\alpha}), \qquad \xi^-_\alpha(z) = (q-q^{-1})^{-1}F_\alpha(zq^{-\alpha})$$

$$\psi_\alpha(z) = K^+_{\alpha+1}(zq^{-\alpha})/K^+_\alpha(zq^{-\alpha}),\ \varphi_\alpha(z) = K^-_{\alpha+1}(zq^{-\alpha})/K^-_\alpha(zq^{-\alpha}). \ \square$$

Thus we have defined the algebras $\mathbf{U}(R)$ and $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$. The connection between the two was studied in detail in [DF], and the result is

Theorem 2.10 [DF]. *The assignment* (see the notation of theorem 2.5)

$$F_\alpha(z) \mapsto A^+_{\alpha,\alpha+1}(C^{-1} z^{-1}) - A^-_{\alpha,\alpha+1}(C\, z^{-1}),$$

$$E_\alpha(z) \mapsto A^+_{\alpha+1,\alpha}(C\, z^{-1}) - A^-_{\alpha+1,\alpha}(C^{-1} z^{-1}),$$

$$K^\pm_\beta(z) \mapsto A^\mp_{\beta\beta}(z^{-1}),$$

extends uniquely to an algebra isomorphism $\mathbf{U}_q(\widehat{\mathfrak{gl}_n}) \xrightarrow{\sim} \mathbf{U}(R)$. □

Let $\mathbf{U}$ be the quotient of the algebra $\mathbf{U}_q(\widehat{\mathfrak{gl}_n}) \simeq \mathbf{U}(R)$ modulo relations $C^{\pm 1} = 1$.

Remark 2.11. The assignment

$$K^\pm_\beta(z) \mapsto K^\pm_\beta(z), \qquad E_\alpha(z) \rightleftarrows F_\alpha(z), \qquad C \rightleftarrows C^{-1},$$

gives rise to an anti-involution on the algebra $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$ preserving both the subalgebra $\mathbf{U}_q(\widehat{\mathfrak{sl}_n})$ and the quotient algebra $\mathbf{U}$. □

3. Quantum groups and Hecke algebras via K-theory

This section is to a large extent a review of the constructions of [Gi], [GV] and [KL 2]; we refer the reader to [CG] for further details.

Given a complex linear algebraic group A and an algebraic A-variety Z, let $K^A(Z)$ stand for the *complexified Grothendieck group* of A-equivariant coherent sheaves on Z. Observe that $K^A(pt) = \mathbf{R}(A)$ is the complexified Representation ring of A, and for any A-variety Z the group $K^A(Z)$ has a natural $\mathbf{R}(A)$-module structure.

Let M_1, M_2, M_3 be smooth quasi-projective A-varieties. Let $p_{ij} : M_1 \times M_2 \times M_3 \longrightarrow M_i \times M_j$ denote the projection along the factor not named. The projections p_{ij} commute with the A-actions.

Let $Z_{ij} \subset M_i \times M_j$, $(i,j) = (1,2)$ or $(2,3)$, be A-stable closed subvarieties. Assume, in addition, that the map

$$p_{13}: \ p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) \to M_1 \times M_3 \qquad \textbf{is proper.} \tag{3.1}$$

Then we let $Z_{12} \circ Z_{23}$ denote its image, a closed A-stable subvariety of $M_1 \times M_3$. In that case there is a *convolution* on the equivariant K-groups

$$\star : \quad K^A(Z_{12}) \otimes K^A(Z_{23}) \ \longrightarrow \ K^A(Z_{12} \circ Z_{23}) \tag{3.2}$$

defined as follows. Let $[\mathcal{F}_{ij}] \in K^A(Z_{ij})$ be the classes of certain sheaves $\mathcal{F}_{ij}$ on Z_{ij}. Set (cf. [CG])

$$\mathcal{F}_{12} \star \mathcal{F}_{23} = (Rp_{13})_* \left(p_{12}^* \mathcal{F}_{12} \overset{L}{\underset{\mathcal{O}_{M_1 \times M_2 \times M_3}}{\bigotimes}} p_{23}^* \mathcal{F}_{23} \right)$$

In this formula the upper star stands for the pull-back morphism, well-defined for smooth maps (e.g., $p_{12}^* \mathcal{F}_{12} = \mathcal{F}_{12} \boxtimes \mathcal{O}_{M_3}$), and $\overset{L}{\otimes}$ for the derived tensor product.

Given an integer $d \geq 1$, let $\mathcal{F}$ denote (cf. [BLM]) the set of all n-step partial flags in $\mathbb{C}^d$ of the form $F = (0 = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = \mathbb{C}^d)$. The set $\mathcal{F}$ is a smooth projective variety whose connected components are parametrized by partitions $\mathbf{d} = (0 = i_0 \leq i_1 \leq \ldots \leq i_n = d)$, of the segment $[1,d]$ into n possibly empty segments $]i_0, i_1]$, $]i_1, i_2], \ldots$, $]i_{n-1}, i_n]$. The component $\mathcal{F}_{\mathbf{d}}$ associated to the partition $\mathbf{d}$ consists of all flags $F = (F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n)$ such that $\dim F_k = i_k$, $k = 0, 1, \ldots, n$. The group $G := GL(\mathbb{C}^d)$ acts naturally on each component $\mathcal{F}_{\mathbf{d}}$. The cotangent bundle on $\mathcal{F}$ can be identified in a natural way with the set $M = \{(F, x) \in \mathcal{F} \times \mathfrak{gl}_d \mid x(F_i) \subset F_{i-1}, \forall i = 1, 2, \ldots, n\}$.

Let N be the variety of all linear maps $x : \mathbb{C}^d \to \mathbb{C}^d$ such that $x^n = 0$. Define a morphism $\pi : M \to N$ to be the second projection $\pi : (F, x) \mapsto x$. Put $A_d := \mathbb{C}^* \times G$. Observe that M and N are A_d-varieties (the group G acts by conjugation and $z \in \mathbb{C}^*$ acts by multiplication by z^{-2} on $x \in N$). Furthermore, the variety M is smooth and π is a proper A_d-equivariant morphism.

Put $Z_d = M \times_N M \subset M \times M$. If there is no ambiguity we will simply write Z and A instead of Z_d and A_d. There are natural isomorphisms: $M \times M \simeq T^*\mathcal{F} \times T^*\mathcal{F} \simeq T^*(\mathcal{F} \times \mathcal{F})$ where the first isomorphism acts as multiplication by (-1) on the second factor (the sign of the standard symplectic 2-form on the second factor $T^*\mathcal{F}$ is also changed). The variety Z, viewed as a subvariety of $T^*(\mathcal{F} \times \mathcal{F})$, can be shown (see [CG]) to be the union of conormal bundles on all the G-orbits in $\mathcal{F} \times \mathcal{F}$. Furthermore, we have

$$Z \circ Z = (M \times_N M) \circ (M \times_N M) = M \times_N M = Z$$

It is clear also that Z is stable with respect to the diagonal A-action on $M \times M$. Thus the convolution makes $K^A(Z)$ an associative (non-commutative) algebra with unit. We recall one of the main results of [GV].

Theorem 3.3 [GV]. *There is a surjective algebra homomorphism:*

$$\mathbf{U} \twoheadrightarrow K^A(Z). \qquad \square$$

Given any semisimple element $a = (t, s) \in A$, let $\epsilon : \mathbf{R}(A) \to \mathbb{C}$ be the algebra homomorphism given by evaluation of characters of the group A at the point a, and let $\mathbb{C}_a$ denote the 1-dimensional $\mathbf{R}(A)$-module with $\chi \in \mathbf{R}(A)$ acting via the multiplication by $\epsilon(\chi)$. Further, let A^a denote the centralizer of a in A. We shall often view $\mathbb{C}_a$ as an $\mathbf{R}(A^a)$-module, by restriction. Given an A-variety Y, let Y^a denote the a-fixed point subvariety. In particular Y^a is an A^a-stable subvariety in Y.

Observe that, for $Z = M \times_N M$ and any semisimple element $a \in A$, we have $Z^a \circ Z^a = Z^a$. Thus, the K-group $K^{A^a}(Z^a)$ has again a natural associative algebra structure via convolution. Furthermore, the Bivariant Localization theorem (cf. [CG, ch.4]) yields a natural algebra homomorphism

$$r_a : \mathbb{C}_a \otimes_{\mathbf{R}(A^a)} K^A(Z) \to \mathbb{C}_a \otimes_{\mathbf{R}(A^a)} K^{A^a}(Z^a) \tag{3.4}$$

Fix a semisimple (diagonal) matrix $s \in G = GL_d$ with k distinct eigenvalues with multiplicities $d_1, d_2, \ldots, d_k$, and write $a = (1, s) \in \mathbb{C}^* \times G$. Then there are natural isomorphisms

$$G^s \simeq GL_{d_1} \times \ldots \times GL_{d_k} \quad , \quad Z^a \simeq Z_{d_1} \times \ldots \times Z_{d_k}$$

The algebra $\mathbf{U}(R)$ (see sect.2) has a *Hopf algebra* structure with *coproduct* morphism Δ given by the formula

$$\Delta : E^{\pm}_{\alpha\gamma}(z) \mapsto \sum_{\beta=1}^{n} E^{\pm}_{\alpha\beta}(C^{\pm1}z) \otimes E^{\pm}_{\beta\gamma}(C^{\mp1}z).$$

The result below provides a geometric interpretation of the coproduct in the quotient $\mathbf{U} = \mathbf{U}(R)/(C^{\pm1} = 1)$; cf. [Gr] for a closely related result.

Theorem 3.5. (i) *There is a natural Kunneth isomorphism*

$$K^{A^a}(Z^a) \simeq \Big(\bigotimes_{\mathbf{R}(A^a)}\Big)_{i=1}^{k} K^{A_{d_i}}(Z_{d_i})$$

(ii) *Moreover, the following natural diagram of algebra homomorphisms commutes*

$$\begin{array}{ccc}
\mathbf{U} & \xrightarrow{\qquad\qquad\Delta^k\qquad\qquad} & \mathbf{U}\otimes\ldots\otimes\mathbf{U} \\
\Big\downarrow{\scriptstyle thm.\ 3.3} & & \Big\downarrow{\scriptstyle thm.\ 3.3} \\
\mathbb{C}_a\otimes K^A(Z) \to \mathbb{C}_a\otimes K^{A^a}(Z^a) & \underset{part\ (i)}{=\!=\!=} & \mathbb{C}_a\otimes K^{A_{d_1}}(Z_{d_1})\otimes\ldots\otimes K^{A_{d_k}}(Z_{d_k})
\end{array}$$

Proof of the first claim can be deduced from a general Kunneth formula in equivariant algebraic K-theory of cellular fibrations (cf. [CG]). The second claim is verified by a direct computation based on explicit formulas for the vertical maps in the diagram, given in [GV]. $\square$

Theorem 3.3 is in fact an analogue of a geometric construction discovered earlier for *affine Hecke algebras.* Recall that the affine Hecke algebra $\mathbf{H} = \mathbf{H}_d$ of type GL_d is defined algebraically as the $\mathbb{C}[q, q^{-1}]$-algebra on generators $T_1, T_2, \ldots, T_{d-1}$, and $X_1^{\pm1}, \ldots, X_d^{\pm1}$ subject to the relations

$$(T_a - q^2)(T_a + 1) = 0,$$

$$T_a T_{a+1} T_a = T_{a+1} T_a T_{a+1} \quad , \quad [T_a, T_b] = 0 \quad \text{if} \quad |a - b| \geq 2, \qquad (3.6)$$

$$T_a X_a T_a = q^2 X_{a+1} \quad , \quad [X_a, T_b] = 0 \quad \text{if} \quad a \neq b,\, b+1.$$

Now let $\dot{\mathcal{F}}$ be the variety of all complete flags in $\mathbb{C}^d$ and $\dot{N}$ the variety of all nilpotent $d \times d$-matrices. The cotangent bundle on $\dot{\mathcal{F}}$ can be identified naturally (cf. [CG]) with the set $\dot{M} = \{(F, x) \in \dot{\mathcal{F}} \times \dot{N} \mid x(F_i) \subset F_{i-1}, \forall i = 1, 2, \ldots, d\}$. The second projection gives a proper morphism $\dot{M} \to \dot{N}$. Mimicing the construction at the beginning of the section, we put $\dot{Z} = \dot{M} \times_{\dot{N}} \dot{M} \subset \dot{M} \times \dot{M}$. The variety $\dot{Z}$, the so-called Steinberg variety, is clearly stable with respect to the diagonal action of A in $\dot{M} \times \dot{M}$. Furthermore, we have $\dot{Z} \circ \dot{Z} = \dot{Z}$. Thus the convolution makes $K^A(\dot{Z})$ an associative algebra. Here is an analogue of theorem 3.3:

Theorem 3.7 (cf. [Gi], [KL 2], [CG]). *There is an algebra isomorphism*

$$\mathbf{H} \simeq K^A(\dot{Z}). \qquad \square$$

4. The Polynomial Tensor Representation

For a finite dimensional vector space V, there is a canonical isomorphism $\mathrm{End}(V \otimes V) \simeq V^* \otimes V^* \otimes V \otimes V \simeq \mathrm{Hom}(\mathrm{End}V, \mathrm{End}V)$. Using this isomorphism, one may view the R-matrix (2.1) as a linear map

$$\hat{R} : \mathfrak{g} \to \mathbb{C}(X) \otimes \mathfrak{g}, \ \text{ i.e.,} \quad E_{ij} \mapsto \hat{R}(E_{ij})(X) = \sum_{k,l} R^{ijkl}(X)\, E_{kl}$$

where $R^{ijkl}(X)$ are rational $\mathbb{C}$-valued functions given by (2.1):

$$R(X) = \sum_{i,j,k,l} R^{ijkl}(X)\, E_{ij} \otimes E_{kl}$$

Define a linear map $\pi : \mathbf{U} \to \mathfrak{g}[X^{\pm 1}, q^{\pm 1}] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[X, X^{-1}, q, q^{-1}]$ by the following assignment of generating functions:

$$A^{\pm}(z) \mapsto \hat{R}(A)(z\,X), \qquad A \in \mathfrak{h}\,, \mathfrak{n}^+\,, \mathfrak{n}^- \tag{4.1}$$

where the right hand side is understood as a power series expansion in z^{-1} in the "$A^+(z)$" case, and in z in the "$A^-(z)$" case, respectively. In particular

$$(Id \otimes \pi)(L^{\pm}(z)) = R(z\,X). \tag{4.2}$$

Using the Yang-Baxter equation (2.3) and the unitarity condition, one deduces the following result.

Proposition 4.3. *The map π defined by (4.1) is an algebra homomorphism.*$\square$

The standard $\mathfrak{g}$-action on the vector space $\mathbb{C}^n$ gives rise to a $\mathbb{C}[X^{\pm 1}, q^{\pm 1}]$-linear action of the algebra $\mathfrak{g}[X^{\pm 1}, q^{\pm 1}]$ on the vector space $\mathbf{P} := \mathbb{C}^n[X^{\pm 1}, q^{\pm 1}]$. This makes $\mathbf{P}$ an $\mathbf{U}(R)$-module via the homomorphism π, cf. (4.1). In a similar way, there is an algebra homomorphism $\rho : \mathbf{U}_q(\widehat{\mathfrak{gl}_n}) \to \mathfrak{g}[X^{\pm 1}, q^{\pm 1}]$, which is a special case of the general construction [GV, (4.1)] in the particular case $d = 1$ (in [GV] the notation $K = \mathbb{C}^n[X^{\pm 1}, q^{\pm 1}]$ was used). The homomorphism ρ is given by the following formulas

$$E_\alpha(z) \mapsto (q - q^{-1})\,\delta(\tfrac{X}{z})\,E_{\alpha\,\alpha+1} \tag{4.4.1}$$

$$F_\alpha(z) \mapsto (q - q^{-1})\,\delta(\tfrac{X}{z})\,E_{\alpha+1\,\alpha} \tag{4.4.2}$$

$$K^{\pm}_{\beta}(z) \mapsto \sum_{\gamma<\beta} \theta^{\pm}_{-1}(q^{-1}\tfrac{X}{z})\,E_{\gamma\gamma} + E_{\beta\beta} + \sum_{\gamma>\beta} \theta^{\pm}_{-1}(q\tfrac{X}{z})\,E_{\gamma\gamma} \tag{4.4.3}$$

where $\theta^{\pm}_{m}(z)$ is the power series expansion in $z^{\mp 1}$ of the rational function (2.2).

Observe that the homomorphism ρ makes $\mathbf{P}$ an $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$-module via the pull-back of the natural $\mathfrak{g}[X^{\pm 1}, q^{\pm 1}]$-module structure. A straightforward computation based on formulas (2.10) and (4.4.1-3) yields the following result.

Proposition 4.5. *The above defined $\mathbf{U}(R)$- and $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$-module structures on $\mathbf{P} = \mathbb{C}[X^{\pm 1}, q^{\pm 1}]$ are compatible with the isomorphism of theorem 2.10, i.e. the following algebra homomorphisms form a com-*

mutative triangle

$$
\begin{tikzcd}
\mathbf{U}_q(\widehat{\mathfrak{gl}_n}) \arrow[rr, equal, "thm.\ 2.10"] \arrow[dr, "\rho"'] & & \mathbf{U}(R) \arrow[dl, "\pi"] \\
& \mathfrak{g}[X^{\pm 1}, q^{\pm 1}] &
\end{tikzcd}
$$

The coproduct on the Hopf algebra $\mathbf{U}$ provides, for each integer $d \geq 1$, the composition of algebra homomorphisms:

$$\pi^d : \mathbf{U} \to \mathbf{U}^{\otimes d} \to (\mathfrak{g}[X^{\pm 1}, q^{\pm 1}])^{\otimes d} \simeq \mathfrak{g}^{\otimes d}[X_1^{\pm 1}, \dots, X_d^{\pm 1}, q^{\pm 1}] \quad (4.6)$$

Here the first arrow is given by the iterated coproduct map (tensor product is taken over $\mathbb{C}[q, q^{-1}]$), and the second arrow is a tensor power of the homomorphism π introduced above. Formula (4.2) yields the following property of the map (4.6)

$$(Id \otimes \pi^d)(L^{\pm}(z)) = R_{0,d}(z\, X_d)\, R_{0,d-1}(z\, X_{d-1}) \dots R_{0,1}(z\, X_1). \quad (4.7)$$

Set

$$\mathbf{P}^{\otimes d} = \mathbf{P} \otimes_{\mathbb{C}[q,q^{-1}]} \dots \otimes_{\mathbb{C}[q,q^{-1}]} \mathbf{P} \simeq (\mathbb{C}^n)^{\otimes d}[X_1^{\pm 1}, \dots, X_d^{\pm 1}, q^{\pm 1}]$$

Observe that the rightmost vector space above has a natural action of the associative algebra $\mathfrak{g}^{\otimes d}[X_1^{\pm 1}, \dots, X_d^{\pm 1}, q^{\pm 1}] \simeq (\mathfrak{g}[X^{\pm 1}, q^{\pm 1}])^{\otimes d}$. That makes $\mathbf{P}^{\otimes d}$ an $\mathbf{U}$-module via the homomorphism (4.6). It will be referred to as the *Polynomial Tensor Representation.*

Let $e_1, \dots, e_n$ be the standard base of $\mathbb{C}^n$ and $[1, n]^d$ the set of d-tuples of integers from the segment $[1, n]$. Then the vectors

$$e_{\mathbf{j}} = e_{j_1} \otimes \dots \otimes e_{j_d} \quad , \quad \mathbf{j} = (j_1, j_2, \dots, j_d) \in [1, n]^d$$

form the standard base of $(\mathbb{C}^n)^{\otimes d}$. The Symmetric group S_d acts naturally on the set $[1, n]^d$ by permutations of elements of a d-tuple $(j_1, j_2, \dots, j_d)$. That induces the standard S_d-action $\sigma : e_{\mathbf{j}} \mapsto e_{\sigma(\mathbf{j})}$ on $(\mathbb{C}^n)^{\otimes d}$. Let further $P \mapsto P^\sigma$ be the natural S_d-action on the polynomial ring $\mathbb{C}[X_1^{\pm 1}, \dots, X_d^{\pm 1}, q^{\pm 1}]$ by permutation of the variables $X_1, \dots, X_d$. The simultaneous actions on $(\mathbb{C}^n)^{\otimes d}$ and on the polynomial ring give rise to an S_d-module structure on the tensor product $\mathbf{P}^{\otimes d} = (\mathbb{C}^n)^{\otimes d}[X_1^{\pm 1}, \dots, X_d^{\pm 1}, q^{\pm 1}]$. There is also a natural action of the additive group $\mathbb{Z}^d$ on the same space $\mathbf{P}^{\otimes d}$ by multiplication $\mathbb{Z}^d \ni (m_1, \ldots, m_d) : P \mapsto P \cdot X_1^{m_1} \cdot \ldots \cdot X_d^{m_d}$. The S_d- and $\mathbb{Z}^d$-module structures fit together making $\mathbf{P}^{\otimes d}$ a module over the semi-direct product $W_{af} := S_d \ltimes \mathbb{Z}^d$, the affine Weyl group.

We now 'quantize' the above defined W_{af}-action by introducing a right $\mathbf{H}$-module structure on $\mathbf{P}^{\otimes d}$, where $\mathbf{H}$ is the affine Hecke algebra. For $\alpha = 1, 2, \ldots, n-1$, let $s_\alpha \in S_d$ be the transposition $(\alpha, \alpha+1)$ and T_α the corresponding generator of the Hecke algebra (cf. (3.6)). The vector space $\mathbf{P}^{\otimes d}$ is spanned over $\mathbb{C}$ by elements of the form $e_{\mathbf{j}} \otimes P$, $P \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}, q^{\pm 1}]$. For each $\alpha = 1, 2, \ldots, n-1$ and $P \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}, q^{\pm 1}]$, we put

$$(e_{\mathbf{j}} \otimes P) \cdot X_\alpha = e_{\mathbf{j}} \otimes (P \cdot X_\alpha) \tag{4.8}$$

$$(e_{\mathbf{j}} \otimes P) \cdot T_\alpha = \begin{cases} (1-q^2) e_{\mathbf{j}} \otimes \frac{X_{\alpha+1}(P^{s_\alpha} - P)}{X_{\alpha+1} - X_\alpha} + e_{s_\alpha(\mathbf{j})} \otimes P^{s_\alpha} & \text{if } j_\alpha < j_{\alpha+1}, \\ (1-q^2) e_{\mathbf{j}} \otimes \frac{X_{\alpha+1}(P^{s_\alpha} - P)}{X_{\alpha+1} - X_\alpha} + q^2 e_{\mathbf{j}} \otimes P^{s_\alpha} & \text{if } j_\alpha = j_{\alpha+1}, \\ (1-q^2) e_{\mathbf{j}} \otimes \frac{X_\alpha P^{s_\alpha} - X_{\alpha+1} P}{X_{\alpha+1} - X_\alpha} + q^2 e_{s_\alpha(\mathbf{j})} \otimes P^{s_\alpha} & \text{if } j_\alpha > j_{\alpha+1}, \end{cases}$$

In the next section we will prove the following result.

Theorem 4.9. *The above formulas give rise to a well-defined right $\mathbf{H}_d$-module structure on $\mathbf{P}^{\otimes d}$.*

Recall that the Representation ring $\mathbf{R}(A)$ is isomorphic to

$$\mathbf{R}(A) = \mathbf{R}(\mathbb{C}^* \times GL_d) = \mathbb{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}, q^{\pm 1}]^{S_d},$$

the algebra of symmetric polynomials. Formulas (4.8) yield

Corollary 4.10. (i) *The $\mathbf{H}$-action on $\mathbf{P}^{\otimes d}$ is $\mathbf{R}(A)$-linear.*

(ii) *Let $\mathbf{j} \in [1, n]^d$ be a non-decreasing d-tuple, and $\mathbf{j}'$ a d-tuple obtained from it by permutation. Let $w \in S_d$ be a permutation of minimal length such that $\mathbf{j}' = w(\mathbf{j})$. Then we have* $(e_{\mathbf{j}} \otimes 1) \cdot T_w = e_{\mathbf{j}'} \otimes 1$. □

The right $\mathbf{H}$-module structure on $\mathbf{P}^{\otimes d}$ enables us to describe the left $\mathbf{U}$-module structure on $\mathbf{P}^{\otimes d}$ given by the Polynomial Tensor Representation in a rather explicit way. We have the following result.

Theorem 4.11. *The $\mathbf{U}$-action arising from homomorphism (4.6) is the unique $\mathbf{R}(A)$-linear left $\mathbf{U}$-module structure on $\mathbf{P}^{\otimes d}$ that commutes with the right $\mathbf{H}$-action and such that, for any non-decreasing d-tuple $\mathbf{j}$, we have*

$$E_\alpha(z) \cdot e_{\mathbf{j}} = (q^2 - 1)q^{-\#\mathbf{j}^{-1}(\alpha+1)} e_{\mathbf{j}^-_{s+1}} \cdot \delta(\tfrac{X_{s+1}}{z})(1 + \sum_{m=s+1}^{t-1} T_{s+1}T_{s+2}\cdots T_m) \tag{4.12.1}$$

$$F_\alpha(z) \cdot e_{\mathbf{j}} = (q^2 - 1)\, q^{-\#\mathbf{j}^{-1}(\alpha)} e_{\mathbf{j}^+_s} \cdot \delta(\tfrac{X_s}{z})(1 + \sum_{m=r+1}^{s-1} T_{s-1}T_{s-2}\cdots T_m) \tag{4.12.2}$$

$$K^\pm_\alpha(z) \cdot e_{\mathbf{j}} = \prod_{l_m<\alpha} \theta^\pm_{-1}\left(q^{-1}\tfrac{X_m}{z}\right) \prod_{l_m>\alpha} \theta^\pm_{-1}\left(q\tfrac{X_m}{z}\right) e_{\mathbf{j}}, \tag{4.12.3}$$

where $]r,s] := \mathbf{j}^{-1}(\alpha)$, $]s,t] := \mathbf{j}^{-1}(\alpha+1)$, and one puts

$$\begin{cases} e_{\mathbf{j}^-_{s+1}} = 0 & \text{if } \mathbf{j}^{-1}(\alpha+1) = \emptyset, \\ \mathbf{j}^-_{s+1} = (j_1, \dots, j_s, \alpha, j_{s+2}, \dots, j_d) & \text{else}, \end{cases}$$

$$\begin{cases} e_{\mathbf{j}^+_s} = 0 & \text{if } \mathbf{j}^{-1}(\alpha) = \emptyset, \\ \mathbf{j}^+_s = (j_1, \dots, j_{s-1}, \alpha+1, j_{s+1}, \dots, j_d) & \text{else}. \end{cases}$$

5. The Polynomial Tensor Representation via K-theory

In this section and in section 7 we will give three geometric constructions of the Polynomial Tensor Representation. The first construction, presented here, is based on the equivariant K-theory. The other two, of section 7, involve affine flag varieties over a finite field and are dual to the first one in the sense of Langlands. Those constructions may be viewed as an extension to the affine case of the Grojnowski-Lusztig construction [GL] in the finite case.

Recall the n-step flag variety $\mathcal{F}$ and the complete flag variety $\dot{\mathcal{F}}$ introduced in section 3. The corresponding cotangent bundles $M = T^*\mathcal{F}$ and $\dot{M} = T^*\dot{\mathcal{F}}$ have natural $A = \mathbb{C}^* \times GL_d$-actions. Following [GL], we introduce the A-stable subvariety $W = M \times_N \dot{M} \subset M \times \dot{M}$. The variety W 'links' Z with $\dot{Z}$ and it will play a major role in the rest of the paper. Observe that we have

$$Z \circ W = W\,, \quad W \circ \dot{Z} = W$$

Hence there are natural convolution morphisms

$$K^A(Z) \otimes K^A(W) \to K^A(W)\,, \quad K^A(W) \otimes K^A(\dot{Z}) \to K^A(W) \quad (5.1)$$

Thus the group $K^A(W)$ has natural *left* $K^A(Z)$-module and *right* $K^A(\dot{Z})$-module structures respectively. Here is one of the main results of this section.

Theorem 5.2. *There is an* $\mathbf{R}(A)$*-linear isomorphism* $\psi : \mathbf{P}^{\otimes d} \simeq K^A(W)$ *that intertwines both the left* $\mathbf{U}$*- and the right* $\mathbf{H}$*-module structures, i.e., makes the following diagram commute*

$$\begin{tikzcd}
\mathbf{U} \arrow[r, "(4.6)"] \arrow[d, "thm.\ 3.3"'] & \operatorname{End} \mathbf{P}^{\otimes d} \arrow[d, equal, "\psi"] & \mathbf{H} \arrow[l, "thm.\ 4.8"'] \arrow[d, equal, "thm.\ 3.7"] \\
K^A(Z) \arrow[r, "(5.1)"'] & \operatorname{End} K^A(W) & K^A(\dot{Z}) \arrow[l, "(5.1)"]
\end{tikzcd}$$

Sketch of Proof. Following [GL] we observe that the G-orbits in $\mathcal{F} \times \dot{\mathcal{F}}$ are parametrized by the set $[1, n]^d$. To a d-tuple $\mathbf{j} = (j_1, j_2, \ldots, j_d)$ one associates the G-orbit $\mathcal{W}_\mathbf{j}$ of the pair $(F, \dot{F})$ of flags

$$F = (\{0\} \subseteq \bigoplus_{k_i \leq 1} \mathbb{C} \cdot e_i \subseteq \bigoplus_{k_i \leq 2} \mathbb{C} \cdot e_i \subseteq \ldots \subseteq \mathbb{C}^d)$$

$$\dot{F} = (\{0\} \subset \mathbb{C} \cdot e_1 \subset \mathbb{C} \cdot e_1 \oplus \mathbb{C} \cdot e_2 \subset \ldots \subset \mathbb{C}^d)$$

where $e_1, \ldots, e_d$ is the standard base of $\mathbb{C}^d$. Further, set

$$\mathbf{a}(\mathbf{j}) = \sum_{1 \leq i < k \leq n} \#\mathbf{j}^{-1}(i) \cdot \#\mathbf{j}^{-1}(k) \quad (5.3)$$

The integer $\mathbf{a}(\mathbf{j})$ is shown to be the dimension of the fiber of the first projection $\mathcal{F} \times \dot{\mathcal{F}} \supset \mathcal{W}_\mathbf{j} \to \mathcal{F}$.

Now let $\mathbf{j} \in [1,n]^d$ be a *nondecreasing* d-tuple. We show that the corresponding G-orbit $\mathcal{W}_\mathbf{j}$ is closed. Let $\Omega_\mathbf{j}$ of the sheaf of maximal rank relative forms along the first projection $\mathcal{W}_\mathbf{j} \to \mathcal{F}$ (see [V, §2]). Let $[\pi^*\Omega_\mathbf{j}] \in K^A(W)$ be the class of the pull-back of $\Omega_\mathbf{j}$ via the natural conormal bundle projection $\pi : T^*_{\mathcal{W}_\mathbf{j}}(\mathcal{F} \times \dot{\mathcal{F}}) \to \mathcal{W}_\mathbf{j}$. Thus, the sheaf $\pi^*\Omega_\mathbf{j}$ is supported on $T^*_{\mathcal{W}_\mathbf{j}}(\mathcal{F} \times \dot{\mathcal{F}})$, and we set

$$\mathcal{E}_\mathbf{j} = q^{\mathbf{a}(\mathbf{j})} \cdot [\pi^*\Omega_\mathbf{j}] \in K^A(W)$$

For a general $\mathbf{j} \in [1,n]^d$, which is not necessarily non-decreasing, we define a class $\mathcal{E}_\mathbf{j} \in K^A(W)$ by means of the isomorphism of theorem 3.7 as follows. First, find a *non-decreasing* d-tuple $\mathbf{j}_+$ obtained from $\mathbf{j}$ by a permutation. Let $w \in S_d$ be a permutation of minimal length such that $\mathbf{j} = w(\mathbf{j}_+)$ and $T_w \in \mathbf{H}$ the corresponding element of the Hecke algebra. Let $[T_w] \in K^A(\dot{Z})$ be the image of T_w under the isomorphism of theorem 3.7. The K-group $K^A(\dot{Z})$ acts on $K^A(W)$ on the right via convolution, and we put by definition $\mathcal{E}_\mathbf{j} := \mathcal{E}_{\mathbf{j}_+} \star [T_w]$.

Next let $\Delta \subset \dot{\mathcal{F}} \times \dot{\mathcal{F}}$ be the diagonal and $\dot{Z}_\Delta \subset \dot{Z}$ its conormal bundle. The group $K^A(\dot{Z}_\Delta)$ is a subalgebra of $K^A(\dot{Z})$. Moreover, this subalgebra is isomorphic to $\mathbb{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}, q^{\pm 1}]$ in a canonical way (cf. [CG]). Hence, the right $K^A(\dot{Z}_\Delta)$-action on $K^A(W)$ gives an $\mathbb{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}, q^{\pm 1}]$-module structure on $K^A(W)$. We define an $\mathbf{R}(A)$-linear map $\psi : \mathbf{P}^{\otimes d} \to K^A(W)$ by the following assingment

$$(\mathbb{C})^{\otimes d}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}, q^{\pm 1}] \ni e_\mathbf{j} \otimes P \;\mapsto\; \mathcal{E}_\mathbf{j} \star P$$

This map can be shown to be bijective. Furthermore, one verifies by a lengthy but straightforward computation that the $K^A(Z)$- and $K^A(\dot{Z})$-actions on $K^A(W)$ correspond via the bijection ψ to the action given by the formulas (4.8) and (4.12) respectively. That completes the proof of the theorem. □

Remark 5.4. The commutativity of the right square in the diagram of theorem 5.2, combined with theorem 3.7, implies theorem 4.9. □

Next fix a diagonal matrix $s \in GL_d$ with k distinct eigenvalues with multiplicities $d_1, d_2, \ldots, d_k$. Put $a = (s, 1) \in GL_d \times \mathbb{C}^*$. Then $Z^a \circ W^a = W^a$, so that the group $K^{A^a}(W^a)$ acquires a $K^{A^a}(Z^a)$-module structure. Moreover, we have natural isomorphisms

$$Z^a \simeq Z_{d_1} \times \ldots \times Z_{d_k} \qquad\qquad W^a \simeq W_{d_1} \times \ldots \times W_{d_k}.$$

Proposition 5.5. (i) *There is a Künneth isomorphism*

$$K^{A^a}(W^a) \simeq \bigotimes_{i=1}^{k} K^{A_{d_i}}(W_{d_i})$$

here the tensor product is taken over $\mathbb{C}[q, q^{-1}]$.

(ii) *The following diagram of algebra homomorphisms commutes:*

$$\begin{tikzcd}
K^A(Z)_a \arrow[r] \arrow[d, "(5.1)"] & K^{A^a}(Z^a)_a \arrow[r, equal, "3.5(i)"] \arrow[d, "(5.1)"] & \left(\bigotimes_{\mathbf{R}(A^a)}\right)_{i=1}^{k} K^{A_{d_i}}(Z_{d_i})_a \arrow[d, "(5.1)"] \\
\mathrm{End}_{\mathbf{R}(A)} K^A(W)_a \arrow[r] \arrow[d, equal] & \mathrm{End}_{\mathbf{R}(A^a)} K^{A^a}(W^a)_a \arrow[r, equal] & \left(\bigotimes_{\mathbf{R}(A^a)}\right)_{i=1}^{k} \mathrm{End}_{\mathbf{R}(A_{d_i})} K^{A_{d_i}}(W_{d_i})_a \arrow[d, equal] \\
\mathrm{End}_{\mathbf{R}(A)} \mathbf{P}_a^{\otimes d} \arrow[r] & \mathrm{End}_{\mathbf{R}(A^a)} \mathbf{P}_a^{\otimes(d_1+\ldots+d_k)} \arrow[r, equal] & \left(\bigotimes_{\mathbf{R}(A^a)}\right)_{i=1}^{k} \mathrm{End}_{\mathbf{R}(A_{d_i})} \mathbf{P}_a^{\otimes d_i}
\end{tikzcd}$$

Where the subscript 'a' indicates specialization at the point $a \in \operatorname{Specm} \mathbf{R}(A)$, e.g., $K^A(Z)_a := \mathbb{C}_a \otimes_{\mathbf{R}(A)} K^A(Z)$.

6. Geometric Weyl correspondence

In this section we establish geometrically an equivalence between the category of finite dimensional $\mathbf{H}$-module and a certain category of finite dimensional $\mathbf{U}$-modules (cf. [Ch 2], [Dr 3]).

We first remind the results of [Gi],[GV] (see also [KL 2] and [CG]) about simple finite dimensional modules over the algebras $\mathbf{U}$ and $\mathbf{H}$. They have a similar geometric description. Observe that in the setup of (4.1), (4.2) the K-theoretic convolution has a counterpart in *Borel-Moore homology* with complex coefficients defined e.g. in [CG, ch. 2] :

$$\star : \ H_k(Z_{12}) \times H_l(Z_{23}) \longrightarrow H_{l+k-\dim M_2}(Z_{12} \circ Z_{23})$$

and there is a *bivariant Riemann-Roch* map (see [CG]):

$$c : \ K(Z_{ij}) \longrightarrow H_*(Z_{ij}) \tag{6.1}$$

which commutes with convolution. Moreover, for varieties like Z and $\dot{Z}$, which are built out of complex cells, the Riemann-Roch map (6.1) is known (see [CG, Cellular Fibration Lemma]) to be an isomorphism.

Fix a semisimple element $a = (t, s) \in A$ and the corresponding 1-dimensional $\mathbf{R}(A)$-module $\mathbb{C}_a$ given by evaluation at a. We have a chain of natural morphisms :

$$\mathbb{C}_a \otimes_{\mathbf{R}(A)} K^A(Z) \xrightarrow{(3.4)} \mathbb{C}_a \otimes_{\mathbf{R}(A)} K^{A^a}(Z^a) \xrightarrow{forget} K(Z^a) \xrightarrow{(6.1)} H_*(Z^a) \tag{6.2}$$

all of which commute with convolution. Furthermore, for varieties built out of complex cells, each of the morphisms above is actually an isomorphism (cf. [CG] for details). It follows that the composite map yields *algebra isomorphisms*

$$\mathbb{C}_a \otimes_{\mathbf{R}(A)} K^A(Z) \xrightarrow{\sim} H_*(Z^a)$$

$$\mathbb{C}_a \otimes_{\mathbf{R}(A)} K^A(\dot{Z}) \xrightarrow{\sim} H_*(\dot{Z}^a).$$

Thus from Theorem 4.3 and (6.1) we have *surjective* algebra homomorphisms

$$\mathbf{U} \twoheadrightarrow H_*(Z^a) \quad , \quad \mathbf{H} \twoheadrightarrow H_*(\dot{Z}^a) \tag{6.3}$$

Next, let $\dot{N} \subset \mathrm{End}(\mathbb{C}^d)$ denote the variety of all nilpotent endomorphisms, and $\dot{N}^a$ the fixed point subvariety. Explicitly, we have $\dot{N}^a = \{x \in \dot{N} \mid s \cdot x \cdot s^{-1} = t^2 \cdot x\}$ and $N^a = \{x \in \dot{N}^a \mid x^n = 0\}$. The centralizer G^s of s in G acts on M^a, N^a (resp. on $\dot{M}^a$, $\dot{N}^a$) in such a way that the projection $\pi : M^a \longrightarrow N^a$ (resp. $\dot{\pi} : \dot{M}^a \longrightarrow \dot{N}^a$) is G^s-equivariant. We form the following G^s-equivariant constructible complexes

$$\mathcal{L} = \pi_* \mathbb{C}_{M^a}[\dim M^a] \in D^b(N^a), \qquad \dot{\mathcal{L}} = \dot{\pi}_* \mathbb{C}_{\dot{M}^a}[\dim \dot{M}^a] \in D^b(\dot{N}^a)$$

The varieties N^a and $\dot{N}^a$ are known to consist of finite union of G^s-orbits each. Moreover, any G^s-equivariant local system on any G^s-orbit $\mathbf{O}$ is constant, for its isotropy group is connected. Thus, $\mathcal{L}$ and $\dot{\mathcal{L}}$ decompose in the following way (Decomposition theorem [BBD]):

$$\mathcal{L} = \bigoplus_{\mathbf{O}\subset N^a,\, i\in\mathbb{Z}} L_{\mathbf{O}}[i] \otimes IC_{\mathbf{O}}[i] \tag{6.4.1}$$

$$\dot{\mathcal{L}} = \bigoplus_{\mathbf{O}\subset \dot{N}^a,\, i\in\mathbb{Z}} \dot{L}_{\mathbf{O}}[i] \otimes IC_{\mathbf{O}}[i] \tag{6.4.2}$$

where $IC_{\mathbf{O}}$ is the *Intersection cohomology complex* associated with the constant sheaf on $\mathbf{O}$ and $L_{\mathbf{O}}[i]$, $\dot{L}_{\mathbf{O}}[i]$, are certain finite dimensional vector spaces. Put $L_{\mathbf{O}} := \oplus_i L_{\mathbf{O}}[i]$ and $\dot{L}_{\mathbf{O}} := \oplus_i \dot{L}_{\mathbf{O}}[i]$. Then (6.4.1-2) yield

$$\operatorname{Ext}(\mathcal{L},\mathcal{L}) \simeq \bigoplus_{\mathbf{O}\subset N^a} \operatorname{Hom}(L_{\mathbf{O}}, L_{\mathbf{O}}) \bigoplus \bigoplus_{\mathbf{O},\mathbf{O}'\subset N^a,\, k>0} \operatorname{Hom}(L_{\mathbf{O}}, L_{\mathbf{O}'}) \otimes \operatorname{Ext}^k(IC_{\mathbf{O}}, IC_{\mathbf{O}'}) \tag{6.5.1}$$

$$\operatorname{Ext}(\dot{\mathcal{L}},\dot{\mathcal{L}}) \simeq \bigoplus_{\mathbf{O}\subset \dot{N}^a} \operatorname{Hom}(\dot{L}_{\mathbf{O}}, \dot{L}_{\mathbf{O}}) \bigoplus \bigoplus_{\mathbf{O},\mathbf{O}'\subset \dot{N}^a,\, k>0} \operatorname{Hom}(\dot{L}_{\mathbf{O}}, \dot{L}_{\mathbf{O}'}) \otimes \operatorname{Ext}^k(IC_{\mathbf{O}}, IC_{\mathbf{O}'}). \tag{6.5.2}$$

We now introduce the fixed point subvarieties Z^a and $\dot{Z}^a$. Observe that we have

$$Z^a = M^a \times_{N^a} M^a \quad , \quad \dot{Z}^a = \dot{M}^a \times_{\dot{N}^a} \dot{M}^a$$

It follows (see [Gi],[CG]) that there are natural algebra isomorphisms (not grading-preserving):

$$H_*(Z^a) \simeq \operatorname{Ext}(\mathcal{L},\mathcal{L}) \quad , \quad H_*(\dot{Z}^a) \simeq \operatorname{Ext}(\dot{\mathcal{L}},\dot{\mathcal{L}}) \tag{6.6}$$

We see that the second sums in both (6.5.1) and (6.5.2) are the *radicals* $\operatorname{Rad}(H_*(Z^a))$, $\operatorname{Rad}(H_*(\dot{Z}^a))$ of the algebras $H_*(Z^a)$ and $H_*(\dot{Z}^a)$ respectively, while the first sums are, respectively, their maximal semisimple quotients. Thus, formulas (6.3) and (6.6) yield the following result, cf. [CG].

Theorem 6.7 [Gi], [GV]. (i) *Each vector space $L_{\mathbf{O}}$ (resp. $\dot{L}_{\mathbf{O}}$) has a natural* $\mathbf{U}$*-module (resp.* $\mathbf{H}$*-module) structure.*

(ii) *Fix $t \in \mathbb{C}^*$ which is not a root of unity. Then the set $\{L_{\mathbf{O}}\}_{a,\mathbf{O}}$ is a collection of non-isomorphic simple finite dimensional*

$\mathbf{U}_{|q=t}$*-modules, and the set* $\{\dot{L}_{\mathrm{O}}\}_{a,\mathrm{O}}$ *is a complete collection of isomorphism classes of simple* $\mathbf{H}_{|q=t}$*-modules.*

We now state the main result of the section, which is a quantum affine counterpart of the classical Weyl correspondence between simple GL_n- and S_d-modules in the tensor representation.

Theorem 6.8. *Fix a positive integer* $d \leq n$, *and* $t \in \mathbb{C}^*$ *which is not a root of unity. Then the functor*

$$V \mapsto \mathbf{P}^{\otimes d} \otimes_{\mathbf{H}_d} V$$

provides an equivalence between the category of finite dimensional $\mathbf{H}_d$*-modules and the category of finite dimensional* $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$*-modules whose simple components are among the* $\{L_{\mathrm{O}}\}_{a,\mathrm{O}}$ *with fixed d.*

Sketch of proof: The arguments are similar to those leading to theorem 6.7 with the variety Z being replaced by W. First, given any semisimple element $a \in A$, one has

$$W^a = M^a \times_{N^a} \dot{M}^a. \tag{6.9}$$

It follows from (6.3) and from theorem 4.3 that both the K-group $K^A(W)$ and the homology group $H_*(W^a)$ have left $\mathbf{U}$-module structures and right $\mathbf{H}$-module structures each. Moreover, it was shown in [CG] that those structures commute with the localization isomorphism, cf. [CG] and (6.2) :

$$c \circ r_a : \mathbb{C}_a \otimes_{\mathbf{R}(A)} K^A(W) \xrightarrow{\sim} H_*(W^a).$$

Assume from now on that $d \leq n$. Then, for any nilpotent operator x in $\mathbb{C}^d$ we have $x^n = 0$, hence $x^d = 0$. It follows that $N = \dot{N}$. Formula (6.9) and the general results of [CG, ch. 7] yield a natural isomorphism

$$H_*(W^a) \simeq \operatorname{Ext}(\dot{\mathcal{L}}, \mathcal{L}).$$

Thus, using the decompositions (6.4.1-2) one finds

$$H_*(W^a) \simeq \bigoplus_{\mathbf{O}\subset N^a} \mathrm{Hom}(\dot{L}_{\mathbf{O}}, L_{\mathbf{O}}) \quad \bigoplus \tag{6.10}$$
$$\bigoplus_{\mathbf{O},\mathbf{O}'\subset N^a,\, k>0} \mathrm{Hom}(\dot{L}_{\mathbf{O}'}, L_{\mathbf{O}}) \otimes \mathrm{Ext}^k(IC_{\mathbf{O}'}, IC_{\mathbf{O}}).$$

Similarly, as we have observed earlier, formulas (6.5.2) and (6.6) yield an algebra isomorphism

$$H_*(\dot{Z}^a) \simeq \Big(\bigoplus_{\mathbf{O}\subset N^a} \mathrm{End}\, \dot{L}_{\mathbf{O}}\Big) \bigoplus \Big(\bigoplus_{\mathbf{O}.\mathbf{O}'\subset N^a,\, k>0} \mathrm{Hom}(\dot{L}_{\mathbf{O}}, \dot{L}_{\mathbf{O}'}) \otimes \mathrm{Ext}^k(IC_{\mathbf{O}}, IC_{\mathbf{O}'})\Big)$$

Let $p_{\mathbf{O}}$ denote the identity element of the simple algebra $\mathrm{End}\,\dot{L}_{\mathbf{O}}$, viewed as the projector to the corresponding simple direct summand in the maximal semisimple subalgebra

$$\mathcal{A} := \bigoplus_{\mathbf{O}\subset N^a} \mathrm{End}\, \dot{L}_{\mathbf{O}} \quad \subset \quad H_*(\dot{Z}^a)$$

The projectors form the canonical central decomposition $1 = \sum_{\mathbf{O}} p_{\mathbf{O}} \in \mathcal{A}$. It follows from the formulas above that, for any $\mathbf{O} \subset N^a$, the right ideal $p_{\mathbf{O}} \cdot H_*(\dot{Z}^a) \subset H_*(\dot{Z}^a)$ has the form

$$p_{\mathbf{O}} H_*(\dot{Z}^a) = \mathrm{End}\, \dot{L}_{\mathbf{O}} \;\bigoplus\; \Big(\bigoplus_{\mathbf{O}'\subset N^a,\, k>0} \mathrm{Hom}(\dot{L}_{\mathbf{O}'}, \dot{L}_{\mathbf{O}}) \otimes \mathrm{Ext}^k(IC_{\mathbf{O}'}, IC_{\mathbf{O}})\Big) \tag{6.11}$$

Hence, we find

$$\mathrm{Hom}(\dot{L}_{\mathbf{O}}, L_{\mathbf{O}}) \bigoplus \Big(\bigoplus_{\mathbf{O}'\subset N^a,\, k>0} \mathrm{Hom}(\dot{L}_{\mathbf{O}'}, L_{\mathbf{O}}) \otimes \mathrm{Ext}^k(IC_{\mathbf{O}'}, IC_{\mathbf{O}})\Big) =$$
$$= \mathrm{Hom}(\dot{L}_{\mathbf{O}}, L_{\mathbf{O}}) \bigotimes_{\mathcal{A}} \Big(\mathrm{End}\, \dot{L}_{\mathbf{O}} \bigoplus_{\mathbf{O}'\subset N^a,\, k>0} \mathrm{Hom}(\dot{L}_{\mathbf{O}'}, \dot{L}_{\mathbf{O}}) \otimes \mathrm{Ext}^k(IC_{\mathbf{O}'}, IC_{\mathbf{O}})\Big)$$

The last expression can be rewritten, by (6.11), as

$$\mathrm{Hom}(\dot{L}_{\mathbf{O}}, L_{\mathbf{O}}) \bigotimes_{\mathcal{A}} p_{\mathbf{O}} \cdot H_*(\dot{Z}^a) \quad ,$$

which implies that it is a projective right $H_*(\dot{Z}^a)$-module, for $p_{\mathbf{O}} \cdot H_*(\dot{Z}^a)$ is a direct summand of the free module $H_*(\dot{Z}^a)$. Observe now that the RHS of (6.10) is a direct sum of right $H_*(\dot{Z}^a)$-modules of that form. Thus, we conclude that

$$H_*(W^a) \simeq \bigoplus_{\mathbf{O} \subset N^a} \left(\mathrm{Hom}(\dot{L}_{\mathbf{O}}, L_{\mathbf{O}}) \bigotimes_{\mathcal{A}} p_{\mathbf{O}} \cdot H_*(\dot{Z}^a)\right) \tag{6.12}$$

is a projective right $H_*(\dot{Z}^a)$-module. Hence, the functor $H_*(W^a) \otimes_{H_*(\dot{Z}^a)} (\bullet)$ is exact.

Finally, for any orbit $\mathbf{O}'$, we calculate using (6.12)

$$\begin{aligned} H_*(W^a) \otimes_{H_*(\dot{Z}^a)} \dot{L}_{\mathbf{O}'} \\ &= \left(\bigoplus_{\mathbf{O} \subset N^a} \mathrm{Hom}(\dot{L}_{\mathbf{O}}, L_{\mathbf{O}}) \bigotimes_{\mathcal{A}} p_{\mathbf{O}} H_*(\dot{Z}^a)\right) \otimes_{H_*(\dot{Z}^a)} \dot{L}_{\mathbf{O}'} = \\ &= \left(\bigoplus_{\mathbf{O} \subset N^a} \mathrm{Hom}(\dot{L}_{\mathbf{O}}, L_{\mathbf{O}})\right) \bigotimes_{\mathcal{A}} \dot{L}_{\mathbf{O}'} = L_{\mathbf{O}'} \end{aligned}$$

Thus the functor is exact and takes simple $H_*(\dot{Z}^a)$-modules into the corresponding simple $H_*(Z^a)$-modules. The result now follows easily from theorem 5.2. $\square$

Remark 6.13. Similar arguments yield the duality theorem of [Dr 3] between the *Yangian* $Y(\widehat{\mathfrak{gl}_n})$ and the *degenerate affine Hecke algebra* D_d of type GL_d. To that end one uses, instead of (5.1) and Theorem 4.3, analogously constructed algebra morphisms (see [Lu 2], [GV, Remark 8.7]) :

$$Y(\widehat{\mathfrak{gl}_n}) \twoheadrightarrow H^A(Z) \qquad\qquad D_d \simeq H^A(\dot{Z})$$

where H^A stands for the *equivariant Borel-Moore homology* with complex coefficients. $\square$

7. Dual approach to the Polynomial Tensor Representation

Fix a finite field $\mathbb{F}$ with p elements and let $\mathbb{K} = \mathbb{F}((z))$ be the field of Laurent power series. By a *lattice* we mean a free $\mathbb{F}[[z]]$-submodule in $\mathbb{K}^d$ of rank d. Let $\mathcal{B}$ be the flag variety of all *n-periodic* sequences of lattices $F_i \subset \mathbb{K}^d$, $i \in \mathbb{Z}$, of the form

$$F = (\cdots \subseteq F_{i-1} \subseteq F_i \subseteq F_{i+1} \subseteq \cdots), \qquad \text{where} \quad F_{i+n} = z^{-1} \cdot F_i, \ \forall i \in \mathbb{Z}.$$

Similarly, let $\dot{\mathcal{B}}$ be the flag variety of all *complete* sequences of lattices

$F_i \subset \mathbb{K}^d$, $i \in \mathbb{Z}$, of the form

$$F = (\cdots \subset F_{i-1} \subset F_i \subset F_{i+1} \subset \cdots),$$

where $F_{i+d} = z^{-1} \cdot F_i$, and $\dim(F_{i+1}/F_i) = 1$, $\forall i \in \mathbb{Z}$.

Let $G(\mathbb{K}) = GL_d(\mathbb{K})$ be the group of invertible $\mathbb{K}$-valued $d \times d$-matrices. The group $G(\mathbb{K})$ acts naturally on the set of lattices, inducing a $G(\mathbb{K})$-action on $\mathcal{B}$ and on $\dot{\mathcal{B}}$. Let $\mathbb{C}_{G(\mathbb{K})}[\mathcal{B} \times \mathcal{B}]$ be the complex vector space formed by all $G(\mathbb{K})$-invariant functions on $\mathcal{B} \times \mathcal{B}$ whose support is a finite union of $G(\mathbb{K})$-orbits. For any two orbits Z, $Z' \subset \mathcal{B} \times \mathcal{B}$ the map (3.1) has finite fibers (where $M_1 = M_2 = M_3 = \mathcal{B}$). Hence, the convolution makes $\mathbb{C}_{G(\mathbb{K})}[\mathcal{B} \times \mathcal{B}]$ an associative algebra.

Let $\mathbf{d} = (0 = d_0 \leq d_1 \leq \ldots \leq d_n = d)$ denote a partition of the set $\{1, \ldots, d\}$ into n (possibly empty) segments of lengths $d_1 - d_0, \ldots, d_n - d_{n-1}$, respectively. Introduce the set $\mathcal{P} = \{$*partitions of d into n segments*$\} \cup \{\nabla\}$ where ∇ is a formal element, the *ghost-partition*. For each $\alpha = 1, \ldots, n$, define two transformations $\mathbf{d} \mapsto \mathbf{d}_\alpha^{\pm}$ of the set $\mathcal{P}$ by the following rules:

(i) The element ∇ is kept fixed by both transformations;

(ii) Given a partition $\mathbf{d}$ and $\alpha \neq n$, set $\mathbf{d}_\alpha^{\pm} = (0 = d_0 \leq \ldots \leq d_{\alpha-1} \leq d_\alpha \pm 1 \leq d_{\alpha+1} \leq \ldots \leq d_n = d)$ unless $d_\alpha = d_{\alpha \pm 1}$, in which case we set $\mathbf{d}_\alpha^{\pm} = \nabla$;

(iii) Set $\mathbf{d}_n^{\pm} = (d_0 \leq d_1 \mp 1 \leq \cdots \leq d_{n-1} \mp 1 \leq d_n = d)$ unless $d_0 = d_1$ or $d_{n-1} = d_n$, in which case we let $\mathbf{d}_n^{\pm}$ to be ∇.

Given a partition $\mathbf{d} \in \mathcal{P}$, let $\mathcal{B}_{\mathbf{d}}$ be the subset of $\mathcal{B}$ formed by all flags F such that $\dim(F_k/F_0) = d_k$, for any $k = 1, \ldots, n$. Put $\mathcal{B}_\nabla = \emptyset$. For each $\alpha = 1, \ldots, n$ and each $\mathbf{d} \in \mathcal{P}$ such that $\mathbf{d}_\alpha^{+} \neq \nabla$, resp. $\mathbf{d}_\alpha^{-} \neq \nabla$, set:

$$\hat{Y}_{d_\alpha^+, d} = \{(F, F') \in \mathcal{B}_{d_\alpha^+} \times \mathcal{B}_{\mathbf{d}} \mid \begin{array}{ll} F_i = F'_i & \forall i \in \{1, \ldots, n\} \setminus \{\alpha\} \\ F'_\alpha \subset F_\alpha & \& \quad \dim(F_\alpha/F'_\alpha) = 1 \end{array} \}$$

$$\hat{Y}_{d_\alpha^-,d} = \{(F,F') \in \mathcal{B}_{d_\alpha^-} \times \mathcal{B}_{\mathbf{d}} \mid \begin{array}{lll} F_i = F'_i & \forall i \in \{1,\dots,n\} \setminus \{\alpha\} \\ F_\alpha \subset F'_\alpha & \& \quad \dim(F'_\alpha/F_\alpha) = 1 \end{array}\}$$

Each of the sets $\hat{Y}_{\mathbf{d}_\alpha^\pm,\mathbf{d}}$ is a single $G(\mathbb{K})$-orbit. Let $\mathbf{1}_{\mathbf{d}_\alpha^\pm,\mathbf{d}} \in \mathbb{C}_{G(\mathbb{K})}[\mathcal{B}\times\mathcal{B}]$ denote the characteristic function of the corresponding orbit. Further, let $\mathbf{1}_{\mathbf{d},\mathbf{d}}$ denote the characteristic function of the diagonal in $\mathcal{B}_{\mathbf{d}} \times \mathcal{B}_{\mathbf{d}}$. Analogous to the construction of [BLM] in the finite case, for $\alpha = 1,\dots,n$, define the following elements of the algebra $\mathbb{C}_{G(\mathbb{K})}[\mathcal{B}\times\mathcal{B}]$:

$$\mathbf{e}_\alpha = \sum_{\mathbf{d}\in\mathcal{P}} p^{(d_{\alpha-1}-d_\alpha)/2}\, \mathbf{1}_{\mathbf{d}_\alpha^+,\mathbf{d}} \quad , \quad \mathbf{f}_\alpha = \sum_{\mathbf{d}\in\mathcal{P}} p^{(d_\alpha - d_{\alpha+1})/2}\, \mathbf{1}_{\mathbf{d}_\alpha^-,\mathbf{d}}$$

(where $d_{n+1} := d + d_1$) and also

$$\mathbf{k}_\alpha = \sum_{\mathbf{d}\in\mathcal{P}} p^{(d_\alpha - d_{\alpha-1})/2}\, \mathbf{1}_{\mathbf{d},\mathbf{d}}.$$

Now, let $K_\alpha^{\pm1}, E_\alpha, F_\alpha$, $\alpha = 1,\dots,n,$ be the images in $\mathbf{U}$ of the standard Kac-Moody generators of the algebra $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$.

Theorem 7.1.[GV] *The assignment:*

$$q \mapsto p^{1/2} \quad , \quad K_\alpha \mapsto \mathbf{k}_\alpha \quad , \quad E_\alpha \mapsto \mathbf{e}_\alpha \quad , \quad F_\alpha \mapsto \mathbf{f}_\alpha$$

can be extended (uniquely) to a surjective algebra homomorphism: $\mathbf{U} \twoheadrightarrow \mathbb{C}_{G(\mathbb{K})}[\mathcal{B}\times\mathcal{B}]$. $\square$

We turn now to the Polynomial Tensor Representation. Let $\mathbb{C}_{G(\mathbf{K})}[\mathcal{B}\times\dot{\mathcal{B}}]$ be the complex vector space formed by $G(\mathbb{K})$-invariant functions on $\mathcal{B}\times\dot{\mathcal{B}}$ whose support is a finite union of $G(\mathbb{K})$-orbits. The convolution product endows the space $\mathbb{C}_{G(\mathbf{K})}[\mathcal{B}\times\dot{\mathcal{B}}]$ with a left $\mathbb{C}_{G(\mathbf{K})}[\mathcal{B}\times\mathcal{B}]$-module structure. Hence, by Theorem 7.1, the vector space $\mathbb{C}_{G(\mathbf{K})}[\mathcal{B}\times\dot{\mathcal{B}}]$ acquires a left $\mathbf{U}$-module structure.

Theorem 7.2 *The* $\mathbf{U}$*-module* $\mathbb{C}_{G(\mathbb{K})}[\mathcal{B}\times\dot{\mathcal{B}}]$ *is isomorphic to the Polynomial Tensor Representation, i.e., there is an isomorphism* ψ : $\left(\mathbf{P}^{\otimes d}|_{q=p}\right) \xrightarrow{\sim} \mathbb{C}_{G(\mathbb{K})}[\mathcal{B}\times\dot{\mathcal{B}}]$ *making the following diagram commute*

$$\begin{tikzcd}
\mathbf{U} \arrow[r, "(4.6)"] \arrow[d, "thm.\ 7.1"] & \mathrm{End}\left(\mathbf{P}^{\otimes d}|_{q=p}\right) \arrow[d, equal, "\psi"] \\
\mathbb{C}_{G(\mathbb{K})}[\mathcal{B}\times\mathcal{B}] \arrow[r] & \mathrm{End}\,\mathbb{C}_{G(\mathbb{K})}[\mathcal{B}\times\dot{\mathcal{B}}]
\end{tikzcd}$$

In order to construct the isomorphism ψ of the theorem (that will not be done here) one has do describe first the $\mathbf{U}$-action on $\mathbb{C}_{G(\mathbb{K})}[\mathcal{B}\times\dot{\mathcal{B}}]$ in an explicit way. To that end we introduce some combinatorial objects.

For any $s \in \mathbb{Z}$, write $s = \underline{s}\cdot d + \overline{s}$, where $\underline{s}$ is a certain integer and $\overline{s} \in \{1,\dots,d\}$, is the remainder. Let $\mathcal{J}$ be the set of all functions

$$\mathcal{J} = \{\mathbf{j} : \mathbb{Z} \to \mathbb{Z} \quad \text{such that} \quad \mathbf{j}(s+d) = \mathbf{j}(s) + n, \quad \forall s \in \mathbb{Z}\}$$

To such a function we assign the d-tuple $(\mathbf{j}(1),\mathbf{j}(2),\dots,\mathbf{j}(d)) \in \mathbb{Z}^d$. This way one gets a bijection $\mathcal{J} \simeq \mathbb{Z}^d$. We shall often identify a function $\mathbf{j}$ with the corresponding d-tuple, an element of $\mathbb{Z}^d$. Further, given an integer s, we define two transformations $\mathbf{j} \mapsto \mathbf{j}_s^{\pm}$ of the set $\mathbb{Z}^d$ by the following formulas

$$\mathbf{j}_s^{\pm} = (\mathbf{j}(1),\dots,\mathbf{j}(\overline{s}-1),\mathbf{j}(\overline{s}) \pm 1,\mathbf{j}(\overline{s}+1),\dots,\mathbf{j}(d))$$

We claim next that there is a bijection between the set $\mathcal{J}$ and the set of $G(\mathbb{K})$-orbits in $\mathcal{B}\times\dot{\mathcal{B}}$. If $e_1,\dots,e_d$ denotes the standard base of $\mathbb{K}^d$, then to a function $\mathbf{j} \in \mathcal{J}$ we assign the $G(\mathbb{K})$-orbit that contains the pair $(F,\dot{F}) \in \mathcal{B}\times\dot{\mathcal{B}}$ where $F = (\dots \subset F_i \subset \dots)$ and $\dot{F} = (\dots \subset \dot{F}_i \subset \dots)$ are given by

$$F_i = \bigoplus_{\mathbf{j}(k+jd)\le i} z^{-j}\mathbb{F}[[z]]\cdot e_k \quad , \quad \dot{F}_i = \bigoplus_{k+jd\le i} z^{-i}\mathbb{F}[[z]]\cdot e_k$$

Let $\mathbf{1}_\mathbf{j}$ denote the characteristic function of this $G(\mathbb{K})$-orbit. The family $\{\mathbf{1}_\mathbf{j}\}_{\mathbf{j}\in\mathcal{J}}$ forms a $\mathbb{C}$-basis of $\mathbb{C}_{G(\mathbb{K})}[\mathcal{B}\times\dot{\mathcal{B}}]$. The $\mathbf{U}$-action in that basis can be computed explicitly. It is given by the following formulas (compare with [GL] in the finite case).

$$E_\alpha \cdot \mathbf{1}_{\mathbf{j}} = q^{-\#\mathbf{j}^{-1}(\alpha)} \sum_{s \in \mathbf{j}^{-1}(\alpha+1)} q^{2\#\{z \in \mathbf{j}^{-1}(\alpha): z > s\}} \mathbf{1}_{\mathbf{j}_s^-},$$

$$F_\alpha \cdot \mathbf{1}_{\mathbf{j}} = q^{-\#\mathbf{j}^{-1}(\alpha+1)} \sum_{s \in \mathbf{j}^{-1}(\alpha)} q^{2\#\{z \in \mathbf{j}^{-1}(\alpha+1): z < s\}} \mathbf{1}_{\mathbf{j}_s^+},$$

$$K_\alpha \cdot \mathbf{1}_{\mathbf{j}} = q^{\#\mathbf{j}^{-1}(\alpha)} \mathbf{1}_{\mathbf{j}}.$$

Here $\alpha \in \{1, 2, \dots, n\}$. $\square$

Finally, we introduce yet another realization of the Polynomial Tensor Representation in terms of functions on the *periodic flag manifold.* This manifold was independently introduced (over $\mathbb{C}$) by Feigin-Frenkel [FF] and less explicitly by Lusztig [Lu 3]. Mimicing their construction over the finite field, we define $\mathring{\mathcal{B}}$ to be the set of pairs:

$$\mathring{\mathcal{B}} = \{(\mathring{F}, L) \mid \mathring{F} = (\{0\} = \mathring{F}_0 \subset \mathring{F}_1 \subset \cdots \subset \mathring{F}_d = \mathbb{K}^d), \\ L = (L_1 \subset \mathring{F}_1/\mathring{F}_0, \dots, L_d \subset \mathring{F}_d/\mathring{F}_{d-1})\}$$

where $\mathring{F}_i$ is an i-dimensional $\mathbb{K}$-vector space and $L_i \subset \mathring{F}_i/\mathring{F}_{i-1}$ is a 1-dimensional $\mathbb{F}$-vector subspace, for any $i = 1, \dots, d$.

Replacing the set $\dot{\mathcal{B}}$ by $\mathring{\mathcal{B}}$ in the previous constructions, we obtain a new left $\mathbf{U}$-module $\mathbb{C}_{G(\mathbb{K})}[\mathcal{B} \times \mathring{\mathcal{B}}]$. Similarly, to any d-tuple $\mathbf{j} \in \mathcal{J}$, one associates the $G(\mathbb{K})$-orbit in $\mathcal{B} \times \mathring{\mathcal{B}}$ of the pair $\big(F, (\mathring{F}, L)\big)$ where $F = (\dots \subset F_i \subset \dots)$, $\mathring{F} = (\dots \subset \dot{F}_i \subset \dots)$ and $L = (\dots \subset L_i \subset \dots)$ are given by

$$F_i = \bigoplus_{\mathbf{j}(k+jd) \le i} z^{-j}\, \mathbb{F}[[z]] \cdot e_k, \quad \mathring{F}_i = \bigoplus_{k=1}^{i} \mathbb{K} \cdot e_k, \quad L_i = \mathbb{F} \cdot e_i \pmod{\mathring{F}_{i-1}}$$

Let $\varepsilon_{\mathbf{j}}$ be the characteristic function of this $G(\mathbb{K})$-orbit.

Recall the function $\mathbf{a}$ defined by (5.3) and other notation involved in the proof of theorem 5.2. Make the renormalization: $\hat{X}_s = q^{2s-d-1} X_s$, where $s = 1, \dots, d$, and , for any $\mathbf{j} \in \mathbb{Z}^d$, write

$$\mathbf{j} = \bar{\mathbf{j}} + n \cdot \underline{\mathbf{j}} \quad \text{where} \quad \bar{\mathbf{j}} \in [1, n]^d \ , \ \underline{\mathbf{j}} \in \mathbb{Z}^d$$

Theorem 7.3 *The assignment*

$$\phi : e_{\bar{\mathrm{j}}} \otimes \hat{X}_1^{-\mathrm{j}_1} \cdot \ldots \cdot \hat{X}_d^{-\mathrm{j}_d} \mapsto q^{\mathbf{a}(\bar{\mathrm{j}})} \varepsilon_{\mathrm{j}}$$

gives an isomorphism of $\mathbf{U}$*-modules* $\phi : \left(\mathbf{P}^{\otimes d}{}_{|q=p}\right) \xrightarrow{\sim} \mathbb{C}_{G(\mathbb{K})}[\mathcal{B} \times \mathring{\mathcal{B}}]$, *i.e., the following diagram commutes*

$$\begin{tikzcd} \mathbf{U} \arrow[r, "(4.6)"] \arrow[d, "thm.\ 7.1"] & \mathrm{End}\left(\mathbf{P}^{\otimes d}{}_{|q=p}\right) \arrow[d, equal, "\phi"] \\ \mathbb{C}_{G(\mathbb{K})}[\mathcal{B} \times \mathcal{B}] \arrow[r] & \mathrm{End}\, \mathbb{C}_{G(\mathbb{K})}[\mathcal{B} \times \mathring{\mathcal{B}}] \end{tikzcd}$$

Proof of theorem 7.3 is entirely analogous to that of theorem 7.2. We observe first that the $G(\mathbb{K})$-orbits in $\mathcal{B} \times \mathring{\mathcal{B}}$ are parametrized by the set $\mathcal{J}$ so that the characteristic functions ε_{j} form a base of the $\mathbb{C}$-vector space $\mathbb{C}_{G(\mathbb{K})}[\mathcal{B} \times \mathring{\mathcal{B}}]$. In that basis, the $\mathbf{U}$-action can be computed explicitly, and is given by the following formulas:

$$E_\alpha \cdot \varepsilon_{\mathrm{j}} = q^{-\#\mathrm{j}^{-1}(\alpha)} \sum_{s \in \mathrm{j}^{-1}(\alpha+1)} q^{2\#\{z \in \mathrm{j}^{-1}(\alpha) : \bar{z} > \bar{s}\}} \varepsilon_{\mathrm{j}_s^-},$$

$$F_\alpha \cdot \varepsilon_{\mathrm{j}} = q^{-\#\mathrm{j}^{-1}(\alpha+1)} \sum_{s \in \mathrm{j}^{-1}(\alpha)} q^{2\#\{z \in \mathrm{j}^{-1}(\alpha+1) : \bar{z} < \bar{s}\}} \varepsilon_{\mathrm{j}_s^+},$$

$$K_\alpha \cdot \varepsilon_{\mathrm{j}} = q^{\#\mathrm{j}^{-1}(\alpha)} \varepsilon_{\mathrm{j}}.$$

The rest of the proof is a straightforward exercise. □

References

[BBD] A. Beilinson, J. Bernstein, P. Deligne. Faisceaux Pervers. *Astérisque* **100** (1981).

[BLM] A. Beilinson, G. Lusztig, R. MacPherson. A geometric setting for quantum groups. *Duke Math. J.* **61** (1990), 655-675.

[CP] V. Chari, A. Pressley. Quantum affine algebras. *Commun. Math. Phys.* **142** (1991), 261-283.

[Ch 1] I. Cherednik. Quantum groups as hidden symmetries of the classical Representation theory. *Proceedings of the 17-th. Intern. conf. of Differential Geometry methods in Theoretical Physics* (1989).

[Ch 2] I. Cherednik. A new interpretation of Gelfand-Tzetlin bases. *Duke Math. J.* **54** (1987), 563-578.

[CG] N. Chriss, V. Ginzburg. Representation theory and Complex Geometry (Geometric technique in Representation theory of Reductive groups). Progress in Mathem., Birkhäuser (1994), to appear.

[DF] J. Ding, I. Frenkel. Isomorphism of two realizations of quantum affine algebra $\mathbf{U}_q(\widehat{\mathfrak{gl}_n})$. *Commun. Mathem. Phys.* (1993).

[Dr 1] V. Drinfeld. Quantum Groups. *Proceedings of the ICM* , Berkeley 1986.

[Dr 2] V. Drinfeld. A new realization of Yangians and Quantum affine algebras. *Soviet Math. Dokl.* **36** (1988), 212 - 216.

[Dr 3] V. Drinfeld. Degenerate affine Hecke algebras and Yangians. *Funct. Anal. and Appl.* **20:1** (1986), 69-70.

[FRT] L. Faddeev, N. Reshetikhin, L. Takhtajan. Quantization of Lie groups and Lie algebras, Yang-Baxter equation in Integrable Systems. *Advanced Series in Mathem. Physics.* **10** (1989), 299-309. World Scientific.

[FF] B. Feigin, E. Frenkel. Affine Lie algebras and semi-infinite flag manifold. *Commun. Math. Phys.* **128** (1990), 16.

[Gi] V. Ginzburg. Deligne - Langlands conjecture and Representations of affine Hecke algebras. *Preprint*, Moscow 1985.

[GV] V. Ginzburg, E. Vasserot. Langlands Reciprocity for Affine Quantum groups of type A_n . *Intern. Mathem. Research Notices. (Duke Math. J.)* **3** (1993), 67-85.

[Gr] I. Grojnowski. The coproduct for quantum GL_n. *Preprint* 1992.

[GL] I. Grojnowski, G. Lusztig. On bases of irreducible representations of quantum GL_n. *Contemp. Mathem.* (1992).

[J] M. Jimbo. A q-Analogue of $U(\mathfrak{g}l(n+1))$, Hecke Algebra, and the Yang-Baxter Equation. *Lett. in Mathem. Phys.* **11** (1986), 247-252.

[KKM] S.-J. Kang, et al. Perfect crystals of quantum affine algebra. *Duke Math. Journ.* **68** (1992), 499-607.

[KL 1] D. Kazhdan, G. Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.* **53** (1979), 165-184.

[KL 2] D. Kazhdan, G. Lusztig. Proof of the Deligne - Langlands conjecture for affine Hecke algebras. *Invent. Math.* **87** (1987), 153-215.

[KR] A. Kirillov, N. Reshetikhin. The Yangian, Bethe ansatz and Combinatorics. *Lett. in Mathem. Phys.* **12** (1986), 199-208.

[Lu 1] G. Lusztig. Canonical bases arising from Quantized enveloping algebras. *Journ. A.M.S.* **3** (1990), 447-498.

[Lu 2] G. Lusztig. Cuspidal local systems and graded Hecke algebras. *Publ. Mathem. I.H.E.S.* **67** (1988), 145-212.

[Lu 3] G. Lusztig. Hecke algebras and Jantzen generic decomposition pattern. *Advances Math.* (1981).

[RS 1] N. Reshetikhin, M. Semenov-Tian-Shansky. Central extensions of quantum current groups. *Lett. Mathem. Phys.* **19** (1990), 133-142.

[V] E. Vasserot. Représentations de groupes quantiques et permutations. *Annales Sci. ENS*, **26**, 747–773, (1993).

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