# Perverse Sheaves and Fundamental Lemmas

These are my live-TeXed notes for the course *Math G6761: Perverse Sheaves and Fundamental Lemmas* taught by Wei Zhang at Columbia, Fall 2015. The final part of the course discusses the recent breakthrough *Shtukas and the Taylor expansion of L-functions* by Zhiwei Yun and Wei Zhang.

Any mistakes are the fault of the notetaker. Let me know if you notice any mistakes or have any comments!

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## Motivation

A *fundamental lemma* is an identity between orbital integrals on two different groups. For example, the *endoscopic fundamental lemma* arises from the stabilization of the Arthur-Selberg trace formula and endoscopic functoriality in the Langlands program. The *Jacquet-Rallis fundamental lemma* arises from the W. Zhang's relative trace formula approach to the Gan-Gross-Prasad conjecture for unitary groups. Both sides of the fundamental lemma can be thought of as counting the number of lattices satisfying certain properties.

In the equal characteristic case, the endoscopic fundamental lemma was proved by Ngo and the Jacquet-Rallis fundamental lemma was proved by Yun. These proofs use geometric methods (*perverse sheaves*) in an essential way. The advantage in the equal characteristic is one can endow the space of lattices in question with a *geometric structure* (e.g., the $\mathbb{F}_q$ -rational points of an algebraic variety, usually realized as the fiber of an invariant map from a certain moduli space of vector bundles to an affine space). These $\mathbb{F}_q$ -rational points then can be counted using Lefschetz trace formula.

In order to prove the identity between orbital integrals ("functions on orbits"), one instead proves the identity between perverse sheaves ("sheaves on orbits"). The miracle is that one can prove the identity between perverse sheaves by only verifying it over certain open dense subsets ("very regular" orbits), which can be easier. This reflects a uniqueness principle of orbital integrals: the "very regular" orbital integral determines the more degenerate ones. This principle, however, is hard to see directly from orbital integrals.

## Intersection homology

The idea of perverse sheaves begins from Goresky-MacPherson's theory of intersection homology, which is a homology theory for *singular* manifolds with an analogue of *Poincare duality* and *Hodge decomposition*. More precisely, we define

**Definition 1** A Hausdorff space $X$ is a *pseudo-manifold* if it admits a stratification $X = X_n \supseteq X_{n-1} \supseteq X_0$ by closed subspaces such that

a. $X - X_{n-1}$ is dense in $X$.
b. $X_{n-1} = X_{n-2}$. (Notice this condition is automatic when $X$ is the complex points of an algebraic variety)
c. For each $i$, and $x \in X_i - X_{i-1}$, there exists a neighborhood $N \subseteq X$ of $x$ such that there is a stratification-preserving homeomorphism $N \cong \mathbb{R}^i \times \operatorname{cone}^0 L$. Here $L$ is a compact pseudo-manifold of dimension $n - i - 1$ and $\operatorname{cone}^0 L$ is the open cone of $L$ (of dimension $n - i$).

**Remark 1**

a. We call $L$ the *link* of $x \in X_i - X_{i-1}$. It is a fact that $L$ (up to homeomorphism) depends only on $X_i - X_{i-1}$, not on $x$.
b. $N$ is called a distinguished neighborhood of $x$. It is a fact that distinguished neighborhoods form a basis around $x$.

**Example 1** $X = S^2 \vee S^2$ (complex points of the union of two lines $xy = 0$), or $X$ a pinched torus (complex points of a nodal curve)), is a pseudo-manifold. Near the singular point, $X$ looks like the open cone of two circles.

a. For $X = S^2 \vee S^2$, we have $h_0(X) = 1$, $h_1(X) = 0$, $h_2(X) = 2$. The Poincare duality fails.
b. For $X$ a pinched torus, we have $h_0(X) = 1$, $h_1(X) = 1$, $h_2(X) = 1$. The Hodge decomposition fails.

**Remark 2** It is a theorem that the complex points of any quasi-projective varieties (with singularities) is a pseudo-manifold. This forms the main example considered in this course.

To get a homology theory for pseudo-manifold with desired properties, one needs to consider only the chains that has nice intersection properties with the stratification.

**Definition 2** To measure how "perverse" these intersections are, we introduce the *perversity*, which is a function $p : \mathbb{Z}_{\geq 2} \to \mathbb{Z}_{\geq 0}$ such that

a. $p(2) = 0$,
b. $p(j) - p(j-1) \in \{0, 1\}$.

**Example 2** There are three special perversity functions:

a. the *zero perversity* $p = 0$,
b. the *top perversity* $p(j) = j - 2$,
c. the *middle perversity* $p(j) = \lfloor \frac{j-2}{2} \rfloor$.

**Definition 3** Given a perversity $p$, we define a $i$-chain $\sigma$ to be $p$-*allowable* if

$$\dim(\sigma \cap X_{n-j}) \leq i - j + p(j).$$

We then define the *intersection homology* groups $I_pH_i(X)$ by considering only the $p$-allowable locally finite $i$-chains. When $p$ is the middle perversity, $I_pH_i(X)$ is often written as $IH_i(X)$ for short. Similarly one can define the *intersection homology with compact support* $I_pH_i^c(X)$ by considering only the $p$-allowable finite $i$-chains.

**Remark 3** Notice the $p$-allowable condition is most strict when $p = 0$. For the top perversity, the $p$-allowable condition is nothing but the condition that $\dim(\sigma \cap X_{n-2}) \leq i - 2$, i.e., "most" of $\sigma$ lies in the "non-singular" part $X - X_{n-2}$.

**Example 3** Suppose $X$ (of dimension $n$) has only one singular point $x$. Then one can compute

$$IH_i(X) = \begin{cases} H_i(X) & i > n, \\ \mathrm{im}\,(H_i(X - x) \to H_i(X)) & i = n, \\ H_i(X - x) & i < n. \end{cases}$$

Using this we find that:

a. For $X = S^2 \vee S^2$, we have $ih_0(X) = 2$, $ih_1(X) = 0$, $ih_2(X) = 2$.

b. For $X$ a pinched torus, we have $ih_0(X) = 1$, $ih_1(X) = 0$, $ih_2(X) = 1$.

Notice that both Poincare duality and Hodge decomposition hold for $IH_*(X)$!

**Theorem 1**

a. All intersection homology groups are finitely generated, independent of the choice the stratification. It gives the usual homology groups for manifolds without singularities.

b. $I_pH_0^c(X) = \mathbb{Q}$ if $X - X_{n-2}$ is connected.

c. (intersection product) Suppose $p, q$ are perversities such that $p + q$ is still a perversity. There is an intersection product

$$I_pH_i(X) \times I_qH_j^c(X) = I_{p+q}H_{i+j-n}^c(X),$$

where $n$ is the real dimension of $X$. When $p + q = p_{\mathrm{top}}$, we have a non-degenerate pairing

$$I_pH_i(X) \times I_qH_{n-i} \to I_{p_{\mathrm{top}}}H_0^c(X) \to \mathbb{Q}.$$

In particular, when $X$ has even dimensional strata, we have the duality between $IH_i(X)$ and $IH_{n-i}(X)$.

**Remark 4** The intersection homology is a topological invariant but is *not* a homotopy invariant, unlike the usual homology theory. In the 70's, the only proof of the topological invariance is sheaf-theoretic.

Let $\mathrm{Sh}(X)$ be the category of sheaves of $\mathbb{Q}$-vector spaces on $X$. Two important examples:

**Example 4**

a. The constant sheaf on $X$, denoted by $\underline{\mathbb{Q}}$.

b. Locally constant sheaves on $X$. Recall that $\mathcal{E}$ is *locally constant* if for any $x \in X$, there exists an open $U$ such that the restriction map $\Gamma(U, \mathcal{E}) \to \mathcal{E}_x$ is an isomorphism. Locally constant sheaves with stalks of finite rank are called *local systems*.

Repeating the construction of $p$-allowable $i$-chains for each open $U \subseteq X$ we obtain a sheaf $\mathcal{I}_p\mathcal{C}_i(X)$. We define a complex of sheaves $\mathcal{I}_p\mathcal{C}^{\cdot}$ given by $\mathcal{I}_p\mathcal{C}^{-i} = I_pC_i$ (so $\mathcal{I}_p\mathcal{C}^{\cdot}$ is concentrated in the negative degrees). In particular, $\mathcal{I}_p\mathcal{C}^{\cdot}$ is concentrated in negative degrees.

**Theorem 2** Denote $I_pH^i(X) = I_pH_{-i}(X)$. Then there is an isomorphism

$$I_pH^i(X) \cong \mathbb{H}(X, \mathcal{I}_p\mathcal{C}^{\cdot}(X)).$$

It turns out that the IC sheaf $\mathcal{I}_p\mathcal{C}^{\cdot}(X)$ (as an object in the derived category $D^b(X)$) is topologically invariant. This topological invariance on the level of sheaves is more flexible and easier to prove.

**Theorem 3** (Deligne, Goresky-MacPherson) Let $U_i = X - X_{n-i}$. Let $j_k : U_k \hookrightarrow U_{k+1}$ ($k \geq 2$). Let $p$ be a perversity. Then in the bounded derived category $D^b(X)$, we have

$$\mathcal{I}_p\mathcal{C}^{\cdot} \cong \cdots \tau_{\leq p(3)-n}(Rj_3)_*\tau_{\leq p(2)-n}(Rj_2)_*\underline{\mathbb{Q}}_{U_2}[n].$$

Here $\tau_{\leq k}$ is the usual truncation functor:

$$\tau_{\leq k}\mathcal{E}^{\cdot} = \begin{cases} \mathcal{E}^i & i \leq k, \\ \ker(\mathcal{E}^k \to \mathcal{E}^{k+1}) & i = k, \\ 0 & i > k. \end{cases}$$

so that $\mathcal{H}^i(\mathcal{E}^{\cdot}) = \mathcal{H}^i(\tau_{\leq k}(\mathcal{E}^{\cdot}))$ for $i \leq k$. Inductively, let $\mathbb{P}_2 = \underline{\mathbb{Q}}_{U_2}[n]$ and

$$\mathbb{P}_{k+1} = \tau_{\leq p(k)-n}(Rj_k)_*\mathbb{P}_k.$$

Then

$$\mathcal{I}_p\mathcal{C}^{\cdot} \cong \mathbb{P}_k.$$

**Remark 5** When $X$ is a manifold, $X_{n-1}, \dots, X_0$ are all empty, so it follows from the theorem that $\mathcal{IC}^{\cdot} = \underline{\mathbb{Q}}_X[n]$. This is clear since there is a quasi-isomorphisms of complex sheaves

$$0 \to \mathcal{C}^{-n} \to \mathcal{C}^{-n+1} \to \cdots$$

and

$$0 \to \mathbb{Q}_X \to 0 \to 0 \cdots.$$

The quasi-isomorphism can be checked locally: the Borel-Moore homology (allowing infinite, locally finite chains) of an Euclidean space is nontrivial only in the top degree.

**Proof** To prove this theorem we only need to do local calculation for $N = \mathbb{R}^{n-k} \times \mathrm{cone}^0(L)$, where $L$ is compact and has dimension $k-1$. It boils down to the following two parts:

a. Kunneth formula gives $IH^i(X \times \mathbb{R}^k) = IH^{i+k}(X)$.

b. Suppose $X$ is a compact pseudo-manifold of dimension $n$. Then

$$IH_i(\mathrm{cone}^0(X)) = \begin{cases} IH_{i-1}(X) & i \geq n - p(n). \\ 0 & i < n - p(n). \end{cases}$$

It then follows that $\mathbb{P}_{k+1} = \tau_{\leq p(k)-n}(Rj_k)_*\mathbb{P}_k$ inductively. □

References:

- *An introduction to perverse sheaves*, Rietsch (brief)
- *Intersection homology*, edited by Borel (detailed)

**Remark 6** The IC sheaf is *constructible* (not far from being a local system). More precisely, we say $\mathcal{E} \in \mathrm{Sh}(X)$ is constructible if $\mathcal{E}|_{X_k - X_{k-1}}$ is a local system for any $k$. We say $\mathcal{E}^{\cdot} \in D^b(X)$ is constructible if $\mathcal{H}^i(\mathcal{E}^{\cdot})$ is constructible for any $i$ (for the given stratification on $X$). The category of constructible sheaves (for a given stratification) is denoted by $D^b_\sigma(X)$. We denoted by $D^b_c(X)$ the larger category of sheaves which are constructible for some stratification. Notice that $\mathcal{IC}^{\cdot}|_{X_k - X_{k-1}}$ is locally constant (but not necessarily constant, depending on the link $L$).

By checking at stalks, one finds that the cohomology of IC sheaves has support of dimension *in a particular range* depending on the perversity,

$$\mathcal{H}^i(\mathbb{P}^{\cdot})_x = 0, \quad i > -n + p(k), x \in X_{n-k} - X_{n-k-1}.$$

It turns out that we can *characterize* Deligne's sheaf $\mathbb{P}_p$ using the following four axioms:

a. $\mathbb{P}$ is constructible with respect to the given stratification.

b. $\mathbb{P}|_{U_2} \cong \underline{\mathbb{Q}}_{U_2}[n]$ in $D^b(U_2)$.

c. The stalk $H^i(i_x^*\mathbb{P}) = 0$ if $i > -n + p(k)$ and $x \in X_{n-k} - X_{n-k-1} = U_k - U_{k-1} =: S_k$.

d. The attachment map $\alpha_k : \mathbb{P}_{U_{k+1}} \to (Rj_k)_*(\mathbb{P}|_{U_k})$ (given by $\mathrm{Id} \to Rj_*j^*$) is a quasi-isomorphism up to degree $i \leq -n + p(k)$.

Our next goal is to rephrase axiom d as an axiom c' similar to c for the *dual* of $\mathbb{P}$ and use Deligne's construction to prove the topological invariance of IC sheaves.

## Operations on sheaves and duality

We would like to define a dualizing functor $\mathbb{D} : D^b(X) \to D^b(X)$. We hope $\mathbb{D}$ to satisfy the following properties. Suppose $f : X \to Y$ is a morphism. Then

$$Rf_! \circ \mathbb{D}_X = \mathbb{D}_Y \circ R_f*$$

and

$$Rf^! \circ \mathbb{D}_Y = \mathbb{D}_X \circ Rf^*.$$

Moreover, when $X$ is a point, we should have $\mathbb{D}(\mathcal{E}^{\cdot}) = \mathcal{H}om(\mathcal{E}^{\cdot}, \mathbb{Q})$. Moreover, the natural transformation $\mathrm{Id} \to \mathbb{D}_X \circ \mathbb{D}_X$ becomes a bi-duality when restricted the constructible sheaves $D^b_c(X)$.

**Remark 7** Here $Rf^!$ is the pull-back with compact support satisfying the adjoint property with respect to $Rf_!$ (*Verdier duality*):

$$Rf_*R\mathcal{H}om(\mathcal{E}^\cdot, Rf^!\mathcal{F}^\cdot) = R\mathcal{H}om(Rf_!\mathcal{E}^\cdot, \mathcal{F}^\cdot).$$

However, $Rf^!$ in general is not the right derived functor of any functor $f^!$. It can be defined as follows: for $\mathcal{E}^\cdot \in D^b(Y)$ and $U \subseteq X$ open, we define

$$Rf^!(\mathcal{E}^\cdot)(U) = \mathrm{Hom}(f_!(K^\cdot|_U), \mathcal{E}^\cdot|_U),$$

where $K^\cdot$ is any soft resolution of $\underline{\mathbb{Q}}_X$. Recall that a sheaf is *soft* if any of its section supported on a compact subset can be extended to a global section.

**Remark 8** Suppose $j : U \hookrightarrow X$ is an open immersion and $i : Z = U - X \hookrightarrow X$ is a closed immersion. We have a distinguished triangle

$$j_!j^* \to \mathrm{Id} \to i_!i^*.$$

By composing with $\pi_!$ (the structure morphism $\pi : X \to \mathrm{pt}$), we obtain a long exact sequence in cohomology with compact support

$$\to \mathcal{H}^i_c(U, \mathcal{E}^\cdot) \to \mathcal{H}^i_c(X, \mathcal{E}^\cdot) \to \mathcal{H}^i_c(Z, \mathcal{E}^\cdot) \to \cdots$$

Notice by definition one can check that $j^*$ is right adjoint of $j_!$, hence $j^! = j^*$. By definition we also have $i_* = i_!$. We have the following adjoint relations

$$(i^*, i_* = i_!, i^!), \quad (j_!, j^! = j^*, j_*).$$

**Remark 9** Dually we have a distinguished triangle

$$i_*i^! \to \mathrm{Id} \to Rj_*j^* \to .$$

In this case $i^!$ can be actually defined as a derived functor of $\gamma_Z : \mathrm{Sh}(X) \to \mathrm{Sh}(Z)$: $\gamma_Z(\varepsilon)(Z) = \ker(\Gamma(X, \mathcal{E}) \to \Gamma(X - Z, \mathcal{E}))$. By composing with $\pi_*$, we obtain a long exact sequence in cohomology

$$\cdots \to \mathcal{H}^i(Z, i^!\mathcal{E}^\cdot) \to \mathcal{H}^i(X, \mathcal{E}^\cdot) \to \mathcal{H}^i(U, \mathcal{E}^\cdot) \to \cdots.$$

The first term recovers the cohomology with support in $Z$.

**Remark 10** (Purity) Suppose $X$ is a manifold, $i : x \to X$ and $\mathcal{E}^\cdot \in D^b(X)$. Then

$$i^!(\mathcal{E}^\cdot) \cong i^*(\mathcal{E}^\cdot)[-\dim X].$$

**Remark 11** Recall that for $\mathcal{E}^\cdot, \mathcal{F}^\cdot \in \mathrm{Sh}(X)$, we define

$$\mathcal{H}om(\mathcal{E}^\cdot, \mathcal{F}^\cdot) = \bigoplus_{j \in \mathbb{Z}} \mathcal{H}om(\mathcal{E}^j, \mathcal{F}^{j+i}),$$

where $\mathcal{H}om(\mathcal{E}^\cdot, \mathbb{Q})^i = \mathcal{H}om(\mathcal{E}^{-i}, \mathbb{Q})$. Let $R\mathcal{H}om : D^b(X) \times D^b(X) \to D^b(X)$ be its derived functor and $RHom = R\Gamma \circ R\mathcal{H}om$.

It turns out there exists a dualizing sheaf $\omega^\cdot_X \in D^b(X)$ such that (at least for constructible sheaves)

$$\mathbb{D}_X(\mathcal{E}^\cdot) = R\mathcal{H}om(\mathcal{E}^\cdot, \omega^\cdot_X).$$

It turns out that one can define $\omega^i_X$ to be the sheaf of locally finite $i$-chains $\mathcal{C}_{-i}$. In particular, when $X$ is a manifold, $\omega^\cdot_X$ is quasi-isomorphic to $\mathbb{Q}_X[n]$.

**Remark 12** We get the following the relationship

$$\omega^\cdot_X = \mathbb{D}_X(\underline{\mathbb{Q}}_X) = Rf^!(\underline{\mathbb{Q}}_{\mathrm{pt}}),$$

where $f : X \to \mathrm{pt}$. Moreover, for a morphism $f : X \to Y$, we have

$$\omega_Y = Rf^!\omega_X.$$

**Remark 13** In the case of manifolds of dimension $n$, we have $\mathbb{D}_X(\mathcal{L}) = \mathcal{L}[n]$ for any local system $\mathcal{L}$.

**Theorem 4** Let $\mathcal{L}_{U_2}$ be a local system on $U_2$ and $\mathcal{L}^\vee_{U_2}$ be the dual local system. Let $p$ and $q$ be complementary perversity (i.e., $p + q$ is the top perversity). Then

$$\mathbb{D}_X(\mathbb{P}^\cdot_p(\mathcal{L}_{U_2}))) = \mathbb{P}^\cdot_q(\mathcal{L}^\vee_{U_2}[-n]).$$

**Remark 14** This theorem follows from the uniqueness of the characterizing properties of IC sheaves $\mathbb{P}^\cdot_p(\mathcal{L}_{U_2})$ (see Theorem 5). When $X$ is a manifold, this theorem recovers the usual Poincare duality.

## Reformulation and topological invariance

Using the theorem that $\mathbb{D}_X(\mathbb{P}^\cdot_p)[n] = \mathbb{P}_q$ and the properties of the dualizing functor. We find the axiom d for $\mathbb{P}_q$ can be formulated as a similar axiom on the *costalk* $i^!_x\mathbb{P}_p$.

c'. $H^i(i^!_x\mathbb{P}_p) = 0$ if $i < -q(k)$ for $x \in S_k$.

In fact, by Remark 9 we have the exact sequence

$$\to H^i((i^!\mathbb{P})_x) \to H^i((\mathbb{P}|_{U_{k+1}})_x) \to H^i((Rj_*\mathbb{P}|_{U_k})_x) \to \cdots$$

One sees that axiom d is equivalent to

$$H^i((i^!\mathbb{P})_x) = 0, \quad i \le -n + p(k) + 1.$$

Let $i'_x : x \hookrightarrow S_k$ and $i : S_k \to U_{k+1}$, then

$$i_x^!\mathbb{P} = (i'_x)^! i^! \mathbb{P}.$$

Using the purity (since $S_k$ is a manifold), we have $(i'_x)^! = (i'_x)^*[-\dim S_k]$. Hence axiom d is equivalent to axiom c'.

Now suppose we are in the case of middle perversity. Let

$$\operatorname{supp}\mathcal{H}^i = \{x : H^i(i_x^*\mathbb{P}) \neq 0\} \subseteq X,$$

and

$$\operatorname{cosupp}\mathcal{H}^i = \{x : H^i(i_x^!\mathbb{P}) \neq 0\} \subseteq X.$$

We can (by shifting) translate the axioms c and c' to the axioms which don't mention the stratification:

c2. $\dim_{\mathbb{C}} \operatorname{supp}\mathcal{H}^{-i}(\mathbb{P}) < i$.

c2'. $\dim_{\mathbb{C}} \operatorname{cosupp}\mathcal{H}^{-i}(\mathbb{P}) < i$.

Notice that a, b still depend on the stratification $\sigma$. For example, when $X$ is a manifold and $\sigma$ is the trivial stratification. Then the sheaves satisfying a, b, c2, c2' are of the form $\mathcal{L}[\dim_{\mathbb{C}} X]$ for local systems $\mathcal{L}$ on $X$. The shift by the complex dimension ensures that the dualizing functor preserves $\mathbb{P}_\sigma(X)$.

**Theorem 5** Given a local system $\mathcal{L}$ on an open $U$ of real codimension at least 2, there exists a unique $\mathbb{P} \in D_c^b(X)$ such that $\mathbb{P}|_U = \mathcal{L}$ satisfying the axioms a, b, c2, c2' for any stratification $\sigma$.

The uniqueness follows easily from the following lemma.

**Lemma 1** Suppose $U \hookrightarrow X$ has codimension at least 2. If $X$ is connected, then $U$ is connected. Moreover, the natural map $\pi_1(U, x) \to \pi_1(X, x)$ is surjective. In particular, a local system on $U$ has *at most* one extension to $X$.

**Proof** The connectedness of $U$ follows from the long exact sequence in cohomology with compact support. The surjectivity follows from the connectedness and that any path can be deformed in codimension 2. □

We can use Deligne's construction to prove the existence of $\mathbb{P}$ inductively. Suppose $\mathcal{L}$ is a local system on $U$. Then there exists a maximal open $U_{\max} =: U_2$ such that $\mathcal{L}$ extends to $U_{\max} = U_2$. We define $S_2$ to be the maximal open of $Z = X - U$ such that $i^*\tau_{\le} Rj_*\mathcal{L}|_S$ is a local system. Refining this idea, we obtain a coarser stratification by also requiring $i^!\tau_{\le} Rj_*\mathcal{L}|_S$ is also a local system. Iterating Deligne's construction, we obtain a stratification given by $S_k$. The stratification thus obtained is the "coarsest" in some sense. This allows us to remove the dependence on the stratification and to show the topological invariance of IC sheaves.

## Perverse sheaves

We now relax the axiom c2, c2' by allowing non-strict inequality. This modification is intended to include sheaves like $i_!\mathbb{P}(\mathcal{L})$ for $\mathbb{P}(\mathcal{L})$ an IC sheaf on a closed $Z$ (e.g., $\dim\mathcal{H}^{-i}(i_!\mathbb{P}(\mathcal{L})) = i$ if $i = \dim Z$).

**Definition 4** A sheaf $\mathcal{E} \in D_c^b(X)$ is *perverse* if

$$\dim_{\mathbb{C}} \operatorname{supp}\mathcal{H}^{-i}(\mathcal{E}) \le i$$

and

$$\dim_{\mathbb{C}} \operatorname{cosupp}\mathcal{H}^{-i}(\mathcal{E}) \le i.$$

The category of perverse sheaves is denoted by $\operatorname{Perv}(X)$.

**Remark 15** If $\mathcal{L}$ is an irreducible local system on an open of $Z$. Then $i_! IC_Z(\mathcal{L})$ is called a Deligne-Goresky-MacPherson (DGM) complex, which is an simple object in $\operatorname{Perv}(X)$.

**Theorem 6**

a. $\operatorname{Perv}(X)$ is an abelian category (the heart of perverse $t$-structure on $D_c^b(X)$ defined by the support and cosupport condition), stable under Verdier dual.

b. All simple objects are DGM complexes.

c. Every object in $\operatorname{Perv}(X)$ is a successive extension of simple objects.

**Definition 5** A proper surjective morphism between two varieties $f : X \to Y$ is *small* (resp. *semi-small*) if $\operatorname{codim}\{y \in Y : \dim f^{-1}(y) \geq r\} > 2r$ (resp. $\geq 2r$) for any $r \geq 1$. In particular, $f$ is a generically finite morphism. For example, when $\dim Y \leq 2$, small is equivalent to finite. When $\dim Y = 3$, small morphism can only have dimension 1 fibers above finitely many points. One can check that a morphism is semi-small if and only if $\dim X \times_Y X = \dim X$ and small if and only $\dim(X \times_Y X - \Delta_X) < \dim X$.

Small morphisms are compatible with IC sheaves.

**Theorem 7** If $f : X \to Y$ is small and has generic degree 1, then $f_! \mathcal{IC}_X = \mathcal{IC}_Y$. In particular, $IH^i(Y) = H^i(X)$.

**Remark 16** This is especially useful when $X$ is smooth (i.e., a small resolution of $Y$).

**Proof** Consider the case $X$ is smooth. Then $f_! \mathbb{Q}[n]|_U = \mathbb{Q}[n]$. So by Theorem 5 it suffices to check the support and cosupport axioms for $\mathcal{E} = f_! \mathbb{Q}[n]$. By base change, $\mathcal{H}^{-i}(\mathcal{E})_y = H^{-i}(X_y, \mathbb{Q}[n]) = H^{-i+n}(X_y, \mathbb{Q})$. If $\mathcal{H}^{-i}(\mathcal{E})_y \neq 0$, then $-i + n \leq 2 \dim X_y$, i.e., $i \geq n - 2 \dim X_y$. The support and cosupport axioms then follow exactly by the smallness assumption. □

The above theorem can be generalized to the following:

**Theorem 8** Suppose $f : X \to Y$ is small. $\mathbb{P}(\mathcal{L}) \in \operatorname{Perv}(X)$. Then $Rf_*(\mathbb{P}(\mathcal{L})) = \mathbb{P}(f_* \mathcal{L})$.

**Remark 17** Notice $f_* \mathcal{L}$ can be viewed as a local system on a smaller open subset $V \subseteq U$ where $f$ is unramified. In terms of the monodromy representation, $f_* \mathcal{L}$ corresponds to the induction of of $\mathcal{L}$.

**Remark 18** This is the key theorem used in the proof of fundamental lemma: one can check the identity between perverse sheaves on $Y$ by throwing away bad fibers. Ngo called this "perverse continuation".

**Example 5** Suppose $f : X \to Y$ is a blow-up along a point $z \in Y$ with fiber $\mathbb{P}^{n-1}$. Then $Rf_* \mathbb{Q}_X = \mathbb{Q}_Y \bigoplus \oplus_{i=0}^{n-1} \mathbb{Q}_z[-i]$. This is no longer perverse and the extra terms $\mathbb{Q}_z[-i]$ measures the failure of $f$ being small.

In general we have the following decomposition theorem.

**Theorem 9** (Beilinson-Bernstein-Deligne + Gaber) If $f : X \to Y$ is proper and surjective, then in $D_c^b(X)$ we have $Rf_*(\mathbb{Q}_X)$ is a direct sum of DGM complexes shifted by certain degrees:

$$Rf_*(\mathbb{Q}_X) = \bigoplus_{(\mathcal{L}, Z)} \mathcal{IC}_Z(\mathcal{L})[d_Z].$$

**Remark 19** The proof is rather indirect: it relies on reducing to the positive characteristics and use Deligne's Weil II and also Gaber's improvement on purity. One can replace $\mathbb{Q}_X$ by more general complexes of "geometric origin" (coming from the pushforward of a constant sheaf): this contains local systems of finite monodromy and also IC sheaves, but is still quite restrictive.

**Remark 20** The key issue is that the $Z$'s appearing in the decomposition are hard to determine. A famous theorem (the support theorem) of Ngo further determines the possible $Z$'s under more restrictions on $f$.

(I was out of town and missed the two lectures on Oct 20 and 22. Thank Pak-Hin Lee for sending his notes to me. )

## Springer fibers

Reference: PCMI 2015 lectures by Ngo, Yun and Zhu, transcription available on Tony Feng's website.

A important class of semi-small maps comes from holomorphic symplectic varieties.

**Definition 6** A *holomorphic symplectic variety* (over $\mathbb{C}$) is a nonsingular variety such that there exists a closed holomorphic 2-form $\omega \in \Gamma(X, \Omega^{\otimes 2})$ satisfying that $\omega^{\dim_{\mathbb{C}} X}$ is nonzero everywhere (i.e., $\omega$ is non-degenerate).

**Example 6** Let $X$ be a nonsingular variety over $\mathbb{C}$ with $\dim_{\mathbb{C}} X = n$. Then the cotangent bundle $T^*X$ is naturally a holomorphic symplectic manifold. In fact, let $x_i$ be the local coordinates on $X$ and $y_i$ be the dual basis of $dx_i$ in the cotangent space, then $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ is a non-degenerate closed 2-form.

**Theorem 10** (Symplectic resolution) Let $f : X \to Y$ be a proper surjective birational map. If $X$ is holomorphic symplectic, then $f$ is semi-small.

**Example 7** (Hilbert scheme of surfaces) Let $S$ be a surface. Then symmetric power $S^{(n)}=S^n/S_n$ becomes singular when some of the $n$ -points on $S$ collide. However, the Hilbert scheme $S^{[n]}$ of $n$ -points on $S$ (which remembers infinitesimal information of the collision as well) is nonsingular. Moreover, one can show that $S^{[n]}$ is a symplectic resolution of $S^{(n)}$ Notice when $S$ is a 3-fold, both the $S^{(n)}$ an $S^{[n]}$ are singular.

Let $G$ be a semisimple connected linear group over $\mathbb{C}$ . Fix a Borel subgroup $B$ . Let $\mathcal{N}\subseteq\mathfrak{g}=\mathrm{Lie}\, G$ be the nilpotent cone. Then $\mathcal{N}$ is singular.

**Example 8** Let $G=SL(V)=SL(n)$ . Then $\mathfrak{g}=\mathfrak{sl}(V)=\{\phi\in\mathrm{End}(V):\mathrm{tr}(\phi)=0\}$ . Define the invariant map given by the coefficients of the characteristic polynomial of $\phi:\ \mathfrak{g}\to\mathbb{A}^{n-1}, \phi\mapsto\mathrm{tr}(\wedge^i\phi)_{i=2}^n$ . Then $\mathcal{N}$ is the preimage of $0\in\mathbb{A}^{n-1}$ , i.e., the set of all traceless matrices $\phi$ such that $\phi^n=0$ . When $n=2$ , $\mathcal{N}=\{\left(\begin{smallmatrix}x & y\\ z & -x\end{smallmatrix}\right):x^2-yz=0\}\subseteq\mathbb{A}^3$ , which is simply a 2-dimensional cone, with a simple singularity at 0.

In general the nilpotent cone $\mathcal{N}$ may have very bad singularities away from the regular nilpotent elements. Springer found a systematic way of resolving the singularities.

**Definition 7** Let $\mathbb{B}=G/B$ be the flag variety, parametrizing all the Borel subgroups of $G$ . Define

$$\tilde{\mathcal{N}}=\{(\phi,b):\phi\in\mathrm{rad}(\mathrm{Lie}\, b)\}\subseteq\mathcal{N}\times\mathbb{B}.$$

Then the natural map $\tilde{\mathcal{N}}\to\mathcal{N}$ is an isomorphism on the regular nilpotent elements. It turns out that the natural map $\tilde{\mathcal{N}}\to\mathcal{N}$ is a symplectic resolution, known as the *Springer resolution*. In fact, $\tilde{\mathcal{N}}\cong T^*\mathbb{B}$ . The fiber $\tilde{\mathcal{N}}_\phi$ is called a *Springer fiber*.

**Example 9** For $G=SL(2)$ , $\tilde{\mathcal{N}}_{\left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)}=\mathrm{pt}$ and $\tilde{\mathcal{N}}_0=\mathbb{P}^1$ . The simple singularity of the 2-dimensional cone at 0 is resolved.

**Remark 21** It follows that $\dim\tilde{\mathcal{N}}=2\dim\mathbb{B}=2(\dim G-\dim B)$ . Therefore $\mathcal{N}\subseteq\mathfrak{g}$ has codimension $2\dim B-\dim G=r$ , where $r$ is the rank of $G$ . One can also see this from that $\mathcal{N}$ is the preimage at 0 of the invariant map $\mathfrak{g}\to\mathfrak{t}//W=\mathbb{A}^r$ , where $\mathfrak{t}$ is the Cartan subalgebra and $W$ is the Weyl group.

Using the fact that $\tilde{\mathcal{N}}_\phi=\frac{1}{2}(\dim\mathcal{N}-G\cdot\phi)$ , one can check that $\tilde{\mathcal{N}}\to\mathcal{N}$ is indeed semi-small. One can generalize the construction of $\tilde{\mathcal{N}}$ and obtain a *small* map $\tilde{\mathfrak{g}}\to\mathfrak{g}$ .

**Definition 8** The *Grothendieck-Springer fibration* is defined to be

$$\tilde{\mathfrak{g}}=\{(\phi,b):\phi\in\mathrm{Lie}(b)\}\to\mathfrak{g}.$$

The Springer fiber $\tilde{\mathcal{N}}_\phi$ is always reduced. However, for the Grothendieck-Springer resolution $\tilde{\mathfrak{g}}\to\mathfrak{g}$ , the fibers may be non-reduced. Therefore we have a commutative (but *not* Cartesian) diagram

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[r, hook] \arrow[d] & \tilde{\mathfrak{g}} \arrow[d] \\
\mathcal{N} \arrow[r, hook] & \mathfrak{g}.
\end{tikzcd}$$

Explicitly,

$$\tilde{\mathcal{N}}_\phi=\{b\in\mathbb{B}:\phi\in\mathrm{rad}(b)\},$$

while

$$\tilde{\mathfrak{g}}_\phi=\{b\in\mathbb{B}:\phi\in b\}.$$

**Example 10** Let $G=SL(V)$ . Let $0\subsetneq V_1\subsetneq\cdots\subsetneq V_n=V$ be the complete flag associated to $b$ . Then for the regular element $\phi$ , the Springer fiber satisfies $\phi V_i\subseteq V_{i-1}$ while the Grothendieck-Springer fiber satisfies the weaker condition $\phi V_i\subseteq V_i$ . For example, when $V$ is 2-dimensional, $\mathbb{B}\cong\mathbb{P}^1$ . Let $R=k[t]/t^2$ , then $\tilde{\mathfrak{g}}_\phi(R)$ has more than one points: it has an obvious point corresponding to $0\subseteq W=R\oplus 0\subseteq R\oplus R$ . It also has an extra $0\subseteq R(1,t)\subseteq R\oplus R$ because $t(1,t)=(t,0)$ . We can compute the scheme-theoretical fixed points of $\phi=\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$ on $\mathbb{B}=\mathbb{P}^1$ : since $\phi(x,y)=(x+y,y)$ , we have $xy=(x+y)y$ , hence $y^2=0$ . Therefore $\tilde{\mathfrak{g}}_\phi=\mathrm{Spec}\, k[y]/y^2$ .

**Remark 22** In general, $\tilde{\mathcal{N}}_\phi$ is the reduced scheme associated to $\tilde{\mathfrak{g}}_\phi$ .

## Affine Springer fibers

Let $k$ be a finite field. One can easily see that the number of $k$ -points in the Springer fiber is the same as fixed points of $\phi$ on the set $G(k)/B(k)$ , which can be rewritten as a simple orbital integral. Moreover, the finite set

$G(k)/B(k)$ can be realized as the $k$-rational points $\mathbb{B}(k)$ of the flag variety $\mathbb{B}$. Now we want to upgrade this to an "affine" version, i.e., for local fields of equal characteristic.

Let $F=k((t))$ and $\mathcal{O}_F=k[[t]]$. We want to geometrize the infinite set $G(F)/G(\mathcal{O}_F)$ by an *affine Grassmannian* $\mathrm{Gr}$. The analogue of the full flag variety $G/B$ should be given by the *affine flag variety* whose $k$ points gives $G(F)/Iw(\mathcal{O}_F)$, where $Iw$ is the Iwahori subgroup.

In terms of moduli interpretation, when $G=SL(n)$, $G(F)/G(\mathcal{O}_F)$ is the set of $\mathcal{O}_F$-lattices in $F^n$, where the identity coset corresponds to the standard lattice $\mathcal{O}_F^n$. $G(F)/Iw(\mathcal{O}_F)$ is the set of chain of lattices in $V\otimes_k F$,

$$\cdots\subseteq\Lambda_0\subseteq\Lambda_1\subseteq\cdots\subseteq\Lambda_i\subseteq\cdots$$

where $\Lambda_i/\Lambda_{i-1}$ is length one $\mathcal{O}_F$-module such that $\Lambda_{i+n}=t^{-1}\Lambda_i$.

**Definition 9** Let $R$ be a $k$-algebra. Let $R[[t]]=\varprojlim R\otimes\mathcal{O}_F/(t^n)$ be the ring of power series in $R$. Let $R((t))=R[[t]][1/t]$ be the field of Laurent sereis. Due to the completion process, $R((t))$ is larger than the naive base change $R\otimes_k F^n$ when $R$ is not finitely generated.

**Remark 23** Notice $D_k=\operatorname{Spec}\mathcal{O}_F$ geometrically is a formal disc (the formal completion of a curve over $k$ at a point) and $D_k^*=\operatorname{Spec}F$ is a punctured disk. In general, $D_R=\operatorname{Spec}R[[t]]$ is the formal completion of a curve $X/R$ along the graph $\Gamma_x$ of $x:\operatorname{Spec}R\to X$ (so the completion is only along the $t$-direction).

**Definition 10** We define an *$R$-family of lattices* in $R((t))^n$ is a finitely generated projective $R[[t]]$-submodule $\Lambda\subseteq R((t))^n$ such that $\Lambda\otimes_{\mathcal{O}_F}F=R((t))^n$. This is equivalent to the data $(\mathcal{E},\beta)$, where $\mathcal{E}$ is a vector bundle over $D_R$ of rank $n$ and $\beta:\mathcal{E}|_{D_R^*}\cong\mathcal{E}_0|_{D_R^*}$ is a trivialization of $\mathcal{E}$ over the punctured disk.

We define the functor $\mathrm{Gr}:\mathbf{Alg}_k\to\mathbf{Sets}$. Then $\mathrm{Gr}(k)\cong G(F)/G(\mathcal{O}_F)$. $\mathrm{Gr}$ has a reasonable geometric structure (though infinite dimensional).

**Theorem 11** $\mathrm{Gr}$ is represented by an ind-scheme, i.e., $\mathrm{Gr}=\bigcup_{N\geq0}\mathrm{Gr}^{(N)}$, where each $\mathrm{Gr}^{(N)}$ is a projective scheme of finite type and each $\mathrm{Gr}^{(N)}\hookrightarrow\mathrm{Gr}^{(N+1)}$ is a closed immersion.

Here $\mathrm{Gr}^{(N)}(R)$ consists of $R[[t]]$-lattices $\Lambda$ (projective as $R[[t]]$-modules) such that

$$t^NV_{R[[t]]}\subseteq\Lambda\subseteq t^{-N}V_{R[[t]]}.$$

Due to this boundness, it can be viewed as the set of quotient $R[[t]]$-modules of $t^{-N}V_{R[[t]]}/t^NV_{R[[t]]}$ (projective as $R$-modules). In other words, let $V_N=t^{-N}\mathcal{O}_F^n/t^N\mathcal{O}_F^n$ (so $\dim_k V_N=2Nn$). Then $\mathrm{Gr}^{(N)}(R)$ is the set of quotient projective $R$-modules $W$ of $V_N\otimes R$, such that the $t$-action (given by a nilpotent operator $\phi_N$) satisfies $\phi_NW\subseteq W$. Observe that $\mathrm{Gr}^{(N)}$ is nothing but a union of generalized version of Springer fibers $\mathrm{Gr}(i,V_n)^{\phi_N}$:

$$\mathrm{Gr}^{(N)}=\coprod_{0\leq i\leq 2Nn}\mathrm{Gr}(i,V_n)^{\phi_N}.$$

Here $\mathrm{Gr}(i,V_n)$ is a Grassmannian of one-step flags (instead of full flags).

**Example 11** For $n=1$, we see $V_N=k^{2N}$. Let $\phi_N$ be a regular nilpotent operator on $V_N$. When $N=1$, we have

$$\mathrm{Gr}^{(N)}=\mathrm{pt}\coprod\operatorname{Spec}k[y]/y^2\coprod\mathrm{pt}.$$

In general $\mathrm{Gr}^{(N)}(k)=2N+1$ (compare: $\mathrm{Gr}(k)=F^\times/\mathcal{O}_F^\times\cong\mathbb{Z}$).

**Remark 24** One can also try to geometrize $G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ and obtain the mixed characteristic affine Grassmannian. In terms of moduli, its $R$-points should be given by the $W(R)$-lattices in $W(R)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p^n$. It turns out the Witt ring $W(R)$ only behaves well when restricting to a subcategory of *perfect* $\mathbb{F}_p$-algebras (these algebras are large but still useful for many purposes, e.g., for answering topological questions). We now know that the mixed characteristic affine Grassmannian is an ind-projective algebraic space (union of projective schemes), after the recent work of X. Zhu, Scholze and Bhatt.

More generally,

**Definition 11** (Affine Grassmannians) Suppose $k$ is algebraically closed. Let $G$ be a group scheme over $\mathcal{O}_F$. We define the functor $\mathrm{Gr}_G:\mathbf{Alg}_k\to\mathbf{Sets}$ such that $\mathrm{Gr}_G(R)$ is the set of pairs $(\mathcal{E},\beta)$, where $\mathcal{E}$ is a $G$-torsor over $D_R$ and $\beta$ is a trivialization $\mathcal{E}|_{D_R^*}\cong\mathcal{E}_0\times_F D_R^*$, where $\mathcal{E}_0$ is the trivial $G_F$-torsor. In other words, $\mathrm{Gr}_G$ is the moduli space of $G$-torsor together with a rigidification. This may remind you of the definition of a Rapoport-Zink space, which is the moduli space of certain $p$-divisible groups together with a rigidification.

**Theorem 12** IF $G$ is a flat group scheme over $\mathcal{O}_F$. Then

a. The presheaf (under the fpqc topology) $\mathrm{Gr}_G$ is an ind-scheme (ind of finite type).
b. If $G$ is reductive, then $\mathrm{Gr}_G$ is ind-projective.
c. If $G$ is parabolic (i.e., a smooth group scheme whose general fiber is reductive but the special fiber may fail to be reductive, e.g., the Iwahori model of $G$), then $\mathrm{Gr}_G$ is ind-projective.

Notice $\mathrm{Aut}_F(\mathcal{E}_0) = G(F)$. So $\gamma \in G(F)$ acts on $\mathrm{Gr}_G$ by $\gamma(\mathcal{E}, \beta) = (\mathcal{E}, \gamma \circ \beta)$.

**Definition 12** (Affine Springer fibers) Let $\gamma \in G(F)$, we define $\tilde{X}_\gamma := \mathrm{Gr}_G^\gamma$ to be the fixed points of $\gamma$ and $X_\gamma := \tilde{X}_\gamma^{\mathrm{red}}$. It turns out that $X_\gamma$ is a closed subscheme of $\mathrm{Gr}_G$.

**Remark 25** The number of the $k$-points of the affine Springer fiber more or less gives the orbital integrals $\mathcal{O}(\gamma, \mathbf{1}_{G(\mathcal{O}_F)})$.

**Remark 26** One can also define a Lie algebra version of affine Springer fibers. If $G = SL(V)$ and $\phi \in \mathfrak{g} = \mathrm{End}(V)$ (not necessarily an automorphism), we define $X_\phi$ as the intersection of the graph of $\phi$ with the diagonal. Namely, $X_\phi$ sits in the Cartesian diagram

$$\begin{tikzcd} X_\phi \arrow[r] \arrow[d] & \Gamma_\phi \arrow[d] \\ \mathrm{Gr}_G \arrow[r, "\Delta"] & \mathrm{Gr}_G \times \mathrm{Gr}_G . \end{tikzcd}$$

We provide two more analogous constructions.

**Example 12** (Affine Schubert varieties) By the Bruhat decomposition (over $k$)

$$G = \coprod_{w \in W} BwB.$$

The orbits of $B$ on $G/B$ are parametrized by elements in the Weyl group $W$. Let

$$\mathrm{inv} : G/B \times G/B \to W, \quad ([g], [h]) \mapsto B(g^{-1}h)B$$

be the invariant map. Define the orbit associated to $w$ to be the *Schubert variety*, so

$$X_w = \mathrm{inv}^{-1}(w) \cap (\{1\} \times G/B).$$

These are locally closed subvariety of $G/B$ (defined by incidence relations) . Then $X_w$ sits in the Cartesian diagram

$$\begin{tikzcd} X_w \arrow[r] \arrow[d] & \mathrm{inv}^{-1}(w) \arrow[d] \\ G/B \arrow[r, "1\times \mathrm{Id}"] & G/B \times G/B. \end{tikzcd}$$

One can analogously define *affine Schubert varieties*. By the Cartan decomposition

$$G(\mathcal{O}_F)\backslash G(F)/G(\mathcal{O}_F) \cong X_*(T)//W \cong X_*(T)^+,$$

where $X_*(T)^+$ consists of the dominant co-characters of $G$, we have an invariant map

$$\mathrm{inv} : \mathrm{Gr}_G(k) \to G(\mathcal{O}_F)\backslash G(F)/G(\mathcal{O}_F).$$

Then for $w \in X_*(T)^+$, define the affine Schubert variety by

$$X_w(R) = \{(\mathcal{E}, \beta) : \forall \text{geometric points } x \in \mathrm{Spec}\, R, \mathrm{inv}_x(\mathcal{E}, \beta) = w\}.$$

**Example 13** (Affine Deligne-Lusztig varieties) Assume $k$ is finite. Define the *Deligne-Lusztig variety* $X_w^{\mathrm{DL}}$ to be the subvariety of $G/B$ given by

$$X_w^{\mathrm{DL}} = \{b \in G/B : \mathrm{inv}(b, \mathrm{Frob}(b)) = w\}.$$

In other words, the Deligne-Lusztig variety sits in the Cartesian diagram

$$\begin{tikzcd} X_w^{\mathrm{DL}} \arrow[r] \arrow[d] & \mathrm{inv}^{-1}(w) \arrow[d] \\ G/B \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & G/B \times G/B. \end{tikzcd}$$

Deligne-Lusztig constructed all the irreducible representations of finite reductive groups in the cohomology (with local systems as coefficients) of Deligne-Lusztig varieties. The computation of the Deligne-Lusztig characters are naturally related to counting points of Springer fibers. Deligne-Lusztig varieties form one of the starting point of the geometric approach to representation theory initiated by Kazhdan and Lusztig.

One can then analogously define affine Deligne-Lusztig varieties using the point-wise condition. Affine Grassmannians and affine Deligne-Lusztig varieties are fundamental objects in geometric representation theory and in the study of local models of Shimura varieties.

An alternative definition of affine Grassmannians uses loop spaces and arc spaces. Let $k$ be a field and $F = k((t))$. Let $X/k$ be a scheme. One would like to geometrize the sets $X(F)$ and $X(\mathcal{O}_F)$. We define the *loop space* functors

$$\mathrm{L}X : \mathbf{Alg}_k \to \mathbf{Sets}, \quad R \mapsto X(R((t))) = \mathrm{Hom}_k(D_R^*, X).$$

Similarly we define the *arc space* (or *positive loop space*) functor

$$\mathrm{L}^+X : \mathbf{Alg}_k \to \mathbf{Sets}, \quad R \mapsto X(R[[t]]) = \mathrm{Hom}_k(D_R, X).$$

These are presheaves under the fpqc topology.

**Example 14** When $X = \mathbb{G}_m$, we have $\mathrm{L}^+X(R) = (R[[t]])^\times$. Therefore $\mathrm{L}^+X$ is in fact represented by a scheme $\mathbb{G}_m \times \mathbb{A}^\infty$, given by the leading coefficient $a_0$ and rest of the coefficients $(a_i)_{i \geq 1}$. The points $\mathrm{L}X(R) = R((t))^\times$ are more complicated: they are Laurent series of the form

$$\sum_{i \gg -\infty} a_i t^i : a_{i_0} \in R^\times, a_i \text{ nilpotent}, i < i_0.$$

When taking the reduced structure, we find that $\mathrm{L}X^{\mathrm{red}} = \mathrm{L}^+X \times \mathbb{Z}$ is an infinite copies of $\mathrm{L}^+X$.

More generally, we have

**Theorem 13**

a. $\mathrm{L}^+X$ is represented by a scheme. It is affine if $X$ is affine.
b. $\mathrm{L}X$ is represented by an ind-scheme.
c. The affine Grassmannian $\mathrm{Gr}_G = \mathrm{L}X/\mathrm{L}^+X$ (as the quotient sheave under the fpqc topology).

## Orbital integrals

**Definition 13** Suppose $n$ is a local field. For $\phi \in C_c^\infty(\mathfrak{g}(F))$ and $\gamma \in \mathfrak{g}(F)$. Define the orbital integral

$$\mathrm{Orb}(\gamma, \phi) = \int_{G_\gamma(F)\backslash G(F)} \phi(g^{-1}\gamma g) dg.$$

Notice that the convergence of this orbital integral is already an issue.

**Example 15** Let us consider the case $G = SL_n$, $\mathfrak{g} = \mathfrak{sl}_n$, and $\gamma$ is regular semisimple. In this case the centralizer $G_\gamma$ of $\gamma$ is the diagonal torus $A \subseteq G$. Since $\phi(g^{-1}\gamma g)$ is $K = GL_n(\mathcal{O}_F)$-bi-invariant and $K$ is compact, to show the convergence of the orbital integral, it suffices to show the convergence of

$$\int_{A(F)\backslash GL_n(F)/K} \phi(g^{-1}\gamma g) dg.$$

By the Iwasawa decomposition

$$G(F) = A(F)N(F)K,$$

where $N$ is the unipotent radical of the Borel subgroup of $G$ (i.e., group of upper triangular unipotent matrices for $G = GL_n$), we know that the convergence is equivalent to

$$\int_{N(F)} \phi(n^{-1}\gamma n) dn.$$

The key observation is that

$$n^{-1}\gamma n = n^{-1}(\gamma n \gamma^{-1})\gamma$$

and one easily compute $\gamma n \gamma^{-1}$ since each root group is an eigenvector under the adjoint action of the semisimple element $\gamma$. For example, when $G = SL_2$ and $\gamma = \mathrm{diag}(\alpha_1, \alpha_2)$, we have

$$n^{-1}(\gamma n \gamma^{-1})\gamma = \begin{pmatrix} 1 & (\frac{\alpha_1}{\alpha_2} - 1)n \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix}.$$

Take $\phi = \mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}$, we find the integrand is nonzero only when the above matrix has $\mathcal{O}_F$-entries. This means that $n \in (\alpha_1 - \alpha_2)^{-1}\mathcal{O}_F$, hence $n$ is bounded and the integral converges.

**Remark 27** In general, when $F = k((t))$, the above argument shows that the integral

$$\int_{G_\gamma(F)\backslash G(F)/G(\mathcal{O}_F)} \mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}(g^{-1}\gamma g) dg$$

essentially counts the set

$$G_\gamma(F)\backslash(G(F)/G(\mathcal{O}_F))^\gamma = \mathrm{L}G_\gamma(k)\backslash X_\gamma(k) \approx (\mathrm{L}G_\gamma\backslash X_\gamma)(k),$$

where $\mathrm{L}G_\gamma$ is the loop space of $G_\gamma$ and $X_\gamma$ is the affine Springer fiber. The last inequality is problematic in general since the rational points of the quotient is not necessarily the quotient of the rational points.

Ngo showed that the action of $\mathrm{L}G_\gamma$ on $X_\gamma$ is not faithful. The action factors through the quotient $\mathcal{P}_\gamma$ known as the *local Picard group*.

**Example 16** For $G = SL_2$, $\gamma = \mathrm{diag}(t, -t) \in \mathfrak{g}$. Then $G_\gamma = \mathbb{G}_m \otimes_k F$ and $\mathcal{P}_\gamma = \mathbb{G}_m \times \mathbb{Z}$ (i.e., the $\mathbb{A}^\infty$ -part acts trivially on $X_\gamma$ ).

**Theorem 14** (Goresky-Kottwitz-MacPherson, Ngo) Assume $\gamma$ is regular semisimple. Then

$$[\mathcal{P}_\gamma\backslash X_\gamma](k) = (*)\cdot \mathrm{SO}_\gamma(\mathbf{1}_{\mathfrak{g}(\mathcal{O}_F)}).$$

Namely, the $k$ -points of the stacky quotient is in fact a stable orbital integral (i.e., a sum of orbital integrals over conjugacy classes $\gamma'$ which are stably conjugate to $\gamma$ ).

Instead of the loop group action, Kazhdan-Lusztig also considered the discrete action of the lattice $\Lambda_\gamma = t^{X_*(G_\gamma)/F^{\mathrm{ur}}} \subseteq G(F^{\mathrm{ur}})$ on the affine Springer fiber.

**Theorem 15** (Kazhdan-Lusztig) $\Lambda_\gamma\backslash X_\gamma$ is proper and of finite type. $X_\gamma$ is a finite dimensional ind-scheme and is locally of finite type.

## Hitchin fibers

The quotient $\Lambda_\gamma\backslash X_\gamma^{\mathrm{red}}$ is a projective variety. It has singularities, but people still expect certain "purity" of its cohomology, which implies the fundamental lemma for regular semisimple elements in the maximal torus.

This purity is still unknown. To prove the fundamental lemma, one instead consider a global version of the affine Springer fibers. Suppose $X/k$ is a smooth projective curve over a finite field with $F = k(X)$. Consider $G = GL_n$. The global analogue of affine Grassmannian $\mathrm{Gr}_G$ is $\mathrm{Bun}_G$, the moduli stack of rank $n$ vector bundles on $X$. It represents the functor

$$\mathbf{Sch}_k \to \mathbf{Groupoid}, \quad \mathrm{Bun}_G(S) = \{\text{rank } n \text{ vector bundles over } X\times_k S\}.$$

It has rational points

$$\mathrm{Bun}_G(k) = G(F)\backslash G(\mathbb{A}_F)/\prod_x G(\mathcal{O}_x).$$

**Remark 28** It turns out $\mathrm{Bun}_G$ is an Artin stack, smooth over $k$ and is locally finite type. It has a stratification given by the instability and it is of finite type on any part with bounded instability. For $\mathcal{E}\in \mathrm{Bun}_G(k)$, by the smoothness we have $H^2(X, \mathrm{End}(\mathcal{E})) = 0$, so

$$\dim \mathrm{Bun}_G = h^1(X, \mathrm{End}(\mathcal{E})) - h^0(X, \mathrm{End}(\mathcal{E})) = -\chi(\mathrm{End}(\mathcal{E})) = n^2(g_X - 1).$$

When $\mathcal{E}$ is stable, $h^0(X, \mathrm{End}(\mathcal{E})) = 1$. So the coarse moduli space of $\mathrm{Bun}_G$ has dimension $1+\dim \mathrm{Bun}_G$ (at least for the stable part).

**Remark 29** The the fiber of the tangent bundle of $\mathrm{Bun}_G$ at $\mathcal{E}\in \mathrm{Bun}_G(k)$ is given by $H^1(X, \mathrm{End}(\mathcal{E}))$. Since $\mathrm{End}(\mathcal{E}) = \mathcal{E}\oplus\mathcal{E}^\vee$ is self-dual, we know that by Serre duality this is isomorphic to $H^0(X, \mathrm{End}(\mathcal{E})\otimes\Omega_X)^\vee$. Hence the fiber of the cotangent bundle at $\mathcal{E}$ is given by the vector space $H^0(X, \mathrm{End}(\mathcal{E})\otimes\Omega_X) = \mathrm{Hom}(\mathcal{E}, \mathcal{E}\otimes\Omega_X)$.

**Definition 14** A pair $(\mathcal{E}, \phi : \mathcal{E}\to\mathcal{E}\otimes\Omega_X)$ is called a *Higgs bundle*. The Hitchin moduli space $T^*\mathrm{Bun}_G$ is defined to be moduli space of Higgs bundles:

$$T^*\mathrm{Bun}_G(S) = \{(\mathcal{E}, \phi) : \mathcal{E}\in \mathrm{Bun}_G(S), \phi\in \mathrm{Hom}_{X\times_k S}(\mathcal{E}, \mathcal{E}\otimes\Omega_X)\}.$$

**Definition 15** For the *Higgs field* $\phi\in \mathrm{Hom}(\mathcal{E}, \mathcal{E}\otimes\Omega_X)$, define its invariants $a_i(\phi) := \mathrm{tr}\wedge^i\phi \in H^0(X, \Omega_X^{\otimes i})$. So $a_1(\phi) = \mathrm{tr}\,\phi\in H^0(X, \Omega_X)$ and $a_n(\phi) = \det\phi \in H^0(X, \Omega_X^{\otimes n})$. We call the affine space $\bigoplus_{i=1}^n H^0(X, \Omega_X^{\otimes i})$ the *Hitchin base*. We have the invariant map to the Hitchin base

$$T^*\mathrm{Bun}_G \to \bigoplus_{i=1}^n H^0(X, \Omega_X^{\otimes i}).$$

**Remark 30** Since

$$\dim H^1(X, \Omega_X^{\otimes i}) = \begin{cases} 1 & i = 1, \\ 0 & i \geq 2, \end{cases}$$

by Riemann-Roch we have

$$\dim H^0(X, \Omega_X^{\otimes i}) = \begin{cases} g_X & i = 1, \\ (2i-1)(g_X - 1) & i \geq 2. \end{cases}$$

So the Hitchin base has dimension $n^2(g_X - 1) + 1$, which becomes the same dimension of the coarse moduli space of $\mathrm{Bun}_G$ and hence half of the dimension of $T^* \mathrm{Bun}_G$. This was one of Hitchin's key observations. Using this Hitchin was able to define certain integrable hamiltonian system, which has applications to differential geometry and physics.

**Definition 16** More generally, we can replace $\Omega_X$ by any vector bundle $\mathcal{L}$. The resulting moduli space of Higgs bundles $\mathcal{M}_{G,\mathcal{L}}$ is called the $\mathcal{L}$*-twisted Hitchin moduli space*. The affine space $\mathcal{A}_\mathcal{L} = \bigoplus_{i=1}^n H^0(X, \mathcal{L}^{\otimes i})$ is called the *Hitchin base*.

When $\mathcal{L} = \mathcal{O}_X(D)$, then $H^0(X, \mathcal{L}^{\otimes i}) = \{f \in k(X) : \mathrm{div}(f) \geq -iD\}$, which is a finite dimensional $k$-subspace of $F = k(X)$. So one can view $\mathcal{A}_\mathcal{L}$ as a finite dimensional $k$-subspace of an infinite dimensional $k$-space $F^n$. On the other hand, when varying $\mathcal{L}$ (allowing more poles) these finite dimensional $k$-subspaces will exhaust all elements of $F^n$. More precisely, one can define a family version of the Hitchin base by considering

$$\mathcal{A}_d = \{(\mathcal{L}, s) : \mathcal{L} \in \mathrm{Pic}^d(X), s \in H^0(X, \mathcal{L})\}.$$

Let $\mathcal{A}_d^0 \subseteq \mathcal{A}_d$ be the open substack with $s \neq 0$. It turns out $\mathcal{A}_d^0$ is the same as $X^d/S_d$ (the effective divisors of degree $d$ on $X$), hence is indeed a scheme. The complement of $\mathcal{A}_d^0 \subseteq \mathcal{A}_d$ is isomorphic to $\mathrm{Pic}_X^d$ (given by the zero section). More generally,

**Definition 17** Define

$$\mathcal{A}_d^i = \{(\mathcal{L}, s) : \mathcal{L} \in \mathrm{Pic}^d(X), s \in H^0(X, \mathcal{L}^{\otimes i})\}.$$

The *universal Hitchin base* is defined to be

$$\mathcal{A}_d^1 \times_{\mathrm{Pic}_X^d} \mathcal{A}_d^2 \cdots \times_{\mathrm{Pic}_X^d} \mathcal{A}_d^n,$$

a family of Hitchin bases over $\mathrm{Pic}_X^d$.

The fibers $\mathcal{M}_a$ of the invariant map (*Hitchin fibers*) $\mathcal{M}_{G,\mathcal{L}} \to \mathcal{A}_\mathcal{L}$ are the global analogue of affine Springer fibers.

**Theorem 16** For $\gamma \in G(F)$ with characteristic polynomial $\mathrm{char}(\gamma) = P_a(T) := \sum_{i=0}^n (-1)^i a_i T^{n-i} \in F[T]$. Assume that $\gamma$ is elliptic (i.e. $P_a \in F[T]$ is irreducible, equivalently $G_\gamma$ is an anisotropic torus). Then

$$[\mathrm{Pic}_X \backslash \mathcal{M}_a](k) = (*) \, \mathrm{Orb}(\gamma, \mathbf{1}_{\mathfrak{g}(\mathcal{O})}).$$

**Remark 31** Since the global orbital integral factors into the product of local orbital integrals, we would expect the Hitchin fiber $\mathcal{M}_a$ "factors" as the product of affine Springer fibers. To do this one needs more explicit description of the Hitchin fiber. We will define a *spectral curve* $Y_a \to X$ such that $M_a \cong \overline{\mathrm{Pic}}(Y_a)$, the compactified Picard variety of $Y_a$ (at least when $Y_a$ is reduced). This allows us to define a relative group scheme $\overline{\mathrm{Pic}}(Y) \to \mathcal{A}_\mathcal{L}$. Moreover, $Y_a$ is in fact smooth or has only mild singularity over a large open of the Hitchin base. One can then prove a sheaf-theoretic version of the fundamental lemma on this large open and use perverse continuity to deduce the rest. This help us to avoid the difficulty in dealing with the complicated singularities of the affine Springer fibers.

**Remark 32** Why does this globalization help to prove the fundamental lemma? Here is a more philosophical explanation. The fundamental lemma is easy if $\gamma_x \in G(\mathcal{O}_x)$ and $\gamma_x \mod t \in G(k_x)$ is regular semisimple. However, these conditions are quite restrictive: i.e., the discriminant $\Delta(\gamma_x)$ is required to be a unit. But globally, $\Delta(\gamma)$ is a unit at *almost all* places. The idea is then to use the fact that the fundamental lemma is easy for all most all places to deduce the fundamental lemma at the remaining places.

## Spectral curves

Today we will discuss a bit more on the spectral curve $Y_a$ mentioned last time. Starting next time we will do a concrete example: use the perverse continuation principle to prove Waldspurger's theorem for central values of $L$-functions on $GL_2$ in the function field setting.

**Definition 18** Consider the total space of the line bundle $\mathcal{L}$,

$$\mathbb{L} = \mathrm{Spec}(\mathrm{Sym}\,\mathcal{L}^\vee) = \mathrm{Spec}(\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L}^{\vee\otimes 2} \oplus \cdots).$$

It is a $\mathbb{A}^1$-fibration over $X$. This total space $\mathbb{L}$ sits in the projective bundle

$$\pi : \mathbb{P} := \mathbb{P}(\mathcal{O}_X \otimes \mathcal{L}^\vee) \to X$$

(so $\pi_*\mathcal{O}(1) = \mathcal{O}_X \otimes \mathcal{L}^\vee$). We have two affine charts given by the two coordinates $x \in H^0(\mathbb{P}, \mathcal{O}(1))$ and $y \in H^0(\mathbb{P}, \pi^*(\mathcal{L})(1))$. Then $\mathbb{L} \subseteq \mathbb{P}$ is given by the $x \neq 0$. Let

$$P_a(x,y) = \sum_{i=1}^n (-1)^i a_i x^i y^{n-i} \in H^0(\mathbb{P}, \pi^*\mathcal{L}^{\otimes n}(n)).$$

We define the *spectral curve* $Y_a \subseteq \mathbb{L}$ to be the zero locus of $P_a(x,y)$.

**Remark 33** Locally $Y_a \to X$ is a finite cover of degree $n$ given by the equation $P_a(t)$, where $t = y/x$. The generic fiber of $Y_a$ has function field $F[t]/P_a(t)$. Hence the generic fiber is *reduced* if and only if $P_a(t)$ has no repeated roots over $\bar{F}$, i.e., $a \in (F^n)$ is *regular semisimple*. We define the locus $A_{\mathcal{L}}^{\heartsuit} \subseteq \mathcal{A}_{\mathcal{L}}$ (resp. $\mathcal{A}_{\mathcal{L}}^{\diamondsuit}$), where the spectral curve is reduced (resp. smooth).

**Lemma 2** Suppose $\mathrm{char}\, k > n$ and $\deg \mathcal{L} \gg 0$.

a. $Y_a$ is reduced (i.e., $a \in \mathcal{A}_{\mathcal{L}}^{\heartsuit}$) if and only if $a \in \mathcal{A}_{\mathcal{L}}$ is regular semisimple.

b. $Y_a \to X$ is etale over $x \in X$ if and only if $\Delta_a(x) \neq 0$, where $\Delta_a \in H^0(X, \mathcal{L}^{n(n-1)})$ is the discriminant section.

c. If $\mathrm{div}(\Delta_a)$ is a multiplicity-free effective divisor, then $Y_a$ is smooth (i.e., $a \in \mathcal{A}_{\mathcal{L}}^{\diamondsuit}$).

**Definition 19** Suppose $Y$ is a reduced curve. The *compactified Picard (or Jacobian) stack* $\overline{\mathrm{Pic}}_Y$ of $Y$ is defined to be the stack of torsion-free coherent sheaves of rank 1 on $Y$. When $Y$ is smooth, $\overline{\mathrm{Pic}}_Y = \mathrm{Pic}_Y$. By a theorem Altman-Iarrobino-Kleinman, for reduced curves with only *planar singularities* (which by definition is satisfied by the spectral curves), the usual Picard scheme $\mathrm{Pic}_Y$ is always open dense in $\overline{\mathrm{Pic}}_Y$. Hence $\overline{\mathrm{Pic}}_Y$ is naturally a compactification of $\mathrm{Pic}_Y$. Notice that in general $\overline{\mathrm{Pic}}_Y$ may have singularities.

For $\mathcal{F} \in \overline{\mathrm{Pic}}_Y$ a torsion-free coherent sheaf on $Y_a$, the pushforward $\pi_*\mathcal{F}$ under $\pi : Y_a \to X$ is a torsion-free sheaf of rank $n$ on the smooth curve $X$, hence is indeed a vector bundle of rank $n$. One can further construct an $\mathcal{O}_X$-linear endomorphism of $\pi_*\mathcal{F}$ with the given characteristic polynomial $P_a$ using the action of $\mathcal{O}_{Y_a}$ on $\mathcal{F}$. In this way one can describe a Hitchin fiber as the compactified Picard of the spectral curve.

**Theorem 17** For $a \in \mathcal{A}_{\mathcal{L}}^{\heartsuit}$, we have $\mathcal{M}_a = \overline{\mathrm{Pic}}_{Y_a}$.

## Waldspurger's formula via Jacquet's relative trace formula

Let $F'/F$ be a quadratic extension of function fields, corresponding to an etale double cover of curves $X' \to X$ over a finite field $k$. Let $G = PGL_2/F$ and $T$ be an anisotropic torus $T = \mathrm{Res}_{F'/F}\,\mathbb{G}_m/\mathbb{G}_m$ (with a fixed embedding $T \subseteq G$). *Waldspurger's formula* relates the toric automorphic period

$$\mathcal{P}_T : \mathcal{A}_0(G) \to \mathbb{C}, \quad \phi \mapsto \int_{[T]} \phi(t)dt$$

to central values of automorphic $L$-functions on $GL_2$. We state a very special (unramified everywhere) case.

**Theorem 18** (Waldspurger) Let $\pi$ be an automorphic cuspidal representation of $G$ that unramified everywhere. Let $\phi \in \pi^K$, where $K = \prod_{x \in |X|} G(\mathcal{O}_x)$ (so $\phi$ is unique up to scaling). Then up to some explicit constants we have an equality

$$\frac{|\mathcal{P}_T(\phi)|^2}{(\phi, \phi)} \sim L(\pi_{F'}, 1/2).$$

**Remark 34** Recall the construction of the $L$-function on $GL_2$ due to Hecke (and Jacquet-Langlands in modern language). Consider the diagonal torus $A \subseteq G$. Then for $\phi \in \pi$, we have up to local factors

$$\int_{[A]} \phi\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \chi(a)|a|^s da \sim L(\pi \otimes \chi, s + 1/2).$$

Taking $s = 0$ we know that Waldspurger's formula is equivalent to an equality of two different toric integrals

$$|\mathcal{P}_T(\phi)|^2 = \mathcal{P}_A(\phi) \cdot \overline{\mathcal{P}_A(\phi \otimes \eta_{F'/F})},$$

here $\eta_{F'/F}$ is the quadratic Hecke character on $[F^\times]$ associated to the quadratic extension $F'/F$.

Now we use the well known procedure of relative trace formula to remove the dependence on the automorphic representations $\pi$. Consider the distribution

$$\mathbb{J}(f)=\iint_{[A\times A]} K_f(a,b)\eta(b)da db,$$

where $f\in C_c^\infty(G(\mathbb{A})\mathbin{/\!/} K)$ and the kernel function is given by

$$K_f(x,y)=\sum_{\gamma\in G(F)} f(x^{-1}\gamma y).$$

The kernel function has a spectral decomposition

$$K_f(x,y)=\sum_{\phi} R(f)\phi(x)\overline{\phi(y)}+\text{non-cusp part},$$

where $\phi$ runs over an orthonormal basis of level one cusp forms on $G$. So we obtain the spectral decomposition

$$\mathbb{J}(f)=\sum_{\pi}\sum_{\substack{\phi\in\pi^K\\(\phi,\phi)=1}}\mathcal{P}_A(\pi(f)\phi)\cdot\overline{\mathcal{P}_A(\phi\otimes\eta_{F'/F})}=\sum_{\pi}\lambda_\pi(f)\mathcal{P}_A(\phi)\overline{\mathcal{P}_A(\phi\otimes\eta_{F'/F})},$$

where $\lambda_\pi:\mathcal{H}:=C_c^\infty(G(\mathbb{A})\mathbin{/\!/} K)\to\mathbb{C}$ is the character determined by $\pi$.

One can repeat the same story for the period on the anisotropic torus. Define

$$\mathbb{I}(f)=\iint_{[T\times T]} K_f(x_1,x_2)dx_1dx_2.$$

Then similarly we have a spectral decomposition

$$\mathbb{I}(f)=\sum_{\pi}\lambda_\pi(f)\mathcal{P}_T(\phi)\overline{\mathcal{P}_T(\phi)}.$$

By the previous remark, Waldspurger's certainly implies the relative trace formula identity

$$\mathbb{J}(f)=\mathbb{I}(f).$$

Conversely, using the linear independence of the automorphic representations, this identity is in fact also sufficient to prove Waldspurger's formula.

To prove this identity of two distributions, we use the geometric decomposition

$$\mathbb{J}(f)=\sum_{\gamma\in T(F)\backslash G(F)/T(F)}\iint_{T(\mathbb{A})\times T(\mathbb{A})} f(a^{-1}\gamma b)dadb.$$

Notice the generic stabilizer $T_\gamma$ is trivial and so the double integral is over $T(\mathbb{A})\times T(\mathbb{A})$ and factors as a product of local orbital integrals. One has a similar geometric decomposition for $\mathbb{I}(f)$.

We can parametrize the orbits $A\backslash G/A$ and $T\backslash G/T$ in a similar way. Consider the invariant map

$$\mathrm{inv}:G\to\mathbb{P}^1-\{1\},\quad \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\mapsto\frac{bc}{ad}.$$

Then $\mathrm{inv}^{-1}(u)$ consists of exactly one orbit when $u\in\mathbb{P}^1-\{0,1,\infty\}$. We call these $u$ *regular semisimple* and the corresponding orbital integral is automatically convergent (regularization process is needed for other $u$). Write

$$\mathbb{J}(f)=\sum_{u\in\mathbb{P}^1(F)-\{1\}}\mathbb{J}(u,f),\quad \mathbb{I}(f)=\sum_{u\in\mathbb{P}^1(F)-\{1\}}\mathbb{I}(u,f).$$

It remains to compare the orbital integrals

$$\mathbb{J}(u,f)=\mathbb{I}(u,f).$$

**Remark 35** Since we are only concerned with spherical Hecke algebra, this identity of orbital integral is nothing but the *fundamental lemma* in this setting. One can prove this fundamental lemma by explicit calculation on both sides (which is what Jacquet did). We are going to prove this fundamental lemma without doing explicit calculation (at least when $u$ is regular semisimple), using perverse continuation.

**Remark 36** For $D=\sum m_x\cdot x$ an effective divisor on $X$, let $h_D=\otimes_x h_{m_x,x}\in\mathcal{H}$, where $h_{m,x}=\mathbf{1}_{GL_2(\mathcal{O}_x)_{\mathrm{val}\det=m}}$. As a consequence of Cantan decomposition, we have $C_c^\infty(G(F_x)\mathbin{/\!/} G(\mathcal{O}_x))=\mathbb{C}[T_x]$, where $T_x=\mathbf{1}_{G(\mathcal{O})_x\left(\begin{smallmatrix}\varpi&\\&1\end{smallmatrix}\right)G(\mathcal{O})_x}$. So $\{h_D\}$ forms a basis of the spherical Hecke algebra $\mathcal{H}$.

## Geometrization for the split torus

Let us ignore the quadratic character $\eta_{F'/F}$ for the moment. So

$$\mathbb{J}(u,f)=\iint_{A(\mathbb{A})\times A(\mathbb{A})} f(a^{-1}\gamma(u)b)dadb,$$

where $\gamma(u)=\left(\begin{smallmatrix}1&u\\1&1\end{smallmatrix}\right)$. It is now convenient to lift the situation to $GL_2$ and consider for $f=h_D$,

$$\mathbb{J}(u,h_D)=\iint_{\tilde{A}\times\tilde{A}/\mathbb{G}_m(\mathbb{A})} h_D(a^{-1}\gamma(u)b)dadb,$$

where $\tilde{A}$ is the diagonal torus in $GL_2$.

In order to geometrize this orbital integral, we define an analogue of Hitchin moduli space.

**Definition 20** Let $\mathbf{d}=(d_{ij})_{1\le i,j\le 2}$ such that $d_{ij}\ge 0$ and $d_{11}+d_{22}=d_{12}+d_{21}=:d$. Define the moduli space of pairs of rank two vector bundles together with a morphism:

$$\mathcal{N}_{\mathbf{d}}(S)=\{(K_1\oplus K_2, K_1'\oplus K_2', \phi_{ij}:K_i\to K_j'):\deg K_j'-\deg K_i=d_{ij}\}/\operatorname{Pic}_X,$$

where $K_i, K_j'$ are line bundles on $X\times S$. For simplicity (since we only consider *regular semisimple* orbits) we also impose the non degeneracy condition that $\phi_{ij}\ne 0\in H^0(K_j'\otimes K_i^\vee)$ (which strictly speaking defines an open subset of $\mathcal{N}_{\mathbf{d}}$). Let $\mathcal{N}_d$ be the union of all such $\mathcal{N}_{\mathbf{d}}$'s with $d_{11}+d_{22}=d$.

Now we define an analogue of the invariant map to the Hitchin base and an analogue of Hitchin fibers.

**Definition 21** Let $\hat{X}_d$ be the the moduli space of pairs $(\mathcal{L},s)$, where $\mathcal{L}\in\operatorname{Pic}_X^d$, $s\in H^0(\mathcal{L})$. Let $\mathcal{A}_d$ be the moduli space of triples $(\mathcal{L},\alpha,\beta)$, where $\alpha,\beta\in H^0(\mathcal{L})$. We have a natural map $\mathcal{A}_d\to\hat{X}_d$ given by $(\mathcal{L},\alpha,\beta)\mapsto(\mathcal{L},\alpha-\beta)$.

We have an invariant map $\mathcal{N}_d\to\mathcal{A}_d$, given by $\mathcal{L}=K_1'\otimes K_2'\otimes K_1^\vee\otimes K_2^\vee$, $\alpha=\phi_{11}\otimes\phi_{22}$ and $\beta=\phi_{12}\otimes\phi_{21}$. Let $\mathcal{N}_{d,s}$ be the fiber of this invariant map above $s\in\mathcal{A}_d(k)$.

**Theorem 19** Let $D\in X_d(k)$ be an effective divisor on $X$ of degree $d$. Let $\mathcal{A}_D(k)\cong H^0(\mathcal{O}(D))$ (viewed as a $k$-subspace of $F$) be the fiber of $\mathcal{A}_d\to\hat{X}_d$ above $D$. Then

$$\mathbb{J}(h_D)=\sum_{s\in\mathcal{A}_D(k)}\#\mathcal{N}_{d,s}(k).$$

**Remark 37** Let $\mathrm{Hk}_d$ be the *Hecke stack*, i.e., the moduli stack of morphism of vector bundles $\phi:\mathcal{E}\to\mathcal{E}'$ such that $\deg\operatorname{div}(\det\phi)=d$. Then $\mathcal{N}_d$ is closely related to the fiber product

$$\begin{array}{ccc}\mathcal{N}_d^* & \longrightarrow & \mathrm{Hk}_d \\ \downarrow & & \downarrow \\ \mathrm{Bun}_{\tilde{A}}\times\mathrm{Bun}_{\tilde{A}} & \longrightarrow & \mathrm{Bun}_G\times\mathrm{Bun}_G\end{array}.$$

Notice $\tilde{A}\cong GL_1\times GL_1$ and $\mathrm{Bun}_{\tilde{A}}$ is the moduli of pairs of line bundles on $X$. Therefore the distribution $\mathbb{J}(h_D)$ can be thought of as an intersection number.

Now sending a point in $\mathcal{N}_{\mathbf{d}}$ to $(K_j'\otimes K_i^\vee,\phi_{ij})$ defines a map

$$\mathcal{N}_{\mathbf{d}}\to(\hat{X}_{d_{11}}\times\hat{X}_{d_{22}})\times_{\operatorname{Pic}_X^d}(\hat{X}_{d_{12}}\times\hat{X}_{d_{21}}).$$

By the non-degeneracy assumption on $\phi_{ij}$, this induces an isomorphism

$$\mathcal{N}_{\mathbf{d}}\hookrightarrow(X_{d_{11}}\times X_{d_{22}})\times_{\operatorname{Pic}_X^d}(X_{d_{12}}\times X_{d_{21}}).$$

Now let $\mathcal{A}_d$ the moduli space of triples $(\mathcal{L},\alpha,\beta)$ such that $\alpha\ne 0,\beta\ne 0$. The we have a commutative diagram

$$\begin{array}{ccc}\mathcal{N}_{\mathbf{d}} & \xrightarrow{\cong} & (X_{d_{11}}\times X_{d_{22}})\times_{\operatorname{Pic}_X^d}(X_{d_{12}}\times X_{d_{21}}) \\ \downarrow & & \downarrow \\ \mathcal{A}_d & \xrightarrow{\cong} & X_d\times_{\operatorname{Pic}_X^d}X_d\end{array}.$$

Here the right vertical map is induced by the addition map $X^a\times X^b\to X^{a+b}$. In this ways the analogue of Hitchin moduli space $\mathcal{N}_{\mathbf{d}}$ becomes a simple construction using symmetric powers of the curve $X$.

**Remark 38** By Riemann-Roch, when $d\ge\deg\omega_X+1$, we know the dimension of $X_d\times_{\operatorname{Pic}_X^d}X_d$ is $2d-(g-1)$. Moreover, in this case $\mathcal{N}_{\mathbf{d}}$ is *smooth* over $k$ and $\pi:\mathcal{N}_{\mathbf{d}}\to\mathcal{A}_d$ is a *finite* map. So we are in the simplest situation to use the machinery of perverse sheaves (Remark 16).

By the previous theorem, we would like to study

$$\sum_{s\in\mathcal{A}_D(k)}\#\mathcal{N}_{d,s}(k)=\sum_s\operatorname{Tr}(\operatorname{Frob}_s(R\pi_*\mathbb{Q}_\ell)_s).$$

Therefore we can forget about the orbital integrals and focus on the sheaf $R\pi_*\mathbb{Q}_\ell$. At this stage one can also insert the character $\eta_{F'/F}$ by taking a nontrivial local system

$$\mathbb{L} = \mathbb{L}_{d_{11}} \boxtimes \mathbb{Q}_\ell \boxtimes \mathbb{L}_{d_{12}} \boxtimes \mathbb{Q}_\ell$$

on $(X_{d_{11}} \times X_{d_{22}}) \times_{\mathrm{Pic}^d_X} (X_{d_{12}} \times X_{d_{21}})$ and then take $R\pi_*$. Here $\mathbb{L}_d = (v_{d,*}\mathbb{L}_\eta^{\boxtimes d})^{S_d}$, $\mathbb{L}_\eta$ is the local system on $X$ associated the the double cover $X' \to X$ and $v_d : X^d \to X_d$ is the natural quotient map by $S_d$.

Now it it remains to study the simpler object:

$$R\mathrm{add}_*(\mathbb{L}_a \boxtimes \mathbb{Q}_\ell)$$

where $\mathrm{add} : X_a \times X_b \to X_{a+b}$ is the addition map. This is nothing but the push-forward of a local system under a finite map, a simplest example of a perverse sheaf (after shifting by the dimension).

- Since $(X^d)^\circ \to X_d^\circ$ (the multiplicity free locus) is a Galois covering with Galois group $S_d$, we know that $R\mathrm{add}_*(\mathbb{Q}_\ell)$ is the middle extension $j_{!*}(\mathbb{L}_{a,b})$ (by the perverse continuation principle). Here $j : X^\circ_{a+b} \hookrightarrow X_{a+b}$ and the local system $\mathbb{L}_{a,b}$ on $X^\circ_{a+b}$ corresponds to the induced representation $\mathrm{Ind}_{S_a\times S_b}^{S_{a+b}} \mathbf{1}$ (of dimension $(a+b)!/a!b!$).
- To deal with the nontrivial coefficient, we need to go to the double covering to trivialize the local system. So we have a Galois covering $(X')^d \to X_d$ which is Galois with Galois group $\Gamma_d = (\mathbb{Z}/2)^d \rtimes S_d$. Here $S_d$ permutes $(\mathbb{Z}/2)^d$ in a natural way, in other words, $\Gamma_d$ is the wreath product $\mathbb{Z}/2 \wr S_d$. Let $\eta_{a,b} : (\mathbb{Z}/2)^{a+b} \to \{\pm 1\}$ be the character that is nontrivial on the first $a$ factors and trivial on the last $d$ factors. The action of $S_{a+b}$ on $\eta_{a,b}$ has stabilizer exactly $S_a \times S_b$. Hence we can extend $\eta_{a,b}$ to $\Gamma_{a,b} = \Gamma_a \times \Gamma_b \to \{\pm 1\}$. Then the local system on $X^\circ_{a+b}$ corresponds to the representation $\rho_{a,b} = \mathrm{Ind}_{\Gamma_{a,b}}^{\Gamma_{a+b}} \eta_{a+b}$. It is irreducible of dimension $(a+b)!/a!b!$ (one check the irreducibility by computing the endormophism algebra to be a division algebra).

**Remark 39** The group $\Gamma_d$ is also known as the *Hyperoctahedral group* (symmetry group of a hypercube), which is also the Weyl group of type BC.

## Geometrization for the nonsplit torus

Today we will geometrize the distribution $\mathbb{I}(h_D)$ on the nonsplit torus as well and verify the identity $\mathbb{J}(h_D) = \mathbb{I}(h_D)$ for $d = \deg D \gg 0$ (at least for the regular semisimple orbits).

**Remark 40** By a density argument, to prove the identity on the spectral side (equivalent to Waldspurger's theorem), it suffices to verify $\mathbb{J}(h_D) = \mathbb{I}(h_D)$ when $d = \deg D$ is *sufficiently large*. The reason is that the action of the Hecke algebra $\mathcal{H}$ on the space of automorphic forms has a large kernel and becomes finitely generated.

In order to geometrize the orbital integral $\mathbb{I}(u, h_D)$, we define an analogue of the space $\mathcal{N}_d$.

**Definition 22** Let $\nu : X' \to X$ be an etale double cover. Define $\mathcal{M}_d$ to be the moduli space $\{(K, K', \phi) : \deg K' - \deg K = d\}/\mathrm{Pic}_X$, where $K, K' \in \mathrm{Pic}_{X'}$ and the map $\phi$ is an element of

$$\mathrm{Hom}_{\mathcal{O}_X}(\nu_*K, \nu_*K') = \mathrm{Hom}_{\mathcal{O}_{X'}}(\nu^*\nu_*K, K') = \mathrm{Hom}_{\mathcal{O}_{X'}}(K, K') \oplus \mathrm{Hom}_{\mathcal{O}_{X'}}(\sigma^*K, K'),$$

where $\sigma$ is the nontrivial Galois involution. Since we only consider regular semisimple orbits, we further impose the non-degeneracy condition $\phi = (\alpha, \beta)$ where $\alpha \neq 0, \beta \neq 0$.

Now sending a point in $\mathcal{M}_d$ to $(K' \otimes K^\vee, \alpha)$ and $(K' \otimes (\sigma^*K)^\vee, \beta)$, we obtain an isomorphism (by an analogue of Hilbert 90)

$$\mathcal{M}_d \to X'_d \times_{\mathrm{Pic}^d_X} X'_d,$$

where the map $X'_d \to \mathrm{Pic}^d_X$ is induced by the norm map

$$\mathrm{Pic}^d_{X'} \xrightarrow{\mathrm{Nm}} \mathrm{Pic}^d_X, \quad \mathcal{L} \mapsto \det(\nu_*\mathcal{L}).$$

We also have the invariant map induced by the norm map $X'_d \to X_d$:

$$\begin{tikzcd} \mathcal{M}_d \arrow[r, "\cong"] \arrow[d] & X'_d \times_{\mathrm{Pic}^d_X} X'_d \arrow[d] \\ \mathcal{A}_d \arrow[r, "\cong"] & X_d \times_{\mathrm{Pic}^d_X} X_d \end{tikzcd}.$$

**Remark 41** By taking the split double cover $X'' = X \coprod X$, we recover the space $\mathcal{N}_d$ defined last time. Since

$$X''_d = \coprod_{d=a+b} X_a \times X_b,$$

the isomorphism $\mathcal{N}_d = X''_d \times_{\mathrm{Pic}^d_X} X''_d$ recovers the decomposition $\mathcal{N}_d = \coprod \mathcal{N}_{\mathbf{d}}$. In this case one can view $M_2(F) = F' \oplus (F')^{\perp}$ and the invariant map is simply $\mathrm{Nm}(\alpha, \beta) = \mathrm{Nm}(\alpha)/\mathrm{Nm}(\beta) = bc/ad$.

**Remark 42** The non-split torus is easier for analysis since the adelic quotient $T(F)\backslash T(\mathbb{A})$ is already compact. Geometrically this means there is no need to do compactification for the moduli spaces in question.

Analogous to Theorem 19, we have

**Theorem 20** Let $D \in X_d(k)$ be an effective divisor on $X$ of degree $d$. Let $\mathcal{A}_D \cong H^0(\mathcal{O}(D))$ (viewed as a $k$-subspace of $F$) be the fiber of $\mathcal{A}_d \to \hat{X}_d(k)$ above $D$. Then

$$\mathbb{I}(h_D) = \sum_{s \in \mathcal{A}_D(k)} \#\mathcal{M}_{d,s}(k).$$

Similarly to the split case, we are now interested in the sheaf $R\pi_{\mathcal{M},*}\mathbb{Q}_\ell$, where $\pi_{\mathcal{M}} : \mathcal{M}_d \to \mathcal{A}_d$. When $d \geq 2g_{X'} - 1$, $X'_d$ is smooth, $X'_d \to \mathrm{Pic}^d_{X'}$ is a projective bundle and the norm map $\mathrm{Pic}^d_{X'} \to \mathrm{Pic}^d_X$ is smooth with kernel a Prym variety of dimension $g - 1$. Therefore $\mathcal{M}_d$ is in fact smooth and hence $R\pi_{\mathcal{M},*}\mathbb{Q}_\ell$ is perverse (after shifting by the dimension). The local system underlying $R\pi_{\mathcal{M},*}\mathbb{Q}_\ell$ is the induced representation $\mathrm{Ind}_{S_d}^{\Gamma_d} \mathbf{1}$.

## Orbital integral identity for regular semisimple orbits

Now we have tow invariant maps with a common base

$$\begin{tikzcd} \mathcal{M}_d \arrow[dr, "\pi_{\mathcal{M}}"'] & & \mathcal{N}_d \arrow[dl, "\pi_{\mathcal{N}}"] \\ & \mathcal{A}_d & \end{tikzcd}$$

The identity $\mathbb{J}(h_D) = \mathbb{I}(h_D)$ now becomes a statement purely about two perverse sheaves.

**Theorem 21** There is an isomorphism between perverse sheaves

$$R\pi_{\mathcal{M},*}\mathbb{Q}_\ell = R\pi_{\mathcal{N},*}\mathbb{L}_d.$$

This will follow from the even stronger claim.

**Theorem 22** Let $\nu' : X'_d \to X_d$ and $\nu'' : X''_d \to X_d$. Then

$$R\nu'_*\mathbb{Q}_\ell = R\nu''_*\mathbb{L}_d.$$

By the perverse continuation principle, the proof of this theorem essentially boils down to representation theory of finite groups because the local system underlying both perverse sheaves have finite monodromy (trivialized after a finite covering). Namely,

**Theorem 23**

$$\mathrm{Ind}_{S_d}^{\Gamma_d} \mathbf{1} = \bigoplus_{a+b=d} \mathrm{Ind}_{\Gamma_{a,b}}^{\Gamma_d} \eta_{a,b}.$$

This is much simpler statement to prove! Notice that both sides have dimension $2^d$. By Frobenius reciprocity, it remains to show that there is a $\Gamma_{a,b}$-equivariant embedding

$$\eta_{a,b} \hookrightarrow \mathbb{Q}[S_d \backslash \Gamma_d],$$

which can be explicitly written down.

**Remark 43** For non-regular semisimple orbits, one needs to consider the larger moduli space where the non-degeneracy condition is removed. In this case the invariant map is no longer small and one needs to verify the support condition directly and use the full strength of Deligne's uniqueness principle. But in the end, it boils down to the same finite group theoretic identity as above.

## Orbital integral identity for non regular semisimple orbits

Now consider the case of non regular semisimple orbits, i.e., when the invariant $u \in \{0, \infty\}$. Let us only consider the case $u = 0$. The case $u = 0$ corresponds to three $A \times A$-orbits, the identity orbit and two unipotent orbits represented by $n_+ = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $n_- = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. The case $u=0$ corresponds to one $T\times T$-orbit: the identity orbit (i.e., $\mathrm{Nm}(\alpha)=0$) under the decomposition $(\alpha,\beta)\in M_2(F)=F'\oplus(F')^\perp$.

Now let us consider the moduli spaces for the non regular semisimple orbits. The moduli space for the nonsplit torus is again simpler. Let $\mathcal{M}_d$ be the space as in Definition 22 but only requiring that $\alpha,\beta$ are not zero simultaneously, i.e., $(\alpha,\beta)\neq(0,0)$. Our old nondegenerate moduli space is thus an open $\mathcal{M}_d^\circ\subseteq\mathcal{M}_d$. By definition we have

$$\mathcal{M}_d\cong\hat{X}'_d\times_{\mathrm{Pic}^d}\hat{X}'_d-Z',$$

where $Z'$ is the closed locus where $(\alpha,\beta)=(0,0)$. Since $\hat{X}'_d=X'_d\coprod\mathrm{Pic}^d_{X'}$, we have

$$\mathcal{M}_d-\mathcal{M}_d^\circ=\mathrm{Pic}^d_{X'}\times_{\mathrm{Pic}^d_X}X'_d\coprod X'_d\times_{\mathrm{Pic}^d_X}\mathrm{Pic}^d_{X'},$$

Now consider the invariant map

$$f_{\mathcal{M}_d}:\mathcal{M}_d\to\mathcal{A}_d=\hat{X}_d\times_{\mathrm{Pic}^d_X}\hat{X}_d-Z.$$

When $d$ is sufficiently large, $\mathcal{M}_d$ is smooth and $f$ is proper. By the same logic for the regular semisimple orbits, it remains to consider the norm map $\nu:\hat{X}'_d\to\hat{X}_d$ and check if $R\nu_!\mathbb{Q}_\ell$ is still perverse. Its restriction on $\hat{X}'_d-X'_d$ is given by the norm map $\mathrm{Pic}^d_{X'}\to\mathrm{Pic}^d_X$, whose fiber is certainly not finite (the kernel is the Prym variety of dimension $g-1$). But when $d$ is sufficiently large, this map is still *small*. In fact, the smallness in this case means $2(g-1)+(g-1)<\dim X_d=d$, i.e, $d>3(g-1)$. Now by the perverse continuation principle for small maps, $R\nu\mathbb{Q}_\ell$ still decomposes as IC sheaves associated to the earlier finite group representation $\mathrm{Ind}_{S_d}^{\Gamma_d}\mathbf{1}$.

**Remark 44** Suppose $f:X\to Y$ is a proper dominant map between varieties and $X$ is smooth. Then $Rf_!\mathbb{Q}_\ell[\dim X]$ is perverse if and only if $f$ is semismall. Moreover this is the IC sheaf associated to the local system over the locus where $f$ is etale (i.e., there is only one irreducible component in its perverse sheaf decomposition) if and only if $f$ is small. The reason is that the nontrivial cohomology of the constant sheaf is accounted for by the top dimension of the bad fibers. Notice this only applies to the constant sheaf. For example, consider the local system $\mathbb{L}$ associated to an etale double covering. Then $H^i(X,\mathbb{L})$ is 0 when $i=0,2$ and $2(g-1)$ when $i=1$ by Euler characteristic formula.

Now consider the moduli space for the split torus. The situation is slightly more complicated. In this case $\hat{X}_d=\coprod_{d=a+b}\hat{X}_a\times_{\mathrm{Pic}^d_X}\hat{X}_b$, which has infinitely many components (when $a<0$, $\hat{X}_d=\mathrm{Pic}^a_X$ since there is only the zero section for a line bundle of negative degree).

**Remark 45** This infiniteness corresponds is in fact natural from the point of view of orbital integrals. There is an integral (already familiar from Tate's thesis) appearing in the unipotent orbital integral:

$$\int_{\mathbb{A}^\times/\mathbb{O}^\times}\mathbf{1}_{\mathbb{O}}(x)\eta(x)|x|^s\,dx.$$

Though $\mathbb{A}^\times/\mathbb{O}^\times=\mathrm{Div}\,X$ is infinite, the Tate integral essentially a finite sum. This is due to the unusual phenomenon for function fields that the Fourier transform $\hat{\phi}$ of a compactly supported function $\phi$ is still *compactly supported*. So by the Poisson summation (ignoring the contribution of $\phi(0)$ and $\hat{\phi}(0)$), we have

$$\sum_{a\in F^\times}\phi(xa)=\sum_{a\in F^\times}\hat{\phi}(a/x).$$

It follows from Riemann-Roch that when $x$ has large enough degree both sides are identically zero!

Notice the the identity orbit gives no contribution to the orbital integral since we are inserting the nontrivial quadratic character $\eta$. So we require the four sections in $\mathcal{N}_d$ has at most one zero, which corresponds to the two unipotent orbits $n_+$ and $n_-$. We impose further assumptions that $\phi_{11}\neq0$ if $d_{11}\le d_{22}$; $\phi_{22}\neq0$ if $d_{11}>d_{22}$; $\phi_{12}\neq0$ if $d_{12}\le d_{21}$ and $\phi_{21}\neq0$ if $d_{12}>d_{21}$.

By these further assumptions if $\mathcal{N}_{\mathbf{d}}$ is nonempty, then $d_{ij}\ge0$. Again $\mathcal{N}_d$ is smooth when $d$ is sufficiently larger and $f_{\mathcal{N}_d}:\mathcal{N}_d\to\mathcal{A}_d$ is proper. For a point $(\alpha,0)$ in $\mathcal{A}_d-\mathcal{A}_d^\circ=0\times X_d\coprod X_d\times0$. Assume $d_{11}\le d_{22}$ (so $\phi_{11}\neq0,\phi_{22}=0$), the fiber at $\alpha=(0,D)$ is then

$$(X_{d_{11}}\times0)\times\mathrm{add}^{-1}_{d_{12},d_{21}}(D).$$

The second term is finite (since the addition map is finite). Therefore the fiber has dimension $d_{11}$. From this one can see that $f_{\mathcal{N}_d}$ is no longer small:

$$d_{11}+2d=\dim\mathcal{N}_{\mathbf{d},(0,D)}+2\dim(0\times X_d)>\dim\mathcal{A}_d=2d-(g-1).$$

Even though $f_{\mathcal{N}_d}$ is no longer small, we can check that the sheaf in question still satisfies the strict support condition in Deligne's uniqueness principle.

**Lemma 3** Let $\mathcal{E} := Rf_{\mathcal{N}_d,!}\mathbb{L}_{\mathbf{d}}[\dim \mathcal{A}_d]$. Then $\dim \operatorname{supp} \mathcal{H}^{-i}(\mathcal{E}) < i$.

**Proof** Notice $H^*(\mathcal{N}_{\mathbf{d},(0,D)}, \mathbb{L}_{\mathbf{d}}) = H^*(X_{d_{11}}, \mathbb{L}_{d_{11}}) \otimes V$, where $V$ from the finite part $\mathrm{add}^{-1}(D)$. The claim follows from the fact that

$$H^*(X_d, \mathbb{L}_d) = (H^*(X, \mathbb{L})^{\otimes d})^{S_d}.$$

By Remark 44, the right hand side has only one possibly nonzero term $(\wedge^d H^1(X, \mathbb{L}))[-d]$, which becomes zero when $d > 2(g-1)$. □

Hence the orbital integral identity for non regular semisimple orbits follows by same finite group representations identity (Theorem 23)!

## Moduli spaces of shtukas

In the final part of the course, we are going to generalize the previous trace formula identity to *higher* derivatives. For this we need to introduce the moduli space of shtukas. We begin with a rather general construction.

**Definition 23** Consider $\mathcal{M}$ defined over a finite field $k = \mathbb{F}_q$ (usually a certain moduli space). Suppose $C \to \mathcal{M} \times \mathcal{M}$ is a correspondence. We define the the moduli space of shutaks associated to $C$ to be the fiber product

$$\begin{tikzcd} \mathrm{Sht}_C \arrow[r] \arrow[d] & C \arrow[d] \\ \mathcal{M} \arrow[r, "(\mathrm{Id}, \mathrm{Frob}_q)"] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

More generally, suppose there are $r$ correspondences $C_i \to \mathcal{M} \times \mathcal{M}$, we define $\mathrm{Sht}_{\{C_i\}}$ to be the moduli space of shtukas associated to the convolution correspondence

$$\begin{tikzcd}[column sep=small] & C_1 \arrow[dl] \arrow[dr] & & C_2 \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & C_r \arrow[dl] \arrow[dr] & \\ \mathcal{M} & & \mathcal{M} & & \mathcal{M} & \cdots & \mathcal{M} & & \mathcal{M} \end{tikzcd}$$

**Definition 24** Consider $G = GL_n$. For $\mathcal{M} = \mathrm{Bun}_G$, we define the Hecke stack $\mathrm{Hk}_D \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ to be the moduli space of arrows $\phi : \mathcal{E} \to \mathcal{E}'$ of vector bundles of rank $n$ such that $\operatorname{div} \det \phi = D$. Similarly define $\mathrm{Hk}_d$ using the condition that $\deg \det \phi = d$.

**Definition 25** Define $\mathrm{Hk}^+$ to be the Hecke stack of *upper (increasing) modifications*, i.e., $\phi : \mathcal{E} \to \mathcal{E}'$ (over $X \times S$) is an injection such that $\operatorname{coker} \phi$ is a line bundle on the graph of a marked point $S \to X$. Similarly define $\mathrm{Hk}^-$ to be the Hecke stack of *lower (decreasing) modifications*.

We have two natural projections $\mathrm{Hk}_d \to \mathrm{Bun}_G \times \mathrm{Bun}_G \xrightarrow{\overleftarrow{p}, \overrightarrow{p}} \mathrm{Bun}_G$ and also a natural map $\mathrm{Hk}^{\pm} \to X$ given by the location of modification.

**Lemma 4** Both $\overrightarrow{p}, \overleftarrow{p} : \mathrm{Hk}_d \to \mathrm{Bun}_G$ are representable and proper. When $d = 1$, both have relative dimension $(n-1) + 1 = n$ (Here 1 comes from the choice the location of modification and $n-1$ comes from the choice of the modification with at a fixed location, i.e., a line in an $n$-dimensional vector space).

**Definition 26** Let $r$ be an even integer. Let $\mu = (\mu_i)$ be a $r$-tuple of signs. Let $\mathrm{Sht}_G^\mu$ be the moduli space of shtukas associated to the convolution of $\{\mathrm{Hk}^{\mu_i}\}$. In other words, an $S$-point of $\mathrm{Sht}_G^\mu$ corresponds to a $r$-tuple of modification of vector bundles $\mathcal{E}_0 \to \mathcal{E}_1 \cdots \to \mathcal{E}_r$ such that $\mathcal{E}_r \cong (\mathrm{Id} \otimes \mathrm{Frob}_S)^* \mathcal{E}_0$.

**Remark 46** Notice $\mathrm{Sht}_G^\mu$ is empty unless $\mu$ has the same number of plus and minus signs, since the degree is preserved under Frobenius.

Recall that (Remark 28) $\mathrm{Bun}_G$ itself is only an Artin stack (which has a lot of automorphism). The moduli of shtukas has better properties.

**Theorem 24** (Drinfeld $r = 2$, Varshavsky in general)

a. $\mathrm{Sht}_G^\mu$ is a Deligne-Mumford stack, locally of finite type.

b. The projection map $\pi_G : \mathrm{Sht}_G^\mu \to X^r$ is separated, smooth of relative dimension $(n-1)r$ (in fact, an $r$-iterated $\mathbb{P}^{n-1}$-bundle).

**Remark 47** To show the smoothness of $\pi_G$, one uses the fact that the relative cotangent bundle of $\mathrm{Hk}_G^\mu \to \mathrm{Bun}_G$ (to the first factor) restricts to the absolute cotangent bundle of $\mathrm{Sht}_G^\mu$ (using that the differential of the Frobenius is zero).

**Example 17** When $r=0$, the Hecke stack simply consists of isomorphisms $\mathcal{E} \xrightarrow{\cong} \mathcal{E}'$. So $\mathrm{Sht}_G$ consists of vector bundles on $X \times S$ such that $\mathcal{E} \cong (\mathrm{Id} \times \mathrm{Frob}_S)^* \mathcal{E}1$, which must come from pullback of vector bundles on $X$ itself. Hence $\mathrm{Sht}_G$ is the discrete group $\mathrm{Bun}_G(k) = G(F) \backslash G(\mathbb{A}) / \prod G(\mathcal{O}_x)$. This exactly puts us in the earlier situation of Waldspurger's formula when $G=GL_2$. From this point of view, the study of automorphic forms (over function fields) is nothing but the study of degree 0 cohomology of the moduli of shtukas with $r=0$ marked points.

**Example 18** When $n=1$, we have $\mathrm{Hk}^\mu = \mathrm{Pic}_X \times X^r$ given by the first line bundle $\mathcal{L}_0$ and the location of modification $\{x_i\}$. So we have the fiber diagram

$$\begin{tikzcd} \mathrm{Sht}^\mu \arrow[r] \arrow[d] & X^r \arrow[d] \\ \mathrm{Pic}\, X \arrow[r, "\mathrm{Id} - \mathrm{Frob}"] & \mathrm{Pic}_X^0 \end{tikzcd}$$

Here the right vertical arrow is given by $(x_i) \mapsto \sum \mu_i x_i$. In particular, considering the degree zero part (and rotating the previous diagram) we obtain the fiber diagram

$$\begin{tikzcd} \mathrm{Sht}^{\mu,0} \arrow[r] \arrow[d] & \mathrm{Pic}_X^0 \arrow[d] \\ X^r \arrow[r] & \mathrm{Pic}_X^0 \end{tikzcd}$$

The right vertical arrow is exactly Lang's isogeny, whose kernel is the class group $\mathrm{Pic}_X^0(k)$ of the function field $k(X)$. This is a generalization of unramified geometric class field theory: when $r=1$, the etale map $\mathrm{Sht}^{\mu,d} \to X$ has Galois group the class group $\mathrm{Pic}_X^0(k)$ and realizes the Hilbert class field of $k(X)$ geometrically.

Next time we will introduce the Hecke algebra action on the moduli of shtukas and see how the equality of *higher* derivatives of $L$-functions and intersection numbers of certain cycles on the moduli of shtukas becomes a refined structure on the perverse sheaves we constructed using Hitchin moduli spaces.

## Heegner-Drinfeld cycles and higher derivatives

Let $G=GL_2$, $T=\mathrm{Res}_{X'/X}\mathbb{G}_m$ with an embedding $\theta: T \to G$. Fix a $r$-tuple of signs $\mu$. We have an induced morphism

$$\theta_*: \mathrm{Bun}_T \to \mathrm{Bun}_G, \quad \mathcal{L} \mapsto \nu_* \mathcal{L},$$

where $\nu: X' \to X$ is the etale double cover. This induces a map of moduli of shtukas $\mathrm{Sht}_T \to \mathrm{Sht}_G$ and we have commutative diagram

$$\begin{tikzcd} \mathrm{Sht}_T \arrow[r] \arrow[d] & \mathrm{Sht}_G \arrow[d] \\ (X')^r \arrow[r] & X^r \end{tikzcd}$$

Notice the right vertical arrow has relative dimension $r$, whereas left right vertical arrow has relative dimension 0 (generically etale with Galois group the class group). Though $\mathrm{Sht}_G$ is not of finite type due to the instability, we can still talk about intersection number

$$(\mathrm{Sht}_T, \mathrm{Sht}_T)_{\mathrm{Sht}_G}$$

since $\mathrm{Sht}_T$ is a *proper* smooth Deligne-Mumford stack (at least after dividing $T$ by $\mathbb{G}_m$).

Now let us define Hecke correspondence on $\mathrm{Sht}_G^\mu$.

**Definition 27** Let $\mathcal{E}_0 \to \mathcal{E}_1 \to \cdots \mathcal{E}_r \cong (\mathrm{Id} \times \mathrm{Frob})^* \mathcal{E}_0$ and $\mathcal{E}'_0 \to \mathcal{E}'_1 \to \cdots \mathcal{E}'_r \cong (\mathrm{Id} \times \mathrm{Frob})^* \mathcal{E}'_0$ be two points in $\mathrm{Sht}_G^\mu$. We define a degree $d$ Hecke correspondence to be the collection of injections $\mathcal{E}_i \to \mathcal{E}'_i$ such that $\deg \mathcal{E}'_i - \deg \mathcal{E}_i = d$ and the natural diagram

$$\begin{tikzcd} \mathcal{E}_0 \arrow[r] \arrow[d] & \mathcal{E}_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{E}_r \arrow[d] \\ \mathcal{E}'_0 \arrow[r] & \mathcal{E}'_1 \arrow[r] & \cdots \arrow[r] & \mathcal{E}'_r \end{tikzcd}$$

commutes. The stack of such degree $d$ Hecke correspondences on $\mathrm{Sht}_G^\mu$ is denoted by $\mathrm{Hk}_d^\mu$.

**Remark 48** Remembering the first column and the last column gives a map

$$\mathrm{Hk}_d^\mu \to \mathrm{Hk}_d \times \mathrm{Hk}_d,$$

and remembering the two rows gives a map

$$\mathrm{Hk}_d^\mu \to \mathrm{Hk}^\mu \times \mathrm{Hk}^\mu .$$

So the Hecke correspondence $\mathrm{Hk}_d^\mu$ is a hybrid version of $\mathrm{Hk}_d$ and $\mathrm{Hk}^\mu$.

**Definition 28** We define a Hecke correspondence version of $\mathrm{Sht}_G^\mu$ by taking the fiber product

$$\begin{tikzcd} \mathrm{Sht}(h_d) \arrow[r] \arrow[d] & \mathrm{Hk}_d^\mu \arrow[d] \\ \mathrm{Hk}_d \arrow[r, "(\mathrm{Id},\mathrm{Frob})"] & \mathrm{Hk}_d \times \mathrm{Hk}_d \end{tikzcd}$$

Then $\mathrm{Sht}(h_d)$ is indeed a correspondence on $\mathrm{Sht}_G$ and thus defines a compactly supported cycle class of dimension $2r$

$$\mathrm{Sht}(h_d) \in \mathrm{Ch}_{c,2r}(\mathrm{Sht}_G \times \mathrm{Sht}_G).$$

In particular, $\mathrm{Sht}(h_D)$ acts on $\mathrm{Ch}_{c,r}(\mathrm{Sht}_G)$. One can similarly define a more refined correspondence $\mathrm{Sht}(h_D)$ for any effective divisor $D$. Recall the spherical Hecke algebra $\mathcal{H}$ is generated by $h_D$, where $D$ runs over all effective divisors.

**Theorem 25** The map

$$h : \mathcal{H} \to \mathrm{Ch}_{c,2r}(\mathrm{Sht}_G \times \mathrm{Sht}_G), \quad h_D \mapsto \mathrm{Sht}(h_D)$$

is a ring homomorphism.

**Remark 49** This can be proved easily on the generic fiber using the method of Drinfeld (in his work on the global Langlands for $GL_2$). Extra work need to be done on the integral level (i.e., over the entire base $X^r$).

**Definition 29** The map $\theta : \mathrm{Sht}_T \to \mathrm{Sht}_G$ induces a map $\theta : \mathrm{Sht}_T \to \mathrm{Sht}'_G := \mathrm{Sht}_G \times_{X^r} (X')^r$ (like Heegner points are imaginary quadratic points of modular curves). We define the *Heegner-Drinfeld cycle* to be the direct image of $[\mathrm{Sht}_T^\mu]$ in $\mathrm{Ch}_{c,r}(\mathrm{Sht}'_G)$ under $\theta$.

**Theorem 26** Let $f \in \mathcal{H}$. Then

$$\mathbb{I}_r(f) = \mathbb{J}_r(f).$$

Here $\mathbb{I}_r(f)$ is the intersection number of the Heegner-Drinfeld cycles

$$(\mathrm{Sht}_T, h(f)\,\mathrm{Sht}_T)_{\mathrm{Sht}'_G}$$

and

$$\mathbb{J}_r(f) = \frac{d^r}{ds^r}\Big|_{s=0} \mathbb{J}(f,s), \quad \mathbb{J}(f,s) = \int_{[A\times A]} K_f(a,b)|ab|^s \eta(b)\,da\,db.$$

**Remark 50** We have already proved the case $r = 0$ (see Example 17).

The ride hand side essentially corresponds to $\sum_\pi L^{(r)}(\pi_{X'}, 1/2)$. After spectral decomposition it follows that the intersection number of the $\pi$-isotypic component of the Heegner-Drinfeld cycle $[\mathrm{Sht}_T^\mu]$ (turns out to be independent of the choice of $\mu$) is essentially the $r$-th derivatives $L^{(r)}(\pi_{X'}, 1/2)$ at the center. More precisely, even though that we don't yet know the action of $\mathcal{H}$ on the entire Chow group $\mathrm{Ch}_{c,r}(\mathrm{Sht}_G)_\mathbb{C}$ is automorphic, we can consider $\tilde V = \mathcal{H}[\mathrm{Sht}_T]$ the subspace of the Chow group generated by the Heegner-Drinfeld cycle. Let $V = \tilde V/(,)_{\tilde V}$ be its quotient by the kernel of the intersection pairing.

**Theorem 27** We have $V = \bigoplus_{\pi \text{ cusp}} V_\pi \bigoplus V_{\mathrm{Eis}}$.

It then makes sense to talk about the $\pi$-isotypic component and using the Theorem 26 one can show that

**Theorem 28** For $\pi$ an everywhere unramified cuspidal automorphic representation of $GL_2$, we have up to a simpler factor

$$([\mathrm{Sht}_T]_\pi, [\mathrm{Sht}_T]_\pi) \sim L^{(r)}(\pi_{X'}, 1/2).$$

## Orbital integral identity for higher derivatives

Our remaining goal is to prove that for $d$ sufficiently large, we have

$$\mathbb{I}_r(h_d) = \mathbb{J}_r(h_d).$$

Notice the intersection number in question is given by the degree of the 0-dimensional scheme (in the proper intersection case) of the fiber product

$$\begin{tikzcd}
([\mathrm{Sht}_T, \mathrm{Sht}(h_d)\,\mathrm{Sht}_T]) \arrow[r] \arrow[d] & \mathrm{Sht}(h_d) \arrow[d] \\
\mathrm{Sht}_T \times \mathrm{Sht}_T \arrow[r] & \mathrm{Sht}_G \times \mathrm{Sht}_G
\end{tikzcd}$$

The key observation is that this fiber product can be viewed in an alternative way involving the Hitchin moduli space $\mathcal{M}_d$. Look at the following $3 \times 3$ commutative diagram:

$$\begin{tikzcd}
(\mathrm{Hk}_T^\mu)^2 \arrow[r] \arrow[d] & (\mathrm{Hk}_G^\mu)^2 \arrow[d] & \mathrm{Hk}_d^\mu \arrow[l] \arrow[d] & \mathrm{Hk}_{\mathcal{M}_d}^\mu \arrow[d] \\
(\mathrm{Bun}_T \times \mathrm{Bun}_T)^2 \arrow[r] & (\mathrm{Bun}_G \times \mathrm{Bun}_G)^2 & \mathrm{Hk}_d \times \mathrm{Hk}_d \arrow[l] & \mathcal{M}_d \times \mathcal{M}_d \\
\mathrm{Bun}_T^2 \arrow[r] \arrow[u] & \mathrm{Bun}_G^2 \arrow[u] & \mathrm{Hk}_d \arrow[l] \arrow[u] & \mathcal{M}_d \arrow[u] \\
 & & & \\
(\mathrm{Sht}_T^\mu)^2 \arrow[r] & (\mathrm{Sht}_G^\mu)^2 & \mathrm{Sht}(h_d) \arrow[l] & \mathrm{Sht}_{\mathcal{M}_d}^\mu
\end{tikzcd}$$

Here all the vertical upward arrows are given by $(\mathrm{Id}, \mathrm{Frob})$). The bottom row shows the fiber products of the three columns and the right column shows the fiber product of the three rows. The intersection in question is the fiber product of the bottom row, which should also equal to the fiber product the right column! (Of course this needs extra work to check after defining the intersection number in the right way, like the change of order of integration). We denote this common fiber product by $\mathrm{Sht}_{\mathcal{M}_d}^\mu$, which is a Hitchin version of moduli of shutaks. One can further decompose $\mathrm{Hk}_{\mathcal{M}_d}^\mu$ into pieces, i.e., the convolution of $\mathrm{Hk}_{\mathcal{M}_d}^\pm$ (consisting of only *2 by 2* diagram).

We have the following general Lefschetz trace formula for computing the intersection of a correspondence $C \to \mathcal{M}_d \times \mathcal{M}_d$ with the graph of the Frobenius morphism.

**Theorem 29** (Lefschetz trace formula)
$$(C, \Gamma_{\mathrm{Frob}})_{\mathcal{M}_d \times \mathcal{M}_d} = \mathrm{Tr}([C] \circ \mathrm{Frob} : H_c^*(\mathcal{M}_d)).$$

Let $f_{\mathcal{M}_d} : \mathcal{M}_d \to \mathcal{A}_d$ be the invariant map to the Hitchin base. Notice a correspondence $C$ over $\mathcal{A}_d$ defines an endomorphism $Rf_{\mathcal{M}_d,!}\mathbb{Q}_\ell \to Rf_{\mathcal{M}_d,!}\mathbb{Q}_\ell$. One can refine the Lefschetz trace formula relative to $f_{\mathcal{M}_d}$ (take $C = \mathrm{Hk}_{\mathcal{M}_d}^r$):

**Theorem 30**
$$\mathbb{I}_r(h_d) = \sum_{a \in \mathcal{A}_d(k)} \mathrm{Tr}(\mathrm{Hk}_{\mathcal{M}_d}^r \circ \mathrm{Frob}_a : (Rf_{\mathcal{M}_d,!}\mathbb{Q}_\ell)_{\bar{a}})$$

**Remark 51** Notice $\mathrm{Hk}^\mu \cong \mathrm{Hk}^r$ is independent of $\mu$ since any upper modification can also be realized as a lower modification (this is special to our rank 2 situation).

This reduces the intersection number of Heegner-Drinfeld cycles to the study of the action of $\mathrm{Hk}_{\mathcal{M}_d}$ on the cohomology the Hitchin moduli spaces, which one can then compare to the $r$-th derivative of the orbital integral on the split torus!

**Theorem 31**
$$\mathbb{J}_r(h_D, s) = \sum_{a \in \mathcal{A}_D(k)} \sum_{\mathbf{d}} q^{(d_{12}-d_{21})s}\, \mathrm{Tr}(\mathrm{Frob}_a, (Rf_{\mathcal{N}_d,!}\mathbb{L}_{\mathbf{d}})_{\bar{a}}).$$
Therefore,
$$\mathbb{J}_r(h_D) = (\log q)^r \sum_{a \in \mathcal{A}_D(k)} \sum_{\mathbf{d}} (d_{12} - d_{21})^r\, \mathrm{Tr}(\mathrm{Frob}_a, (Rf_{\mathcal{N}_d,!}\mathbb{L}_{\mathbf{d}})_{\bar{a}}).$$

**Remark 52** Since $|\mathrm{diag}\{a_1, a_2\}| = |a_1/a_2| = q^{\deg a_1 - a_2}$. The extra factor should be $\deg K_1 - \deg K_2 + \deg K_1' - \deg K_2' = d_{12} - d_{21}$.

Recall that
$$Rf_{\mathcal{M}_d,!}\mathbb{Q}_\ell = Rf_{\mathcal{N}_d,!}\mathbb{L}_d = \bigoplus_{i,j=0}^{d} K_i \boxtimes K_j,$$

where $K_i$ is a perverse sheaf on $\hat{X}_d$ with generic rank $\binom{d}{i}$. Now the final key thing is that each such perverse sheaf $K_i \boxtimes K_j$ is an *Hecke eigensheaf* whose eigenvalue exactly matching up the extra factor $d_{12} - d_{21} = d - 2d_{21}$ in $\mathbb{J}_r(h_d)$.

**Theorem 32** $[\mathrm{Hk}_{\mathcal{M}_d}]$ acts on $K_i \boxtimes K_j$ by the constant $d - 2j$.

**Proof** Notice the Hecke stack $\mathrm{Hk}_{\mathcal{M}_d}$ consists of commutative diagrams of line bundles on $X'$

$$\begin{tikzcd} \mathcal{L}_0 \arrow[r, "f"] \arrow[d, "\alpha"] & \mathcal{L}_1 \arrow[d] \\ \mathcal{L}_0' \arrow[r] & \mathcal{L}_1' \end{tikzcd}, \quad \begin{tikzcd} \mathcal{L}_0 \arrow[r, "f"] \arrow[d, "\beta"] & \mathcal{L}_1 \arrow[d] \\ \sigma^*\mathcal{L}_0' \arrow[r] & \sigma^*\mathcal{L}_1' \end{tikzcd}$$

Here $f$ is an upper modification (at the same location $y$), the vertical arrows are morphisms of $\mathcal{O}_{X'}$-modules and $\sigma$ is the nontrivial automorphism of $X'/X$. Notice these two diagrams are uniquely determined by the data $(\mathcal{L}_0, \mathcal{L}_0', \alpha, \beta, y)$. Moreover, $y$ must satisfy that $y \in D := \mathrm{div}\,\beta$ in order to complete the right vertical arrow (and the latter necessarily has divisor $D - y + \sigma(y)$). Define the incidence stack

$$\begin{tikzcd} I_d' := \{(D, y) : y \in D\} \arrow[r] \arrow[d] & X_d' \arrow[d] \\ X_d' \arrow[r] & X_d \end{tikzcd}$$

Here the top map is given by $(D, y) \mapsto D$ and the left map given by $(D, y) \mapsto D - y + \sigma(y)$. It then follows that the $\mathrm{Hk}_{\mathcal{M}_d}$ is given by the base change the incidence stack along the projection map $X_d' \times_{\mathrm{Pic}_X^d} X_d \to X_d$. It remains to check that $[I_d']$ acts on $K_j$ by $d - 2j$. Because of the perversity, we only need to check this on the generic fiber.

Now let $D = x_1 + \cdots + x_d$. Let $y_i^{\pm}$ be the preimage of $x_i$ in $X'$. A basis of the $2^d$ dimensional space $\mathrm{Ind}_{S_d}^{\Gamma_d} \mathbf{1}$ consists of monomials of the form $\phi = \prod_{i=1}^d y_i^{\pm}$. Moreover, the elements

$$\prod_{i=1}^d (y_i^+ \pm y_i^-)$$

form the basis of the representation corresponding to $K_j$, where $j$ is the number of minus signs. Then

$$[I_d'](\prod_{i=1}^d y_i^{\varepsilon_i}) = \sum_{i=1}^d y_1^{\varepsilon_1} \cdots y_i^{-\varepsilon_i} \cdots y_d^{\varepsilon_d},$$

in particular $[I_d']$ behaves like a derivation. Now notice $[I_d']$ keeps $y_i^+ + y_i^-$ and negates $y_i^+ - y_i^-$. It then follows that $[I_d']$ acts on $K_j$ by the scalar $+(d-j) - j = d - 2j$ ! □