Elliptic Surfaces and Mordell-Weil Lattices

An elliptic surface is a surface that admits an elliptic fibration. Its generic fiber is an elliptic curve over a function field but special fibers may be singular. After giving the Kodaira-Neron classification of possible singular fibers and related geometric invariants, we build the neat connection between the geometry of elliptic surfaces and arithmetic of elliptic curves. This allows us to reprove the finite generation of the Mordell-Weil group of an elliptic curve and to further classify its possible rank and torsion using techniques from intersection theory and lattice theory. The theory of Mordell-Weil lattices plays an important role in finding elliptic curves over $\mathbb{Q}$ of high rank via specialization on elliptic surfaces. Our main sources are [1] and [2]. See also [3], [4] and [5]. This is a note prepared for the Baby Algebraic Geometry Seminar at Harvard.

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Elliptic Surfaces

Definition 1 Let k be an algebraically closed field and k be a smooth projective curve over k. An *elliptic surface* S over k is a smooth projective surface S with an elliptic fibration over k, i.e., a surjection $\pi: S \to C$ such that almost all fibers are smooth curves of genus 1.

Example 1 The trivial family $C \times E$ is an elliptic surface.

Example 2 The Weierstrass equation

$$y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t), \quad a_i(t) \in k[t]$$

defines an elliptic surface over $\mathbb{P}^1$ as long as it is smooth.

Example 3 By the classification of (minimal) algebraic surfaces, all surfaces of Kodaira dimension 1, all Enriques surfaces and all hyperelliptic surfaces are elliptic. Some K3 surfaces, abelian surfaces and ruled surfaces are elliptic.

By definition, the generic fiber E of an elliptic surface S is a smooth curve of genus 1 over the function field $k(C)$. Let $f: C \to S$ be a section of $\pi: S \to C$. Then $f(C) \cap E$ gives a rational point $P \in E(k(C))$. Conversely, let $P \in E(k(C))$. Let Γ be the closure of P in S. We obtain a surjective *birational* morphism $\pi|_\Gamma: \Gamma \to C$, which is an isomorphism since k is smooth. In this way we have exhibited a bijection

$$\{\text{sections of } \pi: S \to C\} \longleftrightarrow \{k(C)\text{-rational points of } E\}.$$

Remark 1 In the sequel, we make the convention that every elliptic surface has a zero section. So taking the generic fiber of $S \to C$ gives an elliptic curve E over $k(C)$. We also make the convention that every elliptic surface has a singular fiber. So E is a genuine elliptic curve over $k(C)$ and cannot be defined over k. In particular, this excludes trivial families of elliptic curves.

Given an elliptic curve E over $k(C)$, there are different ways to extend E to an elliptic surface S over k giving rise to the generic fiber. However, all these models are birational, so if we require that S is *relatively minimal*, i.e., the fibers do not contain (-1)-curves, then S is unique up to isomorphism (the uniqueness will follow from the classification of singular fibers, see [4, II.1.2]). We obtain the following correspondence:

$$\{\text{relatively minimal elliptic surfaces } S \text{ over } C\} \leftrightarrow \{\text{elliptic curves } E \text{ over } k(C)\}.$$

The explicit description of the relatively minimal model is given by Kodaira in characteristic 0 and by Neron in general. The elliptic surface thus associated to E is sometimes called the *Kodaira-Neron model* of E.

As already alluded, the theme of this talk is to relate the geometry of the elliptic surface S and the arithmetic of the elliptic curve E.

The Kodaira-Neron Model

How do the singular fibers of an elliptic surface look like? There are many ways to classify the possible singular fibers. Here we use the explicit equations.

Example 4 Let E be the elliptic curve over $k(C)$. We work locally and take a local parameter t. Assume $\mathrm{char}(k) \neq 2$ for simplicity, then E is given by the Weierstrass equation

$$y^2 = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t).$$

Assume we have a singular fiber at $t = 0$. Moving the singularity to $(0, 0)$, we know that $t \mid a_4$ and $t \mid a_6$. We change notation and write

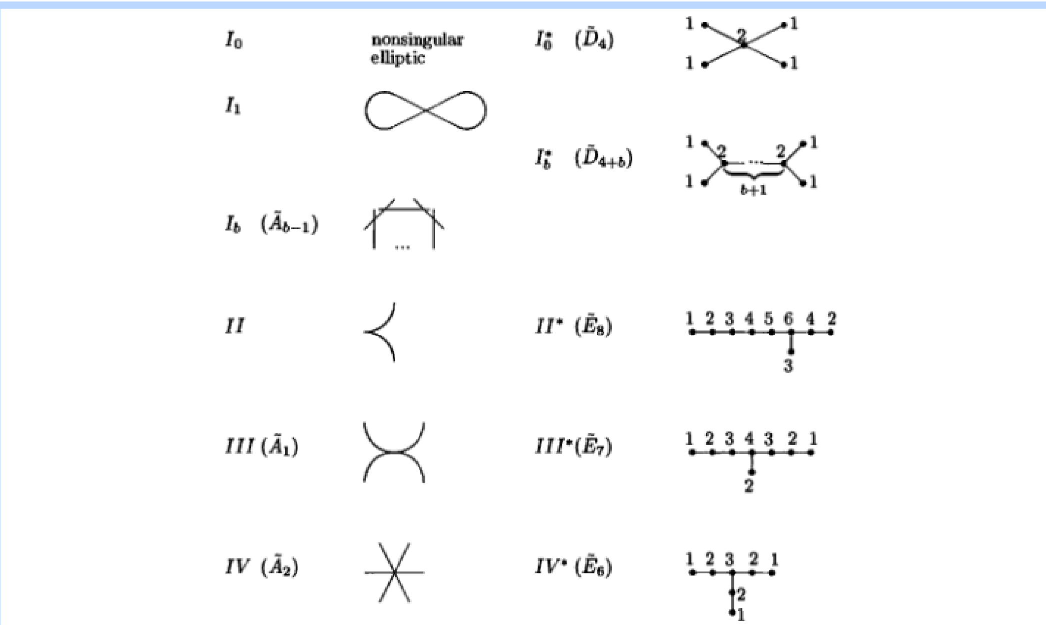
$$y^2 = x^3 + a_2(t)x^2 + ta_4(t) + ta_6(t).$$

So if this equation defines an elliptic surface, then we know the singular fiber is either a nodal cubic curve (if $t \mid a_2$) or a cusp cubic curve (if $t \nmid a_2$). These are called *multiplicative* and *additive reduction* (due to their group structures) and the Kodaira symbols are I_0 and II respectively.

Example 5 It is possible that the equation in the previous example does not define an elliptic surface, in other words, the surface is not smooth. This happens if and only if $t \mid a_6(t)$.

- Suppose $t \nmid a_2$. Then $t^2 \mid \Delta$. Let us consider the case $t^2 || \Delta$ for simplicity, then translating P gives
  $$y^2 = x^3 + a_2(t)x^2 + t^2a_4(t)x + t^2a_6(t),$$
  where $t \nmid a_6(t)$. Now we blow up $(0, 0)$ with $y = y't$, $x = x't$, to get
  $$y'^2 = tx'^3 + a_2(t)x'^2 + ta_4(t)x' + a_6(t).$$
  One can check that the surface singularity is resolved and the fiber at $t = 0$ is the strict transform of the nodal cubic curve (a rational curve) together with the rational exceptional divisor given by
  $$y'^2 = a_2(0)x'^2 + a_6(0).$$
  This singular fiber type is denoted by I_1. More generally, when higher powers of t divide Δ, we need to blow-up multiple times to resolve the surface singularity. The resulting singular type is denoted by I_n, an n-polygon of rational curves.
- Suppose $t \mid a_2$. Then the first blow-up has three possibilities:
  - a rational point meeting the strict transform of the cuspidal curve tangentially at 1 point (Type III);
  - two lines meeting the strict transform of the cuspidal curve at 1 point (Type IV);
  - a double line (Type $*$)

It may not be a good idea if I keep on blowing-up for 3 hours. Let me tell you the result instead. In fact, one can determine the singular type from the equation using the so-called *Tate's algorithm* over any perfect field, as demonstrated above.



Kodaira-Neron classification

Remark 2 For those number theorists, the Neron model (a smooth group scheme over a DVR satisfying the Neron mapping property) of an elliptic curve is obtained by simply taking the smooth part of the relatively minimal model. The group structure extends to each singular fiber F_v, which is either $\mathbb{G}_m \times G_v$ or $\mathbb{G}_a \times G_v$. Here G_v is the component group associated to a singular fiber F_v, which is a finite abelian group.

Example 6 One can see that the dual graphs of the singular fibers are exactly the extended Dynkin diagrams $\tilde{A}_n, \tilde{D}_n, \tilde{E}_n$. Removing the simple component meeting the zero section, we obtain exactly the Dynkin diagrams A_n, D_n, E_n. These are root systems with roots of the same length $\sqrt{2}$ and correspond to even positive definite lattices, with determinants $n+1, 4, 3, 2, 1$ respectively. Miraculously, the determinants are exactly equal to the number of simple components, i.e., the order of G_v. Notice that E_8 is *unimodular* which ends up to be important for us. In fact, a basic result from lattice theory is that E_8 is the *unique* unimodular even positive definite lattice with rank ≤ 15.

Neron-Severi groups and Mordell-Weil groups

Write $K = k(C)$ for short. A point $P \in E(K)$ determines a section of S. Denote the image curve by $\bar{P}$. A point $v \in C$ also gives us a divisor F_v, the fiber above $v \in C$. Every divisor on the elliptic surfaces S can be written as the sum of such *horizontal* and *vertical* divisors. Let $D \in \mathrm{Div}(S)$ be an divisor, then we can decompose $D = D_{\mathrm{hor}} + D_{\mathrm{ver}}$. The horizontal part D_{hor} intersects E at a divisor on E, which gives a point $P \in E(K)$ using the group law. So we have a map

$$\psi : \mathrm{Div}(S) \to E(K) = \mathrm{Pic}^0(E).$$

The kernel of ψ is

$$\ker \psi = \mathrm{Div}_{\mathrm{a}}(S) + \mathbb{Z}\bar{O} + \mathrm{Div}_{\mathrm{ver}}(S),$$

where $\mathrm{Div}_{\mathrm{a}}(S)$ is the group of divisors algebraically equivalent to 0. To see that $\mathrm{Div}_{\mathrm{a}}(S) \subseteq \ker \phi$, we use the fact ([4, VII.1.1]) that $\pi^* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(S)$ is an isomorphism to conclude that any divisor in $\mathrm{Div}_{\mathrm{a}}$ is linearly equivalent to a vertical divisor in $\mathrm{Div}_{\mathrm{ver}}$. So we obtained that

Theorem 1 Let $\mathrm{NS}(S) = \mathrm{Pic}(S)/\mathrm{Pic}^0(S)$ be the Neron-Severi group of S, i.e., the divisors modulo algebraic equivalence. Let $T \subseteq \mathrm{NS}(S)$ be the subgroup generated by $\bar{O}$ and vertical divisors. Then

$$E(K) \cong \mathrm{NS}(S)/T.$$

T is called the *trivial lattice* for the obvious reason.

This theorem relates the arithmetic of E and the geometry of S. It is well-known that the Neron-Severi group $\mathrm{NS}(S)$ is finitely generated for any smooth projective variety (the theorem of the base). Consequently, we have reproved the Mordell-Weil theorem using an argument of geometric nature.

Corollary 1 (Mordell-Weil) $E(K)$ is finitely generated.

Remark 3 The finite generation of the Neron-Severi group is proved via the Mordell-Weil theorem of abelian varieties over global fields. So the above argument is a bit circular. However, we can use intersection theory in our case to bypass it. There is a natural cycle map $\mathrm{Pic}(S) \to H^2(S)$. We have an intersection pairing on $\mathrm{Pic}(S)$ and a cup product pairing on $H^2(S)$. The cycle map preserves the pairings, hence its kernel is the group of divisors numerically equivalent to 0. Using Riemann-Roch, one can show that numerical equivalence and algebraic equivalence are the same on an elliptic surface, hence $\mathrm{NS}(S)$ itself embeds into the finite dimensional vector space $H^2(S)$, In particular. $\mathrm{NS}(S)$ is finitely generated and even torsion-free.

Our next goal is to further study the structure of $E(K)$ (e.g., its rank and torsion) using the geometry of S.

The number $\rho(S) = \mathrm{rank}\, \mathrm{NS}(S)$ is called the *Picard number* of S. By the Hodge index theorem, the lattice $\mathrm{NS}(S)$ equipped with the intersection pairing has signature $(1, \rho(S) - 1)$. We immediately find the following bound on the rank of $E(K)$:

Corollary 2 $\mathrm{rank}\, E(K) = \rho(S) - \mathrm{rank}\, T$.

As discussed above, $\rho(S)$ is bounded above by $b_2(S)$, the second Betti number (it is even bounded by $h^{1,1}$ if $k = \mathbb{C}$). So we need more knowledge about the trivial lattice T.

Definition 2 We introduce the following notation:

- F: a generic fiber
- F_v: the fiber above $v \in C$
- m_v: the number of components of F_v

- $F_{v,0}$: the identity component of F_v
- $F_{v,i}$: the i -th non-identity component of F_v , $i=1,\dots,m_v-1$
- $T_v=\bigoplus_{1\le i\le m_v-1}F_{v,i}$: the subgroup of T generated non-identity components of F_v
- $R=\{v\in C: F_v \text{ is reducible}\}$

Because all fibers are algebraically equivalent, we know that

$$T=\langle\bar{O},F\rangle+\sum_{v\in R}T_v.$$

Note that the intersection matrix of $\langle\bar{O},F\rangle$ is $\begin{bmatrix}\bar{O}.\bar{O} & 1\\ 1 & 0\end{bmatrix}$ which is non-degenerate with determinant -1 and signature $(1,1)$. Also, T_v is a root lattice of type A,D,E , hence the intersection matrix, denoted by A_v , is negative definite. Therefore, the above decomposition of T is actually a direct sum. It follows that

Proposition 1 $\operatorname{rank}T=2+\sum_{v\in R}(m_v-1)$.

Therefore we can really compute the rank of $E(K)$ as long as we know all the singular fibers. Conversely, knowing the possible rank of T will help us to classify configurations of singular fibers of elliptic surfaces.

Height Pairings

How about the torsion? The crucial idea is to endow $E(K)$ a *height pairing*. We already know that $E(K)\cong \mathrm{NS}(S)/T$ and $\mathrm{NS}(S)$ possesses an intersection pairing. So it is natural to construct a splitting of this isomorphism so that we can embed $E(K)$ into $\mathrm{NS}(S)$.

We need the following theorem due to Kodaira.

Theorem 2 (Canonical bundle formula) $K_S=(2g(C)-2+\chi)F$, $K_s.K_s=0$ and for any $P\in E(K)$, $\bar{P}.\bar{P}=-\chi$, where $\chi=\chi(\mathcal{O}_S)>0$.

Proof It may not be enlightening to tell you a long proof using spectral sequences. □

The following is not quite a ``splitting'', nonetheless is good enough for our purpose.

Proposition 2 There is a map $\phi: E(K)\to \mathrm{NS}(S)_{\mathbb{Q}}$ such that

$$\phi(P)\equiv\bar{P}\pmod{T(S)_{\mathbb{Q}}}.$$

Proof Explicitly,

$$\phi(P)=\bar{P}-\bar{O}-(\bar{P}.\bar{O}+\chi)F-\sum_{v\in R}(F_{v,1},\dots,F_{v,m_v-1})A_v^{-1}(\bar{P}.F_{v,1},\dots\bar{P}.F_{v,m_v-1})^t.$$

This is easily verified by taking the intersection pairing of $\phi(P)$ and the basis elements of the trivial lattice T together with the canonical bundle formula. The reason we need to tensor with $\mathbb{Q}$ is that the coefficients may not be integers. □

One can check that ϕ is also a group homomorphism. So we can define a pairing on $E(K)$ using the pairing on $\mathrm{NS}(S)$.

Definition 3 Define the *height pairing* on $E(K)$ by $\langle P,Q\rangle=-\phi(P).\phi(Q)$. We add a sign in order to make the height pairing positive definite.

The following is easily deduced.

Proposition 3 Let $P\in E(K)$. Then $\langle P,P\rangle=0$ if and only if $P\in\ker\phi=E_{\mathrm{tor}}$.

Now from the explicit formula for $\phi(P)$, we can also write down the height pairing explicitly,

$$\langle P,Q\rangle=\chi+\bar{P}.\bar{O}+\bar{Q}.\bar{O}-\bar{P}.\bar{Q}-\sum_{v\in R}c_v(P,Q),$$

where $c_v(P,Q)$ is a positive number only depending on the the fiber components of F_v meeting $\bar{P}$ and $\bar{Q}$.

We can apply the height pairing to deduce some information about torsion groups. Recall that G_v is the component group of the singular fiber F_v . The map $\psi: E(K)\to\prod_{v\in R}G_v$ sending a section to the simple fiber components it meets is a group homomorphism.

Theorem 3 When restricted to torsion points, $\psi : E_{\mathrm{tor}} \to \prod_{v \in R} G_v$ is injective.

Proof Suppose two torsion points P, Q have the same image. Then P, Q meets the same simple components, hence $c_v(P,Q) = c_v(P,P) = c_v(Q,Q)$. Because $\langle P,P\rangle = \langle Q,Q\rangle = \langle P,Q\rangle = 0$, from the above explicit formula we conclude that $\bar{P}.\bar{O} = \bar{Q}.\bar{O}$ and $\bar{P}.\bar{Q} = -\chi < 0$. Hence $P = Q$. □

In this way, the singular types of the elliptic surface S impose very strong constraints on the torsion group of $E(K)$, and vice versa.

Mordell-Weil Lattices

Definition 4 The height pairing induces $E(K)/E(K)_{\mathrm{tor}}$ a lattice structure. We call it the *Mordell-Weil lattice*.

How can we identify $E(K)/E(K)_{\mathrm{tor}}$ with a sublattice of $\mathrm{NS}(S)_{\mathbb{Q}}$? A bit of lattice-theoretic and intersection-theoretic computation gives the precise answer as long as $\mathrm{NS}(S)$ is unimodular.

Theorem 4 (Shioda) Let $M = T^{\perp}$ be the complementary lattice of T in $\mathrm{NS}(S)$ (called the *essential lattice*) and L be the opposite lattice of M. Then L is an even positive definite lattice. Suppose $\mathrm{NS}(S)$ is unimodular. Then $E(K)/E(K)_{\mathrm{tor}} \cong L$ as lattices.

We now step toward the case study of rational elliptic surfaces, where the lattice-theoretic method has achieved huge success in classifying all possible structures of $E(K)$.

Theorem 5 Suppose S is a rational elliptic surface. Then $\mathrm{NS}(S)$ is a unimodular lattice of rank $\rho(S) = 10$.

Proof Note that $\mathrm{NS}(\mathbb{P}^2) \cong \mathbb{Z}$ is unimodular. Since each blow-up changes the determinant of $\mathrm{NS}(S)$ by -1 due to the exceptional divisor, we know that $\mathrm{NS}(S)$ is also unimodular. Because $\rho = b_2$ for a rational surface, it suffices to check that $b_2 = 10$. This follows from the middle row (0,10,0) of the Hodge diamond. □

Here comes a clever way to classify $E(K)/E(K)_{\mathrm{tor}} \cong L$ relying on the fact there is only one *unimodular* even positive definite lattice of rank 8, namely the root lattice E_8. Consider the complementary lattice W of the rank 2 sublattice $\langle \bar{O}, F\rangle \subseteq \mathrm{NS}(S)$. Then W is unimodular, even and positive definite, so it must be E_8! Since $L \cong W/\bigoplus_{v\in R} T_v$ and each T_v is a root lattice. We only need to find all possible embeddings of a root lattice into E_8. If you know E_8, it is just so simple — there are only 74 cases. All the possible shapes of $E(K)$ are beautifully classified by Oguiso and Shioda [6].

Theorem 6 (Oguiso-Shioda) For rational elliptic surfaces, $E(K)$ is one of the following types:

$E(K)_{\mathrm{tor}}$	$r = \mathrm{rank}\, E(K)$
0	$0 \le r \le 8$
$\mathbb{Z}/2\mathbb{Z}$	$0 \le r \le 4$
$\mathbb{Z}/3\mathbb{Z}$	$0 \le r \le 2$
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$0 \le r \le 2$
$\mathbb{Z}/4\mathbb{Z}$	$0 \le r \le 1$
$\mathbb{Z}/5\mathbb{Z}$	0
$\mathbb{Z}/6\mathbb{Z}$	0
$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	0
$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$	0

Remark 4 However, the existence is a problem of different type. People found many examples using explicit equations.

Remark 5 Elliptic surfaces are also useful to find elliptic curves over $\mathbb{Q}$ of high rank via specialization. A theorem of Neron ensures that the ranks of infinitely many specializations do not go down. Indeed, all rank records so far are obtained this way (together with searching techniques). The maximal rank of the Mordell-Weil lattice of a rational elliptic surface is 8, as we have seen, and Neron used it to find infinitely many elliptic curves with rank at least 11. Elkies stepped further and used elliptic K3 surfaces with maximal Mordell-Weil rank 17 to find the current record curve with rank ≥ 28. This story probably is the theme of a different talk.

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