The Hodge-Tate period map and the cohomology of Shimura varieties

These are expanded notes prepared for a talk in a learning seminar on Caraiani-Scholze's paper *On the generic part of the cohomology of compact unitary Shimura varieties*, Spring 2016 at Columbia. We summarize the major ingredients of the proof, explain the preservation of perversity under the Hodge-Tate period map and deduce the main theorems: 1) the existence of Galois representations associated to the torsion classes in betti cohomology of certain compact unitary Shimura varieties; 2) after localized at a maximal ideal of the Hecke algebra satisfying a genericity assumption, the $\mathbb{Z}_\ell$ -cohomology is concentrated in the middle degree and torsion-free.

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Torsion Galois representations

Recall our set-up. Let $(B,*,V,\langle,\rangle)$ be a PEL datum of type A:

- B is a finite dimensional simple $\mathbb{Q}$ -algebra with center a CM field F .
- $*$ is positive involution of second kind on B (so $F^{*=1}=F^+$ is totally real).
- V is a B -module.
- $\langle,\rangle : V\times V\to\mathbb{Q}$ is an alternating form that is $*$ -Hermitian, i.e., $\langle bv,w\rangle=\langle v,b^*w\rangle$.

The PEL datum models the first cohomology of a polarized abelian variety of dimension $\frac{1}{2}\dim_{\mathbb{Q}}V$ with endomorphism by B . Let G be the group of the automorphisms of V (as a B -module) that preserves $\langle,\rangle$ up to a similitude factor. Let $S_K=\mathrm{Sh}_K(G,X)$ be the associated unitary Shimura variety, which is a moduli space of such abelian varieties with K -level structure.

Assume we are in one of the following two extremal cases:

- B is a division algebra and $V=B$ as a B -module. In this case G is an anisotropic unitary group (in $n=\dim_F V$ variables) and has no endoscopy. S_K is an example of Kottwitz's simple Shimura varieties (including those considered by Harris-Taylor).
- $B=F$, $V=F^n$ (equivalently $B=M_n(F)$, $V=B$). Assume $F^+\ne\mathbb{Q}$ to ensure S_K is compact. Assume that G is quasi-split at all finite places. In this case G has most endoscopy and causes most difficulty in the stable trace formula. Wei may like this case more (S_K appears in his arithmetic fundamental lemma).

Assume F , G , K are unramified outside a finite set of primes S (and $S\subseteq\mathrm{Spl}_{F/F^+}$ in the second case). Let $\mathbb{T}^S=\mathbb{Z}[G(\mathbb{A}_f^S)\mathbin{/\!\!/}K^S]$ be the unramified Hecke algebra. Let $\mathfrak{m}\subseteq\mathbb{T}^S$ be a maximal ideal. The first main result constructs an associated torsion Galois representation.

Theorem 1 Assume $H^i(S_K,\mathbb{F}_\ell)_{\mathfrak{m}}\ne0$ for some i . Thee there is a semisimple Galois representation $\rho_{\mathfrak{m}}:G_F\to GL_n(\overline{\mathbb{F}}_\ell)$ unramified outside $S\cup\{\ell\}$ associated to $\mathfrak{m}$.

The proof requires three major ingredients:

(1) Sug Woo Shin has constructed Galois representations attached to the system of (characteristic 0) Hecke eigenvalues appearing in the cohomology of the Shimura varieties and Igusa varieties (in the above two extremal cases), by stable trace formula. In particular, there exists a Galois representation associated to system of Hecke eigenvalues in $[H_c(\mathrm{Ig}^b,\overline{\mathbb{Q}_\ell})]$.

(2) The construction of the (Hecke equivalent) Hodge-Tate period map from the infinite level Shimura variety to the flag variety

$$\pi_{\mathrm{HT}} : \mathcal{S}_{K^p} \to \mathcal{F}\ell_{G,\mu}$$

(for any Shimura varieties of Hodge type), whose fibers are related to the Igusa varieties (for any Shimura varieties of PEL type). In particular,the main result of Chap. 4 shows that for any geometric point $\bar{x} \in \mathcal{F}\ell^b_{G,\mu} \subseteq \mathcal{F}\ell_{G,\mu}$, the fiber above $\bar{x}$,

$$(R\pi_{\mathrm{HT},*}\mathbb{F}_\ell)_{\bar{x}} \cong R\Gamma(\mathrm{Ig}^b, \mathbb{F}_\ell).$$

Remark 1 Recall that the locally closed subspaces $\mathcal{F}\ell^b_{G,\mu}$ of the adic space $\mathcal{F}\ell_{G,\mu}$ form a stratification and becomes the Newton stratification on the Shimura variety via pullback along π_{HT} . To define $\mathcal{F}\ell^b_{G,\mu}$, one identifies $\mathcal{F}\ell_{G,\mu}$ with the B^+_{dR} -affine Grassmannian $\mathrm{Gr}^{B^+_{\mathrm{dR}}}_{G,\mu}$ (using μ is minuscule) and notices that a $(C, \mathcal{O}_C)$ -point of $\mathcal{F}\ell_{G,\mu} \cong \mathrm{Gr}^{B^+_{\mathrm{dR}}}_{G,\mu}$ gives a G -bundle on the Fargues-Fontaine curve by modifying the trivial bundle along ∞ , hence corresponds to an element $b \in B(G, \mu^{-1})$ by classification of G -bundles on the Fargues-Fontaine curve. Since a Newton stratum of the Shimura variety is a product of the corresponding Igusa variety and the Rapoport-Zink space, passing to the infinite level π_{HT} then can be realized as its local analogue for the infinite level Rapoport-Zink space (after Scholze-Weinstein), hence the fiber becomes the Igusa variety.

(3) The perversity of $R\pi_{\mathrm{HT},*}\mathbb{F}_\ell$ (for any compact Shimura variety of PEL type), which has the following consequence:

Theorem 2 (Perversity) Suppose $b \in B(G, \mu)$ is minimal (i.e., d= $\langle 2\rho, \nu_b\rangle$ is minimal, i.e., $\dim \mathcal{F}\ell^b_{G,\mu} = \langle 2\rho, \mu\rangle - d$ is maximal) such that

$$H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$$

for some $i \in \mathbb{Z}$. Then $H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}}$ is concentrated in degree $i = d$.

Using these 3 ingredients, now we can finish the proof of Theorem 1 .

Proof Let $p \in \mathrm{Spl}_{K/\mathbb{Q}} - S \cup \{\ell\}$ be any prime. We have a Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(K_p, H^q(\mathcal{S}_{K^p}, \mathbb{F}_\ell)) \Rightarrow H^{p+q}(S_K, \mathbb{F}_\ell),$$

which computes the cohomology of the classical Shimura variety S_K using the cohomology of the perfectoid Shimura variety $\mathcal{S}_{K^p}$. It follows that

$$H^i(\mathcal{S}_{K^p}, \mathbb{F}_\ell) \neq 0$$

for some i . On the other hand, the Leray spectral sequence for the Hodge-Tate period map

$$\pi_{\mathrm{HT}} : \mathcal{S}_{K^p} \to \mathcal{F}\ell_{G,\mu},$$

gives

$$E_2^{p,q} = H^p(\mathcal{F}\ell_{G,\mu}, R^q\pi_{\mathrm{HT},*}\mathbb{F}_\ell) \Rightarrow H^{p+q}(\mathcal{S}_{K^p}, \mathbb{F}_\ell).$$

Therefore

$$R\pi_{\mathrm{HT},*}\mathbb{F}_\ell \neq 0.$$

Since everything is compatible with the Hecke action away from p , using the Ingredient (2) we know that there exists some b , such that

$$H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_{\mathfrak{m}} \neq 0$$

for some i . Pick minimal such b , by the perversity (Ingredient (3)), we know that $H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)$ is concentrated in one degree. Therefore $\mathbb{Z}_\ell$ -coefficient cohomology is also concentrated in one degree and torsion-free. By Poincare duality (applied to the dual system of Hecke eigenvalues), this is also true for $H_c(\mathrm{Ig}^b, \mathbb{Z}_\ell)$. Hence $\mathfrak{m}$ shows up in $[H_c(\mathrm{Ig}^b, \overline{\mathbb{Q}_\ell})]$, where the Galois representation lifting $\rho_{\mathfrak{m}}$ exists by Ingredient (1). □

Remark 2 As one can see from the proof, (as expected) the Galois representation $\rho_{\mathfrak{m}}$ is not found in $H^i(S_K, \overline{\mathbb{Q}_\ell})$, but rather goes through the cohomology of the perfectoid Shimura variety and Igusa tower, which secretly constructs congruences between automorphic forms of different weights and levels at p .

Vanishing of torsion

Now let us come to the second main result, which asserts the "generic part" of the cohomology of our compact unitary Shimura varieties vanishes outside the middle degree.

Theorem 3 Assume $H^i(S_K, \mathbb{F}_\ell)_\mathfrak{m} \neq 0$ for some i. Assume there is a prime p which splits completely in F and $\rho_\mathfrak{m}$ is unramified and decomposed generic at all places of F above p. Then $H^i(S_K, \mathbb{F}_\ell)_\mathfrak{m}$ is concentrated in the middle degree $i = \dim S_K$. In particular, $H^i(S_K, \mathbb{Z}_\ell)_\mathfrak{m}$ is concentrated in the middle degree and torsion-free.

Remark 3 The existence of torsion Galois representations and vanishing results of similar sort (for locally symmetric spaces of GL_n) are useful for proving modularity lifting results for GL_n over general number fields (after Calegari-Geraghty), where numerical coincidence in the usual Taylor-Wiles method fails.

Remark 4 Recall that a local Galois representation is decomposed generic means that its Frobenius eigenvalues α_i are distinct and for any i, j $\alpha_j/\alpha_i \neq q$. Any characteristic 0 lift is then a direct sum of characters χ_i such χ_j/χ_i is not a cyclotomic character (since there is no nontrivial extension between χ_i 's by the Euler characteristic formula). Such a local Galois representation corresponds to a generic principal series representation of GL_n under local Langlands (hence its name).

Generic principal series are mapped to zero under the Jacquet-Langlands correspondence to any group that is not quasi-split. As we saw last time, using this one deduces that $\mathfrak{m}$ only contributes to the "most ordinary part" of the Igusa variety:

Theorem 4 (Genericity) If $b \in B(G, \mu^{-1})$ is not μ -ordinary, then $[H_c(\mathrm{Ig}^b, \overline{\mathbb{Q}_\ell})]_\mathfrak{m} = 0$.

Remark 5 Since the shape of μ depends on the signature of G, the assumption in Theorem 3 can be relaxed according the the signature of G in order to ensure that J_b is not quasi-split unless b is μ -ordinary. For example, when the signature is $(0, n)$ for all but one infinite place, one needs to require that $\rho_\mathfrak{m}$ is unramified and decomposed generic at only one place of F. In any case, when $\rho_\mathfrak{m}$ has sufficiently large image, the assumption is always satisfied by Chebotarev's density.

Theorem 3 now follows easily from the genericity and the three main ingredients.

Proof Since S_K is compact, by Poincare duality, it suffices to show that

$$H^i(S_K, \mathbb{F}_\ell)_\mathfrak{m} = 0, \quad i < \dim S_K.$$

By the same argument as in proof of Theorem 1, it suffices to show that for any b,

$$H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_\mathfrak{m} = 0, \quad i < \dim S_K.$$

Again take any minimal b as in the proof of Theorem 1. By genericity, we know b must be μ -ordinary, hence $\mathcal{F}\ell^b_{G,\mu}$ is 0 *-dimensional* (notice the order-reversing!) and $d = \dim S_K$. By perversity, $H^i(\mathrm{Ig}^b, \mathbb{F}_\ell)_\mathfrak{m}$ vanishes for $i < \dim S_K$. □

The Hodge-Tate period map preserves perversity

Finally, let us explain the proof of the perversity result. Let S_K be any compact Shimura variety of PEL type with hyperspecial level at p.

To motivate, recall two useful results for perverse sheaves in algebraic geometry:

- Any simple perverse sheaf $\mathcal{F}$ on a scheme X of finite type is of the form $j_{!*}(\mathcal{L}[\dim Y])$ for some locally closed subscheme $j: Y \to X$ and a local system $\mathcal{L}$ on Y. In particular, $\mathcal{F}|_Y$ is concentrated in one degree.
- Any finite (more generally, any small) map $f: X \to Y$ preserves perversity: if $\mathcal{F}$ is a perverse sheaf on X, then $Rf_*\mathcal{F}$ is a perverse sheaf on Y.

By the minimality, we know that $(R\pi_{\mathrm{HT},*}\mathbb{F}_\ell)_\mathfrak{m}$ has support on the union

$$\bigcup_{b'} \mathcal{F}\ell^{b'}_{G,\mu}, \quad \dim \mathcal{F}\ell^{b'}_{G,\mu} \leq d.$$

So it has support in a closed subset of dimension equal to $\dim \mathcal{F}\ell^b_{G,\mu}$. The result then would follow if $(R\pi_{\mathrm{HT},*}\mathbb{F}_\ell)_\mathfrak{m}$ is "perverse". Why should it be? The intuition is that π_{HT} is an affine and partially proper (i.e., satisfies valuative criterion in the category of adic spaces). If we were working with schemes, this would mean π_{HT} is affine and proper, hence finite, and finite morphisms preserve perversity. Of course all the beauty of π_{HT} lies in its very non finite-type behavior, so we cannot literally say this. On the other hand, because of Ingredient (2), we only need to show the sheaf is perverse when restricted on an affinoid etale neighborhood U of x. Then we can pass to the special fiber (by the perfectoidness), where π_{HT} indeed becomes a finite map between affine schemes of finite type over the residual field.

The other issue is that $(R\pi_{\mathrm{HT},*}\mathbb{F}_\ell)_\mathfrak{m}$ admits the action of $G(\mathbb{Q}_p)$ and is infinite dimensional. So it can only be perverse (or just constructible) after taking K_p -invariants for $K_p \subseteq G(\mathbb{Q}_p)$ open compact. So the idea for proving the perversity Theorem 2 is then to pass to finite levels and special fibers. Let us be more precise.

Proof Scholze showed that there exists a basis U_i of affinoid etale neighborhoods $U_i = \mathrm{Spa}(A_i, A_i^\circ)$ of $\bar{x}$ such that its pullback under π_{HT} (denoted by $\mathcal{S}_{K^p,U_i}$) is affinoid perfectoid. For each such U, we have a formal model $\mathfrak{U} = \mathrm{Spf}\, A^\circ$ and correspondingly a formal model $\mathfrak{S}_{K^p,U}$ for $\mathcal{S}_{K^p,U}$. For any such U, for K_p a sufficiently small pro- p open compact subgroup of $G(\mathbb{Q}_p)$, we have a continuous action of K_p on U, which induces the trivial on the special fiber $\mathfrak{U}_s = \mathrm{Spec}(A^\circ/p)$.

We choose such $K_{p,i}$ sufficiently small for each U_i so that $K_{p,i}$'s shrink to 1. Then we know that the fiber at $\bar{x} \in \mathcal{F}\ell_{G,\mu}$ can be computed using the cohomology of the special fiber using the right upper corner of the following diagram,

$$\begin{tikzcd} S_{K^p,U}/K_p \arrow[r, "\pi_{\mathrm{HT}}"] \arrow[d, "\lambda"] & U/K_p \arrow[d, "\lambda"] \\ \mathfrak{S}_{K_pK^p,U,s} \arrow[r, "\pi_{\mathrm{HT}}"] & \mathfrak{U}_s. \end{tikzcd}$$

When shrinking $K_{p,i}$, we obtain

$$(R\pi_{\mathrm{HT},*}\mathbb{F}_\ell)_{\mathfrak{m},\bar{x}} = \varinjlim_i (R\lambda_{i,*}R\pi_{\mathrm{HT},*}\mathbb{F}_\ell)_{\mathfrak{m},\bar{x}_i}.$$

Here $\bar{x}_i \in \mathfrak{U}_{i,s}$ is the specialization of $\bar{x}$. Now we can compute using the left lower corner as well, namely, first specialize, then apply π_{HT} (which now becomes a finite map between schemes of finite type). Because the general fact that specialization (aka, nearby cycle) preserves perversity, we know that each individual term in the direct limit is concentrated in degree d. Hence the direct limit itself is also concentrated in degree d. □