1.

1.1.

This course is about some uses of the variable q.

The funny thing about q is that different people throughout history used it in descriptions of phenomena that were a priori unrelated. Then, later, it was discovered that all these disparate roles for q did, in fact, have something to do with each other.

1.2.

Many of us first encounter q as the order of a finite field, a prime power. We denote the field by $\mathbf{F}_q$.

When we do linear algebra over $\mathbf{F}_q$, we quickly notice: The number of lines through the origin in an n-dimensional vector space over $\mathbf{F}_q$ is

$$[n]_q := \frac{q^n - 1}{q - 1} = 1 + q + \cdots + q^{n-1}.$$

More generally the number of k-dimensional (linear) subspaces turns out to be

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad \text{where } [n]_q]! = [n]_q \cdots [2]_q[1]_q. \tag{1.1}$$

Certainly, this expression would become the binomial coefficient $\frac{n!}{k!(n-k)!}$ if we could treat q as an indeterminate rather than a number and send $q \to 1$. But that is surprising, because there is no field $\mathbf{F}_1$.

This is the first of several bridges: The role of q as the order of a finite field is related to the role of q as a deformation parameter in combinatorics.

1.3.

Let's prove the assertion about (1.1). It will be convenient to assume the following fact that does not involve finite fields:

Lemma 1. *Write*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\alpha \geq 0} c_\alpha q^\alpha.$$

Then c_α is the number of integer partitions of α having at most k parts each of size at most $n-k$: equivalently, Young diagrams of size α that fit into an $k \times (n-k)$ box.

Proof sketch. Use the fact that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is determined for all integers n, k by these properties:

(1) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}_q = 1$.
(2) $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ when $n < 0$ or $k < 0$.
(3) $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$. □

Let $\mathcal{G}_{n,k}(\mathbf{F}_q)$ be the set of k-dimensional subspaces of $\mathbf{F}_q^n$. The following result was probably known to Gauss in a premodern form, and is usually attributed to Schubert. Donald Knuth seems to have discovered it on his own in 1971.

Theorem 2. *There is a partition*

$$\mathcal{G}_{n,k}(\mathbf{F}_q) = \coprod_Y \mathcal{G}_{n,k,Y}(\mathbf{F}_q),$$

where the right-hand side runs over all Young diagrams that fit into a $k \times (n-k)$ box. Moreover, $|\mathcal{G}_{n,k,Y}(\mathbf{F}_q)| = q^{|Y|}$ for all Young diagrams Y.

Proof. Given any k-dimensional subspace of $\mathbf{F}_q^n$, we can pick a basis for it, then write the basis as a list of row vectors to get a $k \times n$ matrix with entries in $\mathbf{F}_q$. Applying row reduction operations, we find that the matrix is equivalent under left multiplication by $\mathrm{GL}_k(\mathbf{F}_q)$ to one in reduced row-echelon form, like the one below for $(n, k) = (10, 3)$ stolen from Sara Billey[1]:

$$\begin{pmatrix} * & * & 0 & * & * & * & 0 & * & 1 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The asterisks show how this reduced row-echelon matrix corresponds to a Young diagram Y, whose size is the total number of asterisks. Let $\mathcal{G}_{n,k,Y}(\mathbf{F}_q)$ be the set of all subspaces that produce this matrix. Then the elements of $\mathcal{G}_{n,k,Y}(\mathbf{F}_q)$ are classified by the labelings of the asterisks with elements of $\mathbf{F}_q$. □

Note that $\mathcal{G}_{n,k}(\mathbf{F}_q)$ is the set of $\mathbf{F}_q$-points of a projective algebraic variety $\mathcal{G}_{n,k}$ defined over $\mathbf{F}_q$ called the *(n, k) Grassmannian*. The pieces $\mathcal{G}_{n,k,Y}(\mathbf{F}_q)$ similarly arise from algebraic varieties $\mathcal{G}_{n,k,Y}$ known as *Schubert cells*. The enumeration of $\mathcal{G}_{n,k,Y}(\mathbf{F}_q)$ can be upgraded to an isomorphism $\mathcal{G}_{n,k,Y} \simeq \mathbf{A}^{|Y|}$.

In particular, this final statement does not involve q at all. We can lift the isomorphism to any field. Over the complex numbers, the Euler characteristic of any affine space is always 1. This gives a sort of topological meaning to the $q \to 1$ limit of $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Remark 3. In general, $\mathbf{F}_q$-point counts need not specialize to the Euler characteristics of corresponding complex algebraic varieties. The simplest counterexample is any sufficiently varied family of algebraic curves over $\mathbf{F}_q$ of constant genus.

[1] See "Tutorial on Schubert Varieties and Schubert Calculus" online.

1.4.

In this course, we will pay more attention to a close cousin of the Grassmannian called the flag variety.

Fix an integer tuple $\vec{k} = (k_1, \ldots, k_l)$, where $0 < k_1 < \cdots < k_l < n$. A *partial flag* of type $\vec{k}$ in an n-dimensional vector space V is a filtration $0 \subset V_1 \subset \cdots \subset V_l \subset V$, where V_i is a (linear) subspace of dimension k_i for all i. The partial flags of type $\vec{k}$ in $\mathbf{F}_q^n$ form the $\mathbf{F}_q$-points of a projective algebraic variety defined over $\mathbf{F}_q$ called the associated *partial flag variety*.

When $\vec{k}$ consists of a single integer k, the partial flag variety is the (n, k) Grassmannian. When $\vec{k} = (1, 2, \ldots, n)$, we instead speak of a *complete flag*, or *flag* for short, and of the *(complete) flag variety* $\mathcal{B}_n$.

It turns out that the numerology of $\mathcal{B}_n(\mathbf{F}_q)$ is similar to that of $\mathcal{G}_{n,k}(\mathbf{F}_q)$. To see how, observe that the outer border of a Young diagram that fits in a $k \times (n-k)$ box forms a lattice path with n steps, k of which go upward and $n-k$ of which go rightward. The symmetric group S_n acts transitively on such lattice paths by permuting the steps, and the stabilizer of any given path is isomorphic to $S_k \times S_{n-k}$. Up to choosing one of them as a "basepoint", we can identify the set of such Young diagrams with the coset space $S_n/(S_k \times S_{n-k})$ for a chosen embedding $S_k \times S_{n-k} \subseteq S_n$. Now the partition of $\mathcal{G}_{n,k}(\mathbf{F}_q)$ indexed by Young diagrams has an analogue

$$\mathcal{B}_n(\mathbf{F}_q) = \coprod_{w \in S_n} \mathcal{B}_{n,w}(\mathbf{F}_q).$$

The pieces $\mathcal{B}_{n,w}(\mathbf{F}_q)$ again arise from varieties $\mathcal{B}_{n,w}$ that we again call Schubert cells, as it turns out that $\mathcal{B}_{n,w} \simeq \mathbf{A}^{\ell(w)}$ for some function ℓ on S_n. In fact, this whole story has an analogue for the partial flag variety of any allowed tuple $\vec{k}$, in which we replace $S_k \times S_{n-k}$ with $S_{k_1} \times S_{k_2-k_1} \times \cdots \times S_{k_l-k_{l-1}}$.

1.5.

One way to construct the Schubert decomposition of $\mathcal{B}_n(\mathbf{F}_q)$ involves the general linear group $\mathrm{GL}_n(\mathbf{F}_q)$. Observe that $\mathrm{GL}_n(\mathbf{F}_q)$ acts transitively on flags in $\mathbf{F}_q^n$, and that the stabilizer of the standard flag is the subgroup of upper- or lower-triangular matrices, depending on whether one uses column or row notation for $\mathbf{F}_q^n$. Either way, let $B(\mathbf{F}_q)$ denote the subgroup. We obtain a bijection

$$\mathrm{GL}_n(\mathbf{F}_q)/B(\mathbf{F}_q) \xrightarrow{\sim} \mathcal{B}_n(\mathbf{F}_q).$$

Bruhat decomposition shows that

$$\mathrm{GL}_n(\mathbf{F}_q) = \bigsqcup_{w \in S_n} B(\mathbf{F}_q)\dot{w}B(\mathbf{F}_q),$$

where $\dot{w} \in \mathrm{GL}_n(\mathbf{F}_q)$ is the permutation matrix corresponding to w. This suggests that we take $\mathcal{B}_{n,w}(\mathbf{F}_q) = B(\mathbf{F}_q)\dot{w}B(\mathbf{F}_q)/B(\mathbf{F}_q)$ as a definition.

To promote this to a definition of the algebraic variety $\mathcal{B}_{n,w}$, we need to make sense of coset spaces in a geometric, not set-theoretic, setting. It turns out to be easier to work over the algebraic closure $\bar{\mathbf{F}}_q$, then recover the story on $\mathbf{F}_q$-points by passing to fixed points under so-called Frobenius maps.

This discussion will lead to the first main theme of the course: The structure of reductive algebraic groups, which behave like GL_n, and their Weyl groups, which behave like S_n; and the role of flag varieties in the representation theory of associated finite groups.

1.6.

As for the function ℓ such that $\ell(w) = \dim \mathcal{B}_{n,w}$, its easiest definition is as the number of *inversions* of w: that is, pairs $i < j$ such that $w(i) > w(j)$.

A more sophisticated definition uses the fact that S_n is a *Coxeter group*, or real reflection group. For $i = 1, 2, \ldots, n-1$, let $s_i \in S_n$ be the transposition of i and $i+1$. Then S_n has a presentation

$$S_n = \left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{l} s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \\ s_i s_j = s_j s_i \text{ for } |i-j| > 1, \\ s_i^2 = e \end{array} \right\rangle.$$

We may define $\ell(w)$ as the length of the shortest word in the s_i needed to express w.

It is helpful to picture the relations above using *wiring diagrams*. If we refine the wiring diagrams by replacing crossings with over- and under-crossings, then we arrive at *braid diagrams*, which satisfy analogues of the first two relations but not the third. In this way we arrive at the *braid group*

$$Br_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| > 1 \end{array} \right\rangle.$$

We have already seen that S_n is related to GL_n via the map $w \mapsto \dot{w}$. We now see that Br_n is related to S_n via the map $\sigma_i \mapsto s_i$.

1.7.

The last story of today is a separate, direct relationship between GL_n and Br_n that also brings us back to the theme of the variable q.

Let E_n be the set of $\mathbf{C}$-valued functions on $\mathcal{B}_n(\mathbf{F}_q)$. Then $\mathrm{GL}_n(\mathbf{F}_q)$ acts linearly on E_n, by precomposition with functions. The resulting $\mathrm{GL}_n(\mathbf{F}_q)$-module may be viewed as the induction of the trivial representation from $B(\mathbf{F}_q)$ to $\mathrm{GL}_n(\mathbf{F}_q)$. Iwahori discovered:

Theorem 4. *Br_n acts on E_n by $\mathrm{GL}_n(\mathbf{F}_q)$-equivariant linear operators. Moreover, the action factors through the algebra*

$$H_n(q) := \frac{\mathbf{C}[Br_n]}{\langle \sigma_i^2 - (q^{1/2} - q^{-1/2})\sigma_i - 1 \mid i = 1, \ldots, n-1\rangle},$$

and the map $H_n(q) \to \mathrm{End}_{\mathbf{C}\mathrm{GL}_n(\mathbf{F}_q)}(E_n)$ is an algebra isomorphism.

We refer to $H_n(q)$ as the *Iwahori–Hecke algebra*, or just *Hecke algebra*, of S_n at q. The reason we say S_n is the observation that, if we could treat q as an indeterminate and send $q \to 1$, then $H_n(q)$ would become the group ring $\mathbf{Z}S_n$. Indeed, this leads us to introduce a "generic" Hecke algebra

$$H_n(x) := \frac{\mathbf{C}[x^{\pm 1}][Br_n]}{\langle \sigma_i^2 - (x - x^{-1})\sigma_i - 1 \mid i = 1, \ldots, n-1\rangle},$$

One of the earliest applications of $H_n(x)$ came from a totally different area of math: namely, knot theory.

A knot is a circle (tamely) embedded into 3-space, and a link is a disjoint union of finitely many such circles. Vaughan Jones and Adrian Ocneanu used trace functions on the algebras $H_n(x)$ to construct polynomial invariants of braids, which then give rise to invariants of knots and links after normalization. Here the variable x becomes the square root of an indeterminate q, whose relationship to the prime power q is completely explicit, yet remains magical.