MODULAR REPRESENTATION THEORY AND GEOMETRY

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The representation theory of algebraic groups is both a classical subject and one of active investigation. Over an algebraically closed field of characteristic zero we have a reasonably complete picture – the category of representations is semisimple, the simple objects are parametrized by highest weights, and their characters are given by Weyl's formula. Indeed the story is an algebraic version of the even more classical theory of representations of compact Lie groups. Nevertheless, Lusztig's discovery [5] of canonical bases a mere seventeen years ago showed that this theory still contained unexpected structure.

Given that Chevalley's classification of connected simply-connected simple algebraic groups over an algebraically closed field is, remarkably, independent of characteristic, it is natural to seek a description of the category of representations of an algebraic group over a field k of characteristic $p > 0$. Here, immediately, many properties familiar from characteristic zero collapse, as even elementary calculations with SL_2 demonstrate. The category is no longer semisimple, and while the simple objects are again classified by highest weights, their character is no longer easy to obtain – even a formula for their dimension is not known in complete generality.

The fundamental new structure for an algebraic group over k is the Frobenius homomorphism: Given an affine scheme X over k, one can define a new k-scheme $X^{(1)}$ by composing the map $\mathsf{k} \to \mathcal{O}(\mathsf{X})$ with the map $\lambda \mapsto \lambda^{\frac{1}{p}}$. There is a natural finite map of k-schemes $Fr\colon X \to X^{(1)}$ which is the identity on points and the map $f \mapsto f^p$ on elements of $\mathcal{O}(X)$. If X is in fact defined over $\mathbb{F}_p$, then $X^{(1)}$ is isomorphic to X, and composing with this isomorphism we get a map $Fr\colon X \to X$ which we call the Frobenius morphism. Since reductive algebraic groups can be defined over $\mathbb{Z}$, they are always equipped with a Frobenius morphism, which is readily seen to be a group homomorphism. The action of Fr on representations is crucial: for example, a theorem of Steinberg shows that every irreducible can be obtained as a tensor product of the Frobenius twists of a finite set of irreducible representations.

1. Quantum Groups

In the 1980s a remarkable observation of Lusztig introduced a new character to this story. He showed that it is possible to define a form of the Drinfeld-Jimbo quantum group over the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$, and that when the deformation parameter v in this integral form is specialized to ζ a root of unity, the representations of the resulting algebra $\mathbf{U}_\zeta$ behave much like those of an algebraic group in positive characteristic. This was all the more remarkable for the fact that the quantum group at a root of unity is defined over the ring of integers of a cyclotomic field, and so is a characteristic zero object. If we take ζ to be a p-th root of unity, a field k of characteristic p is naturally a $\mathbb{Z}[\zeta]$-algebra via the map $\zeta \mapsto 1$, and in this way the quantum group over $\mathbb{Z}[\zeta]$ can be thought of as an integral lift of the characteristic p situation (it is, however the analogue of the algebra of distributions, or hyperalgebra, $\mathcal{U}$ of the group, rather than the group itself).

One of the fundamental discoveries made by Lusztig was that there is an analogue of the Frobenius morphism for $\mathbf{U}_\zeta$. Its definition is somewhat more subtle than in the classical characteristic p setting, as we now recall.

Each quantum group, by analogy with Chevalley's classification of reductive groups, has an associated root datum. Given a positive integer ℓ, and the root datum of $\mathbf{U}$, one can define a new root datum with associated quantum group $\mathbf{U}^*$. In the case where the initial datum is simply-laced, $\mathbf{U}^*$ is just $\mathbf{U}$ again, but in other cases, the new datum is "between" $\mathbf{U}$ and the Langlands' dual datum (if $\mathbf{U}$ is of finite type and ℓ is divisible by 6 then $\mathbf{U}^*$ is precisely the Langlands' dual).

Let $\mathcal{U}^*_{\mathbb{Z}[\zeta]}$ be the Kostant-Chevalley integral form of the classical enveloping algebra with scalars extended from $\mathbb{Z}$ to $\mathbb{Z}[\zeta]$. Lusztig constructed a map

$$Fr\colon \mathbf{U}_\zeta \to \mathcal{U}^*_{\mathbb{Z}[\zeta]},$$

which he called the quantum Frobenius map. When $\ell = p$, if we base change Fr we obtain the classical Frobenius map (or rather its transpose on the hyperalgebra).

In later work, Lusztig observed that it is better to work with a slight modification of $\mathbf{U}$ when dealing with questions of integral structures, and it is this modified algebra one really works with. Working with this form, a simple extension of Lusztig's work shows that it is possible to define a right inverse c to Fr, that is, a map c so that $Fr \circ c$ is the identity on the modified form of the classical enveloping algebra. This map is intimately related to the notion of Frobenius splitting in algebraic geometry, as recent work of Kumar and Littelmann shows.

In the case of $\mathfrak{sl}_n$, Beilinson, Lusztig and MacPherson [1] have given a construction of the quantum group, or rather larger and larger quotients of it known as q-Schur algebras, using the geometry of finite fields. The q-Schur algebras are a family of algebras $S_q(n,d)$ over $\mathbb{Z}[q]$ for integers n, d (where normally $d \geq n$). After extending scalars to $\mathcal{A}$ by setting $q = v^2$, each $S_q(n,d)$ is a quotient of $\mathbf{U}(\mathfrak{sl}_n)$, and indeed the algebras $S_q(n,d)$ naturally form an inverse system so that the quantum group $\mathbf{U}(\mathfrak{sl}_n)$ embeds into the limit of this system.

Given any ring R and an invertible element ε we may base change $S_q(n,d)$ to obtain an algebra $S_R(n,d)$. (The specialization where v is sent to 1 is referred to as a Schur algebra). I have observed [7] that the map Fr descends to give a map from the q-Schur algebra at a root of unity to the Schur algebra:

$$F_d\colon S_{\mathbb{Z}[\zeta]}(n,\ell d) \to \mathbb{Z}[\zeta] \otimes_{\mathbb{Z}} S_{\mathbb{Z}}(n,d).$$

(In fact my proof works in the context of a "generalized q-Schur algebra"). I have also shown that the contraction map c descends to a map c_d on q-Schur algebras, giving a right inverse of the map F_d.

Once it is known that the quantum Frobenius descends to the q-Schur algebra, it is natural to ask for a construction of the map F_d in the context of the geometry of finite fields. This appears, *a priori* to be somewhat unlikely, as the parameter $v^2 = q$ is supposed to be a prime power, and thus rather far from a root of unity. However it turns out that one can give such an interpretation, using the construction of $S_q(n,d)$ for the fields $\mathbb{F}_q$ and $\mathbb{F}_{q^\ell}$. Moreover using the limit construction mentioned above, this produces a construction of the map Fr for $\mathfrak{sl}_n$. Returning to the context of an arbitrary quantum group, the map Fr is constructed by first producing its retriction to the "plus part" $\mathbf{U}^+$, and then extending using the triangular decomposition of $\mathbf{U}$. Thus the critical part of the construction of Fr is demonstrating its existence on $\mathbf{U}^+$. Using the Hall algebra construction of $\mathbf{U}^+$, a more elaborate version of the same technique [8] that constructs the map F_d for q-Schur algebras yields the map $Fr_{|\mathbf{U}^+}$, and so we get a geometric construction of (at least half) of the quantum Frobenius in general.

2. Research aims

Geometrization: Once it is known that Fr can be interpreted in the context of the geometry of finite fields, it is natural to try to perform the "faisceaux-fonctions" lift, and obtain a realization of Fr on the level of perverse sheaves on the moduli of quiver representations. I am currently working on this lifting. The nature of the quantum Frobenius in the divisible case suggests that the action of Frobenius in [8] should be replaced with equivariance with respect to cyclic groups (to some extend this is of course anticipated by [6]). The existence of such a lifting would establish a connection between the action of the quantum Frobenius and the canonical basis.

Quiver varieties: There are natural analogues of generalized q-Schur algebras for affine quantum groups, and I can show that the quantum Frobenius is also compatible with these affine algebras. Nakajima has given a geometric construction of a version of these affine generalized q-Schur algebras in his work on finite dimensional representations of affine quantum groups. I hope that an analogue of my construction with the Hall algebra is possible in this geometry, giving an interpretation of the quantum Frobenius in this context. I suspect that this work will be closely related to Nakajima's "t-analogues of q-characters" [11].

The affine Grassmannian: Recent work of Mirkovic and Vilonen [10] gives a remarkable construction (in characteristic zero), of the tensor category of representations of a reductive algebraic group, in any characteristic, in terms of perverse sheaves on the affine Grassmannian of the Langlands dual group. Thus one can hope to understand aspects of modular representation theory in this context. Indeed it is classical that the affine Weyl group plays a prominent role in this representation theory, so a context where the affine Weyl group is utterly intrinsic should be a great advantage. The most basic issue here is to obtain an intrinsic understanding of the action of Frobenius in this realization, with the Steinberg tensor product theorem mentioned above as the natural first result to interpret. Another reason for seeking a geometric lifting of the quantum Frobenius is that I hope it will provide a model for these questions. Indeed one can use the techniques of [8] for the affine Grassmanian of type A to study Hall-Littlewood functions at a root of unity [9].

Other interests: I am interested in the theory of representations of finite reductive groups, and its connection to perverse sheaves. In the geometric study of these groups, there are two distinct strands. One is the study of the construction of the representations themselves, through the seminal work of Deligne and Lusztig. The other, motivated by Springer's definition of Green functions, is Lusztig's theory of character sheaves, which seeks to calculate the characters of a finite reductive group G, by producing from the geometry of the group, a basis of the space of class functions on G. Although work of Lusztig shows that these two theories are deeply related – both produce, for example, from quite different starting points, the *almost characters* of the group, a basis of the class functions of G which is closely related, but not precisely equal, to the irreducible characters of G – however the nature of the connection remains somewhat mysterious. There are two topics I wish to investigate:

The first is the relation of the classification of character sheaves, due to Lusztig [4], to the geometric Langlands program for G-bundles on $\mathbb{P}^1$ in the monodromic case (this is related to sheaves on the "basic affine space" as discussed in the work of Bezrukavnikov, Finkelberg and Ostrik, who have made precise conjectures relating Lusztig's work to this context). One of the interesting questions here is to understand how to see the finite Hecke algebra as a "natural" subalgebra of the affine Hecke algebra – the larger algebra arises more clearly in the geometric Langlands picture, while the smaller algebra arises in the classification of character sheaves. I hope to study this in collaboration with David Ben-Zvi.

Secondly, Gurevich and Hadani [3] have recently given a geometric construction of the Weil representation as a perverse sheaf. Their sheaf is a geometric version of the algebra of operators on the representation, thus a "geometrization" of the representation, not just its character. Along with Hadani, I would like to use this sheaf to investigate the behaviour of character sheaves under a geometric version of Howe duality, and perhaps give examples of other "representation sheaves" (rather than character sheaves) for finite reductive groups. Such examples might shed some light on the relation between the representation theory and character theory mentioned above.

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