Notes for 'An introduction to character sheaves'

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Abstract

These notes were written in preparation for a talk given by the author in the Universtiy of Texas at Austin's graduate student geometry seminar on 15 October 2013.

1 Introduction

Character sheaves were invented by Lusztig in order to adapt classical constructions from the representation theory of finite groups to the setting of reductive groups. Section 2 gives reminders about characters of finite-dimensional representations of finite groups, leading to a discussion of the desired properties of character sheaves in Section 3. After a (somewhat rushed) summary of the theory perverse sheaves in Section 4, we state the definition of character sheaves in terms of the horocycle correspondence in Section 5. Extra topics, depending on the audience's interests, include Section 6 on Grothendieck's sheaf-function correspondence as a way to convert character sheaves into characters of a finite group, and Section 7 on the notion of the center of a monoidal category and how certain categories of character sheaves realize such categorical centers.

We emphasize that these lecture notes do not provide a thorough treatment of character sheaves and the necessary background material; the author apologies for any unillustrative sections that lack sufficient detail.

2 Characters of finite groups

In this section, we review a selection of concepts from the basic representation theory of finite groups, and restate constructions in ways that will motivate later topics.

Let G be a finite group. The **group algebra** $\mathbb{C}[G]$ is the space of complex-valued functions on G

$$\mathbb{C}[G] = \{f : G \to \mathbb{C}\}$$

equipped with multiplication given by convolution:

$$f * g(x) = \sum_{yz=x} f(y)f(z) = \sum_{y\in G} f(y)f(y^{-1}x).$$

Representations of G are the same as modules for $\mathbb{C}[G]$. The space of conjugation-invariant functions, or **class functions**,

$$\{f \in \mathbb{C}[G] \mid f(yxy^{-1}) = f(x) \text{ for all } x, y \in G\}$$

forms a subalgebra of the group algebra. This subalgebra is denoted $\mathbb{C}[\frac{G}{G}]$ because it can be identified with the space of functions on the set of conjugacy classes $\frac{G}{G}$ on G. Throughout these notes, horizontal bars will symbolize quotients by conjugation actions.

Exercise: Show that the center of the group algebra coincides with class functions:

$$Z(\mathbb{C}[G]) = \mathbb{C}[\frac{G}{G}].$$

Let V be a finite-dimensional representation of G over $\mathbb{C}$. Recall that the **character** $\chi_V \in \mathbb{C}[G]$ of V is defined as

$$\begin{aligned} \chi_V : G &\to \mathbb{C} \\ g &\mapsto \mathrm{tr}(g; V). \end{aligned}$$

In words, the character χ_V is the complex-valued function on G that assigns to each $g \in G$ the trace of the corrseponding linear operator $g : V \to V$. Elementary properties of the trace function imply that each character is a class function. If $\{V_i\}$ is the set of isomorphism classes of irreducible finite-dimensional representations of G, then the set $\{\chi_{V_i}\}$ forms a basis for $\mathbb{C}[\frac{G}{G}]$.

To place the character of V in a more general framework, consider the function

$$\begin{aligned} \phi : \mathrm{End}_{\mathbb{C}}(V) \simeq V^* \otimes V &\to \mathbb{C}[G] \\ v^* \otimes v &\mapsto [g \mapsto \langle v^*, g \cdot v \rangle]. \end{aligned}$$

Here, V^* denotes the dual vector space to V and $\langle , \rangle$ denotes the evaluation pairing on $V^* \otimes V$. The map ϕ is equivariant for the usual action of G on the left-hand side and the conjugation action of G on the right-hand side. Taking invariants on both sides, we obtain the so-called 'generalized trace map' for the representation V:

$$\mathrm{End}_G(V) \longrightarrow \mathbb{C}\left[\frac{G}{G}\right].$$

Observe that the identity endomorphism maps to the character χ_V.

Exercises: Let H be a subgroup of G and $V = \mathrm{Ind}_H^G(\mathbb{C}_{\mathrm{triv}}) = \mathbb{C}[G/H]$ be representation of G induced from the trivial representation of H.

1. Show that the space ${}^H\mathbb{C}[G]^H$ of H-bi-invariant functions on G is a subaglebra of $\mathbb{C}[G]$. Observe that ${}^H\mathbb{C}[G]^H$ can be identified with the space of functions $\mathbb{C}[H\backslash G/H]$ on the double cosets of H in G.

2. There is an isomorphism $\mathrm{End}_G(V) \simeq \mathbb{C}[H\backslash G/H]$. (Hint: the orbits of the diagonal action of G on $G/H \times G/H$ can be identified with the double cosets $H\backslash G/H$.) This common algebra is known as the **Hecke algebra** for the pair (G, H).

3. Consider the maps (a special case of the horocycle correspondence):

$$\frac{G}{G} \xleftarrow{q} \frac{G}{H} \xrightarrow{r} H\backslash G/H.$$

Show that the pullback-pushforward of functions $q_* r^* : \mathbb{C}[H\backslash G/H] \to \mathbb{C}[\frac{G}{G}]$ can be identified with the generalized trace map for V. (If $\alpha : X \to Y$ is a function between finite sets and $f \in \mathbb{C}[X]$, then $\alpha_*(f) \in \mathbb{C}[Y]$ is defined as $\alpha_*(f)(y) = \sum_{x \in \alpha^{-1}(y)} f(x)$.)

3 Motivation for character sheaves

Let K be an algebraically closed field and G a reductive group over K. For example, take $K = \mathbb{C}$ or $K = \bar{\mathbb{F}}_p$ and $G = \mathrm{GL}_n(K)$. Our motivating question is:

Can we develop a theory of characters for G?

To achieve this goal, we consider G as an affine algebraic variety and thus access powerful techniques in algebraic geometry. As a consequence, we are forced to work with categorical versions of objects in classical representation theory. The 'characters' we seek will not be functions on G, but rather sheaves on G, called character sheaves. What properties should these sheaves satisfy? Ideally:

- Character sheaves would appear as the center of the appropriate categorical version of the group algebra.
- Every sufficiently 'small' categorical representation of G would be assigned a character sheaf.
- If $K = \bar{\mathbb{F}}_p$ is the algebraic closure of a finite field $\mathbb{F}_p$, and G is defined over $\mathbb{F}_p$, then we can hope to recover information about the characters of the finite group $G(\mathbb{F}_p)$. This point is perhaps the main motivation for Lusztig to invent character sheaves.

This talk will introduce the theory of character sheaves, as developed by Lusztig. The ideal properties above are realized in the following ways:

- Certain categories of character sheaves are the centers of a Hecke category and its twisted versions; these Hecke categories can be regarded as approximations to a categorical analogue of the group algebra of G.
- Moreover, any dualizable module category for one of these Hecke categories is assigned a character sheaf.
- One of the crowning achievements of the theory is an organization of the character tables of finite groups of Lie type using characters sheaves. This approach is based on the Grothendieck sheaf-function correspondence.

4 Perverse sheaves

Character sheaves are certain perverse sheaves; this section provides an overview of the theory of perverse sheaves that emphasizes properties relevant to the definition of character sheaves. We begin with basic definitions and examples, introduce the derived category of sheaves on X, mention functoriality properties, define perverse sheaves, and consider equivariant constructions when an algebraic group G acts on X. For more thorough treatments of the theory of perverse sheaves, we refer the reader to [4], [3, Chapter 8], and [7].

Definition. 1. A sheaf $\mathcal{F}$ on a topolgical space X is called a **local system** if there is a cover $\{U_i\}$ of X such that the restriction $\mathcal{F}|_{U_i}$ is the constant sheaf on U_i for all i.

2. A sheaf $\mathcal{F}$ on a variety X is **constructible**[1] if, for every closed subvariety $Y \subset X$, there is an open subvariety $U \subset Y$ such that the restriction $\mathcal{F}|_U$ is a local system on U.

3. A complex $\mathcal{F}^\bullet$ of sheaves on a variety X is **constructible** if each cohomology sheaf $\mathcal{H}^i(\mathcal{F})$ is constructible.

Observe that any local system is a constructible sheaf, and any construtible sheaf can be regarded as a constructible complex concentrated in degree 0. We give examples of each of these objects.

1. Suppose $p : Y \to X$ is a covering map. Taking sections of p gives a local system:

$$\mathcal{F}(U) = \{s : U \to Y \mid p \circ s = \mathrm{Id}_U\}.$$

This a local system of sets; to obtain a local system of $\mathbb{C}$-vector spaces, take functions on these discrete sets. If X is connected and has a universal cover, then there is an equivalence of categories between local systems on X and representations of the fundamental group of X.

2. A skyscraper sheaf at a closed point of X is a constructible sheaf that is generally not a local system.

3. The prototypical example of a constructible complex is the de Rham complex for a smooth variety over $\mathbb{C}$ with the analytic topology.

Let $D(X) = D^b(\mathsf{Shv}(X))$ denote the bounded derived category of sheaves of $\mathbb{C}$-vector spaces on X. For the purposes of this talk, we do not explain the construction of the derived category in detail; we only give a summary:

- Start with the category of complexes of sheaves on X.
- Adjoin inverses to quasi-isomorphisms in a controlled way.
- Obtain a triangulated category whose objects are still complexes of sheaves on X, but whose morphisms are not as straightforward to describe.
- By slightly abuse of terminology, we refer the the objects of $D(X)$ as just 'sheaves' rather than 'complexes of sheaves'.

A map $f : X \to Y$ of varieties induces the familiar functors of pushforward f_*, proper pushforward[2] $f_!$, and pullback f^* of sheaves:

$$f_*, f_! : \mathsf{Shv}(X) \to \mathsf{Shv}(Y) \qquad f^* : \mathsf{Shv}(Y) \to D(X).$$

The functors (f^*, f_*) form an adjoint pair. There are derived functors (which we denote with the same symbols):

$$f_*, f_! : D(X) \to D(Y) \qquad f^* : D(Y) \to D(X).$$

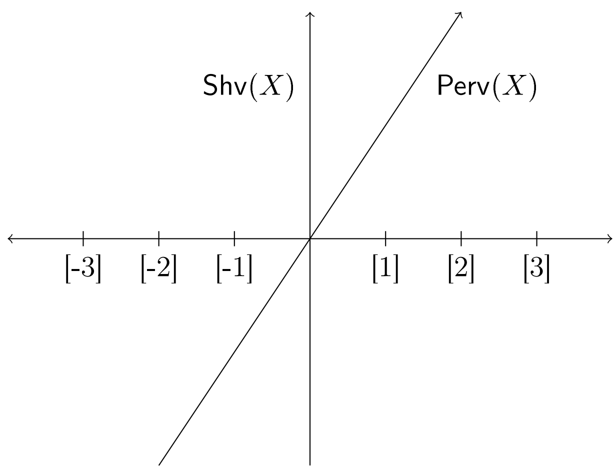
[1]For varieties over $\mathbb{C}$ with an analytic topology, a different definition of a constructible sheaf is in use. Namely, a sheaf $\mathcal{F}$ is constructible if there is a stratification of X such that the restriction of $\mathcal{F}$ to each stratum is a local system. See [4, page 11].

[2]For a sheaf $\mathcal{F}$ on X, the sections of the proper pushforward $f_!\mathcal{F}$ over an open subset $U \subset Y$ are defined as the sections of $\mathcal{F}(f^{-1}(U))$ with proper support.

In addition, for reasonable maps f, the functor $f_!$ has a right adjoint $f^! : D(Y) \to D(X)$ on the derived level. Part of the advantage of the derived category is that these functors have nicer properties – or indeed only exist – at the derived level, rather than at the abelian level.

Local systems are relatively simple types of sheaves, but they are not preserved by the functors $f_*, f_!, f^*, f^!$ in general. However, constructible sheaves are preserved by these functors (for reasonable maps f). In fact, constructible complexes are the smallest subcategory containing local systems and closed under these functors. Therefore, we work with the full subcategory $D_c(X) \subset D(X)$ of constructible complexes[3].

Recall that the abelian category $\mathsf{Shv}(X)$ of sheaves on X, regarded as complexes concentrated in degree 0, is a full subcategory of $D(X)$. In fact, $\mathsf{Shv}(X)$ is a special type of abelian subcategory, called a 'heart of t-structure' on $D(X)$. One can think of a t-structure a 'coordinate system' for $D(X)$, and the heart as the basis of the coordinate system. A schematic illustrating this analogy is given below, where the x-axis is labeled by iterations of the shift functor on $D(X)$. A t-structure defines a notion of cohomology for objects of $D(X)$ and taking cohomology is analogous to measuring where we are in $D(X)$ relative to a coordinate system.



The category $\mathsf{Shv}(X)$ is a natural heart, but there may be other hearts that work better for other purposes. It turns out that $D_c(X)$ has a perverse coordinate system, or 'heart', that is particularly nice (despite its name). More precisely, $D_c(X)$ contains a full abelian subcategory $\mathsf{Perv}(X)$ whose objects are

$$\{\mathcal{F} \in D_c(X) \mid \dim(\mathrm{supp}(\mathcal{H}^i(\mathcal{F}))) \leqslant -i \text{ and } \dim(\mathrm{supp}(\mathcal{H}^i(\mathbb{D}\mathcal{F}))) \leqslant -i\},$$

where $\mathbb{D}$ is the Verdier duality functor which we do not define here since we plan not to linger on this definition. Some of the nice properies of $\mathsf{Perv}(X)$ are

- It is an abelian category all of whose objects have finite length.
- The irreducible objects are given in terms of local systems on closed subvarieties.
- It leads to beautiful categorifications of important strucutres in representation theory.
- There is an intimate connection between perverse sheaves and $\mathcal{D}$-modules via the Riemann-Hilbert correspondence.

[3] As an aside, the Verdier duality functor $\mathbb{D}$ squares to the identity for constructible complexes.

Finally, all constructions of this section can be done equivariantly when group actions are invovled. That is, suppose an algebraic group G acts on a variety X. There is a category of $D_c^G(X)$ of equivariant constructible complexes on X, and it contains a heart $\mathsf{Perv}^G(X)$ of equivariant perverse sheaves on X. The forgetful functor $D_c^G(X) \to D_c(X)$ is compatible with the pervserse t-structures.

Notational remark: For the remainder of these notes, we write $D(X)$ instead of $D_c(X)$, $D(X/G)$ intead of $D_c^G(X)$, and $\mathsf{Perv}(X/G)$ instead of $\mathsf{Perv}^G(X)$.

5 Definition of character sheaves

Let K be an algebraically closed field and G a reductive group over K. For example, take $K = \mathbb{C}$ or $K = \bar{\mathbb{F}}_p$ and $G = GL_n(K)$. Fix a Borel subgroup B with unipotent radical $N = R_u(B)$. Fix a maximal torus $T \subset B$ and let $W = N_G(T)/T$ be the Weyl group. For example, in the case that $G = \mathrm{GL}_n(K)$, the upper triangular matrices form a Borel subgroup whose unipotent radical is the subgroup of upper triangular matrices with 1's along the diagonal. A maximal torus is the subgroup of diagonal matrices. The Weyl group of $\mathrm{GL}_n(K)$ is the symmetric group S_n.

The Borel subgroup B has several valuable properties. For example, B is self-normalizing and the space of Borel subgroups can be identified with the flag variety G/B. The Borel subgroup B is a minimal subgroup such that the quotient G/B has the structure of a projective variety. Finally, B admits a semi-direct product decomposition $B = N \rtimes T$.

Let's recall to the horocycle correspondence for the pair (G, B):

$$\frac{G}{G} \xleftarrow{q} \frac{G}{B} \xrightarrow{r} B\backslash G/B.$$

In the finite group case, the quotients were just sets; now they are stacks, but we largely ignore this technical point[4]. The category $D(B\backslash G/B)$ of B-equivariant sheaves on the flag variety G/B has a convolution structure, making it a monoidal category known as the **Hecke category**. The Hecke category can be regarded as a substitute for the group algebra. In particular, many fundamental categories related to the representation theory of G are module categories for the Hecke category. Similar to the case of finite groups, we consider the pullback-pushforward functor along the horocycle correspondence:

$$q_! r^* : D(B\backslash G/B) \longrightarrow D\left(\frac{G}{G}\right).$$

Intuition from the setting of finite groups suggests that the image of this functor gives some sort of 'characters' of G. In fact, a certain class of character sheaves, called unipotent character sheaves, are the irreducible perverse constituents of objects in the image of this functor.

Definition. An irreducible perverse sheaf $\mathcal{F}$ in $D(\frac{G}{G})$ is a **unipotent character sheaf** if $\mathcal{F}$ appears as a constituent of $q_! r^* A$ for some $A \in D(B\backslash G/B)$.

To obtain all character sheaves, we consider the space $Y = N\backslash\frac{G}{T}/N$, that is, the quotient of G by the action of $N \times T \times N$ given by $(n_1, t, n_2) \cdot g = n_1 t g t^{-1} n_2$. This action is well-defined since T normalizes N. We have a more general version of the horocycle correspondence:

$$\frac{G}{G} \xleftarrow{q} \frac{G}{B} \xrightarrow{r} Y = N\backslash\frac{G}{T}/N.$$

[4] Alternatively, one can work with equivariant sheaves on varieties rather than sheaves on stacks.

Observe that T acts on Y via $t \cdot [g] = [tg]$, and the quotient is identified with $B\backslash G/B$. All character sheaves will be obtained by pulling back and pushing forward along this correspondence, but we restrict our attention to sheaves on Y that interact nicely with the action of T. First we introduce the notion of a monodromic sheaf on a space with a T-action.

Definition. Suppose $a : T \times X \to X$ is an action of T on X. Let $\mathcal{L}$ be a 1-dimensional local system on T. An **$\mathcal{L}$-monodromic sheaf** on X is a sheaf $\mathcal{F} \in D(X)$ equipped with an isomorphism $a^*\mathcal{F} \simeq \mathcal{L} \boxtimes \mathcal{F}$ satisfying the evident associativity and unital conditions.

Let $D^{\mathcal{L}}(Y)$ denote the category of $\mathcal{L}$-monodromic sheaves on Y. Observe that there is an equivalence of categories $D^{\underline{K}_T}(Y) = D(B\backslash G/B)$, where $\underline{K}_T$ is the trivial local system on T.

Definition. An irreducible perverse sheaf $\mathcal{F} \in D(\frac{G}{G})$ is a **character sheaf** if it appears as a constituent of $q_! r^* A$ for some $A \in D^{\mathcal{L}}(Y)$ for some 1-dimensional local system $\mathcal{L}$ on T.

6 Grothendieck's sheaf-function correspondence

Let p be a prime and let X be a variety over $K = \bar{\mathbb{F}}_p$ that is defined over $\mathbb{F}_p$. Our ongoing example $\mathrm{GL}_n(K)$ satisfies this condition. Fix a power $q = p^r$ of p and consider the usual Frobenius endomorphism of $\bar{\mathbb{F}}_p$ gvien by $\alpha \mapsto \alpha^q$. This map extends to a Frobenius endomorphism Fr on X:

$$Fr : X \to X$$

whose fixed points are the $\mathbb{F}_q$-rational points of X:

$$X(\mathbb{F}_q) = X^{Fr}.$$

For example, on $\mathrm{GL}_n(K)$, the Frobenius map corresponding to q raises every entry of a matrix to the q-th power: $(a_{ij}) \mapsto (a_{ij}^q)$. The set of fixed points of Fr on $\mathrm{GL}_n(K)$ is the finite group $\mathrm{GL}_n(\mathbb{F}_q)$.

Technical Remark: We work in the étale topology on X and with coefficients in the algebraic closure $\bar{\mathbb{Q}}_\ell$ of the field $\mathbb{Q}_\ell$ for a prime ℓ different from p. The reason for doing so is that resulting sheaf theory resembles the classical one for complex analytic varieties. Note that $\bar{\mathbb{Q}}_\ell \simeq \mathbb{C}$ as fields. The constructions and results we describe are independent of the choice of ℓ (so long as $\ell \neq p$).

The idea of the Grothendieck sheaf-function correspondence is that functions on the set X^{Fr} are related to sheaves fixed by the Frobenius endomorphism on X. Suppose that $\mathcal{F}$ is a complex of sheaves fixed by Fr, i.e. there is an isomorphism

$$\phi : Fr^*\mathcal{F} \xrightarrow{\sim} \mathcal{F}.$$

Such an isomorphism is called a **Weil structure** on $\mathcal{F}$. Throughout this discussion, we work with a fixed a Weil structure ϕ on $\mathcal{F}$. If $\mathcal{F}$ is irreducible, then there is a unique Weil structure up to a non-zero scalar. Observe that a Weil structure defines a linear endomorphism of the stalk $\mathcal{F}_x$ for every $x \in X^{Fr} = X(\mathbb{F}_q)$:

$$\phi_x : (Fr^*\mathcal{F})_x = \mathcal{F}_{Fr(x)} = \mathcal{F}_x \xrightarrow{\sim} \mathcal{F}_x.$$

More relevant for the derived setting, we obtain linear endomorphisms of the stalks of the cohomology sheaves at points $x \in X(\mathbb{F}_q)$:

$$\phi_x : \mathcal{H}^i(\mathcal{F})_x \xrightarrow{\sim} \mathcal{H}^i(\mathcal{F})_x.$$

Define a function $\chi_{\mathcal{F},\phi}$ on $X^{Fr} = X(\ F_q)$ by taking an alternating sum of the traces of these functions:

$$\chi_{\mathcal{F},\phi} : X(\mathbb{F}_q) \to \bar{\mathbb{Q}}_\ell$$
$$x \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \mathrm{tr}(\phi_x; \mathcal{H}^i(\mathcal{F})_x).$$

The function $\chi_{\mathcal{F},\phi}$ is known as the **characteristic function** of the sheaf $\mathcal{F}$ with Weil structure ϕ. A 'meta-theorem', known as Grothendieck's sheaf-function correspondence, states that:

Interesting functions on $X(\mathbb{F}_q)$ arise as characteristic functions of perverse sheaves on X with a Weil structure.

In particular, when $X = G$ is a group, the expectation is the characteristic functions of certain perverse sheaves on G should coincide with the characters of the finite group $G(\mathbb{F}_q)$. One of Lusztig's remarkable results asserts that this is precisely the case for $G = \mathrm{GL}_n$, where the perverse sheaves are the character sheaves.

Theorem 1. *Suppose $\mathcal{F}$ is a character sheaf on $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$ equipped with a Weil structure. Then the characteristic function $\chi_{\mathcal{F},\phi}$ is (up to a nonzero scalar) an irreducible character of $GL_n(\mathbb{F}_q)$. Moreover, there is a bijection*

$$\left\{ \begin{matrix} \textit{character sheaves on } \mathrm{GL}_n(\bar{\mathbb{F}}_p) \\ \textit{with Weil structure} \\ \textit{(up to isomorphism)} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{irreducible characters} \\ \textit{of } \mathrm{GL}_n(\mathbb{F}_q) \\ \textit{(up to nonzero scalar)} \end{matrix} \right\}.$$

given by $(\mathcal{F}, \phi) \mapsto \chi_{\mathcal{F},\phi}$.

For other reductive groups G, there is still a bijection between Frobenius-fixed character sheaves and irreducible characters of the corresponding finite group $G(\mathbb{F}_q)$, but it is not given in all cases by taking characteristic functions. However, the characteristic functions form an orthonormal basis for the class fuctions on $G(\mathbb{F}_q)$, and there is a 'small' change-of-basis matrix that relates the basis of characteristic functions $\chi_{\mathcal{F},\phi}$ to the basis of irreducible characters χ_V. Here, 'small' means that the number of irreducible constituents of each characteristic function is bounded independently of q and the coefficients are also bounded independently of q. The precise statement involves almost characters and Lusztig's Fourier transform matrices; see [6] for more details.

7 Vector bundles on a finite group and Drinfeld centers

Let G be a finite group and $\mathrm{VB}(G)$ the monoidal category of vector bundles (i.e. G-graded vector spaces) on G. The monoidal structure is given by

$$(V \otimes W)_g = \bigoplus_{xy=g} V_x \otimes W_y.$$

There is a notion of the 'center' of a monoidal category (which we will discuss momentarily) and the center of $\mathrm{VB}(G)$ is the category $\mathrm{VB}(\frac{G}{G})$ of adjoint-equivariant vector bundles on G. To be explicit, an adjoint equivariant vector bundle on G is a vector bundle V on G equipped with isomorphisms $\psi_{g,x} : V_{gxg^{-1}} \xrightarrow{\sim} V_x$ for every $x, g \in G$ satisfying

$$\psi_{gh,x} = \psi_{h,x} \circ \psi_{g,hxh^{-1}} \quad \text{and} \quad \psi_{e,x} = \mathrm{Id}_{V_x}$$

for all $x, g, h \in G$, where $e \in G$ is the identity.

Question: What is the center of a monoidal category $(\mathcal{C}, \otimes)$?

To motivate the construction, let A be a ring and consider the two maps

$$s_1, s_2 : A \rightrightarrows \mathrm{Hom}(A, A)$$

$$s_1(a)(b) = ab, \qquad s_2(a)(b) = ba,$$

that send $a \in A$ to the algebra endomorphisms of right and left multiplication by a. The center $Z(A)$ can be described as the equalizer of s_1 and s_2: $Z(A) = \{a \in A \mid s_1(a) = s_2(a)\}$.

Now let $(\mathcal{C}, \otimes)$ be a tensor category and consider the two maps from $\mathcal{C}$ to tensor endfunctors of $\mathcal{C}$

$$\mathcal{C} \rightrightarrows \mathrm{Hom}_{\otimes}(\mathcal{C}, \mathcal{C})$$

given by $X \mapsto X \otimes (-)$ and $X \mapsto (-) \otimes X$. The **Drinfeld center** $Z(\mathcal{C})$ of $\mathcal{C}$ is defined as the 'categorical equalizer' of these two maps. Explicity, the objects of $Z(\mathcal{C})$ are pairs (X, σ_X) consisting of an object of X and a natural transformation of functors $\sigma_X : X \otimes (-) \to (-) \otimes X$ satisfying $\sigma_X(Y \otimes Z) = (\mathrm{Id}_Y \otimes \sigma_X(Z)) \circ (\sigma_X(Y) \otimes \mathrm{Id}_Z)$. For more details, including definitions of the morphisms and the braided monoidal structure of $Z(\mathcal{C})$, see section XIII.4 of Kassel's book [5].

A remark to make at this point is that we can extend the maps above to form a simplicial object

$$\mathcal{C} \rightrightarrows \mathrm{Hom}_{\otimes}(\mathcal{C}, \mathcal{C}) \equiv\!\!> \mathrm{Hom}_{\otimes}(\mathcal{C} \times \mathcal{C}, \mathcal{C}) \ldots.$$

The limit of this diagram (taken in the appropriate category) is known as the Hochschild cohomology of $\mathcal{C}$. This construction is relvant in the context of ∞-categories. We have that $Z(\mathcal{C}) = HH^0(\mathcal{C})$.

Let's return to vector bundles on a finite group. Recall that the center of a matrix algebra over $\mathbb{C}$ (of any size) consists of the scalar matrices, i.e. $Z(\mathrm{Mat}_n \mathbb{C}) \simeq \mathbb{C}$. A categorical version of this result (known as Müger's Morita invariance of the Drinfeld center) implies the following two facts:

- $Z(\mathrm{VB}(G)) = \mathrm{VB}(\frac{G}{G})$.
- $Z(\mathrm{VB}(H \backslash G / H)) = \mathrm{VB}(\frac{G}{G})$ for any subgroup H of G.

A theorem of Bezrukavnikov, Finkelberg, and Ostrik [1] states that the category of character sheaves can be identified with the Drinfeld center of a monoidal category of Harish-Chandra bimodules. On the other hand, Ben-Zvi and Nadler [2] have shown that the ∞-category of λ-monodromic character sheaves can be identified with both the Hochschild homology and cohomology of the Hecke category $\mathcal{D}_\lambda(B \backslash G / B)$ of λ-twitsted $\mathcal{D}$-modules on the finite orbit stack $B \backslash G / B$. A discussion of either of these results is beyond the scope of these notes; the only thing to say is that both sets of authors work on the $\mathcal{D}$-modules side of the Riemann-Hilbert correspondence.

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