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MILNOR FIBERS AND ALEXANDER POLYNOMIALS OF PLANE CURVES

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Let C be an irreducible algebraic curve of degree n in the projective plane $\mathbf{P}^2$. Since C has real codimension two, it is natural to try to study the "knot theory" of the pair $(\mathbf{P}^2, C)$. From an algebraic point of view, an immediate difficulty is that the first homology of $\mathbf{P}^2 - C$ is cyclic of order n instead of infinite cyclic as in the case of classical knots. Thus, there is not immediately a theory of Alexander invariants. There are at least two possible ways to avoid this difficulty. In this note we introduce the second such way, and show that it generically yields the same answer as the first, more classical way, at least for the Alexander polynomial.

The first approach is used in several recent papers of A. Libgober [**1, 2, 3**]. One removes a line L to form $\mathbf{P}^2 - (C \cup L)$. It is easily seen that the first homology of the latter space is $\mathbf{Z}$, as desired. We will denote the Alexander polynomial resulting from this approach by $\Delta_C^L(t)$. This approach is inspired by work of O. Zariski, though he works with branched covers and not explicitly with the Alexander polynomial.

An alternative approach is to consider the Hopf bundle over $\mathbf{P}^2$. This bundle has total space S^5; if we restrict to $\mathbf{P}^2 - C$, we obtain a bundle with fiber S^1 and total space $S^5 - K$, where K is the link of the singularity of the homogeneous polynomial f which defines the curve C. Now K is a three-dimensional complex, and it is not difficult to show that $H_1(S^5 - K) \cong \mathbf{Z}$. Thus, there is an Alexander polynomial $\Delta_K(t)$. The following two results show that $\Delta_K(t)$ is of interest.

THEOREM 1. *If the line L is in general position with respect to the curve C, then $\Delta_K(t) = \Delta_C^L(t)$. For any L, $\Delta_K(t)$ divides $\Delta_C^L(t)$.*

REMARK. In the first case, we actually show that $\pi_1(\mathbf{P}^2 - (C \cup L)) \cong \pi_1(S^5 - K)$.

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Now let $F = f^{-1}(1)$ be the Milnor fiber of the singularity of f at the origin. That is, there is a bundle $F \to S^5 - K \to S^1$. Furthermore, there is a monodromy map $h\colon F \to F$.

THEOREM 2. *$\Delta_K(t)$ is the characteristic polynomial of the monodromy*

$$h_*\colon \quad \frac{H_1(F; \mathbf{Z})}{torsion} \to \frac{H_1(F; \mathbf{Z})}{torsion}.$$

We will write $\Delta(t)$ for $\Delta_K(t)$. The advantages of our approach basically stem from Theorem 2, which shows that the current theory is analogous to the theory of classical *fibered* knots. Later we will state several corollaries of these results.

EXAMPLE 1. Let C be defined by $\{x^2z = y^3\} \subset \mathbf{P}^2$, and take L to be the line $\{z = 0\}$. Then, $L \cap C$ is a single point, so L and C are not in general position. In [**1**] it is noted that $\mathbf{P}^2 - (C \cup L)$ deformation retracts to the complement of a trefoil knot in S^3. Thus, $\Delta_C^L(t) = t^2 - t + 1$. But it is easily seen that $\Delta_K(t) = 1$ in this case (using the usual techniques for computing fundamental groups, for example). So, in general, $\Delta_K(t) \neq \Delta_C^L(t)$.

EXAMPLE 2. Let C be the sextic with six cusps on a conic given by the equation $(x^2 + z^2)^3 + (y^3 + z^3)^2$. Then $\Delta(t) = t^2 - t + 1$. We defer until later a direct computation which uses Theorem 2 and avoids fundamental groups.

PROOFS. Theorem 2 is analogous to the folklore result about fibered knots in classical knot theory, and is proved in the same way. Note that we have an exact sequence

$$(1) \qquad \to H_1(F) \xrightarrow{h_* - I_*} H_1(F) \to H_1(S^5 - K) \to H_0(F) \cong \mathbf{Z} \to 0.$$

We prove Theorem 1 by considering presentations of the fundamental groups in question. The classical presentation for $\mathbf{P}^2 - C$ is

$$(2) \qquad \pi_1(\mathbf{P}^2 - C) \cong \langle a_1, \ldots, a_n \mid a_1 a_2 \cdots a_n, R_i \rangle.$$

Here we pick a pencil of lines in $\mathbf{P}^2$. A general line L' of the pencil intersects C in n points. The generators $a_1, \ldots, a_n$ are just loops on L' which circle these intersections once. Then on $L' \cong S^2$, we have the relation $a_1 a_2 \cdots a_n$. The relations R_i are obtained from lines of the pencil which intersect C in fewer than n points.

Next, Zariski has shown that if L is in general position with respect to C, one has

$$(3) \qquad \pi_1\big(\mathbf{P}^2 - (C \cup L)\big) \cong \big\langle a_1, \ldots, a_n, b \mid a_1 a_2 \cdots a_n b, R_i, [a_i, b] \big\rangle.$$

The notation here is as for (2). The generators a_j and relations R_i are identical to the above, while the generator b is represented by a loop in L' which circles $L \cap L' = \{x\}$ once.

Next we wish to obtain a presentation of $\pi_1(S^5 - K)$ which we can relate to (2) and (3). To do this, we pull the pencil of $\mathbf{P}^1$'s in $\mathbf{P}^2$ back to a "pencil" of S^3's in

S^5. That is, the inverse image of a $\mathbf{P}^1$ in $\mathbf{P}^2$ is an S^3 in S^5. In $\mathbf{P}^2$, the pencil consists of all $\mathbf{P}^1$'s containing a fixed point. In S^5, the inverse image consists of all S^3's containing a fixed circle.

To get the desired presentation we first consider $L' - (L' \cap C) \cong S^2 - \{n \text{ points}\}$. The inverse image of this set is the complement of n circles in S^3. These circles form an (n, n) torus link in S^3. (This link consists of n parallel $(1, 1)$ curves on a torus in S^3.) In terms of fundamental groups we have

$$\begin{aligned}\pi_1(S^3\text{-link}) &\cong \langle a_1',\ldots,a_n' \mid a_1' \cdots a_n' = a_2' \cdots a_n' a_1' = \cdots = a_n' a_1' \cdots a_{n-1}' \rangle \\ &\cong \langle a_1',\ldots,a_{n-1}' \rangle \times \langle C \rangle, \qquad C = a_1' a_2' \cdots a_n', \\ \pi_1(S^2\text{-points}) &\cong \langle a_1,\ldots,a_n \mid a_1 \cdots a_n = 1 \rangle.\end{aligned}$$

Furthermore, there is the commutative diagram

$$\begin{tikzcd}
 & & 1 & 1 & \\
1 \arrow[r] & \pi_1(S^1) \cong \mathbf{Z} \arrow[r] & \pi_1(S^5 - K) \arrow[r] \arrow[u] & \pi_1(\mathbf{P}^2 - C) \arrow[r] \arrow[u] & 1 \\
1 \arrow[r] & \pi_1(S^1) \cong \mathbf{Z} \arrow[r] & \pi_1(S^3\text{-link}) \arrow[r] \arrow[u, "i_*"'] & \pi_1(S^2\text{-pts}) \arrow[r] \arrow[u, "i_*"'] \arrow[l, bend left, "\sigma"] & 1 \\
 & & G = \sigma(G') \arrow[u] & G' \arrow[u] & \\
 & & 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}$$

where the rows and columns are exact, and i_* are inclusion-induced. The map σ takes a_i to a_i', $i = 1,\ldots,n-1$, $\sigma(a_n) = (a_1' \cdots a_{n-1}')^{-1}$. Here G' is the normal subgroup generated by the relations R_i. It then follows that G is the kernel of i_*: $\pi_1(S^3\text{-link}) \to \pi_1(S^5 - K)$.

Thus, a presentation of $\pi_1(S^5 - K)$ is

$$\text{(4)}\quad \begin{aligned}&\pi_1(S^5 - K) \\ &\quad\cong \langle a_1', a_2',\ldots,a_n' \mid a_1' \cdots a_n' = a_2' \cdots a_n' a_1' = \cdots = a_n' a_1' \cdots a_{n-1}', R_i' \rangle,\end{aligned}$$

where the set R_i' is just the set R_i with all a_i's primed.

Then we define $\psi(a_i) = a_i'$, $\psi(b) = (a_1' \cdots a_n')^{-1}$ and obtain an isomorphism ψ: $\pi_1(\mathbf{P}^2 - (C \cup L)) \to \pi_1(S^5 - K)$.

This concludes the proof when L and C are in general position. If not, we can consider $L = L_0$ as the limit of a family L_u of lines, where for $u \neq 0$, L_u is in general position with C. The result follows since $\pi_1(\mathbf{P}^2 - (C \cup L))$ maps onto $\pi_1(\mathbf{P}^2 - (C \cup L_u))$, implying $\Delta_K(t) = \Delta_C^{L_u}(t)$ divides $\Delta_C^L(t)$, $u \neq 0$.

We gather several immediate consequences of the above theorems. These results were also proved by Libgober in [**1**].

COROLLARIES. $\Delta(t)$ *satisfies*

(i) $\Delta(1) = \pm 1$.

(ii) $\Delta(0) = \pm 1$.

(iii) $\deg \Delta(t) = 2q$, *where q is the irregularity of the cyclic n-fold cover of* $\mathbf{P}^2$ *branched along* C.

(iv) $\Delta(t)$ *is monic and cyclotomic.*

(v) $\Delta_C^L(t)$ *is independent of L as long as L is in general position with respect to C.*

Furthermore, $\Delta(t)$ can sometimes be calculated without reference to fundamental groups. Consider Example 2. Here C is defined by

$$C = \left\{ \left(x^2 + z^2\right)^3 + \left(y^3 + z^3\right)^2 = 0 \right\} \subset \mathbf{P}^2.$$

Thus, $F = \{(x^2 + z^2)^3 + (y^3 + z^3)^2 = 1\} \subset \mathbf{C}^3$.

Define $g\colon F \to G$ by $g(x, y, z) = (u, v)$, where $u = x^2 + z^2$, $v = y^3 + z^3$, and $G = \{(u, v) \in \mathbf{C}^2 \mid u^3 + v^2 = 1\}$. We want to use the Leray spectral sequence of g to identify the monodromy on $H^1(F; \mathbf{Z})$. This spectral sequence has

$$E_2^{p,q} \cong H^p(G; \mathcal{H}^q(g)),$$

where $\mathcal{H}^q(g)$ is the sheaf on G which associates $H^q(g^{-1}(U))$ to the open set U of G. Also, $E_2^{p,q} \Rightarrow H^{p+q}(F)$. We will sketch a proof that $H_1(F) \cong E_2^{1,0}$.

First, for dimensional reasons, there are no nonzero differentials d_r involving $E_r^{1,0}$, for $r \geqslant 2$. Thus $E_2^{1,0} \cong E_\infty^{1,0}$. Because of this, $H^1(F) \cong E_2^{1,0}$ follows from $E_2^{0,1} \cong 0$. We give a sketch of the proof of this latter fact.

We may identify $E_2^{0,1} \cong H^0(G; \mathcal{H}^1(g))$ with $\Gamma(\mathcal{H}^1(g))$, the set of sections of $\mathcal{H}^1(g)$. To show $\mathcal{H}^1(g)$ has no nontrivial sections requires some effort, since $\mathcal{H}^1(g)$ is not trivial.

Fix $(u_0, v_0) \in G$, and let $E_{(u_0,v_0)} = g^{-1}(u_0, v_0) \subset F$. Let p be the projection of $E_{(u_0,v_0)} \subset \mathbf{C}^3$ to the z-axis. Then $p^{-1}(z_0) = \{(x, y, z) \in E_{(u_0,v_0)} \mid z = z_0,\ x^2 = u_0 - z_0^2,\ y^3 = v_0 - z_0^3\}$. Thus, p represents $E_{(u_0,v_0)}$ as a 6-fold branched cover of $\mathbf{C}$, branched over the points $z_0^2 = u_0$, $z_0^3 = v_0$. From this it follows that $E_{(u_0,v_0)}$ is an open 2-manifold. For most points (u_0, v_0), the Euler characteristic of $E_{(u_0,v_0)}$ is -12. For special values $u_0 = 0$; $v_0 = 0$; or $v_0^2 = u_0^3 = \frac{1}{2}$ the first betti number of $E_{(u_0,v_0)}$ drops. Analysis of the monodromy around these special points shows that it is impossible to extend local sections to a nontrivial global section.

Thus, the spectral sequence of the map g yields $H^1(F; \mathbf{Z}) \cong E_2^{1,0}$ and the action of the monodromy h on $E_2^{1,0}$ is just the well-known monodromy for G as the fiber of the trefoil knot; $h_* = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. Thus,

$$\Delta(t) = \mathrm{Det}\begin{pmatrix} t & -1 \\ 1 & t-1 \end{pmatrix} = t^2 - t + 1.$$

REMARK. The invariants considered here are, of course, rational invariants. There are interesting torsion (or integral) invariants also. For example, one has the torsion in $H_1(F; \mathbf{Z})$, which is precisely the torsion of the abelianization of the commutator subgroup of $\pi_1(\mathbf{P}^2 - C)$. The possibilities for this torsion are limited as follows: For each singular point p_i of C, one has a link $L_i \subset S^3$ in the usual way. Let K_i be the n-fold cyclic cover of S^3 branched along L_i. Then we always have [**4**]

$$\operatorname{Tor}(H_1(F; \mathbf{Z})) \subset \bigoplus_i \operatorname{Tor} H_1(K_i; \mathbf{Z}).$$

Furthermore, for certain L_i and n, sharper results may be obtained [**4**]. Similar results have recently been obtained by Libgober [**2**].

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