Stable vector bundles on curves

This is an introduction to a GIT construction of the moduli space of stable vector bundles on curves, presented at the GIT seminar . Our main sources are [1] and [2].

[-] Contents

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Fix a smooth projective algebraic curve C (over $\mathbb{C}$) of genus g. Unlike the case of line bundles, it has been observed that in general all vector bundles are not classifiable. For example, one can construct a family of vector bundles of rank ≥ 2 parametrized by the disk where all the fibers apart from the origin are mutually isomorphic, but not isomorphic to the fiber at the origin ([1, 11.32]). This "jump phenomenon" illustrates that the set of all vector bundles on curves is not even separated. In other words, even the coarse moduli space does not exist. To construct well-behaved moduli spaces of vector bundles, Mumford's geometric invariant theory hints at finding stable conditions on vector bundles and restricting our attention to *stable* vector bundles.

Definition 1 The *slope* of a vector bundle E is the ratio $\mu(E)=\deg E/\operatorname{rank} E$. E is called *stable* (resp. *semistable*) if every subbundle $F\subseteq E$ satisfies $\mu(F)<\mu(E)$ (resp. $\mu(F)\leq\mu(E)$). Equivalently, E is stable (resp. semistable) if every quotient bundle $G=E/F$ satisfies $\mu(G)>\mu(E)$ (resp. $\mu(G)\geq\mu(E)$).

By definition, every line bundle is stable. The following implications partially explain that why stable bundles are "nice".

Proposition 1 Stable $\implies$ simple $\implies$ indecomposable.

Proof The definition of stability implies that every nonzero endomorphism $E\to E$ is an isomorphism, hence is a scalar by looking at one fiber. □

The moduli space $M(r,d)$ of stable vector bundles over C of rank r and degree d was first given by Mumford [3] and Seshadri [4]. Later, Gieseker gave a different construction which generalized to higher dimensions. Simpson invented a more natural and general method using Grothendieck's Quot scheme which also extends to singular curves and higher dimensions (see [5]).

$M(1,d)$ is simply the Picard variety Pic^d we have constructed. We have the natural map $\det: M(r,d)\to M(1,d)$ sending E to its determinant bundle $\det E$. We fix a line bundle L and study the fiber of this map. In other words, we are going to construct the space of stable vector bundles

$$S(r,L)=\{E:\operatorname{rank}E=r,\det E=L, E\text{ stable}\}$$

using GIT.

Similarly to the case of Picard varieties, we will assume $\deg L\gg 0$ so that Riemann-Roch brings us some convenience.

Proposition 2 If E is semistable with $\mu(E)>2g-2$, then $H^1(E)=0$.

Proof By assumption, every quotient line bundle of E has degree $>2g-2$. So there is no nonzero morphism $E\to K_C$ as $\deg K_C=2g-2$. Hence $H^1(E)=0$ by Serre duality. □

Proposition 3 If E is semistable with $\mu(E)>2g-1$, then E is generated by global sections.

Proof Since for any $p\in C$, $\mu(E(-p))>2g-2$, we know that $H^1(E(-p))=0$ as $E(-p)$ is semistable (tensoring with a line bundle does not change stability). By the exact sequence

$$0 \to E(-p) \to E \to E|_p \to 0$$

We know that $H^0(E) \to H^0(E|_p) \cong E_p \otimes \kappa(p)$ is surjective. By Nakayama's lemma we know that $H^0(E) \otimes \mathcal{O}_C \to E$ is surjective. □

Vector bundles of Rank 2

Now let us concentrate on the case $r = 2$. Similarly to the case of Picard varieties, we will associate to each isomorphism class of vector bundles of rank 2 a $GL(N)$-orbit of a matrix and study its stability. Let E be a vector bundle of rank 2 with $H^1(E) = 0$ and generated by global sections. Let $N = h^0(E) = \deg L + 2(1-g)$ and $S = \{s_1, \dots, s_N\} \subseteq H^0(E)$ be a basis. Since E is generated by global sections, we have a surjection

$$\mathcal{O}_C^{\oplus N} \to E, \quad (f_1, \dots, f_N) \mapsto f_1 s_1 + \cdots f_N s_N.$$

The pairing

$$E \times E \to \det E = L, \quad (s,t) \mapsto s \wedge t$$

induces a map

$$E \to L^{\oplus N}, \quad t \mapsto (s_1 \wedge t, \dots, s_N \wedge t),$$

which is injective since E is generated by global sections. The composition map

$$\mathcal{O}_C^{\oplus N} \to E \to L^{\oplus N}$$

is given by the $N \times N$ matrix

$$T_{E,S} = \begin{bmatrix} s_1 \\ \vdots \\ s_N \end{bmatrix} \wedge \begin{bmatrix} s_1, \dots, s_N \end{bmatrix} = \begin{bmatrix} s_i \wedge s_j \end{bmatrix}_{1 \le i,j \le N}.$$

Let $A_N(V)$ be the set of skew-symmetric $N \times N$ matrices with entries in a vector space V. Then $T_{E,S} \in A_N(H^0(L))$ and we call it the *Gieseker point* of (E, S). The different choices of the marking $S \subseteq H^0(E)$ correspond to the $GL(N)$-orbit of $T_{E,S}$ under the action

$$T \mapsto XTX^t, \quad T \in A_N(H^0(L)), X \in GL(N).$$

Moreover, E is isomorphic to the image of $T_{E,S}$, hence one can recover E from its Gieseker points. So we have proved:

Proposition 4 Sending E to the $GL(N)$-orbit of its Gieseker points gives an injection

$$\phi : \left\{ E : \begin{array}{c} \mathrm{rank}\, E = 2, \det E = L, H^1(E) = 0, \\ \text{generated by global sections} \end{array} \right\} / \text{iso.} \hookrightarrow \{GL(N)\text{-orbits in } A_N(H^0$$

Since E is a vector bundle of rank 2, the matrix $T_{E,S}$ have rank 2 over the function field $k(C)$. Denote $A_{N,2}(H^0(L)) \subseteq A_N(H^0(L))$ to be set of matrices having rank ≤ 2 over $k(C)$. $A_{N,2}(H^0(L))$ is a subvariety of $A_N(H^0(L))$ and the image of the above map ϕ lies in it.

Stability of Gieseker points

To apply GIT to construct $S(2, L)$, we need to study the stability of $A_{N,2}(H^0(L))$ under the action of $GL(N)$. This consists of two steps: for $\deg L > 4g - 2$,

$\boxed{\text{Step 1}}$ $T_{E,S}$ is semistable (stable) if and only if E is semistable (stable).

$\boxed{\text{Step 2}}$ Every semistable $T \in A_{N,2}(H^0(L))$ is the Gieseker point for some E.

Assuming these two steps, we can construct $S(2, L)$ as a GIT quotient immediately.

Theorem 1 For $\deg L > 4g - 2$, $S(2, L) \cong A^s_{N,2}(H^0(L))/GL(N)$ is a smooth quasi-projective variety.

Proof The only thing to check is that the orbits in $A^s_{N,2}(H^0(L))$ are free. Suppose $XT_{E,s}X^t = T_{E,S}$. We then have the following commutative diagram

$$\begin{tikzcd} \mathcal{O}_C^{\oplus N} \arrow[r, "T_{E,S}"] \arrow[d, "X^t"] & L^{\oplus N} \\ \mathcal{O}_C^{\oplus N} \arrow[r, "T_{E,S}"] & L^{\oplus N} \arrow[u, "X"'] \end{tikzcd}$$

This gives an endomorphism of E. Since E is stable, we know that E is simple. Hence $X = c\,\mathrm{Id}$ with $c^2 = 1$. Namely $X = \pm\,\mathrm{Id}$, which acts trivially on $A^s_{N,2}(H^0(L))$. □

For the first step, we need the following observation. This phenomenon did not appear in the case of line bundles.

Lemma 1 If $T_{E,S}$ is semistable (stable), then for any sub-line bundle $M \subseteq E$, we have $\dim(H^0(M)) \leq N/2$ ($\dim(H^0(M)) < N/2$).

Proof Let $a = \dim H^0(M)$, $b = N - a$. We may assume $s_1, \ldots, s_a$ generates $H^0(M)$ over $\mathbb{C}$. Since M is a line bundle, we know $T_{E,S}$ contains a top left $a \times a$ block consisting of only zeros. Write

$$T_{E,S} = \begin{bmatrix} 0 & B \\ -B^t & C \end{bmatrix}.$$

Consider a 1-parameter subgroup

$$g(t) = \begin{bmatrix} \operatorname{diag}(t^{-b}) & 0 \\ 0 & \operatorname{diag}(t^a) \end{bmatrix} \subseteq SL(N).$$

Then

$$g(t) T_{E,S} g(t)^t = \begin{bmatrix} 0 & t^{a-b}B \\ -t^{a-b}B^t & t^{2a}C \end{bmatrix}.$$

If $a > N/2$, then $a > b$. Letting $t \to 0$, we know $T_{E,S}$ is unstable. If $a = N/2$, then $a = b$. Letting $t \to 0$, we obtain the matrix

$$T_0 = \begin{bmatrix} 0 & B \\ -B^t & 0 \end{bmatrix}.$$

T_0 is not stable since it does not have finite stabilizer. So $T_{E,S}$ is not stable. □

Definition 2 Let E be a vector bundle. We call E H^0*-semistable* (resp. H^0*-stable*) if $\frac{h^0(M)}{\operatorname{rank}(M)} \leq \frac{h^0(E)}{rank(E)}$ (resp. d) for any subbundle $M \subseteq E$.

So we have shown:

Proposition 5 If $T_{E,S}$ is semistable (stable), then E is H^0-semistable (stable).

In fact, the converse is also true:

Proposition 6 If E is H^0-semistable (stable), then $T_{E,S}$ is semistable (stable).

We omit the proof of this fact since it is a bit long (see [1, 10.70]). The key idea is to construct $GL(N)$-semi-invariant polynomials using the Pfaffians or the radical vectors of skew symmetric matrices depending on whether N is even or odd.

The condition for H^0-semistability is already quite similar to semistability. One can use Riemann-Roch to prove the following, which finishes the proof of the first step.

Proposition 7 If $\deg E > 4g - 2$, then E is H^0-semistable (stable) if and only if it is semistable (stable).

Proof By Riemann-Roch, for any $M \subseteq E$ subbundle,

$$h^0(E) - h^1(E) = \deg E + 2(1-g), \quad h^0(M) - h^1(M) = \deg M + (1-g).$$

Since $h^1(E) = 0$, we know that

$$\frac{h^0(E)}{2} - h^0(M) + h^1(M) = \frac{\deg E}{2} - \deg M.$$

Since the left-hand-side is $\geq \frac{h^0(E)}{2} - h^0(M)$, H^0-(semi)stability implies (semi)stability. Now suppose $h^0(M) \geq h^0(E)/2$. Since $\deg E > 4g - 2$, we know that $h^0(E) > 2g$, hence $h^0(M) > g$. Riemann-Roch implies that $h^1(E) = 0$. Therefore $\frac{\deg E}{2} - \deg M \geq 0$. The converse is proved. □

Now let us come to the second step.

Lemma 2 If $T \in A_{N,2}(H^0(L))$ is semistable, then T has rank N over $\mathbb{C}$.

Proof If not, by a suitable change of basis, we may assume T has only zeros in the first row and column. Then choosing the 1-parameter subgroup $g(t) = \operatorname{diag}(t^{-N+1}, t, \ldots, t) \subseteq SL(N)$. Then $g(t)Tg(t)^t$ goes to 0 as $t \to 0$. Then T is unstable, a contradiction. □

Proposition 8 Suppose $\deg L > 4g - 2$. Every semistable $T \in A_{N,2}(H^0(L))$ is the Gieseker point for some E.

Proof Since $T \neq 0$ is skew-symmetric, it has even rank, so $\operatorname{rank} T = 2$ over $k(C)$. Let E be the image of $T : \mathcal{O}_C^{\oplus N} \to L^{\oplus N}$, then E is a vector bundle of rank 2.

Let us first show $H^1(E)=0$. Since T is semistable, we know that $h^0(E)\geq N$ by the last lemma. Suppose $H^1(E)\neq 0$, then by Serre duality, we have a nonzero morphism $E\to K_C$, which gives a map $H^0(E)\to H^0(K_C)$. Let $M=\ker(f)$, then $\dim H^0(M)\geq N-g>N/2$ as $N=\deg L+2(1-g)>2g$. So T is unstable, a contradiction.

Next we need to show that $\det E=L$. The map

$$\mathcal{O}_C^{\oplus N}\times\mathcal{O}_C^{\oplus N}\to L,\quad (u,v)\mapsto u^tTv$$

is skew-symmetric and vanishes on $\ker T$, hence induces a sheaf morphism $\det E\to L$. From

$$\deg L+2(1-g)=N\leq h^0(E)=\deg E+2(1-g),$$

we know that $\deg L\leq\deg E$. So $\det E\cong L$.

Finally we conclude E is semistable as T is the Gieseker point of E generated by global sections and $H^1(E)=0$. □

Remark 1 For general rank r, Gieseker similarly considered the action of $GL(N)$ on the space $\mathrm{Hom}(\bigwedge^r k^{\oplus N}, H^0(L))$ and deduced the stability condition.

Examples

$M(r,d)$ is smooth and if $M(r,d)$ is nonempty (when $g\geq 2$, or $g=1$, r,d coprime, or $g=0$, $r=1$), then $\dim M(r,d)=r^2(g-1)+1$. Moreover, $M(r,d)$ is a fine moduli space if and only if r,d are coprime. The dimension of $S(r,L)$ is $r^2(g-1)+1-g=(r^2-1)(g-1)$ as expected. In particular, $\dim S(2,L)=3g-3$ if it is nonempty.

Example 1 For $g=0$, $C=\mathbb{P}^1$, Grothendieck's theorem asserts that each vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles. So there is no stable vector bundle of rank 2 and $S(2,L)$ is empty.

Example 2 For $g=1$, C an elliptic curve, Atiyah's theorem asserts that for any line bundle L of odd degree, there exists a unique isomorphism class of stable vector bundles of rank 2 and $S(2,L)$ is a single point. For any line bundle L of even degree, there is no such stable bundle and $S(2,L)$ is empty.

Example 3 Let C be a curve of genus $g=2$. Its canonical bundle K_C has degree 2. Take $L=K_C^3$, then $\deg L=6\geq 4g-2$. By Riemann-Roch, $h^0(L)=5$, so L defines an embedding $C\hookrightarrow\mathbb{P}^4$, whose image is the intersection of four quartics. A stable bundle E has $h^0(E)=N=4$. An element of $A_{4,2}(K_C^3)$ is a 4×4 skew-symmetric matrix of linear forms in $x_0,\dots,x_4$ which has rank 2 on $C\subseteq\mathbb{P}^4$, hence its Pfaffian is a linear combination of the four quartics. The ring of semi-invariants of $A_4(x_0,\dots,x_4)$ has 15 generators and $A_{4,2}(x_0,\dots,x_4)//GL(4)$ is the linear span of the four quartics $\cong\mathbb{P}^3\subseteq\mathbb{P}^{14}$. Moreover, the stable moduli $S(2,L)$ is the complement $\mathbb{P}^3\setminus\mathcal{K}_4$, where $\mathcal{K}_4$ is known as the *Kummer quartic surface*.

References

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