1. Classification of finitely generated field extensions

The following problem will be the main focus of these lectures:

Problem 1.1. *Fix a field K.*

Classify all finitely generated field extensions L/K.

We will be mostly interested in the case when $K = \mathbb{C}$, which is not to say that either the case when K is not algebraically closed or the case when K has characeristic zero is not interesting.

Let n be the transcendence degree of L/K. The complexity of (1.1) increases as n increases. If $n = 0$ then this is essentially the subject of Galois theory (in this case, to make things interesting, we would relax the condition that K is algebraically closed). When $n > 0$ it is necessary to think about the whole problem in a completely different way. Recall that a rational map $\phi \colon X \dashrightarrow Y$ between two quasi-projective varieties is a morphism $f \colon U \longrightarrow Y$ defined on some open subset U of X.

Definition 1.2. *Let $X \subset \mathbb{P}^n$ be a quasi-projective variety over the field K. The **function field** $K(X)/K$ of X is the set of all rational functions $\phi \colon X \dashrightarrow \mathbb{A}^1$.*

Perhaps the most direct way to compute the function field of X is to pick any open affine subset U of X. Then $K(X)/K$ is simply the field of fractions of the coordinate ring $A(U)$.

Lemma 1.3. *Let L/K be a finitely generated field extension, where K has characteristic zero. Suppose either that K is algebraically closed or L/K is not finite.*

Then we may find a quasi-projective variety X such that the field extension $K(X)/K$ is isomorphic to L/K.

Proof. We may find n algebraically independent elements $x_1, x_2, \dots, x_n$ of L. Let $M = K(x_1, x_2, \dots, x_n)$. Then M/K is a purely transcendental extension and L/M is a finite extension. By the theorem of the primitive element (here is where we use the fact that the characteristic is zero) there is an element $x_{n+1} \in L$ such that $L = M(x_{n+1})$. Let $m(x) \in M[x]$ be the minimum polynomial of x_{n+1}. Then the coefficients of m are rational functions of $x_1, x_2, \dots, x_n$ with coefficients in K. Clearing denominators, it follows that we can find a polynomial

$$f(y_1, y_2, \dots, y_{n+1}) \in K[y_1, y_2, \dots, y_{n+1}] \qquad \text{such that} \qquad f(x_1, x_2, \dots, x_{n+1}) = 0.$$

Let $X \subset \mathbb{A}^{n+1}$ be the affine variety defined by f. Then the coordinate ring of X is isomorphic to

$$\frac{K[y_1, y_2, \dots, y_{n+1}]}{\langle f(y_1, y_2, \dots, y_{n+1}) \rangle}.$$

It follows that the function field of X is isomorphic to L. $\square$

Note that one can compose dominant rational maps, so that there is a category of quasi-projective varieties and dominant rational maps.

Definition-Lemma 1.4. *Let $\phi\colon X \dashrightarrow Y$ be a dominant rational map between two quasi-projective varieties.*

Then there is a natural ring homorphism $\phi^*\colon K(Y) \longrightarrow K(X)$, which fixes K.

Proof. If $f\colon Y \dashrightarrow \mathbb{A}^1$ is a rational function then let $\phi^*(f) = f \circ \phi\colon X \dashrightarrow \mathbb{A}^1$. The rest is clear. $\square$

Theorem 1.5. *There is an equivalence of categories between the category of quasi-projective varieties and dominant rational maps over K and the category of finitely generated field extensions of K.*

Proof. Define a functor by sending a quasi-projective variety X over K to the field extension $K(X)/K$ and sending a dominant rational map $\phi\colon X \dashrightarrow Y$ to the ring homomorphism $\phi^*\colon K(Y) \dashrightarrow K(X)$. We have to show that this functor is fully faithful (that is, the morphisms in the two categories are the same) and that this functor is essentially surjective. (1.3) says precisely that this functor is essentially surjective. To show that the functor is fully faithful it suffices to show that given a ring homomorphism $f\colon K(Y) \longrightarrow K(X)$ there is a rational map $\phi\colon X \dashrightarrow Y$ such that $f = \phi^*$. We may assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ are both affine. Suppose that coordinates on $\mathbb{A}^m$ are $y_1, y_2, \dots, y_m$. Let $q_i = f(y_i)$, so that q_i are rational functions of $x_1, x_2, \dots, x_n$, coordinates on $\mathbb{A}^n$. Define a rational map

$$\phi\colon X \dashrightarrow \mathbb{A}^m,$$

by the rule

$$(x_1, x_2, \dots, x_n) \longrightarrow (q_1, q_2, \dots, q_m).$$

It is easy to check that $\phi^* = f$. $\square$

By virtue of (1.5) we may restate (1.1) as:

Problem 1.6. *Classify quasi-projective varieties up to birational equivalence.*

Note that the transcendence degree n of $L = K(X)/K$ is precisely the dimension of X. Note also that (1.3) proves a much stronger result than stated. In fact every non-trivial field extension L/K can be realised as the function field of an affine hypersurface. However this is not the right way to look at all of this.

Lemma 1.7. *Let $\phi\colon C \dashrightarrow Y$ be a rational map, where C is a smooth curve and $Y \subset \mathbb{P}^n$ is a projective variety.*

Then ϕ extends to a morphism $\phi\colon C \longrightarrow Y$.

Proof. By assumption we are given a morphism $\phi\colon U \longrightarrow Y$ defined on a dense open subset U of C. Since $Y \subset \mathbb{P}^n$ is closed we may assume that $Y = \mathbb{P}^n$. The complement of U is a finite set of points $p_1, p_2, \dots, p_k$. Since this result is local about any one of these points, we may assume that $U = C - \{p\}$.

We give a proof of this fact which is only valid over $\mathbb{C}$. It is easy to adapt this proof to the general case using the language of DVR's. Working analytically locally, we may assume that $C = \Delta$ and p is the origin. By assumption ϕ is given locally by a collection of meromorphic functions f_i,

$$z \longrightarrow [f_0(z) : f_1(z) : \cdots : f_n(z)].$$

Now each meromorphic function $f_i(z)$ has a Laurent expansion, $\sum a_j z^j$, so that we may write

$$f_i(z) = z^{m_i} g_i(z),$$

where each $m_i \in \mathbb{Z}$ and $g_i(z)$ is a holomorphic function. Let

$$m = \min_i (m_0, m_1, \dots, m_n).$$

Then ϕ is equally well given by

$$z \longrightarrow [f_0(z) : f_1(z) : \cdots : f_n(z)] = [z^{-m} f_0(z) : z^{-m} f_1(z) : \cdots : z^{-m} f_n(z)].$$

By our choice of m each

$$z^{-m} f_i(z),$$

is holomorphic and one of them is non-zero at the origin. $\square$

Using (1.7) and (1.5) it is possible to restate (1.1) once again, in the case when the transcendence degree $n = 1$ is one. The classification of field extensions of transcendence degree one is equivalent to the classification of smooth curves. By virtue of (1.7) the issue of rational maps which are not morphisms does not play any role.

So now let us turn our attention to the study of smooth curves:

Definition 1.8. *Let C be a projective curve. The* ***arithmetic genus*** *p_a of C is the dimension $h^1(C, \mathcal{O}_C)$ of $H^1(C, \mathcal{O}_C)$. The* ***geometric genus*** *is the arithmetic genus of the normalisation.*

Without going into too many details (one can easily fill a whole course on the topic of the classification of smooth curves) unfortunately the picture is quite bad. The point is that there is a natural parameter space for the set of all curves of genus g, called the **moduli space of curves of genus** g, $\mathcal{M}_g$, which is a quasi-projective variety. $\mathcal{M}_g$ is natural in the following sense. The points of $\mathcal{M}_g$ are in bijection with the set of isomorphism classes of smooth curves of genus g. Let

$$U = \{\, [C] \in \mathcal{M}_g \,|\, C \text{ has only the trivial automorphism} \,\},$$

be the set of points $[C] \in \mathcal{M}_g$ corresponding to curves C with no automorphisms. Then U is a dense open subset of $\mathcal{M}_g$, for $g \geq 3$ and there is a universal curve over U, $\pi\colon \mathcal{C} \dashrightarrow U$, whose fibre $\pi^{-1}([C])$ over the point $[C]$ is isomorphic to the curve C. The unfortunate aspect of all of this is that U and hence $\mathcal{M}_g$ has dimension $3g-3$. A moment's thought will convince the reader that this means that the classification of smooth curves is quite hard. In fact, in a sense we will make precise later on, the general curve of genus g is essentially unknowable.

2. The canonical divisor

In this section we will introduce one of the most important invariants in the birational classification of varieties.

Definition 2.1. *Let X be a normal quasi-projective variety of dimension n.*

*A **Weil divisor** is a formal linear combination of codimension one subvarieties. The set of all divisors with integer coefficients forms a group $Z_{n-1}(X)$, the free abelian group with generators the irreducible and reduced divisors (aka the **prime divisors**). A $\mathbb{Q}$**-divisor** is a divisor with rational coefficients and an $\mathbb{R}$**-divisor** is a divisor with real coefficients.*

Divisors on smooth curves are very easy to understand. A divisor $D = \sum_{p \in C} n_p p$ on a curve is nothing more than a formal sum of points, where all but finitely many of the coefficients n_p are zero.

Definition 2.2. *Let $D = \sum_{p \in C} n_p p$ be a divisor on a smooth curve C. The **degree** of D is the sum*

$$\sum n_p.$$

There are two very natural ways to construct integral divisors:

Definition 2.3. *Let X be a normal quasi-projective variety and let $f \in K(X)$ be a rational function. We associate to f the divisor of zeroes minus the divisor of poles:*

$$\begin{aligned}(f) &= (f)_0 - (f)_\infty \\ &= \sum_{V \subset X} \operatorname{mult}_V f,\end{aligned}$$

where the sum ranges over prime divisor $V \subset X$.

*We say that two divisors D_1 and D_2 are **linearly equivalent**, denoted $D_1 \sim D_2$ if*

$$D_1 = D_2 + (f),$$

where f is a rational function. The group of integral Weil divisors (that is, those Weil divisors with integer coefficients) modulo linear equivalence is denoted $A_{n-1}(X)$.

Example 2.4. *Let $X = \mathbb{P}^n$. Then the group of integral Weil divisors modulo linear equivalence is equal to $\mathbb{Z}$. Indeed define a map*

$$\pi \colon \mathbb{Z} \longrightarrow A_{n-1}(\mathbb{P}^n),$$

by sending

$$d \longrightarrow dH,$$

where H is the hyperplane defined by X_0. We first show that ρ is surjective. Suppose that $V \subset \mathbb{P}^n$ is a divisor. Then V is a hypersurface and it is defined by a homogeneous polynomial F of degree d. Then

$$f = \frac{F}{X_0^d},$$

is a rational function so that

$$V \sim dH,$$

where H is the hyperplane defined by X_0. Thus ρ is surjective.

On the other hand, since the degree of any principal divisor is zero ρ is also injective.

The following easy result will be used so often it is useful to record it as a:

Lemma 2.5. *Let X be a normal variety and let U be an open subset whose complement has codimension at least two.*

Then every Weil divisor on X is determined by its restriction to U.

Proof. Suppose that $B = \sum a_i B_i$ is a Weil divisor on U. Let D_i be the closure of B_i. Then $D = \sum a_i D_i$ is a Weil divisor on X whose restriction to U is equal to B. Uniqueness is equally clear. $\square$

Definition-Lemma 2.6. *Let X be a normal variety. We are going to associate a divisor to X. Note that the singular locus of X has codimension at least two. Thus by (2.5) we may assume that X is smooth. Let ω be a rational n-form. Then the zeroes minus the poles of ω determine a divisor, K_X, called the* ***canonical divisor****. The canonical divisor is well-defined up to linear equivalence.*

Proof. Suppose that η is any other rational n-form, with zeroes minus poles K'_X. The key point is that the ratio $f = \omega/\eta$ is a rational function. Thus

$$K_X = K'_X + (f). \qquad \square$$

There are two reasons that the canonical divisor is so useful as an invariant. One is that it is relatively easy to compute:

Example 2.7. *We will show that $K_{\mathbb{P}^n} = -(n+1)H$. To specify a rational n-form, it suffices to start with a rational n-form on an open affine subset, and compute what it looks like on the other open charts. Consider*

$$\omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \wedge \cdots \wedge \frac{dx_n}{x_n},$$

on the affine chart U_0, given by $X_0 \neq 0$, with coordinates

$$x_i = \frac{X_i}{X_0}.$$

Then this has a pole along every hyperplane $x_i = 0$, $i > 0$. Thus the hyperplane H_i given by $X_i = 0$ occurs with multiplicity -1 for the corresponding canonical divisor. Since U_0 does not have codimension two, it remains to check that the multiplicity of H_0 is also -1. Assuming this we have

$$K_{\mathbb{P}^n} = -(H_0 + H_1 + \cdots + H_n) \sim -(n+1)H.$$

Typically any formula for computing the canonical divisor comes with a fancy name:

Theorem 2.8 (Adjunction formula)**.** *Let X be a smooth variety and let S be a smooth divisor.*

Then

$$(K_X + S)|_S = K_S.$$

Proof. The easiest way to prove this is to realise the canonical divisor as the first chern class of the cotangent bundle T_X^*. There is a short exact sequence

$$0 \longrightarrow T_S \longrightarrow T_X|_S \longrightarrow N_{S/X} \longrightarrow 0.$$

Since the first chern class is additive on exact sequences, we have

$$-K_X|_S = c_1(T_X|_S) = c_1(T_S) + c_1(N_{S/X}) = -K_S + c_1(N_{S/X}).$$

It remains to determine the normal bundle $N_{S/X}$.

Claim 2.9. $N_{S/X} = \mathcal{O}_S(S)$.

Proof of (2.9). Suppose that U_α is an open cover of X and that $S \cap U_\alpha$ is defined by f_α. On overlaps, we have

$$f_\beta = u_{\alpha\beta} f_\alpha,$$

where $u_{\alpha\beta} \in \mathcal{O}_{U_{\alpha\beta}}$ is a unit. Thus the ideal sheaf

$$\mathcal{I}_{S/X} = \mathcal{O}_S(-S),$$

is the line bundle with transition functions $u_{\alpha\beta}$.

Consider the differential form $\mathrm{d} f_\alpha$. This is a section of $T_X^*|_{U_\alpha}$, and by restriction we get a section of the conormal bundle $N_{S/X}^*$. We have

$$\mathrm{d} f_\beta = \mathrm{d}(u_{\alpha\beta} f_\alpha) = f_\alpha \,\mathrm{d} u_{\alpha\beta} + u_{\alpha\beta} \,\mathrm{d} f_\alpha.$$

Now the first term vanishes on S, due to the factor f_α. Thus $N_{S/X}^*$ is a line bundle with the same transition functions as $\mathcal{O}_S(-S)$. Thus the two line bundles $N^*_{S/X}$ and $\mathcal{O}_S(-S)$ are isomorphic. Dualising establishes the claim. $\square$

Thus
$$c_1(N_{S/X}) = S|_S,$$
and rearranging we get the adjunction formula. $\square$

One interesting feature of the adjunction formula is that it suggests that instead of working with canonical divisors we ought to work with canonical divisors plus other divisors.

Definition 2.10. *Let X be a normal variety. We say that a divisor D is* ***Cartier*** *if D is locally defined by a single equation.*

The key point of Cartier divisors is that given a morphism $\pi\colon Y \longrightarrow X$ whose image does not lie in D, then we can pullback a Cartier divisor to Y. Indeed, just pull back local defining equations. In particular suppose that we are given a curve $C \subset X$ or more generally a morphism $f\colon C \longrightarrow X$, whose image does not lie in D. Then we can define the intersection number of D and C,
$$D \underset{f}{\cdot} C = \deg f^* D.$$
More generally one can intersect a Cartier divisor with any subvariety and get a Cartier divisor on the subvariety, again provided the subvariety is not contained in the Cartier divisor. Unfortunately using this, it is all too easy to give examples of integral Weil divisors which are not Cartier:

Example 2.11. *Let $X \subset \mathbb{P}^3$ be the quadric cone, which is given locally as $X_0 = (xy - z^2) \subset \mathbb{A}^3$. Then the line $L = (x = z = 0) \subset \mathbb{A}^3$ is a Weil divisor which is not Cartier. Indeed, let us compute the self-intersection $L^2 = L \cdot L$. First note that L is linearly equivalent to the line $M = (y = z = 0) \subset \mathbb{A}^3$. Thus*
$$L^2 = M \cdot L.$$
Now note that the hyperplane $H = (Y = 0) \subset \mathbb{P}^3$ cuts out twice the line $2M$. Indeed the hyperplane is everywhere tangent to X along M. If L were Cartier then
$$2(M \cdot L) = (2M) \cdot L = H \cdot L = 1,$$
a contradiction.

Definition 2.12. *Let X be a normal variety, and let $D \subset X$ be a $\mathbb{Q}$-divisor. We say that D is* ***$\mathbb{Q}$-Cartier*** *if nD is Cartier for some integer n.*

*We say that a normal variety is $\mathbb{Q}$-**factorial** if every Weil divisor is $\mathbb{Q}$-Cartier.*

In the example above, $2L$ is Cartier. In fact it is defined by the equation $x = 0$ on the quadric. In fact the quadric cone is $\mathbb{Q}$-factorial. Indeed if D is any integral Weil divisor then $2D$ is always Cartier.

As an aside, one can always define the intersection number of a curve C with a Cartier divisor. The clumsy way to do this is to proceed as above, and deform the divisor to a linearly equivalent divisor, which does not contain the curve. A more sophisticated approach is as follows. If the image of the curve lies in the divisor, then instead of pulling the divisor back, pullback the associated line bundle and take the degree of that

$$D \underset{f}{\cdot} C = \deg f^* \mathcal{O}_X(D).$$

Definition 2.13. *We say that (X, Δ) is a **log pair** if X is a normal variety and $\Delta \geq 0$ is a $\mathbb{Q}$-divisor such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. We say that $\Delta = \sum a_i \Delta$ is a **boundary** if $a_i \in [0, 1]$.*

Part of the reason for the justification for this definition is given by:

Theorem 2.14 (Riemann-Hurwitz formula)**.** *Let $f \colon Y \longrightarrow X$ be a finite morphism between normal quasi-projective varieties. Let (X, Δ) be a log pair. Assume that $\Delta = \sum a_i \Delta_i$ contains the support of the branch locus (to achieve this, one might have to throw in components with coefficient zero).*

Then we may write

$$K_Y + \Gamma = f^*(K_X + \Delta),$$

where $\Gamma = \sum_i b_i \Gamma_i$, the sum runs over the prime components of the ramification divisor and the strict transform of the components of Δ and if $f(\Gamma_i) = \Delta_j$ then

$$b_i = r_i a_j - (r_i - 1),$$

*where r_i is a positive integer, known as the **ramification index** of f at Γ_i.*

Proof. Throwing away a subset of codimension at least two, we may assume that X and Y are smooth. Since we are asserting an equality of Weil divisors, it suffices to check that the coefficients are correct. If the dimension of Y is greater than one, to check this we can restrict to a general hyperplane S in X.

Suppose that T is the inverse image of S. Since X and Y are smooth and S and T are general, S and T are smooth (Bertini's Theorem). By adjunction we have

$$(K_X + S)|_S = K_S \qquad \text{and} \qquad (K_Y + T)|_T = K_T.$$

Repeatedly replacing Y by a hyperplane and replacing X by the inverse image of a hyperplane, we may assume that X and Y are smooth curves (equivalently, Riemann surfaces).

Let q be a point in Y with image p in X. Then the result is local about p and q. But any map between two Riemann surfaces is locally given as

$$z \longrightarrow z^k = t,$$

for some positive integer k, which is the ramification index, where $z = 0$ defines q and $t = 0$ defines p. As

$$f^* t = z^k,$$

it follows that

$$f^* p = kq.$$

On the other hand,

$$f^*(\mathrm{d}t) = \mathrm{d}z^k = kz^{k-1}\,\mathrm{d}z.$$

Thus,

$$f^* K_X + (k-1)q = K_Y.$$

locally about p and q. Putting all of this together gives the result. $\square$

There are some very interesting special cases of (2.14). For example suppose that the ramification index r only depends on the branch divisor (for example if the map f is Galois, so that $X = Y/G$, for some finite group $G \subset \operatorname{Aut}(Y)$; in this case the ramification index is simply the order of the stabiliser). In this case if

$$\Delta = \sum_i \frac{r_i - 1}{r_i} \Delta_i,$$

then

$$K_Y = f^*(K_X + \Delta).$$

For this reason, coefficients of the form $(r-1)/r$ play a central role in log geometry. A very special case of this is when the map f is unramified (aka étale) in codimension one. In this case

$$K_Y = f^* K_X.$$

Note that (unfortunately) it is often the case that Δ is a boundary and yet Γ is not. The problem is that some of the coefficients of Γ might be negative. In fact it is necessary and sufficient that the coefficient of the branch divisor is at least the coefficient $(r-1)/r$, where r is the largest ramification index lying over this divisor. Again a very special but interesting case of this is when the coefficients of Δ are all one. In this case the coefficients of Γ are also all one.

We end this section with one of the most fundamental properties of the canonical divisor:

Theorem 2.15 (Serre Duality)**.** *Let L be a line bundle on a smooth (or more generally Cohen-Macaulay) variety X of dimension n. Then there are natural isomorphisms*

$$H^i(X, L)^* \simeq H^{n-i}(X, L^*(K_X)).$$

One normally states this result by saying that there is an isomorphism $\omega_X \simeq \mathcal{O}_X(K_X)$, where ω_X is the dualising sheaf. The Cohen-Macaulay condition is just the condition that there is a dualising sheaf (that is, a sheaf which makes Serre duality work). Thus (2.15) is very strong; whenever Serre duality is true, the duality is given by the canonical divisor.

We also recall:

Theorem 2.16 (Riemann-Roch)**.** *Let C be a smooth curve of genus g and let D be an integral divisor on X of degree d. Then*

$$h^0(C, \mathcal{O}_C(D)) - h^0(C, \mathcal{O}_C(K_C - D)) = d - g + 1.$$

3. Ample and Semiample

We recall some very classical algebraic geometry. Let D be an integral Weil divisor. Provided $h^0(X, \mathcal{O}_X(D)) > 0$, D defines a rational map:

$$\phi = \phi_D \colon X \dashrightarrow Y.$$

The simplest way to define this map is as follows. Pick a basis $\sigma_0, \sigma_1, \dots, \sigma_m$ of the vector space $H^0(X, \mathcal{O}_X(D))$. Define a map

$$\phi \colon X \longrightarrow \mathbb{P}^m \qquad \text{by the rule} \qquad x \longrightarrow [\sigma_0(x) : \sigma_1(x) : \dots : \sigma_m(x)].$$

Note that to make sense of this notation one has to be a little careful. Really the sections don't take values in $\mathbb{C}$, they take values in the fibre L_x of the line bundle L associated to $\mathcal{O}_X(D)$, which is a 1-dimensional vector space (let us assume for simplicity that D is Cartier so that $\mathcal{O}_X(D)$ is locally free). One can however make local sense of this morphism by taking a local trivialisation of the line bundle $L|_U \simeq U \times \mathbb{C}$. Now on a different trivialisation one would get different values. But the two trivialisations differ by a scalar multiple and hence give the same point in $\mathbb{P}^m$.

However a much better way to proceed is as follows.

$$\mathbb{P}^m \simeq \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*).$$

Given a point $x \in X$, let

$$H_x = \{\, \sigma \in H^0(X, \mathcal{O}_X(D)) \,|\, \sigma(x) = 0 \,\}.$$

Then H_x is a hyperplane in $H^0(X, \mathcal{O}_X(D))$, whence a point of

$$\phi(x) = [H_x] \in \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*).$$

Note that ϕ is not defined everywhere. The problem in either description is that the sections of $H^0(X, \mathcal{O}_X(D))$ might all vanish at x.

Definition 3.1. *Let X be a normal variety and let D be an integral Weil divisor. The* ***complete linear system***

$$|D| = \{\, D' \,|\, D' \geq 0, D' \sim D \,\}.$$

The ***base locus*** *of $|D|$ is the intersection of all of the elements of $|D|$. We say that $|D|$ is* ***base point free*** *if the base locus is empty.*

In fact, given a section $\sigma \in H^0(X, \mathcal{O}_X(D))$ the zero locus D' of σ defines an element of $|D|$ and every element of $|D|$ arises in this fashion. Thus the base locus of $|D|$ is precisely the locus where ϕ_D is not defined. Thus $|D|$ is base point free if and only if ϕ_D is a morphism (or better everywhere defined). In this case the linear system can be recovered by pulling back the hyperplane sections of $Y \subset \mathbb{P}^{m-1}$ and in fact $\mathcal{O}_X(D) = \phi^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Definition 3.2. *We say that a $\mathbb{Q}$-divisor D on a normal variety is* ***semiample*** *if $|mD|$ is base point free for some $m \in \mathbb{N}$. We say that an integral divisor D is* ***very ample*** *if $\phi\colon X \longrightarrow \mathbb{P}^n$ defines an embedding of X. We say that D is* ***ample*** *if mD is very ample for some $m \in \mathbb{N}$.*

Theorem 3.3 (Serre-Vanishing)**.** *Let X be a normal projective variety and let D be a Cartier divisor on X.*

TFAE

1. *D is ample.*
2. *For every coherent sheaf $\mathcal{F}$ on X, there is a positive integer m such that*
$$H^i(X, \mathcal{F}(mD)) = 0,$$
for all $m \geq m_0$ and $i > 0$ (and these cohomology groups are finite dimensional vector spaces).
3. *For every coherent sheaf $\mathcal{F}$ on X, there is a positive integer m_0 such that the natural map*
$$H^0(X, \mathcal{F}(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{F}(mD),$$
is surjective, for all m divisible by m_0.

Proof. Suppose that D is ample. Then $\phi_{kD} = i\colon X \longrightarrow \mathbb{P}^n$ defines an embedding of X into projective space, for some $k \in \mathbb{N}$. Let $\mathcal{F}$ be a coherent sheaf on X. Let $\mathcal{F}_j = \mathcal{F}(jD)$, $0 \leq j \leq k-1$. Let $m \in \mathbb{N}$. Then we may write $m = m'k + j$, for some $0 \leq j \leq k-1$. In this case

$$H^i(X, \mathcal{F}(mD)) = H^i(X, \mathcal{F}_j(m'kD)) = 0,$$

Thus replacing D by kD we may assume that $D = i^*H$ is very ample. In this case

$$H^i(X, \mathcal{F}(mD)) = H^i(\mathbb{P}^n, i_*\mathcal{F}(mH)).$$

Thus replacing X by $\mathbb{P}^n$, $\mathcal{F}$ by $i_*\mathcal{F}$ and D by H we may assume that $X = \mathbb{P}^n$.

We proceed by descending induction on i. If $i > n$ the result is clear (either use Grothendieck's theorem or the fact that $\mathbb{P}^n$ is covered by $n+1$ open affines).

Now every coherent sheaf on $\mathbb{P}^n$ is a quotient of a direct sum of line bundles. Thus we get a short exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{E}$ is a direct sum of line bundles and $\mathcal{K}$ is coherent. Since we have shown that $A_{n-1}(\mathbb{P}^n) = \mathbb{Z}$ it follows that each line bundle on $\mathbb{P}^n$ is of the form $\mathcal{O}_{\mathbb{P}^n}(aH)$ for some $a \in \mathbb{Z}$ (it is customary to drop the

H). Twisting by a large multiple of H we may assume that each $a > 0$ is large. By direct computation

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a)) = 0,$$

for $i > 0$. Taking the long exact sequence associated to the short exact sequence gives

$$H^i(\mathbb{P}^n, \mathcal{F}) \simeq H^{i+1}(\mathbb{P}^n, \mathcal{K}),$$

and so (1) implies (2) by descending induction on i.

Suppose that (2) holds. Let $p \in X$ be a point of X and let $\mathbb{C}_p$ be the skyscraper sheaf supported at p, with stalk $\mathbb{C}$. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I}(mD) \longrightarrow \mathcal{F}(mD) \longrightarrow \mathcal{F}(mD) \otimes \mathbb{C}_p \longrightarrow 0,$$

where $\mathcal{I}(mD)$ is a coherent sheaf. Since by assumption

$$H^1(X, \mathcal{I}(mD)) = 0,$$

for m sufficiently large, taking the long exact sequence of cohomology, it follows that

$$H^0(X, \mathcal{F}(mD)) \longrightarrow H^0(X, \mathcal{F}(mD) \otimes \mathbb{C}_p),$$

is surjective for m sufficiently large. But then

$$H^0(X, \mathcal{F}(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{F}(mD),$$

is surjective in a neighbourhood of p by Nakayama's Lemma. As X is compact, it follows that (2) implies (3).

Suppose that (3) holds. Pick m_0 such that

$$H^0(X, \mathcal{O}_X(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(mD),$$

is surjective for all $m \geq m_0$. Since there is a surjection

$$\mathcal{O}_X(mD) \longrightarrow \mathbb{C}_p(mD),$$

it follows that there is a surjection,

$$H^0(X, \mathcal{O}_X(mD)) \otimes \mathcal{O}_X \longrightarrow \mathbb{C}_p(mD).$$

But then we may find a section $\sigma \in H^0(X, \mathcal{O}_X(mD))$ not vanishing at x. This gives an element $D' \in |mD|$ not containing x. Thus $x \notin B_m$. It follows that B_m is empty, so that mD is base point free and so D is semiample.

Let $x \neq y$ be two points of X and let $\mathcal{I}_x$ be the ideal sheaf of x. Pick a positive integer m_0 such that

$$H^0(X, \mathcal{I}_x(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{I}_x(mD),$$

is surjective for all $m \geq m_0$. As above, it follows that

$$H^0(X, \mathcal{I}_x(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{I}_x \otimes \mathbb{C}_y(mD) = \mathbb{C}_y(mD),$$

is surjective. But then we may an element of $|mD|$ passing through x but not through y. Thus $\phi_{mD}(x) \neq \phi_{mD}(y)$. The same m_0 works for all $y \neq x$. Given x and y we may find a neighbourhood of $(x, y) \in X \times X$ such that ph_{mD} is injective on this neighbourhood. By compactness of $X \times X$ we may find m_0 that works for all $x \neq y$. It follows that ϕ_{mD} is injective, that is $|mD|$ separates points.

Finally pick a length two scheme $z \subset X$ supported at x. This determines a surjection,

$$\mathcal{I}_x(mD) \longrightarrow \mathbb{C}_x(mD)$$

Composing, we get a surjection

$$H^0(X, \mathcal{I}_x(mD)) \otimes \mathcal{O}_X \longrightarrow \mathbb{C}_x(mD),$$

By a repeat of the Noetherian argument given above, if m is sufficiently large and divisible then this map is surjective for all irreducible length two schemes. On the other hand, the Zariski tangent space of any scheme is the union of the irreducible length two subschemes of X, and so the map ϕ_{mD} is an injective immersion (that is it is injective on tangent spaces). But then ϕ_{mD} is an embedding (it is a straightforward application of Nakayama's Lemma to check that the implicit function argument extends to the case of singular varieties). Thus (3) implies (1). □

More generally if X is a scheme, we will use (2) of (3.3) as the definition of ample:

Lemma 3.4. *Let $f: X \longrightarrow Y$ be a morphism of projective schemes. Let D be a $\mathbb{Q}$-Cartier divisor on Y.*

1. *If D is ample and f is finite then f^*D is ample.*
2. *If f is surjective and f^*D is ample (this can only happen if f is finite) then D is ample.*

Proof. We may suppose that D is Cartier.

Suppose that D is ample and let $\mathcal{F}$ be a coherent sheaf on X. As f is finite, we have

$$H^i(X, \mathcal{F}(mf^*D)) = H^i(Y, f_*\mathcal{F}(mD)),$$

which is zero for m sufficiently large. Thus f^*D is ample. This proves (1).

Now suppose that f is surjective and f^*D is ample. We first prove (2) in the special case when $X = Y_{\text{red}}$ and f is the natural inclusion. Suppose that f^*D is ample. Let $\mathcal{I}_X$ be the ideal sheaf of X in Y. Then $\mathcal{I}_X^k = 0$ for some $k > 0$, which we may assume to be minimal. We proceed by induction on k. If $k = 1$ then f is an isomorphism and there is nothing to prove. Otherwise $k > 1$. Let $\mathcal{F}$ be a coherent sheaf on Y. Then there is an exact sequence

$$0 \longrightarrow \mathcal{H}(mD) \longrightarrow \mathcal{F}(mD) \longrightarrow \mathcal{G}(mD) \longrightarrow 0,$$

where $\mathcal{G} = \mathcal{F} \otimes \mathcal{O}_X$ and $\mathcal{H}(mD)$ is coherent. Then for m sufficient large,

$$h^i(Y, \mathcal{F}(mD)) = h^i(Y, \mathcal{H}(mD)).$$

On the other hand, $\mathcal{H}$ is a coherent sheaf supported on the proper subscheme $Y' \subset Y$ defined by $\mathcal{I}^{k-1}$, and we are done by induction on k.

Let $X' = X_{\mathrm{red}}$ and $Y' = Y_{\mathrm{red}}$. Since taking the reduction is a functor, there is a commutative square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"'] \\ Y' \arrow[r] & Y, \end{tikzcd}$$

where $f' = f_{\mathrm{red}}$. We may pullback D to X' either way around the square and we get the same answer both ways. It follows that to prove that D is ample, it suffices to prove that $D' = D_{\mathrm{red}}$ is ample. Thus, replacing $f\colon X \longrightarrow Y$ by $f'\colon X' \longrightarrow Y'$, we may assume that X and Y are projective varieties.

Now suppose that $Y = Y_1 \cup Y_2$ is the union of two closed subvarieties, let X be their disjoint union and let $f\colon X \longrightarrow Y$ be the obvious morphism. If $\mathcal{F}$ is any coherent sheaf, then there is an exact sequence

$$0 \longrightarrow \mathcal{F}(mD) \longrightarrow f^*\mathcal{F}(mD) = \mathcal{F}_1(mD) \oplus \mathcal{F}_2(mD) \longrightarrow \mathcal{G}(mD) \longrightarrow 0,$$

where $\mathcal{F}_i = \mathcal{F} \otimes \mathcal{O}_{Y_i} = \mathcal{F}_{Y_i}$ is the restriction of $\mathcal{F}$ to Y_i and $\mathcal{G}$ is a sheaf supported on $Z = Y_1 \cap Y_2$. By (1) applied to either of the inclusions $Z \longrightarrow Y_i$, $D|_Z$ is ample. Moreover for m sufficiently large both of the natural maps,

$$H^0(Y_i, \mathcal{F}_i(mD)) \longrightarrow H^0(Z, \mathcal{G}(mD)),$$

are surjective. Taking the long exact sequence of cohomology, we see that

$$H^i(X, \mathcal{F}(mD)) = 0,$$

is zero, for $i > 0$ and m sufficiently large, since it is easy to see directly that

$$H^0(Y_1, \mathcal{F}_1(mD)) \oplus H^0(Y_2, \mathcal{F}_2(mD)) \longrightarrow H^0(Z, \mathcal{G}(mD)),$$

is surjective, for m sufficiently large.

Using the same argument and the same square as before (but where X' and Y' are now the irreducible components of X and Y) we may assume that X and Y are integral. Now the proof proceeds by induction on the dimension of X (and so of Y).

Let $\mathcal{F}$ be a sheaf on Y.

Claim 3.5. *There is a sheaf $\mathcal{G}$ on X and a map*

$$f_*\mathcal{G} \longrightarrow \bigoplus_{i=1}^{k} \mathcal{F},$$

which is an isomorphism over an open subset of Y, where k is the degree of the map f.

Proof of (3.5). Note that k is the degree of the field extension $K(X)/K(Y)$. Pick an affine subset U of X and pick sections $s_1, s_2, \dots, s_k \in \mathcal{O}_U$ which generate the field extension $K(X)/K(Y)$. Let M be the coherent sheaf generated by the sections $s_1, s_2, \dots, s_k \in K(Y)$. Then there is a morphism of sheaves

$$\bigoplus_{i=1}^{k} \mathcal{O}_Y \longrightarrow f_*\mathcal{M},$$

which is an isomorphism over an open subset of Y. Taking $\mathrm{Hom}_{\mathcal{O}_Y}$, we get a morphism of sheaves

$$\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{M}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}(\bigoplus_{i=1}^{k} \mathcal{O}_Y, \mathcal{F}) \simeq \bigoplus_{i=1}^{k} \mathcal{F}.$$

Finally observe that

$$\mathrm{Hom}(f_*\mathcal{M}, \mathcal{O}_Y) = f_*\mathcal{G},$$

for some coherent sheaf $\mathcal{G}$ on X. □

Thus there is an exact sequence

$$0 \longrightarrow \mathcal{K}(mD) \longrightarrow f_*\mathcal{G}(mD) \longrightarrow \bigoplus_{i=1}^{k} \mathcal{F}(mD) \longrightarrow \mathcal{Q}(mD) \longrightarrow 0,$$

where $\mathcal{K}$ and $\mathcal{Q}$ are defined to preserve exactness. We may break this exact sequence into two parts,

$$0 \longrightarrow \mathcal{H}(mD) \longrightarrow \bigoplus_{i=1}^{k} \mathcal{F}(mD) \longrightarrow \mathcal{Q}(mD) \longrightarrow 0$$
$$0 \longrightarrow \mathcal{K}(mD) \longrightarrow f_*\mathcal{G}(mD) \longrightarrow \mathcal{H}(mD) \longrightarrow 0,$$

where $\mathcal{H}$ is a coherent sheaf. Note that the support of the sheaves $\mathcal{K}$ and $\mathcal{Q}$ is smaller than the support of $\mathcal{F}$. But then by induction any twist of their higher cohomology must vanish, for m sufficiently large. Since the higher cohomology of $f_*\mathcal{G}(mD)$ also vanishes, as f^*D is ample, looking at the first exact sequence, we have that

$$H^i(Y, \mathcal{H}(mD)) = 0,$$

for $i > 0$ and m sufficiently large. Looking at the second exact sequence, it follows that

$$H^i(X, \mathcal{F}(mD)) = 0,$$

for all m sufficiently large. □

Lemma 3.6. *Let X be a projective variety and let D be a $\mathbb{Q}$-Cartier divisor.*

If H is an ample divisor then there is an integer m_0 such that $D+mH$ is ample for all $m \geq m_0$.

Proof. By induction on the dimension n of X. By (3.4) we may assume that X is normal. It is straightforard to check that a divisor on a curve is ample if and only if its degree is positive. In particular this result is easy for $n \leq 1$.

So suppose that $n > 1$. Replacing D and H by a multiple we may assume that D is Cartier, so that it is in particular an integral Weil divisor and we may assume that H is very ample. Let $x \neq y \in X$ be any two points of X and let $Y \in |H|$ be a general element containing x and y. Then Y is a projective variety and there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(kD+(m-1)H) \longrightarrow \mathcal{O}_X(kD+mH) \longrightarrow \mathcal{O}_Y(kD+mH) \longrightarrow 0.$$

By induction on the dimension there is an integer m_0 such that $(D+mH))|_Y$ is ample for all $m \geq m_0$. Pick a positive integer k such that $k(D+mH)|_Y$ is very ample. By Serre vanishing, possibly replacing m_0 by a larger integer, we may assume that $(kD+mH)|_Y$ is very ample and that

$$H^i(X, \mathcal{O}_X(kD+mH)) = 0,$$

for all $m \geq m_0$ and $i > 0$. In particular

$$H^0(X, \mathcal{O}_X(kD+mH)) \longrightarrow H^0(Y, \mathcal{O}_Y(kD+mH)),$$

is surjective. By assumption we may find $B \in |(kD+mH)|_Y|$ containing x but not containing y and we may lift this to $B' \in |(kD+mH)|$ containing x by not containing y. It follows that $\phi_{kD+mH} \colon X \longrightarrow \mathbb{P}^N$ is an injective morphism, whence it is a finite morphism. But then $kD+mH$ is the pullback of an ample divisor under a finite morphism and so it must be ample. □

Definition-Lemma 3.7. *Let X be a variety and let $L/K(X)$ be a finite field extension.*

We may find a normal variety Y and a finite morphism $f\colon Y \longrightarrow X$ such that $K(Y)/K(X)$ is isomorphic to $L/K(X)$.

Proof. We may cover X finitely many open affine varieties $U_i = \operatorname{Spec} A_i$. Let B_i be the integral closure of A_i in L/K. Then $V_i = \operatorname{Spec} B_i$ is an affine variety there is a finite morphism $V_i \longrightarrow U_i$ and $K(V_i)/K(U_i) \simeq L/K(X)$. The variety Y is obtained by gluing V_i together. $\square$

Definition-Theorem 3.8 (Stein factorisation)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of normal varieties. There is a factorisation of $f = h \circ g$ into a morphism $g\colon X \longrightarrow Z$ with connected fibres and a finite morphism $h\colon Z \longrightarrow Y$.*

Proof. Let $Z = \operatorname{Spec}_Y f_*\mathcal{O}_X$. Then $h\colon Z \longrightarrow Y$ is finite and there is a morphism $g\colon X \longrightarrow Z$. By construction $g_*\mathcal{O}_X = \mathcal{O}_Z$. It follows by a result of Zariski that the fibres of g are connected. $\square$

Lemma 3.9. *Let $f\colon X \longrightarrow Y$ be a morphism of projective varieties and let D be a $\mathbb{Q}$-Cartier divisor on Y.*

(1) If D is semiample then f^*D is semiample.
(2) If Y is normal, f is surjective and f^*D is semiample then D is semiample.

Proof. Suppose that D is semiample. Then $|mD|$ is base point free for some $m \in \mathbb{N}$. Pick $x \in X$. By assumption we may find $D' \in |mD|$ not containing $y = f(x)$. But then $f^*D' \in |mf^*D|$ does not contain x. Thus $|mf^*D|$ is base point free and so f^*D is semiample. This proves (1).

Now suppose that Y is normal, f is surjective and f^*D is semiample. Considering the Stein factorisation of f, we only need to prove (2) in the two special cases when f is finite and when f has connected fibres.

First assume that f is a finite morphism of normal varieties. Then we may find a morphism $g\colon W \longrightarrow X$ such that the composition $h = f \circ g\colon W \longrightarrow Y$ is Galois (take W to be the normalisation of X in the Galois closure of $K(X)/K(Y)$). By what we already proved g^*D is semiample. Replacing f by h, we may assume that f is Galois, so that $Y = X/G$ for some finite group G. Pick m so that mf^*D is base point free. Let $y \in Y$ and $x_1, x_2, \dots, x_k \in X$ be the finitely many points lying over y. Pick $\sigma \in H^0(X, \mathcal{O}_X(mf^*D))$ not vanishing at any one of the points $x_1, x_2, \dots, x_k$. Let $\sigma_1, \sigma_2, \dots, \sigma_l$ be the Galois conjugates of σ. Then each σ_i does not vanish at each x_j and $\prod_{i=1}^{l} \sigma_i$ is G-invariant, so that it is the pullback of a section $\tau \in H^0(Y, \mathcal{O}_Y(mlD))$ not vanishing at y. But then D is semiample.

Now suppose that the fibres of f are connected. Suppose that f^*D is semiample. Then $|mf^*D|$ is base point free for some $m > 0$. In this

case
$$H^0(X, \mathcal{O}_X(mf^*D)) = H^0(Y, \mathcal{O}_Y(mD)).$$
Pick $y \in Y$. Let x be a point of the fibre $X_y = f^{-1}(y)$. Then there is a divisor $D' \in |mf^*D|$ not containing x. But $D' = f^*D''$ where $D'' \in |mD|$. Since D' does not contain x, D'' does not contain y. But then D is semiample. $\square$

4. Asymptotic Riemann-Roch

Theorem 4.1 (Asymptotic Riemann-Roch)**.** *Let X be a normal projective variety and let D and E be two integral Weil divisors.*

If D is Cartier then

$$P(m) = \chi(\mathcal{O}_X(mD + E)) = \frac{D^n m^n}{n!} + \frac{D^{n-1} \cdot (K_X - 2E)m^{n-1}}{2(n-1)!} \dots,$$

is a polynomial of degree at most $n = \dim X$, where dots indicate lower order terms.

Since the case of curves is a little bit special we treat this case separately:

Lemma 4.2. *Let C be a smooth curve of genus g and let D be a divisor of degree d.*

Then

$$\chi(\mathcal{O}_C(D)) = d - g + 1.$$

Proof. Let E be any divisor of degree e and let p be any point. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_C(-p) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

Here $\mathcal{O}_p$ is a skyscraper sheaf, supported at the single point p. Twisting by the divisor $E + p$ we have

$$0 \longrightarrow \mathcal{O}_C(E) \longrightarrow \mathcal{O}_C(E + p) \longrightarrow \mathcal{O}_p(E) \longrightarrow 0.$$

Taking the long exact sequence associated to the short exact sequence and using the additivity of the Euler characteristic we have:

$$\chi(\mathcal{O}_C(E + p)) = \chi(\mathcal{O}_C(E)) + \chi(\mathcal{O}_p) = \chi(\mathcal{O}_C(E)) + 1,$$

where we used the fact that $h^1(C, \mathcal{O}_p) = 0$. Since the formula on the RHS of Riemann-Roch is linear it follows that the Riemann-Roch formula holds for E if and only if the Riemann-Roch formula holds for $E + p$.

Any divisor is the difference of two effective divisors $D = D_1 - D_2$, $D_i \geq 0$. If p is a point of the support of D_2 then it suffices to prove the formula for $D + p$. By induction on the degree of D_2 we reduce to the case $D = D_1 \geq 0$. If p is a point of the support of D it suffices to prove the result for $D - p$. By induction on the degree of D it suffices to prove the result when the degree is zero. But then $D = 0$ so that

$$\chi(\mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C) = 1 - g. \qquad \square$$

Lemma 4.3. *Let X be a normal variety and let H be a very ample divisor.*

If $Y \in |H|$ is general then Y is normal.

Proof. X is normal if and only if it is regular in codimension one and S_2. Y is smooth in codimension one by Bertini. As X is S_2 the set of points where X is not Cohen-Macaulay is of codimension three or more. As Y does not contain any of the generic points of this set, Y is S_2. $\square$

Proof of (4.1). By induction on the dimension n of X. Suppose that $n = 1$. Then X is a smooth curve. Riemann-Roch for $mD + E$ then reads

$$\chi(\mathcal{O}_X(mD + E)) = md + e - g + 1 = am - b,$$

where

$$a = d = \frac{\deg D}{1!} \qquad \text{and} \qquad b = g - 1 - e = \frac{\deg(K_X - 2E)}{2 \cdot 1!}.$$

Now suppose that $n > 1$. Pick a very ample divisor H, which is a general element of the linear system $|H|$, such that $H + D$ is very ample and let $G \in |D + H|$ be a general element. Then G and H are normal projective varieties and there are two exact sequences

$$0 \longrightarrow \mathcal{O}_X(mD+E) \longrightarrow \mathcal{O}_X(mD+E+H) \longrightarrow \mathcal{O}_H(mD+E+H) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}_X((m-1)D+E) \longrightarrow \mathcal{O}_X(mD+E+H) \longrightarrow \mathcal{O}_G(mD+E+H) \longrightarrow 0.$$

Hence

$$\begin{aligned} \chi(X, \mathcal{O}_X(mD + E)) - \chi(X, \mathcal{O}_X(mD + E + H)) &= -\chi(H, \mathcal{O}_H(mD + E + H)) \\ \chi(X, \mathcal{O}_X((m-1)D + E)) - \chi(X, \mathcal{O}_X(mD + E + H)) &= -\chi(G, \mathcal{O}_G(mD + E + H)), \end{aligned}$$

and taking the difference we get

$$\begin{aligned} P(m) - P(m-1) &= \chi(G, \mathcal{O}_G(mD + E + H)) - \chi(H, \mathcal{O}_H(mD + E + H)) \\ &= \frac{(D^{n-1} \cdot G - D^{n-1} \cdot H)m^{n-1}}{(n-1)!} + \dots \\ &= \frac{D^n m^{n-1}}{(n-1)!} + \dots, \end{aligned}$$

is a polynomial of degree $n-1$, by induction on the dimension. The result follows by standard results on the difference polynomial $\Delta P(m) = P(m+1) - P(m)$. $\square$

It is fun to use similar arguments to prove special cases of Riemann-Roch.

Theorem 4.4 (Riemann-Roch)**.** *Let C be a smooth curve of genus g and let D be an integral divisor on X of degree d. Then*

$$h^0(C, \mathcal{O}_C(D)) = d - g + 1 + h^0(C, \mathcal{O}_C(K_C - D)).$$

Proof. Follows from Serre duality and (4.1). □

Theorem 4.5 (Riemann-Roch for surfaces)**.** *Let S be a smooth projective surface of irregularity q and geometric genus p_g over an algebraically closed field of characteristic zero. Let D be a divisor on S.*

$$\chi(S, \mathcal{O}_S(D)) = \frac{D^2}{2} - \frac{K_S \cdot D}{2} + 1 - q + p_g.$$

Proof. Pick a very ample divisor H such that $H+D$ is very ample. Let C and Σ be general elements of $|H|$ and $|H+D|$. Then C and Σ are smooth. There are two exact sequences

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D+H) \longrightarrow \mathcal{O}_C(D+H) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(D+H) \longrightarrow \mathcal{O}_\Sigma(D+H) \longrightarrow 0.$$

As the Euler characteristic is additive we have

$$\begin{aligned}\chi(S, \mathcal{O}_S(D+H)) &= \chi(S, \mathcal{O}_S(D)) + \chi(C, \mathcal{O}_C(D+H)) \\ \chi(S, \mathcal{O}_S(D+H)) &= \chi(S, \mathcal{O}_S) + \chi(\Sigma, \mathcal{O}_\Sigma(D+H)).\end{aligned}$$

Subtracting we get

$$\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) = \chi(\Sigma, \mathcal{O}_\Sigma(D+H)) - \chi(C, \mathcal{O}_C(D+H)).$$

Now

$$\begin{aligned}\chi(\Sigma, \mathcal{O}_\Sigma(D+H)) &= (D+H) \cdot \Sigma - \deg K_\Sigma/2 \\ \chi(C, \mathcal{O}_C(D+H)) &= (D+H) \cdot C - \deg K_C/2,\end{aligned}$$

applying Riemann-Roch for curves to both C and Σ. We have

$$(D+H) \cdot \Sigma = (D+H) \cdot H + (D+H) \cdot D,$$

and by adjunction

$$K_\Sigma = (K_S + \Sigma) \cdot \Sigma \qquad \text{and} \qquad K_C = (K_S + C) \cdot C.$$

So putting all of this together we get

$$\begin{aligned}\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) &= (D+H) \cdot D + \frac{1}{2}((K_S + C) \cdot C - (K_S + \Sigma) \cdot \Sigma) \\ &= (D+H) \cdot D + \frac{1}{2} K_S \cdot (C - \Sigma) + \frac{1}{2}(H \cdot H - (H+D) \cdot (H+D)) \\ &= \frac{D \cdot D}{2} - \frac{1}{2} K_S \cdot D.\end{aligned}$$

We have

$$c = \chi(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = 1 - q + p_g.$$

Here we used the highly non-trivial fact that

$$h^1(S, \mathcal{O}_S) = h^0(S, \Omega^1_S) = q,$$

from Hodge theory and Serre duality

$$h^2(S, \mathcal{O}_S) = h^0(S, \omega_S) = p_g. \qquad \square$$

Remark 4.6. *One can turn Riemann-Roch for surfaces around and use the arguments in the proof of* (4.5) *to prove basic properties of the intersection number.*

5. Effective results

It is natural to ask for effective versions of very ampleness and vanishing. Such results exist if we work with particular divisors:

Theorem 5.1 (Kodaira vanishing). *Let X be a smooth projective variety and let D be an ample divisor. Then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0,$$

for $i > 0$.

Kodaira vanishing is one of the most fundamental and important results in higher dimensional geometry. To prove it one needs to use some analytic methods (some Hodge theory and positivity of various metrics) or some very deep results of Deligne and Illusie on Hodge theory in characteristic p. In fact Kodaira vanishing fails in general in characteristic p.

Using Kodaira vanishing one can prove a technically more powerful result, Kawamata-Viehweg vanishing, whose proof is purely algebraic and whose form dictates most of the definitions and shape of the whole subject of higher dimensional geometry.

Conjecture 5.2 (Fujita's conjecture). *Let X be a smooth projective variety of dimension n and let D be an ample divisor.*

Then

(1) $K_X + (n+1)D$ is base point free.
(2) $K_X + (n+2)D$ is very ample.

It is easy to give examples which show that Fujita's conjecture is sharp:

Example 5.3. *Let $X = \mathbb{P}^n$ and $D = H$. Then*

$$K_X + dD \sim (d - n - 1)H.$$

Thus $K_X + dH$ is base point free if and only if $d \geq n+1$ and it is ample if and only if $d \geq n+2$.

Example 5.4. *Let C be a smooth curve and let $D = p$. Then $K_C + p$ is never base point free.*

Indeed I claim that p is a base point of $|K_C + p|$. To see this consider

$$|K_C| \longrightarrow |K_C + p| \qquad \text{given by} \qquad D \longrightarrow D + p.$$

This map is linear and injective. By Riemann-Roch

$$\dim |K_C| = (2g - 2 - g + 1) - 1 + h^1(C, \mathcal{O}_C(K_C)) = g - 2 + h^0(C, \mathcal{O}_C) = g - 1$$
$$\dim |K_C + p| = (2g - 1 - g + 1) - 1 + h^1(C, \mathcal{O}_C(K_C + p)) = g - 1 + h^0(C, \mathcal{O}_{C(-p)}) = g - 1.$$

Thus the map is surjective and

$$|K_C + p| = |K_C| + p.$$

But then p is a base point of the linear system $|K_C + p|$. On the other hand, similar calculations show that $|K_C+p+q|$ is a free linear system, but that

Example 5.5. *ϕ_{K_C+p+q} is not injective if $p \neq q$ and not an embedding if $p = q$, so that $K_C + p + q$ is never very ample.*

Here is a baby version of (5.2):

Lemma 5.6. *Let X be a smooth projective variety.*

If D is a very ample divisor then $K_X + (n+1)D$ is free, and $K_X + (n+2)D$ is very ample.

Proof. We first show (1). We proceed by induction on the dimension of X. Let $x \in X$ be a point of X and let $Y \in |D|$ be a general element containing $x \in X$. Then Y is smooth and

$$(K_X + (n+1)D)|_Y \sim (K_X + Y + nD)|_Y = K_Y + nD|_Y.$$

By induction $K_Y + nD|_Y$ is free. Thus we may find $D' \in |K_Y + nD|_Y|$ not containing x. There is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X+nD) \longrightarrow \mathcal{O}_X(K_X+(n+1)D) \longrightarrow \mathcal{O}_Y(K_Y+nD) \longrightarrow 0.$$

The obstructions to surjectivity of the lifting map

$$H^0(X, \mathcal{O}_X(K_X + (n+1)D) \longrightarrow H^0(Y, \mathcal{O}_Y(K_Y + nD)),$$

live in

$$H^1(X, \mathcal{O}_X(K_X + nD)) = 0,$$

which vanishes by Kodaira vanishing. Thus D' lifts to an element D'' of $|K_X + (n+1)D|$ not containing x. Hence $|K_X + (n+1)D|$ is free.

It is not hard to check that (free) + (very ample) is very ample. Thus $K_X + (n+2)D$ is very ample. □

(5.2) is known in dimension two (Reider's Theorem) and freeness is known in dimensions three (Ein-Lazarsfeld) and four (Kawamata). The best general results are due to Anghern and Siu:

Theorem 5.7. *Let X be a smooth projective variety and let D be an ample divisor.*

(1) $K_X + mD$ is free for $m > \binom{n+1}{2}$, and
(2) $K_X + mD$ separates points, for $m \geq \binom{n+2}{2}$.

Even to show that $K_X + 5D$ is very ample on a smooth threefold X seems very hard at the moment.

6. Ampleness criteria

We return to the problem of determining when a line bundle is ample.

Theorem 6.1 (Nakai-Moishezon)**.** *Let X be a normal projective variety and let D be a $\mathbb{Q}$-Cartier divisor.*

TFAE

(1) D is ample.
(2) For every subvariety $V \subset X$ of dimension k,

$$D^k \cdot V > 0.$$

Proof. Suppose that D is ample. Then mD is very ample for some $m > 0$. Let $\phi\colon X \longrightarrow \mathbb{P}^N$ be the corresponding embedding. Then $mD = \phi^* H$, where H is a hyperplane in $\mathbb{P}^N$. Then

$$D^k \cdot V = \frac{1}{m^k} H^k \cdot \phi(V) > 0,$$

since intersecting $\phi(V)$ with H^k corresponds to intersecting V with a linear space of dimension $N - k$. But this is nothing more than the degree of $\phi(V)$ in projective space.

Now suppose that D satisfies (2). Let H be a general element of a very ample linear system. Then we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(pD+(q-1)H) \longrightarrow \mathcal{O}_X(pD+qH) \longrightarrow \mathcal{O}_H(pD+qH) \longrightarrow 0.$$

By induction, $D|_H$ is ample. It is straightforward to prove that

$$h^i(H, \mathcal{O}_H(pD + qH)) = 0,$$

for $i > 0$, p sufficiently large and any $q > 0$, by induction on the dimension. In particular,

$$h^i(X, \mathcal{O}_X(pD + (q-1)H)) = h^i(X, \mathcal{O}_X(pD + qH)),$$

for $i > 1$, p sufficiently large and any $q \geq 1$. By Serre vanishing the last group vanishes for q sufficiently large. Thus by descending induction

$$h^i(X, \mathcal{O}_X(pD + qH)) = 0,$$

for all $q \geq 0$. Thus by (4.1) it follows that

$$h^0(X, \mathcal{O}_X(mD)) \neq 0,$$

for m sufficiently large, that is, $|mD|$ is non-empty. As usual, this means that we may assume that $D \geq 0$ is Cartier. Let $\nu\colon \tilde{D} \longrightarrow D_{\mathrm{red}}$ be the normalisation of D_{red}, the reduced subscheme associated to D. Then $\nu^* D|_{D_{\mathrm{red}}}$ is ample by induction. It follows by (3.4) that $D|_D$ is ample.

I claim that the map

$$\rho_m \colon H^0(X, \mathcal{O}_X(mD)) \longrightarrow H^0(D, \mathcal{O}_D(mD)),$$

is surjective for m sufficiently large. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X((m-1)D) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_D(mD) \longrightarrow 0.$$

As $D|_D$ is ample,

$$h^i(D, \mathcal{O}_D(mD)) = 0,$$

for $i > 0$ and m sufficiently large, by Serre vanishing. Thus

$$h^1(X, \mathcal{O}_X(mD)) \leq h^1(X, \mathcal{O}_X((m-1)D)) \qquad \text{and} \qquad h^i(X, \mathcal{O}_X(mD)) = 0,$$

for $i > 1$, with equality if and only if ρ_m is surjective. Since

$$h^1(X, \mathcal{O}_X(mD)),$$

is finite dimensional, its dimension cannot drop infinitely often, and so ρ_m is surjective as claimed.

As $D|_D$ is ample, $(mD)|_D$ is very ample. As we can lift sections, it follows that $|mD|$ is base point free, that is, D is semiample. Let $\phi = \phi_{mD} \colon X \longrightarrow \mathbb{P}^N$ be the corresponding morphism. Then $D = \phi^* H$. Suppose that C is a curve contracted by ϕ. Then

$$D \cdot C = \phi^* H \cdot C = H \cdot \phi_* C = 0,$$

a contradiction. But then ϕ_{mD} is a finite morphism and $D = \phi^* H$ is ample by (3.4). $\square$

Definition 6.2. *Let X be a normal projective variety and let D be a $\mathbb{Q}$-Cartier divisor. We say that D is **nef** if $D \cdot C \geq 0$ for all curves $C \subset X$.*

Lemma 6.3. *Let X be a normal variety and let D be a $\mathbb{Q}$-Cartier divisor.*

If D is semiample then D is nef.

Proof. By assumption there is a morphism $\phi \colon X \longrightarrow Y \subset \mathbb{P}^n$ such that

$$mD = \phi^* H.$$

But then

$$D \cdot C = \frac{1}{m} \phi^* C \cdot H \geq 0.$$

$\square$

Lemma 6.4. *Let X be a normal projective variety and let D be a $\mathbb{Q}$-Cartier divisor.*

TFAE

(1) $D + H$ is ample for any ample $\mathbb{Q}$-divisor H.

(2) If $V \subset X$ is any subvariety of X then

$$D^k \cdot V \geq 0,$$

where V has dimension k.

(3) D is nef.

Proof. (1) implies (2) and (2) implies (3) are clear. (2) implies (1) follows from Nakai-Moishezon.

Suppose that D is nef. By induction on $n = \dim X$ it suffices to prove that

$$D^n \geq 0.$$

Pick an ample divisor H. Then $D + tH$ is nef for all $t \geq 0$. We have

$$f(t) = (D + tH)^n = \sum \binom{n}{i} D^i H^{n-i} t^{n-i},$$

is a polynomial in t, all of whose terms are non-negative, except maybe the constant term, which tends to infinity as t tends to infinity. Suppose that $D^n \leq 0$. Then there is a real number $t_0 \in [0, \infty)$ such that

$$f(t_0) = 0,$$

and $f(t) > 0$ for all $t > t_0$. Pick $t > t_0$ rational. Then $D + tH$ is ample, by Nakai's criteria. In particular we may find a divisor $B \in |k(D+tH)|$ for some positive integer k. We may write

$$f(t) = t^n H^n + \sum \binom{n-1}{i} t^i H^i (H + tD)^{n-i-1} D.$$

Consider the product $H^i (H + tD)^{n-i-1}$. Pick k such that kH is very ample and pick l such that $l(H + tD)$ is very ample. Pick general elements $H_1, H_2, \dots, H_i \in |kH|$ and $G_1, G_2, \dots, G_{n-i-1} \in |l(H + tD)|$. Then the intersection

$$C = H_1 \cdot H_2 \cdots H_i \cdot G_1 \cdot G_2 \cdots G_{n-i-1} \equiv \frac{1}{k^i l^{n-i-1}} H^i \cdot (H + tD)^{n-i-1},$$

is a smooth curve (here $\equiv$ denotes numerical equivalence, meaning that both sides dot with any Cartier divisor the same). Thus every term is non-negative, as $D \cdot C \geq 0$. But then

$$0 = f(t_0) = \lim_{t \to t_0} f(t) \geq t_0^n H^n.$$

Thus $t_0 = 0$, and $D^n \geq 0$. $\square$

Lemma 6.5. *Let X be a normal projective variety and let $\pi \colon Y \longrightarrow X$ blow up a smooth point p of X.*

Then $E^n = (-1)^{n-1}$.

Proof. Since this result is local in the analytic topology, we may as well assume that $X = \mathbb{P}^n$. Choose coordinatees $x_1, x_2, \dots, x_n$ about the point p. Then coordinates on $Y \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ are given by the equations

$$x_i Y_j = x_j Y_i.$$

(These equations simply express the fact that $(x_1, x_2, \dots, x_n)$ defines the point $[Y_1 : Y_2 : \dots : Y_n] \in \mathbb{P}^{n-1}$.). On the coordinate chart $Y_n \neq 0$, we have affine coordinates $y_i = Y_i/Y_n$ on $\mathbb{P}^{n-1}$ and since

$$x_i = x_n y_i,$$

it follows that x_n, $y_1, y_2, \dots, y_{n-1}$ are coordinates on Y, and the exceptional divisor is given locally by $x_n = 0$. Let H be the class of a hyperplane in $\mathbb{P}^n$ which passes through p. Then we may assume that H is given by $x_1 = 0$. Since $x_1 = x_n y_1$ it follows that

$$\pi^* H = G + E,$$

where G defined by $y_1 = 0$, is the strict transform of H. Now $G|_E$ restricts to a hyperplane in E. Thus

$$E|_E = -G|_E,$$

since E pushes forward to zero. But then

$$E^n = (E|_E)^{n-1} = (-1)^{n-1}. \qquad \square$$

Definition-Lemma 6.6 (Kodaira's Lemma)**.** *Let X be a normal projective variety of dimension n and let D be a $\mathbb{Q}$-Cartier divisor.*

TFAE

(1) $h^0(X, \mathcal{O}_X(mD)) > \alpha m^n$, for some constant $\alpha > 0$, for any m which is sufficiently divisible.

(2) $D \sim_{\mathbb{Q}} A + E$, where A is an ample divisor and $E \geq 0$.

If further D is nef then these conditions are equivalent to

(3) $D^n > 0$.

*If any of these conditions hold we say that D is **big**.*

Proof. Let $H \geq 0$ be any ample Cartier divisor. If m is sufficiently large, then

$$h^i(X, \mathcal{O}_X(mH)) = 0,$$

so that by Asymptotic Riemann Roch there are positive constants α_i such that

$$\alpha_1 m^n < h^0(X, \mathcal{O}_X(mH)) < \alpha_2 m^n,$$

for all m. Now let G be any divisor. Pick $k > 0$ such that $G + kH$ is ample. Then

$$h^0(X, \mathcal{O}_X(mG)) \leq h^0(X, \mathcal{O}_X(mG + mkH)) \leq \beta_1 m^n,$$

for some constant β_1.

Suppose that (1) holds. Let H be an ample Cartier divisor. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(mD - H) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_H(mD) \longrightarrow 0.$$

Now

$$h^0(H, \mathcal{O}_H(mD)) \leq \beta m^{n-1},$$

for some constant β. It follows that

$$h^0(X, \mathcal{O}_X(mD - H)) > \alpha m^n.$$

In particular we may find B such that

$$B \in |mD - H|.$$

But then

$$D \sim_{\mathbb{Q}} H/m + B/m = A + E.$$

Thus (1) implies (2).

How suppose that (2) holds. Replacing D by a multiple, we may assume that $D \sim A + E$. But then

$$h^0(X, \mathcal{O}_X(mD)) \geq h^0(X, \mathcal{O}_X(mA)) > \alpha m^n,$$

for some constant $\alpha > 0$. Thus (2) implies (1).

Now suppose that D is nef. Assume that (2) holds. We may assume that A is very ample and a general element of $|A|$. Then

$$D^n = A \cdot D^{n-1} + E \cdot D^{n-1} \geq (D|_A)^{n-1} > 0,$$

by induction on the dimension. Thus (2) implies (3).

Finally suppose that (3) holds. Let $\pi \colon Y \longrightarrow X$ be a birational morphism such that Y is smooth. Since $G = \pi^* D$ is nef, $G^n = D^n$ and

$$h^0(Y, \mathcal{O}_Y(mG)) = h^0(X, \mathcal{O}_X(mD)),$$

replacing X by Y and D by G, we may assume that X is smooth. Pick a very ample divisor H, a general element of $|H|$, such that $H + K_X$ is also very ample and let $G \in |H + K_X|$ be a general element. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_X(mD + G) \longrightarrow \mathcal{O}_G(mD + G) \longrightarrow 0.$$

Now

$$h^i(X, \mathcal{O}_X(mD + G)) = h^i(X, \mathcal{O}_X(K_X + H + mD)) = 0$$

for $i > 0$ and $m \geq 0$ by Kodaira vanishing, as $H + mD$ is ample. Thus

$$\chi(X, \mathcal{O}_X(mD + G)) > \alpha m^n,$$

for some constant $\alpha > 0$. Since

$$h^0(G, \mathcal{O}_G(mD + G)) < \beta m^{n-1},$$

for some constant β, (3) implies (1). $\square$

Theorem 6.7 (Seshadri's criteria)**.** *Let X be a normal projective variety and let D be a $\mathbb{Q}$-divisor.*

TFAE

(1) D is ample.
(2) For every point $x \in X$, there is a positive constant $\epsilon = \epsilon(x) > 0$ such that for every curve C,

$$D \cdot C > \epsilon \operatorname{mult}_x C,$$

where $\operatorname{mult}_x C$ is the multiplicity of the point x on C.

Proof. Suppose that D is ample. Then mD is very ample for some positive integer m. Let C be a curve with a point x of multiplicity k. Pick y any other point of C. Then we may find $H \in |mD|$ containing x and not containing y. In this case

$$(mD) \cdot C = H \cdot C \geq k,$$

so that $\epsilon = 1/m$ will do. Thus (1) implies (2).

Now assume that (2) holds. We check the hypotheses for Nakai's criteria. By induction on the dimension n of X it suffices to check that $D^n > 0$. Let $\pi \colon Y \longrightarrow X$ be the blow up of X at x, a smooth point of X, with exceptional divisor E. Consider $\pi^*D - \eta E$, for any $0 < \eta < \epsilon$. Let $\Sigma \subset Y$ be any curve on Y. If Σ is contained in E, then

$$(\pi^*D - \eta E) \cdot \Sigma = -E \cdot \Sigma > 0.$$

Otherwise let C be the image of Σ. If the multiplicity of C at x is m, then $E \cdot \Sigma = m$. Thus

$$(\pi^*D - \eta E) \cdot \Sigma = D \cdot C - \eta m > 0,$$

by definition of ϵ. It follows that $\pi^*D - \eta E$ is nef and so $\pi^*D - \epsilon E$ is nef. By (6.4) it follows that the polynomial

$$f(t) = (\pi^*D - tE)^n,$$

of degree n in t is non-negative. On the other hand, note that $E^n = \pm 1 \neq 0$. Thus the polynomial $f(t)$ is not constant. Thus $f(\eta) > 0$, some $0 < \eta < \epsilon$. It follows that

$$h^0(X, \mathcal{O}_X(mD)) \geq h^0(Y, \mathcal{O}_X(m\pi^*D - m\eta E)) > 0,$$

for m sufficiently large and divisible. It follows easily that $D^n > 0$. $\square$

One of the most interesting aspects of Seshadri's criteria is that gives a local measure of ampleness:

Definition 6.8. *Let X be a normal variety, and let D be a nef $\mathbb{Q}$-Cartier divisor. Given a point $x \in X$, let $\pi\colon Y \longrightarrow X$ be the blow up of X at x. The real number*

$$\epsilon(D, x) = \inf\{\, \epsilon \,|\, \pi^* D - \epsilon E \text{ is nef}\}$$

is called the ***Seshadri constant*** *of D at x.*

It seems to be next to impossible to calculate the Seshadri contant in any interesting cases. For example there is no known example of a smooth surface S and a point $x \in S$ such that the Seshadri constant is irrational, although this is conjectured to happen nearly all the time. One of the first interesting cases is a very general smooth quintic surface S in $\mathbb{P}^3$ (so that S belongs to the complement of a countable union of closed subsets of the space of all quintics $\mathbb{P}^{55}$). Suppose that $p \in S$ is a very general point . Let $\pi\colon T \longrightarrow S$ blow up the point p. As S is very general,

$$\operatorname{Pic}(T) = \mathbb{Z}[\pi^* H] \oplus \mathbb{Z}[E],$$

where H is the class of a hyperplane and E is the exceptional divisor. Since p is very general, it seems reasonable to expect that the only curve of negative self-intersection on T is E. If this is the case then $\pi^* H - aE$ is nef if and only if its self-intersection is non-negative. Now

$$0 = (\pi^* H - aE)^2 = H^2 - a^2 = 5 - a^2.$$

So if there are no curves of negative self-intersection other than E, then the Seshadri constant is $\sqrt{5}$.

7. Closed cone of curves

Finally let us turn to one of the most interesting ampleness criteria.

Definition 7.1. *Let X be a smooth projective variety. The* ***cone of curves*** *of X, denoted $\mathrm{NE}(X)$, is the cone spanned by the classes $[C] \in H_2(X,\mathbb{R})$ inside the vector space $H_2(X,\mathbb{R})$. The* ***closed cone of curves*** *of X, denoted $\overline{\mathrm{NE}}(X)$, is the closure of $\mathrm{NE}(X)$ inside $H_2(X,\mathbb{R})$.*

More generally, if X is normal but not necessarily smooth, one can define the cone of curves as the cone in the real vector space of curves modulo numerical equivalence.

The significance of the closed cone of curves is given by:

Theorem 7.2 (Kleiman's criterion)**.** *Let X be a normal projective variety.*

TFAE:

(1) *D is ample.*
(2) *D defines a positive linear functional on*

$$\overline{\mathrm{NE}}(X) - \{0\} \longrightarrow \mathbb{R}^+$$

defined by extending the map $[C] \longrightarrow H \cdot C$ linearly.

In particular $\overline{\mathrm{NE}}(X)$ does not contain a line and if H is ample then the set

$$\{\, \alpha \in \overline{\mathrm{NE}}(X) \,|\, H \cdot C \leq k \,\},$$

is compact, where k is any positive constant.

Proof. Suppose that H is ample. Then H is certainly positive on $\mathrm{NE}(X)$. Suppose that it is not positive on $\overline{\mathrm{NE}}(X)$. Then there is a non-zero class $\alpha \in \overline{\mathrm{NE}}(X)$ such that $H \cdot \alpha = 0$, which is a limit of classes α_i, where $\alpha_i = \sum a_{ij}[C_{ij}]$, $a_{ij} \geq 0$ are positive real numbers. Pick a $\mathbb{Q}$-Cartier divisor M such that $M \cdot \alpha < 0$. Then there is a positive integer m such that $mH + M$ is ample. But then

$$0 > (mH + M) \cdot \alpha = \lim_i (mH + M) \cdot \alpha_i \geq 0,$$

a contradiction. Thus (1) implies (2).

Suppose that $\overline{\mathrm{NE}}(X)$ contains a line. Then it contains a line through the origin, which is impossible, since X contains an ample divisor H, which is positive on this line outside the origin. Pick a basis $M_1, M_2, \dots, M_k$ for the the space of Cartier divisors modulo numerical equivalence, which is the vector space dual to the space spanned by $\overline{\mathrm{NE}}(X)$. Then we may pick m such that $mH \pm M_i$ is ample for all $1 \leq i \leq k$. In particular

$$|M_i \cdot \alpha| \leq mH \cdot \alpha.$$

Thus the set above is compact as it is a closed subset of a big cube.

Now suppose that (2) holds. By what we just proved, the set of $\alpha \in \overline{\mathrm{NE}}(X)$ such that $H \cdot \alpha = 1$ forms a compact slice of $\overline{\mathrm{NE}}(X)$ (meaning that this set is compact and $\overline{\mathrm{NE}}(X)$ is the cone over this convex set). Thus $B = D - \epsilon H$ is nef, for some $\epsilon > 0$. But then

$$D = (D - \epsilon H) + \epsilon H = B + \epsilon H,$$

is ample, by (6.4) and Nakai's criterion. □

Mori realised that Kleiman's criteria presents a straightforward way to classify projective varieties.

Definition 7.3. *Let $f\colon X \longrightarrow Y$ be a morphism of varieties. We say that f is a* ***contraction*** *morphism if $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

A contraction morphism always has connected fibres; if Y is normal and f has connected fibres then f is a contraction morphism. Consider the category of projective varieties and contraction morphisms. Any morphism $f\colon X \longrightarrow Y$ is determined by the curves contracted by f. Indeed let $\sim$ be the equivalence relation on the points of X, generated by declaring two points to be equivalent if they both belong to an irreducible curve contracted by f. Thus $x \sim y$ if and only if x and y belong to a connected curve C which is contracted to a point.

I claim that $x \sim y$ determines f. First observe that $x \sim y$ if and only if $f(x) = f(y)$. It is clear that if $x \sim y$ then $f(x) = f(y)$. Conversely suppose that $f(x) = f(y) = p$. If $d = \dim f^{-1}(p) \leq 1$ there is nothing to prove. If $d > 1$ then since the fibres of f are connected, we may assume that x and y belong to the same irreducible component. Pick an ample divisor $H \subset f^{-1}(p)$ containing both x and y. Then $\dim H = d - 1$. Repeating this operation we find a curve containing by x and y. This determines $Y = X/\sim$ as a topological space, and the condition $\mathcal{O}_Y = f_*\mathcal{O}_X$ determines the scheme structure.

Note also that there is then a partial correspondence

(1) faces of the Mori cone $\overline{\mathrm{NE}}(X)$.
(2) contraction morphisms $\phi\colon X \longrightarrow Y$.

Given a contraction morphism ϕ, let

$$F = \{\, \alpha \in \overline{\mathrm{NE}}(X) \,|\, D \cdot \alpha = 0 \,\},$$

where $D = \phi^*H$. Kleimans' criteria then says that F is a face. The problem is that given a face F of the closed cone it is in general impossible to contract F. For a start F must be rational (that is, spanned by integral classes).

Equivalently and dually, the problem is that there are nef divisors which are not semiample.

8. Some examples

We present some examples of varieties, mainly surfaces, with interesting Mori cones.

Definition 8.1. *Let $\mathcal{C} \subset V \simeq \mathbb{R}^n$ be a subset of a finite dimensional real vector space. We say that $\mathcal{C}$ is a* ***cone*** *(respectively* ***convex subset****) if whenever α and $\beta \in \mathcal{C}$ then*

$$\lambda\alpha + \mu\beta \in \mathcal{C} \qquad \text{for all} \qquad \lambda \geq 0, \mu \geq 0,$$

(respectively such that $\lambda + \mu = 1$). We say that $\mathcal{C}$ is ***salient*** *if $\mathcal{C}$ contains no positive dimensional linear subspaces.*

We say that $R \subset \mathcal{C}$ is a ***ray*** *of a cone $\mathcal{C}$ if $R = \mathbb{R}^+\alpha$, for some non-zero vector $\alpha \in \mathcal{C}$. We say that R is an* ***extremal ray*** *if whenever $\beta + \gamma \in R$, where β and $\gamma \in \mathcal{C}$, then β and $\gamma \in R$.*

It follows by Kleiman's criteria that the closed cone of curves of a projective variety is salient.

Lemma 8.2. *Let S be a smooth projective surface and let $\alpha = \sum a_i[C_i]$ and $\beta = \sum b_j[C_j]$ be two cycles, where $a_i > 0$ and $b_j > 0$.*

If $\alpha \cdot \beta < 0$ then $C = C_i = C_j$ for some i and j, where $C^2 < 0$. If i and j are the only two indices with this property then $a_i b_j C^2 \leq \alpha \cdot \beta$.

Proof. Clear, since

$$\alpha \cdot \beta = \sum a_i b_j C_i \cdot C_j,$$

and $C_i \cdot C_j \geq 0$ unless $C_i = C_j$. $\square$

Lemma 8.3. *Let S be a smooth projective surface.*

1. *If R is $R = \mathbb{R}^+\alpha \subset \overline{\mathrm{NE}}(S)$ is an extremal ray of the closed cone of curves of S then $\alpha^2 \leq 0$.*
2. *If C is an irreducible curve such that $C^2 < 0$ then $R = \mathbb{R}^+[C]$ is extremal.*

Proof. We will first show that $R = \mathbb{R}^+\alpha$ is never extremal if $\alpha^2 > 0$. Let H be an ample divisor. Then $H \cdot \alpha > 0$ by Kleiman's criteria. Pick a small neighbourhood U of $\alpha \in V$ such that

- $\beta^2 > 0$
- $\beta \cdot H > 0$,

for all $\beta \in U$. Suppose that $\beta \in U$ is rational. Pick $k \in \mathbb{N}$ such that $D = k\beta$ is integral. By Asymptotic Riemann-Roch,

$$\begin{aligned} h^0(S, \mathcal{O}_S(mD)) + h^0(S, \mathcal{O}_S(K_S - mD)) &= h^0(S, \mathcal{O}_S(mD)) + h^2(S, \mathcal{O}_S(mD)) \\ &\geq \chi(S, \mathcal{O}_S(mD)) > 0, \end{aligned}$$

for large m. Thus either $|mD|$ or $|K_S - mD|$ is non-empty for large m. But $(K_S - mD) \cdot H < 0$ so that $|(K_S - mD)|$ is empty and so $|mD|$ is non-empty. Thus $[D] \in \mathrm{NE}(S)$ and so $U \subset \overline{\mathrm{NE}}(S)$. In particular R is not extremal.

Now suppose that C is an irreducible curve such that $C^2 < 0$. Then we may write

$$[C] = \sum \beta_i$$

where the β_i generate extremal rays of $\overline{\mathrm{NE}}(S)$. As

$$0 > C^2 = [C] \cdot (\sum \beta_i).$$

it follows that $[C] \cdot \beta_i < 0$ for some i. Since β_i is a limit of $\beta(j) \in \mathrm{NE}(S)$, (8.2) implies that $\beta_i = \lambda[C] + \beta'$, where $\beta' \in \overline{\mathrm{NE}}(S)$ and $\lambda > 0$. As β_i generates an extremal ray, it follows that $\mathbb{R}^+\beta_i = \mathbb{R}^+[C] = R$ is extremal. $\square$

Let $\pi \colon X \longrightarrow C$ be a $\mathbb{P}^r$-bundle over a smooth curve. We recall the classification of such bundles. We have that $S = \mathbb{P}(E)$, for some rank $r+1$ vector bundle over C and the two $\mathbb{P}^r$-bundles $S_i = \mathbb{P}(E_i)$ are isomorphic over C if and only if there is a line bundle L and an isomorphism of vector bundles $E_1 \otimes L \simeq E_2$.

Now the Picard group of X has rank two. The space of curves modulo numerical equivalence is generated by the class of a line in a fibre and the class of any section. Thus $\overline{\mathrm{NE}}(X) \subset \mathbb{R}^2$. Taking a compact slice, we get a closed interval, so that topologically the situation is an open and closed book. To get a complete description, we have two rays R_1 and R_2 and it suffices to determine generators for each ray.

First suppose that $C = \mathbb{P}^1$, so that $X = S$ is a rational surface, a $\mathbb{P}^1$-bundle over C. In this case any rank two vector bundle E has the form $\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ by a Theorem of Grothendieck. We may normalise so that $E = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. The resulting surface is denoted $\mathbb{F}_n$. Let f denote the class of a fibre and e the class of a section of minimal self-intersection $-n$. Then

$$\overline{\mathrm{NE}}(X) = \mathbb{R}^+ f + \mathbb{R}^+ e.$$

But now suppose that C has higher genus. We need to say something about all sections and multisections of π. Fortunately in characteristic zero we only need to keep track of the sections. We recall some of the theory of vector bundles:

Definition 8.4. *Let E be a vector bundle over a smooth curve C. The* ***slope*** *of E is the rational number*

$$\mu(E) = \frac{\deg E}{\operatorname{rk} E}.$$

We say that E is ***stable*** *(respectively* ***semi-stable****) if for all quotient vector bundles $E \longrightarrow F$ of E, we have*

$$\mu(E) > \mu(F) \qquad \text{(respectively} \qquad \mu(E) \geq \mu(F)).$$

We say that E is ***unstable*** *if it is not semistable. We say that F* ***destabilises*** *E if F is a quotient $E \longrightarrow F$ of E and*

$$\mu(F) < \mu(E).$$

The ***maximal destabilising quotient*** *is a quotient vector bundle with the smallest slope and the largest rank amongst quotients with the same slope.*

Example 8.5. *Suppose that*

$$E = \bigoplus_{i=0}^{r} L_i,$$

is a direct sum of line bundles.

If $\deg L_i = d_i$, then

$$\mu(E) = \frac{d_0 + d_1 + \cdots + d_r}{r+1} \qquad \text{and} \qquad \mu(L_i) = d_i.$$

As $F = L_i$ is a quotient of E, E is never stable and it is semistable if and only if $\deg L_i$ is independent of i. Let $m = \min_i d_i$. If $m \neq \mu(E)$, then the maximal destabilising quotient is

$$\bigoplus_{i:d_i=m}^{r} L_i.$$

Lemma 8.6. *Let E be a vector bundle over a smooth curve C.*

Then E is semi-stable if and only if $E' = f^*E$ is semi-stable for all covers $f\colon C' \longrightarrow C$.

Proof. One direction is clear; if $E \longrightarrow F$ destabilises E then $E' \longrightarrow F' = f^*F$ destabilises E'.

Suppose that E' is not semi-stable. By what we already proved, passing to a finite cover of C' we may assume that f is Galois, with Galois group $G \subset \operatorname{Aut}(C')$. Let $E' \longrightarrow F'$ be a maximal destabilising subsheaf. Then F' is canonical, whence invariant under the action of $G \subset \operatorname{Aut}(C')$. But this means that $F' = f^*F$ for some vector bundle F and F destabilises E. □

Remark 8.7. *It is not true that if E is stable then f^*E is stable.*

It can happen that f^*E is semistable. Also it is not true that if V is an arbitrary vector bundle on C' which is invariant under G then $V = f^*W$, for some vector bundle W on C. In fact this can fail even for line bundles. We need the fact F' is a quotient of $E' = f^*E$.

Lemma 8.8. *Let C be a smooth curve and let $\pi\colon X = \mathbb{P}(E) \longrightarrow C$ be a $\mathbb{P}^r$-bundle over C.*

TFAE

(1) E is stable (respectively semi-stable).
(2) $-K_{X/C} \cdot \Sigma > 0$ (respectively ≥ 0) for all curves $\Sigma \subset X$, where $K_{X/C} = K_X - \pi^* K_C$ is the relative canonical divisor.

Example 8.9. *Let C be a curve of genus at least two. Suppose that E is a general rank two stable vector bundle, which admits an exact sequence*

$$0 \longrightarrow \mathcal{O}_C \longrightarrow E \longrightarrow L \longrightarrow 0,$$

where L is a line bundle of positive degree $d \leq g$ (such exist by general theory). Let $S = \mathbb{P}(E)$.

If E is chosen generically (see for example [1]) then there is a section C_0 of minimal self-intersection d. Suppose that Σ is a multi-section of S, so that $f\colon D = \Sigma \longrightarrow C$ is dominant. I claim that $\Sigma^2 > 0$. Let $Y = \mathbb{P}(f^*E)$. Then

$$f^*\Sigma = \Sigma_0 + \Sigma_1,$$

where Σ_0 is a section of $Y \longrightarrow D$. But then

$$\Sigma^2 = \Sigma_0 \cdot f^*\Sigma \geq \Sigma_0^2.$$

As f^*E is stable we may replace X by Y and Σ by Σ_0^2, so that we may assume that Σ is a section of $X \longrightarrow C$. But then

$$(K_X + \Sigma) \cdot \Sigma = K_\Sigma = \pi^* K_C.$$

Thus

$$\Sigma^2 = -K_{X/C} \cdot \Sigma > 0,$$

by assumption. Consider $\overline{\mathrm{NE}}(S)$. One edge is given by f the class of a fibre. What about the other edge? Suppose that this is generated by α. All curves other than F have positive self-intersection. By (8.3) we must have $\alpha^2 = 0$. Thus $\mathrm{NE}(S)$ is not a closed cone.

Rescaling we may suppose that $\alpha = \sigma - af$, where σ is the class of Σ and $a \geq 0$. We have

$$d - 2a = \alpha^2 = 0.$$

Thus the divisor $D = 2\Sigma - dF = -K_{X/C}$ is a divisor which intersects every curve positively, but which is not ample. This gives an example in characteristic zero where Kleiman's criteria is sharp.

Example 8.10. *Let $S = E \times E$ the product of a general elliptic curve.*

Then $\rho(S) = 3$. Let f_i be the class of a fibre and let δ be the class of the diagonal. Suppose that $\delta = a_1 f_1 + a_2 f_2$. Then

$$a_i = (a_1 f_1 + a_2 f_2) \cdot f_{2-i} = \delta \cdot f_{2-i} = 1.$$

But

$$0 = \delta^2 \neq (f_1 + f_2)^2 = 2.$$

Thus f_1 and f_2 and δ define independent classes, which actually span the Néron-Severi group.

On the other hand let $D \geq 0$ be a $\mathbb{Q}$-Cartier divisor. Then $D^2 \geq 0$ with equality if and only if $D = 0$, as can be seen by acting by a general translation. Thus $\overline{\mathrm{NE}}(S)$ is half of a classical cone. There are uncountably many extremal rays, and at most countably many contractions. Most rays are not rational.

Example 8.11. *Let S be obtained from $\mathbb{P}^2$ by blowing up nine points $p_1, p_2, \dots, p_9$.*

Suppose first that these points are the nine points of the intersection of two general smooth cubics. Then S is the total space of the pencil, and there is a morphism $f \colon S \longrightarrow \mathbb{P}^1$ whose fibres are the elements of the pencil. The nine exceptional divisors $E_1, E_2, \dots, E_9$ are then sections of this fibration. The generic fibre C of f is an elliptic curve (over the function field $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t)$) and the nine sections define nine points $e_1, e_2, \dots, e_9$. Since the pencil is general, it follows that these points generate a subgroup of C, isomorphic to $\mathbb{Z}^8$. This subgroup then corresponds to a subgroup of the automorphism group of S over the base. The orbit of the nine exceptional divisors gives infinitely many exceptional divisors. Each exceptional divisor generates an extremal ray of the closed cone of curves.

What is worse, -1-curves persist under small deformations. If we therefore perturb these nine points to nine very general points, infinitely many of these -1-curves survive. The resulting surface does not have any automorphisms, and yet its Mori cone is still very complicated.

Let us end these series of examples with Zariski's famous example:

Example 8.12. *Pick a smooth cubic curve C in $\mathbb{P}^2$ and let S be the blow up of $f \colon S \longrightarrow \mathbb{P}^2$ at ten very general points $p_1, p_2, \dots, p_{10}$ of C.*

Let $E_1, E_2, \dots, E_{10}$ be the ten exceptional divisors and let Σ be the strict transform of C.

Then Σ is a curve of self-intersection $9 - 10 = -1$. By a result due to Artin, there is a contraction morphism $\pi \colon S \longrightarrow T$ contracting Σ, where T is a normal algebraic space (or if you will an analytic space). I claim that T is not a projective variety.

Suppose it were. Then T would have an ample divisor D. But I claim it has no non-zero Cartier divisors at all. We have

$$\pi^* D \in A_1(S) = \mathbb{Z}^{11} = \mathbb{Z}[f^* H] + \sum_{i=1}^{10} \mathbb{Z}[E_i],$$

where H is a line in $\mathbb{P}^2$. So

$$\pi^* D \sim a\pi^* H - \sum a_i E_i.$$

Now

$$\pi^* D \cdot \Sigma = D \cdot \pi_* \Sigma = 0.$$

Thus

$$3a = \sum a_i.$$

Moreover $\pi^* D|_\Sigma$ would be linearly equivalent to zero. Thus there would be some curve B of degree d in $\mathbb{P}^2$ such that

$$B|_C \sim \sum b_i p_i.$$

But this contradicts the fact that our ten points of C are general.

Note that there are then plenty of nef divisors D on S which are zero on C but which are not semiample (since if D were semiample, it would descend to T).

Definition 8.13. *A* ***stable*** n***-pointed curve of genus*** g *is a connected curve of arithmetic genus g, with only nodes as singularities, with n marked points $p_1, p_2, \dots, p_n$ contained in the smooth locus, such that the normalisation of every component isomorphic to $\mathbb{P}^1$ has at least three special points (either a node or a marked point).*

The **moduli space of genus g, n-pointed stable curves** $\overline{M}_{g,n}$ is a projective variety whose points are in natural correspondence with isomorphism classes of stable n-pointed curves of genus g.

Note that $\overline{M}_{1,1} \simeq \overline{M}_{0,4} \simeq \mathbb{P}^1$. Indeed, the second isomorphism is given by the j-invariant, and the first is a consequence since an elliptic curve double covers $\mathbb{P}^1$ over 4 points.

These gives finitely many rational curves in $\overline{M}_{g,n}$, which we call **vital curves**. Indeed take a stable curve of genus $g-1$ with n points (respectively a stable curve of genus g with $n-3$ points) whose components are copies of $\mathbb{P}^1$ with 3 labelled points, together with one component with only two labelled points. Now attach an elliptic to the special component (or a copy of $\mathbb{P}^1$ with three marked points). The resulting curve is a point of $\overline{M}_{g,n}$. Varying the moduli of the elliptic curve (or of the four points), gives a curve $C \subset \overline{M}_{g,n}$.

Conjecture 8.14 (Faber, Fulton, Mumford). *$\overline{\mathrm{NE}}(\overline{M}_{g,n})$ is spanned by the classes of the vital curves.*

Theorem 8.15 (Keel, Gibney, Morrison). *To prove* (8.14) *it suffices to prove the case when $g = 0$.*

References

[1] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Notes written in collaboration with C. Musili. Lecture Notes in Mathematics, Vol. 156, Springer-Verlag, Berlin, 1970.

9. Birational invariants

Definition 9.1. *Let X be a normal projective variety and let D be a $\mathbb{Q}$-divisor. The* ***Iitaka dimension*** *of D is equal to the maximum dimension of the image of X under the linear systems $|mkD|$,*

$$\kappa(X, D) = \max_{m \in \mathbb{N}} \dim \phi_m(X),$$

where $\phi_m = \phi_{|mkD|} \colon X \longrightarrow \mathbb{P}^N$. Equivalently,

$$\kappa(X, D) = \limsup_{m \in \mathbb{N}} \frac{\log(h^0(X, \mathcal{O}_X(mkD)))}{\log m}.$$

The ***Kodaira dimension*** *of X is the Iitaka dimension of the canonical divisor, $\kappa(X) = \kappa(X, K_X)$.*

The ***plurigenera*** *of X are the dimensions of the space of global n-forms of weight m,*

$$P_m(X) = h^0(X, \mathcal{O}_X(mK_X)).$$

Just to be confusing $p_g(X) = P_1(X)$, the number of independent global n-forms.

The ***irregularity*** *of X is the dimension of the space of 1-forms,*

$$q(X) = h^0(X, \Omega^1_X).$$

Theorem 9.2. *Let $\phi \colon X \dashrightarrow Y$ be two smooth projective varieties which are birational.*

Then X and Y have the same invariants defined in (9.1).

Definition 9.3. *Let X be a variety. We say that X is*

- ***rational*** *if X is normal and birational to $\mathbb{P}^n$, some n.*
- ***unirational*** *if X is normal and there is a dominant rational map $\phi \colon \mathbb{P}^n \dashrightarrow X$, some n.*
- ***rationally connected*** *if for every two points x and y there is a morphism $f \colon \mathbb{P}^1 \longrightarrow X$ such that $f(0) = x$ and $f(\infty) = y$.*
- ***rationally chain connected*** *if for every two points x and y there are morphisms $f_i \colon \mathbb{P}^1 \longrightarrow X$, $0 \leq i \leq k$, such that $f_0(0) = x$ and $f_i(\infty) = f_{i+1}(0)$, $0 \leq i \leq k-1$ and $f_k(\infty) = y$.*
- ***uniruled*** *if for every point $x \in X$ there is a non-constant morphism $f \colon \mathbb{P}^1 \longrightarrow X$ such that $f(0) = x$.*

Remark 9.4. *It is clear that rational implies unirational, unirational implies rationally connected (indeed given x and $y \in \mathbb{P}^n$, the line connecting them shows that $\mathbb{P}^n$ is rationally connected and the image of a rationally connected variety is rationally connected), rationally connected implies rationally chain connected and rationally chain connected implies uniruled, unless X is a point.*

Note that X is uniruled if and only if there is a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$. If $X = \mathbb{P}^1 \times C$, where C is a curve of genus at least one, then X is uniruled but not rationally chain connected. If $X = \mathbb{P}^1 \cup \mathbb{P}^1$, where the two copies of $\mathbb{P}^1$ are joined at a point, then X is rationally chain connected but not rationally connected. More generally, if X is rationally connected then it is irreducible. Let S be the cone over an elliptic curve. Then S is rationally chain connected but not rationally connected. Let $\pi\colon T \longrightarrow S$ blow up the vertex. Then T is not rationally chain connected. On the other hand rationally connected is a birational invariant and to check that X is rationally connected it suffices to show that any two points belonging to an open subset can be connected by the image of $\mathbb{P}^1$. It is conjectured that rationally connected does not imply unirational. In fact every smooth quartic $X \subset \mathbb{P}^4$ is rationally connected but it is conjectured that a general smooth quartic is not unirational. This seems to be one of the hardest outstanding problems in birational geometry. Finally it is known that there are unirational varieties which are not rational (in fact every smooth cubic in $\mathbb{P}^4$ is unirational; the general one is not unirational. Further every smooth quartic in $\mathbb{P}^4$ is not rational and some of these are unirational. Many other examples are now known).

Lemma 9.5. *Let X be a smooth projective variety.*

If X is uniruled then X is covered by rational curves C, such that $K_X \cdot C < 0$.

In particular $\kappa(X) = -\infty$.

Proof. Let $\operatorname{Hom}(\mathbb{P}^1, X)$ denote the moduli space of morphisms from $\mathbb{P}^1$ to X. Then there is a natural evaluation map,

$$\operatorname{Hom}(\mathbb{P}^1, X) \times \mathbb{P}^1 \longrightarrow X,$$

which sends the pair (f, t) to the point $f(t) \in X$. By assumption there is a component B of $\operatorname{Hom}(\mathbb{P}^1, X)$, which parametrises non-constant morphisms, for which this map is dominant. In characteristic zero, the differential of the map $\alpha\colon B \times \mathbb{P}^1 \longrightarrow X$ is surjective at some point (f, t) of B. On the other hand, there is a natural action of $\operatorname{Aut}(\mathbb{P}^1)$ on $\operatorname{Hom}(\mathbb{P}^1, X)$ and on B. It follows that the differential of α is surjective on the whole orbit of $\operatorname{Aut}(\mathbb{P}^1)$. In particular the differential of α is surjective on the whole fibre $\mathbb{P}^1$ (since the action of $\operatorname{Aut}(\mathbb{P}^1)$ is transitive). But then there is an open subset U of B for which the differential of α is surjective on $U \times \mathbb{P}^1$. Cutting by hyperplanes, we may assume that α is finite, in which case the differential is then an isomorphism on the whole fibre over the open set U. Then

$$K_X \underset{f}{\cdot} C \leq K_C = K_{\mathbb{P}^1} = -2.$$

Suppose to the contrary that $B \in |mK_X|$. Pick $C \not\subset B$. Then

$$0 \leq B \cdot C = (mK_X) \cdot C < 0,$$

a contradiction. $\square$

Remark 9.6. *If X is rationally connected then $q(X) = 0$, and in fact all symmetric powers of all p-forms vanish.*

10. Log resolutions

Definition 10.1. *Let X be normal variety and let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf on X. We say that $\mathcal{I}$ is **principal** if X is smooth and every point of X has a neighbourhood with coordinates $x_1, x_2, \dots, x_n$ so that $\mathcal{I}$ is locally given by a single monomial.*

We have the following celebrated result of Hironaka:

Theorem 10.2 (Principalisation of Ideals). *Let M be a smooth variety and let $\mathcal{I}$ be an ideal sheaf on X.*

Then there is a composition of smooth blow ups $\pi\colon Y \longrightarrow X$ along smooth centres, with support contained in the support of $\mathcal{O}_X/\mathcal{I}$, such that $\pi^*\mathcal{I}$ is a principal ideal.

Definition 10.3. *Let (X, Δ) be a log pair.*

*We say that (X, Δ) is **log smooth**, if the pair (X, D) has global normal crossings (that is every irreducible component of D is smooth and locally (in the analytic or étale topology) about any point of X, $(X, D = \sum \Delta_i)$ is isomorphic to $(\mathbb{C}^n, H_1 + H_2 + \dots + H_k)$ where $H_1, H_2, \dots, H_n$ are the coordinate hyperplanes).*

*A **log resolution** of (X, Δ) is a birational morphism $\pi\colon Y \longrightarrow X$ such that $(Y, \Gamma = f_*^{-1}\Delta + E)$ is log smooth, where $f_*^{-1}\Delta$ is the strict transform of Δ and E is the sum of the exceptional divisors, and there is a divisor F, supported on the exceptional locus, such that F is π-ample.*

Remark 10.4. *Note that in the definition of a log resolution, we make no requirement that the locus where π is not an isomorphism is concentrated over any special locus in X (such as where X is singular).*

Corollary 10.5. *Every log pair (X, Δ) has a log resolution.*

Proof. Embed $X \subset M$ inside a smooth variety, where X has codimension at least two. Let $\pi\colon N \longrightarrow M$ be a birational morphism which principalises $\mathcal{I}_X \subset \mathcal{O}_M$. Then the inverse image of X is a divisor. Then at some stage X must have been contained in a centre of some blow up. But the first such time this happens, the centre must be X itself.

In particular, we can resolve the singularities of X. So replacing X by its resolution, and Δ by the strict transform of Δ plus the exceptional locus, we may assume that X is log smooth. Now apply (10.2) to $\mathcal{I}_D \subset \mathcal{O}_X$. □

Theorem 10.6 (Elimination of indeterminancy). *Let $\phi\colon X \dashrightarrow Y$ be a rational map between projective varieties.*

Then there are morphisms $p\colon W \longrightarrow X$ and $q\colon W \longrightarrow Y$, where p is a composition of smooth blow ups along smooth centres, W is smooth and there is a commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "\phi"] & & Y. \end{tikzcd}$$

Moreover if X is smooth and $y \in Y$ is in the image of the indeterminancy locus of ϕ then there is a non-constant morphism $f\colon \mathbb{P}^1 \longrightarrow Y$ such that $f(0) = y$.

Proof. Pick an embedding of $Y \subset \mathbb{P}^n$ into projective space and let H be a hyperplane section. Let $\phi^*H = M + F$ be the decomposition of ϕ^*H into its fixed and mobile parts. Then the linear system $|M|$ defines ϕ. Let B the scheme theoretic base locus of $|M|$. Then $B = 0$ if and only if ϕ is a morphism (or perhaps better, extends to a morphism). Note that the codimension of B is at least two.

Let $\mathcal{I}_B$ be the ideal sheaf of B. Let $p\colon X \longrightarrow Y$ be the birational morphism, whose existence is guaranteed by (10.2). Let $q\colon W \dashrightarrow Y$ be the induced rational map. Then $q^*H = M_1 + F_1$ is the decomposition of q^*H into its mobile and fixed parts, where $F_1 = p^*B$ and $M_1 = p^*M - F_1$. But then $|M_1|$ is base point free, so that q is a morphism.

Let $V \subset X$ be the indeterminancy locus of ϕ, and let $Z = qp^{-1}(V)$. If $x \in V$, them the $qp^{-1}(x)$ is positive dimensional. Since the image of a rationally chain connected variety is rationally chain connected it suffices to prove that the fibres of p are rationally chain connected. We prove this by induction on the number of blow ups. Suppose that p factors as $p_1\colon W_1 \longrightarrow X$ and $\pi\colon W \longrightarrow W_1$, where π is a smooth blow up of $B \subset W_1$. By induction the fibres of p_1 are rationally chain connected. Let $E_1 \subset W$ be the intersection of the exceptional divisor E with a fibre of p and let $B_1 \subset W_1$ be the image of E_1. Then the fibres of E_1 over B_1 are projective spaces, which are rationally connected. If f_1 and f_2 are two points of two fibres F_1 and F_2 then pick x_1 and x_2 belonging to the fibres and the strict transform G of B_1. Then we can find a rational curve connecting f_1 to x_1 in F_1, a chain of rational curves connecting x_1 to x_2 in G and a rational curve connecting x_2 to f_2 in F_2. The resulting chain connects f_1 to f_2 in the fibre E_1. $\square$

To get some idea of the proof of (10.6), consider the case of smooth projective surfaces. To emphasize this point, we change notation and consider $\phi\colon S \dashrightarrow Y$, where S is a smooth projective surface. As M is mobile it is nef (here is one important place where we use the fact that

S is a surface). We proceed by induction on $d = M^2 \geq 0$. Suppose that ϕ is not defined at $x \in |M|$. Then x is a base point of $|M|$. Let $\pi\colon S_1 \longrightarrow S$ be the blow up. Let $\phi_1\colon S_1 \dashrightarrow Y$ be the induced rational map. If ϕ_1 is given by M_1, then

$$M_1 = \pi^* M - mE,$$

where $m > 0$ is a positive integer (in fact $\pi^* M = M_1 + mE$ gives the decomposition into fixed and mobile parts). Now

$$M_1^2 = (\pi^* M - mE)^2 = d - m^2 < d.$$

Thus we are done by induction on d.

Let $X = \mathbb{C}^3$ and let X_1 be the blow up of the origin of X. The exceptional divisor is then a copy of $\mathbb{P}^2$. Let X_2 be the blow up of X_1 along an smooth cubic in the exceptional divisor. Then the exceptional locus is a copy E of $\mathbb{P}^2$ joined to a $\mathbb{P}^1$-bundle F over an elliptic curve, joined along a section and a cubic. Then $E \cup F$ is a rationally chain connected variety, and yet F is not rationally connected. To connect two points of F, f_1 and f_2, let F_1 and F_2 be the two fibres which contain them. Now let x_1 and x_2 be the points in E which meets these two fibres. Let l be the line connecting x_1 to x_2. Then $F_1 \cup l \cup F_2$ connects f_1 to f_2.

Example 10.7. *Let*

$$X = C \times \mathbb{P}^2 \cup \mathbb{P}^2 \times C \subset \mathbb{P}^2 \times \mathbb{P}^2.$$

Then X is rationally chain connected, but neither component is rationally connected.

11. The log discrepancy

Definition 11.1. *Let (X, Δ) be a log pair. If $\pi\colon Y \longrightarrow X$ is any birational morphism such that $K_Y + \Gamma$ is $\mathbb{Q}$-Cartier, and $E_1, E_2, \dots, E_k$ are the exceptional divisors, then we may write*

$$K_Y + \Gamma = K_X + \pi_*^{-1}\Delta + E = \pi^*(K_X + \Delta) + \sum a_i E_i,$$

for rational numbers $a_1, a_2, \dots, a_k$, where $\pi_*^{-1}\Delta$ is the strict transform of Δ and $E = \sum E_i$ is the sum of the exceptional divisors. The number $a_i = a(E_i, X, \Delta)$ is called the ***log discrepancy*** *of the divisor E_i.*

The ***log discrepancy*** *$a = a(X, \Delta)$ of (X, Δ) is the infimum of the log discrepancies over all exceptional divisors of all birational morphisms.*

Note that it is not necessary to assume that $\Delta \geq 0$ to define the log discrepancy. We only need that X is normal and $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

We run through one computation of the log discrepancy. Let X be the cone over a rational normal curve of degree d. If we blow up $\pi\colon Y \longrightarrow X$ the vertex of the cone then π is a log resolution and the exceptional divisor E is a copy of $\mathbb{P}^1$; $E^2 = -d$. We may write

$$K_T + E = \pi^* K_S + aE,$$

for some rational number a. If we do both sides with respect to E we get

$$-2 = \deg K_{\mathbb{P}^1} = \deg K_E = (K_T + E) \cdot E = \pi^* K_S \cdot E + aE^2 = -ad.$$

Thus

$$a = \frac{2}{d}.$$

Definition 11.2. *Let K/k be a field extension. A* ***valuation*** *ν of K/k is a map*

$$\nu\colon K \longrightarrow \mathbb{Z} \cup \{\infty\},$$

such that

(1) $\nu(f) = \infty$ if and only if $f = 0$.
(2) $\nu(fg) = \nu(f) + \nu(g)$.
(3) $\nu(f + g) \geq \max(\nu(f), \nu(g))$.
(4) $\nu(k^*) = \{0\}$.

Example 11.3. *Let X be a normal projective variety and let $D \subset X$ be a prime divisor. Then the order of vanishing of a rational function along D determines a valuation,*

$$\nu_D(f) = \operatorname{mult}_f D.$$

*If ν is a valuation such that $\nu = \nu_E$ for some divisor E, possibly exceptional, then we will call ν an **algebraic valuation**. The **centre** of ν is the image of E in X.*

The language of valuations provides a convenient way to refer to the same exceptional divisors, on different models. Note that if $E_1 \subset Y_1$ and $E_2 \subset Y_2$ are two divisors on birational varieties Y_1 and Y_2, then $\nu_{E_1} = \nu_{E_2}$ if and only if there is a birational map $\phi \colon Y_1 \dashrightarrow Y_2$ which is an isomorphism in a neighbourhood of the generic points of E_1 and E_2.

The log discrepancy is a birational invariant, in the following weak sense:

Lemma 11.4. *Let (X, Δ) be a log pair and let ν be a valuation.*

The log discrepancy only depends on ν.

Proof. Suppose that we are given $\pi_i \colon Y_i \longrightarrow X$ two birational morphisms on which the centre of ν_i is a divisor E_i. If $\phi \colon Y_1 \dashrightarrow Y_2$ is the induced birational map then ϕ is an isomorphism at the generic point of E_1. We may write

$$K_{Y_i} + \Gamma_i = \pi_i^*(K_X + \Delta) + a_i E_i + J_i,$$

where J_i does not involve E_i and we want to show that $a_1 = a_2$. Pick a meromorphic differential form ω_2 on Y_2 and let $\omega_1 = \phi^*\omega_2$. Then

$$a_i = 1 - \operatorname{mult}_{E_i} \pi_i^*(K_X + \Delta) + \operatorname{mult}_{E_i} \omega_i,$$

which is independent of i by construction. □

Definition 11.5. *We say that a log pair (X, Δ) is **canonical** if the log discrepancy is at least one.*

Lemma 11.6. *Let $\phi \colon X \dashrightarrow Y$ be a birational map between two projective varieties with canonical singularities and let m be a positive integer, such that both mK_X and mK_Y are Cartier.*

Then there is a natural isomorphism

$$H^0(X, \mathcal{O}_X(mK_X)) \simeq H^0(Y, \mathcal{O}_X(mK_Y)).$$

Proof. Let $p \colon W \longrightarrow X$ and $q \colon W \longrightarrow Y$ be a common resolution of ϕ. Then we just have to prove the result for p and q. Replacing ϕ by p we may assume that ϕ is a morphism, a log resolution of X.

Let V the indeterminancy locus of ϕ^{-1}. Suppose that ω is a pluricanonical form on X. Then $\eta = \phi_*\omega$ is a rational form on Y whose poles are concentrated on V, which is a closed subset of codimension at least two. But then η is in fact regular. Thus there is a natural map

$$H^0(X, \mathcal{O}_X(mK_X)) \longrightarrow H^0(Y, \mathcal{O}_Y(mK_Y)).$$

Conversely suppose that η is a pluricanonical form on Y. By assumption,

$$K_X = \pi^* K_Y + E,$$

where $E \geq 0$ is exceptional. Then

$$\begin{aligned}
\pi^*\eta &\in H^0(X, \mathcal{O}_X(m\pi^* K_Y)) \\
&\subset H^0(X, \mathcal{O}_X(m\pi^* K_Y + mE)) \\
&= H^0(X, \mathcal{O}_X(mK_X)).
\end{aligned}$$

□

Lemma 11.7. *Let $\pi\colon X \longrightarrow Y$ blow up a smooth variety V of codimension k, with exceptional divisor E.*

Then the log discrepancy of E is equal to k.

Proof. We have

$$K_X + E = \pi^* K_Y + aE,$$

where a is the log discrepancy. Restricting to E, we have

$$K_E = (K_X + E)|_E = \pi^* K_Y|_E + aE|_E = aE|_E.$$

Let F be a general fibre. Restricting to F, we have

$$-kH = K_{\mathbb{P}^{k-1}} = K_F = aE|_F = -aH,$$

where H is the class of a hyperplane. But then $a = k$. □

Lemma 11.8. *Let $(X, \Delta = \sum a_i \Delta_i)$ be a log smooth pair, where we allow some of the coefficients of Δ to be negative.*

If Δ has a component of coefficient greater than one, then set $a = -\infty$. Otherwise, let

$$a = \min_Z (k - \sum a_i),$$

where Z ranges over the irreducible components of the strata of the support of Δ, k is the codimension of Z and we sum over those components of Δ which contain Z.

Then the log discrepancy of $K_X + \Delta$ is a. In particular the log discrepancy of any pair is either at least zero, or it is $-\infty$ and if X is smooth and $\Delta = 0$ then X is canonical.

Proof. Suppose that Δ has a component C of coefficient $1 + \epsilon$, where $\epsilon > 0$. We are going to successively blow up X along a general smooth codimension two subset of C. Thus we might as well suppose that $S = X$ is a smooth surface and $\Delta = (1 + \epsilon)C$, where C is a smooth curve. Suppose that we blow up $\pi\colon T \longrightarrow S$ the point $p \in C$, with exceptional divisor E. As the log discrepancy of E with respect to K_S is 2, we have

$$K_T + E = \pi^* K_S + 2E,$$

where E is the exceptional divisor. Let D be the strict transform of C. As $\pi^*C = D + E$, it follows that

$$K_T + (1+\epsilon)D + E = \pi^*(K_S + (1+\epsilon)C) + (1-\epsilon)E.$$

The log discrepancy of E is then $1-\epsilon$. On the other hand,

$$K_T + (1+\epsilon)D + \epsilon E = \pi^*(K_S + (1+\epsilon)C).$$

Note that D and E are now two smooth curves, intersecting transversally at a smooth point, where D has coefficient $1+\epsilon$ and E has coefficient ϵ. Now suppose that we blow up the intersection of D and E on T. Mutatis mutandis, a similar calculation shows that the exceptional divisor E_1 has log discrepancy $1-2\epsilon$ with respect to $K_T+(1+\epsilon)D+\epsilon E$ and so also with respect to $K_S + (1+\epsilon)C$. Moreover now we have two smooth curves intersecting transversally at a point, one with coefficient 2ϵ the other with coefficient $1+\epsilon$. If we blow up the intersection point, then we get an exceptional divisor with log discrepancy $1-3\epsilon$ and so on. Continuing in this way we get exceptional divisors of log discrepancy $1-k\epsilon$, for all $k \geq 0$. Thus the log discrepancy is $-\infty$.

Now suppose that Δ is a boundary. If we blow up Z, with exceptional divisor E, then we have

$$K_Y + E = \pi^*K_X + kE,$$

since the log discrepancy is k. Since

$$\pi_*^{-1}\Delta + (\sum a_i)E = \pi^*\Delta,$$

it follows that E has log discrepancy $k - \sum a_i$ with respect to $K_X + \Delta$.

Finally suppose that ν is some algebraic valuation. By (10.6), we may realise ν by blowing up smooth centres which intersect the support of Δ transversally. If we rewrite the equation above as

$$K_Y + \pi_*^{-1}\Delta + (\sum a_i + 1 - k)E = \pi^*(K_X + \Delta),$$

and observe that

$$\sum a_i + 1 - k = (a_1 - 1) + \cdots + (a_i - 1) + a_i + (a_{i+1} - 1) + \cdots + (a_k - 1) \leq \min a_i,$$

since we are assuming that $a_i \leq 1$, it is easy to see that the log discrepancy is computed after one blow up. □

Proof of (9.2). We will only show that the plurigenera are birational invariants; a similar argument applies to the irregularity $q(X)$. By (11.8) it follows that X and Y are canonical and we may apply (11.6). □

12. Classification of Surfaces

The key to the classification of surfaces is the behaviour of the canonical divisor.

Definition 12.1. *We say that a smooth projective surface is **minimal** if K_S is nef.*

Warning: This is not the classical definition of a minimal surface.

Definition 12.2. *Let S be a smooth projective surface. We say that a curve $C \subset S$ is a -1**-curve** if*

$$K_S \cdot C = C^2 = -1.$$

Theorem 12.3 (Cone Theorem)**.** *Let S be a smooth projective surface.*

Then there are countably many extremal rays $R_1, R_2, \dots$ of the closed cone of curves of S on which K_S is negative, such that

$$\overline{\mathrm{NE}}(S) = \overline{\mathrm{NE}}(S)_{K_X \geq 0} + \sum R_i.$$

Further, if $R = R_i$ is any one of these K_S-extremal rays then there is a birational morphism $\pi \colon S \longrightarrow Z$ which contracts a curve C if and only if C spans the ray R. There are three possibilities:

(1) Z is a point and $S \simeq \mathbb{P}^2$.
(2) $\pi \colon S \longrightarrow Z$ is a $\mathbb{P}^1$-bundle over a smooth curve Z.
(3) $\pi \colon S \longrightarrow Z$ is a birational morphism contracting a -1-curve C, where Z is a smooth surface.

In particular the relative Picard number of π is one, each extremal ray R_i is spanned by a rational curve and if H is any ample divisor, there are only finitely many extremal rays R_i such that $(K_X + H) \cdot R < 0$.

Remark 12.4. *The last two statements are sometimes informally stated as saying that the closed cone of curves is locally rational polyhedral on the K_S-negative side of the cone.*

Theorem 12.5 (Castelnuovo)**.** *Let S be a smooth projective surface and let $C \subset S$ be a curve.*

Then C is a -1-curve if and only if there is birational morphism $\pi \colon S \longrightarrow T$, which blows up a smooth point $p \in T$, with exceptional divisor C.

Theorem 12.6 (Abundance)**.** *Let S be a smooth projective surface.*

Then K_S is nef if and only if K_S is semiample.

Theorem 12.7 (Kodaira-Enriques)**.** *Let T be a smooth projective surface, with invariants $\kappa = \kappa(T)$, $p_g = p_g(T)$ and $q = q(T)$. Then T is birational to a surface S which falls into the following list:*

$\kappa = -\infty$**:**

Ruled surface *$S \simeq \mathbb{P}^1 \times B$ where B is a smooth curve of genus $q(S) = g(B)$.*

$\kappa = 0$**:**

Abelian surface *$p_g = 1$, $q = 2$. $S \simeq \mathbb{C}^2/\Lambda$.*

Bielliptic *$p_g = 0$, $q = 1$. There is a Galois cover of $\tilde{S} \longrightarrow S$ of order at most 12 such that $\tilde{S} \simeq E \times F$, where E and F are elliptic curves.*

K3 surface *$p_g = 1$, $q = 0$.*

Enriques surface *$p_g = 0$, $q = 0$. There is an étale cover $\tilde{S} \longrightarrow S$ of order two, such that $\tilde{S}$ is a K3-surface.*

$\kappa = 1$**:**

Elliptic fibration *there is a contraction morphism $\pi \colon S \longrightarrow B$ with general fibre a smooth curve of genus one. $P_m(S) > 0$ for all m divisible by 12.*

$\kappa = 2$**:**

General type *ϕ_m is birational for all $m \geq 5$. In particular $P_m(X) > 0$ for all $m \geq 5$.*

In particular $\kappa \geq 0$ if and only if $P_{12} \geq 0$.

Definition 12.8. *Let X be a normal projective variety, let D be a nef divisor and let E be any divisor. The **nef threshold** is the largest multiple of E we can add to D, whilst preserving the nef condition:*

$$\lambda = \sup\{\, t \in \mathbb{R} \mid D + tE \text{ is nef}\}.$$

Definition 12.9. *Let X be a projective scheme and let D be a nef divisor. The **numerical dimension** $\nu(X, D)$ of D is the largest positive integer such that $D^k \cdot H^{n-k} > 0$, where H is an ample divisor.*

Note that if D is semiample then $\kappa(D) = \nu(D)$. We will need the following easy:

Lemma 12.10. *Let X be a normal projective variety and let D be a nef $\mathbb{Q}$-Cartier divisor.*

(1) If $\nu(D) = 0$ then D is semiample if and only if $\kappa(D) = 0$.

(2) If $\nu(D) = 1$ then D is semiample if and only if $h^0(X, \mathcal{O}_X(mD)) \geq 2$, for some $m > 0$.

In particular if $\nu(D) \leq 1$ then D is semiample if and only if $\nu(D) = \kappa(D)$.

Proof. Suppose that $\nu(D) = 0$. Then D is numerically trivial and it is semiample if and only if it is torsion. As $\kappa(D) = 0$, $D \sim_{\mathbb{Q}} B \geq 0$ and since B is numerically trivial, in fact $B = 0$.

Suppose that $\nu(D) = 1$. Pick m so that $|mD|$ contains a pencil. We may as well assume that $m = 1$. We may decompose this linear system into mobile and fixed part:

$$|D| = |M| + F.$$

Let $B_i \in |D|$, $B_1 \neq B_2$. Then we may write $B_i = C_i + F$. By assumption D is not numerically trivial but

$$0 = D^2 \cdot H^{n-2} = (C_1 + F) \cdot D \cdot H^{n-2} \geq C_1 \cdot D \cdot H^{n-2}.$$

As C_1 moves we must have

$$C_1 \cdot C_2 \cdot H^{n-2} = 0 \qquad \text{and} \qquad C_1 \cdot F \cdot H^{n-2} = 0.$$

In particular $C_1 \cap C_2 = \varnothing$. Thus $|M|$ is base point free and we get a morphism $X \longrightarrow \mathbb{P}^1$. Let $f\colon X \longrightarrow \Sigma$ be the Stein factorisation. By assumption C_1 and C_2 are two different fibres. We have $C_1 \cap F = \varnothing$. Thus F is supported on the fibres of f. As it is Cartier and nef, it must be a multiple of a fibre. But then F is semiample and so D is semiample. □

Definition 12.11. *Let $\pi\colon X \longrightarrow U$ be a projective morphism.*

*The **relative cone of curves** is the cone generated by the classes of all curves contracted by π,*

$$\overline{\mathrm{NE}}(X/U) = \{\, \alpha \in \mathrm{NE}(X) \,|\, \pi_*\alpha = 0 \,\}.$$

*We say that a $\mathbb{Q}$-Cartier divisor H is π**-ample** (aka **relatively ample**, aka **ample over** U) if mH is relatively very ample (that is, there is an embedding i of X into $\mathbb{P}^n_U = \mathbb{P}^n \times U$ over U such that $\mathcal{O}_X(mH) = i^*\mathcal{O}(1)$).*

We say that an $\mathbb{R}$-divisor is relatively ample if and only if it is a positive linear combination of relatively ample $\mathbb{Q}$-divisors.

Note that if U is projective, then H is relatively ample if and only if there is an ample divisor G on U, such that $H + \pi^*G$ is ample. Note also that an $\mathbb{R}$-divisor is relatively ample if and only if it defines a positive linear functionall on $\overline{\mathrm{NE}}(X/U) - \{0\}$. Note that many of the definitions for $\mathbb{Q}$-divisors extend to $\mathbb{R}$-divisors. In particular, the property of being nef and the numerical dimension.

Proof of (12.3). Pick an extremal ray $R = \mathbb{R}^+\alpha$ of the closed cone of curves. We may pick an ample $\mathbb{R}$-divisor H such that

$$(K_S + H) \cdot \beta \geq 0,$$

for all $\beta \in \overline{\mathrm{NE}}(S)$ with equality if and only if $R = \mathbb{R}^+\beta$. In particular $D = K_S + H$ is a nef $\mathbb{R}$-Cartier divisor. The key technical point is to establish that R is rational, so that we may choose H to be an ample

$\mathbb{Q}$-divisor. In fact we will prove much more, we will prove that R is spanned by a curve. Let $\nu = \nu(S, D)$. There are three cases:

- $\nu = 0$,
- $\nu = 1$, and
- $\nu = 2$.

If $\nu = 0$, then $K_S + H$ is numerically trivial, and $-K_S$ is numerically equivalent to H. In other words $-K_S$ is ample. Moreover every curve C spans R. Thus S is a Fano surface of Picard number one and it follows that $S \simeq \mathbb{P}^2$. Note that R is rational in this case.

If $\nu = 1$, then we will defer the proof that R is rational. So assume that H is rational. We first prove that D is semiample. Consider asymptotic Riemann-Roch. $D^2 = 0$, by assumption.

$$D \cdot (-K_S) = D \cdot H > 0,$$

also by assumption. Thus $\chi(X, \mathcal{O}_X(mD))$ grows linearly. Since

$$h^2(S, \mathcal{O}_S(mD)) = h^0(S, \mathcal{O}_S(K_S - mD)) = 0,$$

for m sufficiently large, it follows that there is a positive integer $m > 0$ such that $|mD|$ contains a pencil. By (12.10) it follows that D is semiample. Let F by the general fibre of the corresponding morphism $\pi\colon S \longrightarrow C$. Then F is a smooth irreducible curve, $F^2 = 0$ and $-K_S \cdot F > 0$. By adjunction,

$$0 > (K_S + F) \cdot F = K_F = 2g - 2.$$

It follows that $g = 0$ so that $F \simeq \mathbb{P}^1$. Moreover since $R = \mathbb{R}^+[F]$ is extremal, the relative Picard number is one and so there are no reducible fibres. By direct classification it follows that there are no singular fibres. Thus π is a $\mathbb{P}^1$-bundle.

If $\nu = 2$ then D is big but not ample. As D is nef $D^2 > 0$. By continuity there is an ample $\mathbb{Q}$-divisor G such that $(K_S + G)^2 > 0$ and $(K_S + G) \cdot G > 0$, where $H - G$ is ample. Thus $K_S + G$ is big. By Kodaira's Lemma, $K_S + G \sim_{\mathbb{Q}} A + E$, where A is ample and $E \geq 0$. Now

$$(K_S + G) \cdot \alpha = D \cdot \alpha - (H - G) \cdot \alpha < 0.$$

On the other hand

$$0 > D \cdot \alpha = A \cdot \alpha + E \cdot \alpha > E \cdot \alpha.$$

It follows that α is spanned by a component of E so that $R = \mathbb{R}^+[C]$, for some component C of E. In particular R is rational and we may choose H to be a $\mathbb{Q}$-divisor. We have

$$0 > (K_S + C) \cdot C = K_C = 2g - 2.$$

Thus $g = 0$ and C is a -1-curve.

Replacing H by a multiple, we may assume that H is very ample (for the time being, we only need that it is Cartier) and $K_S + H$ is ample. Suppose that $H \cdot C = m > 0$. We may always assume that $m > 1$ (simply replace H by a multiple). If $G = H + (m-2)C$, then G is big and it is nef, since $G \cdot C = 2$. In particular G is ample by Nakai-Moishezon. Let $D = K_S + C + G$. Since we may write

$$D = (K_S + H) + C,$$

the stable base locus of D is contained in C. In particular for every curve $\Sigma \subset S$,

$$D \cdot \Sigma \geq 0,$$

with equality if and only if $\Sigma = C$. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_S(D - C) \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_C(D) \longrightarrow 0.$$

Now $\mathcal{O}_C(D) = \mathcal{O}_C$, since $D|_C$ is a divisor of degree zero on $\mathbb{P}^1$. On the other hand,

$$H^1(S, \mathcal{O}_S(D - C)) = H^1(S, \mathcal{O}_S(K_S + G)) = 0,$$

by Kodaira vanishing. Thus there are no base points of D on C, so that D is semiample, and the resulting morphism $\pi \colon S \longrightarrow Z$ contracts C.

It remains to prove that Z is smooth. Consider the ample divisor $K_S + G$. We may always pick H very ample. In this case, I claim that $K_S + G$ is base point free (in fact it is very ample). The base locus is supported on C. Consider the exact

$$0 \longrightarrow \mathcal{O}_S(K_S + G - C) \longrightarrow \mathcal{O}_S(K_S + G) \longrightarrow \mathcal{O}_C(K_S + G) \longrightarrow 0.$$

Then $\mathcal{O}_C(K_S + G) \simeq \mathcal{O}_{\mathbb{P}^1}(1)$. As before,

$$H^1(S, \mathcal{O}_S(K_S + H + (m-3)C) = 0,$$

by Kodaira vanishing. Thus $K_S + G$ is base point free. Pick a general curve $\Sigma' \in |K_S + G|$. Then this must intersect C transversally at a single smooth point. But then $\Sigma = \pi_*\Sigma'$ is a smooth curve in Z. On the other hand $\Sigma + C \in |D|$. Since $|D|$ defines the contraction, the image of $\Sigma + C$, which is again Σ' is Cartier.

But any variety which contains a smooth Cartier divisor, is smooth in a neighbourhood of the divisor. Thus Z is smooth. $\square$

13. MMP FOR SURFACES

(12.3) allows us to define the K_S-MMP for surfaces. The aim of the minimal model program is to try to make K_S nef.

(1) Start with a smooth projective surface S.
(2) Is K_S nef? Is yes, then stop.
(3) Otherwise there is an extremal ray R of the cone of curves $\overline{\mathrm{NE}}(S)$ on which K_S is negative. By (12.3) there is a contraction $\pi\colon S \longrightarrow Z$ of R.
  Mori fibre space: If $\dim Z \leq 1$ then the fibres of π are Fano varieties.
  Birational contraction: In this case replace S by Z and return to (2).

In other words, the K_S-MMP produces a sequence of smooth surface $\pi_i\colon S_{i-1} \longrightarrow S_i$, where each π_i blows down a -1-curve (conversely each π_i blows up a smooth point of S_i), starting with $S_0 = S$. This process must terminate, since the Picard number of S_i is one less than the Picard number of S_{i-1}. At the end we have a smooth surface $T = S_k$, such that either K_T is nef or $\pi\colon T \longrightarrow C$ is $\mathbb{P}^1$-bundle over a curve, or $T \simeq \mathbb{P}^2$.

Definition 13.1. *Let X be a normal variety.*

*We say that X is a **Fano variety** if X is projective and $-K_X$ is ample.*

*We say that a projective morphism $\pi\colon X \longrightarrow Z$ is a **Fano fibration** if $-K_X$ is π-ample.*

*Let R be an extremal ray of the closed cone of curves of X. We say that R is K_X-**extremal** if $K_X \cdot R < 0$. We say $\pi\colon X \longrightarrow Z$ is the **contraction associated to** R if π is a contraction morphism and C is contracted if and only if $R = \mathbb{R}^+ C$.*

Lemma 13.2. *Let S be a smooth surface. Then the log discrepancy of S is equal to 2 and the only valuations of log discrepancy 2 are given by blowing up a point.*

Proof. Easy calculation. □

Proof of (12.5). One direction is clear. If $\pi\colon S \longrightarrow T$ blows up p, and C is the exceptional divisor, then we have already seen that $C^2 = -1$, and $C \simeq \mathbb{P}^1$. But then by adjunction

$$2g - 2 = -2 = K_C = (K_S + C) \cdot C = K_S \cdot C - 1.$$

Thus $K_S \cdot C = -1$ and C is a -1-curve.

Now suppose that C is a -1-curve. Then $R = \mathbb{R}^+[C]$ is a K_S-extremal ray of the cone of curves. Let $\pi\colon S \longrightarrow T$ be the associated contraction morphism. Then T is smooth.

Now suppose that we write

$$K_S + C = \pi^* K_T + aC.$$

Dotting both sides by C, we see that $a = 2$, and we are done by (13.2). $\square$

The MMP for surfaces can be extended in two interesting (but essentially trivial) ways. The first way we restrict the choice of extremal rays to contract and the second way we group together extremal rays and contract them simultaneously.

First suppose we are given a projective morphism $g\colon S \longrightarrow U$. Then one can ensure that every step of the MMP lies over U, simply by only contracting rays of the relative cone of curves, $\overline{\mathrm{NE}}(S/U)$. At the end, either K_S is nef over U (meaning that it is nef on every curve contracted over U) or we get a Mori fibre space over U.

Secondly suppose we have a group G acting on S. By simultaneously contracting whole faces of the cone of curves, which are orbits of a single extremal ray, the resulting contraction is then G-equivariant. This gives us a K_S-MMP which preserves the action of G. Note though, that the relative Picard number of each step can be larger than one (in fact the relative Picard number of the G-invariant part is always one).

A particularly interesting case, is when S is a smooth surface defined over the real numbers. In this case, we let G be the Galois group of $\mathbb{C}$ over $\mathbb{R}$ (namely $\mathbb{Z}_2$, generated by complex conjugation). The resulting steps of the MMP respect the action of complex conjugation, so that the MMP is defined over $\mathbb{R}$. Clearly similar remarks hold for other non-algebraically closed fields.

Theorem 13.3 (Hodge Index Theorem)**.** *Let S be a smooth projective surface.*

Then the intersection pairing

$$\mathrm{NS}(S) \times \mathrm{NS}(S) \longrightarrow \mathbb{R},$$

has signature $(+, -, -, -, \dots, -)$.

In particular if $D^2 > 0$ and $D \cdot E = 0$ then $E^2 \leq 0$ with equality if and only if E is numerically trivial.

Proof. It suffices to prove the last statement for any D such that $D^2 > 0$. So we may assume that D is ample. Suppose that $E^2 \geq 0$.

Suppose that $E^2 > 0$. Consider $H = D + mE$, where m is large. As $H \cdot E > 0$, it follows that $\kappa(S, E) > 0$, by Asymptotic Riemann-Roch. But then $D \cdot E > 0$, a contradiction.

Now suppose that $E^2 = 0$ but that E is not numerically trivial. Then there is a curve C such that $E \cdot C \neq 0$. Let $C' = (D \cdot C)D - (D^2)C$. Then $C' \cdot D = 0$ and $E \cdot C' \neq 0$. Replacing C by C' we may assume that $D \cdot C' = 0$.

Let $E' = mE + C$. Then $D' \cdot E' = 0$ and

$$E'^2 = 2mE \cdot C + C^2.$$

Since $D \cdot E \neq 0$ we can choose m so that $E'^2 \neq 0$. Thus replacing E by E' we are reduced to the case when $E^2 > 0$. ☐

Lemma 13.4 (Negativity of Contraction)**.** *Let $\pi\colon X \longrightarrow U$ be a proper birational morphism of varieties and let B be an $\mathbb{R}$-Cartier divisor.*

If $-B$ is π-nef then $B \geq 0$ if and only if $\pi_* B \geq 0$.

Proof. One direction is clear, if $B \geq 0$ then $\pi_* B \geq 0$.

Otherwise, we may assume that X and U are normal, and U is affine. Cutting by hyperplanes, we may assume that U is a surface. Passing to a resolution of X, we may assume that X is a smooth surface. Compactifying X and U we may assume that X and U are projective. Let $D = \pi^* H$ and list the exceptional divisors $E_1, E_2, \dots, E_k$. Then $D^2 > 0$ and $D \cdot E_i = 0$. It follows that the intersection matrix $(E_i \cdot E_j)$ is negative definite. Suppose that $B = \sum b_i E_i + B'$, where no component of $B' \geq 0$ is exceptional. Then

$$(\sum b_i E_i) \cdot E_j \leq B \cdot E_j < 0.$$

Thus $b_i \geq 0$. ☐

Theorem 13.5 (Strong Factorisation)**.** *Let $\phi\colon S \dashrightarrow S'$ be a birational map between two smooth projective surfaces.*

Then there are two birational maps $p\colon T \longrightarrow S$ and $q\colon T \longrightarrow S'$ which are both compositions of smooth blow ups of smooth points (and isomorphisms) and a commutative diagram

$$\begin{tikzcd} & T \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S \arrow[rr, dashed, "\phi"] & & S'. \end{tikzcd}$$

Proof. By elimination of indeterminancy we may assume that q is a composition of smooth blow ups. Replacing ϕ by p, we may therefore assume that $\phi\colon T \longrightarrow S$ is a birational morphism.

Consider running the K_W-MMP over S'. This terminates with a relative minimal model, $\pi\colon W \longrightarrow T$ over S'. The morphism π contracts -1-curves, and so π is a composition of smooth blow ups. It suffices to show that $T = S'$; we will only use the fact that K_T is nef over S'.

Suppose not. We may write

$$K_T + E = \pi^* K_{S'} + \sum a_i E_i.$$

Since S' is smooth it has log discrepancy two and so each $a_i \geq 2$. But then if we write

$$K_T = \pi^* K_{S'} + F,$$

then

$$F = \sum (a_i - 1) E_i \geq 0,$$

contains the full exceptional locus. By negativity of contraction, there is an exceptional divisor E_i such that $B \cdot E_i < 0$. But then K_T is not nef, a contradiction. $\square$

Lemma 13.6. *Let (X, Δ) be a log pair.*

If X is a curve and $K_X + \Delta$ is nef then it is semiample.

Proof. Let ν be the numerical dimension of $K_X + \Delta$, and let d be the degree. There are two cases:

(1) $d = 0$ and $\nu = 0$.
(2) $d > 0$ and $\nu = 1$.

If $d > 0$ then $K_X + \Delta$ is ample and there is nothing to prove. If $d = 0$ there are two cases. If $g = 1$ then Δ is empty and $K_X \sim 0$ as X is an elliptic curve. If $g = 0$ then $X \simeq \mathbb{P}^1$. Pick $m > 0$ such that $D = m(K_X + \Delta)$ is integral. Then $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^1}$, so that $|D|$ is base point free. $\square$

Definition 13.7. *Let X be a smooth projective variety. Then there is a morphism $\alpha\colon X \longrightarrow A$ to an abelian variety, which is universal amongst all such morphisms in the following sense:*

Let $f\colon X \longrightarrow B$ be another morphism to an abelian variety. Then there is a morphism $\tilde{f}\colon A \longrightarrow B$ and a commutative diagram

$$\begin{tikzcd} X \arrow[d, "\alpha"'] \arrow[dr, "f"] & \\ A \arrow[r, "\tilde{f}"] & B. \end{tikzcd}$$

In characteristic zero, α induces an isomorphism

$$H^1(X, \mathcal{O}_X) \simeq H^1(A, \mathcal{O}_A).$$

In particular $\dim A = q(X)$.

Lemma 13.8 (Kodaira's Formula)**.** *Let $\pi\colon S \longrightarrow C$ be a contraction morphism, where S is a smooth projective surface and C is a smooth projective curve and the generic fibre is an elliptic curve.*

If K_S is nef over C then there is a divisor $\Delta \geq 0$ on C such that

$$K_S = \pi^*(K_C + \Delta).$$

Sketch of proof of (12.6). Let $\nu = \nu(S, K_S)$ be the numerical dimension. There are three cases.

If $\nu = 0$ then K_S is numerically trivial. Let $\alpha\colon S \longrightarrow A$ be the Albanese morphism. Let Z be the image. There are three cases, given by the dimension of Z.

If $q = 0$, equivalently Z is a point, then every numerically trivial divisor is torsion and there is nothing to prove. Suppose that $Z = C$ is a curve. Let F be a general fibre. Then

$$2g - 2 = K_F = (K_S + F) \cdot F = 0,$$

so that $g = 1$ and F is an elliptic curve. But then the result follows by (13.8) and (13.6). Finally suppose that Z is a surface. With some work, one shows that $Z = A$, and that α is birational, whence an isomorphism.

If $\nu = 1$ we first assume that $q = 0$. But then,

$$\chi(S, \mathcal{O}_S) = 1 - q + p_g > 0.$$

Riemman-Roch then reads

$$h^0(S, \mathcal{O}_S(mK_S)) \geq \chi(S, \mathcal{O}_S(mK_S)) = \chi(S, \mathcal{O}_S)) > 0.$$

It follows that $|mK_S| \neq \varnothing$, for $m > 0$. Let $C \in |mK_S|$. Then

$$2g - 2 = K_C = (K_S + C) \cdot C = (m + 1)C|_C = 0.$$

But then C is a smooth curve of genus one (or a rational curve with a single node or cusp). Moreover since $K_C \sim 0$ it follows that $C|_C$ is torsion. Now there is an exact sequence,

$$0 \longrightarrow \mathcal{O}_S((k-1)C) \longrightarrow \mathcal{O}_S(kC) \longrightarrow \mathcal{O}_C(kC) \longrightarrow 0.$$

Note that $\mathcal{O}_C(kC) = \mathcal{O}_C$ infinitely often. Therefore $h^1(C, \mathcal{O}_C(kC)) \neq 0$ infinitely often. It follows that $h^1(S, \mathcal{O}_S(kC))$ is an unbounded function of k. Since

$$\chi(S, \mathcal{O}_S(kC)) \geq 0 \qquad \text{and} \qquad h^2(S, \mathcal{O}_S(kC)) = 0,$$

for $k \geq 2$, it follows that $h^0(S, \mathcal{O}_S(kC))$ is an unbounded function of k and we are done by (12.10). If $\nu = 1$ and $q > 0$, then we again have to carefully analyse the map α.

If $\nu = 2$ then $K_S^2 > 0$. If K_S is ample there is nothing to prove. Otherwise, by Nakai-Moishezon, there is a curve C such that $K_S \cdot C = 0$.

By Kodaira's Lemma, $K_S \sim A+E$, where A is ample and E is effective. But then C is a component of E and C has negative self-intersection. We have

$$2g-2 = K_C = (K_S + C) \cdot C < 0.$$

But then $g = 0$, $C \simeq \mathbb{P}^1$, and $C^2 = -2$ (C is then called a -2-curve).

With some work, we can contract C, as before, $\pi \colon S \longrightarrow T$. In fact $K_S = \pi^* K_T$. Repeating this process, as in the K_S-MMP, we reduce to the case when K_S is ample. The only twist is that if we contract any curves, the resulting surface is necessarily singular. □

Proof of (12.7). Modulo some interesting details which we will skip, this essentially follows by applying the K_S-MMP and considering the maps given by abundance and the Albanese. □

14. Bend and Break

In this section we will indicate how to prove an interesting consequence of Mori's famous bend and break result:

Theorem 14.1 (Mori-Miyaoka)**.** *Let X be a normal projective variety of dimension n, let H be a nef $\mathbb{R}$-divisor and let C a curve contained in the smooth locus of X.*

If $K_X \cdot C < 0$ then through every point $x \in C$ there passes a rational curve L_x such that

$$M \cdot L_x \leq 2n \frac{M \cdot C}{-K_X \cdot C}.$$

In fact we will only prove:

Theorem 14.2 (Mori)**.** *Let X be a smooth Fano variety of dimension n.*

Then X is covered by rational curves C such that $-K_X \cdot C \leq n+1$.

However once one sees the proof of (14.2) it is not hard to at least imagine how the proof of (14.1) goes. In both cases we start with a morphism $f \colon C \longrightarrow X$, such that $-K_X \underset{f}{\cdot} C < 0$. Pick a point $p \in C$ and let $x = f(p)$. The basic idea is to bend C, meaning that we will deform the morphism f, whilst preserving the condition that p maps to x. This will break C:

Lemma 14.3 (Rigidity-Lemma)**.** *Let $f \colon X \longrightarrow Y$ and $g \colon X \longrightarrow Z$ be morphisms of varieties.*

If $f_*\mathcal{O}_X = \mathcal{O}_Y$, f is proper and there is a point $y \in Y$ such that the whole fibre $f^{-1}(y)$ is contracted to a point by g, then there is an open neighbourhood U of y in Y, and a factorisation

$$\begin{tikzcd} f^{-1}(U) \arrow[r, "f"] \arrow[dr, "g"'] & U \arrow[d, "h"] \\ & Z. \end{tikzcd}$$

Proof. Let $\Gamma \subset Y \times Z$ be the image of (f, g). Then the projection morphism $p \colon \Gamma \longrightarrow Y$ is proper, and $p^{-1}(y) = (y, z)$, is a single point, by assumption. It follows that p is finite over an open neighbourhood U of $y \in Y$. But

$$f_*\mathcal{O}_{f^{-1}(U)} \supset p_*\mathcal{O}_{p^{-1}U} \supset \mathcal{O}_U = f_*\mathcal{O}_{f^{-1}(U)}.$$

It follows that $p|_{p^{-1}(U)}$ is an isomorphism. Let h be the composition of the inverse map with the projection down to Z. $\square$

Lemma 14.4. *Let $F\colon C\times B_0 \longrightarrow Y$ be a morphism such that $F(p,u) = y$, for all $b \in B_0$, where C is a smooth proper curve and B_0 is an open subset of a smooth proper curve B.*

If there is a point $q \in C$ such that $F|_{\{q\}\times B_0}$ is not constant then the rational map $F\colon C\times B \dashrightarrow Y$ is not defined at some point of $\{p\}\times B$. In particular Y contains a rational curve passing through x.

Proof. If F were a morphism on the whole of $C \times B$, then by the rigidity Lemma, we could find an open neighbourhood U of p and an open neighbourhood V of $\{p\}\times B$ such that $F(p_1,b_1)$ is independent of b_1, for all $(p_1,b_1) \in V$. But then $F(q,b)$ is independent of b, a contradiction. $\square$

So it remains to show that we can bend f.

Theorem 14.5. *Let $f\colon C \longrightarrow X$ be a morphism of a smooth curve into a variety X, such that $f(C)$ is contained in the smooth locus.*

Then

$$T_f \operatorname{Hom}(C,X) = H^0(C,f^*T_X) \quad \textit{and} \quad T_f \operatorname{Hom}(C,X,p,x) = H^0(C,f^*T_X\otimes\mathcal{I}_p).$$

Moreover,

$$\dim_f \operatorname{Hom}(C,X) \geq \chi(C,f^*T_X) \quad \textit{and} \quad \dim_f \operatorname{Hom}(C,X,p,x) \geq \chi(C,f^*T_X\otimes\mathcal{I}_p).$$

Here the second space represents the set of morphisms which send p to x. The point here is that the spaces $\operatorname{Hom}(C,X)$ are naturally schemes not varieties. So even if the spaces were smooth at f, there might be obstructions to deforming f, beyond the first level (which are represented by the tangent space). But the obstructions to deforming f live in $H^1(C,f^*T_X)$. Thus the difference

$$\chi(C,f^*T_X) = h^0(C,f^*T_X) - h^1(C,f^*T_X),$$

represents deformations which can be lifted to any level, whence the inequality on dimensions.

We need to recall Hirzebruch-Riemman-Roch for curves:

Theorem 14.6. *Let E be a vector bundle of rank r and degree d over a smooth curve C.*

Then

$$\chi(C,E) = h^0(C,E) - h^1(C,E) = d - r(g-1).$$

In our case, $-K_X \underset{f}{\cdot} C$ is positive and the degree is either $-K_X \underset{f}{\cdot} C$ or $-K_X \underset{f}{\cdot} C - n$ (if one wants to fix a point). But there seems to no way to ensure that the degree is more than $n(g-1)$. In other words, why should the genus be small in relation to the degree? Note also, that we want to deform f in a non-trivial way (that is, not simply by applying an automorphism of C). In other words we want more deformations than the dimension the automorphism group (equal to $h^0(C, T_C)$).

If C is a copy of $\mathbb{P}^1$, one can compose f with the morphism $z \longrightarrow z^n$. This has the effect of multiplying the degree by n, without changing the genus.

If C is an elliptic curve, we can still play the same trick. Multiplication by n defines an isogeny of degree n^2. This has the same effect of increasing the degree by a factor of n^2, whilst leaving the genus unchanged.

But now suppose that the genus is at least 2. Let $\pi\colon D \longrightarrow C$ be a generically smooth morphism of degree e. Then

$$2h - 2 = e(2g - 2) + b.$$

The best one can hope for is that π is étale, that is, $b = 0$. The problem is that then the genus increases by the same factor as the degree (they both go up by a factor of e).

Suppose for a minute that the characteristic is p. Consider Frobenius $F\colon C \longrightarrow C$. Composing f with Frobenius has the effect of increasing the degree, without changing the genus. So, applying a high enough power of Frobenius, we may assume that the morphism f deforms, fixing the fact that p maps to x. But then X must be covered by rational curves.

Now the case when C is a rational curve is a little special. Since $\mathbb{P}^1$ has automorphisms, to break C, we need to deform it keeping two points fixed. This wastes a little more of the degree. In fact

$$\chi(\mathbb{P}^1, f^*T_X \otimes \mathcal{O}_{\mathbb{P}^1}(-p-q)) = -K_X \underset{f}{\cdot} C - 2n.$$

Playing around with this a little, one sees that one can break a rational curve on a Fano variety until its degree is no more than $n + 1$.

To summarise. If the characterstic is not zero, then we can find rational curves C of degree $-K_X \cdot C \leq n + 1$.

In the general case, we realise X and C as schemes over finitely generated extension K of $\mathbb{Q}$ (just embed X into projective space, and let K be the field generated any set of defining equations). Pick an integral domain R, with field of fractions K, a finitely generated extension of $\mathbb{Z}$ and realise X over $\operatorname{Spec} R$ (that is, clear denominators from the definining equations).

For every prime p, consider the reduction X_p of X modulo p (just reduce the equations modulo this prime). The residue field is a finitely generated extension of $\mathbb{F}_p$, whence a finite field. For all but finitely many primes, X_p is smooth along C_p and $-K_{X_p} \cdot C_p < 0$, and if X is a smooth Fano, then so is X_p.

Thus by what we have already proved, X_p is covered by rational curves of bounded degree, for all but finitely many primes. Since the Hilbert scheme of subvarieties of bounded degree is of finite type, the same must hold at the generic point, that is, in characteristic zero.

Note that it is absolutely crucial that the rational curves we produce are of bounded degree, else we could never return from characteristic p to characteristic zero. To prove (14.1) one needs to work a little harder. As X is not smooth and $-K_X$ is not ample, one cannot assume that the rational curves we produce are K_X-negative. To compensate, one composes with a larger power of Frobenius, but at the same time fixes as many points as possible. This way we break off many curves. Playing around with the Hodge Index Theorem, one gets the indicated bound, in characteristic p, at least when M is an ample $\mathbb{Q}$-divisor. One then lifts this result to all characteristics. The general case follows easily from the case when M is ample by an easy limiting argument.

Corollary 14.7. *Let X be a normal projective variety of dimension n. Let M be any nef divisor. Suppose that we may find nef $\mathbb{R}$-divisors $D_1, D_2, \dots, D_n$ with the following two properties:*

(1) $D_1 \cdot D_2 \cdot \dots \cdot D_n = 0$, and

(2) $-K_X \cdot D_2 \cdot \dots \cdot D_n > 0$.

Then X is swept out by rational curves Σ, such that $D_1 \cdot \Sigma = 0$ and

$$M \cdot \Sigma \leq 2n \frac{M \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n}{-K_X \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n}.$$

Proof. Let $H_1, H_2, \dots, H_n$ be ample $\mathbb{Q}$-divisors. If we pick H_2, H_3, $\dots H_n$ close enough to D_2, D_3, $\dots$, D_n, we have

$$-K_X \cdot H_2 \cdot H_3 \cdots \cdot H_n > 0.$$

Pick positive integers m_i such that $m_i H_i$ is very ample, $2 \leq i \leq n$ and let C be the intersection of general elements of $|m_i H_i|$. Then C is contained in the smooth locus of X and $-K_X \cdot C > 0$. By (14.1) there is a rational curve Σ such that

$$\begin{aligned}(kD + H) \cdot \Sigma &\leq 2n \frac{(kD + H) \cdot (m_2 H_2) \cdot (m_3 H_3) \cdot \dots \cdot (m_n H_n)}{-K_X \cdot (m_2 H_2) \cdot (m_3 H_3) \cdot \dots \cdot (m_n H_n)} \\ &= 2n \frac{(kD + H) \cdot H_2 \cdot H_3 \cdot \dots \cdot H_n}{-K_X \cdot H_2 \cdot H_3 \cdot \dots \cdot H_n}.\end{aligned}$$

where k is a positive integer. As H_2, H_3, $\dots H_n$ approach D_2, D_3, $\dots$, D_n, the numerator and denominator approach positive constants. Thus the left hand side is bounded, and as we vary k, $\Sigma = \Sigma_k$ belongs to a bounded family. Thus we may as well assume that Σ is fixed. Letting k go to infinity and H approach M gives the result. $\square$

Let me end this section by mentioning three fabulous results:

Theorem 14.8 (Mori). *Let X be a smooth projective variety.*

If T_X is ample then $X \simeq \mathbb{P}^n$.

Theorem 14.9 (Cho-Miyaoka-Shepherd-Barron; Kebekus). *Let X be a smooth projective Fano variety of dimension n.*

If the smallest degree of a covering family of rational curves is $n+1$ then $X \simeq \mathbb{P}^n$.

Note that (14.9) implies (14.8). Indeed, if C is a rational curve, then

$$T_X|_C = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2) \oplus \dots \mathcal{O}_{\mathbb{P}^1}(a_n),$$

by a result of Grothendieck (every vector bundle on $\mathbb{P}^1$ splits) and if $T_X|_C$ is ample then $a_i \geq 1$. But then

$$-K_X \cdot C = 2 + \sum a_i \geq n+1.$$

Theorem 14.10 (Bogomolov-McQuillan; Kebekus-Solá Conde-Toma). *Let $\mathcal{F} \subset T_X$ be a possibly singular foliation on a normal projective variety.*

If C is any curve contained in the smooth locus of X along which $\mathcal{F}$ is a regular foliation, then the leaves of $\mathcal{F}$ through any point $p \in C$ are algebraic and if $x \in C$ is general, or $\mathcal{F}$ is regular, then this leaf is rationally connected.

15. Kawamata log terminal and all that

Definition 15.1. *We say that a log pair (X,Δ) is **kawamata log terminal** if there is a log resolution $\pi\colon Y \longrightarrow X$ such that if we write*

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E,$$

where $\Gamma \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma = \Delta$, and $\pi_* E = 0$ then $\llcorner \Gamma \lrcorner = 0$.

We could rephrase this definition as saying that the coefficients of Δ lie between zero and one, and that this condition continues to hold on Y. If we rewrite the equation above as

$$K_Y + \Gamma = \pi^*(K_X + \Delta),$$

note that the kawamata log terminal condition becomes $\lfloor \Gamma \rfloor \leq 0$.

In fact this condition holds on any birational model and we have:

Lemma 15.2. *A log pair (X,Δ) is kawamata log terminal if and only if the log discrepancy is greater than zero and $\lfloor \Delta \rfloor = 0$.*

Proof. Suppose that (X,Δ) is kawamata log terminal. We have to check that the log discrepancy of every valuation ν is greater than zero. If ν is exceptional for π then this is clear. Replacing (X,Δ) by (Y,Γ) it suffices to check that a log smooth pair (X,Δ) has log discrepancy greater than zero, if $\lfloor \Delta \rfloor = 0$. This follows from the formula for the log discrepancy of a blow up.

Now suppose that the log discrepancy is greater than zero. Let $\pi\colon Y \longrightarrow X$ be a log resolution. If we write

$$K_Y + \tilde{\Delta} + \sum E_i = \pi^*(K_X + \Delta) + \sum a_i E_i$$

then $a_i > 0$. Thus $\lfloor \Gamma \rfloor \leq 0$. □

Kawamata log terminal pairs behave very well with respect to finite morphisms:

Lemma 15.3. *Let $\pi\colon Y \longrightarrow X$ be a finite morphism and let (X,Δ) and (Y,Γ) be log pairs such that*

$$K_Y + \Gamma = \pi^*(K_X + \Delta).$$

Then (X,Δ) is kawamata log terminal if and only if (Y,Γ) is kawamata log terminal.

Proof. The trick is to prove a much stronger result. Let us drop the condition that Δ and Γ are effective. We will then prove that (X,Δ) has log discrepancy at least zero if and only if (Y,Γ) has log discrepancy at least zero, with simultaneous equality.

We first prove that if (Y, Γ) is kawamata log terminal and π is Galois, with Galois group G, then (X, Δ) is kawamata log terminal. Let ν be a valuation of X. Pick a G-equivariant log resolution $g\colon V \longrightarrow Y$ of Y, which extracts the valuations $\mu_1, \mu_2, \dots, \mu_k$ corresonding to ν. Let $f\colon W \longrightarrow X$ be the quotient of g, so that there is a commutative square

$$\begin{tikzcd} V \arrow[r, "g"] \arrow[d, "\psi"'] & Y \arrow[d, "\pi"] \\ W \arrow[r, "f"] & X. \end{tikzcd}$$

Now f is not necessarily a log resolution. However it will extract ν and W is $\mathbb{Q}$-factorial. By assumption the log discrepancy of μ_i is at least zero. The Riemann-Hurwitz formula for log pairs then says that the log discrepancy of ν is at least zero, with equality if and only if we have equality for each μ_i.

Thus (X, Δ) is kawamata log terminal.

Now suppose that (X, Δ) is kawamata log terminal. Let $\psi\colon Z \longrightarrow X$ be the Galois closure of π. Then the induced morphism $Z \longrightarrow Y$ is Galois. Replacing Z by Y we may assume that π is Galois. The result is easy in this case.

Finally if (Y, Γ) is Galois then going up we may assume that π is Galois. $\square$

Suppose that $\mathbb{Z}_r$ acts on $\mathbb{C}^n$. It turns out that we can always diagonalise the action:

$$(x_1, x_2, \dots, x_n) \longrightarrow (y_1, y_2, \dots, y_n),$$

where $y_i = \omega^{a_i} x_i$, and ω is a primitive rth root of unity. We can encode this by the datum:

$$\frac{1}{r}(a_1, a_2, \dots, a_n).$$

As usual, we can assume that $0 \leq a_i \leq r - 1$. Also, since we get to choose ω, if the action is faithful, then we can partialise normalise, and we may assume that $a_1 = 1$. Finally, we always assume that the action is unramified in codimension one, so that the gcd of all but one of the a_i, for any i, is always one. The number r is called the index of the quotient singularity.

For surfaces there are two interesting extreme cases:

$$\frac{1}{r}(1, r-1) \qquad \text{and} \qquad \frac{1}{r}(1, 1).$$

In the first case,

$$\mathbb{C}[x,y]^{\mathbb{Z}_r} = \mathbb{C}[x^r, y^r, xy] = \frac{\mathbb{C}[a,b,c]}{\langle ac = b^n \rangle}.$$

Using different coordinates, we have

$$(x^2 + y^2 - z^n = 0) \subset \mathbb{C}^3.$$

Suppose that we blow up the origin. We introduce coordinates s and t such that $x = sz$ and $y = tz$. Then we get

$$(s^2 + t^2 - z^{n-2}) \subset \mathbb{C}^3,$$

and an exceptional divisor, which is the union of two copies of $\mathbb{P}^1$, where the new singular point is the intersection of the two copies of $\mathbb{P}^1$. Continuing in this way, we get a chain of $n-1$, -2-curves. This is called an A_{n-1}-singularity.

We can encode the resolution by using a graph. The vertices are the exceptional divisors, and edges correspond to intersection of two exceptional divisors. We further label the vertices by minus the self-intersection of the exceptional divisors. An A_n-singularity corresponds therefore to a chain of n vertices, all labelled with 2.

At the opposite extreme, consider $1/r(1,1)$. Then

$$\mathbb{C}[x,y]^{\mathbb{Z}_r} = \mathbb{C}[x^r, x^{r-1}y, x^{r-2}y^2, \dots, y^r],$$

which is the coordinate ring of the cone over a rational normal curve of degree r. The minimal resolution consists of a single copy of $\mathbb{P}^1$, with self-intersection $-r$. The corresponding graph is a single vertex labelled by r.

Let S be any singular surface. The **minimal resolution** of S is a (the) relatively minimal model $\pi\colon T \longrightarrow S$ over S. That is, take any log resolution of S, and run a relatively minimal model program over S. The resulting morphism π is characterised by the property that it does not contract any -1-curves.

Theorem 15.4. *Let S be a cyclic quotient singularity of type $1/r(1,a)$.*

Then the graph of the minimal resolution of S is a chain of $\mathbb{P}^1$'s, labelled by $(a_1, a_2, \dots, a_k)$, where

$$\frac{r}{a} = a_1 - \frac{1}{a_2 - \dots},$$

a continued fraction.

For example, consider $1/11(1,5)$. We have

$$\begin{aligned}
\frac{11}{5} &= 3 - \frac{4}{5} \\
&= 3 - \frac{1}{5/4} \\
&= 3 - \frac{1}{2 - 3/4} \\
&= 3 - \frac{1}{2 - \frac{1}{4/3}} \\
&= 3 - \frac{1}{2 - \frac{1}{2 - 2/3}} \\
&= 3 - \frac{1}{2 - \frac{1}{2 - \frac{1}{3/2}}} \\
&= 3 - \frac{1}{2 - \frac{1}{2 - \frac{1}{2 - 1/2}}}
\end{aligned}$$

Thus the minimal resolution is a chain of 5 $\mathbb{P}^1$'s, of self-intersection $(-3,-2,-2,-2,-2)$.

Theorem 15.5. *Let S be a surface, let C be a smooth curve on S, and suppose that S has cyclic quotient singularities of index $r_1, r_2, \dots, r_k$, such that the strict transform of C always intersects one end of the chain at a single point.*

Then

$$(K_S + C)|_C = K_C + \sum \frac{r_i - 1}{r_i} p_i,$$

where p_i are the points of C where S is singular, and the log discrepancy of the pair (S, C) is greater than zero (in fact equal to the minimum of $1/r_i$).

Proof. One can prove this in two ways. One is by direct computation, on the minimal resolution. The second is to use the Riemann-Hurwitz formula. □

Kawamata log terminal singularities are completely classified for surfaces.

Theorem 15.6. *Let S be a kawamata log terminal surface.*

Then the resolution graph of S is either a chain, or has one vertex of degree three, attached to three chains. If the indices of the chains *are p, q and r, then*

$$(p, q, r) = (2, 2, m), (2, 3, 3), (2, 3, 4), (2, 3, 5).$$

The log discrepancy is one if and only if each self-intersection is -2. The corresponding singularities are known as Du Val singularities, and the corresponding graphs are known as A_n, D_n, E_6, E_7 and E_8.

Proof. Suppose that there is a vertex of degree at least three. Let ν be the corresponding valuation. Then we can find a morphism $\pi\colon T \longrightarrow S$ which extracts precisely the exceptional divisor associated to ν (in other words, contract all other divisors on the minimal resolution). By assumption we may write

$$K_T + E = \pi^* K_S + aE,$$

where $a > 0$. It follows that $K_T + E$ is π-negative. Suppose that the singular points along E are cyclic quotient singularities $p_1, p_2, \dots, p_k$, with indices $r_1, r_2, \dots, r_k$. By adjunction, we have

$$0 > (K_T + E) \cdot E = K_E + \sum \frac{r_i - 1}{r_i} p_i = K_E + \Delta.$$

But then (E, Δ) is a log Fano pair, and the only possibilities have been listed.

With a little more work one can show that the only possibility is that each p_i is cyclic quotient, ie that otherwise $(S, (1-a)E)$ is not kawamata log terminal. □

Corollary 15.7. *Let S be a normal surface.*

S is kawamata log terminal if and only if S has quotient singularities.

Proof. We already know that if S has quotient singularities then it is kawamata log terminal. Now suppose that S is kawamata log terminal. Then the resolution graph is given by (15.6). In characteristic zero the resolution graph determines the singularity and it is not hard to check that any graph in the list is a quotient singularity. □

Definition 15.8. *We say that a log pair (X, Δ) is **log canonical** if the log discrepancy is at least zero.*

Theorem 15.9. *Let S be a log canonical surface which is not kawamata log terminal.*

Then the minimal resolution of S is a smooth elliptic curve, a cycle of $\mathbb{P}^1$'s, a tree with a vertex of degree 3, and indices $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$, or two vertices of degree 3, connected by an interior chain, with two sets of -2-curves at the end, or a vertex with degree 4, attached to 4, -2-curves (the last case is really a degenerate case of the penultimate case).

Proof. In this case,

$$K_T + E = \pi^* K_S + \sum a_i E_i,$$

where $a_i \geq 0$ with equality at least once, and the result follows as in the proof of (15.6). $\square$

16. Cone and Contraction Theorem

The cone and contraction theorem are valid for kawamata log terminal pairs. These results are due principally to Kawamata and Shokurov:

Definition 16.1. *Let $\pi\colon X \longrightarrow Z$ be a proper morphism and let D be an $\mathbb{R}$-Cartier divisor. We say that D is π-**big** if its restriction to the general fibre is big.*

*Let D be an $\mathbb{R}$-divisor. We say that D is π-**semiample** if there is a contraction $\psi\colon X \longrightarrow Y$ over Z such that $D = \psi^* H$, where H is an ample over Z, $\mathbb{R}$-divisor on Y.*

Note that if $\pi\colon X \longrightarrow Z$ is birational then every divisor is big over Z as the generic fibre is a point.

Theorem 16.2 (Kawamata-Viehweg vanishing)**.** *Let $\pi\colon X \longrightarrow Z$ be a projective morphism and let D be an integral $\mathbb{Q}$-Cartier divisor.*

If (X,Δ) kawamata log terminal, $D-(K_X+\Delta)$ is π-nef and π-big then $R^i\pi_*\mathcal{O}_X(D) = 0$ for $i > 0$.

Theorem 16.3 (Base point free theorem)**.** *Let $\pi\colon X \longrightarrow Z$ be a projective morphism.*

If (X,Δ) kawamata log terminal, $K_X+\Delta$ is π-nef and Δ is π-big then $K_X+\Delta$ is π-semiample.

Corollary 16.4. *Let $\pi\colon X \longrightarrow Z$ be a projective morphism.*

If (X,Δ) is kawamata log terminal and $K_X+\Delta$ is π-nef and π-big then $K_X+\Delta$ is π-semiample.

We indicate how (16.4) is derived from (16.3). We will need a simple result about kawamata log terminal pairs:

Lemma 16.5. *Let (X,Δ) be a kawamata log terminal pair and let D be any $\mathbb{R}$-Cartier divisor.*

If $D \geq 0$ then we may find $\delta > 0$ such that $(X,\Delta+\delta D)$ is kawamata log terminal.

Proof. Pick a log resolution of $(X,\Delta+D)$, $\pi\colon Y \longrightarrow X$. By assumption if we write

$$K_Y+\Gamma = \pi^*(K_X+\Delta)$$

then $\lfloor \Gamma \rfloor \leq 0$. If $G = \pi^*D$ then

$$\pi^*(\delta D) = \delta\pi^* D = \delta G.$$

and so

$$K_Y+\Gamma+\delta G = \pi^*(K_X+\Delta+\delta D). \qquad \square$$

Proof of (16.4). By assumption $K_X + \Delta \sim_{\mathbb{R}} D \geq 0$. Pick $\delta > 0$ such that $(X, \Delta + \delta D)$ is kawamata log terminal. As $\Delta + \delta D$ is π-big we may apply (16.3) to

$$K_X + \Delta + \delta D \sim_{\mathbb{R}} (1+\delta)(K_X + \Delta)$$

to conclude that $K_X + \Delta$ is π-semiample. □

Theorem 16.6 (Cone Theorem)**.** *Let (X, Δ) be a kawamata log terminal pair and let $\pi\colon X \longrightarrow Z$ be a projective morphism.*

Then

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta \geq 0} + \sum_i R_i = \mathbb{R}^+[C_i],$$

where R_i are countably many extremal rays spanned by rational curves C_i contracted by π, such that $0 < -(K_X + \Delta) \cdot C_i \leq 2n$.

In particular if H is any π-ample divisor, then there are only finitely many of these curves such that $(K_X + \Delta + H) \cdot C_i < 0$.

We sketch a proof of a stronger version of (16.6). We will need some preliminary definitions and results:

Definition 16.7. *Let (X, Δ) be a log pair.*

*A **non kawamata log terminal place** is a valuation of log discrepancy at most zero. A **non kawamata log terminal centre** is the centre of a non kawamata log terminal place. We say that a non kawamata log terminal centre is **minimal** if it is minimal with respect to inclusion.*

*The **non kawamata log terminal locus*** $\mathrm{Nklt}(X, \Delta)$ *is the union of the non kawamata log terminal centres.*

In the case when (X, Δ) is log canonical we will also refer to a non kawamata log terminal place (respectively centre, respectively locus) as a log canonical place (respectively centre, respectively locus).

Example 16.8. *Let $(X = \mathbb{P}^2, \Delta = C)$ where C is a nodal cubic. Then (X, Δ) is log canonical and the non kawamata log terminal centres are C and the node. The node is minimal and the non kawamata log terminal locus is the C.*

We will need a basic result about the calculus of log canonical centres:

Theorem 16.9. *Let (X, Δ) be a log canonical pair.*

1. *There are only finitely many log canonical centres.*
2. *The intersection of two log canonical centres is a union of log canonical centres.*
3. *A minimal log canonical centre is normal.*

Theorem 16.10. *Let (X, Δ) be a log pair and let $\pi\colon X \longrightarrow Z$ be a projective morphism.*

Then

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta\geq 0} + i_*\overline{\mathrm{NE}}(Z_{-\infty}) + \sum_i R_i = \mathbb{R}^+[C_i],$$

where $i\colon Z_\infty \longrightarrow X$ is the inclusion of the non kawamata log terminal locus and R_i are countably many extremal rays spanned by rational curves C_i contracted by π, such that $0 < -(K_X + \Delta) \cdot C_i \leq 2n$.

In particular if H is any π-ample divisor, then there are only finitely many of these curves such that $(K_X + \Delta + H) \cdot C_i < 0$.

Corollary 16.11. *Let (X, Δ) be a log pair and let $\pi\colon X \longrightarrow Z$ be a projective morphism.*

If (X, Δ) is log canonical outside finitely many points then

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X+\Delta\geq 0} + \sum_i R_i = \mathbb{R}^+[C_i],$$

where R_i are countably many extremal rays spanned by rational curves C_i contracted by π, such that $0 < -(K_X + \Delta) \cdot C_i \leq 2n$

In particular if H is any π-ample divisor, then there are only finitely many of these curves such that $(K_X + \Delta + H) \cdot C_i < 0$.

Proof. Immediate from (16.10), since $Z_{-\infty}$ contains no curves. $\square$

The following key result is due to Kawamata:

Theorem 16.12. *Let (X, Δ) be a log pair where X is projective and kawamata log terminal. Let H be an ample divisor and let V be the normalisation of a non kawamata log terminal centre W.*

If (X, Δ) is log canonical at the generic point of W then we may write

$$(K_X + \Delta + H)|_V = K_V + \Theta,$$

where (V, Θ) is a log pair and the non kawamata log terminal locus of (V, Θ) is the restriction of the non kawamata log terminal locus of (X, Δ).

Definition 16.13. *Let (X, Δ) be a log canonical pair and let $D \geq 0$ be an $\mathbb{R}$-Cartier divisor. The* ***log canonical threshold*** *of (X, Δ) with respect to D is*

$$\lambda = \sup\{\, t \in \mathbb{R} \,|\, (X, \Delta + tD) \text{ is log canonical}\}.$$

Proof of (16.10). We just prove the absolute case, that is, when Z is a point. As usual pick an ample divisor A such that if μ is the nef threshold of (X, Δ) with respect to A then $D = K_X + \Delta + \mu A = K_X + \Delta + H$ is zero on only one $(K_X + \Delta)$-extremal ray R.

Let $\nu = \nu(X, D)$ be the numerical dimension. There are two cases. If $\nu < n$, that is, if D is not big then we are looking for rational curves which cover X. We apply (14.7) to $D_1, D_2, \dots, D_n$,

$$D_i = \begin{cases} D & \text{if } i \leq \nu + 1 \\ H & \text{otherwise.} \end{cases}$$

With this choice, we have

$$D_1 \cdot D_2 \cdot \ldots D_n = 0$$

and

$$\begin{aligned} -K_X \cdot D_2 \cdot \ldots D_n &= -D_1 \cdot D_2 \cdot \ldots D_n + \Delta \cdot D_2 \cdot \ldots D_n + H \cdot D_2 \cdot \ldots D_n \\ &> 0. \end{aligned}$$

Thus (14.7) implies that X is covered by rational curves Σ such that

$$D \cdot \Sigma = 0 \qquad \text{and} \qquad H \cdot \Sigma \leq 2n \frac{H \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n}{-K_X \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n}.$$

The first condition implies that Σ spans the extremal ray R. Using the first equality, we can rewrite the second inequality as

$$\begin{aligned} -(K_X + \Delta) \cdot \Sigma &= H \cdot \Sigma \\ &\leq 2n \frac{H \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n}{-K_X \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n} \\ &= 2n \frac{-(K_X + \Delta) \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n}{-K_X \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n} \\ &\leq 2n \frac{-K_X \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n}{-K_X \cdot D_2 \cdot D_3 \cdot \dots \cdot D_n} \\ &= 2n. \end{aligned}$$

Now suppose that D is big. Pick G such that $H - G$ is ample, close enough to H such that G is ample and $K_X + \Delta + G$ is big. Then we may find $B \geq 0$ such that

$$B \sim_{\mathbb{R}} K_X + \Delta + G.$$

Consider the closed sets

$$Z_t = \mathrm{Nklt}(X, \Delta + G + tB).$$

If $t = 0$ then we get $Z_{-\infty}$ and if

$$t \leq s \qquad \text{then} \qquad Z_t \subset Z_s.$$

If t is large then Z_t is equal to the support of B and by Noetherian induction

$$\{ Z_t \,|\, t \in [0, \infty) \}$$

is a finite set. Let W be a closed irreducible subset with normalisation V and let $j\colon V \longrightarrow X$ be the composition of the normalisation and inclusion. We say that R *comes from* V if there is a ray S of $\overline{\mathrm{NE}}(V)$ such $i_*S = R$. In this case note that we can choose S extremal.

By construction $B \cdot R < 0$. It follows that $R = \mathbb{R}_{\geq 0}\alpha$ and $\beta \in \mathrm{NE}(X)$ is close enough to α then $B \cdot \beta < 0$ and we may write

$$\beta = \sum a_i [C_i] \qquad \text{where} \qquad B \cdot C_i < 0.$$

It follows that $C_i \subset B$ so that β comes from the normalisation V of a component W of B. But then R comes from the normalisation of a component V of B.

Pick V with the property that it is the normalisation of a component W of some Z_t, R comes from V and W is minimal with this property. If V is the normalisation of a component of $Z_0 = Z_{-\infty}$ then there is nothing to prove. Otherwise let λ be the log canonical threshold of $(X, \Delta + G)$ with respect to B at the generic point of V. By (16.12) we may find (V, Θ) such that

$$(K_X + \Delta + \lambda B + G)|_V = K_V + \Theta,$$

and

$$\mathrm{Nklt}(V, \Theta) = Z_{-\infty}|_V.$$

Clearly $(K_V + \Theta) \cdot S < 0$ and by assumption S does not come from $\mathrm{Nklt}(V, \Theta)$. Therefore we are done by induction on the dimension. $\square$

17. The MMP

Using the cone (16.6) and contraction theorem (16.3), one can define the MMP in all dimensions:

(1) Start with a kawamata log terminal pair (X, Δ).
(2) Is $K_X + \Delta$ nef? Is yes, then stop.
(3) Otherwise by (16.6) there is an extremal ray R of the cone of curves $\overline{\mathrm{NE}}(X)$ on which $K_X + \Delta$ is negative. By (16.3) there is a contraction $\pi\colon X \longrightarrow Z$ of R.
  - **Mori fibre space:** If $\dim Z \leq \dim X$ then the fibres of π are Fano varieties.
  - **Birational contraction:** There are two cases:
    - **Divisorial:** π contracts a divisor. Replace X by Z and return to (2).
    - **Small:** π is small. In this case we cannot replace X by Z.

Lemma 17.1. *Let $\pi\colon X \longrightarrow Z$ be a proper small contraction and let D be $\mathbb{R}$-Cartier.*

If $\pi_* D$ is $\mathbb{R}$-Cartier then $D \cdot C = 0$ for all curves C contracted by π.

Proof. If $E = \pi_* D$ is $\mathbb{R}$-Cartier, then we can pull it back, $\pi^* E = \pi^* \pi_* D$. Now outside of the exceptional locus, D and $\pi^* \pi_* D$ are equal. But then they must be equal, since the exceptional locus is of codimension two or more. Thus

$$D = \pi^* \pi_* D.$$

In particular $D \cdot C = 0$ for all curves C contracted by π. □

Definition 17.2. *Let $\pi\colon X \longrightarrow Z$ be a small extremal contraction, such that $-(K_X + \Delta)$ is π-ample and X is $\mathbb{Q}$-factorial. The **flip** of π, $\pi^+\colon X^+ \longrightarrow Z$ is a small extremal contraction such that $K_{X^+} + \Delta^+$ is π^+-ample and X^+ is $\mathbb{Q}$-factorial.*

Theorem 17.3 (Existence)**.** *Flips exist.*

Using (17.3) we simply replace X by X^+. This raises another issue:

Conjecture 17.4 (Termination)**.** *There is no infinite sequence of flips.*

The problem is that there is no obvious topological reason why we cannot have an infinite sequence of flips.

However if we aim for less than we do get termination. We define the MMP with scaling:

(1) Start with a kawamata log terminal pair (X, Δ), where C is π-big and $K_X + \Delta + C$ is π-nef.

(2) Let $\lambda \geq 0$ be the nef threshold, so that $K_X + \Delta + \lambda C$ is π-nef. Is $\lambda = 0$? Is yes, then stop. $K_X + \Delta$ is nef.
(3) Otherwise by (16.6) there is an extremal ray R of the cone of curves $\overline{\mathrm{NE}}(X)$ on which $K_X + \Delta$ is negative and $K_X + \Delta + C$ is zero. By (16.3) there is a contraction $\pi\colon X \longrightarrow Z$ of R.
    Mori fibre space: If $\dim Z \leq \dim X$ then the fibres of π are Fano varieties.
    Birational contraction: There are two cases:
        Divisorial: π contracts a divisor. Replace X by Z and return to (2).
        Flip: π is small. In this case we replace X by X^+ the flip.

Note that the MMP with scaling is somewhat similar to the relative MMP; using the divisor C we narrow down the set of extremal rays we pick as we run the MMP.

Theorem 17.5 (Termination)**.** *The MMP with scaling always terminates.*

Note that there is an implied big condition which is crucial to the proof. We end with two applications of the MMP with scaling. In the appendix to Hartshorne there is an example due to Hironaka of a smooth proper threefold which is not projective. It is rational, birational to $\mathbb{P}^3$.

Definition 17.6. *Let M be a smooth compact complex manifold.*

We say that M is ***Moishezon*** *if the transcendence degree of the space of meromorphic functions is equal to the dimension.*

In fact M is Moishezon if and only if it is birational to a projective variety. More generally one can extend the notion of Moishezon manifolds to analytic spaces. By a result due to Artin these are the same as algebraic spaces.

Theorem 17.7. *Let $f\colon X \longrightarrow Z$ be a proper morphism of algebraic spaces (or Moishezon spaces).*

If $K_X + \Delta$ is kawamata log terminal and X is $\mathbb{Q}$-factorial, and f does not contract any rational curves then f is a minimal model. In particular f is projective.

Proof. Let $\pi\colon Y \longrightarrow X$ be a log resolution. We may write
$$K_Y + \Gamma' = \pi^*(K_X + \Delta) + E,$$
where $\Gamma' \geq 0$ and $E \geq 0$ have no common components, $\pi_*\Gamma' = \Delta$ and $\pi_*E = 0$. Then $\llcorner \Gamma' \lrcorner = 0$. Pick $F \geq 0$ supported on the exceptional

locus such that $\llcorner \Gamma' + F \lrcorner = 0$. Then $(Y, \Gamma = \Gamma' + F)$ is kawamata log terminal, and

$$K_Y + \Gamma = \pi^*(K_X + \Delta) + E + F.$$

Pick H ample such that $K_Y + \Gamma + H$ is nef and kawamata log terminal. We run the $(K_Y + \Gamma)$-MMP with scaling over Z.

Suppose that some step of this MMP is not over X. Then we get $g\colon Y \longrightarrow Y'$ and $Y' \dashrightarrow X$ is not a morphism. But then X contains a rational curve, the image of a rational curve contracted by f, a contradiction. So the MMP over Z is automatically a MMP over X. As π is birational, the MMP with scaling over X always terminates (every divisor is big over X).

At the end, negativity of contraction implies that $Y \longrightarrow X$ is the identity. But then f is a log terminal model. □

Definition 17.8. *We say that G is **Jordan** if there is a an integer m such that if $H \subset G$ is any finite subgroup then there is a normal abelian subgroup $A \triangleleft H$ of index at most m.*

Note that it suffices to exhibit an abelian subgroup $A \subset H$ of bounded index. The kernel of the natural action of H on the left cosets of A in H is an abelian normal subgroup of index at most $m!$.

Theorem 17.9 (Jordan)**.** $\mathrm{GL}_n(\mathbb{C})$ *is Jordan.*

Corollary 17.10. $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbb{C})$ *is Jordan.*

Proof. $\mathrm{PGL}_{n+1}(\mathbb{C})$ is a subgroup of $\mathrm{GL}_m(\mathbb{C})$, some m. □

Conjecture 17.11 (Serre)**.** $\mathrm{Gal}(K(x_0, x_1, \dots, x_n)/K) = \mathrm{Bir}(\mathbb{P}^n)$ *is Jordan.*

Conjecture 17.12 (Borisov-Alexeev-Borisov)**.** *Fix n and $\epsilon > 0$.*

The family of all projective varieties of dimension n such that $-K_X$ is ample and the log discrepancy is at least ϵ is bounded.

Example 17.13. *Let S be the cone over a rational normal curve of degree n.*

Then S is a projective surface with quotient singularities. Varying n we certainly don't get a bounded family. Let $\pi\colon T \longrightarrow S$ be the minimal resolution. Then $T \simeq \mathbb{F}_n$, the unique $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, with a section E of self-intersection $-n$. T has Picard number two so that S has Picard number one. If we write

$$K_T + E = \pi^* K_S + aE,$$

then $a = 2/n$ so that
$$-K_T - \frac{n-2}{n}E = \pi^*(-K_S).$$
Let L be the image of a fibre F. We have
$$\begin{aligned} -K_S \cdot L &= -\pi^* K_S \cdot F \\ &= -K_T \cdot F - \frac{n-2}{n} E \cdot F \\ &= 2 - \frac{n-2}{n} > 0. \end{aligned}$$
As the Picard rank of S is 1 it follows that $-K_S$ is ample. Of course the log discrepancy is going to zero.

We have the following result of Prokhorov and Shramov:

Theorem 17.14. $(17.12)_n$ *implies* $(17.11)_n$.

We sketch the proof. We will in fact prove much more. We will prove the same result provided X is rationally connected and in this case we will exhibit a universal m, depending only on the dimension. As part of the induction we will prove that there is a point fixed by a subgroup of bounded index.

Lemma 17.15. *Let X be a quasi-projective variety and let G be a finite subgroup of the birational automorphism group of X.*

Then we may find a smooth projective variety Y birational to X such that $G \subset \operatorname{Aut}(Y)$.

Proof. We may assume that X is projective. Let $L = K(X)$ be the function field of X and let $M = L^G$ be the fixed field. Then $Z = X/G$ is a variety with functional field M. Let Y be the normalisation of Z in L. □

According to (17.15) we may assume that if G is a finite group then it is a subgroup of the automorphism group. We run the G-equivariant K_X-MMP, with scaling of an ample divisor H. Since X is rationally connected and X is smooth we will never get to a minimal model. Therefore we never get to the case $t = 0$. Every step of the K_X-MMP with scaling of H is also a step of the $(K_X + \Delta)$-MMP with scaling of H, where $\epsilon > 0$ is sufficiently small, so that $(X, \Delta = \epsilon H)$ is kawamata log terminal. Thus this MMP always terminates.

At the end we have a Mori fibre space, $\pi\colon X \longrightarrow Z$. G acts on both X and Z and π is equivariant. There are two cases. If $\dim Z > 0$ then let F be the generic fibre. There is an exact sequence
$$0 \longrightarrow G_0 \longrightarrow G \longrightarrow G_1 \longrightarrow 0$$

where $G_0 \subset \mathrm{Aut}(F)$ and $G_1 \subset \mathrm{Aut}(Z)$. F is a Fano variety and so it is rationally connected. Z is rationally connected as it is the image of X. It is not hard to check we are done by induction on the dimension.

Otherwise Z is a point and X is a Fano variety of Picard number one. As we ran the K_X-MMP, X has terminal singularities. By $(17.12)_n$, applied with $\epsilon = 1$, X belongs to a bounded family. It follows that there is some fixed m and N such that $-mK_X$ is very ample and embeds X in $\mathbb{P}^N$. In this case $G \subset \mathrm{PGL}_{N+1}(\mathbb{C})$.

The major unsolved conjecture would seem to be:

Conjecture 17.16 (Abundance)**.** *Let* (X, Δ) *be a kawamata log terminal pair.*

Then $K_X + \Delta$ *is nef if and only if it is semiample.*

Challenge case: Prove (17.16) when $\Delta = 0$ and X is a smooth surface S, $q = 0$ and $\nu = 1$, without cheating (e.g using the fact that $\chi(S, \mathcal{O}_S) > 0$). The key point is to exhibit elliptic curves through every point.

The main point is to prove:

Conjecture 17.17. *Let* X *be a smooth projective variety.*

Then either

(1) $\kappa(X) \geq 0$, *or*

(2) X *is uniruled.*