1. Rational maps

It is often the case that we are given a variety X and a morphism defined on an open subset U of X. As open sets in the Zariski topology are very large, it is natural to view this as a map on the whole of X, which is not everywhere defined.

Definition 1.1. *A* ***rational map*** $\phi\colon X \dashrightarrow Y$ *between quasi-projective varieties is a pair* (f, U) *where* U *is a dense open subset of* X *and* $f\colon U \longrightarrow Y$ *is a morphism of varieties. Two rational maps* (f_1, U_1) *and* (f_2, U_2) *are considered equal if there is a dense open subset* $V \subset U_1 \cap U_2$ *such that the two functions* $f_1|_V$ *and* $f_2|_V$ *are equal.*

It is customary to avoid using the pair notation and to leave U unspecified. We often say in this case that ϕ is defined on U. Note that if U and V are two dense open sets, and (f, U), (g, V) represent the same rational map, then $(h, U \cup V)$ also represents the same map, where h is defined in the obvious way. By Noetherian induction, it follows that there is a largest open set on which ϕ is defined, which is called the **domain of** ϕ. The complement of the domain is called the **locus of indeterminacy**.

One way to get a picture of a rational map, is to consider the graph.

Definition 1.2. *Let* $\phi\colon X \dashrightarrow Y$ *be a rational map.*

The ***graph of*** ϕ *is the closure of the graph of* f*, where the pair* (f, U) *represents* ϕ*.*

The ***image of*** ϕ *is the image of the graph of* ϕ *under the second projection.*

Note that the domain of ϕ is precisely the locus where the first projection map is an isomorphism.

Definition 1.3. *Let* $\phi\colon X \dashrightarrow Y$ *and* $\psi\colon Y \dashrightarrow Z$ *be two rational maps. Suppose that* $\phi = (f, U)$ *and* $\psi = (g, V)$ *and that* $f(U) \cap V$ *is non-empty. Then we may define the composition of* ϕ *and* ψ *by taking the pair* $(g \circ f, f^{-1}(V))$*.*

Note that in general, we cannot compose rational maps. The problem might be that the image of the first map might lie in the locus where the second map is not defined. However there will never be a problem if X is irreducible and ϕ is dominant:

Definition 1.4. *We say that* ϕ *is* ***dominant*** *if the closure of the image of* ϕ *is the whole of* Y*.*

Note that this gives us a category, the category of irreducible varieties and dominant rational maps.

Definition 1.5. *We say that a dominant rational map $\phi\colon X \dashrightarrow Y$ of irreducible quasi-projective varieties is birational if it has an inverse. In this case we say that X and Y are* ***birational***. *We say that X is* ***rational*** *if it is birational to $\mathbb{P}^n$.*

It is interesting to see an example. Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the map

$$[X : Y : Z] \longrightarrow [YZ : XZ : XY].$$

This map is clearly a rational map. It is called a **Cremona transformation**. Note that it is a priori not defined at those points where two coordinates vanish. To get a better understanding of this map, it is convenient to rewrite it as

$$[X : Y : Z] \longrightarrow [1/X : 1/Y : 1/Z].$$

Written as such it is clear that this map is an involution, so that it is in particular a birational map.

It is interesting to check whether or not this map really is well defined at the points $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$. To do this, we need to look at the graph.

To get a better picture of what is going on, consider the following map,

$$\mathbb{A}^2 \dashrightarrow \mathbb{A}^1,$$

which assigns to a point $p \in \mathbb{A}^2$ the slope of the line connecting the point p to the origin,

$$(x, y) \longrightarrow x/y.$$

Now this map is not defined along the locus where $y = 0$. Replacing $\mathbb{A}^1$ with $\mathbb{P}^1$ we get a map

$$(x, y) \longrightarrow [x : y].$$

Now the only point where this map is not defined is the origin. We consider the graph,

$$\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Consider how the graph sits over $\mathbb{A}^2$. Outside the origin the first projection is an isomorphism. Over the origin the graph is contained in a copy of the image, that is, $\mathbb{P}^1$. Consider any line $y = tx$ through the origin. Then this line, minus the origin, is sent to the point with slope t. It follows that the closure of this line is sent to the point with slope t. Varying t, it follows that any point of the fibre over $\mathbb{P}^1$ is a point of the graph.

Thus the morphism $\pi\colon \Gamma \longrightarrow \mathbb{A}^2$ is an isomorphism outside the origin and contracts a whole copy of $\mathbb{P}^1$ to a point. For this reason, we call π a blow up:

Definition-Lemma 1.6. *Let $\phi\colon X \dashrightarrow Y$ be a rational map between quasi-projective varieties. We say that ϕ is* ***proper*** *if both of the induced morphisms $\Gamma \longrightarrow X$ and $\Gamma \longrightarrow Y$ are proper.*

If I is an ideal in the homogeneous coordinate ring of $X \subset \mathbb{P}^n$ and ϕ is given locally by generators of I, then we say that π is the ***blow up of the ideal*** *I.*

Proof. We have to check that the blow up does not depend on the choice of generators. The result is local on X, so that we may assume that X is affine. Suppose that $f_1, f_2, \dots, f_k$ and $g_1, g_2, \dots, g_l$ are two sets of generators of I. Then $f_1, f_2, \dots, f_k$ and $f_1, f_2, \dots, f_k, g = g_1$. are two sets of generators of I and it suffices, by a standard induction, to prove that these two give the same blow up. By assumption we may find polynomials $h_1, h_2, \dots, h_k$ such that

$$g = h_1 f_1 + h_2 f_2 + \dots + h_k f_k.$$

The rational map

$$\phi = [f_1 : f_2 : \dots : f_k : g]\colon X \dashrightarrow \mathbb{P}^k,$$

factors through the rational map

$$\psi = [f_1 : f_2 : \dots : f_k]\colon X \dashrightarrow \mathbb{P}^{k-1},$$

via the rational map

$$\alpha\colon \mathbb{P}^k \dashrightarrow \mathbb{P}^{k-1} \qquad \text{where} \qquad \alpha[y_1 : y_2 : \dots : y_k] = [y_1 : y_2 : \dots : y_{k-1}],$$

is projection from the point $p = [0 : 0 : \dots : 0 : 1]$. The only point of indeterminacy of α is the point p. Since the equation

$$y_{k+1} = h_1 y_1 + h_2 y_2 + \dots + h_k y_k,$$

is valid on Γ_ϕ, it follows that $p \notin \Gamma_\phi$. But then the induced rational map

$$\Gamma_\phi \dashrightarrow \Gamma_\psi,$$

is in fact a morphism. The inverse map is induced by the standard inclusion

$$\mathbb{P}^{k-1} \longrightarrow \mathbb{P}^k,$$

and so the morphism

$$\Gamma_\phi \longrightarrow \Gamma_\psi,$$

is an isomorphism, which is all we had to prove. □

Clearly π is always birational, as it is an isomorphism outside $V(I)$. Note that (1.6) is somewhat ad hoc (to say the least). Later on, we will give a much cleaner definition using schemes. In practice, (1.6) is quite close to how we compute using the blow up.

In our case $I = \langle x, y \rangle$, the maximal ideal of the origin, so that we call p the blow up of a point. Suppose we have coordinates $[S : T]$ on $\mathbb{P}^1$. Then outside of the origin, the graph satisfies the equation $xT = yS$. Thus the closure must satisfy the same equation. Since this equation determines the graph outside the origin, in fact the graph is defined by this equation (as the whole fibre over the origin lives in the graph, we don't need anymore equations).

The inverse image of the origin is called the **exceptional divisor**.

Definition 1.7. *Let $\pi\colon X \longrightarrow Y$ be a birational morphism. The locus where π is not an isomorphism is called the* ***exceptional locus****. If $V \subset Y$, the inverse image of V is called the* ***total transform****. Let Z be the image of the exceptional locus. Suppose that V is not contained in Z. The* ***strict transform of*** *V is the closure of the inverse image of $V - Z$.*

It is interesting to compute the strict transform of some planar curves. We have already seen that lines through the origin lift to curves that sweep out the exceptional divisor. In fact the blow up separates the lines through the origin. These are then the fibres of the second morphism.

Let us now take a nodal cubic,

$$y^2 = x^2 + x^3.$$

We want to figure out its strict transform, so that we need the inverse image in the blow up. Outside the origin, there are two equations to be satisfied,

$$y^2 = x^2 + x^3 \qquad \text{and} \qquad xT = yS.$$

Passing to the coordinate patch $y = xt$, where $t = T/S$, and substituting for y in the first equation we get

$$x^2t^2 - x^2 - x^3 = x^2(t^2 - x - 1).$$

Now if $x = 0$, then $y = 0$, so that in fact locally $x = 0$ is the equation of the exceptional divisor. So the first factor just corresponds to the exceptional divisor. The second factor will tell us what the closure of our curve looks like, that is, the strict transform. Now over the origin, $x = 0$, so that $t^2 = 1$ and $t = \pm 1$. Thus our curve lifts to a curve which intersects the exceptional divisor in two points. (If we compute in the coordinate patch $x = sy$, we will see that the curve does not meet the point at infinity). These two points correspond to the fact that the nodal cubic has two tangent lines at the origin, one of slope 1 and one of slope -1. We call the closure of the inverse image outside the origin as the strict transform (the total transform being just the whole inverse image).

Now consider what happens for the cuspidal cubic, $y^2 = x^3$. In this case we get

$$(xt)^2 - x^3 = x^2(t^2 - x).$$

Once again the factor of x^2 corresponds to the fact that the inverse image surely contains the exceptional divisor. But now we get the equation $t^2 = 0$, so that there is only one point over the origin, as one might expect from the geometry.

Let us go back to the Cremona transformation. To compute what gets blown up and blown down, it suffices to figure out what gets blown down, by symmetry. Consider the line $X = 0$. If $bc \neq 0$, the point $[0 : b : c]$ gets mapped to $[0 : 0 : 1]$. Thus the strict transform of the line $X = 0$ in the graph gets blown down to a point. By symmetry the strict transforms of the other two lines are also blown down to points. Outside of the union of these three lines, the map is clearly an isomorphism.

Thus the involution blows up the three points $[0 : 0 : 1]$, $[0 : 1 : 0]$, and $[1 : 0 : 0]$ and then blows down the three disjoint lines. Note that the three exceptional divisors become the three new coordinate lines.

One of the most impressive results of the nineteenth century is the following characterisation of the birational automorphism group of $\mathbb{P}^2$.

Theorem 1.8 (Noether)**.** *The birational automorphism group is generated by a Cremona transformation and* PGL(3)*.*

This result is very deceptive, since it is known that the birational automorphism group is, by any standards, very large.

Definition 1.9. *A **rational function** is a rational map to $\mathbb{A}^1$.*

*The set of all rational functions, denoted $K(X)$, is called the **function field**.*

Lemma 1.10. *Let X be an irreducible variety.*

Then the function field is a field. If $U \subset X$ is any open affine subset, then function field is precisely the field of fractions of the coordinate ring.

Proof. Clear, since on an irreducible variety, any rational function is determined by its restriction to any open subset, and locally any morphism is given by a rational function. □

Proposition 1.11. *Let K be an algebraically closed field.*

Then there is an equivalence of categories between the category of irreducible varieties over K with morphisms the dominant rational maps, and the category of finitely generated field extensions of K.

Proof. Define a functor F from the category of varieties to the category of fields as follows. Given a variety X, let $K(X)$ be the function field of X. Given a rational map $\phi\colon X \dashrightarrow Y$, define $F(\phi)\colon K(Y) \longrightarrow K(X)$ by composition. If f is a rational function on Y, then $\phi \circ f$ is a rational function on X.

We have to check that F is essentially surjective and fully faithful. Suppose that L is a finitely generated field extension of K. Then $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$. Let $A = K[\alpha_1, \alpha_2, \dots, \alpha_n]$. Let X be any affine variety with coordinate ring A. Then X is irreducible as A is an integral domain and the function field of X is precisely L as this is the field of fractions of A.

The fact that F is fully faithful is proved in the same way as before. □

Proposition 1.12. *Let X and Y be two irreducible varieties.*

Then the following are equivalent

(1) *X and Y are birational.*
(2) *X and Y contain isomorphic open subsets.*
(3) *The function fields of X and Y are isomorphic.*

Proof. We have already seen that (1) and (3) are equivalent and clearly (2) implies (1) (or indeed (3)). It remains to prove that if X and Y are birational then they contain isomorphic open subsets.

Let $\phi\colon X \dashrightarrow Y$ be a birational map with inverse $\psi\colon Y \dashrightarrow X$. Suppose that ϕ is defined on U and ψ is defined on V. Let $U' = \phi^{-1}(V) \subset U$. Let f be the restriction of ϕ to U'. Then $f\colon U' \longrightarrow f(U') \subset V$. Suppose that ψ is represented by (g, V).

The composition $g \circ f\colon U' \longrightarrow U'$ is the identity morphism, as it is the identity on an open subset. Therefore $f(U') = g^{-1}(U')$ is open and so $g : f(U') \longrightarrow U'$ is the inverse of f. Indeed $f \circ g$ and $g \circ f$ are both morphisms and equal to the identity on dense open subsets, so that they are both the identity morphism. So U' and $f(U')$ are isomorphic open subsets. □

Corollary 1.13. *Let X be an irreducible variety.*

Then the following are equivalent

(1) *X is rational.*
(2) *X contains an open subset of $\mathbb{P}^n$.*
(3) *The function field of X is a purely transcendental extension of K.*

Proof. Immediate from (1.12). □

Let us consider some examples. I claim that the curve $C = V(y^2 - x^2 - x^3)$ is rational. We have already seen that there is a morphism

$\mathbb{A}^1 \longrightarrow C$. We want to show that it is a birational map. One way to proceed is to construct the inverse. In fact the inverse map is $C \dashrightarrow \mathbb{A}^1$ given by $(x, y) \longrightarrow y/x$. Another way to proceed is to prove that the function field is purely transcendental. Now the coordinate ring is

$$K[x, y]/\langle y^2 - x^2 - x^3 \rangle.$$

So the fraction field is $K(x, y)$, where $y^2 = x^2 + x^3$. Consider $t = y/x$. I claim that $K(t) = K(x, y)$. Clearly there is an inclusion one way. Now $t^2 = y^2/x^2 = 1 + x$. So $x = t^2 - 1 \in K(t)$. But $y = tx$, so that we do indeed have equality $K(t) = K(x, y)$. Thus C is rational.

Perhaps a more interesting example is to consider the Segre variety $V \subset \mathbb{P}^3$. Consider projection π from a point p of the Segre variety,

$$\pi \colon V \dashrightarrow \mathbb{P}^2.$$

Clearly the only possible point of indeterminacy is the point p. Since a line, not contained in V, meets the Segre variety in at most two points, it follows that this map is one to one outside p, unless that line is contained in V. On the other hand, if $q \in \mathbb{P}^2$, the line $\langle p, q\rangle$ will meet the Segre variety in at least two points, one of which is p.

Now at the point p, there passes two lines l and m (one line of each ruling). These get mapped to two separate points, say q and r. It follows that p is indeed a point of indeterminacy. To proceed further, it is useful to introduce coordinates. Suppose that $p = [0 : 0 : 0 : 1]$, where $V = V(XW - YZ)$.

Now projection from $p \in \mathbb{P}^3$ defines a rational map

$$\phi \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2,$$

whose exceptional locus is a copy of $\mathbb{P}^2$. Indeed the graph of ϕ lies in $\mathbb{P}^3 \times \mathbb{P}^2$ and as before over the point p, we get a copy of the whole of the image $\mathbb{P}^2$, as can be seen by looking at lines through p. Working on the affine chart $W \neq 0$, V is locally defined as $x = yz$. If $[R : S : T]$ are coordinates on $\mathbb{P}^2$, the equations for the blow up of $\mathbb{P}^3$ are given as

$$xS = yR \qquad xT = zR \qquad yT = zS.$$

The blow up of V at p is given as the strict transform of V in the blow up of $\mathbb{P}^3$. We work in the patch $T \neq 0$. Then $x = rz$ and $y = sz$ so that the we get the equation

$$rz - sz^2 = z(r - sz) = 0.$$

Now $z = 0$ corresponds to the whole exceptional locus so that $r = sz$ defines the strict transform. In this case $z = 0$, means $r = 0$, so that we get a line in the exceptional $\mathbb{P}^2$.

In other words the graph of π is the blow up of p, with an exceptional divisor isomorphic to $\mathbb{P}^1$. The graph of π then blows down the strict transform of the two lines. Note that the image of the exceptional divisor is precisely the line connecting the two points q and r.

To see that π is birational, we write down the inverse,

$$\psi\colon \mathbb{P}^2 \dashrightarrow V.$$

Given $[R : S : T]$, we send this to $[R : S : T : ST/R]$. Clearly this lies on the quadric $XW - YZ$ and is indeed the inverse map. Note that the inverse map blows up q and r then blows down the line connecting them to p.

In fact it turns out that the picture above for rational maps on surfaces is the complete picture.

Theorem 1.14 (Elimination of Indeterminancy)**.** *Let $\phi\colon S \dashrightarrow Z$ be a rational map from a smooth surface.*

Then there is an iterated sequence of blow ups of points $p\colon T \longrightarrow S$ such that the induced rational map $\psi\colon T \longrightarrow Z$ is a morphism.

Theorem 1.15. *Let $\phi\colon S \dashrightarrow T$ be a birational map of smooth surfaces.*

Then there is an iterated sequence of blow ups of points $p\colon W \longrightarrow S$ such that the induced map $q\colon W \longrightarrow T$ is also an iterated sequence of blow up of points.

In fact it turns out that both of these results generalise to all dimensions. In the first result, one must allow blowing up the ideal of any smooth subvariety. In the second result, one must allow mixing up the sequence of blowing up and down, although it is conjectured that the one can perform first a sequence of blow ups and then a sequence of blow downs.

Another way to proceed, is to compute the field of fractions. The coordinate ring on the affine piece $W \neq 0$ is

$$K[x, y, z]/\langle x - yz\rangle = k[y, z].$$

The field of fractions is visibly then $K(y, z)$. However perhaps the easiest way to proceed is to observe that $\mathbb{P}^1 \times \mathbb{P}^1$ contains $\mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{A}^2$, so that the Segre Variety is clearly rational.

In fact it turns out in general to be a vary hard problem to determine which varieties are rational. As an example of this consider Lüroth's problem.

Definition 1.16. *We say that a variety X is **unirational** if there is a dominant rational map $\phi\colon \mathbb{P}^n \dashrightarrow X$.*

Question 1.17 (Lüroth)**.** *Is every unirational variety rational?*

Note that one way to restate Lüroth's problem is to ask if every subfield of a purely transcendental field extension is purely transcendental. It turns out that the answer is yes in dimension one, in all characteristics. This is typically a homework problem in a course on Galois Theory.

In dimension two the problem is already considerably harder, and it is false if one allows inseparable field extensions. The first step is in fact to establish (1.14) and (1.15).

In dimension three it was shown to be false even in characteristic zero, in 1972, using three different methods.

One proof is due to Artin and Mumford. It had been observed by Serre that the cohomology ring of a smooth unirational threefold is indistinguishable from that of a rational variety (for $\mathbb{P}^3$ one gets $\frac{\mathbb{Z}[x]}{\langle x^3 \rangle}$, and the cohomology ring varies in a very predictable under blowing up and down) except possibly that there might be torsion in $H^3(X, \mathbb{Z})$. They then give an reasonably elementary construction of a threefold with non-zero torsion in H^3.

Another proof is due to Clemens and Griffiths. It is not hard to prove that every smooth cubic hypersurface in $\mathbb{P}^4$ is unirational. On the other hand they prove that some smooth cubics are not rational. To prove this consider the family of lines on the cubic. It turns out that this is a two dimensional family, and that a lot of the geometry of the cubic is controlled by the geometry of this surface.

The third proof is due to Iskovskikh and Manin. They prove that every smooth quartic in $\mathbb{P}^4$ is not rational. On the other hand, some quartics are unirational. In fact they show, in an amazing tour de force, that the birational automorphism group of a smooth quartic is finite. Clearly this means that a smooth quartic is never rational.

2. Toric varieties

First some stuff about algebraic groups:

Definition 2.1. *Let G be a group. We say that G is an **algebraic group** if G is a quasi-projective variety and the two maps $m\colon G\times G \longrightarrow G$ and $i\colon G \longrightarrow G$, where m is multiplication and i is the inverse map, are both morphisms.*

It is easy to give examples of algebraic groups. Consider the group $G = \mathrm{GL}_n(K)$. In this case G is an open subset of $\mathbb{A}^{n^2}$, the complement of the zero locus of the determinant, which expands to a polynomial. Matrix multiplication is obviously a morphism, and the inverse map is a morphism by Cramer's rule. Note that there are then many obvious algebraic subgroups; the orthogonal groups, special linear group and so on. Clearly $\mathrm{PGL}_n(K)$ is also an algebraic group; indeed the quotient of an algebraic group by a closed normal subgroup is an algebraic group. All of these are affine algebraic groups.

Definition 2.2. *Let G be an algebraic group. If G is affine then we say that G is a **linear algebraic group**. If G is projective and connected then we say that G is an **abelian variety**.*

Note that any finite group is an algebraic group (both affine and projective). It turns out that any affine group is always a subgroup of a matrix group, so that the notation makes sense.

Definition 2.3. *The group $\mathbb{G}_m$ is $\mathrm{GL}_1(K)$. The group $\mathbb{G}_a$ is the subgroup of $\mathrm{GL}_2(K)$ of upper triangular matrices with ones on the diagonal.*

Note that as a group $\mathbb{G}_m$ is the set of units in K under multiplication and $\mathbb{G}_a$ is equal to K under addition, and that both groups are affine of dimension 1; in fact they are the only linear algebraic groups of dimension one, up to isomorphism.

Note that if we are given a linear algebraic group G, we get a Hopf algebra A. Indeed if A is the coordinate ring of G, then A is a K-algebra and there are maps

$$A \longrightarrow A \otimes A \qquad \text{and} \qquad A \longrightarrow A,$$

induced by the multiplication and inverse map for G.

It is not hard to see that the product of two algebraic groups is an algebraic group.

Definition 2.4. *The algebraic group $\mathbb{G}_m^k$ is called a **torus**.*

So a torus in algebraic geometry is just a product of copies of $\mathbb{G}_m$.

In fact one can define what it means to be a group scheme:

Definition 2.5. *Let $\pi\colon X \longrightarrow S$ be a morphism of schemes. We say that X is a* ***group scheme*** *over S, if there are three morphisms, $e\colon S \longrightarrow X$, $\mu\colon X \underset{S}{\times} X \longrightarrow X$ and $i\colon X \longrightarrow X$ over S which satisfy the obvious axioms.*

We can define a group scheme $\mathbb{G}_{m,\operatorname{Spec}\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$, by taking

$$\operatorname{Spec}\mathbb{Z}[x, x^{-1}].$$

Given any scheme S, this gives us a group scheme $\mathbb{G}_{m,S}$ over S, by taking the fibre square. In the case when $S = \operatorname{Spec} k$, k an algebraically closed field, then $\mathbb{G}_{m,\operatorname{Spec} k}$ is $t(\mathbb{G}_m)$, the scheme associated to the quasi-projective variety $\mathbb{G}_m$. We will be somewhat sloppy and not be too careful to distinguish the two cases.

Similarly we can take

$$\mathbb{G}_{a,\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x].$$

Definition 2.6. *Let G be an algebraic group and let X be a variety acted on by G, $\pi\colon G \times X \longrightarrow X$. We say that the action is* ***algebraic*** *if π is a morphism.*

For example the natural action of $\mathrm{PGL}_n(K)$ on $\mathbb{P}^n$ is algebraic, and all the natural actions of an algebraic group on itself are algebraic.

Definition 2.7. *We say that a quasi-projective variety X is a* ***toric variety*** *if X is irreducible and normal and there is a dense open subset U isomorphic to a torus, such that the natural action of U on itself extends to an action on the whole of X.*

For example, any torus is a toric variety. $\mathbb{A}^n_k$ is a toric variety. The natural torus is the complement of the coordinate hyperplanes and the natural action is as follows

$$((t_1, t_2, \dots, t_n), (a_1, a_2, \dots, a_n)) \longrightarrow (t_1a_1, t_2a_2, \dots, t_na_n).$$

More generally, $\mathbb{P}^n$ is a toric variety. The action is just the natural action induced from the action above. A product of toric varieties is toric.

One thing to keep track of are the closures of the orbits. For the torus there is one orbit. For affine space and projective space the closure of the orbits are the coordinate subspaces.

Definition 2.8. *Let M be a lattice and let $N = \operatorname{Hom}(M, \mathbb{Z})$ be the dual lattice.*

A ***strongly convex rational polyhedral cone*** $\sigma \subset N_{\mathbb{R}} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ *is*

- *a* ***cone****, that is, if* $v \in \sigma$ *and* $\lambda \in \mathbb{R}$, $\lambda \geq 0$ *then* $\lambda v \in \sigma$*;*
- ***polyhedral****, that is,* σ *is the intersection of finitely many half spaces;*
- ***rational****, that is, the half spaces are defined by equations with rational coefficients;*
- ***strongly convex****, that is,* σ *contains no linear spaces other than the origin.*

A ***fan*** *in* N *is a set* F *of finitely many strongly convex rational polyhedra, such that*

- *every face of a cone in* F *is a cone in* F*, and*
- *the intersection of any two cones in* F *is a face of each cone.*

One can reformulate some of the parts of the definition of a strongly rational polyhedral cone. For example, σ is a polyhedral cone if and only if σ is the intersection of finitely many half spaces which are defined by homogeneous linear polynomials. σ is a strongly convex polyhedral cone if and only if σ is a cone over finitely many vectors which lie in a common half space (in other words a strongly convex polyhedral cone is the same as a cone over a polytope). And so on.

We will see that the set of toric varieties, up to isomorphism, are in bijection with fans, up to the action of $\mathrm{SL}(n, \mathbb{Z})$.

We first give the recipe of how to go from a fan to a toric variety. Suppose we start with σ. Form the dual cone

$$\check{\sigma} = \{\, u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0, v \in \sigma \,\}.$$

Now take the integral points,

$$S_\sigma = \check{\sigma} \cap M.$$

Then form the (semi)group algebra,

$$A_\sigma = K[S_\sigma].$$

Finally form the affine variety,

$$U_\sigma = \operatorname{Spec} A_\sigma.$$

Given a semigroup S, to form the semigroup algebra $K[S]$, start with a K-vector space with basis χ^u, as u ranges over the elements of S. Given u and $v \in S$ define the product

$$\chi^u \cdot \chi^v = \chi^{u+v},$$

and extend this linearly to the whole of $K[S]$.

Example 2.9. *For example, suppose we start with $M = \mathbb{Z}^2$, σ the cone spanned by $(1,0)$ and $(0,1)$, inside $N_{\mathbb{R}} = \mathbb{R}^2$. Then $\check{\sigma}$ is spanned by the same vectors in $M_{\mathbb{R}}$. Therefore $S_\sigma = \mathbb{N}^2$, the group algebra is $\mathbb{C}[x,y]$ and so we get $\mathbb{A}^2$. Similarly if we start with the cone spanned by $e_1, e_2, \dots, e_n$ inside $N_{\mathbb{R}} = \mathbb{R}^n$ then we get $\mathbb{A}^n$.*

Now suppose we start with $\sigma = \{0\}$ in $\mathbb{R}$. Then $\check{\sigma}$ is the whole of $M_{\mathbb{R}}$, S_σ is the whole of $M = \mathbb{Z}$ and so $\mathbb{C}[M] = \mathbb{C}[x, x^{-1}]$. Taking Spec *we get the torus $\mathbb{G}_m$.*

More generally we always get a torus of dimension n if we take the origin in $\mathbb{R}^n$. Note that if $\tau \subset \sigma$ is a face then $\check{\sigma} \subset \check{\tau}$ is also a face so that $U_\tau \subset U_\sigma$ is an open subset. In fact

Lemma 2.10. *Let $\tau \subset \sigma \subset N_{\mathbb{R}}$ be a face of the cone σ.*

Then we may find $u \in S_\sigma$ such that

(1) $\tau = \sigma \cap u^{\perp}$,

(2)
$$S_\tau = S_\sigma + \mathbb{Z}^+(-u),$$

(3) *A_τ is a localisation of A_σ, and*

(4) *U_τ is a principal open subset of U_σ.*

Proof. The fact that every face of a cone is cut out by a hyperplane is a standard fact in convex geometry and this is (1). For (2) note that the RHS is contained in the LHS by definition of a cone. If $w \in S_\tau$ then $w + p \cdot u$ is in $\check{\sigma}$ for any p sufficiently large. If we take p to be also an integer this shows that w belongs to the RHS.

Let χ^u be the monomial corresponding to u. (2) implies that A_τ is the localisation of A_σ along χ^u. This is (3) and (4) is immediate from (3). □

Since the cone $\{0\}$ is a face of every cone, the affine scheme associated to a cone always contains a torus, which is then dense. In particular the affine scheme associated to a cone is always irreducible.

Definition 2.11. *Let $S \subset T$ be a subsemigroup of the semigroup T. We say that S is* ***saturated*** *in T if whenever $u \in T$ and $p \cdot u \in S$ for some positive integer p, then $u \in S$.*

Given a subsemigroup $S \subset M$ saturation is always with respect to M.

Lemma 2.12. *Suppose that $S \subset M$.*

Then $K[S]$ is integrally closed if and only if S is saturated.

Proof. Suppose that $K[S]$ is integrally closed.

Pick $u \in M$ such that $v = p \cdot u \in S$ for some positive integer p. Let $b = \chi^u$ and $a = \chi^v \in K[S]$. Then

$$b^p = \chi^{pu} = \chi^v = a,$$

so that b is a root of the monic polynomial $x^p - a \in K[S][x]$. As we are assuming that $K[S]$ is integrally closed this implies that $b \in K[S]$ which implies that $u \in S$. Thus S is saturated.

Now suppose that S is saturated. As $K[S] \subset K[M]$ and the latter is integrally closed, the integral closure L of $K[S]$ sits in the middle, $K[S] \subset L \subset K[M]$. The torus acts naturally on $K[M]$ and this action fixes $K[S]$, so that it also fixes L. L is therefore a direct sum of eigenspaces, which are all one dimensional (a set of commuting diagonalisable matrices are simultaneously diagonalisable) that is L has a basis of the form χ^u, as u ranges over some subset of M. It suffices to prove that $u \in S$.

Since $b = \chi^u$ is integral over $K[S]$, we may find $k_1, k_2, \dots, k_p \in K[S]$ such that

$$b^p + k_1 b^{p-1} + \cdots + k_p = 0.$$

We may assume that every term is in the same eigenspace as b^p. We may also assume that $k_p \neq 0$. As b^p and $k_p \neq 0$ belong to the same eigenspace, which is one dimensional, we get $b^p \in K[S]$. Thus $pu \in S$ and so $u \in S$ as S is saturated. Thus $b \in K[S]$ and $K[S]$ is integrally closed. $\square$

Note that S_σ is automatically saturated, as $\check{\sigma}$ is a rational polyhedral cone. In particular U_σ is normal.

Example 2.13. *Suppose that we start with the semigroup S generated by 2 and 3 inside $M = \mathbb{Z}$. Then*

$$K[S] = K[t^2, t^3] = K[x, y]/\langle y^2 - x^3 \rangle.$$

Note that this does come with an action of $\mathbb{G}_m$;

$$(t, x, y) \longrightarrow (t^2 x, t^3 y).$$

However the curve $V(y^2 - x^3) \subset \mathbb{A}^2$ is not normal.

In fact some authorities drop the requirement that a toric variety is normal.

An action of the torus corresponds to a map of algebras

$$A_\sigma \longrightarrow A_\sigma \underset{K}{\otimes} A_0,$$

which is naturally the restriction of

$$A_0 \longrightarrow A_0 \underset{K}{\otimes} A_0.$$

It is straightforward to check that the restricted map does land in $A_\sigma \underset{K}{\otimes} A_0$.

Lemma 2.14 (Gordan's Lemma)**.** *Let $\sigma \subset \mathbb{M}_{\mathbb{R}}$ be a strongly convex rational cone.*

Then S_σ is a finitely generated semigroup.

Proof. Pick vectors $v_1, v_2, \dots, v_n \in S_\sigma$ which generate the cone $\check{\sigma}$. Consider the set

$$K = \{\, v \in M \mid v = \sum t_i v_i, t_i \in [0, 1] \,\}.$$

Then K is compact. As M is discrete $K \cap M$ is finite. I claim that the elements of $K \cap M$ generate the semigroup S_σ. Pick $u \in S_\sigma$. Since $u \in \check{\sigma}$ and $v_1, v_2, \dots, v_n$ generate the cone, we may write

$$u = \sum \lambda_i v_i,$$

where $\lambda_i \in \mathbb{Q}$. Let $n_i = \llcorner \lambda_i \lrcorner$. Then

$$u - \sum n_i v_i = \sum (\lambda_i - \llcorner \lambda_i \lrcorner) v_i \in K \cap M.$$

As $v_1, v_2, \dots, v_n \in K \cap M$ the result follows. $\square$

Gordan's lemma (2.14) implies that U_σ is of finite type over K. So U_σ is an affine toric variety.

Example 2.15. *Suppose we start with the cone spanned by e_2 and $2e_1 - e_2$. The dual cone is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the semigroup are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. The corresponding group algebra is $A_\sigma = K[x, xy, xy^2]$. Suppose we call $u = x$, $v = xy$ and $w = xy^2$. Then $v^2 = x^2y^2 = x(xy^2) = uw$. Therefore the corresponding affine toric variety is given as the zero locus of $v^2 - uw$ in $\mathbb{A}^3$.*

Given a fan F, we get a collection of affine toric varieties, one for every cone of F. It remains to check how to glue these together to get a toric variety. Suppose we are given two cones σ and τ belonging to F. The intersection is a cone ρ which is also a cone belonging to F. Since ρ is a face of both σ and τ there are natural inclusions

$$U_\rho \subset U_\sigma \qquad \text{and} \qquad U_\rho \subset U_\tau.$$

We glue U_σ to U_τ using the natural identification of the common open subset U_ρ. Compatibility of gluing follows automatically from the fact that the identification is natural and from the combinatorics of the fan.

It is clear that the resulting scheme is of finite type over the groundfield. Separatedness follows from:

Lemma 2.16. *Let σ and τ be two cones whose intersection is the cone ρ.*

If ρ is a face of each then the diagonal map

$$U_\rho \longrightarrow U_\sigma \times U_\tau,$$

is a closed embedding.

Proof. This is equivalent to the statement that the natural map

$$A_\sigma \otimes A_\tau \longrightarrow A_\rho,$$

is surjective. For this, one just needs to check that

$$S_\rho = S_\sigma + S_\tau.$$

One inclusion is easy; the RHS is contained in the LHS. For the other inclusion, one needs a standard fact from convex geometry, which is called the separation lemma: there is a vector $u \in S_\tau \cap S_{-\tau}$ such that simultaneously

$$\rho = \sigma \cap u^\perp \qquad \text{and} \qquad \rho = \tau \cap u^\perp.$$

By the first equality $S_\rho = S_\sigma + \mathbb{Z}(-u)$. As $u \in S_{-\tau}$ we have $-u \in S_\tau$ and so the LHS is contained in the RHS. □

So we have shown that given a fan F we can construct a normal variety $X = X(F)$. It is not hard to see that the natural action of the torus corresponding to the zero cone extends to an action on the whole of X. Therefore $X(F)$ is indeed a toric variety.

Let us look at some examples.

Example 2.17. *Suppose that we start with $M = \mathbb{Z}$ and we let F be the fan given by the three cones $\{0\}$, the cone spanned by e_1 and the cone spanned by $-e_1$ inside $N_\mathbb{R} = \mathbb{R}$. The two big cones correspond to $\mathbb{A}^1$. We identify the two $\mathbb{A}^1$'s along the common open subset isomorphic to K^*. Now the first $\mathbb{A}^1 = \operatorname{Spec} K[x]$ and the second is $\mathbb{A}^1 = \operatorname{Spec} K[x^{-1}]$. So the corresponding toric variety is $\mathbb{P}^1$ (if we have homogeneous coordinates $[X : Y]$ on $\mathbb{P}^1$ coordinates on U_0 are $x = X/Y$ and $y = Y/X = 1/x$).*

Now suppose that we start with three cones in $N_\mathbb{R} = \mathbb{R}^2$, σ_1, σ_2 and σ_3. We let σ_1 be the cone spanned by e_1 and e_2, σ_2 be the cone spanned by e_2 and $-e_1 - e_2$ and σ be the cone spanned by $-e_1 - e_2$ and e_1. Let F be the fan given as the faces of these three cones. Note that the three affine varieties corresponding to these three cones are all copies of $\mathbb{A}^2$. Indeed, any two of the vectors, e_1, e_2 and $-e_1 - e_2$ are a basis not *only of the underlying vector space but they also generate the standard lattice. We check how to glue two such copies of $\mathbb{A}^2$.*

The dual cone of σ_1 is the cone spanned by f_1 and f_2 in $M_{\mathbb{R}} = \mathbb{R}^2$. The dual cone of σ_2 is the cone spanned by $-f_1$ and $-f_1 + f_2$. So we have $U_1 = \operatorname{Spec} K[x, y]$ and $U_2 = \operatorname{Spec} K[x^{-1}, x^{-1}y]$. On the other hand, if we start with $\mathbb{P}^2$ with homogeneous coordinates $[X : Y : Z]$ and the two basic open subsets $U_0 = \operatorname{Spec} K[Y/X, Z/X]$ and $U_1 = \operatorname{Spec} K[X/Y, Z/Y]$, then we get the same picture, if we set $x = Y/X$, $y = Z/X$ (since then $X/Y = x^{-1}$ and $Z/Y = Z/X \cdot X/Y = yx^{-1}$). With a little more work one can check that we have $\mathbb{P}^2$.

More generally, suppose we start with $n+1$ vectors $v_1, v_2, \dots, v_{n+1}$ in $N_{\mathbb{R}} = \mathbb{R}^n$ which sum to zero such that the first n vectors $v_1, v_2, \dots, v_n$ span the standard lattice. Let F be the fan obtained by taking all the cones spanned by all subsets of at most n vectors. One can check that the resulting toric variety is $\mathbb{P}^n$.

Now suppose that we take the four vectors e_1, e_2, $-e_1$ and $-e_2$ in $N_{\mathbb{R}} = \mathbb{R}^2$ and let F be the fan consisting of all cones spanned by at most two vectors. Then we get four copies of $\mathbb{A}^2$. It is easy to check that the resulting toric variety is $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed the top two fans glue together to get $\mathbb{P}^1 \times \mathbb{A}^1$ and so on.

We have already seen that cones correspond to open subsets. In fact cones also correspond (in some sort of dual sense) to closed subsets, the closure of the orbits. First observe that given a fan F, we can associate a closed point x_σ to any cone σ. To see this, observe that one can spot the closed points of U_σ using semigroups:

Lemma 2.18. *Let $S \subset M$ be a semigroup. Then there is a natural bijection,*

$$\operatorname{Hom}(K[S], K) \simeq \operatorname{Hom}(S, K).$$

Here the RHS is the set of semigroup homomorphisms, where $K = \{0\} \cup K^*$ is the multiplicative subsemigroup of K (and not the additive).

Proof. Suppose we are given a ring homomorphism

$$f \colon K[S] \longrightarrow K.$$

Define

$$g \colon S \longrightarrow K,$$

by sending u to $f(\chi^u)$. Conversely, given g, define $f(\chi^u) = g(u)$ and extend linearly. $\square$

Consider the semigroup homomorphism:

$$S_\sigma \longrightarrow \{0, 1\},$$

where $\{0,1\} \subset \{0\} \cup K^*$ inherits the obvious semigroup structure. We send $u \in S_\sigma$ to 1 if $u \in \sigma^\perp$ and send it 0 otherwise. Note that as $\sigma^\perp$ is a face of $\check{\sigma}$ we do indeed get a homomorphism of semigroups. By (2.18) we get a surjective ring homomorphism

$$K[S_\sigma] \longrightarrow K.$$

The kernel is a maximal ideal of $K[S_\sigma]$, that is a closed point x_σ of U_σ, with residue field K.

To get the orbits, take the orbits of these points. It follows that the orbits are in correspondence with the cones in F. In fact the correspondence is inclusion reversing.

Example 2.19. *For the fan corresponding to $\mathbb{P}^1$, the point corresponding to $\{0\}$ is the identity, and the points corresponding to e_1 and $-e_1$ are 0 and ∞. For the fan corresponding to $\mathbb{P}^2$ the three maximal cones give the three coordinate points, the three one dimensional cones give the three coordinate lines (in fact the lines spanned by the points corresponding to the two maximal cones which contain them). As before the zero cone corresponds to the identity point. The orbit is the whole torus and the closure is the whole of $\mathbb{P}^2$.*

Suppose that we start with the cone σ spanned by e_1 and e_2 inside $N_\mathbb{R} = \mathbb{R}^2$. We have already seen that this gives the affine toric variety $\mathbb{A}^2$. Now suppose we insert the vector $e_1 + e_2$. We now get two cones σ_1 and σ_2, the first spanned by e_1 and $e_1 + e_2$ and the second spanned by $e_1 + e_2$ and e_2. Individually each is a copy of $\mathbb{A}^2$. The dual cones are spanned by f_2, $f_1 - f_2$ and f_1 and $f_2 - f_1$. So we get $\operatorname{Spec} K[y, x/y]$ and $\operatorname{Spec} K[x, x/y]$.

Suppose that we blow up $\mathbb{A}^2$ at the origin. The blow up sits inside $\mathbb{A}^2 \times \mathbb{P}^1$ with coordinates (x,y) and $[S : T]$ subject to the equations $xT = yS$. On the open subset $T \neq 0$ we have coordinates s and y and $x = sy$ so that $s = x/y$. On the open subset $S \neq 0$ we have coordinates x and t and $y = xt$ so that $t = y/x$. So the toric variety above is nothing more than the blow up of $\mathbb{A}^2$ at the origin. The central ray corresponds to the exceptional divisor E, a copy of $\mathbb{P}^1$.

A couple of definitions:

Definition 2.20. *Let G and H be algebraic groups which act on quasi-projective varieties X and Y. Suppose we are given an algebraic group homomorphism, $\rho\colon G \longrightarrow H$. We say that a morphism $f\colon X \longrightarrow Y$ is ρ-**equivariant** if f commutes with the action of G and H. If X and Y are toric varieties and G and H are the tori contained in X and Y then we say that f is a **toric morphism**.*

It is easy to see that the morphism defined above is toric. We can extend this picture to other toric surfaces. First a more intrinsic description of the blow up. Suppose we are given a ray, that is a one dimensional cone σ. Then we can describe σ by specifying the unique integral vector $v \in \sigma$ which is closest to the origin. Note that every other integral vector belonging to σ is a positive integral multiple of v, which we call the **primitive generator** v. Suppose we are given a toric surface and a two dimensional cone σ such that the primitive generators v and w of the two one dimensional faces of σ generate the lattice (so that up the action of $\mathrm{SL}(2, \mathbb{Z})$, σ is the cone spanned by e_1 and e_2). Then the blow up of the point corresponding to σ is a toric surface, which is obtained by inserting the sum $v + w$ of the two primitive generators and subdividing σ in the obvious way (somewhat like the barycentric subdivision in simplicial topology).

Example 2.21. *Suppose we start with $\mathbb{P}^2$ and the standard fan. If we insert the two vectors $-e_1$ and $-e_2$ this corresponds to blowing up two invariant points, say $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Note that now $-e_1 - e_2$ is the sum of $-e_1$ and $-e_2$. So if we remove this vector this is like blowing down a copy of $\mathbb{P}^1$. The resulting fan is the fan for $\mathbb{P}^1 \times \mathbb{P}^1$.*

We can generalise this to higher dimensions. For example suppose we start with the standard cone for $\mathbb{A}^3$ spanned by e_1 and e_2 and e_3. If we insert the vector $e_1 + e_2 + e_3$ (thereby creating three maximal cones) this corresponds to blowing up the origin. (In fact there is a simple recipe for calculating the exceptional divisor; mod out by the central $e_1 + e_2 + e_3$; the quotient vector space is two dimensional and the three cones map to the three cones in the quotient two dimensional vector space which correspond to the fan for $\mathbb{P}^2$). Suppose we insert the vector $e_1 + e_2$. Then the exceptional locus is $\mathbb{P}^1 \times \mathbb{A}^1$. In fact this corresponds to blowing up one of the axes (the axis is a copy of $\mathbb{A}^1$ and over every point of the axis there is a copy of $\mathbb{P}^1$).

3. Some naive enumerative geometry

Question 3.1. *How many lines meet four fixed lines in $\mathbb{P}^3$?*

Let us first check that this question makes sense, that is, let us first check that the answer is finite.

Definition 3.2. $\mathbb{G}(k, n)$ *denotes the space of r-dimensional linear subspaces of $\mathbb{P}^n$.*

We will assume that we have constructed the **Grassmannian** as a variety. The first natural question then is to determine the dimension of $\mathbb{G}(1, 3)$. We do so in an ad hoc manner. A line l in $\mathbb{P}^3$ is specified by picking two points p and q. Now the set of choices for two points p and q is equal to $\mathbb{P}^3 \times \mathbb{P}^3 - \Delta$, where Δ is the diagonal. Thus the set of choices of pairs of distinct points is six dimensional.

Fix a line l. Then if we pick any two points p and q of this line, they give us the same line l. Thus the the Grassmannian of lines in $\mathbb{P}^3$ is $6 = 2 = 4$-dimensional.

It might help to look at this differently. Let

$$\Sigma = \{\, (P, l) \mid P \in l \,\} \subset \mathbb{G}(1, 3) \times \mathbb{P}^3.$$

Then Σ is a closed subset of the product $\mathbb{G}(1, 3) \times \mathbb{P}^3$. There are two natural projection maps.

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"] & \mathbb{P}^3 \\ \mathbb{G}(1,3). & \end{tikzcd}$$

In fact Σ (together with this diagram) is called an **incidence correspondence**. It is interesting to consider the two morphisms p and q. First q. Pick a line $l \in \mathbb{G}(1, 3)$. Then the fibre of p over l consists of all points P that are contained in l, so that the fibres of p are all isomorphic to $\mathbb{P}^1$. Now consider the morphism q. Fix a point P. Then the fibre of q over P consists of all lines that contain P. Again the fibres of q are isomorphic.

To compute the dimension of $\mathbb{G}(1, 3)$, we compute the dimension of Σ in two ways. We will need:

Theorem 3.3. *Let $\pi\colon X \longrightarrow Y$ be a morphism of varieties.*

Then there is an open subset U of Y, such that for every $y \in U$, the dimension of the fibre of π over is equal to k, a constant. Moreover the dimension of X is equal to the dimension of Y plus k.

We first apply (3.3) to q. The dimension of the base is 3. As every fibre is isomorphic, to compute k, we can consider any fibre. Pick any point P. Pick an auxiliary plane, not passing through P. Then the set of lines containing P is in bijection with the points of this auxiliary plane, so that the dimension of a fibre is two. Thus the dimension of $\Sigma = 3 + 2 = 5$.

Now we apply (3.3) to p. The dimension of any fibre is one. Thus the dimension of the Grassmannian is $5 - 1 = 4$, as before.

The next question is to determine how many conditions it is to meet a fixed line l_1. I claim it is one condition. The easiest way to see this, is to just to imagine swinging a sword around in space. This will cut any line into two. Thus any one dimensional family of lines meets a given line in finitely many points.

More formally, carry out the computation above, replacing Σ with Σ_1, the space of lines which meet l_1. The fibre of q over a point P is now a copy of $\mathbb{P}^1$ (parametrised by l_1). Thus Σ_1 has dimension 4 and the space of lines which meets l_1 has dimension $4 - 1 = 3$.

Thus we have a threefold in the fourfold $\mathbb{G}(1, 3)$. Clearly we expect that four of them will intersect in a finite set of points. There are two ways to proceed. Here is one which uses the Segre variety:

Lemma 3.4. *Let l_1, l_2 and l_3 and m_1, m_2 and m_3 be two sequences of skew lines in $\mathbb{P}^3$.*

Then there is an element of $\mathrm{PGL}_4(K)$ carrying the first sequence to the second.

Proof. We may as well assume that the first set is given as

$$X = Y = 0 \qquad\qquad Z = W = 0 \qquad \text{and} \qquad X - Z = Y - W = 0.$$

Clearly we may find a transformation carrying m_1 to l_1 and m_2 to l_2. For example, pick four points on both sets of lines, and use the fact that any four sets of points in linear general position are projectively equivalent.

Consider the two planes $X = 0$ and $Y = 0$. m_3 cannot lie in either of these planes, else either the lines m_1 and m_3 or the lines m_1 and m_3 would not be skew. Consider the two points $[0 : a : b : c]$ and $[d : e : f : 0]$ where m_3 intersects the planes $X = 0$ and $Y = 0$.

Clearly m_3 is determined by these points, and in the case of l_3, we may take $a = c = d = f = 1$, $b = e = 0$. Pick an element $\phi \in \mathrm{PGL}_4(K)$ and represent it as a 4×4 matrix. If we decompose this 4×4 matrix in block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where each block is a 2×2 matrix, the condition that ϕ fix l_1 and l_2 is equivalent to the condition that $B = C = 0$. By choosing A and D appropriately, we reduce to the case that $b = e = 0$. In this case the matrix

$$\begin{pmatrix} 1/d & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \\ 0 & 0 & 1/f & 0 \\ 0 & 0 & 0 & 1/c \end{pmatrix}$$

carries the two points to the standard two points, so that it carries m_3 to l_3. $\square$

This result has the following surprising consequence.

Lemma 3.5. *Let l_1, l_2 and l_3 be three skew lines in $\mathbb{P}^3$.*

Then the family of lines that meets all three lines sweeps out a quadric surface in $\mathbb{P}^3$.

Proof. By (3.4) we may assume that the three lines are any set of three skew lines in $\mathbb{P}^3$. Now the Segre variety V in $\mathbb{P}^1 \times \mathbb{P}^1$ contains three skew lines (just choose any three lines of one of the rulings). Moreover any line of the other ruling certainly meets all three lines. So the set of lines meeting all three lines, certainly sweeps out at least a quadric surface.

To finish, suppose we are given a line l that meets l_1, l_2 and l_3. Then l meets V in three points. As V is defined by a quadratic polynomial, it follows that l is contained in V. Thus any line that meets all three lines, is contained in V. $\square$

Theorem 3.6. *There are two lines that meet four general skew lines in $\mathbb{P}^3$.*

Proof. Fix the first three lines l_1, l_2 and l_3. We have already seen that the set l of lines that meets all three of these lines is precisely the set of lines of one ruling of the Segre variety (up to choice of coordinates).

Pick a line l_4 that meets V transversally in two points. Now for a line l of one ruling to meet the fourth line l_4, it must meet l_4 at a point $P = l \cap l_4$ of V. Moreover this point determines the line l. $\square$

Here is an entirely different way to answer (3.1). Consider using the principle of continuity. Take two of the four lines and deform them so they become concurrent (or what comes to exactly the same thing, coplanar). Similarly take the other pair of lines and degenerate them until they also become concurrent.

Now consider how a line l can meet the four given lines.

Lemma 3.7. *Let l be a line that meets two concurrent lines l_1 and l_2 in $\mathbb{P}^3$.*

Then either l contains $l_1 \cap l_2$ or l is contained in the plane $\langle l_1, l_2 \rangle$.

Proof. Suppose that l does not contain the point $l_1 \cap l_2$. Then l meets l_i, $i = 1$, 2 at two points p_i contained in the plane $\langle l_1, l_2 \rangle$. □

Thus if l meets all four lines, there are three possibilities.

(1) l contains both points of intersection.
(2) l is contained in both planes.
(3) l contains one point and is contained in the other plane.

Clearly there is only one line that satisfies (1). It is not so hard to see that there is only one line that satisfies (2), it is the intersection of the two planes. Finally it is not so hard to see that (3) is impossible. Just choose the point outside of the plane.

Thus the answer is two. It is convenient to introduce some notation to compute these numbers, which is known as Schubert calculus. Let l denote the condition that we meet a fixed line. We want to compute l^4. We proceed formally. We have already seen that

$$l^2 = l_p + l_\pi$$

where l_p denotes the condition that a line contains a point, and l_π is the condition that l is contained in π.

Thus

$$\begin{aligned} l^4 &= (l^2)^2 \\ &= (l_p + l_\pi)^2 \\ &= l_p^2 + 2l_p l_\pi + l_\pi \\ &= 1 + 2 \cdot 0 + 1 = 2, \end{aligned}$$

where the last line is computed as before.

4. Grassmannians

We first treat Grassmannians classically. Fix an algebraically closed field K. We want to parametrise the space of k-planes W in a vector space V. The obvious way to parametrise k-planes is to pick a basis $v_1, v_2, \dots, v_k$ for W. Unfortunately this does not specify W uniquely, as the same vector space has many different bases. However, the line spanned by the vector

$$\omega = v_1 \wedge v_2 \wedge \cdots \wedge v_k \in \bigwedge^k V,$$

is invariant under re-choosing a basis.

Definition 4.1. *The* ***Grassmannian*** $G(k, V)$ ***of*** k***-planes in*** V *is the set of rank one vectors in* $\mathbb{P}(\bigwedge^k V)$.

We set $G(k, n) = G(k, K^n)$ and $\mathbb{G}(k, n) = G(k+1, n+1)$. The latter may be thought of as the set of k-planes in $\mathbb{P}^n$.

Lemma 4.2. *The Grassmannian is a constructible subset of* $\mathbb{P}^N$.

Proof. Consider the rational map

$$\prod^k \mathbb{P}(V) \dashrightarrow \mathbb{P}(\bigwedge^k V),$$

which sends $([v_1], [v_2], \dots [v_k])$ to $[v_1 \wedge v_2 \wedge \cdots \wedge v_k]$. This map is defined (at least) on the locus where the vectors $v_1, v_2, \dots, v_k$ are independent, which is an open subset of the product. But the image of a constructible subset is constructible, by Chevalley's Theorem. $\square$

In fact we will see later that the Grassmannian is a closed subset of $\mathbb{P}^N$, so that it is a projective variety. The embedding of the Grassmannian inside $\mathbb{P}(\bigwedge^k V)$ is known as the **Plücker embedding**. If we choose a basis $e_1, e_2, \dots, e_n$ for V, then a general element of $\bigwedge^k V$ is given by

$$\sum_I p_I e_I,$$

where I ranges over all collections of increasing sequences of integers between 1 and n,

$$i_1 < i_2 < \cdots < i_k,$$

and e_I is shorthand for the wedge of the corresponding vectors,

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}.$$

The coefficients p_I are naturally coordinates on $\mathbb{P}(\bigwedge^k V)$, which are known as the **Plücker coordinates**.

There is another way to look at the construction of the Grassmannian, which is very instructive. If we pick a basis $e_1, e_2, \dots, e_n$ for V, then let A be the $k \times n$ matrix whose rows are $v_1, v_2, \dots, v_k$, in this basis. As before, this matrix does not uniquely specify $W \subset V$, since we could pick a new basis for W. However the operation of picking a new basis corresponds to taking linear combinations of the rows of our matrix which, in turn, is the same as multiplying our matrix by a $k \times k$ invertible matrix on the left. In other words the Grassmannian is the set of equivalence classes of $k \times n$ matrices under the action of $\mathrm{GL}_k(K)$ by multiplication on the left.

It is not hard to connect the two constructions. Given the matrix A, then form all possible $k \times k$ determinants. Any such determinant is determined by specifying the columns to pick, which we indicate by a multindex I. In terms of $\bigwedge^k V$, this is the same as picking a basis and expanding our vector as a sum

$$\sum_I p_I e_I,$$

where, as before, e_I is the wedge of the corresponding vectors. For example consider the case $k = 2$, $n = 4$ (lines in $\mathbb{P}^3$). We have a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

The corresponding plane is given as the span of the rows. We can form six two by two determinants. Clearly these are invariant, up to scalars, under the action of $\mathrm{GL}_2(K)$.

The Grassmannian has a natural cover by open affine subsets, isomorphic to affine space, in much the same way that projective space has a cover by open affines, isomorphic to affine space. Pick a linear space U of dimension $n - k$, and consider the set of linear spaces W of dimension k which are complementary to U, that is, which meet U only at the origin. Identify V with the sum $V/U + U$. Then a linear space W complementary to U can be identified with the graph of a linear map

$$V/U \longrightarrow U.$$

It follows that the subset of all linear spaces W complementary to U is equal to

$$\mathrm{Hom}(V/U, U) \simeq K^{k(n-k)} \simeq \mathbb{A}^{k(n-k)}_K.$$

Another way to see this is as follows. Consider the first $k \times k$ minor. Suppose that the corresponding determinant is non-zero, that is, the corresponding vectors are independent. In this case the $k \times k$ minor is equivalent to the identity matrix, and the only element of $\mathrm{GL}_k(K)$

which fixes the identity, is the identity itself. Thus we have a canonical representative of the matrix A for the linear space W. We are free to choose the other $k \times (n-k)$ block of the matrix, which gives us an affine space of dimension $k(n-k)$. The condition that the first $k \times k$ minor has non-zero determinant is an open condition, and this gives us an open affine cover by affine spaces of dimension $k(n-k)$. Note that the condition that the first $k \times k$ minor is invertible is equivalent to the condition that we do not meet the space given by the vanishing of the first k coordinates, which is indeed a linear space of dimension $n-k$.

It is interesting to write down the equations cutting out the image of the Grassmannian under the Plücker embedding, although this turns out to involve some non-trivial multilinear algebra. The problem is to characterise the set of rank one vectors ω in $\bigwedge^k V$.

Definition 4.3. *Let $\omega \in \bigwedge^k V$. We say that ω is* ***divisible*** *by $v \in V$ if there is an element $\phi \in \bigwedge^k V$ such that $\omega = \phi \wedge v$.*

Lemma 4.4. *Let $\omega \in \bigwedge^k V$.*

Then ω is divisible by v iff $\omega \wedge v = 0$.

Proof. This is easy. If $\omega = \phi \wedge v$, then

$$\begin{aligned} \omega \wedge v &= \phi \wedge v \wedge v \\ &= 0. \end{aligned}$$

To see the other direction, extend v to a basis $v = e_1, e_2, \dots, e_n$ of V. Then we may expand ω in this basis.

$$\omega = \sum p_I e_I.$$

On the other hand

$$e_I \wedge v = \begin{cases} e_J & \text{if } 1 \notin I \text{, where } J = \{1\} \cup I \\ 0 & \text{if } 1 \in I. \end{cases}$$

Thus $\omega \wedge v = 0$ iff $p_I \neq 0$ implies $1 \in I$ iff v divides ω. $\square$

Lemma 4.5. *Let $\omega \in \bigwedge^k V$.*

Then ω has rank at most one iff the linear map

$$\phi(\omega) \colon V \longrightarrow \bigwedge^{k+1} V \qquad\qquad v \longrightarrow \omega \wedge v,$$

has rank at most $n-k$.

Proof. Indeed $\phi(\omega)$ has rank at most $n-k$ iff the linear subspace of vectors dividing ω has dimension at least k iff ω has rank one. $\square$

Now the map

$$\phi\colon \bigwedge^k V \longrightarrow \operatorname{Hom}(V, \bigwedge^{k+1} V),$$

is clearly linear. Thus the map ϕ can be interpreted as a matrix whose entries are linear coordinates on $\bigwedge^k V$ and the locus we want is given by the vanishing of the $(n-k+1)\times(n-k+1)$ minors.

Unfortunately the equations we get in this way won't be best possible. In particular they won't generate the ideal of the Grassmannian (they only cut out the Grassmannian set theoretically). To find equations that generate the ideal, we have to work quite a bit harder.

Lemma 4.6. *There is a natural isomorphism, up to scalars, between $\bigwedge^k V$ and $\bigwedge^{n-k} V^*$, which preserves the rank.*

Proof. There is a natural pairing

$$\bigwedge^k V \times \bigwedge^{n-k} V \longrightarrow \bigwedge^n V,$$

which sends

$$(\omega, \eta) \longrightarrow \omega \wedge \eta.$$

As $\bigwedge^n V$ is one dimensional, it is non-canonically isomorphic to K and so $\bigwedge^k V$ is isomorphic to $(\bigwedge^{n-k} V)^*$, up to scalars. But $(\bigwedge^{n-k} V)^*$ is isomorphic to $\bigwedge^{n-k} V^*$. $\square$

Given ω, let ω^* be the corresponding element of $\bigwedge^{n-k} V^*$. Now there is a natural map

$$\psi(\omega^*)\colon V^* \longrightarrow \bigwedge^{n-k+1} V^*$$

which sends

$$v^* \longrightarrow \omega^* \wedge v^*.$$

Further ω has rank one iff ω^* has rank one, which occurs if and only if $\psi(\omega^*)$ has rank k.

Moreover the kernel of $\phi(\omega)$, namely W, is precisely the annihilator of the kernel of $\psi(\omega^*)$. Dualising, we get maps

$$\phi^*(\omega)\colon \bigwedge^{k+1} V^* \longrightarrow V^* \qquad \text{and} \qquad \psi^*(\omega)\colon \bigwedge^{n-k+1} V \longrightarrow V,$$

whose images annihilate each other.

Thus ω has rank one iff for every $\alpha \in \bigwedge^{k+1} V^*$ and $\beta \in \bigwedge^{n-k+1} V$,

$$\Xi_{\alpha,\beta}(\omega) = \langle \phi^*(\omega)(\alpha), \psi^*(\omega)(\beta)\rangle = 0.$$

Now $\Xi_{\alpha,\beta}$ are quadratic polynomials, which are known as the Plücker relations. It turns out that they do indeed generate the ideal of the Grassmannian.

It is interesting to see what happens when $k = 2$:

Lemma 4.7. *Let $\omega \in \bigwedge^2 V$.*

Then ω has rank one iff $\omega \wedge \omega = 0$.

Proof. One direction is clear, in fact for every k, if ω has rank one then $\omega \wedge \omega = 0$.

To see the other direction, we need to prove that if ω has rank at least two, then $\omega \wedge \omega \neq 0$. First observe that if ω has rank at least two, then we may find a projection down to a vector space of dimension four, such that the image has rank two. Thus we may assume that V has dimension four and ω has rank two. In this case, up to change of coordinates,

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4,$$

and by direct computation, $\omega \wedge \omega$ is not zero. $\square$

Now

$$\omega = \sum_{i,j} p_{i,j} e_i \wedge e_j.$$

Suppose that $n = 4$. If we expand

$$\omega \wedge \omega,$$

then everything is a multiple of $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We need to pick a term from each bracket, so that the union is $\{1, 2, 3, 4\}$. In other words, the coefficient of the expansion is a sum over all partitions of $\{1, 2, 3, 4\}$ into two equal parts. By direct computation, we get

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

In particular, $\mathbb{G}(1, 3)$ is a quadric in $\mathbb{P}^5$, of maximal rank. Unfortunately this also makes it clear that the Grassmannian is not a toric variety (if it were, it would be defined by a binomial, not a trinomial). It turns out that the Grassmannian is close to a toric variety (it is a spherical variety). In fact the algebraic group $\mathrm{GL}_n(V)$ acts transitively on $G(k, V)$. The stabiliser subgroup H of the k-plane $W \subset V$ spanned by the first k vectors is given by invertible matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

So

$$G(k, V) = \mathrm{GL}_n(V)/H.$$

As with the space of conics in $\mathbb{P}^2$, the main point of the Grassmannian, is that it comes with a universal family. We first investigate what this means in the baby case of quasi-projective varieties before we move on to the more interesting case of schemes.

Definition 4.8. *A **family of k-planes in $\mathbb{P}^n$ over** B is a closed subset $\Sigma \subset B \times \mathbb{P}^n$ such that the fibres, under projection to the first factor, are identified with k-planes in $\mathbb{P}^n$.*

Definition 4.9. *Let F be the functor from the category of varieties to the category of sets, which assigns to every variety, the set of all (flat) families of k-planes in $\mathbb{P}^n$, up to isomorphism.*

Theorem 4.10. *The Grassmannian $\mathbb{G}(k,n)$ represents the functor F.*

It might help to unravel some of the definitions. Suppose that we are given a variety B. Essentially we have to show that there is a natural bijection of sets,

$$F(B) = \mathrm{Hom}(B, \mathbb{G}(k,n)).$$

The set on the left is nothing more than the set of all families of k-planes in $\mathbb{P}^n$, with base B. In particular given a morphism $f\colon B \longrightarrow \mathbb{G}(k,n)$, we are supposed to produce a family of k-planes over B. Here is how we do this. Suppose that we have constructed the natural family of k-planes over $\mathbb{G}(k,n)$,

$$\begin{tikzcd} \Sigma \arrow[r, hook] & \mathbb{G}(k,n) \times \mathbb{P}^n \arrow[d] \\ & \mathbb{G}(k,n), \end{tikzcd}$$

so that the fibre over $[\Lambda] \in \mathbb{G}(k,n)$ is exactly the set,

$$\{[\Lambda]\} \times \Lambda \subset \{[\Lambda]\} \times \mathbb{P}^n$$

that is, the k-plane Λ sitting inside $\mathbb{P}^n$. Then we obtain a family of k-planes over B, simply by taking the fibre square,

$$\begin{tikzcd} \Sigma' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Sigma \arrow[d] \\ B \arrow[r, "f"] & \mathbb{G}(k,n). \end{tikzcd}$$

For this reason, we call the family $\Sigma \longrightarrow \mathbb{G}(k,n)$ the universal family. Note that we can reverse this process. Suppose that $\mathbb{G}(k,n)$ represents the functor F. By considering the identity morphism $\mathbb{G}(k,n) \longrightarrow \mathbb{G}(k,n)$, we get a family $\Sigma \longrightarrow \mathbb{G}(k,n)$, which is universal, in the sense that to obtain any other family, over any other base, we simply pullback

Σ along the morphism $f\colon B \longrightarrow \mathbb{G}(k,n)$, whose existence is guaranteed by the universal property of $\mathbb{G}(k,n)$ (that is, that it represents the functor). To summarise the previous discussion: to prove (4.10) it suffices to construct the natural family over $\mathbb{G}(k,n)$ and prove that it is the universal family.

We won't prove (4.10) here. We will simply observe that the natural family exists, without proving that it is in fact also universal. Recall that the Grassmannian is by definition the set of all rank one elements ω of $\bigwedge^{k+1} K^{n+1}$. The universal family is then the set

$$\{\, (\omega, v) \in \bigwedge^{k+1} V \times V \,|\, \omega \wedge v = 0 \,\},$$

which is easily seen to be algebraic.

Before we go deeper into the geometry of the Grassmannian, it is interesting to note that the space of conics satisfies the same universal property. Suppose $\mathbb{P}^2 = \mathbb{P}(V)$. Then $\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2(V^*))$ represents the functor G which assigns to every variety B, the set of all (flat) families of conics in $\mathbb{P}^2$, over B. As before the key thing is to show that the natural family of conics in $\mathbb{P}^2$ over $\mathbb{P}^5$, is in fact a universal family. As before we won't show that the natural family is universal, but we observe that the natural family does exist. Indeed,

$$aX^2 + bY^2 + cZ^2 + dYZ + eXZ + fYZ,$$

is bihomogeneous of degree $(1,2)$ and cuts out the natural family.

Using the diagram,

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^n \\ \mathbb{G}(k,n), & \end{tikzcd}$$

one can make some interesting constructions. For example, suppose we are given a closed subset $X \subset \mathbb{P}^n$. Then $p(q^{-1}(X))$ is a closed subvariety of $\mathbb{G}(k,n)$, consisting of all k-planes in $\mathbb{P}^n$ which intersect X. The first interesting case is that of a curve C in $\mathbb{P}^3$. In this case the general line does not meet the curve C. In fact we get a codimension one subvariety of $\mathbb{G}(1,3)$. Conversely suppose we are given a closed subvariety Φ of $\mathbb{G}(k,n)$. Then $q(p^{-1}(\Phi))$ is a closed subvariety of $\mathbb{P}^n$, equal to

$$X = \bigcup_{\Lambda \in \Phi} \Lambda.$$

Note that X has the interesting property that through every point of X there passes a k-plane. Classically such varieties are called **scrolls**.

Perhaps the first interesting example of a scroll is the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$.

Let us give some more constructions of scrolls. Suppose that we are given two subvarieties X and Y of $\mathbb{P}^n$. Define a rational map

$$\phi\colon X \times Y \dashrightarrow \mathbb{G}(1,n),$$

by sending

$$([v],[w]) \longrightarrow [v \wedge w].$$

The subvariety in $\mathbb{P}^n$, corresponding to the image, is called the **join**. It is the closure of the union of all lines obtained by taking the span of a point of X and a point of Y. Note that ϕ is a morphism if X and Y are disjoint and in this case we don't need to take the closure. If we take $X = Y$, then we get the **secant variety of** X, which is the closure of all the lines which join two points of X.

Suppose that we are given a morphism $f\colon X \longrightarrow Y$, with the property that there is a point $x \in X$ such that $f(x) \neq x$. Consider the morphism

$$\psi\colon X \longrightarrow \mathbb{G}(1,n),$$

which is the composition of

$$X \longrightarrow X \times Y \qquad \text{given by} \qquad x \longrightarrow (x, f(x)),$$

and the morphism ϕ above. As before this gives us a scroll in $\mathbb{P}^n$, by taking the image. Note that all of this generalises to products of k varieties.

Definition 4.11. *Pick complimentary linear spaces $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ of dimensions $n_1, n_2, \dots, n_k$ in $\mathbb{P}^n$, where*

$$n+1 = \sum_i (n_i + 1).$$

Pick rational normal curves $C_i \subset \Lambda_i$ in and pick identifications

$$\phi_i\colon \mathbb{P}^1 \longrightarrow C_i.$$

Let

$$X = \bigcup_{p \in \mathbb{P}^1} \langle \phi_1(p), \phi_2(p), \dots, \phi_k(p) \rangle.$$

Then X is called a ***rational normal scroll****.*

It is interesting to give some examples. Suppose that we pick two skew lines l and m in $\mathbb{P}^3$. Then we get a surface in $\mathbb{P}^3$, swept out by lines, meeting l and m. Suppose we pick coordinates such that $l = V(X,Y)$ and $m = V(Z,W)$. Identify $(0,0,a,b)$ with $(a,b,0,0)$. Then it is not hard to see that we get the surface $V(XW - YZ)$.

The next case is when we take a line and a complimentary plane in $\mathbb{P}^4$. The resulting surface in $\mathbb{P}^4$ is called the cubic scroll.

Let us now investigate how to work with the Grassmannian in the case of schemes. As in the case of affine and projective space we can define a scheme over $\operatorname{Spec}\mathbb{Z}$ and use this scheme to define the Grassmannian over any base scheme. In fact the equations defining the Grassmannian over an algebraically closed field have integral coefficients (better still, presumably 0 and ± 1) and this defines the Grassmannian as a closed subscheme of $\mathbb{P}^N_{\mathbb{Z}}$. However this somehow begs the question; what role does the Grassmannian play over an arbitrary base scheme S?. We want to extend the functor F, which is a priori defined only as a functor from varieties over K to (Sets), to a functor from the category of schemes over S to the category (Sets). To answer this question, we need to decide what we mean by a family of k-planes in $\mathbb{P}^n_S$. It turns out to be easier to answer what it means to have a family of vector subspaces of dimension $k+1$.

5. Smoothness and the Zariski tangent space

We want to give an algebraic notion of the tangent space. In differential geometry, tangent vectors are equivalence classes of maps of intervals in $\mathbb{R}$ into the manifold. This definition lifts to algebraic geometry over $\mathbb{C}$ but not over any other field (for example a field of characteristic p).

Classically tangent vectors are determined by taking derivatives, and the tangent space to a variety X at x is then the space of tangent directions, in the whole space, which are tangent to X. Even is this is how we will compute the tangent space in general, it is still desirable to have an intrinsic definition, that is, a definition which does not use the fact that X is embedded in $\mathbb{P}^n$.

Now note first that the notion of smoothness is surely local and that if we want an intrinsic definition, then we want a definition that only uses the functions on X. Putting this together, smoothness should be a property of the local ring of X at p. On the other hand taking derivatives is the same as linear approximation, which means dropping quadratic and higher terms.

Definition 5.1. *Let X be a variety and let $p \in X$ be a point of X. The **Zariski tangent space** of X at p, denoted T_pX, is equal to the dual of the quotient*

$$\mathfrak{m}/\mathfrak{m}^2,$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$.

Note that $\mathfrak{m}/\mathfrak{m}^2$ is a vector space. Suppose that we are given a morphism

$$f\colon X \longrightarrow Y,$$

which sends p to q. In this case there is a ring homomorphism

$$f^*\colon \mathcal{O}_{Y,q} \longrightarrow \mathcal{O}_{X,p}$$

which sends the maximal ideal $\mathfrak{n}$ into the maximal ideal $\mathfrak{m}$. Thus we get an induced map

$$df\colon \mathfrak{n}/\mathfrak{n}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2$$

On the other hand, geometrically the map on tangent spaces obviously goes the other way. Therefore it follows that we really do want the dual of $\mathfrak{m}/\mathfrak{m}^2$. In fact $\mathfrak{m}/\mathfrak{m}^2$ is the dual of the Zariski tangent space, and is referred to as the *cotangent space*.

In particular, given a morphism $f\colon X \longrightarrow Y$ carrying p to q, then there is a linear map

$$df\colon T_pX \longrightarrow T_qY.$$

Definition 5.2. *Let X be a quasi-projective variety.*

*We say that X is **smooth** at p if the local dimension of X at p is equal to the dimension of the Zariski tangent space at p.*

Now the tangent space to $\mathbb{A}^n$ is canonically a copy of $\mathbb{A}^n$ itself, considered as a vector space based at the point in question. If $X \subset \mathbb{A}^n$, then the tangent space to X is included inside the tangent space to $\mathbb{A}^n$. The question is then how to describe this subspace.

Lemma 5.3. *Let $X \subset \mathbb{A}^n$ be an affine variety. Suppose that $f_1, f_2, \dots, f_k$ generate the ideal I of X. Then the tangent space of X at p, considered as a subspace of the tangent space to $\mathbb{A}^n$, via the inclusion of X in $\mathbb{A}^n$, is equal to the kernel of the Jacobian matrix.*

Proof. Clearly it is easier to give the dual description of the cotangent space.

If $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{\mathbb{A}^n,p}$ and $\mathfrak{n}$ is the maximal ideal of $\mathcal{O}_{X,p}$, then clearly the natural map $\mathfrak{m} \longrightarrow \mathfrak{n}$ is surjective, so that the induced map on contangent spaces is surjective. Dually, the induced map on the Zariski tangent space is injective, so that T_pX is indeed included in $T_p\mathbb{A}^n$.

We may as well choose coordinates $x_1, x_2, \dots, x_n$ so that p is the origin. In this case $\mathfrak{m} = \langle x_1, x_2, \dots, x_n \rangle$ and $\mathfrak{n} = \mathfrak{m}/I$. Moreover $\mathfrak{m}/\mathfrak{m}^2$ is the vector space spanned by $dx_1, dx_2, \dots, dx_n$, where dx_i denotes the equivalence class $x_i + \mathfrak{m}^2$, and $\mathfrak{n}/\mathfrak{n}^2$ is canonically isomorphic to $\mathfrak{m}/(\mathfrak{m}^2 + I)$. Now the transpose of the Jacobian matrix, defines a linear map

$$K^k \longrightarrow K^n = T_p^*\mathbb{A}^n,$$

and it suffices to prove that the image of this map is the kernel of the map

$$df : \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{n}/\mathfrak{n}^2.$$

Let $g \in \mathfrak{m}$. Then

$$g(x) = \sum a_i x_i + h(x),$$

where $h(x) \in \mathfrak{m}^2$. Thus the image of $g(x)$ in $\mathfrak{m}/\mathfrak{m}^2$ is equal to $\sum_i a_i dx_i$. Moreover, by standard calculus a_i is nothing more than

$$a_i = \left.\frac{\partial g}{\partial x_i}\right|_p .$$

Thus the kernel of the map df is generated by the image of f_i in $\mathfrak{m}/\mathfrak{m}^2$, which is

$$\sum_j \left.\frac{\partial f_i}{\partial x_j}\right|_p dx_j,$$

which is nothing more than the image of the Jacobian. $\square$

Lemma 5.4. *Let X be a quasi-projective variety. Then the function*

$$\lambda\colon X \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\lambda(x)$ is the dimension of the Zariski tangent space at x.

Proof. Clearly this result is local on X so that we may assume that X is affine. In this case the Jacobian matrix defines a morphism π from X to the space of matrices and the locus where the Zariski tangent space has a fixed dimension is equal to the locus where this morphism lands in the space of matrices of fixed rank. Put differently the function λ is the composition of π and an affine linear function of the rank on the space of matrices. Since the rank function is upper semicontinuous, the result follows. $\square$

Lemma 5.5. *Every irreducible quasi-projective variety is birational to a hypersurface.*

Proof. Let X be a quasi-projective variety of dimension k, with function field L/K. Let $L/M/K$ be an intermediary field, such that M/K is purely transcendental of transcendence degree, so that L/M is algebraic. As L/M is a finitely generated extension, it follows that L/M is finite. Suppose that L/M is not separable. Then there is an element $y \in L$ such that $y \notin M$ but $x_1 = y^p \in M$. We may extend x_1 to a transcendence basis $x_1, x_2, \dots, x_k$ of M/K. Let M' be the intermediary field generated by $y, x_2, x_3, \dots, x_k$. Then M'/K is a purely transcendental extension of K and

$$[L : M] = [L : M'][M' : M] = p[L : M'].$$

Repeatedely replacing M by M' we may assume that L/M is a separable extension.

By the primitive element Theorem, L/M is generated by one element, say α. It follows that there is polynomial $f(x) \in M[x]$ such that α is a root of $f(x)$. If $M = K(x_1, x_2, \dots, x_k)$, then clearing denominators, we may assume that $f(x) \in K[x_1, x_2, \dots, x_k][x] \simeq K[x_1, x_2, \dots, x_{k+1}]$. But then X is birational to the hypersurface defined by $F(X)$, where $F(X)$ is the homogenisation of $f(x)$. $\square$

Proposition 5.6. *The set of smooth points of any variety is Zariski dense.*

Proof. Since the dimension of the Zariski tangent space is upper semicontinuous, and always at least the dimension of the variety, it suffices to prove that every irreducible variety contains at least one smooth point. By (5.5) we may assume that X is a hypersurface. Passing to an affine open subset, we may assume that X is an affine hypersurface. Let f be a definining equation, so that f is an irreducible polynomial. Then the set of singular points of X is equal to the locus of points where every partial derivative vanishes. If g is a non-zero partial derivative of f, then g is a non-zero polynomial of degree one less than f, and so cannot vanish on X.

If all the partial derivatives of f are the zero polynomial, then f is a pth power, where the characteristic is p, which contradicts the fact that f is irreducible. $\square$

A basic result in the theory of C^∞-maps is Sard's Theorem, which states that the set of points where a map is singular is a subset of measure zero (of the base). Since any holomorphic map between complex manifolds is automatically C^∞, and the derivative of a polynomial is the same as the derivative as a holomorphic function, it follows that any morphism between varieties, over $\mathbb{C}$, is smooth over an open subset. In fact by the Lefschetz principle, this result extends to any variety over $\mathbb{C}$.

Theorem 5.7. *Let $f\colon X \longrightarrow Y$ be a morphism of varieties over a field of characteristic zero.*

Then there is a dense open subset U of Y such that if $q \in U$ and $p \in f^{-1}(q) \cap X_{sm}$ then the differential $df_p\colon T_pX \longrightarrow T_qY$ is surjective. Further, if X is smooth, then the fibres $f^{-1}(q)$ are smooth if $q \in U$.

Let us recall the Lefschetz principle. First recall the notion of a first order theory of logic. Basically this means that one describes a theory of mathematics using a theory based on predicate calculus. For example, the following is a true statement from the first order theory of number theory,

$$\forall n \forall x \forall y \forall z \quad n \geq 3 \implies x^n + y^n \neq z^n.$$

One basic and desirable property of a first order theory of logic is that it is complete. In other words every possible statement (meaning anything that is well-formed) can be either proved or disproved. It is a very well-known result that the first order theory of number theory is not complete (Gödel's Incompleteness Theorem). What is perhaps more surprising is that there are interesting theories which are complete.

Theorem 5.8. *The first order logic of algebraically closed fields of characteristic zero is complete.*

Notice that a typical statement of the first order logic of fields is that a system of polynomial equations does or does not have solution. Since most statements in algebraic geometry turn on whether or not a system of polynomial equations have a solution, the following result is very useful.

Principle 5.9 (Lefschetz Principle)**.** *Every statement in the first order logic of algebraically closed fields of characteristic zero, which is true over $\mathbb{C}$, is in fact true over any algebraically closed field of characteristic.*

In fact this principle is immediate from (5.8). Suppose that p is a statement in the first order logic of algebraically closed fields of characteristic zero. By completeness, we can either prove p or not p. Since p holds over the complex numbers, there is no way we can prove not p. Therefore there must be a proof of p. But this proof is valid over any field of characteristic zero, so p holds over any algebraically closed field of characteristic zero.

A typical application of the Lefschetz principle is (5.7). By Sard's Theorem, we know that (5.7) holds over $\mathbb{C}$. On the other hand, (5.7), can be reformulated in the first order logic of algebraically closed fields of characteristic zero. Therefore by the Lefschetz principle, (5.7) is true over algebraically closed field of characteristic zero.

Perhaps even more interesting, is that (5.7) fails in characteristic p. Let $f\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the morphism $t \longrightarrow t^p$. If we fix s, then the equation

$$x^p - s$$

is purely inseparable, that is, has only one root. Thus f is a bijection. However, df is the zero map, since $dz^p = pz^{p-1}dz = 0$. Thus df_p is nowhere surjective. Note that the fibres of this map, as schemes, are isomorphic to zero dimensional schemes of length p.

We now want to aim for a version of the Inverse function Theorem. In differential geometry, the inverse function theorem states that if a function is an isomorphism on tangent spaces, then it is locally an isomorphism. Unfortunately this is too much to expect in algebraic geometry, since the Zariski topology is too weak for this to be true. For example consider a curve which double covers another curve. At any point where there are two points in the fibre, the map on tangent spaces is an isomorphism. But there is no Zariski neighbourhood of any point where the map is an isomorphism.

Thus a minimal requirement is that the morphism is a bijection. Note that this is not enough in general for a morphism between algebraic varieties to be an isomorphism. For example in characteristic

p, Frobenius is nowhere smooth and even in characteristic zero, the parametrisation of the cuspidal cubic is a bijection but not an isomorphism.

Lemma 5.10. *If $f\colon X \longrightarrow Y$ is a projective morphism with finite fibres, then f is finite.*

Proof. Since the result is local on the base, we may assume that Y is affine. By assumption $X \subset Y \times \mathbb{P}^n$ and we are projecting onto the first factor. Possibly passing to a smaller open subset of Y, we may assume that there is a point $p \in \mathbb{P}^n$ such that X does not intersect $Y \times \{p\}$.

As the blow up of $\mathbb{P}^n$ at p, fibres over $\mathbb{P}^{n-1}$ with fibres isomorphic to $\mathbb{P}^1$, and the composition of finite morphisms is finite, we may assume that $n = 1$, by induction on n.

We may assume that p is the point at infinity, so that $X \subset Y \times \mathbb{A}^1$, and X is affine. Now X is defined by $f(x) \in A(Y)[x]$, where the coefficients of $f(x)$ lie in $A(Y)$. Suppose that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

We may always assume that a_n does not vanish at y. Passing to the locus where a_n does not vanish, we may assume that a_n is a unit, so that dividing by a_n, we may assume that $a_n = 1$. In this case the ring B is a quotient of the ring

$$A[x]/\langle f\rangle.$$

But the latter is generated over A by 1, x, $\dots x^{n-1}$, and so is a finitely generated module over A. □

Theorem 5.11. *Let $f\colon X \longrightarrow Y$ be a projective morphism between quasi-projective varieties.*

Then f is an isomorphism iff it is a bijection and the differential df_p is injective.

Proof. One direction is clear. Otherwise assume that f is projective and a bijection on closed points. Then f is finite by (5.10). The result is local on the base, so we may assume that $Y = \operatorname{Spec} C$ is affine, in which case $X = \operatorname{Spec} D$ is affine, where C is a finitely generated D-module. Pick $x \in X$ and let $y = f(x)$. Then $x = \mathfrak{p}$ and $y = \mathfrak{q}$ are two prime ideals in C and D. Let A be the local ring of Y at y, B of X at x. Then A is the localisation of C at the multiplicative subset $S = C - \mathfrak{q}$ and as x is the unique point of the fibre, B is the localisation of D by the multiplicative subset $T = S \cdot D$, so that B is a finitely generated A-module.

Let $\phi\colon A \longrightarrow B$ be the induced ring homomorphism. Then B is a finitely generated A-module and we just need to show that ϕ is an isomorphism.

As f is a bijection on closed points, it follows that ϕ is injective. So we might as well suppose that ϕ is an inclusion. Let $\mathfrak{m}$ be the maximal ideal of A and let $\mathfrak{n}$ be the maximal ideal of B. By assumption

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow \frac{\mathfrak{n}}{\mathfrak{n}^2},$$

is surjective. But then

$$\mathfrak{m}B + \mathfrak{n}^2 = \mathfrak{n}.$$

By Nakayama's Lemma applied to the B-module $\mathfrak{n}/\mathfrak{m}B$, it follows that $\mathfrak{m}B = \mathfrak{n}$. But then

$$B/A \otimes A/\mathfrak{m} = B/(\mathfrak{m}B + A) = B/(\mathfrak{n} + A) = 0.$$

Nakayama's Lemma applied to the finitely generated A-module B/A implies that $B/A = 0$ so that ϕ is an isomorphism. $\square$

Lemma 5.12. *Suppose that $X \subset \mathbb{P}^n$ is a quasi-projective variety and suppose that $\pi\colon X \longrightarrow Y$ is the morphism induced by projection from a linear subspace.*

Let $y \in Y$. Then $\pi^{-1}(y) = \langle \Lambda, y\rangle \cap X$. If further this fibre consists of one point, then the map between Zariski tangent spaces is an isomorphism if the intersection of $\langle \Lambda, x\rangle$ with the Zariksi tangent space to X at x has dimension zero.

Proof. Easy. $\square$

Lemma 5.13. *Let X be a smooth irreducible subset of $\mathbb{P}^n$ of dimension k. Consider the projection Y of X down to a smaller dimensional projective space $\mathbb{P}^m$, from a linear space Λ of dimension $n - m - 1$.*

If the dimension of $m \geq 2k + 1$ and Λ is general (that is belongs to an appropriate open subset of the Grassmannian) then π is an isomorphism.

Proof. Since projection from a general linear space is the same as a sequence of projections from general points, we may assume that Λ is in fact a point p, so that $m = n - 1$.

Now we know that π is a bijection provided that p does not lie on any secant line. Since the secant variety has dimension at most $2k + 1$, it follows that we may certainly find a point away from the secant variety, provided that $n > 2k + 1$. Now since a tangent line is a limit of secant lines, it follows that such a point will also not lie on any tangent lines.

But then π is then an isomorphism on tangent spaces, whence an isomorphism. $\square$

For example, it follows that any curve may be embedded in $\mathbb{P}^3$ and any surface in $\mathbb{P}^5$. Now let us turn to the following classical problem in enumerative geometry.

Question 5.14. *Let C be a curve in $\mathbb{P}^2$ and let $p \in \mathbb{P}^2$.*

How many tangent lines does p lie on?

The first thing that we will need is a natty way to describe the projective tangent space to a variety.

Definition 5.15. *Let $X \subset \mathbb{P}^n$.*

*The **projective tangent space to** X **at** p is the closure of the affine tangent space.*

In other words the projective tangent space has the same dimension as the affine tangent space and is obtained by adding the suitable points at infinity. Suppose that the curve is defined by the polynomial $F(X, Y, Z)$. Then the tangent line to C at p, is

$$\left.\frac{\partial F}{\partial X}\right|_p X + \left.\frac{\partial F}{\partial Y}\right|_p Y + \left.\frac{\partial F}{\partial Z}\right|_p Z.$$

Of course it suffices to check that we get the right answer on an affine piece.

Lemma 5.16. *Let F be a homogeneous polynomial of degree d in $X_0, X_1, \dots, X_n$. Then*

$$dF = \sum X_i \frac{\partial F}{\partial X_i}$$

Proof. Both sides are linear in F. Thus it suffices to prove this for a monomial of degree d, when the result is clear. $\square$

It follows then that the tangent line above does indeed pass through p. The rest is easy.

Finally we will need Bézout's Theorem.

Theorem 5.17 (Bézout's Theorem)**.** *Let C and D be two curves defined by homogenous polynomials of degrees d and e. Suppose that $C \cap D$ does not contain a curve.*

Then $|C \cap D|$ is at most de, with equality iff the intersection of the two tangent spaces at $p \in C \cap D$ is equal to p.

We are now ready to answer (5.14).

Lemma 5.18. *Let $C \subset \mathbb{P}^n$ be a curve in $\mathbb{P}^2$ and let $p \in \mathbb{P}^2$ be a general point.*

Then p lies on $d(d-1)$ tangent lines.

Proof. Fix $p = [a : b : c]$ and let D be the curve defined by

$$G = a\frac{\partial F}{\partial X} + b\frac{\partial F}{\partial Y} + c\frac{\partial F}{\partial Z}.$$

Then G is a polynomial of degree $d - 1$. Consider a point q where C intersects D. Then the tangent line to C at q is given by

$$\left.\frac{\partial F}{\partial X}\right|_q X + \left.\frac{\partial F}{\partial Y}\right|_q Y + \left.\frac{\partial F}{\partial Z}\right|_q Z.$$

But then since p satisfies this equation, as q lies on D, it follows that p lies on the tangent line of C at q. Similarly it is easy to check the converse, that if p lies on the tangent line to C at q, then q is an intersection point of C and D.

Now apply Bézout's Theorem. □

There is an interesting way to look at all of this. In fact one may generalise the result above to the case of curves with nodes. Note that if you take a curve in $\mathbb{P}^3$ and take a general projection down to $\mathbb{P}^2$, then you get a nodal curve. Indeed it is easy to pick the point of projection not on a tangent line, since the space of tangent lines obviously sweeps out a surface; it is a little more involved to show that the space of three secant lines is a proper subvariety. (5.18) was then generalised to this case and it was shown that if δ is the number of nodes, then the number

$$\frac{d(d-1)}{2} - \delta$$

is an invariant of the curve.

Here is another way to look at this. Suppose that we project our curve down to $\mathbb{P}^1$ from a point. Then we get a finite cover of $\mathbb{P}^1$, with d points in the general fibre. Lines tangent to C passing through p then count the number of branch points, that is, the number of points in the base where the fibre has fewer than d points. Since this tangent line is only tangent to p and is simply tangent (that is, there are no flex points) there are $d - 1$ points in this fibre, and the ramification point corresponding to the branch point is where two sheets come together.

The modern approach to this invariant is quite different. If we are over the complex numbers $\mathbb{C}$, changing perspective, we may view the curve C as a Riemann surface covering another Riemann surface D. Now the basic topological invariant of a compact oriented Riemann surface is it's genus. In these terms there is a simple formula that connects the genus of C and B, in terms of the ramification data, known as Riemann-Hurwitz,

$$2g - 2 = d(2h - 2) + b,$$

where g is the genus of C, h the genus of B, d the order of the cover and b the contribution from the ramification points. Indeed if locally on C, the map is given as $z \longrightarrow z^e$ so that e sheets come together, the contribution is $e-1$.

In our case, $B = \mathbb{P}^1$ which is of genus 0, for each branch point, we have simple ramification, so that $e = 2$ and the contribution is one, making a total $b = d(d-1)$. Thus

$$2g - 2 = -2d + d(d-1).$$

Solving for g we get

$$g = \frac{(d-1)(d-2)}{2}.$$

Note that if $d \leq 2$, then we get $g = 0$ as expected (that is $C \simeq \mathbb{P}^1$) and if $d = 3$ then we get an elliptic curve.

It also seems worth pointing out that if we take a smooth variety X and blow up a point p, then the exceptional divisor E is canonically the projectivisation of the Zariski tangent space to X at p,

$$E = \mathbb{P}(T_p X).$$

Indeed the point is that E picks up the different tangent directions to X at p, and this is exactly the set of lines in $T_p X$. Note the difference between the projective tangent space and the projectivisation of the tangent space.

It also seems worth pointing out that one defines the Zariski tangent space to a scheme X, at a point x, using exactly the same definition, the dual of

$$\mathfrak{m}/\mathfrak{m}^2,$$

where $\mathfrak{m} \subset \mathcal{O}_{X,x}$ is the maximal ideal of the local ring. However in general, if we have the equality of dimensions of both the Zariski tangent space and the local dimension, we only call X **regular at** $x \in X$. Smoothness is a more restricted notion in general.

Having said this, if X is a quasi-projective variety over an algebraically closed field then X is smooth as a variety if and only if it is smooth as a scheme over $\operatorname{Spec} k$. In fact an abstract variety over $\operatorname{Spec} k$ is smooth if and only if it is regular. Note that if x is a specialisation of ξ and X is regular at x then X is regular at ξ, so it is enough to check that X is regular at the closed points.

Note that one can sometimes use the Zariski tangent space to identify embedded points. If X is a scheme and $Y = X_{\text{red}}$ is the reduced subscheme then $x \in X$ is an embedded point if

$$\dim T_x X > \dim T_x Y,$$

and X is reduced away from x. For example, if X is not regular at x but Y is regular at x then $x \in X$ is an embedded point. Note however that it is possible that $x \in X$ is an embedded point but the Zariski tangent space is no bigger than it should be; for example if $\mathcal{H}_0^3$ is the punctual Hilbert scheme of a smooth surface then the underlying variety is a quadric cone. The vertex of the cone corresponds to the unique zero dimensional length three scheme which is not curivilinear and this is an embedded point, even though the Zariski tangent space is three dimensional.

It is interesting to see which toric varieties are smooth. The question is local, so we might as well assume that $X = U_\sigma$ is affine. If $\sigma \subset N_{\mathbb{R}}$ does not span $N_{\mathbb{R}}$, then $X \simeq U_{\sigma'} \times \mathbb{G}_m^l$, where σ' is the same cone as σ embedded in the space it spans. So we might as well assume that σ spans $N_{\mathbb{R}}$. In this case X contains a unique fixed point x_σ which is in the closure of every orbit. Since X only contains finitely many orbits, it follows that X is smooth if and only if X is regular at x_σ. The maximal ideal of x_σ is generated by χ^u, where $u \in S_\sigma$. The square of the maximal is generated by χ^{u+v}, where u and v are two elements of S_σ. So a basis for $\mathfrak{m}/\mathfrak{m}^2$ is given by elements of S_σ that are not sums of two elements. Since the elements of S_σ generate the group M, the elements of S_σ which are not sums of two elements, must generate the group. Given an extremal ray of $\check{\sigma}$, a primitive generator of this ray is not the sum of two elements in S_σ. So $\check{\sigma}$ must have n edges and they must generate M. So these elements are a basis of the lattice and in fact $X \simeq \mathbb{A}_k^n$.

6. Divisors

Definition 6.1. *We say that a scheme X is* ***regular in codimension one*** *if every local ring of dimension one is regular, that is, the quotient $\mathfrak{m}/\mathfrak{m}^2$ is one dimensional, where $\mathfrak{m}$ is the unique maximal ideal of the corresponding local ring.*

Regular in codimension one often translates to smooth in codimension one.

When talking about Weil divisors, we will only consider schemes which are

($*$) noetherian, integral, separated, and regular in codimension one.

Definition 6.2. *Let X be a scheme satisfying ($*$). A* ***prime divisor*** *Y on X is a closed integral subscheme of codimension one.*

A ***Weil divisor*** *D on X is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors.*

Thus a Weil divisor is a formal linear combination $D = \sum_Y n_Y Y$ of prime divisors, where all but finitely many $n_Y = 0$. We say that D is **effective** if $n_Y \geq 0$.

Definition 6.3. *Let X be a scheme satisfying ($*$), and let Y be a prime divisor, with generic point η. Then $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field K.*

The ***valuation*** *ν_Y* ***associated to*** *Y is the corresponding valuation.*

Note that as X is separated, Y is determined by its valuation. If $f \in K$ and $\nu_Y(f) > 0$ then we say that f has a **zero of order** $\nu_Y(f)$; if $\nu_Y(f) < 0$ then we say that f has a **pole of order** $-\nu_Y(f)$.

Definition-Lemma 6.4. *Let X be a scheme satisfying ($*$), and let $f \in K^*$.*

$$(f) = \sum_Y \nu_Y(f) Y \in \operatorname{Div} X.$$

Proof. We have to show that $\nu_Y(f) = 0$ for all but finitely many Y. Let U be the open subset where f is regular. Then the only poles of f are along $Z = X - U$. As Z is a proper closed subset and X is noetherian, Z contains only finitely many prime divisors.

Similarly the zeroes of f only occur outside the open subset V where $g = f^{-1}$ is regular. $\square$

Any divisor D of the form (f) will be called **principal**.

Lemma 6.5. *Let X be a scheme satisfying ($*$).*

The principal divisors are a subgroup of $\operatorname{Div} X$.

Proof. The map
$$K^* \longrightarrow \operatorname{Div} X,$$
is easily seen to be a group homomorphism. $\square$

Definition 6.6. *Two Weil divisors D and D' are called* ***linearly equivalent****, denoted $D \sim D'$, if and only if the difference is principal. The group of Weil divisors modulo linear equivalence is called the* ***divisor Class group****, denoted $\operatorname{Cl} X$.*

We will also denote the group of Weil divisors modulo linear equivalence as $A_{n-1}(X)$.

Proposition 6.7. *If k is a field then*
$$\operatorname{Cl}(\mathbb{P}^r_k) \simeq \mathbb{Z}.$$

Proof. Note that if Y is a prime divisor in $\mathbb{P}^r_k$ then Y is a hypersurface in $\mathbb{P}^n$, so that $I = \langle G \rangle$ and Y is defined by a single homogeneous polynomial G. The degree of G is called the degree of Y.

If $D = \sum n_Y Y$ is a Weil divisor then define the degree $\deg D$ of D to be the sum
$$\sum n_Y \deg Y,$$
where $\deg Y$ is the degree of Y.

Note that the degree of any rational function is zero. Thus there is a well-defined group homomorphism
$$\deg\colon \operatorname{Cl}(\mathbb{P}^r_k) \longrightarrow \mathbb{Z},$$
and it suffices to prove that this map is an isomorphism. Let H be defined by X_0. Then H is a hyperplane and H has degree one. The divisor $D = nH$ has degree n and so the degree map is surjective. One the other hand, if $D = \sum n_i Y_i$ is effective, and Y_i is defined by G_i,
$$(\prod_i G^{n_i}/X_0^d) = D - dH,$$
where d is the degree of D. $\square$

Example 6.8. *Let C be a smooth cubic curve in $\mathbb{P}^2_k$. Suppose that the line $Z = 0$ is a flex line to the cubic at the point $P_0 = [0 : 1 : 0]$. If the equation of the cubic is $F(X, Y, Z)$ this says that $F(X, Y, 0) = X^3$. Therefore the cubic has the form $X^3 + ZG(X, Y, Z)$. If we work on the open subset $U_3 \simeq \mathbb{A}^2_k$, then we get*
$$x^3 + g(x, y) = 0,$$
where $g(x, y)$ has degree at most two. If we expand $g(x, y)$ as a polynomial in y,
$$g_0(x)y^2 + g_1(x)y + g_2(x),$$

then $g_0(x)$ must be a non-zero scalar, since otherwise C is singular (a nodal or cuspidal cubic). We may assume that $g_0 = 1$. If we assume that the characteristic is not two, then we may complete the square to get

$$y^2 = x^3 + g(x),$$

for some quadratic polynomial $g(x)$. If we assume that the characteristic is not three, then we may complete the cube to get

$$y^2 = x^3 + ax + ab,$$

for some a and $b \in k$.

Now any two sets of three collinear points are linearly equivalent (since the equation of one line divided by another line is a rational function on the whole $\mathbb{P}^2_k$). In fact given any three points P, Q and P' we may find Q' such that $P + Q \sim P' + Q'$; indeed the line $l = \langle P, Q\rangle$ meets the cubic in one more point R. The line $l' = \langle R, P'\rangle$ then meets the cubic in yet another point Q'. We have

$$P + Q + R \sim P' + Q' + R'.$$

Cancelling we get

$$P + Q \sim P' + Q'.$$

It follows that if there are further linear equivalences then there are two points P and P' such that $P \sim P'$. This gives us a rational function f with a single zero P and a single pole P'; in turn this gives rise to a morphism $C \longrightarrow \mathbb{P}^1$ which is an isomorphism. It turns out that a smooth cubic is not isomorphic to $\mathbb{P}^1$, so that in fact the only relations are those generated by setting two sets of three collinear points to be linearly equivalent.

Put differently, the rational points of C form an abelian group, where three points sum to zero if and only if they are collinear, and P_0 is declared to be the identity. The divisors of degree zero modulo linear equivalence are equal to this group.

It is interesting to calculate the Class group of a toric variety X, which always satisfies $(*)$. By assumption there is a dense open subset $U \simeq \mathbb{G}_m^n$. The complement Z is a union of the invariant divisors.

Lemma 6.9. *Suppose that X satisfies $(*)$, let Z be a closed subset and let $U = X \setminus Z$.*

Then there is an exact sequence

$$\mathbb{Z}^k \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0,$$

where k is the number of components of Z which are prime divisors.

Proof. If Y is a prime divisor on X then $Y' = Y \cap U$ is either a prime divisor on U or empty. This defines a group homomorphism

$$\rho\colon \operatorname{Div}(X) \longrightarrow \operatorname{Div}(U).$$

If $Y' \subset U$ is a prime divisor, then let Y be the closure of Y' in X. Then Y is a prime divisor and $Y' = Y \cap U$. Thus ρ is surjective. If f is a rational function on X and $Y = (f)$, then the image of Y in $\operatorname{Div}(U)$ is equal to $(f|_U)$. If $Z = Z' \cup \bigcup_{i=1}^k Z_i$ where Z' has codimension at least two, then the map which sends $(m_1, m_2, \dots, m_k)$ to $\sum m_i Z_i$ generates the kernel. $\square$

Example 6.10. *Let $X = \mathbb{P}^2_k$ and C be an irreducible curve of degree d. Then $\operatorname{Cl}(\mathbb{P}^2 - C)$ is equal to $\mathbb{Z}_d$. Similarly $\operatorname{Cl}(\mathbb{A}^n_k) = 0$.*

It follows by (6.9) that there is an exact sequence

$$\mathbb{Z}^k \longrightarrow \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(U) \longrightarrow 0.$$

Applying this to $X = \mathbb{A}^n_k$ it follows that $\operatorname{Cl}(U) = 0$. So we get an exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^s \longrightarrow \operatorname{Cl}(X) \longrightarrow 0.$$

We want to identify the kernel K. This is equal to the set of principal divisors which are supported on the invariant divisors. If f is a rational function such that (f) is supported on the invariant divisors then f has no zeroes or poles on the torus; it follows that $f = \lambda \chi^u$, where $\lambda \in k^*$ and $u \in M$.

It follows that there is an exact sequence

$$M \longrightarrow \mathbb{Z}^s \longrightarrow \operatorname{Cl}(X) \longrightarrow 0.$$

Lemma 6.11. *Let $u \in M$. Suppose that X is the affine toric variety associated to a cone σ, where σ spans $N_{\mathbb{R}}$. Let v be a primitive generator of a one dimensional ray τ of σ and let D be the corresponding invariant divisor.*

Then $\operatorname{ord}_D(\chi^u) = \langle u, v \rangle$. In particular

$$(\chi^u) = \sum_i \langle u, v_i \rangle D_i,$$

where the sum ranges over the invariant divisors.

Proof. We can calculate the order on the open set $U_\tau = \mathbb{A}^1_k \times \mathbb{G}_m^{n-1}$, where D corresponds to $\{0\} \times \mathbb{G}_m^{n-1}$. Using this, we are reduced to the one dimensional case. So $N = \mathbb{Z}$, $v = 1$ and $u \in M = \mathbb{Z}$. In this case χ^u is the monomial x^u and the order of vanishing at the origin is exactly u. $\square$

It follows that if $X = X(F)$ is the toric variety associated to a fan F which spans $N_{\mathbb{R}}$ then we have short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Example 6.12. *Let σ be the cone spanned by $2e_1 - e_2$ and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. There are two invariant divisors D_1 and D_2. The principal divisor associated to $u = f_1 = (1,0)$ is $2D_1$ and the principal divisor associated to $u = f_2 = (0,1)$ is $D_2 - D_1$. So the class group is $\mathbb{Z}_2$.*

Note that the dual $\check{\sigma}$ is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the monoid $S_\sigma = \check{\sigma} \cap M$ are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. So the group algebra

$$A_\sigma = k[x, xy, xy^2] = \frac{k[u,v,w]}{\langle v^2 - uw \rangle},$$

and $X = U_\sigma$ is the quadric cone.

Now suppose we take the standard fan associated to $\mathbb{P}^2$. The invariant divisors are the three coordinate lines, D_1, D_2 and D_3. If $f_1 = (1,0)$ and $f_2 = (0,1)$ then

$$(\chi^{f_1}) = D_1 - D_3 \qquad \text{and} \qquad (\chi^{f_2}) = D_2 - D_3.$$

So the class group is $\mathbb{Z}$.

We now turn to the notion of a Cartier divisor.

Definition 6.13. *Given a ring A, let S be the multiplicative set of non-zero divisors of A. The localisation A_S of A at S is called the* ***total quotient ring*** *of A.*

Given a scheme X, let $\mathcal{K}$ be the sheaf associated to the presheaf, which associates to every open subset $U \subset X$, the total quotient ring of $\Gamma(U, \mathcal{O}_X)$. $\mathcal{K}$ is called the ***sheaf of total quotient rings****.*

Definition 6.14. *A* ***Cartier divisor*** *on a scheme X is any global section of $\mathcal{K}^*/\mathcal{O}_X^*$.*

In other words, a Cartier divisor is specified by an open cover U_i and a collection of rational functions f_i, such that f_i/f_j is a nowhere zero regular function.

A Cartier divisor is called **principal** if it is in the image of $\Gamma(X, \mathcal{K}^*)$. Two Cartier divisors D and D' are called **linearly equivalent**, denoted $D \sim D'$, if and only if the difference is principal.

Definition 6.15. *Let X be a scheme satisfying $(*)$. Then every Cartier divisor determines a Weil divisor.*

Informally a Cartier divisor is simply a Weil divisor defined locally by one equation. If every Weil divisor is Cartier then we say that X

is **factorial**. This is equivalent to requiring that every local ring is a UFD; for example every smooth variety is factorial.

Definition-Lemma 6.16. *Let X be a scheme.*

The set of invertible sheaves forms an abelian group $\operatorname{Pic}(X)$*, where multiplication corresponds to tensor product and the inverse to the dual.*

Definition 6.17. *Let D be a Cartier divisor, represented by $\{(U_i, f_i)\}$. Define a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}$ by taking the subsheaf generated by f_i^{-1} over the open set U_i.*

Proposition 6.18. *Let X be a scheme.*

1. *The association $D \longrightarrow \mathcal{O}_X(D)$ defines a correspondence between Cartier divisors and invertible subsheaves of $\mathcal{K}$.*
2. *$\mathcal{O}_X(D_1 - D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$, as abstract $\mathcal{O}_X$-modules.*
3. *Two Cartier divisors D_1 and D_2 are linearly equivalent if and only if $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$, as abstract $\mathcal{O}_X$-modules.*

Let's consider which Weil divisors on a toric variety are Cartier. We classify all Cartier divisors whose underlying Weil divisor is invariant; we dub these Cartier divisors T-Cartier. We start with the case of the affine toric variety associated to a cone $\sigma \subset N_{\mathbb{R}}$. By (6.18) it suffices to classify all invertible subsheaves $\mathcal{O}_X(D) \subset \mathcal{K}$. Taking global sections, since we are on an affine variety, it suffices to classify all fractional ideals,

$$I = H^0(X, \mathcal{O}_X(D)) \subset H^0(X, \mathcal{O}_X) = A_\sigma.$$

Invariance of D implies that I is graded by M, that is, I is a direct sum of eigenspaces. As D is Cartier, I is principal at the distinguished point x_σ of U_σ, so that $I/\mathfrak{m}I$ is one dimensional, where

$$\mathfrak{m} = \sum k \cdot \chi^u.$$

It follows that $I = A_\sigma \chi^u$, so that $D = (\chi^u)$ is principal. In particular, the only Cartier divisors are the principal divisors and X is factorial if and only if the Class group is trivial.

Example 6.19. *The quadric cone Q, given by $xy - z^2 = 0$ in $\mathbb{A}^3_k$ is not factorial. We have already seen* (6.12) *that the class group is $\mathbb{Z}_2$.*

If $\sigma \subset N_{\mathbb{R}}$ is not maximal dimensional then every Cartier divisor on U_σ whose associated Weil divisor is invariant is of the form (χ^u) but

$$(\chi^u) = (\chi^{u'}) \qquad \text{if and only if} \qquad u - u' \in \sigma^\perp \cap M = M(\sigma).$$

So the T-Cartier divisors are in correspondence with $M/M(\sigma)$.

Now suppose that $X = X(F)$ is a general toric variety. Then a T-Cartier divisor is given by specifying an element $u(\sigma) \in M/M(\sigma)$, for every cone σ in F. This defines a divisor $(\chi^{-u(\sigma)})$; equivalently a fractional ideal

$$I = H^0(X, \mathcal{O}_X(D)) = A_\sigma \cdot \chi^{u(\sigma)}.$$

These maps must agree on overlaps; if τ is a face of σ then $u(\sigma) \in M/M(\sigma)$ must map to $u(\tau) \in M/M(\tau)$.

The data

$$\{\, u(\sigma) \in M/M(\sigma) \,|\, \sigma \in F \,\},$$

for a T-Cartier divisor D determines a continuous piecewise linear function ϕ_D on the support $|F|$ of F. If $v \in \sigma$ then let

$$\phi_D(v) = \langle u(\sigma), v \rangle.$$

Compatibility of the data implies that ϕ_D is well-defined and continuous. Conversely, given any continuous function ϕ, which is linear and integral (that is, given by an element of M) on each cone, we can associate a unique T-Cartier divisor D. If $D = \sum a_i D_i$ the function is given by $\phi_D(v_i) = -a_i$, where v_i is the primitive generator of the ray corresponding to D_i.

Note that

$$\phi_D + \phi_E = \phi_{D+E} \qquad \text{and} \qquad \phi_{mD} = m\phi_D.$$

Note also that $\phi_{(\chi^u)}$ is the linear function given by u. So D and E are linearly equivalent if and only if ϕ_D and ϕ_E differ by a linear function.

If X is any variety which satisfies $(*)$ then the natural map

$$\operatorname{Pic}(X) \longrightarrow \operatorname{Cl}(X),$$

is an embedding. It is an interesting to compare $\operatorname{Pic}(X)$ and $\operatorname{Cl}(X)$ on a toric variety. Denote by $\operatorname{Div}_T(X)$ the group of T-Cartier divisors.

Proposition 6.20. *Let $X = X(F)$ be the toric variety associated to a fan F which spans $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \operatorname{Div}_T(X) \arrow[r] \arrow[d, hook] & \operatorname{Pic}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbb{Z}^s \arrow[r] & \operatorname{Cl}(X) \arrow[r] & 0
\end{tikzcd}$$

In particular

$$\rho(X) = \operatorname{rank}(\operatorname{Pic}(X)) \leq \operatorname{rank}(\operatorname{Cl}(X)) = s - n.$$

Further $\operatorname{Pic}(X)$ *is a free abelian group.*

Proof. We have already seen that the bottom row is exact. If L is an invertible sheaf then $L|_U$ is trivial. Suppose that $L = \mathcal{O}_X(E)$. Pick a rational function such that $(f)|_U = E|_U$. Let $D = E - (f)$. Then D is T-Cartier and exactness of the top row is easy.

Finally, $\mathrm{Pic}(X)$ is subgroup of the direct sum of $M/M(\sigma)$ and each of these is a lattice, whence $\mathrm{Pic}(X)$ is torsion free. $\square$

Finally we end with an example to illustrate some of the difficulties of working with varieties which are not regular in codimension one.

Example 6.21. *Let $C \subset \mathbb{P}^2_k$ be the nodal cubic $ZY^2 = X^3 + X^2Z$, so that in the affine piece $U_3 \simeq \mathbb{A}^2_k$, $C \cap U_3$ is given by $y^2 = x^2 + x^3$. Let N be the node. Note that if D is a Weil divisor whose support does not contain N then D is automatically a Cartier divisor. As in the case of the smooth cubic, if P, Q, R and P', Q' and R' are two triples of collinear points on C (none of which are N), then $P + Q + R \sim P' + Q' + R'$.*

Now we already know that the nodal cubic is not isomorphic to $\mathbb{P}^1$. This implies that if P and P' are two smooth points of C then P and P' are not linearly equivalent. It follows, with a little bit of work, that all linear equivalences on C are generated by the linear equivalences above.

The normalisation of C is isomorphic to $\mathbb{P}^1$; on the affine piece where $Z \neq 0$ the normalisation morphism is given as $t \longrightarrow (t^2 - 1, t(t^2 - 1))$. The inverse image of the node N contains two points of $\mathbb{P}^1$ and it follows that $C - \{N\}$ is isomorphic to $\mathbb{G}_m$. In fact one can check that this is an isomorphism of algebraic groups, where the group law on $C - \{N\}$ is given by declaring three collinear points to sum to zero.

There is an exact sequence of groups,

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Pic}(C) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the first map sends P to $P - P_0 = [0 : 1 : 0]$, and the second map is the degree map which sends $D = \sum n_i P_i$ to $\sum n_i$.

Note that even though we can talk about Weil divisors on C, it only makes sense to talk about linear equivalences of Weil divisors supported away from N. Indeed, the problem is that any line through N cuts out $2N + R$, where R is another point of C. Varying the line varies R but fixes $2N$. In terms of Cartier divisors, a line through N (and not tangent to a branch) is equivalent to a length two scheme contained in the line. As we vary the line, both R and the length two scheme vary.

It is interesting to consider what happens at the level of invertible sheaves. Consider an invertible sheaf L on C which is of degree zero, that is, consider an invertible sheaf which corresponds to a Cartier *divisor D of degree zero. If $\pi\colon \mathbb{P}^1 \longrightarrow C$ is the normalisation map then*

$$\pi^* L = \pi^* \mathcal{O}_C(D) = \mathcal{O}_{\mathbb{P}^1}(\pi^* D),$$

has degree zero (to pullback a Cartier divisor, just pullback the defining equations. It is easy to check that this commutes with pullback of the sheaf). Since $\mathrm{Pic}(\mathbb{P}^1) \simeq \mathbb{Z}$, $\pi^* L \simeq \mathcal{O}_{\mathbb{P}^1}$, the trivial sheaf. Now to get a sheaf on C we have to glue the two local rings over the inverse image N_1 and N_2 of N. The only isomorphisms of two such local rings are $\mathbb{G}_m$ acting by scalar multiplication (this is particularly transparent if one thinks of a invertible sheaf as a line bundle; in this case we are just identifying two copies of a one dimensional vector space) and this is precisely the kernel of the degree map on $\mathbb{P}^1$).

There is a similar picture for the cuspidal cubic, given as $Y^2Z = X^3$. The only twist is that $C - \{N\}$, where N is the cusp, is now a copy of $\mathbb{G}_a$.

7. Linear systems

First a word about the base scheme. We would like to work in enough generality to cover the general case. On the other hand, it takes some work to state properly the general results if one works over an arbitrary scheme S. As a compromise we work over an arbitrary affine variety $S = \operatorname{Spec} A$. As most statements are local on the base, we don't lose any generality. It is customary to say X is a scheme over a ring A, as shorthand for X is a scheme over the corresponding scheme $S = \operatorname{Spec} A$.

Theorem 7.1. *Let X be a scheme over a ring A.*

(1) *If $\phi\colon X \longrightarrow \mathbb{P}^n_A$ is an A-morphism then $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ is an invertible sheaf on X, which is generated by the global sections $s_0, s_1, \dots, s_n$, where $s_i = \phi^* x_i$.*
(2) *If $\mathcal{L}$ is an invertible sheaf on X, which is generated by the global sections $s_0, s_1, \dots, s_n$, then there is a unique A-morphism $\phi\colon X \longrightarrow \mathbb{P}^n_A$ such that $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ and $s_i = \phi^* x_i$.*

Proof. It is clear that $\mathcal{L}$ is an invertible sheaf. Since $x_0, x_1, \dots, x_n$ generate the ring $A[x_0, x_1, \dots, x_n]$, it follows that $x_0, x_1, \dots, x_n$ generate the sheaf $\mathcal{O}_{\mathbb{P}^n_A}(1)$. Thus $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. Hence (1).

Now suppose that $\mathcal{L}$ is an invertible sheaf generated by $s_0, s_1, \dots, s_n$. Let

$$X_i = \{\, p \in X \mid s_i \notin \mathfrak{m}_p \mathcal{L}_p \,\}.$$

Then X_i is an open subset of X and the sets $X_0, X_1, \dots, X_n$ cover X. Define a morphism

$$\phi_i\colon X_i \longrightarrow U_i,$$

where U_i is the standard open subset of $\mathbb{P}^n_A$, as follows: Since

$$U_i = \operatorname{Spec} A[y_0, y_1, \dots, y_n],$$

where $y_j = x_j/x_i$, is affine, it suffices to give a ring homomorphism

$$A[y_0, y_1, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}).$$

We send y_j to s_j/s_i, and extend by linearity. The key observation is that the ratio is a well-defined element of $\mathcal{O}_{X_i}$, which does not depend on the choice of isomorphism $\mathcal{L}|_{X_i} \simeq \mathcal{O}_{X_i}$.

It is then straightforward to check that the set of morphisms $\{\phi_i\}$ glues to a morphism ϕ with the given properties. $\square$

Example 7.2. *Let $X = \mathbb{P}^1_k$, $A = k$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_k}(2)$.*

In this case, global sections of $\mathcal{L}$ are generated by S^2, ST and T^2. This morphism is represented globally by

$$[S : T] \longrightarrow [S^2 : ST : T^2].$$

The image is the conic $XZ = Y^2$ inside $\mathbb{P}^2_k$.

More generally one can map $\mathbb{P}^1_k$ into $\mathbb{P}^n_k$ by the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_k}(n)$. More generally still, one can map $\mathbb{P}^m_k$ into $\mathbb{P}^n_k$ using the invertible sheaf $\mathcal{O}_{\mathbb{P}^m_k}(1)$.

Corollary 7.3.

$$\mathrm{Aut}(\mathbb{P}^n_k) \simeq \mathrm{PGL}(n+1,k).$$

Proof. First note that $\mathrm{PGL}(n+1,k)$ acts naturally on $\mathbb{P}^n_k$ and that this action is faithful.

Now suppose that $\phi \in \mathrm{Aut}(\mathbb{P}^n_k)$. Let $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_k}(1)$. Since $\mathrm{Pic}(\mathbb{P}^n_k) \simeq \mathbb{Z}$ is generated by $\mathcal{O}_{\mathbb{P}^n_k}(1)$, it follows that $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(\pm 1)$. As $\mathcal{L}$ is globally generated, we must have $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$. Let $s_i = \phi^* x_i$. Then $s_0, s_1, \dots, s_n$ is a basis for the k-vector space $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$. But then there is a matrix

$$A = (a_{ij}) \in \mathrm{GL}(n+1,k) \qquad \text{such that} \qquad s_i = \sum_{ij} a_{ij} x_j.$$

Since the morphism ϕ is determined by $s_0, s_1, \dots, s_n$, it follows that ϕ is determined by the class of A in $\mathrm{GL}(n+1,k)$. $\square$

Lemma 7.4. *Let $\phi\colon X \longrightarrow \mathbb{P}^n_A$ be an A-morphism. Then ϕ is a closed immersion if and only if*

(1) *$X_i = X_{s_i}$ is affine, and*
(2) *the natural map of rings*

$$A[y_0, y_1, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}) \qquad \textit{which sends} \qquad y_i \longrightarrow \frac{s_i}{s_j},$$

is surjective.

Proof. Suppose that ϕ is a closed immersion. Then X_i is isomorphic to $\phi(X) \cap U_i$, a closed subscheme of affine space. Thus X_i is affine. Hence (1) and (2) follows as we have surjectivity on all of the localisations.

Now suppose that (1) and (2) hold. Then X_i is a closed subscheme of U_i and so X is a closed subscheme of $\mathbb{P}^n_A$. $\square$

Theorem 7.5. *Let X be a projective scheme over an algebraically closed field k and let $\phi\colon X \longrightarrow \mathbb{P}^n_k$ be a morphism over k, which is given by an invertible sheaf $\mathcal{L}$ and global sections $s_0, s_1, \dots, s_n$ which generate $\mathcal{L}$. Let $V \subset \Gamma(X, \mathcal{L})$ be the space spanned by the sections.*

Then ϕ is a closed immersion if and only if

(1) *V **separates points**: that is, given p and $q \in X$ there is $\sigma \in V$ such that $\sigma \in \mathfrak{m}_P \mathcal{L}_p$ but $\sigma \notin \mathfrak{m}_q \mathcal{L}_q$.*

(2) V ***separates tangent vectors***: *that is, given $p \in X$ the set*

$$\{ \sigma \in V \mid \sigma \in \mathfrak{m}_p \mathcal{L}_p \},$$

spans $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$.

Proof. Suppose that ϕ is a closed immersion. Then we might as well consider $X \subset \mathbb{P}^n_k$ as a closed subscheme. In this case (1) is clear. Just pick a linear function on the whole of $\mathbb{P}^n_k$ which vanishes at p but not at q (equivalently pick a hyperplane which contains p but not q). Similarly linear functions on $\mathbb{P}^n_k$ separate tangent vectors on the whole of projective space, so they certainly separate on X.

Now suppose that (1) and (2) hold. Then ϕ is clearly injective. Since X is proper over $\operatorname{Spec} k$ and $\mathbb{P}^n_k$ is separated over $\operatorname{Spec} k$ it follows that ϕ is proper. In particular, $\phi(X)$ is closed and ϕ is a homeomorphism onto $\phi(X)$. It remains to show that the map on stalks

$$\mathcal{O}_{\mathbb{P}^n_k,p} \longrightarrow \mathcal{O}_{X,x},$$

is surjective. But the same piece of commutative algebra as we used in the proof of the inverse function theorem, works here. □

Definition 7.6. *Let X be a noetherian scheme. We say that an invertible sheaf $\mathcal{L}$ is* ***ample*** *if for every coherent sheaf $\mathcal{F}$ there is an integer $n_0 > 0$ such that $\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^n$ is globally generated, for all $n \geq n_0$.*

Lemma 7.7. *Let $\mathcal{L}$ be an invertible sheaf on a Noetherian scheme. The following are equivalent:*

(1) *$\mathcal{L}$ is ample.*
(2) *$\mathcal{L}^m$ is ample for all $m > 0$.*
(3) *$\mathcal{L}^m$ is ample for some $m > 0$.*

Proof. (1) implies (2) implies (3) is clear.

So assume that $\mathcal{M} = \mathcal{L}^m$ is ample and let $\mathcal{F}$ be a coherent sheaf. For each $0 \leq i \leq m-1$, let $\mathcal{F}_i = \mathcal{F} \otimes \mathcal{L}^i$. By assumption there is an integer n_i such that $\mathcal{F}_i \otimes \mathcal{M}^n$ is globally generated for all $n \geq n_i$. Let n_0 be the maximum of the n_i. If $n \geq n_0 m$, then we may write $n = qm + i$, where $0 \leq i \leq m-1$ and $q \geq n_0 \geq n_i$.

But then

$$\mathcal{F} \otimes \mathcal{L}^m = \mathcal{F}_i \otimes \mathcal{M}^q,$$

which is globally generated. □

Theorem 7.8. *Let X be a scheme of finite type over a Noetherian ring A and let $\mathcal{L}$ be an invertible sheaf on X.*

Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample for some $m > 0$.

Proof. Suppose that $\mathcal{L}^m$ is very ample. Then there is an immersion $X \subset \mathbb{P}^r_A$, for some positive integer r, and $\mathcal{L}^m = \mathcal{O}_X(1)$. Let $\bar{X}$ be the closure. If $\mathcal{F}$ is any coherent sheaf on X then there is a coherent sheaf $\overline{\mathcal{F}}$ on $\bar{X}$, such that $\mathcal{F} = \overline{\mathcal{F}}|_X$. By Serre's result, $\overline{\mathcal{F}}(k)$ is globally generated for some positive integer k. It follows that $\mathcal{F}(k)$ is globally generated, so that $\mathcal{L}^m$ is ample, and the result follows by (7.7).

Conversely, suppose that $\mathcal{L}$ is ample. Given $p \in X$, pick an open affine neighbourhood U of p so that $\mathcal{L}|_U$ is free. Let $Y = X - U$, give it the reduced induced strucure, with ideal sheaf $\mathcal{I}$. Then $\mathcal{I}$ is coherent. Pick $n > 0$ so that $\mathcal{I} \otimes \mathcal{L}^n$ is globally generated. Then we may find $s \in \mathcal{I} \otimes \mathcal{L}^n$ not vanishing at p. We may identify s with $s' \in \mathcal{O}_U$ and then $p \in U_s \subset U$, an affine subset of X.

By compactness, we may cover X by finitely many such open affines and we may assume that n is fixed. Replacing $\mathcal{L}$ by $\mathcal{L}^n$ we may assume that $n = 1$. Then there are global sections $s_1, s_2, \dots, s_k \in H^0(X, \mathcal{L})$ such that $U_i = U_{s_i}$ is an open affine cover.

Since X is of finite type, each $B_i = H^0(U_i, \mathcal{O}_{U_i})$ is a finitely generated A-algebra. Pick generators b_{ij}. Then $s^n b_{ij}$ lifts to $s_{ij} \in H^0(X, \mathcal{L}^n)$, for some positive integer n. Again we might as well assume $n = 1$.

Now let $\mathbb{P}^N_A$ be the projective space with coordinates $x_1, x_2, \dots, x_k$ and x_{ij}. Locally we can define a map on each U_i to the standard open affine, by the obvious rule, and it is standard to check that this glues to an immersion. $\square$

Definition 7.9. *Let $\mathcal{L}$ be an invertible sheaf on a smooth projective variety over an algebraically closed field. Let $s \in H^0(X, \mathcal{L})$. The **divisor** (s) **of zeroes of** s is defined as follows: by assumption we may cover X by open subsets U_i over which we may identify $s|_{U_i}$ with $f_i \in \mathcal{O}_{U_i}$. The defines a Cartier divisor $\{(U_i, f_i)\}$.*

It is a simple matter to check that the Cartier divisor does not depend on our choice of trivialisations. Note that as X is smooth the Cartier divisor may safely be identified with the corresponding Weil divisor.

Lemma 7.10. *Let X be a smooth projective variety over an algebraically closed field. Let D_0 be a divisor and let $\mathcal{L} = \mathcal{O}_X(D_0)$.*

1. *If $s \in H^0(X, \mathcal{L})$, $s \neq 0$ then $(s) \sim D_0$.*
2. *If $D \geq 0$ and $D \sim D_0$ then there is a global section $s \in H^0(X, \mathcal{L})$ such that $D = (s)$.*
3. *If $s_i \in H^0(X, \mathcal{L})$, $i = 1$ and 2, are two global sections then $(s_1) = (s_2)$ if and only if $s_2 = \lambda s_1$ where $\lambda \in k^*$.*

Proof. As $\mathcal{O}_X(D_0) \subset \mathcal{K}$, the section s corresponds to a rational function f. If D_0 is the Cartier divisor $\{(U_i, f_i)\}$ then $\mathcal{O}_X(D_0)$ is locally generated by f_i^{-1} so that multiplication by f_i induces an isomorphism with $\mathcal{O}_{U_i}$. D is then locally defined by ff_i. But then

$$D = D_0 + (f).$$

Hence (1).

Now suppose that $D > 0$ and $D = D_0 + (f)$. Then $(f) \geq -D_0$. Hence

$$f \in H^0(X, \mathcal{O}_X(D_0)) \subset H^0(X, \mathcal{K}) = K(X),$$

and the divisor of zeroes of f is D. This is (2).

Now suppose that $(s_1) = (s_2)$. Then

$$D_0 + (f_1) = (s_1) = (s_2) = D_0 + (f_2).$$

Cancelling, we get that $(f_1) = (f_2)$ and the rational function f_1/f_2 has no zeroes nor poles. Since X is a projective variety, $f_1/f_2 = \lambda$, a constant. □

Definition 7.11. *Let D_0 be a divisor. The **complete linear system** associated to D_0 is the set*

$$|D_0| = \{\, D \in \operatorname{Div}(X) \mid D \geq 0, D \sim D_0 \,\}.$$

We have seen that

$$|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Thus $|D|$ is naturally a projective space.

Definition 7.12. *A **linear system** is any linear subspace of a complete linear system $|D_0|$.*

In other words, a linear system corresponds to a linear subspace, $V \subset H^0(X, \mathcal{O}_X(D_0))$. We will then write

$$|V| = \{\, D \in |D_0| \mid D = (s), s \in V \,\} \simeq \mathbb{P}(V) \subset \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Definition 7.13. *Let $|V|$ be a linear system. The **base locus** of $|V|$ is the intersection of the elements of $|V|$.*

Lemma 7.14. *Let X be a smooth projective variety over an algebraically closed field, and let $|V| \subset |D_0|$ be a linear system.*

V generates $\mathcal{O}_X(D_0)$ if and only if $|V|$ is base point free.

Proof. If V generates $\mathcal{O}_X(D_0)$ then for every point $x \in X$ we may find an element $\sigma \in V$ such that $\sigma(x) \neq 0$. But then $D = (\sigma)$ does not contain x, and so the base locus is empty.

Conversely suppose that the base locus is empty. The locus where V is not generated $\mathcal{O}_X(D_0)$ is a closed subset Z of X. Pick $x \in Z$ a closed point. By assumption we may find $D \in |V|$ such that $x \notin D$. But then if $D = (\sigma)$, $\sigma(x) \neq 0$ and σ generates the stalk $\mathcal{L}_x$, a contradiction. Thus Z is empty and $\mathcal{O}_X(D_0)$ is globally generated. □

Example 7.15. *Consider $\mathcal{O}_{\mathbb{P}^1}(4)$. The complete linear system $|4p|$ defines a morphism into $\mathbb{P}^4$, where $p = [0 : 1]$ and $q = [1 : 0]$, given by $\mathbb{P}^1 \longrightarrow \mathbb{P}^4$, $[S : T] \longrightarrow [S^4 : ST^3 : S^2T^2 : ST^3 : T^4]$. If we project from $[0 : 0 : 1 : 0 : 0]$ we will get a morphism into $\mathbb{P}^3$, $[S : T] \longrightarrow [S^4 : ST^3 : ST^3 : T^4]$. This corresponds to the sublinear system spanned by $4p$, $3p + q$, $p + 3q$, $4q$.*

Consider $\mathcal{O}_{\mathbb{P}^2}(2)$ and the corresponding complete linear system. The map associated to this linear system is the Veronese embedding $\mathbb{P}^2 \longrightarrow \mathbb{P}^5$, $[X : Y : Z] \longrightarrow [X^2 : Y^2 : Z^2 : YZ : XZ : XY]$.

Note also the notion of separating points and tangent directions becomes a little clearer in this more geometric setting. Separating points means that given x and $y \in X$, we can find $D \in |V|$ such that $x \in D$ and $y \notin D$. Separating tangent vectors means that given any irreducible length two zero dimensional scheme z, with support x, we can find $D \in |V|$ such that $x \in D$ but z is not contained in D. In fact the condition about separating tangent vectors is really the limiting case of separating points.

Thinking in terms of linear systems also presents an inductive approach to proving global generation. Suppose that we consider the complete linear system $|D|$. Suppose that we can find $Y \in |D|$. Then the base locus of $|D|$ is supported on Y. On the other hand suppose that $\mathcal{I}$ is the ideal sheaf of Y in X. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

As X is smooth D is Cartier and $\mathcal{O}_X(D)$ is an invertible sheaf. Tensoring by locally free preserves exactness, so there are short exact sequences,

$$0 \longrightarrow \mathcal{I}(mD) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_Y(mD) \longrightarrow 0.$$

Taking global sections, we get

$$0 \longrightarrow H^0(X, \mathcal{I}(mD)) \longrightarrow H^0(X, \mathcal{O}_X(mD)) \longrightarrow H^0(Y, \mathcal{O}_Y(mD)).$$

At the level of linear systems there is therefore a linear map

$$|D| \longrightarrow |D|_Y|.$$

It is interesting to see what happens for toric varieties. Suppose that $X = X(F)$ is the toric variety associated to the fan $F \subset N_{\mathbb{R}}$. Recall that we can associate to a T-Cartier divisor $D = \sum a_i D_i$, a continuous piecewise linear function

$$\phi_D \colon |F| \longrightarrow \mathbb{R},$$

where $|F| \subset N_{\mathbb{R}}$ is the support of the fan, which is specified by the rule that $\phi_D(v_i) = -a_i$.

We can also associate to D a rational polyhedron

$$\begin{aligned} P_D &= \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i\rangle \geq -a_i \quad \forall i \,\} \\ &= \{\, u \in M_{\mathbb{R}} \,|\, u \geq \phi_D \,\}. \end{aligned}$$

Lemma 7.16. *If X is a toric variety and D is T-Cartier then*

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{u \in P_D \cap M} k \cdot \chi^u.$$

Proof. Suppose that $\sigma \in F$ is a cone. Then, we have already seen that

$$H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D)) = \bigoplus_{u \in P_D(\sigma) \cap M} k \cdot \chi^u,$$

where

$$P_D(\sigma) = \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i\rangle \geq -a_i \quad \forall v_i \in \sigma \,\}.$$

These identifications are compatible on overlaps. Since

$$H^0(X, \mathcal{O}_X(D)) = \bigcap_{\sigma \in F} H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D))$$

and

$$P_D = \bigcap_{\sigma \in F} P_D(\sigma),$$

the result is clear. □

It is interesting to compute some examples. First, suppose we consider $\mathbb{P}^1$. A T-Cartier divisor is a sum $ap+bq$ (p and q the fixed points). The corresponding function is

$$\phi(x) = \begin{cases} -ax & x > 0 \\ -bx & x < 0. \end{cases}$$

The corresponding polytope is the interval

$$[-a, b] \subset \mathbb{R} = M_{\mathbb{R}}.$$

There are $a+b+1$ integral points, corresponding to the fact that there are $a+b+1$ monomials of degree $a+b$. For $\mathbb{P}^2$ and dD_3, P_D is the convex hull of $(0,0)$, $(-d,0)$ and $(0,-d)$. The number of integral points is

$$\frac{(d+1)^2}{2} + \frac{d+1}{2} = \frac{(d+2)(d+1)}{2},$$

which is the usual formula.

Let D be a Cartier divisor on a toric variety $X = X(F)$ given by a fan F. It is interesting to consider when the complete linear system $|D|$ is base point free. Since any Cartier divisor is linearly equivalent to a T-Cartier divisor, we might as well suppose that $D = \sum a_i D_i$ is T-Cartier. Note that the base locus of the complete linear system of any Cartier divisor is invariant under the action of the torus. Since there are only finitely many orbits, it suffices to show that for each cone $\sigma \in F$ the point $x_\sigma \in U_\sigma$ is not in the base locus. It is also clear that if x_σ is not in the base locus of $|D|$ then in fact one can find a T-Cartier divisor $D' \in |D|$ which does not contain x_σ. Equivalently we can find $u \in M$ such that

$$\langle u, v_i \rangle \geq -a_i,$$

with strict equality if $v_i \in \sigma$. The interesting thing is that we can reinterpret this condition using ϕ_D.

Definition 7.17. *The function $\phi\colon V \longrightarrow \mathbb{R}$ is **upper convex** if*

$$\phi(\lambda v + (1-\lambda)w) \geq \lambda\phi(v) + (1-\lambda)w \qquad \forall v, w \in V.$$

*When we have a fan F and ϕ is linear on each cone σ, then ϕ is called **strictly upper convex** if the linear functions $u(\sigma)$ and $u(\sigma')$ are different, for different maximal cones σ and σ'.*

Theorem 7.18. *Let $X = X(F)$ be the toric variety associated to a T-Cartier divisor D.*

Then

1. *$|D|$ is base point free if and only if ψ_D is upper convex.*
2. *D is very ample if and only if ψ_D is strictly upper convex and the semigroup S_σ is generated by*

$$\{\, u - u(\sigma) \mid u \in P_D \cap M \,\}.$$

Proof. (1) follows from the remarks above. (2) is proved in Fulton's book. □

For example if $X = \mathbb{P}^1$ and

$$\phi(x) = \begin{cases} -ax & x > 0 \\ -bx & x < 0. \end{cases}$$

so that $D = ap + bq$ then ϕ is upper convex if and only if $a + b \geq 0$ in which case D is base point free. D is very ample if and only if $a+b > 0$. When ϕ is continuous and linear on each cone σ, we may restate the upper convex condition as saying that the graph of ϕ lies under the graph of $u(\sigma)$. It is strictly upper convex if it lies strictly under the graph of $u(\sigma)$, for all n-dimensional cones σ.

Using this, it is altogether too easy to give an example of a smooth proper toric variety which is not projective. Let $F \subset N_{\mathbb{R}} = \mathbb{R}^3$ given by the edges $v_1 = -e_1$, $v_2 = -e_2$, $v_3 = -e_3$, $v_4 = e_1 + e_2 + e_3$, $v_5 = v_3 + v_4$, $v_6 = v_1 + v_4$ and $v_7 = v_2 + v_4$. Now connect v_1 to v_5, v_3 to v_7 and v_2 to v_6 and v_5 to v_6, v_6 to v_7 and v_7 to v_5.

It is not hard to check that X is smooth and proper (proper translates to the statement that the support $|F|$ of the fan is the whole of $N_{\mathbb{R}}$). Suppose that ψ is strictly upper convex. Let w be the midpoint of the line connecting v_1 and v_5. Then

$$w = \frac{v_1 + v_5}{2} = \frac{v_3 + v_6}{2}.$$

Since v_1 and v_5 belong to the same maximal cone, ψ is linear on the line connecting them. In particular

$$\psi(w) = \psi(\frac{v_1 + v_5}{2}) = \frac{1}{2}\psi(v_1) + \frac{1}{2}\psi(v_5).$$

Since v_1, v_5 and v_3 belong to the same cone and v_6 does not, by strict convexity,

$$\psi(w) = \psi(\frac{v_3 + v_6}{2}) > \frac{1}{2}\psi(v_3) + \frac{1}{2}\psi(v_6).$$

Putting all of this together, we get

$$\psi(v_1) + \psi(v_5) > \psi(v_2) + \psi(v_6).$$

By symmetry

$$\begin{aligned}
\psi(v_1) + \psi(v_5) &> \psi(v_3) + \psi(v_6) \\
\psi(v_2) + \psi(v_6) &> \psi(v_1) + \psi(v_7) \\
\psi(v_3) + \psi(v_7) &> \psi(v_2) + \psi(v_5).
\end{aligned}$$

But adding up these three inequalities gives a contradiction.

8. Relative proj and the blow up

We want to define a relative version of Proj, in pretty much the same way we defined a relative version of Spec. We start with a scheme X and a quasi-coherent sheaf $\mathcal{S}$ sheaf of graded $\mathcal{O}_X$-algebras,

$$\mathcal{S} = \bigoplus_{d \in \mathbb{N}} \mathcal{S}_d,$$

where $\mathcal{S}_0 = \mathcal{O}_X$. It is convenient to make some simplifying assumptions:

(†) X is Noetherian, $\mathcal{S}_1$ is coherent, $\mathcal{S}$ is locally generated by $\mathcal{S}_1$.

To construct relative Proj, we cover X by open affines $U = \operatorname{Spec} A$. $\mathcal{S}(U) = H^0(U, \mathcal{S})$ is a graded A-algebra, and we get $\pi_U \colon \operatorname{Proj} \mathcal{S}(U) \longrightarrow U$ a projective morphism. If $f \in A$ then we get a commutative diagram

$$\begin{tikzcd} \operatorname{Proj} \mathcal{S}(U_f) \arrow[r] \arrow[d, "\pi_{U_f}"'] & \operatorname{Proj} \mathcal{S}(U) \arrow[d, "\pi_U"'] \\ U_f \arrow[r] & U. \end{tikzcd}$$

It is not hard to glue π_U together to get $\pi \colon \mathbf{Proj}\, \mathcal{S} \longrightarrow X$. We can also glue the invertible sheaves together to get an invertible sheaf $\mathcal{O}(1)$.

The relative consruction is very similar to the old construction.

Example 8.1. *If X is Noetherian and*

$$\mathcal{S} = \mathcal{O}_X[T_0, T_1, \dots, T_n],$$

then satisfies (†) *and* $\mathbf{Proj}\, \mathcal{S} = \mathbb{P}^n_X$.

Given a sheaf $\mathcal{S}$ satisfying (†), and an invertible sheaf $\mathcal{L}$, it is easy to construct a quasi-coherent sheaf $\mathcal{S}' = \mathcal{S} \star \mathcal{L}$, which satisfies (†). The graded pieces of $\mathcal{S}'$ are $\mathcal{S}_d \otimes \mathcal{L}^d$ and the multiplication maps are the obvious ones. There is a natural isomorphism

$$\phi \colon P' = \mathbf{Proj}\, \mathcal{S}' \longrightarrow P = \mathbf{Proj}\, \mathcal{S},$$

which makes the diagram commute

$$\begin{tikzcd} P' \arrow[rr, "\phi"] \arrow[dr, "\pi'"'] & & P \arrow[dl, "\pi"] \\ & X, & \end{tikzcd}$$

and

$$\phi^* \mathcal{O}_P(1) \simeq \mathcal{O}_{P'}(1) \otimes \pi'^* \mathcal{L}.$$

Note that π is always proper; in fact π is projective over any open affine and properness is local on the base.

There are two very interesting family of examples of the construction of relative Proj. Suppose that we start with a locally free sheaf $\mathcal{E}$ of rank $r \geq 2$. Note that

$$\mathcal{S} = \bigoplus \operatorname{Sym}^d \mathcal{E},$$

satisfies (†). $\mathbb{P}(\mathcal{E}) = \mathbf{Proj}\, \mathcal{S}$ is the **projective bundle** over X associated to $\mathcal{E}$. The fibres of $\pi \colon \mathbb{P}(\mathcal{E}) \longrightarrow X$ are copies of $\mathbb{P}^n$, where $n = r - 1$. We have

$$\bigoplus_{l=0}^{\infty} \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) = \mathcal{S},$$

so that in particular

$$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}.$$

Also there is a natural surjection

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Indeed, it suffices to check both statements locally, so that we may assume that X is affine. The first statement is then (4.21) and the second statement reduces to the statement that the sections $x_0, x_1, \dots, x_n$ generate $\mathcal{O}_P(1)$. The most interesting result is:

Proposition 8.2. *Let $g \colon Y \longrightarrow X$ be a morphism.*

Then a morphism $f \colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X is the same as giving an invertible sheaf $\mathcal{L}$ on Y and a surjection $g^* \mathcal{E} \longrightarrow \mathcal{L}$.

Proof. One direction is clear; if $f \colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ is a morphism over X, then the surjective morphism of sheaves

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1),$$

pullsback to a surjective morphism

$$g^* \mathcal{E} = f^*(\pi^* \mathcal{E}) \longrightarrow \mathcal{L} = f^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Conversely suppose we are given an invertible sheaf $\mathcal{L}$ and a surjective morphism of sheaves

$$g^* \mathcal{E} \longrightarrow \mathcal{L}.$$

I claim that there is then a unique morphism $f \colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X, which induces the given surjection. By uniqueness, it suffices to prove this result locally. So we may assume that $X = \operatorname{Spec} A$ is affine and

$$\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}_X,$$

is free. In this case surjectivity reduces to the statement that the images $s_0, s_1, \ldots, s_n$ of the standard sections generate $\mathcal{L}$, and the result reduces to one we have already proved. $\square$

Definition 8.3. *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Let*

$$\mathcal{S} = \bigoplus_{d=0}^{\infty} \mathcal{I}^d,$$

where $\mathcal{I}^0 = \mathcal{O}_X$ and $\mathcal{I}^d$ is the dth power of $\mathcal{I}$. Then $\mathcal{S}$ satisfies (†).

*$\pi\colon \mathbf{Proj}\,\mathcal{S} \longrightarrow X$ is called the **blow up** of $\mathcal{I}$ (or Y, if Y is the subscheme of X associated to $\mathcal{I}$).*

Example 8.4. *Let $X = \mathbb{A}^n_k$ and let P be the origin. We check that we get the usual blow up. Let*

$$A = k[x_1, x_2, \ldots, x_n].$$

As $X = \operatorname{Spec} A$ is affine and the ideal sheaf $\mathcal{I}$ of P is the sheaf associated to $\langle x_1, x_2, \ldots, x_n\rangle$,

$$Y = \mathbf{Proj}\,\mathcal{S} = \operatorname{Proj} S,$$

where

$$S = \bigoplus_{d=0}^{\infty} I^d.$$

There is a surjective map

$$A[y_1, y_2, \ldots, y_n] \longrightarrow S,$$

of graded rings, where y_i is sent to x_i. $Y \subset \mathbb{P}^n_A$ is the closed subscheme corresponding to this morphism. The kernel of this morphism is

$$\langle y_i x_j - y_j x_i\rangle,$$

which are the usual equations of the blow up.

Definition 8.5. *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. We are going to define the **inverse image ideal sheaf** $\mathcal{I}' \subset \mathcal{O}_Y$. First we take the inverse image of the sheaf $f^{-1}\mathcal{I}$, where we just think of f as being a continuous map. Then $f^{-1}\mathcal{I} \subset f^{-1}\mathcal{O}_Y$. Let $\mathcal{I}' = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ be the ideal generated by the image of $f^{-1}\mathcal{I}$ under the natural morphism $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$.*

Theorem 8.6 (Universal Property of the blow up)**.** *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent ideal sheaf.*

If $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ then $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ is an invertible sheaf. Moreover π is universal amongst all such morphisms. If

$f\colon Z \longrightarrow X$ is any morphism such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is invertible then there is a unique induced morphism $g\colon Z \longrightarrow Y$ which makes the diagram commute

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "f"'] & Y \arrow[d, "\pi"] \\ & X. \end{tikzcd}$$

Proof. By uniqueness, we can check this locally. So we may assume that $X = \operatorname{Spec} A$ is affine. As $\mathcal{I}$ is coherent, it corresponds to an ideal $I \subset A$ and

$$X = \operatorname{Proj} \bigoplus_{d=0}^{\infty} I^d.$$

Now $\mathcal{O}_Y(1)$ is an invertible sheaf on Y. It is not hard to check that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_Y = \mathcal{O}_Y(1)$.

Pick generators $a_0, a_1, \dots, a_n$ for I. This gives rise to a surjective map of rings

$$\phi\colon A[x_0, x_1, \dots, x_n] \longrightarrow I,$$

whence to a closed immersion $Y \subset \mathbb{P}^n_A$. The kernel of ϕ is generated by all homogeneous polynomials $F(x_0, x_1, \dots, x_n)$ such that $F(a_0, a_1, \dots, a_n) = 0$.

Now the elements $a_0, a_1, \dots, a_n$ pullback to global sections $s_0, s_1, \dots, s_n$ of the invertible sheaf $\mathcal{L} = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ and $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. So we get a morphism

$$g\colon Z \longrightarrow \mathbb{P}^n_X,$$

over X, such that $g^*\mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{L}$ and $g^{-1}x_i = s_i$. Suppose that $F(x_0, x_1, \dots, x_n)$ is a homogeneous polynomial in the kernel of ϕ. Then $F(a_0, a_1, \dots, a_n) = 0$ so that $F(s_0, s_1, \dots, s_n) = 0$ in $H^0(X, \mathcal{L}^d)$. It follows that g factors through Y.

Now suppose that $f\colon Z \longrightarrow X$ factors through $g\colon Z \longrightarrow Y$. Then

$$f^{-1}\mathcal{I} \cdot \mathcal{O}_Z = g^{-1}(\mathcal{I} \cdot \mathcal{O}_Y) \cdot \mathcal{O}_Z = g^{-1}\mathcal{O}_Y(1) \cdot \mathcal{O}_Z.$$

Therefore there is a surjective map

$$g^*\mathcal{O}_Y(1) \longrightarrow \mathcal{L}.$$

But then this map must be an isomorphism and so $g^*\mathcal{O}_Y(1) = \mathcal{L}$. $s_i = g^*x_i$ and uniqueness follows. $\square$

Note that by the universal property, the morphism π is an isomorphism outside of the subscheme V defined by $\mathcal{I}$. We may put the universal property differently. The only subscheme with an invertible ideal sheaf is a Cartier divisor (local generators of the ideal, give local equations for the Cartier divisor). So the blow up is the smallest morphism which turns a subscheme into a Cartier divisor. Perhaps surprisingly, therefore, blowing up a Weil divisor might give a non-trivial birational map.

If X is a variety it is not hard to see that π is a projective, birational morphism. In particular if X is quasi-projective or projective then so is Y. We note that there is a converse to this:

Theorem 8.7. *Let X be a quasi-projective variety and let $f\colon Z \longrightarrow X$ be a birational projective morphism.*

Then there is an coherent ideal sheaf $\mathcal{I}$ and a commutative diagram

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "\pi"] \\ & X, & \end{tikzcd}$$

where $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ and the top row is an isomorphism.

It is interesting to figure out the geometry behind the example of a toric variety which is not projective. To warm up, suppose that we start with $\mathbb{A}^3_k$. This is the toric variety associated to the fan spanned by e_1, e_2, e_3. Imagine blowing up two of the axes. This corresponds to inserting two vectors, $e_1 + e_2$ and $e_1 + e_3$. However the order in which we blow up is significant. Let's introduce some notation. If we blow up the x-axis $\pi\colon Y \longrightarrow X$ and then the y-axis, $\psi\colon Z \longrightarrow Y$, let's call the exceptional divisors E_1 and E_2, and let E_1' denote the strict transform of E_1 on Z. E_1 is a $\mathbb{P}^1$-bundle over the x-axis. The strict transform of the y-axis in Y intersects E_1 in a point p. When we blow up this curve, $E_1' \longrightarrow E_1$ blows up the point p. The fibre of E_1' over the origin therefore consists of two copies Σ_1 and Σ_2 of $\mathbb{P}^1$. Σ_1 is the strict transform of the fibre of E_1 over the origin and Σ_2 is the exceptional divisor. The fibre Σ of E_2 over the origin is a copy of $\mathbb{P}^1$. Σ and Σ_2 are the same curve in Z.

The example of a toric variety which is not projective is obtained from $\mathbb{P}^3$ by blowing up three coordinate axes, which form a triangle. The twist is that we do something different at each of the three coordinate points. Suppose that $\pi\colon X \longrightarrow \mathbb{P}^3$ is the birational morphism down to $\mathbb{P}^3$, and let E_1, E_2 and E_3 be the three exceptional divisors. Over one point we extract E_1 first then E_2, over the second point we extract first E_2 then E_3 and over the last point we extract first E_3 then E_1.

To see what has gone wrong, we need to work in the homology and cohomology groups of X. Any curve C in X determines an element of $[C] \in H_2(X, \mathbb{Z})$. Any Cartier divisor D in X determines a class $[D] \in H^2(X, \mathbb{Z})$. We can pair these two classes to get an intersection number $D \cdot C \in \mathbb{Z}$. One way to compute this number is to consider the line bundle $\mathcal{L} = \mathcal{O}_X(D)$ associated to D. Then

$$D \cdot C = \deg \mathcal{L}|_C.$$

If D is ample then this intersection number is always positive. This implies that the class of every curve is non-trivial in homology.

Suppose the reducible fibres of E_1, E_2 and E_3 over their images are $A_1 + A_2$, $B_1 + B_2$ and $C_1 + C_3$. Suppose that the general fibres are A, B and C. We suppose that A_1 is attached to B, B_1 is attached to C and C_1 is attached to A. We have

$$\begin{aligned}
[A] &= [A_1] + [A_2] \\
&= [B] + [A_2] \\
&= [B_1] + [B_2] + [A_2] \\
&= [C] + [B_2] + [A_2] \\
&= [C_1] + [C_2] + [B_2] + [A_2] \\
&= [A] + [C_2] + [B_2] + [A_2],
\end{aligned}$$

in $H_2(X, \mathbb{Z})$, so that

$$[A_2] + [B_2] + [C_2] = 0 \in H_2(X, \mathbb{Z}).$$

Suppose that D were an ample divisor on X. Then

$$0 = D \cdot ([A_2] + [B_2] + [C_2]) > D \cdot [A_2] + D \cdot [B_2] + D \cdot [C_2] > 0,$$

a contradiction.

There are a number of things to say about this way of looking at things, which lead in different directions. The first is that there is no particular reason to start with a triangle of curves. We could start with two conics intersecting transversally (so that they lie in different planes). We could even start with a nodal cubic, and just do something different over the two branches of the curve passing through the node. Neither of these examples are toric, of course. It is clear that in the first two examples, the morphism

$$\pi \colon X \longrightarrow \mathbb{P}^3,$$

is locally projective. It cannot be a projective morphism, since $\mathbb{P}^3$ is projective and the composition of projective is projective. It also follows that π is not the blow up of a coherent sheaf of ideals on $\mathbb{P}^3$.

The third example is not even a variety. It is a complex manifold (and in fact it is something called an algebraic space). In particular the notion of the blow up in algebraic geometry is more delicate than it might first appear.

The second thing is to consider the difference between the order of blow ups of the two axes. Suppose we denote the composition of blowing up the x-axis and then the y-axis by $\pi_1\colon X_1 \longrightarrow \mathbb{A}^3$ and the composition the other way by $\pi_2\colon X_1 \longrightarrow \mathbb{A}^3$. Now X_1 and X_2 agree outside the origin. Let $\phi\colon X_1 \dashrightarrow X_2$ be the resulting birational map. If $A_1 + A_2$ is the fibre of π_1 over the origin and $B_1 + B_2$ is the fibre of π_2 over the origin, then ϕ is in fact an isomorphism outside A_2 and B_2. So ϕ is a birational map which is an isomorphism in codimension one, in fact an isomorphism outside a curve, isomorphic to $\mathbb{P}^1$. ϕ is an example of a *flop*. In terms of fans, we have four vectors v_1, v_2, v_3 and v_4, such that

$$v_1 + v_3 = v_2 + v4,$$

and any three vectors span the lattice. If σ is the cone spanned by these four vectors, then $Q = U_\sigma$ is the cone over a quadric. There are two ways to subdivide σ into two cones. Insert the edge connecting v_1 to v_3 or the edge corresponding to $v_2 + v_4$. The corresponding morphisms extract a copy of $\mathbb{P}^1$ and the resulting birational map between the two toric varieties is a (simple) flop. One can also insert the vector $w = v_1 + v_3$, to get a toric variety Y. The corresponding exceptional divisor is $\mathbb{P}^1 \times \mathbb{P}^1$. The toric varieties fit into a picture

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr] & \\ X_1 \arrow[rr, dashed] \arrow[dr] & & X_2 \arrow[dl] \\ & Q. & \end{tikzcd}$$

The two maps $Y \longrightarrow X_i$ correspond to the two projections of $\mathbb{P}^1 \times \mathbb{P}^1$ down to $\mathbb{P}^1$. By (8.7) $\pi_i\colon X_i \longrightarrow Q$ corresponds to blowing up a coherent ideal sheaf. In fact it corresponds to blowing up a Weil divisor (in fact this is a given, as π_i does not extract any divisors), the plane determined by either ruling.

Finally, it is interesting to wonder more about the original examples of varieties which are not projective. Note that in the case when we blow up either a triangle or a conic if we make one flop then we get a projective variety. Put differently, if we start with a projective variety then it is possible to get a non-projective variety by flopping a curve. When does flopping a curve mean that the variety is no longer projective? A variety is projective if it contains an ample divisor. Ample divisors intersect all curve positively. Note that any sum of ample divisors is ample.

Definition 8.8. *Let X be a proper variety. The* ***ample cone*** *is the cone in $H^2(X, \mathbb{R})$ spanned by the classes of the ample divisors.*

The ***Kleiman-Mori cone*** $\overline{\mathrm{NE}}(X)$ *in $H_2(X, \mathbb{R})$ is the closure of the cone spanned by the classes of curves.*

The significance of all of this is the following:

Theorem 8.9 (Kleiman's Criteria)**.** *Let X be a proper variety (or even algebraic space).*

A divisor D is ample if and only if the linear functional

$$\psi\colon H_2(X, \mathbb{R}) \longrightarrow \mathbb{R},$$

given by $\phi(\alpha) = [D] \cdot \alpha$ is strictly positive on $\overline{\mathrm{NE}}(X) - \{0\}$.

Using Kleiman's criteria, it is not hard to show that if $\phi\colon X \dashrightarrow Y$ is a flop of the curve C and X is projective then Y is projective if and only if the class of $[C]$ generates a one dimensional face of $\overline{\mathrm{NE}}(X)$.

9. Sheaf Cohomology

Definition 9.1. *Let X be a topological space. For every $i \geq 0$ there are functors H^i from the category of sheaves of abelian groups on X to the category of abelian groups such that*

(1) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.
(2) *Given a short exact sequence,*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$

there are coboundary maps

$$H^i(X, \mathcal{H}) \longrightarrow H^{i+1}(X, \mathcal{F}).$$

which can be strung together to get a long exact sequence of cohomology.

In short, sheaf cohomology was invented to fix the lack of exactness, and in fact this property essentially fixes the definition.

Example 9.2. *If X is a simplicial complex (or a CW-complex) then $H^i(X, \mathbb{Z})$ agrees with the usual definition. The same goes for any other coefficient ring (considered as a locally constant sheaf).*

Like ordinary cohomology, sheaf cohomology inherits a cup product,

$$H^i(X, \mathcal{F}) \otimes H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes \mathcal{G}),$$

where $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. In particular if X is a projective scheme over A then

$$H^i(X, \mathcal{F}),$$

is an A-module, where $\mathcal{F}$ is an $\mathcal{O}_X$-module, since there is a ring homorphism $A \longrightarrow H^0(X, \mathcal{O}_X)$. In particular if A is a field, then

$$H^i(X, \mathcal{F}),$$

are vector spaces.

$$h^i(X, \mathcal{F}),$$

denotes their dimension.

We would like to have a definition of these groups which allows us to compute. Let X be a topological space and let $\mathcal{U} = \{U_i\}$ be an open cover, which is locally finite. The group of k-cochains is

$$C^k(\mathcal{U}, \mathcal{F}) = \bigoplus_I \Gamma(U_I, \mathcal{F}),$$

where I runs over all $(k+1)$-tuples of indices and

$$U_I = \bigcap_{i \in I} U_i,$$

denotes intersection. k-cochains are skew-commutative, so that if we switch two indices we get a sign change.

Define a coboundary map

$$\delta^k \colon C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^{k+1}(\mathcal{U}, \mathcal{F}).$$

Given $\sigma = (\sigma_I)$, we have to construct $\tau = \delta(\sigma) \in C^{k+1}(\mathcal{U}, \mathcal{F})$. We just need to determine the components τ_J of τ. Now $J = \{j_0, j_1, \dots, j_k\}$. If we drop an index, then we get a k-tuple. We define

$$\tau_J = \left(\sum_{i=0}^{k} (-1)^i \sigma_{J - \{i_i\}} \right) \bigg|_{U_J}.$$

The key point is that $\delta^2 = 0$. So we can take cohomology

$$H^i(\mathcal{U}, \mathcal{F}) = Z^i(\mathcal{U}, \mathcal{F}) / B^i(\mathcal{U}, \mathcal{F}).$$

Here Z^i denotes the group of i-cocycles, those elements killed by δ^i and B^i denotes the group of coboundaries, those cochains which are in the image of δ^{i-1}. Note that $\delta^i(B^i) = \delta^i \delta^{i-1}(C^{i-1}) = 0$, so that $B^i \subset Z^i$.

The problem is that this is not enough. Perhaps our open cover is not fine enough to capture all the interesting cohomology. A **refinement** of the open cover $\mathcal{U}$ is an open cover $\mathcal{V}$, together with a map h between the indexing sets, such that if V_j is an open subset of the refinement, then for the index $i = h(j)$, we have $V_j \subset U_i$. It is straightforward to check that there are maps,

$$H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{V}, \mathcal{F}),$$

on cohomology. Taking the (direct) limit, we get the Čech cohomology groups,

$$\check{H}^i(X, \mathcal{F}).$$

For example, consider the case $i = 0$. Given a cover, a cochain is just a collection of sections, (σ_i), $\sigma_i \in \Gamma(U_i, \mathcal{F})$. This cochain is a cocycle if $(\sigma_i - \sigma_j)|_{U_{ij}} = 0$ for every i and j. By the sheaf axiom, this means that there is a global section $\sigma \in \Gamma(X, \mathcal{F})$, so that in fact

$$H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}).$$

It is also sometimes possible to untwist the definition of H^1. A 1-cocycle is precisely the data of a collection

$$(\sigma_{ij}) \in \Gamma(\mathcal{U}, \mathcal{F}),$$

such that

$$\sigma_{ij} - \sigma_{ik} + \sigma_{jk} = 0.$$

In general of course, one does not want to compute these things using limits. The question is how fine does the cover have to be to compute the cohomology? As a first guess one might require that

$$H^i(U_j, \mathcal{F}) = 0,$$

for all j, and $i > 0$. In other words there is no cohomology on each open subset. But this is not enough. One needs instead the slightly stronger condition that

$$H^i(U_I, \mathcal{F}) = 0.$$

Theorem 9.3 (Leray)**.** *If X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups and $\mathcal{U}$ is an open cover such that*

$$H^i(U_I, \mathcal{F}) = 0,$$

for all $i > 0$ and indices I, then in fact the natural map

$$H^i(\mathcal{U}, \mathcal{F}) \simeq \check{H}^i(X, \mathcal{F}),$$

is an isomorphism.

It is in fact not too hard to prove:

Theorem 9.4 (Serre)**.** *Let X be a Noetherian scheme. TFAE*

(1) *X is affine,*
(2) *$H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves,*
(3) *$H^1(X, \mathcal{I}) = 0$ for all coherent ideals $\mathcal{I}$.*

Finally, we need to construct the coboundary maps. Suppose that we are given a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

We want to define

$$H^i(X, \mathcal{H}) \longrightarrow H^{i+1}(X, \mathcal{F}).$$

Cheating a little, we may assume that we have a commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & C^i(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & C^i(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d] & C^i(\mathcal{U}, \mathcal{H}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{F}) \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{G}) \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{H}) \arrow[r] & 0. \end{tikzcd}$$

Suppose we start with an element $t \in H^i(X, \mathcal{H})$. Then t is the image of $t' \in H^i(\mathcal{U}, \mathcal{H})$, for some open cover $\mathcal{U}$. In turn t' is represented by $\tau \in Z^i(\mathcal{U}, \mathcal{H})$. Now we may suppose our cover is sufficiently fine, so that $\tau_I \in \Gamma(U_I, \mathcal{H})$ is the image of $\sigma_I \in \Gamma(U_I, \mathcal{G})$ (and this fixes the cheat). Applying the boundary map, we get $\delta(\sigma) \in C^{i+1}(\mathcal{U}, \mathcal{G})$. Now the image of $\delta(\sigma)$ in $C^{i+1}(\mathcal{U}, \mathcal{H})$ is the same as $\delta(\tau)$, which is zero, as τ is a cocycle. But then by exactness of the bottom rows, we get $\rho \in C^{i+1}(\mathcal{U}, \mathcal{F})$. It is straightforward to check that ρ is a cocycle, so that we get an element $r' \in H^{i+1}(\mathcal{U}, \mathcal{F})$, whence an element r of $H^{i+1}(X, \mathcal{F})$, and that r does not depend on the choice of σ.

Thus sheaf cohomology does exist (at least when X is paracompact, which is not a problem for schemes). Let us calculate the cohomology of projective space.

Theorem 9.5. *Let A be a Noetherian ring. Let $X = \mathbb{P}^r_A$.*

(1) *The natural map $S \longrightarrow \Gamma_*(X, \mathcal{O}_X)$ is an isomorphism.*

(2)
$$H^i(X, \mathcal{O}_X(n)) = 0 \qquad \text{for all} \qquad 0 < i < r \text{ and } n.$$

(3)
$$H^r(X, \mathcal{O}_X(-r-1)) \simeq A.$$

(4) *The natural map*
$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathcal{O}_X(-r-1)) \simeq A,$$
is a perfect pairing of finitely generated free A-modules.

Proof. Let
$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n).$$
Then $\mathcal{F}$ is a quasi-coherent sheaf. Let $\mathcal{U}$ be the standard open affine cover. As every intersection is affine, it follows that we may compute using this cover. Now
$$\Gamma(U_I, \mathcal{F}) = S_{x_I},$$
where
$$x_I = \prod_{i \in I} x_i.$$
Thus Čech cohomology is the cohomology of the complex
$$\prod_{i=0}^{r} S_{x_i} \longrightarrow \prod_{i<j}^{r} S_{x_i x_j} \longrightarrow \dots \longrightarrow S_{x_0 x_1, \dots x_r}.$$
The kernel of the first map is just $H^0(X, \mathcal{F})$, which we already know is S. Now let us turn to $H^r(X, \mathcal{F})$. It is the cokernel of the map
$$\prod_i S_{x_0 x_1 \dots \hat{x}_i \dots x_r} \longrightarrow S_{x_0 x_1 \dots x_r}.$$
The last term is the free A-module with generators all monomials in the Laurent ring (that is, we allow both positive and negative powers).

The image is the set of monomials where x_i has non-negative exponent. Thus the cokernel is naturally identified with the free A-module generated by arbitrary products of reciprocals x_i^{-1},

$$\{\, x_0^{l_0} x_1^{l_1} \dots x_r^{l_r} \mid l_i < 0 \,\}.$$

The grading is then given by

$$l = \sum_{i=0}^{r} l_i.$$

In particular

$$H^r(X, \mathcal{O}_X(-r-1)),$$

is the free A-module with generator $x_0^{-1} x_1^{-1} \dots x_r^{-1}$. Hence (3).

To define a pairing, we declare

$$x_0^{l_0} x_1^{l_1} \dots x_r^{l_r},$$

to be the dual of

$$x_0^{m_0} x_1^{m_1} \dots x_r^{m_r} = x_0^{-1-l_0} x_1^{-1-l_1} \dots x_r^{-1-l_r}.$$

As $m_i \geq 0$ if and only if $l_i < 0$ it is straightforward to check that this gives a perfect pairing. Hence (4).

It remains to prove (2). If we localise the complex above with respect to x_r, we get a complex which computes $\mathcal{F}|_{U_r}$, which is zero in positive degree, as U_r is affine. Thus

$$H^i(X, \mathcal{F})_{x_r} = 0,$$

for $i > 0$ so that every element of $H^i(X, \mathcal{F})$ is annihilated by some power of x_r.

To finish the proof, we will show that multiplication by x_r induces an inclusion of cohomology. We proceed by induction on the dimension. Suppose that $r > 1$ and let $Y \simeq \mathbb{P}^{r-1}_A$ be the hyperplane $x_r = 0$. Then

$$\mathcal{I}_Y = \mathcal{O}_X(-Y) = \mathcal{O}_X(-1).$$

Thus there are short exact sequences

$$0 \longrightarrow \mathcal{O}_X(n-1) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Now $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < r-1$ and the natural restriction map

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective (every polynomial of degree n on Y is the restriction of a polynomial of degree n on X). Thus

$$H^i(X, \mathcal{O}_X(n-1)) \simeq H^i(X, \mathcal{O}_X(n)),$$

for $0 < i < r-1$, and even if $i = r-1$, then we get an injective map. But this map is the one induced by multiplication by x_r. $\square$

Theorem 9.6 (Serre vanishing)**.** *Let X be a projective variety over a Noetherian ring and let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf.*

(1) *$H^i(X, \mathcal{F})$ are finitely generated A-modules.*
(2) *There is an integer n_0 such that $H^i(X, \mathcal{F}(n)) = 0$ for all $n \geq n_0$ and $i > 0$.*

Proof. By assumption there is an immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. As X is projective, it is proper and so i is a closed immersion. If $\mathcal{G} = i_*\mathcal{F}$ then

$$H^i(\mathbb{P}^r_A, \mathcal{G}) \simeq H^i(X, \mathcal{F}).$$

Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$ we may assume that $X = \mathbb{P}^r_A$.

If $\mathcal{F} = \mathcal{O}_X(q)$ then the result is given by (9.5). Thus the result also holds is $\mathcal{F}$ is a direct sum of invertible sheaves. The general case proceeds by descending induction on i. Now

$$H^i(X, \mathcal{F}) = 0,$$

if $i > r$ (clear, if we use Čech cohomology). On the other hand, $\mathcal{F}$ is a quotient of a direct sum $\mathcal{E}$ of invertible sheaves. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{R}$ is coherent. Twisting by $\mathcal{O}_X(n)$ we get

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{F}(n) \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we get isomorphisms

$$H^i(X, \mathcal{F}(n)) \simeq H^{i+1}(X, \mathcal{R}(n)),$$

for n large enough, and we are done by descending induction on i. $\square$

Theorem 9.7. *Let A be a Noetherian ring and let X be a proper scheme over A. Let $\mathcal{L}$ be an invertible sheaf on X. TFAE*

(1) *$\mathcal{L}$ is ample.*
(2) *For every coherent sheaf $\mathcal{F}$ on X there is an integer n_0 such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for $n > n_0$.

Proof. (1) implies (2) is proved using the division algorithm, as in the proof of (7.7).

Now suppose that (2) holds. Let $\mathcal{F}$ be a coherent sheaf. Let $p \in X$ be a closed point. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \longrightarrow 0,$$

where $\mathcal{I}_p$ is the ideal sheaf of p. If we tensor this exact sequence with $\mathcal{L}^n$ we get an exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p \longrightarrow 0.$$

By hypotheses we can find n_0 such that

$$H^1(X, \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for all $n \geq n_0$. It follows that the natural map

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p),$$

is surjective, for all $n \geq n_0$. It follows by Nakayama's lemma applied to the local ring $\mathcal{O}_{X,p}$ that that the stalk of $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. As $\mathcal{F}$ is a coherent sheaf, for each integer $n \neq n_0$ there is an open subset U, depending on n, such that sections of $H^0(X, \mathcal{F} \otimes \mathcal{L})$ generate the sheaf at every point of U.

If we take $\mathcal{L} = \mathcal{O}_X$ it follows that there is an integer n_1 such that $\mathcal{L}^{n_1}$ is generated by global sections over an open neighbourhood V of p. For each $0 \leq r \leq n_1 - 1$ we may find U_r such that $\mathcal{F} \otimes \mathcal{L}^{n_0+r}$ is generated by global sections over U_r. Now let

$$U_p = V \cap U_0 \cap U_1 \cap \cdots \cap U_{n_1-1}.$$

Then

$$\mathcal{F} \otimes \mathcal{L}^n = (\mathcal{F} \otimes \mathcal{L}^{n_0+r}) \otimes (\mathcal{L}^{n_1})^m,$$

is generated by global sections over the whole of U_p for all $n \neq n_0$.

Now use compactness of X to conclude that we can cover X by finitely many U_p. $\square$

Theorem 9.8 (Serre duality)**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field. Then there is an invertible sheaf ω_X such that*

(1) $h^n(X, \omega_X) = 1$.
(2) *Given any other invertible sheaf $\mathcal{L}$ there is a perfect pairing*

$$H^i(X, \mathcal{L}) \times H^{n-i}(X, \omega_X \otimes \mathcal{L}^*) \longrightarrow H^n(X, \omega_X).$$

Example 9.9. *Let $X = \mathbb{P}^r_k$. Then $\omega_X = \mathcal{O}_X(-r-1)$ is a dualising sheaf.*

In fact, on any smooth projective variety, the dualising sheaf is constructed as the determinant of the cotangent bundle, which is a locally free sheaf. To construct the cotangent bundle, let $i\colon X \longrightarrow X \times X$ be the diagonal embedding. Let $\mathcal{I}$ be the ideal sheaf of the diagonal and let

$$\Omega^1_X = i^*\frac{\mathcal{I}}{\mathcal{I}^2}.$$

Ω^1_X is the dual of the tangent bundle. Ω^1_X is a locally free sheaf of rank n, known as the sheaf of Kähler differentials. The determinant sheaf is then the dualising sheaf,

$$\omega_X = \wedge^n \Omega^1_X.$$

This expresses a remarkable coincidence between the dualising sheaf, which is something defined in terms of sheaf cohomology and the determinant of the sheaf of Kähler differentials, which is something which comes from calculus on the variety.

Theorem 9.10. *Let $X = X(F)$ be a toric variety over $\mathbb{C}$ and let D be a T-Cartier divisor. Given $u \in M$ let*

$$Z(u) = \{\, v \in |F| \,|\, \langle u, v\rangle \geq \psi_D(v) \,\}.$$

Then

$$H^p(X, \mathcal{O}_X(D)) = \bigoplus_{u \in M} H^p(X, \mathcal{O}_X(D))_u \qquad \text{where} \qquad H^p(X, \mathcal{O}_X(D))_u = H^p_{Z(u)}(|F|).$$

Some explanation is in order. Note that the cohomology groups of X are naturally graded by M. (9.10) identifies the graded pieces.

$$H^p_{Z(u)}(|F|) = H^p(|F|, |F| - Z(u), \mathbb{C}).$$

denotes local cohomology. This comes with a long exact sequence for the pair. If X is an affine toric variety then both $|F|$ and $Z(u)$ are convex and the local cohomology vanishes. More generally, if D is ample, then then both $|F|$ and $Z(u)$ are convex and the local cohomology vanishes. This gives a slightly stronger result than Serre vanishing in the case of an arbitrary variety.

It is interesting to calculate the dualising sheaf in the case of a smooth toric variety. First of all note that the dualising sheaf is a line bundle, so that $\omega_X = \mathcal{O}_X(K_X)$, for some divisor K_X, which is called the **canonical divisor**. Note that the canonical divisor is only defined up to linear equivalence.

To calculate the canonical divisor, we need to write down a rational (or meromorphic in the case of $\mathbb{C}$) differential form. Note that if

$z_1, z_2, \dots, z_n$ are coordinates on the torus then

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

is invariant under the action of the torus, so that the associated divisor is supported on the invariant divisor. With a little bit of work one can show that this rational form has a simple pole along every invaraint divisor, that is

$$K_X + D \sim 0,$$

where D is a sum of the invariant divisors. For example,

$$-K_{\mathbb{P}^n} = H_0 + H_1 + \cdots + H_n \sim (n+1)H,$$

as expected.

Even if X is not smooth, it is possible to define the canonical divisor. Suppose that X is normal, so that the singular locus has codimension at least two. Let U be the smooth locus and let K_U be the canonical divisor of U. Let K_X be the divisor obtained by taking the closure of the components of K_U. Note that K_X is only defined as a Weil divisor in this case.

10. Singularities

It is the aim of this section to develop some of the theory and practice of the classification of singularities in algebraic geometry. The classification of singularities is clearly a local problem. Unfortunately the Zariski topology is very weak, and the property of being local in the Zariski topology does not satisfactorily capture the correct notion of classification. In general the correct approach is to work with the formal completion. Since this is somewhat technical, we work instead over $\mathbb{C}$, and we work locally analytically. This has the effect of replacing regular functions by analytic functions (convergent power series).

The most basic invariant of a singular point is the dimension of the Zariski tangent space.

Definition 10.1. *Let (X, p) be a germ of a singularity. The **embedding dimension** is the dimension of the Zariski tangent space of X at p.*

As the name might suggest, we have the following characterisation of the embedding dimension.

Lemma 10.2. *Let (X, p) be a germ of a singularity. The embedding dimension is equal to the smallest dimension of any smooth germ (M, q) such that X embeds in M.*

Proof. Let k be the embedding dimension of X, and suppose that $X \subset M$, where M is smooth. As $T_pX \subset T_pM$, and the dimension of M is equal to the dimension of T_pM, it is clear that the dimension of M is at least k.

Now consider embedding X into a smooth germ N and then projecting down to a smaller subspace M. Clearly we can always choose the projection to be an embedding of the Zariski tangent space to X at p, provided the dimension of M is at least k. Since the property that df is an isomorphism of Zariski tangent spaces is a local condition, it follows that possibly passing to a smaller open subset, we may assume that projection down to M induces an isomorphism of Zariski tangent spaces. But then the projection map is an isomorphism. $\square$

Definition 10.3. *We will say that X has a **hypersurface singularity** if the embedding dimension is one more than the dimension of X; we will say that a curve singularity is **planar** if it is a hypersurface singularity.*

Let $(X,p) \subset (M,p)$ be a hypersurface singularity. Pick coordinates $x_1, x_2, \dots, x_n$ on M and suppose that f defines X. Let m be the maximal ideal of M at p. The ***multiplicity of*** *X* ***at*** *p is equal to the largest integer μ such that $f \in m^\mu$.*

Given X, let Y be the singularity given by $x^2 + f$, where x is a new variable, so that Y is a hypersurface singularity of dimension one more than X. Any singularity obtained by successively replacing f by $x^2 + f$ will be called a ***suspension of*** *X.*

We say that X has type ***A_n-singularities*** *if X is defined by the suspension of x^{n+1}. We say that X has type* ***D_n-singularities***, *for $n \geq 4$, if X is defined by the suspension of $x^2y + y^{n-1}$. We say that X has a type* ***E_6-singularity***, *if X is defined by the suspension of x^3+y^4, a type* ***E_7-singularity***, *if X is defined by the suspension of $x^3 + xy^3$, and a type* ***E_8-singularity***, *if X is defined by the suspension of x^3+y^5.*

Note that the multiplicity of X is independent of the choice of coordinates and that a hypersurface is smooth iff the multiplicity is one. Note that the multiplicity is upper semi-continuous in families.

There are a couple of basic results about power series that we will use time and again. First some basic notation. We say that a monomial m appears in f and write $m \in f$ if the coefficient of m in f is non-zero.

Lemma 10.4. *Let $f \in \mathbb{C}\{x_1, x_2, \dots, x_n\}$ be the germ of an analytic function.*

(1) *If f has non-zero constant term then f is invertible and we may take nth roots.*
(2) *If we write $f = ax_n^k + \dots$, where dots indicate terms divisible by x_n^k of higher degree and $a \neq 0$, then we may change coordinates so that $f = x_n^k$.*
(3) *If $f = ux_n^k + \dots$, where $\dots$ indicate terms other than x_n^k and u is not in the maximal ideal, then we may change coordinates so that $f = x_n^k + f_{n-2}x_n^{k-2} + \dots + f_0$, where f_i are analytic functions in the first $n-1$ variables.*

Proof. (1) is well-known. Consider (2). By assumption we may write $f = ax_n^k + x_n^k g$, where g is an analytic function lying in the maximal ideal. In this case $f = x_n^k(a + g) = x_n^k u$, where by (1) u is a unit. In this case, also by (1), there is an analytic function v such that $v^k = u$. Replacing x_n by vx_n, f now has the correct form. This is (2).

Finally consider (3). Clearly we may expand f as

$$f = \sum_i f_i x_n^i,$$

where f_i are power series in the first $n-1$ variables. By assumption f_k is a unit. As before, we may then assume that $f_k = 1$. By (2) we may assume that $f_i = 0$ for $i > k$. Completing the nth power we may assume that $f_{k-1} = 0$. Now f has the required form. □

Definition-Lemma 10.5. *Let X be a hypersurface singularity of multiplicity μ. Then we may choose coordinates $x_1, x_2, \dots, x_n$ such that X is given by*

$$x_n^\mu + f_{\mu-2}x_n^{\mu-2} + \cdots + f_0,$$

*where f_i are analytic functions of the first $n-1$ variables. Any such polynomial is called a **Weierstrass polynomial**.*

Proof. By assumption f_μ is non-zero. Possibly changing coordinates, we may assume that $x_n^\mu \in f$. The result is now an easy consequence of (10.4). □

Lemma 10.6. *A planar curve singularity has multiplicity two iff it is of type A_n.*

Proof. After putting f into Weierstrass form, the result becomes easy. □

It is interesting to see what happens for small values of n. If $n = 1$, so that $f = y^2+x^2$, then we have a **node**. This corresponds to two smooth curves with distinct tangent directions. If $n = 2$, then $f = y^2+x^3$, then we have a **cusp**. In the case $n = 3$, we have $y^2 + x^4$, this represents two smooth curves which are tangent. We call this a **tacnode**. The case $n = 4$ is called a **ramphoid cusp**, $n = 5$ an **oscnode**, and $n = 6$ a **hyper-ramphoid cusp** and so on.

Definition 10.7. *Let C be a planar singularity of order μ. We say that C is **ordinary** if when we write $f = f_\mu + \dots$, where dots indicate higher order terms, then f_μ factors into μ distinct linear factors.*

It is not hard to show that that if C is ordinary, we may always choose coordinates so that $f = f_\mu$.

Definition 10.8. *Let X be a singular variety, a subset of $\mathbb{A}^n$. The **tangent cone** of X at a point p is the intersection of the strict transform of X with the exceptional divisor.*

If X is a hypersurface singularity, then the tangent cone is given by $f_\mu = 0$ a subset of $\mathbb{P}^{n-1} \simeq E$.

Example 10.9. *Consider ordinary planar curve singularities of multiplicity four. Then each linear factor defines an element of $\mathbb{P}^1$. But four unordered points in $\mathbb{P}^1$ have moduli (the j-invariant). Thus there is a* *one dimensional family of non-isomorphic planar curve singularities of multiplicity four. Indeed, since one can always choose the first three points to be* 0, 1 *and* ∞, *we can write*

$$f = xy(x-y)(x-\lambda y).$$

Definition 10.10. *Let* (X, p) *be the germ of a singularity. A* ***deformation*** *of* X *is a triple* (π, σ, i), *where* $\pi \colon \mathcal{X} \longrightarrow B$ *is a flat morphism,* σ *is a* ***section*** *of* π *(that is,* $\pi \circ \sigma$ *is the identity) such that for every* $t \in B$ *the pair* $(X_t, \sigma(t))$ *is a germ of a singularity and* i *is an* ***isomorphism*** *of the pair* (X, p) *and* $(X_0, \sigma(0))$, *the central fibre of* π.

In practice, it is customary to drop σ and i and refer to a deformation using only π. Note that since the multiplicity is upper semi-continuous in families, it follows that the multiplicity can only go down under deformation.

In other words, a deformation is to the germ of a singularity, as a family is to a variety. As such one might hope that there exists universal deformations, as there exists universal families. Equivalently, one might hope to write down the obvious functor and hope that there is a space which represents this functor. Unfortunately this is not so; the problem is that the central fibre might have more automorphisms, than the typical fibre (and this why we are careful to specify the isomorphism of the central fibre with the space to be deformed). Instead, the best we can hope for is:

Definition 10.11. *Let* (X, p) *be a germ of a singularity. We say that a deformation* π *of* X *is* ***versal*** *if for every other deformation* ψ *there is a morphism* $B' \longrightarrow B$ *such that* ψ *is pulled back from* π *in the obvious way.*

Note that we do not require uniqueness of the versal family, and in fact we cannot, since if there is an automorphism of the central fibre that does not lift to the whole deformation space, for example if it does not lift to every fibre, then we get a different deformation, simply by composing with this automorphism (that is, we change the isomorphism i).

Fortunately, versal deformation spaces are easy to write down.

Definition 10.12. *Let* X *be a hypersurface singularity, defined by the equation* $f = 0$. *Let*

$$T^1_f = \frac{\mathbb{C}[x_1, x_2, \dots, x_n]}{\langle f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \frac{\partial f}{\partial x_n} \rangle}.$$

Theorem 10.13. *Let X be an isolated hypersurface singularity. Pick holomorphic functions $g_1, g_2, \dots, g_k$ such that their images in T^1_f are a basis of T^1_f. Then the deformation given by*

$$f_t = f + \sum_i t_i g_i,$$

where t_i are coordinates on the germ $(\mathbb{C}^k, 0)$ is a versal deformation.

Another way to state (10.13), is that T^1_f is the Zariski tangent space to the versal deformation space given above. We will also need the following basic fact.

Lemma 10.14. *Let $\pi\colon \mathcal{X} \longrightarrow B$ be a versal deformation space, and let B' be a general subvariety of B. Then the restriction of π to B' defines a versal deformation space of the general point of B'.*

It is interesting to see what happens in a series of examples. Suppose we start with planar singularities. The simplest is an A_1-singularity. In this case

$$f = x^2 + y^2 \qquad \text{so that} \qquad \frac{\partial f}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2y.$$

Thus

$$T^1_f = \frac{\mathbb{C}[x, y]}{\langle x, y\rangle}$$

and for g_1 we take 1. Thus the versal deformation space of a node is given as

$$f = x^2 + y^2 + t.$$

In other words, the only thing that we can do with a node is smooth it. Now consider what happens in the case of an A_n-singularity. In this case the derivatives are $(n+1)x^n$ and $2y$, so that we make take $g_i = x^i$, $i = 0 \dots n-1$. Thus the versal deformation space has dimension n, and it is given by

$$y^2 + x^{n+1} + t_0 + t_1 x + t_2 x^2 + \dots + t_{n-1}x^{n-1}.$$

For example consider the case of a cusp. In this case the versal deformation space has dimension two, and it is given by

$$y^2 + x^3 + ax + b,$$

where a and b are coordinates on the base. The point is that now we have two completely different one dimensional families. Either we can smooth the cusp, or we can partially smooth it to a node. In fact the locus of nodes forms a curve in the base.

Note that $y^2 = x^3 + ax + b$ is singular iff the polynomial $x^3 + ax + b$ has a double root. But then the singular locus is given by the discriminant, that is $4b^3 + 27a^2$, so that this locus is not smooth.

Similarly, it is not hard to see that the locus of A_n-singularities contains loci corresponding to the A_k-singularities, for $k \leq n$. In fact this locus will have codimension $n - k$.

In fact the converse is true, that is, one can only deform an A_n-singularity to an A_k singularity, for $k \leq n$. Compare this with the case of an ordinary four-fold point. Suppose that we start with $x^4 + y^4$. Then the derivatives are $4x^3$ and $4y^3$ and so we can take for g_i, 1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2. In this case, there is a one dimensional locus corresponding to all other (nearby) ordinary four-fold singularities, given by $x^4 + y^4 + tx^2y^2$. In other words, there are infinitely many non-isomorphic germs in the versal deformation space.

It is interesting to look at some of these ideas from the point of view of blowing up and resolution of singularities.

Definition 10.15. *Let X be a variety and let $D = \sum D_i$ be a divisor, the sum of distinct prime divisors. We say that the pair (X, D) has normal crossings if X is smooth and locally about every point, the pair (X, D) is equivalent to $\mathbb{C}^n$ union some of the coordinate hyperplanes. We say that (X, D) has* ***global normal crossings*** *if in addition every component of D is smooth.*

A ***resolution of singularities*** *for X is a birational map $\pi\colon Y \longrightarrow X$ with the following properties:*

(1) *π is an isomorphism over the smooth locus of X,*
(2) *Y is smooth, and*
(3) *the exceptional locus is a divisor with global normal crossings.*

Example 10.16. *Let C be the nodal plane cubic $V(Y^2 - X^2 - X^3) \subset \mathbb{P}^2$. Then $(\mathbb{P}^2, C)$ has normal crossings but not global normal crossings, since C is not smooth.*

It follows that the identity map is not a resolution of singularities.

To date there is only one known way to resolve singularities (at this level of generality) and that is to embed X into a smooth variety M and then carefully choose an appropriate sequence of blow ups, at each stage blowing up M and replacing X by its strict transform. In this case we want the exceptional locus of $\psi\colon N \longrightarrow M$ to intersect the strict transform Y of X as transversally as possible. For example if X has a hypersurface singularity, then we want $Y + E$, where E is the exceptional locus, to have normal crossings.

We have already seen some examples of this. Perhaps the easiest example is the case of a nodal curve. In this case C sits inside a smooth surface M, and we simply blow up the singular point of C. At this point C is smooth and meets the exceptional locus smoothly in two points, so that the pair $C + E$ does have normal crossings.

Now suppose that we take a curve with a cusp. Pick local coordinates so that we have $y^2 + x^3$. Blowing up once, we have already seen that C becomes smooth. However C is tangent to the exceptional locus. If we blow up, then the strict transform of C intersects the points where the two exceptional divisors intersect. Thus it is necessary to blow up once more to achieve normal crossings.

It is interesting to see what happens for an ordinary singularity. In this case we have seen that we may choose coordinates so that f is homogeneous. Thus f factors into μ distinct linear factors. Now each of those factors corresponds to a point of the exceptional locus and in fact when we blow up then C is smooth and meets the exceptional divisor at μ points. At this point $C + E$ has normal crossings.

For example, if the multiplicity is four, then C meets E in four points, and we get our j-invariant directly.

Theorem 10.17. *Let C be a planar curve singularity.*

Then the versal deformation space of C contains only finitely many isomorphism types iff C is one of A_n, D_n, E_6, E_7 or E_8.

Proof. Let f be a defining equation for C. Let us show that if there are only finitely many isomorphism types in the versal deformation space, then C must be one of the ADE-singularities. Suppose that the multiplicity of f is at least four. Then we may deform f to an ordinary multiplicity four singularity. But then there are infinitely many non-isomorphic singularities in the versal deformation space. On the other hand, if the multiplicity is two, then by (10.6) C must have type A_n.

Thus we may suppose that f has multiplicity three. Consider f_3. This factors into three linear factors. There are three cases; the three factors are distinct; there are two distinct factors; there is one.

Suppose that there are at least two distinct factors. Then there is a factor which occurs only once. We may assume that this factor is y. Since the multiplicity is three, in fact y must divide f, so that we may write

$$f = h \cdot y,$$

where h only depends on x and y. Now h has multiplicity two and its rank two part is not divisible by y. It follows that there is a change of variable, so that $h = x^2 + y^n$, where $n \geq 2$. So f has the form $(ax + by)(x^2 + y^n)$, where $b \neq 0$. With a little bit of work, we can put this in the form $f = x^2y + y^{n+1}$ and we have a singularity of type D (more precisely, a D_{n+3}-singularity).

This final case is when $f_3 = y^3$, so that $f = y_3 + g$ and the multiplicity of g is at least four. Putting f into Weierstrass form, once again, we may assume that $f = y^3 + yg + h$, where g and h only involve x. Thus f can be put in the form $y^3 + ayx^k + x^l + \dots$, where the dots indicate higher powers of x, a is either zero or one and $k < l$. If $l = 4$, it follows that $k = 3$ so that completing the square we may assume that $a = 0$. In this case, it is not hard to show that we can choose coordinates so that $f = y^3 + x^4$ and we have an E_6-singularity. If $x^3y \in f$ and $l > 4$ then with some manipulation we can put f into the form $y^3 + x^3y$, so that we have an E_7-singularity; similarly if $y^5 \in f$ but we have no lower terms, then we have an E_8-singularity. Otherwise we may assume that $l > 5$ and that $k > 3$. In this case, we may as well assume that $f = y^3 + \lambda yx^4 = y(y^2 + \lambda x^4)$, which represents three smooth curves which are tangent. Suppose we blow up once; we get four curves passing through one point, the strict transform of the three tangent curves and the exceptional divisor. If we blow up the point, we get four curves intersecting the new exceptional divisor and the four points of intersection with the new exceptional gives one dimension of moduli (the j-invariant, which varies as we vary λ).

Now let us consider the converse problem, to show that there are only finitely many isomorphism types in the versal deformation space. Clearly it suffices to prove that we can only deform an ADE-singularity, to an ADE-singularity. This is clear for A_n-singularities, since under deformation the multiplicity can only go down.

Now consider the case of a D_n-singularity. We only need to consider deformations that preserve the multiplicity. In this case, the deformations of f_3 can only increase the number of distinct linear factors, and we cannot lose a term of the form y^k. Thus the deformation of an D_n-singularity is either a D_k-singularity, for some $k \leq n$ or an A_n-singularity.

Finally consider the three exceptional cases. Suppose we start with x^3+y^4. Then the only possible deformation which fixes the multiplicity, deforms to a singularity of type D_n, $n \leq 5$. Now suppose we start with $y^3 + x^3y$. Again we can only pick up a term of the form x^4 or increase the number of linear factors. Similarly for an E_8-singularity. $\square$

Here is a way to restate (10.17):

Proposition 10.18. *The ADE-singularities are the only singularities, which have multiplicity two and three, and such that after blowing up, the multiplicity of the total transform has multiplicity two or three.*

Proof. It is not hard to check that this is all we have used in the proof of (10.17) to characterise ADE-singularities. ∎

The are four other obvious ways of creating examples of singularities other than writing down equations. The first is simply to take the cone over a closed subset of $\mathbb{P}^n$. Note that the cone is a degenerate example of the join of two varieties, where one of the two varieties to be joined is a point. Note also that if I is the ideal of $X \subset \mathbb{P}^n$, then I is also the ideal of the cone Y over X, where Y is the closure of the inverse image of X inside K^{n+1}. In particular, the classification of singularities is at least as hard as the classification of varieties. On the other hand, note that the resolution problem for such singularities is in fact easy. If X is smooth, then simply blowing up the vertex, we get a birational map $\pi\colon W \longrightarrow Y$, whose exceptional locus E is a copy of X, where W is smooth. In fact W is a $\mathbb{P}^1$-bundle over X, and E is simply a section of this bundle.

The next method to construct singularities is to start with a configuration of divisors and contract them. Unfortunately it is quite hard to characterise which configurations are contractible; even in the simplest cases it involves quite a bit of work to prove that certain configurations are contractible.

The third method is to take a quotient:

Definition 10.19. *Let G be an algebraic group acting on a variety X. We say that Y is a **categorical quotient of** X **by** G if there is a morphism $\pi\colon X \longrightarrow Y$ such that $\pi(g\cdot x) = \pi(x)$ for every $g \in G$, which is universal amongst all such morphisms:*

If $\phi\colon X \longrightarrow Z$ is a morphism such that $\psi(g \cdot x) = \psi(x)$ then there is a unique morphism $\psi\colon Y \longrightarrow Z$ which makes the diagram commute,

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ Y \arrow[r, "\psi"] & Z. \end{tikzcd}$$

It is common to denote the categorical quotient by X/G (if it exists at all). Fortunately there is one quite general existence theorem:

Theorem 10.20. *Let $X = \operatorname{Spec} A$ be an affine variety and let G be a finite group acting on X.*

Then the categorical quotient is the affine variety $Y = \operatorname{Spec} A^G$.

Proof. The key fact is that the ring of invariants

$$A^G = \{\, a \in A \mid g \cdot a = a \,\},$$

is a finitely generated k-algebra. ∎

Note that $Y = X/G$ will in general be a singular variety. It is however $\mathbb{Q}$-factorial, that is, every Weil divisor is $\mathbb{Q}$-Cartier, that is, given by any Weil divisor D, some multiple is Cartier (indeed, $r = |G|$ will do).

The final method is to use toric geometry. We start with the canonical example.

Let σ be the cone spanned by e_2 and $2e_1 - e_2$. The dual cone σ is spanned by f_1 and $f_1 + 2f_2$. Generators for the monoid are f_1, $f_1 + f_2$ and $f_1 + 2f_2$, so that

$$X = U_\sigma = \operatorname{Spec} \mathbb{C}[x, xy, xy^2] = \operatorname{Spec} \mathbb{C}[u, v, w]/\langle v^2 = uw\rangle.$$

Thus we have an A_1-singularity.

It is interesting to see how to resolve this singularity. Suppose we insert the vector e_1; this corresponds to a blow up with exceptional divisor isomorphic to $\mathbb{P}^1$. We get two cones σ_1 and σ_2, one spanned by e_1 and e_2 and the other spanned by e_1 and $2e_1 - e_2$. It follows that the blow up is smooth. Note that X is the cone over a conic; it follows once again that X can be resolved in one step.

Let's make this example a little more complicated. Let's start with the cone spanned by e_2, $re_1 - e_2$. The dual cone is the cone spanned by f_1 and $f_1 + rf_2$. Generators for the monoid are f_1, $f_1 + f_2$, $\dots$, $f_1 + rf_2$. We get

$$U_\sigma = \operatorname{Spec} \mathbb{C}[x, xy, \dots, xy^r] = \operatorname{Spec} \mathbb{C}[u^r, u^{r-1}v, \dots, v^r],$$

where $u^r = x$ and $v = y/x$, which is the cone over a rational normal curve of degree r. Note that the embedding dimension is $r + 1$. Note that this is again resolved in one step by inserting the vector e_1.

At the other extreme, consider the cone spanned by e_2 and $re_1 - (r - 1)e_2$. The dual cone is spanned by f_1 and $(r-1)f_1 + rf_2$. Generators for the monoid are f_1, $(r-1)f_1 + rf_2$ and $f_1 + f_2$. We get

$$U_\sigma = \operatorname{Spec} \mathbb{C}[x, xy, x^{r-1}y^r] = \operatorname{Spec} \mathbb{C}[u, v, w]/\langle v^r = uw\rangle,$$

which is an A_{r-1}-singularity. If we insert the vector e_1 then we the resulting blow up has two affine pieces. One is smooth, corresponding to the cone spanned by e_1 and e_2 and the other is the cone given by e_1 and $re_1 - (r-1)e_2$. Switching the sign of e_2 we get e_1, $re_1 + (r-1)e_2$. Switching e_1 and e_2 we get e_2 and $(r-1)e_1 - re_2$. Replacing e_1 by $e_1 - 2e_2$ we get e_2 and $(r-1)e_1 - (r-2)e_2$ which as we have already seen is an A_{r-2}-singularity. Thus an A_r-singularity takes r-steps to resolve. On the resolution we get a chain of r-copies of $\mathbb{P}^1$.

More generally, we could consider the cone spanned by e_2 and $re_1 - ae_2$, where $0 < a < r$, is coprime to r. However the best way to proceed, is to look at all of this a different way.

We start with an example. The cyclic group $G = \mathbb{Z}_r$ acts on $\mathbb{C}^2$ via

$$(u, v) \longrightarrow (\omega u, \omega v),$$

where ω is a primitive rth root of unity. In this case the ring of invariants is precisely

$$\mathbb{C}[u, v]^G = \mathbb{C}[u^r, u^{r-1}v, \dots, v^r].$$

To see this using the toric structure, let $N' \subset N$ be the sublattice spanned by $e'_2 = e_2$ and $e'_1 = re_1 - e_2$. Then the cone σ' spanned by the same vectors e_2 and $re_1 - e_2$ now corresponds to a smooth toric variety. The dual lattice M' is an overlattice of M.

Thinking this way, we should make a basis for N' the standard vectors e'_1 and e'_2. The overlattice N is spanned by N' and the vector e_1 in the old coordinates. As

$$e_1 = 1/r(re_1 - e_2) + 1/re_2,$$

in the old coordinates, in the new coordinates we have that N' is spanned by e'_1, e'_2 and $1/r(e'_1 + e'_2)$. If we insert this vector, we get a basis for the lattice. If the dual lattice M' is the overlattice spanned by f'_1 and f'_2 then M is the sublattice spanned by all $af'_1 + bf'_2$ such that $a + b$ is divisible by r.

In the other example, where we started with e_2 and $re_1 - (r-1)e_2$, then

$$e_1 = 1/r(re_1 - (r-1)e_2) + (r-1)/re_2.$$

So N is the lattice spanned by e_1, e_2 and $1/re_1 + (r-1)/re_2$. This suggests we should look at the action

$$(x, y) \longrightarrow (\omega x, \omega^{r-1}y) = (\omega x, \omega^{-1}y),$$

Indeed, the ring of invariants is $u = x^r$, $w = y^r$ and $v = xy$ and $v^r = uw$, as expected.

More generally still, for the action

$$(x, y) \longrightarrow (\omega x, \omega^a y),$$

we should look at the lattice N spaned by the standard lattice and the vector $1/r(1, a)$. Inserting this vector, gives two cones, one spanned by e_1, $1/r(1, a)$ and the other spanned by $1/r(1, a)$ and e_2. The second one is smooth. For the first, let us make the two vectors $1/r(1, a)$ and e_1 the standard generators for the lattice. As

$$(0, 1) = r/a(1/r, a/r) - 1/a(0, 1),$$

we then have the overlattice generated by $(-1/a, r/a)$. Now

$$r/a = k - b/a,$$

for some unique $0 \leq b < a$. So we get a singularity of type $1/a(1, b)$. Note that resolving the singularity corresponds to computing a continued fraction. The significance of k is the self-intersection of exceptional divisor (on the minimal resolution).

So the resolution graph of any cyclic surface singularity is a chain of $\mathbb{P}^1$'s. Singularities of type A_r correspond to a chain of r such curves, where each curve has self-intersection -2. In fact it is not hard to prove:

Theorem 10.21. *Let $S = \mathbb{C}^2/G$ be a two dimensional quotient singularity. Then $G \subset \mathrm{GL}(2, \mathbb{C})$ and there are three possibilities:*

(1) *G is cyclic and the dual graph of the (minimal) resolution corresponds to the Dynkin diagram A_n. The action is $(x, y) \longrightarrow (\omega x, \omega^a y)$, where ω and ω^a is both primtive roots of unity. S is isomorphic to a toric surface.*
(2) *G is a dihedral group and the dual graph corresponds to the Dynkin diagram for D_n, $n \geq 4$.*
(3) *G is one of three exceptional groups and the dual graph is the Dynkin diagram for E_6, E_7 or E_8.*

If in addition $G \subset \mathrm{SL}(2, \mathbb{C})$ then S has an ADE-singularity and the self-intersections of the exceptional curves are all -2.

More generally suppose that $\sigma \subset N \simeq \mathbb{Z}^n$ is a simplicial cone. As before let $N' \subset N$ be the sublattice spanned by the primitive generators of σ. Let $M \subset M'$ be the corresponding overlattice. Then there is a natural pairing

$$N/N' \times M'/M \longrightarrow \mathbb{Q}/\mathbb{Z}$$

This makes M the invariant sublattice of M', under the action of the finite abelian group $G = N/N'$ and under this action it is not hard to see that

$$A_\sigma = (A_{\sigma'})^G.$$

Note that G is a product of at most $n - 1$ cyclic factors.

11. Resolution of singularities I

We start to consider the problem of resolution of singularities. At it most basic we are given a finitely generated field extension K/k and we would like to find a smooth projective variety X over k with function field K.

Before we get into a proof of resolution of singularities via smooth blow ups, we first describe some other ways to resolve singularities. Even though these methods don't always work, they introduce ideas and techniques which are of considerable independent interest.

Definition 11.1. *Let X be an integral scheme. We say that X is **normal** if all of the local rings $\mathcal{O}_{X,p}$ are integrally closed.*

*The **normalisation of** X is a morphism $Y \longrightarrow X$ from a normal scheme, which is universal amongst all such morphisms. If $Z \longrightarrow X$ is a morphism from a normal scheme Z, then there is a unique morphism $Z \longrightarrow Y$ which make the diagram commute:*

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X. \end{tikzcd}$$

One can always construct the normalisation of a scheme as follows. By the universal property, it suffices to construct the normalisation locally. If $X = \operatorname{Spec} A$, then $Y = \operatorname{Spec} B$, where B is the integral closure of A inside the field of fractions. Note that if X is quasi-projective variety then the normalisation $Y \longrightarrow X$ is a finite and birational morphism.

Definition 11.2. *Let X be a scheme. We say that X satisfies **condition** S_2 if every regular function defined on an open subset U whose complement has codimension at least two, extends to the whole of X.*

Lemma 11.3 (Serre's criterion)**.** *Let X be an integral scheme.*

Then X is normal if and only if it is regular in codimension one (condition R_1) and satisfies condition S_2.

Note that this gives a simple method to resolve singularities of curves. If C is a curve, the normalisation $C' \longrightarrow C$ is smooth in codimension one, which is to say that C' is smooth.

Note that lots of surface singularities are normal. For example, every hypersurface singularity is S_2, so that a hypersurface singularity is normal if and only if it is smooth in codimension one. Similarly, every quotient singularity is normal.

Before we pass on to other methods, it is interesting to write down some example of varieties which are R_1 but not normal, that is, which are not S_2.

Example 11.4. *Let S be the union of two smooth surfaces S_1 and S_2 joined at a single point p. For example, two general planes in $\mathbb{A}^4$ which both contain the same point p. Let $U = S - \{p\}$. Then U is the disjoint union of $U_1 = S_1 - \{p\}$ and $S_2 - \{p\}$, so U is smooth and the codimension of the complement is two. Let $f : U \longrightarrow k$ be the function which takes the value 1 on U_1 and the value 0 on U_2. Then f is regular, but it does not even extend to a continuous function, let alone a regular function, on S.*

Let C be a projection of a rational normal quartic down to $\mathbb{P}^3$, for example the image of

$$[S : T] \longrightarrow [S^4 : S^3T : ST^3 : T^4] = [A : B : C : D].$$

Let S be the cone over C. Then S is regular in codimension one, but it is not S_2. Indeed, $A/B = C/D$ is a regular function whose only pole is along $B = 0$ and $D = 0$, that is, only at $(0, 0, 0, 0)$ of S.

Beyond the dimension of the Zariski tangent space, perhaps the most basic invariant of any singular point is:

Definition 11.5. *Let $X \subset M$ be a subvariety of a smooth variety. The **multiplicity of** X **at** $p \in M$ is the largest μ such that $\mathcal{I}_p \subset \mathfrak{m}^\mu$ where $\mathfrak{m}$ is the maximal ideal of M at p in $\mathcal{O}_{M,p}$ and $\mathcal{I}$ is the ideal sheaf of X in M.*

Note that this generalises the multiplicity of a hypersurface singularity. The multiplicity has two basic properties. X is smooth at p if and only if the multiplicity is one and the multiplicity is upper semi-continuous in families.

We next describe the method of Albanese. Start with $X \subset \mathbb{P}^n$. Now re-embed X by the very ample line bundle $\mathcal{O}_X(m)$, where m is very large, so that $X = X_0 \subset \mathbb{P}^r$, where r is large. Pick a point $p = p_0 \in X_0$, where the multiplicity is largest, to get $X_1 \subset \mathbb{P}^{r-1}$. Now pick a point $p_1 \in X_1$ of largest multiplicity and project down to get $X_2 \subset \mathbb{P}^{r-2}$. Continuing in this way, always projecting from a point of maximal mulitplicity, we construct $X_i \subset \mathbb{P}^{r-i}$.

Theorem 11.6. *If*

$$\deg X_0 < (n! + 1)(r + 1 - n),$$

then the Albanese algorithm stops with a variety X_k and a generically finite map $f_k \colon X_0 \dashrightarrow X_k$, such that either

(1) $\deg f_k \operatorname{mult}_p(X_k) \leq n!$, *or*
(2) X_k *is a cone and* $\deg f_k \leq n!$.

Corollary 11.7. *Assume that every variety of dimension at most $n-1$ is birational to a smooth projective variety.*

Then every projective variety is birational to a projective variety with singularities of multiplicity at most $n!$.

Note that this resolves singularities for curves, since $1! = 1$ and a point of multiplicity one is a smooth point of X. Even for surfaces we get down to points of multiplicity two, which are not so bad. Starting with threefolds, the situation is not nearly so rosy, especially when one realises that if f is a hypersurface singularity of arbitrary multiplicity, then the suspension of f, $x^2 + f$, is a hypersurface singularity of multiplicity two. It is pretty clear that resolving $x^2 + f$ entails resolving f.

Unfortunately it seems impossible to improve the bound given in (11.6).

We will need:

Theorem 11.8. *Let $X \subset \mathbb{P}^r$ be an irreducible projective variety of degree d and dimension n.*

If X is not contained in a hyperplane, then

$$d \geq r + 1 - n.$$

Proof of (11.6). By induction on k. Suppose that

$$\deg f_k \cdot \operatorname{mult}_p(X_k) \leq (n! + 1)(r - k + 1 - n).$$

Suppose that p is a point of maximal multiplicity μ. If X_k is a cone with vertex p, then there is nothing to prove. Otherwise let X_{k+1} be the closure of the image of p under projection, and let $\pi \colon X_k \dashrightarrow X_{k+1}$ be the resulting rational map. As X_k is not a cone over p, π is generically finite. We have

$$\deg \pi \cdot d_{k+1} = d_k - \mu.$$

If

$$\deg f_k \cdot \mu > n!,$$

then

$$\begin{aligned} \deg f_{k+1} \cdot d_{k+1} &= \deg f_k \deg \pi \cdot d_{k+1} \\ &= \deg f_k \cdot d_k - \deg f_k \mu \\ &\leq \deg f_k \cdot d_k - (n! + 1) \\ &\leq (n! + 1)(r - k + 1 - n) - (n! + 1) \\ &\leq (n! + 1)(r - (k + 1) + 1 - n). \end{aligned}$$

It follows that eventually either X_k becomes a cone or we get

$$\deg f_k \cdot \operatorname{mult}_p X_k \leq n!.$$

As $X_k \subset \mathbb{P}^{r-k}$ is not contained in a hyperplane, we have

$$d_k \geq (r-k+1-n).$$

It follows that if X_k is a cone, then

$$\deg f_k \leq n!.$$

□

Notice how truly bizarre this argument is; presumably projecting from a point will introduce all sorts of bad singularities (corresponding to secant lines and so on), but just by projecting from the point of maximal multiplicity works.

Example 11.9. *Let*

$$m_1 \leq m_2 \leq \cdots \leq m_r,$$

be a sequence of positive integers. Let C be the image of

$$t \longrightarrow (t^{m_1}, t^{m_2}, t^{m_3}, \dots, t^{m_r}),$$

inside $\mathbb{A}^r$. If we project from $(1, 0, 0, \dots, 0)$, then we get the image of

$$t \longrightarrow (t^{m_2-m_1}, t_2^{m_3-m_1}, t_3^{m_4-m_1}, \dots, t_r^{m_r-m_1}),$$

inside $\mathbb{A}^{r-1}$. It is intuitively clear that the projection of C is less singular than C, but it is hard to say exactly why; for example the multiplicity might go up.

Let us turn to the proof of (11.7). We will need:

Theorem 11.10 (Asymptotic Riemann-Roch)**.** *Let X be a normal projective variety and let $\mathcal{O}_X(1)$ be a very ample line bundle. Suppose that $X \subset \mathbb{P}^n$ has degree d.*

Then

$$h^0(X, \mathcal{O}_X(m)) = \frac{dm^n}{n!} + \dots,$$

is a polynomial of degree n, for m large enough, with the given leading term.

Proof. Let Y be a general hyperplane section. Then Y is a normal projective variety of degree d; indeed, Y is certainly regular in codimension one and one can check that Y is S_2. The trick is to compute $\chi(X, \mathcal{O}_X(m))$ by looking at the exact sequence

$$0 \longrightarrow \mathcal{O}_X(m-1) \longrightarrow \mathcal{O}_X(m) \longrightarrow \mathcal{O}_Y(m) \longrightarrow 0.$$

The Euler characteristic is additive so that

$$\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m-1)) = \chi(Y, \mathcal{O}_Y(m)).$$

By an easy induction, it follows that $\chi(X, \mathcal{O}_X(m))$ is a polynomial of degree n, with the given leading term. Now apply Serre vanishing. $\square$

Definition 11.11. *Let X be a quasi-projective variety and let K be the function field of X. Let L/K be a finite field extension.*

The ***normalisation of*** X ***in*** L *is a finite morphism $Y \longrightarrow X$, where Y is a normal quasi-projective variety and the function field of Y is L.*

One can consruct Y in much the same way that one constructs the normalisation. It suffices to construct Y locally, in which case we may assume that $X = \operatorname{Spec} A$ is affine. In this case one simply takes $Y = \operatorname{Spec} B$, where B is the integral closure of A inside L.

Lemma 11.12. *Let $\pi \colon Y \longrightarrow X$ be a finite morphism.*

If $\pi(q) = p$, then

$$\operatorname{mult}_q Y = \deg \pi \cdot \operatorname{mult}_p X.$$

Proof of (11.7). By (11.10) we may pick m sufficiently large such that if

$$\deg X_0 \subset \mathbb{P}^r$$

is the embedding given $\mathcal{O}_X(m)$, then

$$d_0 \leq (n! + 1)(r + 1 - n).$$

By (11.6) we may find a generically finite morphism $f \colon X \dashrightarrow W$ such that either

$$\deg f \operatorname{mult}_w W \leq n!,$$

or W is a cone and

$$\deg f \leq n!.$$

If W is a cone, then W is birational to a product $\mathbb{P}^1 \times W'$. By our induction hypothesis, W' is birational to a smooth projective variety W''. Then W is birational to $W'' \times \mathbb{P}^1$. Replacing W by $W'' \times \mathbb{P}^1$, we may assume that W is smooth.

Let $\pi \colon Y \longrightarrow W$ be the normalisation of W in the field $L = K(X)/K(W)$. Then Y is birational to X and $\deg f = \deg \pi$. By (11.12),

$$\operatorname{mult}_y Y \leq n!.$$

$\square$

Another intriguing method was proposed by Nash:

Definition 11.13. *Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension n. The* ***Gauss map*** *is the rational map*

$$X \dashrightarrow \mathbb{G}(n, N) \qquad \text{given by} \qquad x \longrightarrow T_x X,$$

which sends a point to its (projective) tangent space.

The ***Nash blow up*** *is given by taking the graph of this rational map.*

Conjecture 11.14. *We can always resolve any variety by successively taking the Nash blow up and normalising.*

Despite the very appealing nature of this conjecture (consider for example the case of curves, when we don't even need to normalise) we only know (11.14) in very special cases. The one very nice feature of the Nash blow up is that it does not involve any choices. Unfortunately it is known that one needs to normalise, and this messes up any sort of induction.

If X is a toric variety there is a pretty simple method to resolve singularities. First subdivide the cone until X is simplicial. It is not too hard to argue that one can resolve any simplicial toric variety (one only needs to keep track of a simple invariant). One subtle point is to make sure that as one improves one cone, then another cone does not become worse.

There is a pretty straightforward way to resolve the singularities of a quasi-projective surface S. It does not hurt to assume that S is projective. Replacing S by its normalisation, we may assume that S is normal. First embed S into $\mathbb{P}^n$. By repeatedly projecting from a point, we may express S as a large degree cover of $\mathbb{P}^2$, $\pi\colon S \longrightarrow \mathbb{P}^2$. Let $B \subset \mathbb{P}^2$ be the branch locus of π, the locus where two or more points come together.

Take an embedded resolution of $(\mathbb{P}^2, B)$. This is a birational morphism $f\colon N \longrightarrow \mathbb{P}^2$ such that the total transform C of B is a divisor with global normal crossings. Let T be the normalisation of the fibre product $N \underset{\mathbb{P}^2}{\times} S$. We have a commutative diagram,

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "\psi"'] & S \arrow[d, "\pi"] \\ N \arrow[r, "f"] & \mathbb{P}^2. \end{tikzcd}$$

Now $\psi\colon T \longrightarrow N$ only ramifies over C, which is a divisor with normal crossings. Consider the field extension $M = \mathbb{C}(T)/\mathbb{C}(N) = K$. This is not necessarily Galois; let L/M be the Galois closure, so that L/M and L/K are Galois extensions.

Let R be the normalisation of T inside L. If G is the Galois group of L/K, then G acts on R and $N = R/G$. Similarly if $H \subset G$ is the Galois group of L/M, then $T = R/H$. As a warm up, suppose first that R is smooth. Then T has quotient singularities, and we have already seen that it is easy to resolve the singularities of T. In fact, T has cyclic quotient singularities.

Definition 11.15. *Let $f\colon X \longrightarrow Y$ be a finite morphism. We say that f is* ***Galois****, if there is a finite group G acting on X such that f is the quotient map.*

Consider $R \longrightarrow N$. This morphism is Galois. Locally we have a Galois cover of $\mathbb{C}^2$, only ramified over the x and y-axis. Topologically we have an unramified Galois cover of the complement, a torus. Such covers are classified by the fundamental group,

$$\pi_1(\mathbb{C}^{*2}, 1) = \mathbb{Z} \oplus \mathbb{Z}.$$

A finite cover is given by a cylic quotient,

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}_a \oplus \mathbb{Z}_b.$$

Let $n = ab$. Then the map

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2 \qquad \text{given by} \qquad (x, y) \longrightarrow (x^n, y^n),$$

is a cover given by the following quotient of the fundamental group

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}_n \oplus \mathbb{Z}_n.$$

As

$$\mathbb{Z}_n \oplus \mathbb{Z}_n \longrightarrow \mathbb{Z}_a \oplus \mathbb{Z}_b,$$

factors the first map, it follows that any Galois cover

$$X \longrightarrow \mathbb{C}^2,$$

is itself a quotient of

$$\mathbb{C}^2 \longrightarrow X,$$

which only ramifies along the x and y-axis. So R has cyclic quotient singularities. It is easy to resolve R, preserving the action of G. The map $R \longrightarrow R/H$ is Galois and there is a birational morphism $R/H \longrightarrow T$. Finally, if we resolve the cyclic quotient singularities R/H, then we are done.

12. Resolution of singularities II

The only reason why the approach sketched at the end of lecture 11 does not work in complete generality is that the induction breaks down. In dimension n, we use *embedded* resolution of singularities in dimension $n-1$. In other words, we only prove that every quasi-projective surface is birational to a smooth projective surface. But in dimension three, to get resolution of threefolds, we need to know that every divisor in a smooth threefold is birational to a divisor with global normal crossings.

So let's examine the problem of embedded resolution. As a warm up, let's look at the problem of embedded resolution of curves. We start with a smooth surface S and a divisor $B = \sum B_i$ on S (so that the prime components $B_1, B_2, \dots, B_k$ are curves) and we want to find a birational morphism

$$\pi \colon T \longrightarrow S,$$

such that the sum of the strict transform of the prime components of B and of the exceptional divisors is a divisor with global normal crossings.

There is one way that the case of surfaces is significantly easier than in higher dimensions. At every step, we only need to choose which points to blow up on S. This means the problem is entirely local over every point of S (this is very far from being the case in higher dimensions; more about this later).

Given that the problem is local, we may use the Weirstrass polynomial to keep track of the situation. Given any point $p \in S$, we pick local coordinates x and y so that B is given by the zeroes of

$$y^{\mu} + a_{\mu-2}(x)y^{\mu-2} + a_{\mu-3}(x)y^{\mu-3} + \dots + a_1(x)y + a_0(x),$$

where $a_0(x), a_1(x), \dots, a_{\mu-2}(x)$ are analytic functions of the complex variable x. By assumption the multiplicity of $a_i(x)$ at 0 is at least $\mu - i$.

Now consider what happens when we blow up S at the point p. Then

$$S_1 = \operatorname{Bl}_p S \subset S \times \mathbb{P}^1.$$

Suppose that we put coordinates $[S : T]$ on $\mathbb{P}^1$, so that S_1 is defined by $xT = yS$. Then the blow up is covered by two coordinate patches $S \neq 0$ and $T \neq 0$. If $S \neq 0$, then $y = xt$, and coordinates upstairs are given by (x, t). The strict transform of B is given by

$$t^{\mu} + \frac{a_{\mu-2}(x)}{x^2}y^{\mu-2} + \frac{a_{\mu-3}(x)}{x^3}t^{\mu-3} + \dots + a_1(x)t + \frac{a_0(x)}{x^{\mu}}.$$

If we put $y = t$ and

$$b_i(x) = \frac{a_i(x)}{x^{\mu-i}},$$

then the equation of the strict transform of B is given by the zeroes of

$$y^\mu + b_{\mu-2}(x)y^{\mu-2} + b_{\mu-3}(x)y^{\mu-3} + \cdots + b_1(x)y + b_0(x),$$

where the multiplicity of $b_i(x)$ is at most the multiplicity of $a_i(x)$ minus $\mu - i$.

Suppose we consider what happens at a point q lying over the point p, that is, a point q of the exceptional divisor . We first check to see what happens over the unique point $[1 : 0]$ of the other coordinate patch. If $T \neq 0$, then $x = ys$ and it is easy to see that the strict transform of B does not even pass through $[1 : 0]$. So we may assume that $S \neq 0$ and $y = xt$. Note that the multiplicity of the strict transform of B at any point q other than $(x, t) = (0, 0)$ is less than μ. Indeed, let

$$\beta_i = b_i(0).$$

If we differentiate the equation

$$t^\mu + \beta_{\mu-2}y^{\mu-2} + \beta_{\mu-3}y^{\mu-3} + \cdots + \beta_1 t + \beta_0.$$

for the strict transform $\mu - 1$ times with respect to t, we get that

$$(\mu)!t = 0,$$

so that if $t = \alpha$ is a root of multiplicity μ, then

$$t = 0.$$

Notice that this is heavily dependent on the fact the characteristic is zero. On the other hand, it is pretty clear that at the point $(x, t) = (0, 0)$, the situation is better, since the multiplicity of each one of the functions $b_i(x)$ dropped.

By induction on the multiplicity μ and the multiplicity of each $a_i(x)$, after finitely many blow ups, $f_1 \colon S_1 \longrightarrow S$, we reduce to the case when the strict transform B_1 of B is smooth. Let E_1 be the sum of the exceptional divisors of f_1. Note that (S_1, E_1) has global normal crossings. It remains to reduce to the case when $(S_1, E_1 + B_1)$ has normal crossings. Pick a point $p_1 \in S_1$, where $(S_1, B_1 = E_1)$ does not have normal crossings. Then p_1 is contained in exactly one component of B_1, since B_1 is smooth, and at most two components of E_1, since E_1 has global normal crossings. B_1 is tangent to at most component of E_1 (since if there are two components of E_1, then they have different tangents). Blowing up finitely many times, we reduce to the case when no component of B_1 is tangent to a component of E_1. At this point, the only problem is if three components contain the same point. Blowing up each of these points, we are done.

Now let us consider the situation in higher dimensions. From the case of embedded resolution of curves, it is clear that it is a good idea to keep track of some invariants. So what invariants should we consider?

Example 12.1. *Consider the surface*

$$X = (y^2 = zx^2 + x^3) \subset \mathbb{C}^3.$$

This surface is called the ***Whitney umbrella****. If we project,*

$$\pi \colon \mathbb{C}^3 \longrightarrow \mathbb{C},$$

by sending (x, y, z) to z, then we get a family of nodal curves

$$y^2 = ax^2 + x^3,$$

$a \neq 0$, degenerating to a cuspidal curve,

$$y^2 = x^3.$$

The singular locus of X is the z-axis. Clearly the most singular point is the origin. So let's blow up the origin. Suppose that coordinates on the exceptional divisor $\mathbb{P}^2$ are $[A : B : C]$. The most important coordinate patch is $C \neq 0$, so that $x = az$ and $y = bz$.

$$(bz)^2 - z(az)^2 - (az)^3 = z^2(b^2 - a^2z - a^3z).$$

Replacing a by x and b by y, we get

$$y^2 - x^2z - x^3z,$$

which hardly seems like progress.

In fact, we should blow up the z-axis. The blow up of $\mathbb{C}^3$ along the z-axis sits inside $\mathbb{C} \times \mathbb{P}^1$. Let's suppose that $\mathbb{P}^1$ has coordinates $[S : T]$. Then $xT = yS$ and there are two coordinate patches. The most relevant is given by $S \neq 0$, so that $y = xt$, and we get

$$(xt)^2 - zx^2 - x^3 = x^2(t^2 - zx - x),$$

and so the equation of the strict transform is

$$t^2 - zx - x,$$

which is smooth.

There is another way to look at all of this. If we forget the embedding of X into $\mathbb{C}^3$, and consider the normalisation $\nu \colon X^\nu \longrightarrow X$ of X, then X^ν is smooth and fibres over $\mathbb{C}$ as well. The inverse image of the singular locus is a smooth curve C which double covers $\mathbb{C}$. This morphism ramifies over the origin. If $Y \longrightarrow X$ denotes the blow up of X at the origin, then the normalisation $Y^\nu \longrightarrow Y$ is simply given by *blowing up X^ν. It is then clear that no amount of blowing up points on X will ever improve the situation.*

The moral of this example is that we don't need to be so careful to distinguish the most singular points. In fact the only real invariant we need to keep track of is the multiplicity.

Unfortunately it is also clear that we need to be quite careful how to choose the locus to blow up. For example consider

$$z^2 - x^3y^3.$$

The singular locus consists of the x and y-axis. If we blow up either axis it is clear that we are making progress (generically along the y-axis we have $z^2 - x^3$, which is resolved in three steps by blowing up the origin). But we are not allowed to blow up an axis. The problem is that this is only the local analytic picture. Globally the singular locus might be a nodal cubic (for example). In this case it is not possible to blow up one axis, since globally blowing up one axis forces us to blow up the other axis. On the other hand, we cannot blow up both axes, since this locus is not smooth.

The only possible relevant locus we could blow up which is in the singular locus is the origin. On the blow up we have coordinates $(x, y, z) \times [A : B : C]$, and equations expressing the equality $[x : y : z] = [A : B : C]$. On the coordinate patch $A \neq 0$ we have $y = bx$, $z = cx$ so that

$$z^2 - x^3y^3 = c^2x^2 - b^3x^6 = x^2(c^2 - b^3x^4).$$

Changing variables we have $z^2 - x^3y^4$ which is surely worse than before. So how has the situation improved? The key thing is that the singular locus is given by $c = b = 0$ and $c = x = 0$. The locus $c = x = 0$ lies in the exceptional divisor; we created it ourselves, and so we know that this locus is algebraically irreducible and not just locally analytically irreducible. So we are allowed to blow up $c = x = 0$. In fact we are allowed to blow up $c = b = 0$, since the first blow up separated the x and y-axis. Notice though that we must blow up the strict transform of the other axis (again, because globally it might be part of a single irreducible algebraic curve).

It is clear from this example, that we must keep track of the sequence of blow ups. Fortunately, it turns out we don't need to keep track of much of the history. All we really need to distinguish are components of the original variety and the exceptional locus (we also need to order the components by when they appear).

There is one simple way to make sure that we never get into trouble passing from the local picture to the global picture. Note that on the original surface, there is an obvious symmetry between x and y. If every blow up respects this symmetry, we can never go wrong. We cannot chose to blow up the x-axis, since this is not symmetric. If we blow up the x-axis, then are allowed to blow up the strict transform of the x-axis, provided we also blow up the strict transform of the y-axis.

13. Resolution of singularities III

Let us now turn to a detailed explanation of Hironaka's proof of embedded resolution. In large part we follow the proof of Włodarczyk, which consists of a considerable simplification of Hironaka's original proof.

Theorem 13.1 (Embedded resolution)**.** *Let $X = X_0 \subset M = M_0$ be quasi-projective varieties over a field of characteristic zero, where M is smooth.*

Then there is a sequence of blow ups $\sigma_i \colon M_{i+1} \longrightarrow M_i$ of smooth centres, $1 \leq i \leq r-1$, such that if $\Pi \colon N = M_r \longrightarrow M$ denotes the composition, and X_i denotes the strict transform of X, then we have

1. *$Y = X_r$ is smooth,*
2. *the induced birational maps $\tau_i \colon X_{i+1} \longrightarrow X_i$ don't depend on the embedding of X into M, and*
3. *the maps $\sigma_1, \sigma_2, \dots, \sigma_{r-1}$ commute with smooth base change and with any automorphism, not even necessarily over the groundfield.*

Let me comment briefly on the significance of the last statement. Suppose that the groundfield k is not algebraically closed. Let G be the Galois group of the algebraic closure. Let $X' \subset M'$ be what you get by base changing to the algebraic closure. Since Π commutes with the action of G, it follows that resolving $X' \subset M'$ automatically resolves $X \subset M$. So we may safely assume that the groundfield is algebraically closed. So we may assume that the groundfield is $\mathbb{C}$.

Suppose that $X \subset M$ is smooth to begin with. Then it is clear that $r = 0$ and Π is the identity. Let $U \subset X$ be the smooth locus of X. Then U is an open subset of X and we may pick $V \subset M$ open such that $U = X \cap V \subset V$. The map $V \longrightarrow M$ induces a smooth base change. By what we just said, it follows that Π is an isomorphism outside of the singular locus of X.

Unfortunately it is not true that one can find Π which commutes with any base change. For example, suppose that S is a surface with an A_1-singularity. At least locally, S is a quotient singularity,

$$f \colon \mathbb{C}^2 \longrightarrow S = \mathbb{C}^2/\mathbb{Z}_2.$$

But we have already seen that Π does not change a smooth variety, so Π cannot commute with f.

In fact Π does not even commute with taking products. Consider the example of an A_1-singularity

$$S = (x^2 + y^2 + z^2 = 0) \subset \mathbb{C}^3.$$

Then

$$\Pi\colon N = \mathrm{Bl}_p\,\mathbb{C}^3 \longrightarrow \mathbb{C}^3,$$

blows up the origin. But if

$$X = S \times S \subset \mathbb{C}^6,$$

then the singular locus of X is $\{p\} \times S \cup S \times \{p\}$ and the product morphism would try to blow up this locus, which is not even smooth.

One of Hironaka's great ideas is to write down a smooth hypersurface $H \subset M$ and to construct Π inductively by considering $X \cap H \subset H$. If X is a hypersurface itself, which is locally given by a Weirstrass polynomial

$$y^\mu + f_{\mu-2}y^{\mu-2} + \cdots + f_0 = 0,$$

then we can take $H = (y = 0)$. The idea is that no matter how many times we blow up, as long as the multiplicity of the strict transform of X is exactly μ, then the singular locus of the strict transform of X is contained in the strict transform of H. For this reason, H is called a **hypersurface of maximal contact**.

Suppose that $M = \mathbb{C}^2$ and $X = (y^2 + x^{n+1} = 0) \subset \mathbb{C}^2$. In this case we take $H = (y = 0)$, so that $X \cap H \subset H$ is a point. It is clear that we need to keep track of the scheme structure; $z = X \cap H \subset M$ is a zero dimensional scheme, of length $n+1$. Since we have to keep track of the scheme structure, it is in fact better to work with ideal sheaves $\mathcal{I}$ on M and not just subvarieties X. Notice also that, by way of induction, one has to allow the rather silly possibility that we blow up a divisor on H.

Definition 13.2. *Let $\mathcal{I}$ be a coherent sheaf of ideals on a smooth variety M. The **multiplicity** of $\mathcal{I}$ at a point $p \in M$ is the largest μ such that $\mathcal{I}_p \subset \mathfrak{m}^\mu$, where $\mathfrak{m} \subset \mathcal{O}_{M,p}$ is the maximal ideal of $p \in M$.*

*Let $V \subset M$ be a smooth subvariety and let $\sigma\colon N \longrightarrow M$ be the blow up of M along V. The **strict transform** of $\mathcal{I}$ is the ideal sheaf $\mathcal{J} = \sigma^{-1}\mathcal{I} \underset{\mathcal{O}_N}{\otimes} \mathcal{O}_N(-\mu E)$*

Theorem 13.3 (Principalisation of ideals)**.** *Let $\mathcal{I} = \mathcal{I}_0 \subset M = M_0$ be a coherent sheaf of ideal on a quasi-projective variety over a field of characteristic zero, where M is smooth.*

Then there is a sequence of blow ups $\sigma_i\colon M_{i+1} \longrightarrow M_i$ of smooth centres, $1 \leq i \leq r-1$, such that if $\Pi\colon N = M_r \longrightarrow M$ denotes the composition, and $\mathcal{I}_{i+1}$ denotes the strict transform of $\mathcal{I}_i$, then we have

(1) *$\mathcal{I}_r = \mathcal{O}_Y$ is the trivial ideal sheaf,*
(2) *the induced birational maps $\tau_i\colon X_{i+1} \longrightarrow X_i$ don't depend on the embedding of X into M, and*

(3) *the maps $\sigma_1, \sigma_2, \dots, \sigma_{r-1}$ commute with smooth base change and with any automorphism, not even necessarily over the ground-field.*

Lemma 13.4. (13.3) *implies* (13.1).

Proof. Let $\mathcal{I}$ be the ideal sheaf of X. At some point we must blow up the strict transform Y_i of X, either to make Y_i into a divisor, or to cancel off some multiple of the divisor Y_i. Either way, since we only blow up smooth subvarieties of M, Y_i must be smooth. $\square$

Note that we have to take account of the fact that when we restrict to a hypersurface of maximal contact, then we may not be able to cancel off all of the exceptional divisor.

Definition 13.5. *Let $\mathcal{I}$ be a coherent sheaf of ideals and let m be a positive integer. The pair $(\mathcal{I}, m)$ is called a **marked ideal**.*

*The **support** of a marked ideal $(\mathcal{I}, m)$ is the closed subset where the multiplicity of $\mathcal{I}$ is at least m.*

*Let $V \subset M$ be a smooth subvariety and let $\sigma\colon N \longrightarrow M$ be the blow up of M along V. The **strict transform** of $(\mathcal{I}, m)$ is the pair $(\mathcal{J}, m)$, where $\mathcal{J} = \sigma^{-1}\mathcal{I} \underset{\mathcal{O}_N}{\otimes} \mathcal{O}_N(-mE)$.*

We will always suppose that the multiplicity of $\mathcal{J}$ along V is at least m.

Theorem 13.6 (Principalisation of marked ideals)**.** *Let $\mathcal{I} = \mathcal{I}_0 \subset M = M_0$ be a coherent sheaf of ideals on a quasi-projective variety over a field of characteristic zero, where M is smooth. Let m be a positive integer.*

Then there is a sequence of blow ups $\sigma_i\colon M_{i+1} \longrightarrow M_i$ of smooth centres, $1 \leq i \leq r-1$, such that if $\Pi\colon N = M_r \longrightarrow M$ denotes the composition, and $(\mathcal{I}_{i+1}, m)$ denotes the strict transform of $(\mathcal{I}_i, m)$, then we have

(1) *the support of $\mathcal{I}_r$ is empty,*
(2) *the induced birational maps $\tau_i\colon X_{i+1} \longrightarrow X_i$ don't depend on the embedding of X into M, and*
(3) *the maps $\sigma_1, \sigma_2, \dots, \sigma_{r-1}$ commute with smooth base change and with any automorphism, not even necessarily over the ground-field.*

The scheme of the induction is then to show that (13.6) in dimension $n-1$ implies (13.3) in dimension n and that (13.3) in dimension n implies (13.6) in dimension n. The second implication is relatively easy and the first takes some work.

In practice we should also keep track of the exceptional divisors and we should order them. The only thing we really have to worry about is making sure that we never blow up a centre which destroys the fact that these divisors have global normal crossings.

Given the ideal $\mathcal{I}$, how should we define the hypersurface of maximal contact? The answer is relatively easy, just differentiate enough times; if the multiplicity of $\mathcal{I}$ is μ, then the ideal

$$D(I),$$

of all partial derivatives, has multiplicity $\mu - 1$. So pick a general element of

$$f \in D^{\mu-1}(I),$$

and let $H = (f = 0)$. With this choice of a hypersurface of maximal contact, any sequence of blow ups for $\mathcal{I}$ of order m induces a unique choice of blow ups for $(\mathcal{I}|_H, m)$.

There is one more example to illustrate one more complication. Consider

$$X = (x^2 + y^3 - z^6) \subset \mathbb{C}^3.$$

Suppose we pick $H = (x - z^3 = 0)$ as a hypersurface of maximal contact. As

$$x^2 - z^6 + y^3 = (x - z^3)(z + z^3) + y^3,$$

the restriction $Y = X \cap H \subset \mathbb{C}^2 = H$ is a triple line. The original surface has an isolated singularity, so if we are not careful we might have to do more work to resolve $Y \subset H$.

Definition 13.7. *Let $\mathcal{I}$ be a coherent sheaf of ideals on a smooth quasi-projective variety. We say that $\mathcal{I}$ is D-**balanced** if*

$$(D^i(\mathcal{I}))^\mu \subset \mathcal{I}^{\mu-i},$$

for all $i < \mu$, where μ is the maximal multiplicity.

If I is D-balanced and H is a hypersurface of maximal contact, then any sequence of blow ups for $\mathcal{I}$ of order m is the same as a sequence of blow ups for $(\mathcal{I}|_H, m)$.

Example 13.8. *Suppose we start with*

$$I = \langle xy - z^n \rangle \subset \mathbb{C}[x, y, z].$$

If we restrict to $x = 0$, we get an n-fold line, which is not correct. Now consider

$$I + D(I)^2 = \langle xy, x^2, y^2, xz^{n-1}, yz^{n-1}, z^n \rangle.$$

If we restrict to $x = 0$, then we get the ideal

$$\langle y^2, yz^{n-1}, z^n \rangle.$$

It is easy to check that resolving this ideal induces the correct resolution of I.

The final problem is that we need to make sure that the resolution process does not depend on our choice of an element $D^{\mu-1}(I)$, so that we can pass from the local picture to the global picture. Recall that it is enough to make sure that the sequence of blow ups commutes with all automorphisms.

Definition 13.9. *We say that $\mathcal{I}$ is* ***symmetric*** *if*

$$D^{\mu-1}(\mathcal{I}) \cdot \mathcal{I} \subset \mathcal{I}.$$

Note that this is very similar to the D-balanced condition. One can show that if $\mathcal{I}$ is symmetric and D-balanced, then we are done. The idea is to change $\mathcal{I}$, to another ideal sheaf $\mathcal{J}$, so that resolving $\mathcal{J}$ induces a resolution of $\mathcal{J}$, where now $\mathcal{J}$ is symmetric and D-balanced.

The problem is that we need to make $\mathcal{J}$ have very large multiplicity. Suppose that $\mathcal{I}$ has maximal multiplicity μ.

$$W(\mathcal{I}) = \{ \prod_{j=0}^{\mu} D^j(\mathcal{I})^{c_j} \mid \sum(\mu - c_j) \geq \mu! \}.$$

Theorem 13.10. *Let $\mathcal{I}$ be a coherent sheaf of ideals on a smooth quasi-projective variety M of maximal multiplicity μ. Then*

1. *$W(\mathcal{I})$ has maximal multiplicity $\mu!$.*
2. *$W(\mathcal{I})$ is D-balanced.*
3. *$W(\mathcal{I})$ is symmetric.*
4. *Blows up for $(\mathcal{I}, \mu)$ are the same as blow ups for $(W(\mathcal{I}), \mu!)$.*

Let us consider how to resolve a threefold $(f = 0)$ of multiplicity 3 in $\mathbb{C}^4$. The first step is to replace

$$I = \langle f \rangle \qquad \text{by} \qquad W(I),$$

which increases the multiplicity from 3 to $6 = 3!$. The next step is to restrict to a hypersurface of maximal contact. Then we repeat the same operation, replacing 6 by $720 = 6!$. It is clear that we need quite a bit more work to implement resolution of singularities on a computer.