OVERVIEW ON TWO GEOMETRIC REALIZATIONS OF THE AFFINE HECKE ALGEBRA

This story concerns an old puzzle about the affine Hecke algebra. On one hand, it's the Grothendieck group of Iwahori-equivariant mixed ℓ-adic sheaves on the affine flag variety. On the other hand, it's the equivariant K-group of the Steinberg variety (for the Langlands dual group). This raises the question: are these two categorifications related?

This is the work of Bezrukavnikov and collaborators culminating in the full monoidal equivalence in [Bez16].

1. Introduction

The description of the affine Hecke algebra as the equivariant K-group of the Steinberg variety goes back to the work of Kazhdan and Lusztig on the classification of irreducible smooth representations of p-adic groups with an Iwahori-fixed vector. See also the last pages of the introduction to [CG97].

1.1. **Affine Hecke algebra.** Let $G^\vee$ be a semisimple adjoint, I the Iwahori subgroup of $G^\vee(\mathbb{Q}_p)$. It is known that taking the I-fixed subspace gives a bijection

$$\left\{\begin{matrix}\text{smooth irreps of } G^\vee(\mathbb{Q}_p)\\ \text{with } I\text{-fixed vector}\end{matrix}\right\} / \cong \quad \longleftrightarrow \quad \left\{\begin{matrix}\text{irreps of}\\ \mathbb{C}[I\backslash G^\vee(\mathbb{Q}_p)/I]\end{matrix}\right\} / \cong,$$

where $\mathbb{C}[I\backslash G^\vee(\mathbb{Q}_p)/I]$ is the Iwahori–Hecke algebra, the convolution algebra of I-biequivariant compactly supported $\mathbb{C}$-valued functions on $G^\vee(\mathbb{Q}_p)$. There is another description of the convolution algebra as a specialized affine Hecke algebra.

Definition 1.1. The *affine Weyl group* is the semidirect product

$$W_{\text{aff}} := \mathbf{X} \rtimes W$$

where $\mathbf{X}$ is the cocharacter lattice of $G^\vee$ (canonically identified with the character lattice of G).

Remark 1.2. More precisely, this is the *extended* affine Weyl group. It contains a finite-index subgroup as a Coxeter group (generated by the simple reflections of W and an additional simple reflection corresponding to the affine root) but is not itself a Coxeter group. ($G = \mathrm{SL}_2$ is confusing for seeing this. Think about SL_3.)

Let q be an indeterminate. The *affine Hecke algebra* $\mathcal{H}_{\text{aff}}$ is a certain $\mathbb{Z}[q, q^{-1}]$-algebra, a q-deformation of the group algebra $\mathbb{Z}[W_{\text{aff}}]$. For example, it is free as a $\mathbb{Z}[q, q^{-1}]$-module with a basis in bijection with W_{aff}, but the basis element T_s corresponding to simple reflection s satisfies a quadratic relation involving q instead of $T_s^2 = 1$ as in the group algebra. For an actual definition, see [CG97, Definition 7.1.9].

Then it is known (Iwahori–Matsumoto, Bernstein) that

$$\mathbb{C}[I\backslash G^\vee(\mathbb{Q}_p)/I] \cong \mathbb{C} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathcal{H}_{\text{aff}},$$

where the tensor product is via the map $\mathbb{Z}[q, q^{-1}] \to \mathbb{C}$ sending $q \mapsto p$.

Date: July 1, 2016.

1.2. **Geometric realization of $\mathcal{H}_{\mathrm{aff}}$.** As a first step towards the classification of irreducible representations of $\mathcal{H}_{\mathrm{aff}}$, Kazhdan and Lusztig gave a geometric realization of $\mathcal{H}_{\mathrm{aff}}$. Let G be the Langlands dual group of $G^\vee$ over $\mathbb{C}$, and let Z be the Steinberg variety (see the next subsection for the definition). This is a complex variety with a $G \times \mathbb{G}_{\mathrm{m}}$ action, so that we may consider the equivariant K-group $K^{G\times\mathbb{G}_{\mathrm{m}}}(Z)$ of coherent sheaves, a $\mathbb{Z}[q, q^{-1}]$-module where q corresponds to changing the $\mathbb{G}_{\mathrm{m}}$-grading. This admits a convolution product, making it into a $\mathbb{Z}[q, q^{-1}]$-algebra.

Remark 1.3. Note that we consider the K-group of coherent sheaves (which behaves like homology), not the K-group of vector bundles (which behaves like cohomology). Since Z is not smooth, these are actually different, and tensor product is not defined on $K^{G\times\mathbb{G}_{\mathrm{m}}}(Z)$.

Theorem 1.4 (Kazhdan–Lusztig)**.** *There is a $\mathbb{Z}[q, q^{-1}]$-algebra isomorphism*

$$\mathcal{H}_{\mathrm{aff}} \cong K^{G\times\mathbb{G}_{\mathrm{m}}}(Z).$$

In particular, specializing $q \mapsto 1$ and forgetting the $\mathbb{G}_{\mathrm{m}}$-action, we get

$$\mathbb{Z}[W_{\mathrm{aff}}] \cong K^G(Z).$$

Remark 1.5. This contains the classical Satake isomorphism as the center of each side.

1.3. **Upgrade to categorical equivalence.** The sheaf-function correspondence suggests another categorification of $\mathcal{H}_{\mathrm{aff}}$. Consider the group $G^\vee(\mathcal{K})$, where $\mathcal{K} = \mathbb{C}((t))$. There is again an Iwahori I, and the *affine flag variety* $\mathcal{F}\ell$ is the quotient $G^\vee(\mathcal{K})/I$, which has the structure of an ind-scheme over $\mathbb{C}$. As with finite flag varieties, now with I playing the role of a Borel, this has a cell stratification (Schubert stratifictaion) by I-orbits indexed by W_{aff} with dimension given by length. Then $\mathcal{H}_{\mathrm{aff}}$, at least as a $\mathbb{Z}[q, q^{-1}]$-module, is categorified by $\mathrm{Perv}_I^{\mathrm{m}}(\mathcal{F}\ell, \mathbb{Q}_\ell)$, the category of I-equivariant mixed ℓ-adic perverse sheaves on $\mathcal{F}\ell$. This leads to the main question in this whole story:

Question 1.6. Can one relate the two categorifications

$$\mathrm{Perv}_I^{\mathrm{m}}(\mathcal{F}\ell) \quad \text{and} \quad \mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(Z) \quad ?$$

This is the work of Bezrukavnikov and collaborators, culminating in the following result.

Theorem 1.7 ([Bez16])**.** *There is a monoidal equivalence of triangulated categories*

$$\mathrm{D}_I^{\mathrm{b,m}}(\mathcal{F}\ell) \cong \mathrm{D}^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(Z)$$

Forgetting the mixed structure on the left corresponds to forgetting the $\mathbb{G}_{\mathrm{m}}$-action on the right. [Actually, the mixed version doesn't seem to be in [Bez16]. Maybe there's a subtlety I'm missing?]

Remark 1.8.
(1) The convolution product on the left does not preserve perverse sheaves, so we really need to go to the derived category.
(2) $\mathrm{Perv}_I^{\mathrm{m}}(\mathcal{F}\ell)$ is finite length (every object is a finite successive extension of simple perverse sheaves), unlike $\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(Z)$. So the perverse t-structure on the left cannot correspond to the standard t-structure on the right. The corresponding t-structure on the right is related to the exotic t-structure on $\tilde{\mathcal{N}}$ and the perverse coherent t-structure on $\mathcal{N}$.

(3) This is a technical point. $\mathrm{D}^{\mathrm{b,m}}_I(\mathcal{F}\ell)$ is the equivariant derived category in the sense of Bernstein–Lunts. In particular, it is not the derived category of any abelian category, so the right hand side also cannot mean derived category of $\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(Z)$. In this full monoidal equivalence, one needs to replace the fiber product $Z = \tilde{\mathcal{N}}\times_{\mathcal{N}}\tilde{\mathcal{N}}$ with a derived fiber product $\tilde{\mathcal{N}}\times^L_{\mathfrak{g}}\tilde{\mathcal{N}}$, and define its equivariant derived category appropriately.

[Compatibility with geometric Satake]

2. Equivariant K-theory of the Steinberg variety

A textbook reference for this material is Chapter 7 of [CG97]. For now, we merely give a plausibility argument for the specialized isomorphism $\mathbb{Z}[W_{\mathrm{aff}}] \cong K^G(Z)$.

2.1. **Springer resolution and the Steinberg.**

Let $\mathfrak{g}$ be a semisimple Lie algebra, viewed as a variety over $\mathbb{C}$. Let $\mathcal{N}$ be the nilpotent cone, the subvariety of nilpotent elements of $\mathfrak{g}$. This is a singular affine subvariety; viewing elements of $\mathfrak{g}$ as endomorphisms via an embedding $\mathfrak{g}\hookrightarrow \mathfrak{gl}_n$, it is cut out by the vanishing of coefficients of the characteristic polynomial. The nilpotent cone admits a resolution $\tilde{\mathcal{N}}$, the *Springer resolution*, fitting into a correspondence

$$\begin{tikzcd} & \tilde{\mathcal{N}} := \{(x,\mathfrak{b})\in\mathcal{N}\times\mathcal{B} \mid x\in\mathfrak{b}\} \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ \mathcal{N} & & \mathcal{B} \end{tikzcd}$$

where $\mathcal{B}$ is the associated (finite) flag variety, viewed as the variety of Borel subalgebras of $\mathfrak{g}$.

The *Steinberg variety* for G is defined to be $Z := \tilde{\mathcal{N}}\times_{\mathcal{N}}\tilde{\mathcal{N}}$. Consider the composition $Z\hookrightarrow \tilde{\mathcal{N}}\times\tilde{\mathcal{N}} \xrightarrow{\pi\times\pi} \mathcal{B}\times\mathcal{B}$. The fiber over a pair of Borel subalgebras $(\mathfrak{b},\mathfrak{b}')$ consists of nilpotent elements in $\mathfrak{b}\cap\mathfrak{b}'$.

$$\begin{tikzcd} \mathfrak{b}\cap\mathfrak{b}\cap\mathcal{N} \arrow[r, hook] \arrow[d] & Z \arrow[d, "\pi\times\pi"] \\ (\mathfrak{b},\mathfrak{b}') \arrow[r, hook] & \mathcal{B}\times\mathcal{B} \end{tikzcd}$$

To study the geometry of Z, we stratify $\mathcal{B}\times\mathcal{B}$ by relative position. Recall that there is a bijection

$$\begin{aligned} W &\longleftrightarrow \left\{\begin{smallmatrix} G\text{-orbits of } \mathcal{B}\times\mathcal{B} \text{ under} \\ \text{the diagonal action} \end{smallmatrix}\right\} \\ w &\longleftrightarrow Y_w := G.(\mathfrak{b}, w\mathfrak{b}w^{-1}), \end{aligned}$$

so Y_w consists of pairs of Borels with "relative position w." The fiber of $\pi\times\pi\colon Z\to\mathcal{B}\times\mathcal{B}$ only depends on the relative position. In fact, under the identification $\tilde{\mathcal{N}}\cong T^*\mathcal{B}$ and $\tilde{\mathcal{N}}\times\tilde{\mathcal{N}}\cong T^*(\mathcal{B}\times\mathcal{B})$, the map $\pi\times\pi$ becomes the bundle projection, and

$$Z = \bigsqcup_{w\in W} Z_w \cong \bigsqcup_{w\in W} T^*_{Y_w}(\mathcal{B}\times\mathcal{B}),$$

the union of conormal bundles to Y_w.

Example 2.1. Draw some pictures for SL_2.

In this case, $W = \{e, s\}$, and the decomposition $\mathcal{B} \times \mathcal{B} = Y_e \sqcup Y_s$ becomes $\mathbb{P}^1 \times \mathbb{P}^1 = \mathbb{P}^1_\Delta \sqcup U$.

$$\begin{tikzcd} Z \arrow[d, "\pi\times\pi"'] & = & Z_e \arrow[d, "\mathbb{A}^1"] & \sqcup & Z_s \arrow[d, "\sim"] \\ \mathbb{P}^1 \times \mathbb{P}^1 & = & \mathbb{P}^1_\Delta & \sqcup & U \end{tikzcd}$$

2.2. **Equivariant K-theory of the Steinberg.** For now, we merely give a plausibility argument for the existence of an isomorphism

$$\Phi \colon \mathbb{Z}[W_{\mathrm{aff}}] \xrightarrow{\sim} K^G(Z),$$

at least as modules. We will not actually define Φ here, nor talk at all about the product structure.

Recall that the Borel construction of G-equivariant line bundles on the flag variety defines an isomorphism

$$\begin{aligned} \mathbb{Z}[\mathbf{X}] &\xrightarrow{\sim} K^G(\mathcal{B}) \\ \lambda &\mapsto [L(\lambda)] \end{aligned}$$

We identified Z_e as a vector bundle over the diagonal $\mathcal{B}_\Delta$. By the homotopy invariance of K-theory, pullback induces an isomorphism $K^G(\mathcal{B}_\Delta) \xrightarrow{\sim} K^G(Z_e)$. Moreover, it can be shown that the pushforward $K^G(Z_e) \to K^G(Z)$ is injective, and that Φ restricts to the composition

$$\mathbb{Z}[\mathbf{X}] \xrightarrow{\sim} K^G(\mathcal{B}) \xrightarrow{\sim} K^G(\mathcal{B}_\Delta) \xrightarrow{\sim} K^G(Z_e),$$

where the second isomorphism is induced from pullback under the composition $\mathcal{B}_\Delta \hookrightarrow \mathcal{B} \times \mathcal{B} \xrightarrow{\mathrm{pr}_1} \mathcal{B}$ induced by, say, first projection.

More generally, let

$$\mathbb{Z}[\mathbf{X}]_{\leq w} := \bigoplus_{x \leq w} \mathbb{Z}[\mathbf{X}]x, \qquad Z_{\leq w} := \bigsqcup_{x \leq w} Z_x.$$

Then it can be shown that Φ preserves the filtration

$$\mathbb{Z}[\mathbf{X}]_{\leq w} \xrightarrow{\sim} K^G(Z_{\leq w}),$$

and that the induced map on the degree-w piece, identified with an isomorphism

$$\mathbb{Z}[\mathbf{X}]w \xrightarrow{\sim} K^G(Z_w),$$

is the composition

$$\mathbb{Z}[\mathbf{X}]w \xrightarrow{\sim} K^G(\mathcal{B}) \to K^G(Y_w) \cong K^G(Z_w),$$

where the second map is induced from pullback under the composition $Y_w \hookrightarrow \mathcal{B} \times \mathcal{B} \xrightarrow{\mathrm{pr}_1} \mathcal{B}$.

Remark 2.2. Part of the claim here is that each exact sequence

$$K^G(Z_{<w}) \to K^G(Z_{\leq w}) \to K^G(Z_w) \to 0$$

is injective on the left.

Thus, roughly speaking, the equivariant K-group of Z can be seen as coming from two orthogonal parts: a single copy of $\mathcal{B}$ already contributes the lattice part $\mathbb{Z}[\mathbf{X}]$, while relative position under $\pi \times \pi \colon Z \to \mathcal{B} \times \mathcal{B}$ contributes the finite Weyl group part.

The definition of Φ involves a deformation argument like in Springer theory, using the Grothendieck alteration, and using the simpler geometry over the regular semisimple locus together with a specialization map in K-theory to get back to the Steinberg.

2.3. **A philosophical point.** As with the Borel-equivariant geometry of finite flag varieties, the I-equivariant geometry of $\mathcal{F}\ell$ is controlled by the Coxeter-like presentation of $\mathcal{H}_{\mathrm{aff}}$; for example the closure relation of strata is given by the (extended) Bruhat order, and the dimension of each cell is given by the length. Meanwhile, we saw above that the G-equivariant geometry of Z is naturally related to the Bernstein presentation of $\mathcal{H}_{\mathrm{aff}}$, which makes apparent the large commutative subalgebra, the lattice part $\mathbb{Z}[\mathbf{X}]$. These presentations show up on Langlands dual sides.

construtible (automorphic)	coherent/algebraic (Galois?)
$\mathrm{D}^{\mathrm{b,m}}_I(\mathcal{F}\ell)$	$\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(Z)$
Coxeter presentation of $\mathcal{H}_{\mathrm{aff}}$	Bernstein presentation of $\mathcal{H}_{\mathrm{aff}}$

3. Overview of the various equivalences

Bezrukavnikov and collaborators proved a number of related equivalences. We give an overview, with references.

3.1. **Convention on shifts.** On the constructible side, recall that the category $\mathrm{D}^{\mathrm{b,m}}$ of mixed ℓ-adic sheaves has an extra grading coming from weights, so there are two shifts:

$$[1] = \text{cohomological shift}$$
$$\langle 1\rangle := (-\tfrac{1}{2}) = \text{degree 1 Tate twist.}$$

Recall that $[1]$ shifts the weight down by 1, while $\langle 1\rangle$ shifts it up by 1. It will be convenient to also define

$$\{1\} := [1]\langle 1\rangle = \text{weight-preserving shift-twist}$$

The coherent side $\mathrm{D}^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}$ also has two shifts:

$$[1] = \text{cohomological shift}$$
$$\langle 1\rangle := \text{twisting the action of } \mathbb{G}_{\mathrm{m}}.$$

3.2. **Affine Hecke algebra categorification.** For completeness, we restate Bezrukavnikov's result.

Theorem 3.1 ([Bez16]). *There is a monoidal equivalence of triangulated categories*

$$\Phi \colon \mathrm{D}^{\mathrm{b,m}}_I(\mathcal{F}\ell) \cong \mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(Z)$$

satisfying $\Phi \circ \langle 1\rangle \cong \langle 1\rangle \circ \Phi$.

A triangulated equivalence in particular satisfies $\Phi \circ [1] \cong [1] \circ \Phi$. The last line means that forgetting the mixed structure on the left corresponds to forgetting the $\mathbb{G}_{\mathrm{m}}$-equivariance on the right.

In this series of talks, we won't get to the proof of this monoidal equivalence. What we will see are equivalences for categorifications of certain modules and subalgebras of $\mathcal{H}_{\mathrm{aff}}$.

3.3. **Spherical module categorification.** Consider the affine Grassmannian $\mathcal{G}r = G^\vee(K)/G^\vee(\mathcal{O})$, where $\mathcal{O} = \mathbb{C}[[t]]$. The natural projection $\mathcal{F}\ell \to \mathcal{G}r$ is a fibration with fiber the finite flag variety of $G^\vee$. On the constructible side, we now look at $\mathrm{D}^{\mathrm{b,m}}_{(I)}(\mathcal{G}r)$, the category of mixed I-constructible sheaves on $\mathcal{G}r$. The I-orbits of $\mathcal{G}r$ are labeled by $\mathbf{X} \xrightarrow{\sim} W_{\mathrm{aff}}/W$.

At the level of Grothendieck groups, going from $\mathcal{F}\ell$ to $\mathcal{G}r$ corresponds to quotienting on the right by W. More precisely, it turns out that $\mathrm{D}^{\mathrm{b,m}}_{(I)}(\mathcal{G}r)$ is a categorification of the *spherical module* $M_{\mathrm{sph}} := \mathrm{ind}_{\mathcal{H}}^{\mathcal{H}_{\mathrm{aff}}}\mathrm{triv}$, the induction of the trivial representation from the finite Hecke algebra $\mathcal{H}$ to $\mathcal{H}_{\mathrm{aff}}$. As a $\mathbb{Z}[q, q^{-1}]$-module, it is free with a basis indexed by $\mathbf{X} \xrightarrow{\sim} W_{\mathrm{aff}}/W$.

On the coherent side, the corresponding passage takes one from the Steinberg to $\tilde{\mathcal{N}}$. There is again an equivalences of the two categorifications, but intertwining the shifts in a different way.

Theorem 3.2 ([ABG04])**.** *There is an equivalence of triangulated categories*

$$P\colon \mathrm{D}^{\mathrm{b,m}}_{(I)}(\mathcal{G}r) \cong \mathrm{D}^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\tilde{\mathcal{N}})$$

satisfying $P \circ \{1\} \cong \langle 1\rangle \circ P$. Furthermore, P is compatible with geometric Satake.

Note that the mixed structure on the left no longer corresponds to the $\mathbb{G}_{\mathrm{m}}$-equivariance on the right.

Remark 3.3. The constructible side is I-constructible rather than I-equivariant sheaves. Since I-orbits afford a cell stratification of $\mathcal{G}r$, the constructible derived category is equivalent to the bounded derived category of the subcategory of perverse sheaves; in particular, it is the derived category of some abelian category. Correspondings, the coherent side is the honest bounded derived category of the abelian category of equivariant coherent sheaves. There is no technical subtlety about derived fiber product as in the affine Hecke algebra categorifications.

3.4. **Anti-spherical module categorification.** Although we will not discuss this result in detail, we mention for conceptual clarity the equivalence of categorifictions of the anti-spherical module $M_{\mathrm{asph}} := \mathrm{ind}_{\mathcal{H}}^{\mathcal{H}_{\mathrm{aff}}}\mathrm{sgn}$. The constructible side in this case turns out to be $\mathrm{D}^{\mathrm{b,m}}_{\mathrm{IW}}(\mathcal{F}\ell)$, the category of Iwahori–Whittaker sheaves on $\mathcal{F}\ell$.

Theorem 3.4 ([AB09])**.** *There is a triangulated equivalence*

$$Q\colon \mathrm{D}^{\mathrm{b,m}}_{IW}(\mathcal{F}\ell) \cong \mathrm{D}^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\tilde{\mathcal{N}})$$

satisfying $Q \circ \langle 1\rangle \cong \langle 1\rangle \circ Q$.

Note that the mixed structure corresponds to the $\mathbb{G}_{\mathrm{m}}$-equivariance, unlike for the spherical case.

3.5. **Spherical vs. anti-spherical: comparison of t-structures.** The constructible derived category has an important t-structure, the perverse t-structure, which cuts out the abelian heart, the perverse sheaves. We may ask what the various triangulated equivalences to this t-structure. The following diagram, to be gradually explained, summarizes the situation.

$$\begin{tikzcd}
\text{derived} & \mathcal{M}_{\mathrm{sph}} = \mathrm{D}^{\mathrm{b,m}}_{(I)}(\mathcal{G}r) \arrow[r, "P", "\sim"'] \arrow[rr, bend left, "\substack{\text{Koszul duality}\\ \sim}"] & \mathrm{D}^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\tilde{\mathcal{N}}) & \mathrm{D}^{\mathrm{b,m}}_{\mathrm{IW}}(\mathcal{F}\ell) = \mathcal{M}_{\mathrm{asph}} \arrow[l, "Q"', "\sim"] \\
 & \cup & \cup & \cup \\
\text{abelian} & \mathrm{Adv}_{(I)}(\mathcal{G}r) \arrow[r, "\sim"] & \mathrm{ExCoh}(\tilde{\mathcal{N}}) & \mathrm{Perv}^{\mathrm{m}}_{\mathrm{IW}}(\mathcal{F}\ell) \arrow[l, "\sim"'] \\
 & \cup & \cup & \cup \\
\text{tilting} & \mathrm{Parity}_{(I)}(\mathcal{G}r) \arrow[r, "\sim"] & \mathrm{Tilt}(\mathrm{ExCoh}(\tilde{\mathcal{N}})) & \mathrm{Tilt}^{\mathrm{m}}_{\mathrm{IW}}(\mathcal{F}\ell) \arrow[l, "\sim"'] \\
\text{shifts} & \{1\} \arrow[r, leftrightarrow] & \langle 1\rangle \arrow[r, leftrightarrow] & \langle 1\rangle
\end{tikzcd}$$

The middle column is the coherent side, while the left (spherical) and right (aspherical) columns are on the constructible side.

The first row shows the two equivalences, for the spherical module [ABG04] and for the anti-spherical module [AB09]. The last row indicates which shifts are intertwined. Of course, these equivalences, being triangulated, intertwine the cohomological shift [1] in each category.

The second row shows the abelian heart of t-structures. Starting with the perverse t-structure on the right, the corresponding t-structure in the middle column is called the *exotic t-structure*, whose heart $\mathrm{ExCoh}(\tilde{\mathcal{N}})$ consists of *exotic coherent sheaves*; on the left is the *adverse t-structure*, whose heart $\mathrm{Adv}_{(I)}(\mathcal{G}r)$ consists of *adverse sheaves*.

It turns out that these abelian categories are graded highest weight (graded for the shift shown in the last row) for the partially ordered set $(\mathbf{X}, \leq)$. Here, $\leq$ is the closure order of the I-orbits on $\mathcal{G}r$, not the usual order on $\mathbf{X}$ using dominant weights. In particular, it makes sense to speak of the tilting objects in each category; this is the third row. The tilting objects in $\mathrm{Adv}_{(I)}$ turn out to be semisimple perverse sheaves that are pure of weight 0, i.e. a direct sum of shift-twisted IC sheaves. We denote them here by $\mathrm{Parity}_{(I)}(\mathcal{G}r)$.

3.6. **Koszul duality.** The composite equivalence

$$K := P^{-1} \circ Q \colon \mathrm{D}^{\mathrm{b,m}}_{IW}(\mathcal{F}\ell) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b,m}}_{I}(\mathcal{G}r)$$

satisfies $K \circ \langle 1\rangle \cong K \circ \{1\}$ (since K intertwines [1], this is equivalent to $K \circ \langle 1\rangle \cong \{1\} \circ K$). This is a parabolic version of Koszul duality for Kac–Moody groups, proved in [BY13]. In fact, this composite equivalence was the motivation for this Koszul duality; see [Bez06, Section 1.2].

[understanding Koszul duality on the level of Hecke algebra: The quadratic relation, suitably normalized, is $(T_s + q)(T_s - q^{-1})$, so there is an automorphism of the Hecke algebra sending $q \mapsto -q^{-1}$. This exchanges IC and tilting.]

3.7. **Regular module of the spherical subalgebra.** Now we forget the anti-spherical story. From the spherical module, we can take a further quotient on the left by W, so that the index set is now $\mathbf{X}^+ \xrightarrow{\sim} W\backslash W_{\mathrm{aff}}/W$. On the constructible side, this corresponds to consider spherical orbits (i.e. $G^\vee(\mathcal{O})$-orbits) on $\mathcal{G}r$, which are labeled by $\mathbf{X}^+$. On the coherent side, this corresponds to passing from $\tilde{\mathcal{N}}$ and $\mathcal{N}$.

Theorem 3.5. *There is a triangulated equivalence*

$$P_{\mathrm{sph}}\colon \mathrm{D}^{\mathrm{b,m}}_{(G_{\mathcal{O}})}(\mathcal{G}r) \cong \mathrm{D^b Coh}^{G\times\mathbb{G}_\mathrm{m}}_{\mathrm{perf}}(\mathcal{N})$$

satisfying $P_{\mathrm{sph}} \circ \{1\} \cong \langle 1\rangle \circ P_{\mathrm{sph}}$*, again compatible with geometric Satake.*

Here, perf indicates the full subcategory of perfect complexes, i.e. those complexes that have a finite resolution by coherent sheaves of the form $\mathcal{O}_{\mathcal{N}} \otimes V$ for $V \in \operatorname{Rep} G$.

We can again look at t-structures.

$$\begin{array}{ccc}
\text{derived} & \mathrm{D}^{\mathrm{b,m}}_{(G_{\mathcal{O}})}(\mathcal{G}r) \xrightarrow[\sim]{P_{\mathrm{sph}}} \mathrm{D^b Coh}^{G\times\mathbb{G}_\mathrm{m}}_{\mathrm{perf}}(\mathcal{N}) \\
 & \cup \qquad\qquad\qquad \cup \\
\text{abelian} & \mathrm{Adv}_{(G_{\mathcal{O}})}(\mathcal{G}r) \xrightarrow{\sim} \mathrm{PCoh}(\mathcal{N}) \\
 & \cup \qquad\qquad\qquad \cup \\
\text{tilting} & \mathrm{Parity}_{(G_{\mathcal{O}})}(\mathcal{G}r) \xrightarrow{\sim} \mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N})) \\
\text{shifts} & \{1\} \longleftrightarrow \langle 1\rangle
\end{array}$$

The adverse t-structure on the constructible corresponds to the *perverse coherent t-structure* on $\mathrm{D^b Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$. The derived pushforward $\pi_*\colon \mathrm{D^b Coh}^{G\times\mathbb{G}_\mathrm{m}}(\tilde{\mathcal{N}}) \to \mathrm{D^b Coh}^{G\times\mathbb{G}_\mathrm{m}}(\tilde{\mathcal{N}}) \to \mathrm{D^b Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$ is t-exact for the exotic t-structure and the perverse coherent t-structure.

Unlike for spherical modules, the heart is no longer graded highest weight, but only graded properly stratified. However, it can be shown that the tilting and co-tilting objects coincide.

3.8. **Summary.** The various equivalences above (ignoring the anti-spherical side) are summarized in the following table.

constructible side	index set	coherent side	module/subalgebra
$\mathrm{D}^{\mathrm{b,m}}_{I}(\mathcal{F}\ell), \langle 1\rangle$	$W_{\mathrm{aff}} = \mathbf{X} \rtimes W$	$\mathrm{D^b Coh}^{G\times\mathbb{G}_\mathrm{m}}(\tilde{\mathcal{N}} \times^L_{\mathfrak{g}} \tilde{\mathcal{N}}), \langle 1\rangle$	$\mathcal{H}_{\mathrm{aff}}$
$\mathrm{D}^{\mathrm{b,m}}_{(I)}(\mathcal{G}r), \{1\}$	$\mathbf{X} \xrightarrow{\sim} W_{\mathrm{aff}}/W$	$\mathrm{D^b Coh}^{G\times\mathbb{G}_\mathrm{m}}(\tilde{\mathcal{N}}), \langle 1\rangle$	M_{sph}
$\mathrm{D}^{\mathrm{b,m}}_{(G^\vee{}_{\mathcal{O}})}(\mathcal{G}r), \{1\}$	$\mathbf{X}^+ \xrightarrow{\sim} W\backslash W_{\mathrm{aff}}/W$	$\mathrm{D^b Coh}^{G\times\mathbb{G}_\mathrm{m}}_{\mathrm{perf}}(\mathcal{N}), \langle 1\rangle$	$\mathcal{H}_{\mathrm{sph}}$

3.9. **Other related equivalences.** For example, we may consider $\mathrm{D}^{\mathrm{b,m}}_{(I)}(\mathcal{F}\ell)$, categorifying the right regular representation of $\mathcal{H}_{\mathrm{aff}}$. In this case, we get an equivalence of module categories over the monoidal equivalence for the affine Hecke algebra.

4. Motivation from representation theory

4.1. **Representations of quantum group at root of unity.** The spherical module categorification [ABG04] has a third part. Let U_ϵ be the quantum group of G at an ℓ-th root of unity, for ℓ odd. Let $\operatorname{Rep} U_\epsilon$ be the abelian category of finite-dimensional representations of $\operatorname{Rep} U_\epsilon$, and $\operatorname{Rep}_0 U_\epsilon$ the principal block (the block containing the trivial representation).

Theorem 4.1 ([ABG04]). *There are triangulated equivalences*

$$\mathrm{D}^{\mathrm{b}}_{(I)}(\mathcal{G}r,\mathbb{C}) \cong \mathrm{D}^{\mathrm{b}}\mathrm{Coh}^{G}(\tilde{\mathcal{N}}) \cong \mathrm{D}^{\mathrm{b}}\operatorname{Rep}_0 U_\epsilon.$$

Moreover, the perverse t-structure on the left corresponds to the standard t-structure on the right, so

$$\mathrm{Perv}_{(I)}(\mathcal{G}r,\mathbb{C}) \cong \operatorname{Rep}_0 U_\epsilon.$$

Like the BGG category $\mathcal{O}$ of a semisimple complex Lie algebra, $\operatorname{Rep}_0 U_\epsilon$ has a distinguished collection of objects L_w (simples), Δ_w (standards, analogue of Verma), ∇_w (costandards), T_w (tiltings), but now labeled by $w \in W_{\mathrm{aff}}/W$. As with $\mathcal{O}$, the characters of Δ_w and ∇_w are easy to calculate (now given by the Weyl character formula), and a basic problem is to describe the characters of L_w, or what turns out to be stronger, the characters of T_w. The equivalence above and affine parabolic Koszul duality turns this into a question about the dimension of stalks of Iwahori–Whittaker sheaves on $\mathcal{G}r$, which are known to be given by (parabolic) affine Kazhdan–Lusztig polynomials.

Remark 4.2. There is another way to relate $\operatorname{Rep} U_\epsilon$ to perverse sheaves. Yet another result of Kazhdan and Lusztig (usually called Kazhdan–Lusztig equivalence) relates $\operatorname{Rep} U_\epsilon$ to representations of an affine Lie algebra at a negative level, expressed in terms of ℓ. Via affine Beilinson–Bernstein localization (due to Kashiwara–Tanisaki) and Riemann–Hilbert correspondence, this is equivalent to perverse sheaves on a certain $\mathbb{G}_{\mathrm{m}}$-torsor over $\mathcal{F}\ell$, with prescribed monodromy along the fibers corresponding to the level. I'm not sure if it's known how to directly relate these two categories of perverse sheaves. See the introduction to [ABG04] for a bit more detail.

Bezrukavnikov used this to study the cohomlogy of tilting modules oer quantum groups in [Bez06].

4.2. **Lusztig's philosophy.** [Some of what's written here is my guesses based on talking to people, so don't quote me! You can find this philisophy, at least on the character level, in the two short papers [Lus14, LW15].]

According to a recent philosophy by Lusztig, for each G and p, there should be an infinite sequence of representation categories

$$\begin{array}{ccccc} \mathcal{A}_0 & \mathcal{A}_1 & \mathcal{A}_2 & \cdots & \mathcal{A}_\infty \\ \| & \| & \| & & \| \\ \operatorname{Rep} G/\mathbb{C} & \operatorname{Rep} U_\epsilon & ? & \cdots & \operatorname{Rep} G/\overline{\mathbb{F}_p} \end{array}$$

where all categories are abelian, but only $\mathcal{A}_0$ is semisimple. For $\mathcal{A}_1$ and on, the principal block should be highest weight for the poset W_{aff}/W, so that each category contains a distinguished collection of objects L_w, Δ_w, ∇_w, T_w labeled by W_{aff}/W. As before, the characters of Δ_w and ∇_w are easy, and a basic question is to describe the characters of L_w and T_w.

[For the sense in which $\mathcal{A}_\infty$ is the limit of $\mathcal{A}_n$, recall the Steinberg tensor product theorem (for simples) resp. the Donkin tensor product theorem (for tiltings).]

4.3. **Finkelberg–Mirković conjecture.** The equivalence of Arkhipov–Bezrukavnikov–Ginzburg thus solves this basic question for $\mathcal{A}_1$, a category over $\mathbb{C}$, by relating it to the geometry of $\mathcal{G}r$. The Finkelberg–Mirković conjecture states that $\mathcal{A}_\infty$ should be described by the geometry of $\mathcal{G}r$, still over $\mathbb{C}$, but now using sheaves with coefficients in positive characteristic.

Conjecture 4.1 (Finkelberg–Mirković)**.** *Let G be a reductive group over $\overline{\mathbb{F}_p}$. There is an equivalence of highest weight categories*

$$\mathrm{Perv}_{(I)}(\mathcal{G}r, \overline{\mathbb{F}_p}) \cong \mathrm{Rep}_0\, G$$

compatible with geometric Satake.

[need to explain the compatibility with geometric Satake]

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