CENTRALISERS OF SEMISIMPLE ELEMENTS IN $\mathrm{SO}_5(k)$

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1. Introduction and Definition

Let $q = p^a$ for some odd prime p and $K = \overline{\mathbb{F}_q}$ be an algebraically closed field of characteristic p. We recall our definition of the Special Orthogonal Group from Example 1.3.15 of [Gec03]. We start by defining the following matrix, associated to the standard orthogonal form on $\mathbb{R}^{2n}$

$$\Omega = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & 1 & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix} = \begin{bmatrix} & & Q \\ & 1 & \\ Q & & \end{bmatrix}$$

Then we define the Orthogonal Group $\mathrm{O}_5(K)$ as the set $\{X \in \mathrm{Mat}_5(K) \mid X^T\Omega X = \Omega\}$, i.e. all 5×5 matrices over K that preserve our chosen orthogonal form. The Special Orthogonal Group is defined as $\mathrm{SO}_5(K) = \mathrm{O}_5(K) \cap \mathrm{SL}_5(K)$ and is a normal index 2 subgroup of $\mathrm{O}_5(K)$.

The standard split torus in $\mathrm{SO}_5(K)$ has the form

$$T_0 = \left\{ \begin{bmatrix} d_1 & & & & \\ & d_2 & & & \\ & & 1 & & \\ & & & d_2^{-1} & \\ & & & & d_1^{-1} \end{bmatrix} \middle| \, \alpha, \beta \in K^\times \right\}.$$

In this short discussion we would like to determine the centralisers of semisimple elements in T_0 and consider when these centralisers are connected subgroups of $\mathrm{SO}_5(K)$. We recall that the conjugacy classes of semisimple elements in $\mathrm{SO}_5(K)$ are in bijection with the orbits of the Weyl group acting on the maximal torus T_0. If $s, s' \in T_0$ are conjugate semisimple elements, so $s' = s^g$ for some $g \in G$, then we will have that $C_G(s') = C_G(s^g) = C_G(s)^g$. Hence we find it sufficient to determine the centraliser for only one representative from each conjugacy class.

2. Explicitly Determining the Centralisers

We can write an element $s \in T_0$ as a quintuple $(d_1, d_2, 1, d_2^{-1}, d_1^{-1})$. In fact it would be sufficient to only list the first two elements but for clarity we will write all the entries. We can see that there will be infinitely many conjugacy classes of semisimple elements in G. However these fall neatly into certain special cases and we give a representative for each such case below. Note that all the entries in $(d_1, d_2, 1, d_2^{-1}, d_1^{-1})$ are assumed to be pairwise distinct unless otherwise stated.

(i) $(d_1, d_2, 1, d_2^{-1}, d_1^{-1})$.
(ii) $(d_1, d_1, 1, d_1^{-1}, d_1^{-1})$.
(iii) $(1, d_2, 1, d_2^{-1}, 1)$.
(iv) $(-1, d_2, 1, d_2^{-1}, -1)$.
(v) $(1, -1, 1, -1, 1)$.
(vi) $(-1, -1, 1, -1, -1)$.
(vii) $(1, 1, 1, 1, 1)$.

Let s be one of the seven elements listed above. We wish to consider the centraliser of s, denoted $C_G(s)$, in each of these cases. We start by considering a general matrix $a = (a_{ij}) \in \mathrm{Mat}_5(k)$ and determine what conditions the relation $as = sa$ puts on the entries a_{ij}. We then determine extra conditions so that $a \in \mathrm{SO}_5(K)$ and hence $a \in C_G(s)$. Consider $s = (d_1, d_2, 1, d_2^{-1}, d_1^{-1})$ then we will have the condition $as = sa$ is such that

$$(\dagger)\quad \begin{bmatrix} a_{11}d_1 & a_{12}d_2 & a_{13} & a_{14}d_2^{-1} & a_{15}d_1^{-1} \\ a_{21}d_1 & a_{22}d_2 & a_{23} & a_{24}d_2^{-1} & a_{25}d_1^{-1} \\ a_{31}d_1 & a_{32}d_2 & a_{33} & a_{34}d_2^{-1} & a_{35}d_1^{-1} \\ a_{41}d_1 & a_{42}d_2 & a_{43} & a_{44}d_2^{-1} & a_{45}d_1^{-1} \\ a_{51}d_1 & a_{52}d_2 & a_{53} & a_{54}d_2^{-1} & a_{55}d_1^{-1} \end{bmatrix} = \begin{bmatrix} a_{11}d_1 & a_{12}d_1 & a_{13}d_1 & a_{14}d_1 & a_{15}d_1 \\ a_{21}d_2 & a_{22}d_2 & a_{23}d_2 & a_{24}d_2 & a_{25}d_2 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41}d_2^{-1} & a_{42}d_2^{-1} & a_{43}d_2^{-1} & a_{44}d_2^{-1} & a_{45}d_2^{-1} \\ a_{51}d_1^{-1} & a_{52}d_1^{-1} & a_{53}d_1^{-1} & a_{54}d_1^{-1} & a_{55}d_1^{-1} \end{bmatrix}.$$

We note that all the d_i are non-zero as they are elements of $K^\times$. Now considering the individual cases listed above we have the following.

(i) If every entry is distinct then we must have $a_{ij} = 0$ for all $i \neq j$ and hence we have that a is a diagonal matrix. Clearly $a \in \mathrm{SO}_5(K)$ if and only if $a \in T_0$, therefore $C_G(s) = T_0$. A maximal torus is always a connected subgroup and so we have $C_G(s)$ is connected.

(ii) Now $d_2 = d_1 \neq \pm 1$ then from $(\dagger)$ we have the following relations

$$\begin{aligned} a_{12}d_2 &= a_{12}d_1 & \qquad a_{45}d_1^{-1} &= a_{45}d_2^{-1}, \\ a_{21}d_1 &= a_{21}d_2 & \qquad a_{54}d_2^{-1} &= a_{54}d_1^{-1}. \end{aligned}$$

This tells us that a_{12}, a_{21}, a_{45} and a_{54} can be non-zero elements. Hence we need to know when the matrix

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

lies in $\mathrm{SO}_5(k)$. We consider this as a block matrix and see what the defining relation for $\mathrm{O}_5(K)$ tells us. For a to be in $\mathrm{O}_5(K)$ we must have $a^T\Omega a = \Omega$ or in other words

$$\begin{bmatrix} C^T & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & D^T \end{bmatrix} \begin{bmatrix} 0 & 0 & Q \\ 0 & 1 & 0 \\ Q & 0 & 0 \end{bmatrix} \begin{bmatrix} C & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 0 & Q \\ 0 & 1 & 0 \\ Q & 0 & 0 \end{bmatrix},$$
$$\Rightarrow \begin{bmatrix} 0 & 0 & C^TQD \\ 0 & x^2 & 0 \\ D^TQC & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & Q \\ 0 & 1 & 0 \\ Q & 0 & 0 \end{bmatrix}.$$

So this tells us that $C^TQD = Q \Rightarrow D = Q(C^{-1})^TQ$ and $x^2 = 1 \Rightarrow x = \pm 1$. Now the matrix will be an element of $\mathrm{SO}_5(K)$ if and only if $x = 1$ and $C \in \mathrm{GL}_2(K)$. Hence we have that the centraliser will be

$$C_G(s) = C_G(s)^\circ = \left\{ \begin{bmatrix} C & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & Q(C^{-1})^TQ \end{bmatrix} \middle| \; C \in \mathrm{GL}_2(K) \right\}.$$

(iii) Now $d_1 = d_1^{-1} = 1$ then from (†) we have the following relations

$$\begin{aligned} a_{15} &= a_{15} & a_{51} &= a_{51} & a_{13} &= a_{13}, \\ a_{31} &= a_{31} & a_{53} &= a_{53} & a_{35} &= a_{35}. \end{aligned}$$

This tells us that $a_{15}, a_{51}, a_{13}, a_{31}, a_{53}$ and a_{35} can be non-zero entries. Hence we need to know when the following matrix

$$\begin{bmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} \\ 0 & a_{22} & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & a_{35} \\ 0 & 0 & 0 & a_{44} & 0 \\ a_{51} & 0 & a_{53} & 0 & a_{55} \end{bmatrix}$$

lies in $\mathrm{SO}_5(K)$. The defining equation for $\mathrm{O}_5(k)$ is $a^T\Omega a = \Omega$. We first recall that $(a^T\Omega)_{ij} = a_{j,2n+2-i}$. So this defining condition gives us that

$$\begin{bmatrix} a_{51} & 0 & a_{31} & 0 & a_{11} \\ 0 & 0 & 0 & a_{22} & 0 \\ a_{53} & 0 & a_{33} & 0 & a_{13} \\ 0 & a_{44} & 0 & 0 & 0 \\ a_{55} & 0 & a_{35} & 0 & a_{15} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} \\ 0 & a_{22} & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & a_{35} \\ 0 & 0 & 0 & a_{44} & 0 \\ a_{51} & 0 & a_{53} & 0 & a_{55} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Removing redundant equations we obtain a system of equations

$$2a_{11}a_{51} + a_{31}^2 = 0 \qquad a_{11}a_{53} + a_{31}a_{33} + a_{51}a_{13} = 0,$$
$$2a_{55}a_{15} + a_{35}^2 = 0 \qquad a_{13}a_{55} + a_{33}a_{35} + a_{53}a_{15} = 0,$$
$$2a_{53}a_{13} + a_{33}^2 = 1 \qquad a_{51}a_{15} + a_{31}a_{35} + a_{11}a_{55} = 1,$$
$$a_{44}a_{22} = 1.$$

These equations cannot be into independent sets. This means that all the entries of the matrix are dependent upon each other, which tells us that the centraliser will be connected. Therefore we will have

$$C_G(s) = C_G(s)^\circ = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} \\ 0 & a_{22} & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & a_{35} \\ 0 & 0 & 0 & a_{44} & 0 \\ a_{51} & 0 & a_{53} & 0 & a_{55} \end{bmatrix} \right\}.$$

(iv) Now $d_1 = d_1^{-1} = -1$ then from (†) we have the following relations

$$-a_{15} = -a_{15} \qquad -a_{51} = -a_{51} \qquad a_{13} = -a_{13},$$
$$-a_{31} = a_{31} \qquad a_{53} = -a_{53} \qquad -a_{35} = a_{35}.$$

This tells us that a_{15}, a_{51} can be non-zero entries but $a_{13} = a_{31} = a_{53} = a_{35} = 0$. Hence we need to know when the matrix

$$\begin{bmatrix} a_{11} & 0 & 0 & 0 & a_{15} \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ a_{51} & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

lies in $\mathrm{SO}_5(K)$. The defining equation for $\mathrm{O}_5(k)$ is $a^T\Omega a = \Omega$. We again recall that $(a^T\Omega)_{ij} = a_{j,2n+2-i}$. So this defining condition gives us that

$$\begin{bmatrix} a_{51} & 0 & 0 & 0 & a_{11} \\ 0 & 0 & 0 & a_{22} & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & a_{44} & 0 & 0 & 0 \\ a_{55} & 0 & 0 & 0 & a_{15} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 & 0 & a_{15} \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 \\ a_{51} & 0 & 0 & 0 & a_{55} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Removing redundant equations we obtain a system of equations

$$a_{11}a_{51} = 0 \qquad a_{51}a_{15} + a_{11}a_{55} = 1 \qquad a_{33}^2 = 1,$$
$$a_{55}a_{15} = 0 \qquad a_{44}a_{22} = 1.$$

Solving this system of equations tells us that we have either $a_{11} = a_{55} = 0$ or $a_{15} = a_{51} = 0$. This means we have two distinct possibilities for the centraliser. We can see that the centraliser will be disconnected of the form

$$C_G(s) = \left\{ \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_{22}^{-1} & 0 \\ 0 & 0 & 0 & 0 & a_{11}^{-1} \end{bmatrix} \right\} \bigsqcup \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 & a_{15} \\ 0 & a_{22} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & a_{22}^{-1} & 0 \\ a_{15}^{-1} & 0 & 0 & 0 & 0 \end{bmatrix} \right\}.$$

(v) Now $d_1 = d_1^{-1} = 1$ and $d_2 = d_2^{-1} = -1$ then from (†) we have the following relations

$$\begin{array}{llll} a_{13} = a_{13} & a_{31} = a_{31} & a_{53} = a_{53} & a_{35} = a_{35}, \\ a_{23} = -a_{23} & -a_{32} = a_{32} & a_{43} = -a_{43} & -a_{34} = a_{34}, \\ a_{51} = a_{51} & -a_{42} = -a_{42} & a_{15} = a_{15} & -a_{24} = a_{24}. \end{array}$$

This tells us that $a_{13}, a_{31}, a_{53}, a_{35}, a_{51}, a_{42}, a_{15}$ can be non-zero entries but $a_{23} = a_{32} = a_{43} = a_{34} = a_{24} = 0$. Hence we need to know when the following matrix

$$\begin{bmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} \\ 0 & a_{22} & 0 & a_{24} & 0 \\ a_{31} & 0 & a_{33} & 0 & a_{35} \\ 0 & a_{42} & 0 & a_{44} & 0 \\ a_{51} & 0 & a_{53} & 0 & a_{55} \end{bmatrix}$$

lies in $\mathrm{SO}_5(k)$. We check the defining equation $a^T \Omega a = \Omega$ for $\mathrm{O}_5(K)$ for this matrix

$$\begin{bmatrix} a_{51} & 0 & a_{31} & 0 & a_{11} \\ 0 & a_{42} & 0 & a_{22} & 0 \\ a_{53} & 0 & a_{33} & 0 & a_{13} \\ 0 & a_{44} & 0 & a_{24} & 0 \\ a_{55} & 0 & a_{35} & 0 & a_{15} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} \\ 0 & a_{22} & 0 & a_{24} & 0 \\ a_{31} & 0 & a_{33} & 0 & a_{35} \\ 0 & a_{42} & 0 & a_{44} & 0 \\ a_{51} & 0 & a_{53} & 0 & a_{55} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Removing redundant equations we obtain a system of equations

$$\begin{aligned} a_{11}a_{53} + a_{13}a_{51} + a_{31}a_{33} &= 0 & a_{55}a_{11} + a_{35}a_{31} + a_{15}a_{51} &= 1, \\ a_{13}a_{55} + a_{15}a_{53} + a_{33}a_{35} &= 0 & a_{44}a_{22} + a_{24}a_{42} &= 1, \\ 2a_{15}a_{55} + a_{35}^2 &= 0 & 2a_{13}a_{53} + a_{33}^2 &= 1, \\ 2a_{11}a_{51} + a_{31}^2 &= 0, \\ 2a_{42}a_{22} &= 0, \\ 2a_{24}a_{44} &= 0. \end{aligned}$$

The final two equations give us two possibilities. We either have $a_{22} = a_{44} = 0$ and $a_{42} = a_{24}^{-1}$ or $a_{24} = a_{42} = 0$ and $a_{44} = a_{22}^{-1}$. Therefore these split the centraliser of s into two distinct cases. Hence the centraliser will be disconnected of the form

$$C_G(s) = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} \\ 0 & a_{22} & 0 & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 & a_{35} \\ 0 & 0 & 0 & a_{22}^{-1} & 0 \\ a_{51} & 0 & a_{53} & 0 & a_{55} \end{bmatrix} \right\} \bigsqcup \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} & 0 & a_{15} \\ 0 & 0 & 0 & a_{24} & 0 \\ a_{31} & 0 & a_{33} & 0 & a_{35} \\ 0 & a_{24}^{-1} & 0 & 0 & 0 \\ a_{51} & 0 & a_{53} & 0 & a_{55} \end{bmatrix} \right\}.$$

(vi) Now $d_1 = d_2 = d_1^{-1} = d_2^{-1} = -1$ and we can see from (†) that an element of the centraliser will have the form

$$\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} & a_{15} \\ a_{21} & a_{22} & 0 & a_{24} & a_{25} \\ 0 & 0 & a_{33} & 0 & 0 \\ a_{41} & a_{42} & 0 & a_{44} & a_{45} \\ a_{51} & a_{52} & 0 & a_{54} & a_{55} \end{bmatrix}.$$

We do not calculate what it means for a matrix of this form to be an element of $\mathrm{O}_5(K)$ as this would be too messy. In fact it is not easy at all to see from the relations that the centraliser is in fact disconnected. We instead comment that it will be explained in the next section.

(vii) It's clear that $C_G(s) = \mathrm{SO}_5(K)$ in this case and hence is connected.

3. Determining Centralisers From Theory

Let $\Phi = \Phi^+ \sqcup \Phi^-$ be the root system of $\mathrm{SO}_5(K)$. There are eight roots in total for $\mathrm{SO}_5(K)$ and these are given as the morphisms

$$\begin{aligned} \alpha(s) &= d_1 d_2^{-1} & -\alpha(s) &= d_1^{-1} d_2 \\ \beta(s) &= d_2 & -\beta(s) &= d_2^{-1} \\ (\alpha+\beta)(s) &= d_1 & (-\alpha-\beta)(s) &= d_1^{-1} \\ (\alpha+2\beta)(s) &= d_1 d_2 & (-\alpha-2\beta)(s) &= d_1^{-1} d_2^{-1}. \end{aligned}$$

Clearly the left hand column are the positive roots, the right hand column are the corresponding negative roots and we have fixed a system of fundamental roots $\Pi = \{\alpha, \beta\}$. Now to each root $\alpha \in \Phi$ there is a corresponding root subgroup $X_\alpha \leqslant R_U(B_0)$ and reflection of the Weyl group $w_\alpha \in W = N_G(T_0)/T_0$. The root subgroups are minimal unipotent subgroups of the unipotent radical of the Borel subgroup containing T_0, which are normalised by the torus T_0.

The corresponding root subgroups of $\mathrm{SO}_5(k)$ are

$$X_\alpha = \left\{ \begin{bmatrix} 1 & \mu & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\mu^{-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\} \qquad X_{-\alpha} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \mu & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\mu^{-1} & 1 \end{bmatrix} \right\},$$

$$X_\beta = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\mu & -\frac{1}{2}\mu^2 & 0 \\ 0 & 0 & 1 & \mu & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\} \qquad X_{-\beta} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -\mu & 1 & 0 & 0 \\ 0 & -\frac{1}{2}\mu^2 & \mu & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\},$$

$$X_{\alpha+\beta} = \left\{ \begin{bmatrix} 1 & 0 & -\mu & 0 & -\frac{1}{2}\mu^2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \mu \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\} \qquad X_{-\alpha-\beta} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\mu & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{2}\mu^2 & 0 & \mu & 0 & 1 \end{bmatrix} \right\},$$

$$X_{\alpha+2\beta} = \left\{ \begin{bmatrix} 1 & 0 & 0 & \mu & 0 \\ 0 & 1 & 0 & 0 & \mu \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\} \qquad X_{-\alpha-2\beta} = \left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \mu & 0 & 0 & 1 & 0 \\ 0 & \mu & 0 & 0 & 1 \end{bmatrix} \right\},$$

where $\mu \in K$. Recall that the reflection w_α associated to a root α is such that ${}^{w_\alpha}\alpha = -\alpha$ and its representative $\dot{w}_\alpha$ is an element of $N_G(T_0) \cap \langle X_\alpha, X_{-\alpha} \rangle$.

Abstractly we can view the Weyl group as a subgroup of $\mathfrak{S}_5$, which makes it easy to see its action on the split torus T_0. We can show that

$$W \cong \{1, (15), (24), (12)(45), (14)(25), (15)(24), (1254), (1452)\} \leqslant \mathfrak{S}_5.$$

We choose two generators $s_1 = (12)(45)$ and $s_2 = (24)$, which correspond to the fundamental roots α and β. The reflections associated to $\alpha + \beta$ and $\alpha + 2\beta$ are the conjugates $s_1 s_2 s_1 = (15)$ and $s_2 s_1 s_2 = (14)(25)$ of the generators. Recall that the reflection w_α and $w_{-\alpha}$ are always the same. We can now give explicit matrix representatives for the elements of the Weyl group in $N_G(T_0)$.

$$\dot{1} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & m_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & m_1^{-1} \end{bmatrix} \qquad \dot{s_1} = \begin{bmatrix} 0 & m_1 & 0 & 0 & 0 \\ m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2^{-1} \\ 0 & 0 & 0 & m_1^{-1} & 0 \end{bmatrix},$$

$$\dot{s_2} = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2^{-1} & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_1^{-1} \end{bmatrix} \qquad s_1\dot{s_2} = \begin{bmatrix} 0 & m_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 \\ 0 & 0 & -1 & 0 & 0 \\ m_2^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_1^{-1} & 0 \end{bmatrix},$$

$$s_2\dot{s_1} = \begin{bmatrix} 0 & 0 & 0 & m_1 & 0 \\ m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2^{-1} \\ 0 & m_1^{-1} & 0 & 0 & 0 \end{bmatrix} \qquad s_2\dot{s_1}s_2 = \begin{bmatrix} 0 & 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & 0 & m_2 \\ 0 & 0 & 1 & 0 & 0 \\ m_2^{-1} & 0 & 0 & 0 & 0 \\ 0 & m_1^{-1} & 0 & 0 & 0 \end{bmatrix},$$

$$s_1\dot{s_2}s_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & m_1 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & m_2^{-1} & 0 \\ m_1^{-1} & 0 & 0 & 0 & 0 \end{bmatrix} \qquad s_1s_2\dot{s_1}s_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & m_1 \\ 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & m_2^{-1} & 0 & 0 & 0 \\ m_1^{-1} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We recall from the theory that if $s \in G$ is a semisimple element then

$$\begin{aligned} C_G(s) &= \langle T_0, X_\alpha, \dot{w} \mid \alpha \in \Phi, \alpha(s) = 1 \text{ and } s^{\dot{w}} = s \rangle, \\ C_G(s)^\circ &= \langle T_0, X_\alpha \mid \alpha \in \Phi, \alpha(s) = 1 \rangle. \end{aligned}$$

We now examine this for each of the cases we listed in the previous section.

(i) All entries are distinct, which means there exists no root $\alpha \in \Phi$ such that $\alpha(s) = 1$. The only element of W which commutes with s is the identity. Therefore we have

$$C_G(s) = C_G(s)^\circ = T_0.$$

(ii) Now $d_2 = d_1 \neq \pm 1$. Therefore the only roots that vanish on s are $\pm\alpha$. The only Weyl group element to commute with s is the element $\dot{s_1}$. Now $\dot{s_1} \in \langle X_\alpha, X_{-\alpha} \rangle$ and so is an element of $C_G(s)^\circ$. Therefore the centraliser is connected and we have

$$C_G(s) = C_G(s)^\circ = \langle T_0, X_\alpha, X_{-\alpha} \rangle.$$

(iii) Now $d_1 = 1$ and $d_2 \neq d_2^{-1}$. Therefore the only roots to vanish on s are $\pm(\alpha + \beta)$. The only Weyl group element to commute with s is the element $s_1\dot{s_2}s_1$, which is the reflection associated to $\alpha + \beta$. Therefore $s_1\dot{s_2}s_1 \in \langle X_{\alpha+\beta}, X_{-\alpha-\beta} \rangle$ and so is an element of $C_G(s)^\circ$. Therefore the centraliser is connected and we have

$$C_G(s) = C_G(s)^\circ = \langle T_0, X_{\alpha+\beta}, X_{-\alpha-\beta} \rangle.$$

(iv) Now $d_1 = -1$ and $d_2 \neq d_2^{-1}$. It's easy to see that no roots vanish on s. However the Weyl group element $s_1\dot{s_2}s_1$ commutes with s. Therefore the centraliser is disconnected and we have

$$C_G(s) = \langle T_0, s_1\dot{s_2}s_1\rangle \text{ and } C_G(s)^\circ = T_0.$$

(v) Now $d_1 = 1$ and $d_2 = -1$. Therefore the only roots to vanish on s are $\pm(\alpha+\beta)$. There are three Weyl group elements that commute with s namely $\dot{s_2}, s_1\dot{s_2}s_1$ and $s_1s_2\dot{s_1}s_2$. Now $s_1\dot{s_2}s_1$ is the reflection associated to the root $\alpha+\beta$ and hence $s_1\dot{s_2}s_1 \in \langle X_\alpha, X_{-\alpha}\rangle$. Therefore the centraliser will be disconnected and we have

$$C_G(s) = \langle T_0, X_{\alpha+\beta}, X_{-\alpha-\beta}, \dot{s_2}\rangle \text{ and } C_G(s)^\circ = \langle T_0, X_{\alpha+\beta}, X_{-\alpha-\beta}\rangle$$

(vi) Now $d_1 = d_2 = -1$. Therefore the only roots to vanish on s are $\pm\alpha$ and $\pm(\alpha+2\beta)$. However every Weyl group element will commute with s. Therefore the centraliser will be disconnected and we have

$$C_G(s) = \langle T_0, X_\alpha, X_{\alpha+2\beta}, X_{-\alpha}, X_{-\alpha-2\beta}, \dot{s_2}\rangle,$$
$$C_G(s)^\circ = \langle T_0, X_\alpha, X_{\alpha+2\beta}, X_{-\alpha}, X_{-\alpha-2\beta}\rangle.$$

(vii) Clearly every root vanishes on the identity and every Weyl commutes with the identity so we have $C_G(s) = C_G(s)^\circ = \mathrm{SO}_5(K)$.

Representative	Φ_1	Φ_1 Type	$C_G(s)/C_G(s)^\circ$
(d_1, d_2)	$\varnothing$	$\varnothing$	$\{1\}$
(d_1, d_1)	$\{\pm\alpha\}$	A_1	$\{1\}$
$(1, d_2)$	$\{\pm(\alpha+\beta)\}$	B_1	$\{1\}$
$(-1, d_2)$	$\varnothing$	$\varnothing$	$\langle s_1s_2s_1\rangle$
$(1, -1)$	$\{\pm(\alpha+\beta)\}$	B_1	$\langle s_2\rangle$
$(-1, -1)$	$\{\pm\alpha, \pm(\alpha+2\beta)\}$	$\mathrm{A}_1 \times \mathrm{A}_1$	$\langle s_2\rangle$
$(1, 1)$	Φ	B_2	$\{1\}$

Table 1. The Centralisers of Semisimple Elements in $\mathrm{SO}_5(K)$.

By comparing what we have determined in this section with the matrix calculations we did in the previous section, it is easy to see how the two compare. Now $C_G(s)^\circ$ is a connected reductive subgroup of $\mathrm{SO}_5(K)$ and so it has a root system which will be an additively closed subroot system $\Phi_1 \subseteq \Phi$. By additively closed we mean that if $\alpha, \alpha' \in \Phi_1$ such that $\alpha+\alpha' \in \Phi$ then $\alpha+\alpha' \in \Phi_1$. In fact clearly we will have $\Phi_1 = \{\alpha \in \Phi \mid \alpha(s) = 1\}$. We summarise the information given so far in table 1.

Note that when we list the representatives in table 1 we list them in an abbreviated form as a pair (d_1, d_2). We make a distinction between A_1 and B_1 type root systems, using

A_1 if the root is long and B_1 if the root is short. Also the Weyl group elements listed in $C_G(s)/C_G(s)^\circ$ are coset representatives.

References

[Gec03] Meinolf Geck. *An Introduction to Algebraic Geometry and Algebraic Groups*. Number 10 in Oxford Graduate Texts in Mathematics. Oxford University Press, first edition, 2003.