Deligne-Lusztig curves

Drinfeld, at the age of 20, discovered that the discrete series representations of the finite group $SL_2(\mathbb{F}_q)$ can be realized in the ℓ-adic cohomology of the curve defined over $\mathbb{F}_q$,

$$xy^q - x^q y = 1.$$

Deligne and Lusztig were inspired by this result to associate algebraic varieties to any finite group of Lie type and were extremely successful in using them to construct all representations of such a finite group. We will explain their beautiful ideas and supply concrete examples of Deligne-Lusztig curves. These curves themselves also enjoy extremal geometric and arithmetic properties, which, among others, lead us to contemplate on the answer to life, the universe and everything.

This is a note I prepared for my third Trivial Notions talk at Harvard, Fall 2013. Our main sources are [1], [2], [3], [4], [5] and [6].

[-] **Contents**

My talk consists of two parts. In the first part I shall define our trivial notion in the title. In second part I shall explain how this definition makes sense.

Definition 1 A Deligne-Lusztig curve is a Deligne-Lusztig variety of dimension 1.

This completes the first part of my talk. Surprisingly we still have enough time to do the second part. Now let us start with a completely different story.

Finite groups of Lie types

According to the classification of finite simple groups, except 26 sporadic groups, all finite simple groups fit into three infinite series

- a cyclic group $\mathbb{Z}/p\mathbb{Z}$ (of prime order);
- an alternating group A_n ($n \geq 5$);
- a simple finite group of Lie type.

A finite group of Lie type, as you can imagine, is a finite group analogue of Lie groups over real or complex numbers.

Example 1 The most basic finite group of Lie type is $G = SL_2(\mathbb{F}_q)$, where $q = p^r$ is a prime power. It is the fixed points of the standard Frobenius endomorphism $\mathrm{Frob}_q : SL_2(k) \to SL_2(k)$ by raising each entry to the q-th power, where k is an algebraic closure of $\mathbb{F}_q$.

Definition 2 A *finite group of Lie type* is the fixed points G^F of an endomorphism $F : G \to G$, where G is any reductive group over k and a certain power of F is a standard Frobenius Frob_q.

If you never care about algebraic varieties in positive characteristics, this probably serves as a reason that you probably should. After all these finite groups of Lie type form the major bulk of the building blocks of any finite group!

People familiar with Lie groups shall recall that the reductive groups over an algebraically closed field k are classified by certain combinatoric data called *root data*, which are, roughly speaking, instructions telling you how to glue the building blocks SL_2 (and tori) to obtain G. These fit into 4 infinite series A_n, B_n, C_n, D_n and five exceptional ones E_6, E_7, E_8, F_4, G_2 according to the associated Dynkin diagram (please refer to the picture on the wall outside 507 if you have good eyesight). For example, A_{n-1} consists of the general linear group GL_n and its variations SL_n, PGL_n, and so on.

Example 2 Suppose $G=SL_n$. For $F=\mathrm{Frob}_q$, we obtain $G^F=SL_n(q)$. The quotient by its center $PSL_n(q)=SL_n(q)/\mu_n$ is simple except $n=2$ and $q=2,3$.

Example 3 Suppose $G=SL_n$. For $F(g)={}^t\mathrm{Frob}_q(g)^{-1}$, we obtain $F^2=\mathrm{Frob}_{q^2}$ and $G^F=SU_n(q^2)=SU_n(\mathbb{F}_{q^2}/\mathbb{F}_q)$ is the (unique) unitary group over $\mathbb{F}_q$ ($n\ge 3$). It is usually denoted by ${}^2A_{n-1}(q^2)$. The endomorphism F induces the involution of the Dynkin diagram A_{n-1}.

In general, such a endomorphism F induces an automorphism of the Dynkin diagram with arrow disregarded. Besides the $\mathbb{F}_q$-points of the split groups (a.k.a., *Chevalley groups*), we have new series of finite groups of Lie type:

- ${}^2A_n(q^2)$, ${}^2D_n(q^2)$, ${}^3D_4(q^3)$, ${}^2E_6(q^2)$. These are $\mathbb{F}_q$-points of the quasi-split forms of A_n, D_n, D_4, E_6, known as *Steinberg groups*;
- ${}^2B_2(q^2)$: *Suzuki groups*, exist only when $q^2=2^{2k+1}$;
- ${}^2G_2(q^2)$: *Ree groups of type* G_2, exist only when $q^2=3^{2k+1}$;
- ${}^2F_4(q^2)$: *Ree groups of type* F_4, exist only when $q^2=2^{2k+1}$.

For the last three groups, the involution on the Dynkin diagram does not preserve the length of the roots and they are not $\mathbb{F}_q$-points of any reductive groups! (You may want to think of them as points of a reductive group defined over a field of $\sqrt{2^{2k+1}}$ or $\sqrt{3^{2k+1}}$ elements, which of course does not make sense).

Now I have described the classification of finite groups of Lie type. You may find it interesting or simply don't care. But it will certainly become more interesting when a finite group G acts on objects that you care more about, e.g., topological spaces, manifolds, algebraic varieties... Linearizing such an action gives rise to a linear representation of G. So how can we understand all the irreducible representations of G when G is a finite group of Lie type? Let us consider the simplest (but already rich enough) example $G=SL_2(q)$.

Representations of $SL_2(q)$

Over the complex numbers, the representation theory of any finite group G is rather clean: any finite dimensional representation of G decomposes as a direct sum of irreducible subrepresentations. A representation π is characterized by its character $\chi_\pi: G\to\mathbb{C}: g\mapsto \mathrm{tr}\,\pi(g)$. π is irreducible if and only if $\langle\chi_\pi,\chi_\pi\rangle=1$. The number of irreducible representations of G is the same as the number of conjugacy classes of G. And so on.

The conjugacy classes of $SL_2(q)$ can be classified using elementary methods. For simplicity we shall assume that q is odd. The representatives can be chosen as

representatives		number of classes
$\begin{pmatrix}\pm1&0\\0&\pm1\end{pmatrix}$		2
$\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}$	$a\in\mathbb{F}_q^\times-\{\pm1\}, a\sim a^{-1}$	$\frac{q-3}{2}$
$\zeta\in\mu_{q+1}$	$\zeta\in\mu_{p+1}-\{\pm1\},\zeta\sim\zeta^{-1}$	$\frac{q-1}{2}$
$\begin{pmatrix}a&b\\0&a^{-1}\end{pmatrix}$	$a=\pm1,\ b\in\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$	4

Notice $\mathbb{F}_{q^2}$ acts on $\mathbb{F}_{q^2}$ and produces a nonsplit torus $\mathbb{F}_{q^2}^\times\subseteq GL_2(q)$. Restricting to the norm one elements, we obtain $\mu_{q+1}=\ker(\mathbb{N}:\mathbb{F}_{q^2}^\times\to\mathbb{F}_q)\subseteq SL_2(q)$, a nonsplit torus of order $q+1$. Over $\mathbb{F}_{q^2}$, its elements are conjugate to elements of the form $\begin{pmatrix}\zeta&0\\0&\zeta^q\end{pmatrix}$.

So in total we have $2+(q-3)/2+(q-1)/2+4=q+4$ conjugacy classes. How do we construct $q+4$ irreducible representations of $SL_2(q)$? One usual way to build the character table of any finite group G is to try to induce known representations of subgroups of G. A nice subgroup is given by the split diagonal torus $T\cong\mathbb{F}_q^\times\subseteq SL_2(q)$, which is a cyclic group of order $q-1$. The irreducible representations of T are simply the $q-1$ characters $\theta:T\to\mathbb{C}^\times$. But T itself is too small which makes $\mathrm{Ind}_T^G\theta$ huge and far from irreducible. Instead we can view θ as a character on the Borel subgroup $B=\{\begin{pmatrix}a&b\\0&a^{-1}\end{pmatrix}\}$ (trivial on the unipotent subgroup $U=\{\begin{pmatrix}1&b\\0&1\end{pmatrix}\}$). The resulting representation produced this way is called a *parabolic induction*.

- When $\theta^2\ne1$, $\mathrm{Ind}_B^G\theta$ is irreducible of dimension $q+1$ (*principal series representations*).

- When $\theta^2 = \mathbf{1}$ but $\theta \neq \mathbf{1}$, $\mathrm{Ind}_B^G \theta$ is a direct sum of two irreducible representations of dimension $(q+1)/2$ (*half principal series representations*).
- When $\theta = \mathbf{1}$, $\mathrm{Ind}_B^G \theta$ is a direct sum of the trivial representation and an irreducible representation of dimension q (*the Steinberg representation*).

This gives us $(q-3)/2+2+2=(q+5)/2$ irreducible representations and there are $(q+3)/2$ (about half) left to be discovered. It turns out $(q-1)/2$ of them has dimension $q-1$ (*discrete series representations*) and 2 of them has dimension $(q-1)/2$ (*half discrete series representations*). Looking at the number $(q-1)/2$, one is tempted to induce a character of μ_{q+1} to construct the rest, but since there is no Borel subgroup containing it, there is no parabolic induction and the naive induction is more complicated than our expectation. Of course, one can mess around the character table and construct the discrete series representations using brutal force; but life would be harder for groups other than $SL_2(q)$.

The Drinfeld curve

If you are Drinfeld, then at this stage you must have realized that the right thing to look at is the affine curve $Y/\mathbb{F}_q$ defined by the equation

$$xy^q - x^q y = 1.$$

What is nice about it? First of all, the group μ_{q+1} acts on it by $\zeta\cdot(x,y)=(\zeta x,\zeta y)$. More interestingly, it admits the action of $SL_2(q)$ given by the linear transformation $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Indeed, $\left(\begin{smallmatrix} ax+by & (ax+by)^q \\ cx+dy & (cx+dy)^q \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} x & x^q \\ y & y^q \end{smallmatrix}\right)$ by characteristic p miracle and so fixes the determinant $\det\left(\begin{smallmatrix} x & x^q \\ y & y^q \end{smallmatrix}\right) = xy^q - x^q y = 1$! These two actions commute with each other and produce a large group of automorphisms of Y.

Remark 1 The compactification $\overline{Y}$ of the curve Y has genus $g = q(q-1)/2$. But $|SL_2(q)| = q(q+1)(q-1)$ (indeed, a large group of order $\sim q^5$ acts on $\overline{Y}$) grows much quickly than g! So when q is large, Y is an example of curves in positive characteristic violating the Hurwitz bound $\mathrm{Aut}(X) \le 42(2g-2)$ for curves X with $g(X)\ge 2$ in characteristic 0. So the answer to life, the universe and everything may be something different if the universe has positive characteristic.

Let $H_c^i(Y) = H_c^i(Y, \overline{\mathbb{Q}}_l)$ be ℓ-adic etale cohomology groups with compact support of Y. Then the group $\mu_{q+1}\times SL_2(q)$ naturally acts on $H_c^i(Y)$.

Definition 3 Let $\theta : \mu_{q+1} \to \mathbb{C}^\times$ be a character. We define the *Deligne-Lusztig induction* of θ to be $R(\theta) = H_c^*(Y)[\theta] = \sum_i (-1)^i H_c^i(Y)[\theta]$, a virtual character of $SL_2(q)$. Here $H_c^i(Y)[\theta]$ denotes the θ-isotypic component of $H_c^i(Y)$.

So we successfully "induce" a character of μ_{q+1} to obtain a (virtual) character of $SL_2(q)$. It behaves much nicer than the naive induction from μ_{q+1} to $SL_2(q)$.

Example 4 Consider the trivial character $\theta = \mathbf{1}_{\mu_{q+1}}$. Notice $Y \to \mathbb{P}^1, (x,y)\mapsto [x:y]$ gives an isomorphism $Y/\mu_{q+1} \cong \mathbb{P}^1 - \mathbb{P}^1(\mathbb{F}_q)$. So we have

$$H_c^*(Y)[\mathbf{1}] = H_c^*(Y/\mu_{q+1}) = H_c^*(\mathbb{P}^1) - H_c^*(\mathbb{P}^1(\mathbb{F}_q))$$

by excision. $H_c^0(\mathbb{P}^1) = H_c^2(\mathbb{P}^1) = \mathbf{1}_G$ and $H_c^i(\mathbb{P}^1)=0$ for $i\neq 0,2$. So $H_c^*(\mathbb{P}^1) = 2\cdot \mathbf{1}_G$ as a G-representation. $\mathbb{P}^1(\mathbb{F}_q)$ is simply a finite set of points, so $H_C^*(\mathbb{P}^1(\mathbb{F}_q)) = H_c^0(\mathbb{P}^1(\mathbb{F}_q)) = \overline{\mathbb{Q}}_\ell[\mathbb{P}^1(\mathbb{F}_q)]$ and G acts on it by permutation. This permutation action is the same as the left action of G on $G/B = \mathbb{P}^1(\mathbb{F}_q)$, hence $H_c^*(\mathbb{P}^1(\mathbb{F}_q)) = \overline{\mathbb{Q}}_\ell[G/B]$, which is exactly $\mathrm{Ind}_B^G \mathbf{1} = \mathbf{1}_G + \mathrm{St}_G$! In particular, $R(\mathbf{1}) = \mathbf{1}_G - \mathrm{St}_G$ has degree $2-(1+q) = 1-q$.

Example 5 Let $1 \neq \zeta \in \mu_{q+1}$, then a fixed point formula of Deligne-Lusztig shows that $\mathrm{Tr}(\zeta | H_c^*(Y)) = \mathrm{Tr}(1|H_c^*(Y^\zeta))$, whcih is zero since $Y^\zeta = \varnothing$. Therefore the trace of any element ζ on $H_c^*(Y)$ is zero if $\zeta\neq 1$, hence as a μ_{q+1}-virtual representation, $H_c^*(Y)$ is a multiple of the regular representation $\overline{\mathbb{Q}}_\ell[\mu_{q+1}]$. In particular, every θ-isotypic component the same degree, which is $1-q$ for $\theta = 1$. Therefore $-R(\theta)$ is a degree $q-1$ representation of $G = SL_2(q)$. It turns out to be irreducible when $\theta^2\neq 1$ and is a direct sum of two representations of dimension $(q-1)/2$ when $\theta^2 = 1$. These are exactly the discrete series representations of $SL_2(q)$!

The discrete series representations have been realized in the cohomology of Y. Contemplating on this beautiful example, the following geometric picture emerges: the group G acts on $\mathbb{P}^1$ "horizontally" and μ_{q+1} acts on Y

"vertically" by permuting the points in the fiber of $Y \to \mathbb{P}^1$. One should really think of $\mathbb{P}^1$ as the flag variety of SL_2 and $Y \to \mathbb{P}^1$ is a finite covering with the covering group the nonsplit torus μ_{q+1}.

Deligne-Lusztig varieties

We come back to the general consideration of finite groups of Lie type G^F. Deligne and Lusztig generalized Drinfeld's construction to associate varieties to any such G^F.

Let $T \subseteq G$ be a F-stable maximal torus and $B \subseteq G$ be a F-stable Borel containing T (their existence is ensured by Lang's theorem). Let $W = N_G(T)/T$ be the Weyl group. All Borel subgroups of G are conjugate: so the conjugate action of G on the set of Borel subgroups of G is transitive and the stabilizer the action of on B is simply B itself. Therefore we have a bijection

$$X := G/B = \{\text{Borel subgroups of } G\}, \quad g \mapsto {}^gB.$$

Now the Bruhat decomposition $G = \coprod_{w \in W} BwB$ tells us that $W \cong B\backslash G/B.$ We say the two Borel subgroups ${}^gB$ and B are *in relative position* w, where w is the image of g in W. The nice thing is that $X = G/B$ itself is a projective variety over k. We cut out a locally closed subvariety $X(w) \subseteq G/B$ (*Deligne-Lusztig variety*) consisting of Borel subgroups ${}^gB$ such that ${}^gB$ and $F({}^gB)$ that are in relative position w. In other words,

$$X(w) \cong \{gB \in G/B : g^{-1}F(g) \in BwB\} \subseteq X.$$

It is a smooth quasi-projective (indeed, quasi-affine, and conjecturally, affine) variety of dimension $\ell(w)$. This gives a stratification

$$X = \coprod_{w \in W} X(w).$$

Notice the left action of G^F doesn't change the relative position, hence G^F acts on each $X(w)$ from the left, which is what we want.

Let U be the unipotent radical of B. Then $Y = G/U \to X = G/B$ is a T-torsor: T normalizes U and acts on Y from the right. We define similarly a locally closed subvariety

$$Y(w) = \{gU \in G/U : g^{-1}F(g) \in UwU\} \subseteq Y.$$

Then $Y(w) \to X(w)$ is indeed a T^{wF}-torsor, where $T^{wF} = \{t \in T : wF(t)w^{-1} = t\}$. Now we can play the same game by "inducing" a character θ of the torus T^{wF} to obtain a virtual character $R_w(\theta)$ of G^F using the cohomology $H_c^*(Y(w))$ of the Deligne-Lusztig variety $Y(w)$.

Example 6 Consider $G = SL_2$. Then the diagonal torus T and the standard Borel are F-stable. The Weyl group $W \cong S_2 = \{e, w\}$, with the nontrivial element w represented by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and maps the standard Borel to the opposite Borel, the subgroup of lower triangular matrices. The elements of $X = G/B$ can be identified as complete flags $\{0 \subseteq L \subseteq V = k^2\}$, in other words, points in $\mathbb{P}^1/k$. Two flags are in relative position w if and only if $L + L' = V$, in other words, two points in X are in relative position w if and only they are distinct. In particular $X(e) = \mathbb{P}^1(\mathbb{F}_q)$, and $X(w) = \mathbb{P}^1 - \mathbb{P}^1(\mathbb{F}_q)$. An element in $Y = G/U$ is nothing but a complete flag together with two vectors $v \in L$ and $u \in V/L$ such that $\det(v, u) = 1$. Two marked flags are in relative position w if and only if $v' \equiv u \pmod{\langle v \rangle}$. Therefore $Y(w)$ consists of marked flags such that $F(v) \equiv u \pmod{\langle v \rangle}$, i.e., $\det(v, F(v)) = 1$. Writing $v = (x, y)$, this gives exactly $\det\left(\begin{smallmatrix} x & x^q \\ y & y^q \end{smallmatrix}\right) = 1$ and recovers Drinfeld's ingenious construction. In this case T^{wF} is exactly the nonsplit torus $\left\{\left(\begin{smallmatrix} x & 0 \\ 0 & x^q \end{smallmatrix}\right)\right\} \cong \mu_{q+1}$.

Example 7 For $G = SL_n$ and w a permuatation of length F. One similarly obtains that $X(w) = \mathbb{P}^{n-1} - \mathbb{F}_q\text{-rational hyperplanes}$ and $Y(w)$ is given by the equation $\det(x_i^{q^j})_{i,j=0}^{n-1} = 1$.

Example 8 For any G, $Y(e) = G^F/U^F \to X(e) = G^F/B^F$ is a covering map between zero-dimensional varieties. Then $R_e(\theta)$ is simply the usual parabolic induction of a character θ of T^F.

By carefully studying the geometry of the varieties $X(w)$ and $Y(w)$. Deligne-Lusztig proved:

Theorem 1 Every irreducible representation of G^F appears in $R_w(\theta)$ for some w and $\theta : T^{wF} :\to \mathbb{C}^\times$. Moreover, $\langle R_w(\theta), R_w(\theta)\rangle = \#\{x \in W^F : x \text{ fixes } \theta\}$. In particular, when θ is *in general position*, i.e., no nontrivial elements of W^F fixes θ, one of $\pm R_w(\theta)$ is an *irreducible* representation of G^F.

Example 9 In the case $G = SL_2$, the nontrivial element w acts by $\theta \mapsto \theta^{-1}$. The above theorem coincides with the fact that $-R_w(\theta)$ (discrete series) and $R_e(\theta)$ (principle series) is irreducible whenever $\theta^2 \neq 1$.

To complete the construction of all irreducible representations of any finite group of Lie type, it "suffices" to decompose each $R_w(\theta)$ when θ is not in general position. This task is far from trivial but was eventually done by Lusztig in 80's in a series of papers and books.

Deligne-Lusztig curves

Finally, let us consider the special case when dimension of $X(w)$ is one-dimensional. This corresponds to the case w is a simple reflection. The relevant groups are groups of $\mathbb{F}_q$ -rank 1. There are only four such groups: $A_1(q)$, ${}^2A_2(q^2)$, ${}^2B_2(q^2=2^{2k+1})$, ${}^2G_2(q^2=3^{2k+1})$. Let $X=\overline{X(w)}=X(w)\cup X(e)$, then X is a smooth projective curve over $\mathbb{F}_q$. One can compute the Euler characteristic of X (hence the genus) and the number of rational points of X (= $|X(e)|$) from the finite group data using the fixed point formula. We gather the results here (c.f., [7]).

- A_1: $X=\mathbb{P}^1$.
- ${}^2A_2$: The Fermat curve $X: x^{q+1}+y^{q+1}=z^{q+1}$ of degree $q+1$. It has genus $q(q-1)/2$ and $1+q^3$ $\mathbb{F}_{q^2}$ -points.
- ${}^2B_2$: The Deligne-Lusztig curve of Suzuki type (DLS). It has genus $q(q^2-1)/\sqrt{2}$ and $1+q^4$ $\mathbb{F}_{q^2}$ -points.
- ${}^2G_2$: The Deligne-Lusztig curve of Ree type (DLR). It has genus $\sqrt{3}q(q^4-1)/2+q^2(q^2-1)/2$ and $1+q^6$ $\mathbb{F}_{q^2}$ -points.

Now there comes no surprise that these curves admits a large number of automorphisms. A theorem of Stichtenoth asserts that $|\mathrm{Aut}(X)|\le 16g(X)^4$ except X is Fermat curve ${}^2A_2$. A theorem of Henn shows that $|\mathrm{Aut}(X)|\le 8g(X)^3$ except the Fermat curve ${}^2A_2$, the DLS, the hyperelliptic curve $y^2=x^q-x$ ($p\ne 2$) and the curve $y^2+y=x^{2^k+1}$ ($k>1$, $p=2$). Amazingly enough you can go home and check by hand that the Deligne-Lusztig curves ${}^2A_2$, DLS, DLR are all maximal curves: they all achieve the Hasse-Weil bound $\lfloor 1+q^2+2gq\rfloor$ for the number of $\mathbb{F}_{q^2}$-rational points!

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