III-2 GROUP ACTIONS AND TRACE FORMULA

Let H be a finite group acting by automorphisms on X inear action of H on Hi(x) by functionality

Pop:
$$H_c^i(H \setminus X) = H_c^i(X) = \overline{\mathbb{Q}} \otimes_{\mathbb{Q}} H_c^i(X)$$
invariants

winvariants

n gives the contribution of the trivial representation in $H_c(X)$

$$\underline{E_X}: \widetilde{X}(w) = \{g \cup EG/U \mid \overline{g'}F(g) \in UwU\}$$

$$X := \{g \in G \mid \overline{g'}F(g) \in UwU\}$$

The fibers are affine spaces isomorphic to U $\Rightarrow H_c^i(\widetilde{X}(w)) \simeq H_c^{i+2dimU}(X)$

Now
$$X \longrightarrow UwU$$
 induces $X \stackrel{\sim}{\longrightarrow} UwU$
 $g \longmapsto g^{-1}F(g)$ G^{F}

$$\Rightarrow$$
 $H_c^{\bullet}(G^{F} \setminus X) \circ \overline{\mathbb{Q}}_{e}[-2\dim X]$

Corollary: the trivial representation of G^F occurs only in the top degree of $H_c(\tilde{X}(w))$ and with multiplicity one

Pmk: if X, Y are both endowed with an action of H
we can fam the amalgameted poduct $X \times_H Y = (X \times Y)/\Delta H$ $\longrightarrow H_c^*(X \times_H Y) = H_c^*(X) \otimes_{\mathbb{Q}_e H} H_c^*(Y)$

Ex: if $w \in W_{\underline{I}}$ the isomorphism $G^{\underline{F}} \times_{P_{\underline{I}}} X_{L_{\underline{I}}}(w) \xrightarrow{\sim} X(w)$ gives $|H_{\underline{C}}(X(w)) = \overline{\mathbb{Q}}_{\ell}G^{\underline{F}} \otimes_{\overline{\mathbb{Q}}_{\ell}} P_{\underline{T}}^{\underline{F}} H_{\underline{C}}(X_{L_{\underline{I}}}(w))$

= Ind Pr (Inf Pr Hc(XL (w)))

Haush-Chandra induction

Prop: If the action of H extends to an action of $L_c(x) = L_c(x) + L_c(x) = L_c(x) = L_c(x) + L_c(x) = L_c(x)$

~ such actions are "boring".

 $\underline{E_X}: X(w_0) \stackrel{\circ}{\sim} B$ the action of G^F extends to G \Longrightarrow only the trivial representation occurs in $H^{\bullet}(X(w_0))$

2) Consequence of the trace formula

Assume now that we have both

· a Frobenius F: X -> X for some Fg-structure

. an action of the finite group H on X

such that $F(h.x) = h.F(x) \forall h \in H, x \in X$

We can define the class finction

$$\mathcal{L}: h \longrightarrow \sum_{i \in \mathbb{Z}} (-1)^i \operatorname{Trace}(h \mid H_c^i(X))$$

if is a character of a virtual representation hence $\mathcal{L}(h)$ is an algebraic integer for all $h \in H$

proof: hF = Fh so we can simultaneously triangularize with eigenvalues $\lambda_{i,1}, ..., \lambda_{i,r}$ of h on $H_c^i(X)$ $\mu_{i,1}, ..., \mu_{i,r}$ of FSince h F is a Frabenius endomorphism of X $+ \times^{hF^n} = Trace(hF^n \mid H_c(X))$ $= \sum_{i \in \mathbb{Z}} (-1)^i \sum_{i} \lambda_{i,j} \mu_{i,j}^n$ $\Rightarrow \sum_{n\geqslant 1} \# X^{hF^n} t^n = \sum_{i \neq j} (-1)^i \lambda_{ij} \sum_{n\geqslant 1} (\mu_{ij} t)^n$ and $\mathcal{L}(h) = \frac{2}{in}(-1)^i \lambda_{ij}$ Corollary: $Z(h) = Z(1)^i H_c^i(X)$ is an integer independent of lproof: L(h) is an algebraic integer with is a limit of a formal series with integral coefficients indep/l []

Feature from the cohomology over Ite, Ite. It Stabp(a) for all a \in X => \in 1s the lift of a virtual projective representation Corollary: If h is a p'elt then \in X(h) Leapart the Education of X proof: assume first that h order r for some prime number. Since I(h) does not depend on I one an assume r=1 Then every projective representation of \in X is multiple of the regular rep., on which Trace(h) is zero Since \in X acto feely on X\in X = X\in X we deduce that \[Z(1) \in Trace(h H_c(X\in X)) = 0 \in The the general case he decompose h and we induction \[\in X \i	S / Clier of D Contacto
LI Stab _H (x) for all x \in X => L is the lift of a virtual projective representation Corollary: If h is a p'elt then L _X (h) Legrals the Enter char. of X ^h Proof: assume first that h order r for roome prime number. Since L(h) does not depend on l one can assume rel Then every projective representation of (h) is multiple of the regular rep., on which Trave(h) is zero Since <h> acto freely on X \times \tim</h>	3- Action of p-elements
LI Stab _H (x) for all x \in X => L is the lift of a virtual projective representation Corollary: If h is a p'elt then L _X (h) Legrals the Enter char. of X ^h Proof: assume first that h order r for roome prime number. Since L(h) does not depend on l one can assume rel Then every projective representation of (h) is multiple of the regular rep., on which Trave(h) is zero Since <h> acto freely on X \times \tim</h>	Feature from the cohomotory over E. To.
Exception is the lift of a virtual projective representation. Corollary: If h is a p'elt then $Z_X(h)$ Legrals the Euler char of X^h Proof: assume first that h order r for some prime number. Since $Z(h)$ does not depend on l One can assume $r=l$ Then every projective representation of $\langle h \rangle$ is multiple of the regular rep., on which $Trace(h)$ is zero Since $\langle h \rangle$ acto finely on $X \setminus X^h = X \setminus X^h \rangle$ we deduce that $Z(-1)^n$ Trace $(h \mid H_c^n(X^h)) = 0$ from which we get $Z(h) = Z(-1)^n$ Trace $(h \mid H_c^n(X^h))$ $= Z(-1)^n$ dim $H_c^n(X^h)$	
Corollary: If h is a p'elt then $L_X(h)$ Legrals the Eder chair of X^h proof: assume first that h order r for some prime number. Since $L(h)$ does not depend on L one can assume $r=L$ Then every projective representation of L is multiple of the regular rep., on which L trace L is zero. Since L acto finely on L is zero. Since L acto finely on L ac	
proof: assume first that horder r for rome prime number. Since I(h) does not depend on l one can assume rel Then every projective representation of <h> is multiple of the regular rep., on which Trace(h) is zero Since <h> acto finely on X\X^h = X\X^h> we deduce that \[\begin{align*} & \begi</h></h>	=> & is the lift of a virtual projective representation
proof: assume first that horder r for rome prime number. Since 2(h) does not depend on l one can assume r=l Then every projective representation of <h> is multiple of the regular rep., on which Trave(h) is zero Since <h> acto finely on X\X^h = X\X^h> we deduce that \[Z(-1)^i \text{ Trave}(h H_c^*(X\X^h)) =0 \] from which we get \[Z(-1)^i \text{ Trave}(h H_c^*(X^h)) = \frac{1}{2}(-1)^i \text{ dim } H_c^*(X^h) \] \[= Z(-1)^i \text{ dim } H_c^*(X^h) \]</h></h>	Corollaun: If his a pielt then L. (h)
proof: assume first that horder r for rome prime number. Since 2(h) does not depend on l one can assume r=l Then every projective representation of <h> is multiple of the regular rep., on which Trave(h) is zero Since <h> acto finely on X\X^h = X\X^h> we deduce that \[Z(-1)^i \text{ Trave}(h H_c^*(X\X^h)) =0 \] from which we get \[Z(-1)^i \text{ Trave}(h H_c^*(X^h)) = \frac{1}{2}(-1)^i \text{ dim } H_c^*(X^h) \] \[= Z(-1)^i \text{ dim } H_c^*(X^h) \]</h></h>	Tegral the Eder char of Xh
Then every projective representation of <h> is multiple of the regular rep., on which Trace(h) is zero Since <h> acto feely on X \ X \ = X \ X \ we deduce that \[\sum_{(-1)}^i \text{Trace}(h H_c^*(X \ X^h)) = 0 \] from which we get \[\sum_{(-1)}^i \text{Trace}(h H_c^*(X^h)) = 0 \] \[= \sum_{(-1)}^i \text{Trace}(h H_c^*(X^h)) = 0 \]</h></h>	
Then every projective representation of <h> is multiple of the regular rep., on which Trace(h) is zero Since <h> acto feely on X \ X \ = X \ X \ we deduce that \[\sum_{(-1)}^i \text{Trace}(h H_c^*(X \ X^h)) = 0 \] from which we get \[\sum_{(-1)}^i \text{Trace}(h H_c^*(X^h)) = 0 \] \[= \sum_{(-1)}^i \text{Trace}(h H_c^*(X^h)) = 0 \]</h></h>	proof: assume first that horder r by some prime
Then every projective representation of <h> is multiple of the regular rep., on which Trace(h) is zero Since <h> acto feely on X \ X \ = X \ X \ we deduce that \[\sum_{(-1)}^i \text{Trace}(h H_c^*(X \ X^h)) = 0 \] from which we get \[\sum_{(-1)}^i \text{Trace}(h H_c^*(X^h)) = 0 \] \[= \sum_{(-1)}^i \text{Trace}(h H_c^*(X^h)) = 0 \]</h></h>	number. Since 2(h) they not depend on 1
Then every projective representation of $\langle h \rangle$ is multiple of the regular rep., on which $Trace(h)$ is zero Since $\langle h \rangle$ acts freely on $X \setminus X^h = X \setminus X^{\langle h \rangle}$ we deduce that $Z(-1)^i$ $Trace(h \mid H_c^i(X \setminus X^h)) = 0$ from which we get $Z(h) = Z(-1)^i$ $Trace(h \mid H_c^i(X^h))$ $= Z(-1)^i$ $dim(H_c^i(X^h))$	
Since $\langle h \rangle$ acts feely on $X \setminus X^h = X \setminus X^{\langle h \rangle}$ we deduce that	0 /2 Car 4250m2 122
Since $\langle h \rangle$ acts feely on $X \setminus X^h = X \setminus X^{\langle h \rangle}$ we deduce that	Then every rapiective representation of <h> is multiple</h>
Since $\langle h \rangle$ acts feely on $X \setminus X^h = X \setminus X^{\langle h \rangle}$ we deduce that	of the soular men on which Trave (b) is zero
we deduce that $\sum_{(-1)^i} \text{Trace}(h \mid H_c^i(X \setminus X^h)) = 0$ from which we get $\mathcal{L}(h) = \sum_{(-1)^i} \text{Trace}(h \mid H_c^i(X^h))$ $= \sum_{(-1)^i} \text{dim } H_c^i(X^h)$) 1.2 regular 14., 47 miss 1100 (11) 13 630
we deduce that $\sum_{(-1)^i} \text{Trace}(h \mid H_c^i(X \setminus X^h)) = 0$ from which we get $\mathcal{L}(h) = \sum_{(-1)^i} \text{Trace}(h \mid H_c^i(X^h))$ $= \sum_{(-1)^i} \text{dim } H_c^i(X^h)$	
$\sum_{(i)} Trace(h H_c^*(X \setminus X^h)) = 0$ from which neget $\mathcal{L}(h) = \sum_{(i)} Trace(h H_c^*(X^h))$ $= \sum_{(i)} dim H_c^*(X^h)$	Since < h > acto findly on X \ X h - X \ X h >
from which neget $\mathcal{L}(h) = \sum_{(1)^i} \text{Trace}(h \mid H_c(X^h))$ $= \sum_{(1)^i} \text{dim } H_c(X^h)$	Since <h> acto feely on X\X = X\X</h>
$\mathcal{L}(h) = \sum_{(1)^i} \overline{\text{Trake}}(h \mid H_{\mathcal{L}}(X^h))$ $= \sum_{(1)^i} \dim_{\mathcal{L}}(X^h)$	we deduce that
$= \sum_{i=1}^{n} (i)^{i} \dim_{i} H_{i}^{i}(X^{h})$	we deduce that $\sum_{(-1)^i} Trace(h H_c(X \setminus X^h)) = 0$
	we deduce that $\sum_{(-1)^i} Trace(h H_c^i(\times \times^h)) = 0$ from which we get
tor the general case he decompose h and we induction \square	we deduce that $\sum_{(-1)^i} \frac{1}{\text{Trace}(h \mid H_c^i(x \setminus x^h))} = 0$ from which we get $2(h) = \sum_{(-1)^i} \frac{1}{\text{Trace}(h \mid H_c^i(x^h))}$
	we deduce that $\sum_{(-1)^i} \frac{1}{(-1)^i} 1$