# Chromatic and Temperley-Lieb algebras

## A homomorphism

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# Overview

- Introduce
  - Chromatic algebra
  - Temperley-Lieb algebra
- Define a map $\Phi$: Chromatic $\rightarrow$ Temperley-Lieb
- Prove $\Phi$ is well-defined & preserves trace
- Further applications - knotted graph invariant

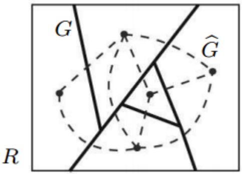
# Terminology

## Definition

A **trivalent** graph is a graph in which all vertices have three associated edges

## Definition

Given a planar graph $G$, the **dual** graph $\hat{G}$, is obtained as shown below.



# Free Algebra

## Definition

### Definition

The **free algebra** in degree $n$, $\mathcal{F}_n$ over $\mathbb{C}[\mathcal{Q}]$

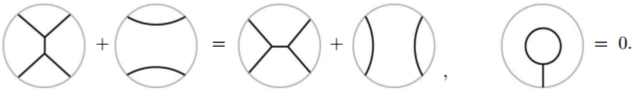
Elements are linear combinations of trivalent graphs contained in a rectangle, with $n$ endpoints on both the top and the bottom.

### Definition

The **chromatic algebra** in degree n, $C_n$ is the algebra over $\mathbb{C}[\mathcal{Q}]$ given by the quotient of the free algebra $\mathcal{F}_n$ by $I_n$

Sometimes denoted $C_n^{\mathcal{Q}}$. Set $C = \cup_n C_n$

The ideal $I_n$ is generated by the "H-I relation" and the fact that "tadpoles" vanish.




Also note, the value of a simple, closed curve is set to $\mathcal{Q} - 1$

# Chromatic Polynomial
## Intuitive definition



**Definition**

The **chromatic polynomial** $\chi_\Gamma(\mathcal{Q})$ of a graph $\Gamma$, $\mathcal{Q} \in \mathbb{Z}^+$

is the number of colorings of the vertices of $\Gamma$ with $\mathcal{Q}$ colors where no adjacent vertices have the same color.

# Chromatic Polynomial
## definition

Properties:

1. $\chi_\Gamma(\mathcal{Q}) = \chi_{\Gamma \backslash e}(\mathcal{Q}) - \chi_{\Gamma / e}(\mathcal{Q})$
2. if $\Gamma$ has no edges and $V$ vertices, then $\chi_\Gamma(\mathcal{Q}) = \mathcal{Q}^V$
3. if $\Gamma$ contains a loop, then $\chi_\Gamma(\mathcal{Q}) = 0$

These properties allow us to generalize the chromatic polynomial from $\mathbb{Z}^+$ to $\mathbb{C}$

# Chromatic Polynomial
## State space formula

An explicit formula can be derived from the preceeding properties. This formula is called the **state space forumla** and is given by:

$$\chi_G(\mathcal{Q}) = \sum_{s \subseteq E(G)} (-1)^{|s|} \mathcal{Q}^{k(s)}$$

where $\mathbf{k(s)}$ is the number of connected components of $G$.

# Chromatic Algebra

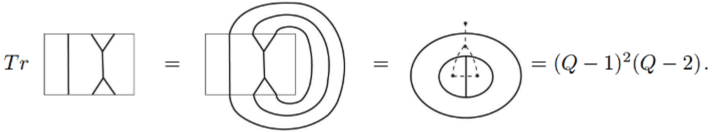
## Trace

### Definition

**Trace** $tr_\chi : \mathcal{C}^{\mathcal{Q}} \to \mathbb{C}$

Defined on the additive generators, i.e. a graph $G$
Connect the endpoints of the rectangle and denote the result $\bar{G}$
Now, evaluate

$$tr_\chi(G) = \mathcal{Q}^{-1} \cdot \chi_{\hat{\bar{G}}}(\mathcal{Q})$$



# Temperley-Lieb
## Algebraic definition

### Definition

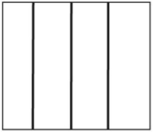
The Temperley-Lieb algebra in degree n,
$TL_n$ is an algebra over $\mathbb{C}[d]$ (d is some complex number)

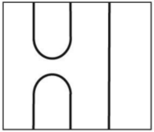
generated by $\{1, E_1, ..., E_{n-1}\}$ with relations:

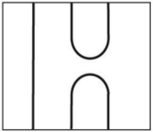
$$E_i^2 = E_i$$
$$E_i E_{i\pm 1} E_i = \frac{1}{d^3} E_i$$
$$E_i E_j = E_j E_i \text{ for } |i-j| \geq 1$$

# Temperley-Lieb
## Geometric

The generators of $TL_3$

$1 =$ 

$E_1 = \frac{1}{d}$ 

$E_2 = \frac{1}{d}$ 

### Definition

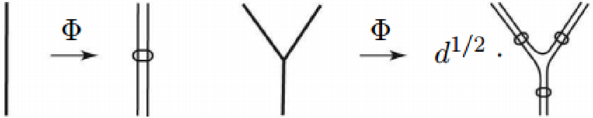
**Trace** $tr_d : TL_n^d \to \mathbb{C}$

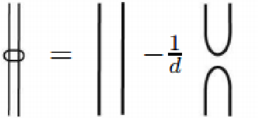
Connect the endpoints and evaluate $d^{\#circles}$

# Homomorphism

## Definition

Define a homomorphism $\Phi : \mathcal{F}_n \to TL_n^d$



# Theorem 1

**Theorem**

*$\Phi$ induces a well-defined algebra homomorphism*

$$C_n^{\mathcal{Q}} \to TL_{2n}^d$$

*where $\mathcal{Q} = d^2$*

Suffices to show that the relations in $C_n^{\mathcal{Q}}$ hold in $TL_{2n}^d$
Namely, check the following

- H-I relation
- Tadpoles vanish
- Simple closed curves go to $\mathcal{Q} - 1$ $(= d^2 - 1)$

# Theorem 2



## Theorem

*Let $G$ be a trivalent planar graph. Then*

$$\mathcal{Q}^{-1}\chi_{\hat{G}}(\mathcal{Q}) = \Phi(G)$$

*where $\mathcal{Q} = d^2$.*

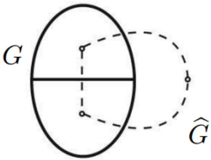
*It follows that the following diagram commutes*

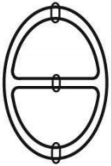
$$\begin{array}{ccc} C_n^{\mathcal{Q}} & \xrightarrow{\Phi} & TL_{2n}^{d} \\ \downarrow {\scriptstyle tr_\chi} & & \downarrow {\scriptstyle tr_d} \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C} \end{array}$$

# Theorem 2
## Example

### Example

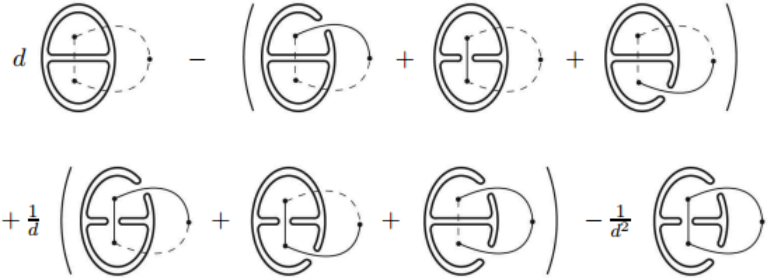
$$Q^{-1}\chi_Q(\widehat{G}) = (Q-1)(Q-2) = d^4 - 3d^2 + 2 = \Phi(G).$$




$\Phi(G) = d \cdot$ 

# Theorem 2

## Example



# Theorem 2
## Proof



First assume $G$ connected, and recall the state sum forumla of the chromatic polynomial, here applied to the dual graph $\hat{G}$

$$\chi_{\hat{G}}(\mathcal{Q}) = \sum_{s \subseteq E(\hat{G})} (-1)^{|s|} \mathcal{Q}^{k(s)}$$

Now, since $\Phi(G)$ maps the edges of $G$ to a binary sum, the total number of summands of $\Phi(G)$ is $2^{|E(G)|}$.
This power of two is precisely the number of terms in the state sum forumla above, and hints toward a potential correspondence.

# Theorem 2
## Proof



Indeed! $\Phi(G)$ can be parameterized by the subsets of the edge set of $\hat{G}$. (i.e. the same indexing of chromatic state-sum)

### State sum of $\Phi$ (v1)

$$\Phi(G) = d^{V(G)/2} \sum_{s \subseteq E(\hat{G})} (-1)^{|s|} \frac{1}{d^{|s|}} d^{k(s)+n(s)}$$

$\mathbf{k(s)}$ is the number of connected components.
$\mathbf{n(s)}$ is "the rank of the first homology of $\hat{G}_s$"
The number of edges remaining after the removal of a spanning tree. Also, called "circuit rank"

# Quick aside
## Euler characteristic

The **Euler characteristic** is a topological invariant.

In particular, for planar connected graphs it is defined to be $V - E + F$ and always equals 2.

# Theorem 2

## Proof

Since we restricted ourselves to trivalent graphs, $\hat{G}$ is a triangulation.

$$\begin{aligned} \implies 2V(\hat{G}) &= F(\hat{G}) + 4, \text{ derived from Euler characteristic} \\ &= V(G) + 4, \text{ by construction of } \hat{G} \end{aligned}$$

$$\implies V(G)/2 = V(\hat{G}) - 2$$

### State sum of Φ (v2)

$$\Phi(G) = \sum_{s \subseteq E(\hat{G})} (-1)^{|s|} d^{V(\hat{G}) - 2 + k(s) + n(s) - |s|}$$

# Theorem 2
## Proof

**Claim:** $V(\hat{G}) = k(s) - n(s) + |s|$
**Proof:** True for $s = 0$, and the addition of one edge either decrements $k(s)$ or increments $n(s)$.

### State sum of $\Phi$ (v3)

$$\Phi(G) = \sum_{s \subseteq E(\hat{G})} (-1)^{|s|} d^{2k(s)-2}$$

# Theorem 2
## Proof

Substituting $Q$ for $d^2$, we get

$$\Phi(G) = \mathcal{Q}^{-1} \sum_{s \subseteq E(\hat{G})} (-1)^{|s|} \mathcal{Q}^{k(s)}$$

$$= \mathcal{Q}^{-1} \chi_{\hat{G}}(\mathcal{Q})$$

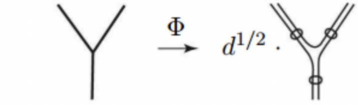
This concludes the proof for connected graphs.

The proof for unconnected graphs is not difficult.
If $G = G_1 \sqcup G_2$ then
$\phi(G) = \phi(G_1)\phi(G_2)$ and
$\chi_{\hat{G}}(\mathcal{Q}) = \mathcal{Q}^{-1} \chi_{\hat{G}_1}(\mathcal{Q}) \chi_{\hat{G}_2}(\mathcal{Q})$ □

# Knotted Graph Invariant



Using the $\Phi$ constructed above,
we can now map knotted graphs to the Jones polynomial.



$$d = -A^2 - A^{-2}$$

This turns out to be an invariant of knotted graphs!
So, with $\Phi$, we can extend the application of the Jones polynomial beyond knots.

# For Further Reading I



P. Fendley, V. Krushkal.
Tutte Chromatic Identities from the Temperley-Lieb Algebra
19 July 2008

L. Kauffman, S. Lins
Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds
1994