## 2.

Today we discuss Borel subgroups of reductive groups and the corresponding quotients. They allow us to relate the structure of the finite reductive groups $G^F$ to that of (finite) reflection groups called Weyl groups.

### *2.1.*

We have discussed how an algebraic group $G$ over $\bar{\mathbf{F}}_q$, equipped with a Frobenius map $F : G \to G$, gives rise to a finite group $G^F$. When $G$ is reductive, the structure of $G$, *resp.* $G^F$, closely resembles that of $\mathrm{GL}_n$, *resp.* $\mathrm{GL}_n(\mathbf{F}_q)$.

Even earlier, we discussed the *Bruhat decomposition*. For now, let $k$ be an arbitrary algebraically closed field of characteristic $p \nmid n$. Let $B \subseteq \mathrm{GL}_n$ be the subgroup of upper-triangular matrices, and for all $w \in S_n$, let $\dot{w} \in \mathrm{GL}_n$ be the permutation matrix of $w$. The Bruhat decomposition on $k$-points is

$$\mathrm{GL}_n(k) = \coprod_{w \in S_n} B(k)\dot{w}B(k).$$

Its proof is similar to how we used row reduction to establish the Schubert cell decomposition of any Grassmannian. Namely, it suffices to show:

**Theorem 2.1.** *The coset space $B(k)\backslash\mathrm{GL}_n(k)$ is the disjoint union of the subsets $B(k)\backslash(B(k)\dot{w}B(k))$ for $w \in S_n$.*

*Proof.* We can identify cosets of $B(k)$ with (complete) flags in $k^n$ via the map $B(k)g \mapsto \vec{V} \cdot g$, where $\vec{V} = (V_i)_i$ is the standard flag in row notation.

Pick an ordered basis $(v_i)_i$ that generates $\vec{V} \cdot g$ in the sense that $V_i \cdot g = \langle v_1, \ldots, v_i \rangle$, then apply row reduction to the associated matrix. The result is an upper-triangular matrix $b \in B(k)$. From the algorithm, we also get a permutation $w^{-1} \in S_n$ that only depends on the flag $\vec{V} \cdot g$: the composition of the row swaps (from the left) used to reduce $g$ to $b$. We have $\vec{V} \cdot g = \vec{V} \cdot \dot{w}b$. □

Note that the expression $B\dot{w}B$ can be reduced even further. For instance, we can always take $b$ unipotent in the proof above. Another way to see this: Recall that $B = TU$, where $T$, *resp.* $U$ is the subgroup of diagonal, *resp.* unipotent matrices, and observe that permutation matrices normalize $T$, meaning $\dot{w}T = T\dot{w}$ for all $w$.

Something stronger is true. For any algebraic groups $H \subseteq G$, Milne Prop. 1.83 exhibits an algebraic group $N_G(H)$ such that $N_G(H)(R) = N_{G(R)}(H(R))$ for any $k$-algebra $R$. It turns out that the connected components $N_{\mathrm{GL}_n}(T)$ are precisely the cosets $\dot{w}T$ for $w \in S_n$. This suggests how Bruhat decomposition ought to generalize beyond $\mathrm{GL}_n$.

Define the *Weyl group* of a maximal torus $T \subseteq G$ to be the normalizer $W = W(G, T) := N_G(T)/T$. Note that for any $w \in W$, and any algebraic subgroup $B \subseteq G$ containing $T$, the notation $wB$ is unambiguous.

**Theorem 2.2.** *Suppose that $G$ is a reductive algebraic group. Let $B = T \ltimes U \subseteq G$ be a Borel subgroup, where $U = [B, B]$. Then $B(k)\backslash G(k)$ is the disjoint union of the subsets $B(k)\backslash(B(k)wB(k))$ for $w \in W(G, T)$.*

*2.2.*

As in the first lecture, we switch notation from left-hand quotients back to right-hand quotients. The set $G(k)/B(k)$ is precisely the set of $k$-points of the fppf sheaf quotient $G/B$, essentially because $\operatorname{Spec} k$ has no nontrivial fppf covers. However, we can be much more concrete about spaces like this. The key idea is a representation-theoretic characterization of algebraic subgroups:

**Theorem 2.3** (Chevalley)**.** *If $G$ is any affine algebraic group with algebraic subgroup $G'$, then there exist a (finite-dimensional) representation $V$ of $G$ and a subspace $V' \subseteq V$ such that*

$$G'(k) = \{g \in G(k) \mid gV' \subseteq V'\}.$$

*We can even choose $V, V'$ so that $V'$ is a line.*

*Proof.* Let $I$ be the kernel of the quotient map $k[G] \to k[G']$. We can pick a finite generating set for $I$ as an ideal. Then we can pick a finite-dimensional $k[G]$-comodule $V^\vee \subseteq k[G]$ containing these generators, just like in the proof of the linearity of affine algebraic groups. This gives the representation $V$. To get $V' \subseteq V$, we take $(V')^\vee = V^\vee \cap I$.

If $g \in G'(k)$, then $gI \subseteq I$, so $gV' \subseteq V'$. Conversely, if $gV' \subseteq V'$, then $g$ sends every generator of $I$ to another element of $I$, but $g$ acts on $k[G]$ by algebra automorphisms, so $gI \subseteq I$ and hence $I$ is also the kernel of the quotient map $k[G] \to k[G'g]$, which forces $g \in G'$.

Finally, once we have such $V, V'$, we see that the same characterization of $G'$ holds when we replace $V, V'$ by $\bigwedge^d V, \bigwedge^d V'$, respectively, where $d = \dim(V')$. $\square$

**Corollary 2.4.** *If $G$ is a* smooth *affine algebraic group with algebraic subgroup $G'$, then there is a locally closed, $G$-equivariant embedding $G/G' \to \mathbf{P}V$ for some representation $V$ of $G$. In particular, $G/G'$ is a quasiprojective variety.*

*Proof.* Take $V, V'$ as in the theorem, with $V'$ a line. Let $G/G' \to \mathbf{P}V$ be induced by the map from $G$ onto the orbit of $[V']$. The smoothness of $G$ ensures that the latter is faithfully flat, allowing us to identify the orbit with $G/G'$. $\square$

*2.3.*

Henceforth, *assume that $G$ is smooth and affine over $k$*. An algebraic subgroup $P \subseteq G$ is *parabolic* if and only if $G/P$ is projective, not merely quasiprojective.

As it turns out, there is a nice characterization of parabolic subgroups. For the proof of the following fixed-point theorem, see Milne Chapter 17.

**Theorem 2.5** (Borel)**.** *If $B$ is a connected, smooth, solvable algebraic group acting on a nonempty proper variety $X$, then $X^B$ is nonempty.*

**Corollary 2.6.** *An algebraic subgroup of (a smooth, affine) $G$ is parabolic if and only if it contains some Borel subgroup of $G$.*

*Proof.* The "only if" direction: First, Borels exist in $G$, because $G$ has finite dimension and $\{1\}$ is connected, smooth, and solvable. If $B \subseteq G$ is a Borel and $P \subseteq G$ is parabolic, then the action of $B$ by left multiplication on $G/P$ must have a fixed point $gP$ by Borel's theorem, in which case $g^{-1}Bg$ is a Borel contained in $P$.

The "if" direction: Since the image of any proper variety is proper, it suffices to show that if $B \subseteq G$ is a Borel, then $G/B$ is proper. We induct on the dimension of $G$. Pick a faithful representation $V$ of $G$. The action of $G$ on $\mathbf{P}V$ must have a closed orbit. The stabilizer of any $k$-point of this orbit is a parabolic subgroup $P \subseteq G$. By the preceding paragraph, it contains some conjugate of $B$, and without loss of generality, we may replace $B$ with this conjugate. Two cases: Either $P$ is smaller than $G$, in which case $P/B$ is proper by the inductive hypothesis, and hence $G/B$ is proper, or else $P = G$, in which case $V^G$ contains a line, and we can replace $V$ with $V/(V^G)$ until we either reach $\{0\}$ or reduce to the previous case. $\square$

**Corollary 2.7.** *Any two Borels in a smooth affine algebraic group are conjugate.*

For most purposes, this characterization is overkill. In matrix groups, we can picture Borels roughly as conjugates of some subgroup of upper-triangular matrices, and parabolics as conjugates of subgroups of *block* upper-triangular matrices.

Instead, the main takeaway for now is the fact that $G/B$ is a *projective variety* for any Borel subgroup $B \subseteq G$. Here is another useful viewpoint on $G/B$, which I may or may not prove later.

**Theorem 2.8.** *If $B$ is a Borel subgroup of a connected (smooth, affine) algebraic group $G$, then $N_G(B) = B$. In particular, $(G/B)(k)$ is in bijection with the set of Borel subgroups of $G$, via the map $gB \mapsto gBg^{-1}$.*

For $G = \mathrm{GL}_n$, this theorem follows from the flag description of $(G/B)(k)$: If $B$ is the stabilizer of $\vec{V}$, then $gBg^{-1}$ is the stabilizer of $g \cdot \vec{V}$.

*2.4.*

The orbit decomposition of $G/B$ under the action of $B$ by left multiplication gives rise, on $k$-points, to the Bruhat decomposition that we discussed at the start.

I have not yet given a proof of Bruhat decomposition for an arbitrary reductive $G$. I can sketch the idea if we assume the following result, which I may or may not prove for general $G$ later. For $G = \mathrm{GL}_n$, it is a consequence of the proof of the Jordan–Hölder theorem, via the identification of $(G/B)(k)$ with the set of flags in $k^n$.

**Theorem 2.9.** *If $G$ is reductive, then any two Borel subgroups of $G$ contain a common maximal torus of $G$.*

*Sketch of Bruhat decomposition.* We will exhibit a map from $B(k)\backslash G(k)/B(k)$ to the Weyl group $W = W(G, T)$. For any $g \in G(k)$, pick a maximal torus $S \subseteq B \cap gBg^{-1}$. By Cartan–Lie–Kolchin, we can write

$$S = bTb^{-1} = (gb'g^{-1})(gTg^{-1})(gb'g^{-1})^{-1}$$

for some $b, b' \in B(k)$. But then $b^{-1}gb'$ normalizes $T$, so we obtain an element $[b^{-1}gb'] \in W$. One has to check that this element only depends on $BgB$. $\square$

*2.5.*

We now explain how the preceding structure theory for $G$ implies consequences for $G^F$. Henceforth, $k = \bar{\mathbf{F}}_q$.

As a starting point, consider the Frobenius map $F : \mathrm{GL}_n \to \mathrm{GL}_n$ given by raising each matrix coordinate to the $q$th power, so that $\mathrm{GL}_n^F$ is the group classically denoted $\mathrm{GL}_n(\mathbf{F}_q)$. Then $F$ stabilizes $B$ and fixes $\dot{w}$ for all $w$. Hence the Bruhat decomposition of $\mathrm{GL}_n(k)$ into double cosets of $B(k)$ implies an analogous decomposition of $\mathrm{GL}_n^F$ into double cosets of $B^F$.

In particular, this immediately proves the identity $(G/B)^F = G^F/B^F$ (on $k$-points). What happens if we replace $\mathrm{GL}_n$ with another $G$? What happens if we choose a different Frobenius map $F$ and $F$-stable Borel?

It turns out that the connectedness of $B$ ensures that we always have $(G/B)^F = G^F/B^F$. On Problem Set 1, you will deduce a generalization of this fact from the following theorem:

**Theorem 2.10** (Lang)**.** *Let $H$ be a connected, smooth algebraic group over $\bar{\mathbf{F}}_q$ and $F : H \to H$ the Frobenius map for some $\mathbf{F}_q$-form. Then the* Lang map

$$h \mapsto h^{-1}F(h) : H \to H$$

*is surjective.*

For the proof, see Wikipedia. The key idea of the proof is to calculate the induced map on Lie algebras, using the fact that the differential of $F$ vanishes to show bijectivity.

Note that the Lang map is finite étale, and its fiber over the identity is precisely $H^F$. For this reason, one can think of the theorem as presenting $H$ as an $H^F$-principal bundle over itself in the étale topology.

*Remark* 2.11. For affine algebraic groups, Steinberg generalized Lang's theorem from Frobenius maps to any surjective map $F$ with finitely many fixed points. For this reason, I sometimes speak of the Lang–Steinberg theorem.

*2.6.*

We now focus on the $G^F$-set $G^F/B^F$, where $G$ is a reductive algebraic group over $\bar{\mathbf{F}}_q$ with Frobenius map $F$, and $B$ is an $F$-stable Borel subgroup of $G$.

As usual, write $B = T \ltimes U$. The