# Perverse sheaves in representation theory

These are my live-TeXed notes for the course Mathematics 262y: *Perverse Sheaves in Representation Theory* taught by Carl Mautner at Harvard (Fall 2011). I eventually put some effort and editted the part before the Verdier duality during the winter. Please let me know if you notice any errors or have any comments!

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## The Lefschetz hyperplane theorem

**Theorem 1** (Lefschetz Hyperplane Theorem) Let $X \subseteq \mathbb{P}^N$ be a projective variety and $Y = H \cap X$ be the intersection of $X$ with a hyperplane $H \subseteq \mathbb{P}^N$ such that $X \setminus Y$ is smooth, then the map $i^* : H^k(X;\mathbb{Z}) \to H^k(Y;\mathbb{Z})$ induced by the inclusion $i : Y \hookrightarrow X$ is an isomorphism for $k < n-1$ and an injection for $k = n-1$.

**Example 1** Let $X \subseteq \mathbb{P}^n$ be a hypersurface of degree $d$. Then $X = \nu_d(\mathbb{P}^n) \cap H$ for some hyperplane $p$, where $\nu_d$ is the Veronese embedding. Therefore by the Lefschetz hyperplane thoerem, for $0 \le k < n-1$, we have

$$H^k(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & k \text{ even}, \\ 0, & k \text{ odd}. \end{cases}$$

Moreover, when $X$ is smooth, we have the same $H^k(X;\mathbb{Z})$ for $k > n$ by Poincare duality. If we replace $\mathbb{Z}$ by a field, this holds except the middle degree $n-1$.

If $Y = H \cap X$ is smooth, by Poincare duality we also have a *Gysin homomorphism* $i_* : H^*(Y) \to H^{*+2}(X)$. The composition

$$H^*(X) \xrightarrow{i^*} H^*(Y) \xrightarrow{i_*} H^{*+2}(X)$$

is given by $\eta = c_1(H) \cup -$, where $c_1(H) \in H^2(X)$ is the first Chern class of the hyperplane section.

**Theorem 2** (Hard Lefschetz) Let $X$ be a smooth projective variety of dimension $n$. Then for any embedding $X \hookrightarrow \mathbb{P}^N$ and $0 \leq i \leq n$, the map

$$\eta^i = c_1(H)^i \cup - : H^{n-i}(X;\mathbb{Q}) \to H^{n+i}(X;\mathbb{Q})$$

is an isomorphism.

The Hard Lefschetz can be generalized to any Kähler manifold. Let $X$ be a complex manifold. One can show that any $X$ can be endowed wit a Hermitian metric $h$ on $X$. Write $h = g - i\omega$, where $g$ is a Riemann metric and $\omega$ is an anti-symmetric 2-form.

**Definition 1** A *Kähler manifold* is a complex manifold with a Hermitian metric $h = g - i\omega$ where $d\omega = 0$.

**Example 2** Any complex smooth projective variety is Kähler. The idea is to pull back the Hermitian metric from $\mathbb{P}^n \times \mathbb{C}^{n+1}$.

**Theorem 3** (Kahler version) Let $X$ be a Kähler manifold of dimension $n$ with Kähler class $\omega$. Then for $0 \leq i \leq n$,

$$\eta^i = [\omega]^i \cup - : H^{n-i}(X;\mathbb{R}) \to H^{n+i}(X;\mathbb{R})$$

is an isomorphism.

**Remark 1** We are going to sketch the proof of this Kähler vesion of the Hard Lefschetz theorem using Hodge theory in the following several sections.

## Hodge theory on Riemannian manifolds

Let $V$ be a real vector space of dimension $m$. Let $\langle , \rangle : V \times V \to \mathbb{R}$ be an inner product. Let $\Lambda(V)$ be the exterior algebra of $V$. Then there is an induced inner product on $\Lambda(V)$ such that $\Lambda^p(V) \perp \Lambda^q(V)$ for $p \neq q$.

**Definition 2** An *orientation* of $V$ is a choice of a vector $dV \in \Lambda^m V$ of length 1. Define an operator $* : \Lambda(V) \to \Lambda(V)$ such that $* : \Lambda^p(V) \to \Lambda^{m-p}(V)$ is characterized by $u \wedge *v = \langle u, v\rangle dV$. One can check that $** = (-1)^{p(m-p)} \operatorname{Id}_{\Lambda^p V}$.

Let $(M, g)$ be a Riemannian manifold. Then $g^*$ is an inner product on $T^*M$, which extends to a smooth inner product on $\Lambda(T^*M)$. Let $A^p(M) = \Gamma(\Lambda^p(T^*M))$ be the smooth $p$-forms.

**Definition 3** An *orientation* on $M$ is a choice of a top form on $M$ with norm 1 in each fiber.

**Definition 4** If $M$ is oriented, we define the *Hodge star operator* $* : A^p(M) \to A^{m-p}(M)$ using the above local construction pointwise.

**Definition 5** For $u, v \in A^p(M)$, we define the inner product on $A^p(M)$ by

$$\langle\langle u, v\rangle\rangle = \int_M \langle u, v\rangle dV = \int_M u \wedge *v.$$

**Definition 6** Define $d^* : A^p(M) \to A^{p-1}(M)$ by $d^* = (-1)^{m(p+1)+1} * d *$. Using Stokes' theorem, one can show that $\langle\langle d^*u, v\rangle\rangle = \langle\langle u, dv\rangle\rangle$.

**Definition 7** Define the *Laplacian* $\Delta : A^p(M) \to A^p(M)$ by $\Delta = d^*d + dd^*$. The kernel $\mathcal{H}(M) \subseteq A^p(M)$ of $\Delta$ is called the space of *harmonic forms*.

**Proposition 1** $u \in A^p(M)$ is harmonic if and only if $du = 0$ and $d^*u = 0$.

**Definition 8** A direct check using definition.

**Theorem 4** (Hodge) Let $(M, g)$ be a compact oriented Riemannian manifold. Then there is an isomorphism (depending on $g$) $\mathcal{H}^p(M;\mathbb{R}) \cong H^p(M;\mathbb{R})$.

The proof of the Hodge theorem involves the analysis of elliptic operators in order to construct *Green's operator*, which we shall not get into here.

**Corollary 1** (Poincare Duality with real coefficients) The pairing

$$H^p(M;\mathbb{R}) \times H^{m-p}(M;\mathbb{R}) \to \mathbb{R}, \quad (u, v) \mapsto \int_M u \wedge v$$

is non-degenerated.

**Proof** Let $u$ be a harmonic form, then $*u$ is also harmonic by Proposition 1 and

$$\int_M (u \wedge *u) = ||u||^2 \neq 0.$$

So the pairing is non-degenerated. □

## Complex manifolds and the $p, q$ -Hodge Theorem

**Definition 9** Let $X$ be a complex manifold of dimension $n$. We denote $T_{\mathbb{R}}(X)$ the *real tangent bundle* (think of $X$ as real manifold). It is a real vector bundle of rank $2n$ with $T_{\mathbb{R},x}(X)=\mathbb{R}\langle\partial_{x_1},\dots,\partial_{x_n},\partial_{y_1},\dots,\partial_{y_n}\rangle$. The *complex tangent bundle* $T_{\mathbb{C}}(X)$ is defined to be $T_{\mathbb{R}}(X)\otimes_{\mathbb{R}}\mathbb{C}$, namely $T_{\mathbb{C},x}(X)=\mathbb{C}\langle\partial_{x_1},\dots,\partial_{x_n},\partial_{y_1},\dots,\partial_{y_n}\rangle$.

A map $f: M\to N$ induces the tangent map $f_*: T_{\mathbb{C}}M\to T_{\mathbb{C}}N$. If the $f$ is holomorphic, then the tangent map $f_*$ (or, its Jacobian matrix) has very restricted form, since the subspaces $\mathbb{C}\langle\partial_{z_1},\dots,\partial_{z_n}\rangle$ and $\mathbb{C}\langle\partial_{\bar z_1},\dots,\partial_{\bar z_n}\rangle$ are preserved under holomorphic change of variables.

**Definition 10** Define the *holomorphic tangent bundle* $T^{(1,0)}(X)$ to be the holomorphic subbundle of $T_{\mathbb{C}}(X)$ with $T_x^{(1,0)}(X)=\mathbb{C}\langle\partial_{z_1},\dots,\partial_{z_n}\rangle$. Similarly define $T^{(0,1)}$ be the *anti-holomorphic tangent* bundle with $T_x^{(0,1)}(X)=\mathbb{C}\langle\partial_{\bar z_1},\dots,\partial_{\bar z_n}\rangle$.

We have $\mathbb{R}$ -isomorphisms between $T_{\mathbb{R}}(X)$ and $T^{(1,0)}(X)$, $T^{(0,1)}(X)$ and also $T_{\mathbb{C}}(X)=T^{(1,0)}(X)\oplus T^{(0,1)}(X)$.

**Definition 11** Using the dual basis $dx_j, dy_j, dz_j, d\bar z_j$, we have similar notions of *real, complex, holomorphic and anti-holomorphic cotangent bundles*.

**Remark 2** Every complex manifold admits a canonical orientation.

**Definition 12** The space of *complex-valued smooth $p$ -forms* is defined to be $A^p_{\mathbb{C}}(X):=A^p_{\mathbb{R}}(X)\otimes_{\mathbb{R}}\mathbb{C}\cong C^\infty(\Lambda^p(T^*_{\mathbb{C}}(X)))$.

For $x\in X$, $p,q$ nonnegative integers, we denote $\Lambda^{p,q}(T^*_{\mathbb{C},x}):=\Lambda^p(T_x^{*(1,0)})\otimes\Lambda^q(T_x^{*(0,1)})\subseteq\Lambda^{p+q}(T^*_{\mathbb{C},x})$. Then

$$\Lambda^l(T^*_{\mathbb{C},x})=\bigoplus_{p+q=l}\Lambda^{p,q}(T^*_{\mathbb{C},x}),\quad \Lambda^l(T^*_{\mathbb{C}})=\bigoplus_{p+q=l}\Lambda^{p,q}(T^*_{\mathbb{C}}).$$

**Definition 13** A $(p,q)$ *-form* is an element of $A^{p,q}(X):=C^\infty(\Lambda^{p,q}(T^*_{\mathbb{C}}))$, which can be locally written as $\sum a_{IJ}dz_Id\bar z_J$.

We have $d(A^{p,q})\subseteq A^{p+1,q}\oplus A^{p,q+1}$. Let $\pi^{p,q}: A^l_{\mathbb{C}}\to A^{p,q}$ be the projection. Define the *Dolbeault operators* $\partial: A^{p,q}\to A^{p+1,q}$ by $\partial=\pi^{p,q}\circ d$ and similarly $\bar\partial$. Then $d=\partial+\bar\partial$ and $\partial\bar\partial+\bar\partial\partial=0$. We denote the corresponding *Dolbeault cohomology groups* of $\partial$ and $\bar\partial$ by $H^{p,q}_{\partial}$ and $H^{p,q}_{\bar\partial}$.

Let $h$ be a Hermitian form, then $g:=\operatorname{Re}h$ is a positive definite quadratic form. Let $X$ be a complex manifold with the Hermitian metric $h$, then we can define $dV$ and $*$ on $\Lambda(T^*_{\mathbb{R}}X)$ using $g^*$. We can extend them $\mathbb{C}$ -linearly to $\Lambda_{\mathbb{C}}(T^*_{\mathbb{C}}X)$. Then $u\wedge\overline{*v}=\langle u,v\rangle dV$ and $*:\Lambda^{p,q}(T^*)\to\Lambda^{n-p,n-q}(T^*)$ is a linear isometry. We can also extend the pairing on $A_{\mathbb{R}}$ to $A_{\mathbb{C}}$ by

$$\langle\langle u,v\rangle\rangle=\int_X\langle u,v\rangle dV=\int_X u\wedge\overline{*v}.$$

**Definition 14** Define $\partial^*=-*\bar\partial*$ and $\bar\partial^*=-*\partial*$. Again using Stokes' theorem, one can show that $(\partial,\partial^*)$ and $(\bar\partial,\bar\partial^*)$ are adjoint pairs.

**Definition 15** Define $\Delta_\partial=\partial\partial^*+\partial^*\partial$ and $\Delta_{\bar\partial}=\bar\partial\bar\partial^*+\bar\partial^*\bar\partial$. Define $\mathcal{H}^{p,q}_{\partial}=\ker(\Delta_\partial: A^{p,q}\to A^{p,q})$. Define similarly for $\mathcal{H}^{p,q}_{\bar\partial}$.

**Theorem 5** ( $p,q$ -Hodge Theorem) Let $(X,h)$ be a Hermitian manifold. Then there are isomorphisms of finite dimensional vector spaces

$$\mathcal{H}^{p,q}_{\partial}(X)\cong H^{p,q}_{\partial}(X),\quad \mathcal{H}^{p,q}_{\bar\partial}(X)\cong H^{p,q}_{\bar\partial}(X).$$

## The Hard Lefschetz theorem on Kähler manifolds

**Definition 16** A Hermitian manifold $(X,h)$ is *Kähler* if the real 2-form $-\operatorname{Im}(h)=\omega$ is closed, i.e., $d\omega=0$.

**Definition 17** Let $X$ be a Kähler manifold. Define the *Lefschetz operator* $\eta: A^l\to A^{l+2}$ by $u\mapsto\omega\wedge u$. As $d\omega=0$, this induces a map $H^l(X)\to H^{l+2}(X)$. Define $\nu: A^l\to A^{l-2}$ to be the adjoint to $\eta$. We have $\nu=*^{-1}\eta*$ since

$$\langle\langle \eta u, v\rangle\rangle = \int_X \omega \wedge u \wedge \overline{*v} = \langle\langle u, (*^{-1}\eta * v)\rangle\rangle.$$

**Lemma 1** $[\eta, \nu] = (l-n)\,\mathrm{Id}$ on $l$-forms.

**Corollary 2** The actions of $\eta, \nu$ and $\oplus_l (l-n)\,\mathrm{Id}|_{A^l}$ form an action of $\mathfrak{sl}_2$ on $\Lambda(X^*)$. This implies $\eta^{n-l} : \Lambda^l T^* \cong \Lambda^{2n-l} T^*$, hence $\eta^{n-l} : A^l \cong A^{2n-l}$.

By Corollary 2, to prove the Kähler version of the Hard Lefschetz Theorem, it suffices to show that $\eta^{n-l}$ sends harmonic forms to harmonic forms.

**Theorem 6** (Kahler identities)

a. $[\nu, \bar\partial] = -i\partial^*$, $[\nu, \partial] = i\bar\partial^*$.
b. $\partial\bar\partial^* + \bar\partial^*\partial = 0$.
c. $\Delta = 2\Delta_\partial = 2\Delta_{\bar\partial}$.

**Proof** For (a), see Griffiths-Harris. For (b), use (a). For (c), using the definition of $\Delta$ and (b), it suffices to check that $\Delta_\partial = \Delta_{\bar\partial}$, which can be shown using (a). □

**Corollary 3** $\mathcal{H}^{p,q}_\partial = \mathcal{H}^{p,q}_{\bar\partial}$. $\mathcal{H}^l_\Delta = \mathcal{H}^l_\partial = \mathcal{H}^l_{\bar\partial}$.

**Corollary 4** $\Delta$ commutes with $\eta$.

As $\Delta$ commutes with $\eta$, we now know that $\eta^{n-l}$ sends harmonic forms to harmonic forms, which implies the Hard Lefschetz Theorem.

## The Lefschetz decomposition and Hodge-Riemann bilinear relations ▲

**Definition 18** For $k \le n$, we define $A^k_{\mathrm{prim}} \subseteq A^k_{\mathbb{R}}$ to be $\ker(\eta^{n-k+1} : A^k \to A^{2n-k+2})$, called the space of *primitive $k$-forms*. We define $H^k_{\mathrm{prim}} \subseteq H^k_{\mathbb{R}}$ to be $\ker(\eta^{n-k+1} : H^k \to H^{2n-k+2})$, called the space of *primitive cohomology classes*.

The Hard Lefschetz Theorem then has the following easy consequence.

**Corollary 5** (Lefschetz Decomposition) Any $\alpha \in A^k_{\mathbb{R}}$ can be written uniquely as a sum $\alpha = \sum_r \eta^r \alpha_r$, where $\alpha_r \in A^{k-2r}_{\mathrm{prim}}$. For $k \le n$, we have the *Lefschetz decomposition*

$$H^k(X;\mathbb{R}) \cong \bigoplus_r \eta^r H^{k-2r}_{\mathrm{prim}}(X;\mathbb{R}).$$

**Remark 3** The same is true for bihomogeneous elements of $A_{\mathbb{C}}$.

**Lemma 2** If $\alpha \in A^{p,q}_{\mathrm{prim}} = A^{p,q} \cap H^k_{\mathrm{prim}}$, then $*\alpha = (-1)^{k(k+1)/2} i^{p-q} \frac{\eta^{n-k}}{(n-k)!}\alpha$, where $k = p+q$.

We omit the proof of the above useful fact.

Let $P : H^k(X;\mathbb{R}) \times H^{2n-k}(X;\mathbb{R}) \to \mathbb{R}$ be the Poincare duality pairing. Define the *intersection form* $Q$ on $H^k(X;\mathbb{R})$ using $\eta^{n-k}$,

$$Q(\alpha,\beta) = P(\eta^{n-k}\alpha, \beta) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta.$$

Then $Q$ is symmetric when $k$ is even and skew-symmetric if $k$ is odd. Define the Hermitian form $\mathbb{H}_k$ on $H^k(X;\mathbb{C})$ by

$$\mathbb{H}_k(\alpha,\beta) = i^k Q(\alpha, \bar\beta).$$

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**Lemma 3** The Lefschetz decomposition on $H^k(X;\mathbb{C})$ is an orthogonal decomposition, i.e., $\eta^r H^{k-2r}_{\mathrm{prim}} \perp \eta^{r'} H^{k-2r'}_{\mathrm{prim}}$ for $r \neq r'$.

**Proof**

$$\mathbb{H}_k(\eta^r\alpha, \eta^{r'}\beta) = \int_X \omega^{n-k} \wedge (\omega^r \wedge \alpha) \wedge (\omega^{r'} \wedge \beta) = \int_X \omega^{n-k+r+r'} \wedge \alpha \wedge \beta = 0$$

as $n-k+r+r' > n-k+2\min(r,r')$. □

**Theorem 7** (Hodge-Riemann bilinear relation) The decomposition

$$H^k(X;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

is orthogonal with respect to $\mathbb{H}_k$. Moreover, $(-1)^{k(k+1)/2} i^{p-q-k} \mathbb{H}_k$ is positive definite on $H^{p,q}_{\text{prim}} = H^{p,q} \cap H^k_{\text{prim}}$.

**Proof** $\omega^{n-k} \wedge \alpha \wedge \bar{\beta}$ is a $(n-k, n-k) + (p,q) + (p',q')$-form in $A^{2n} = A^{n,n}$. It is non-zero only when $p + q' = k$ and $p' + q = k$, hence $(p,q) = (p',q')$, which implies the orthogonality. Let $a \in H^{p,q}_{\text{prim}}$, then there exists $\alpha$ primitive and harmonic with $[\alpha] = a$. So $\bar{\alpha}$ is also primitive and harmonic. Then by Lemma 2,

$$\begin{aligned} \frac{(-1)^{k(k+1)/2} i^{p-q-k}}{(n-k)!} \mathbb{H}_k(\alpha, \alpha) &= \frac{(-1)^{k(k+1)/2} i^{p-q}}{(n-k)!} \int_X \alpha \wedge \omega^{n-k} \wedge \bar{\alpha} \\ &= \int_X \alpha \wedge *\bar{\alpha} = ||\alpha||^2. \end{aligned}$$

The positive definiteness then follows. □

**Remark 4** Using the Hodge-Riemann bilinear relations, one can show the *Hodge index theorem*: suppose $n$ is even, then the signature of the intersection form $Q$ on $H^n(X;\mathbb{R})$ is equal to $\sum_{p+q\le 2n} (-1)^p h^{p,q}$, where $h^{p,q} = \dim H^{p,q}(X)$ are the Hodge numbers.

## Cohomology of sheaves and categories

We construct a sequence of vector bundles $\Lambda^i(T^*M)$ which is an injective ?? resolution of the trivial bundle. The global sections $A^i = C^\infty(\Lambda^i(T^*))$ form a complex and its cohomology is the de Rham cohomology. More generally, we would like to replace the vector bundles $\Lambda^k(T^*M)$ by any sheaves. In abstract language, we would like to define a new category of sheaves such that

a. Any object in $\mathcal{C}$ should be identified with all its resolutions.

b. Functors should only be applied to special representatives in the isomorphism class of an object to obtain its cohomology.

**Definition 19** A morphism of $f : K^\cdot \to L^\cdot$ of complexes in an abelian category is called a *quasi-isomorphism* if the induced morphism on the cohomology $H^\cdot(f)$ is an isomorphism.

**Example 3** Let $X \to I.$ be a resolution. Then the morphism

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & I_0 \arrow[r] & I_1 \arrow[r] & I_2 \arrow[r] & \cdots \end{tikzcd}$$

is an quasi-isomorphism.

**Definition 20** Let $\mathcal{A}$ be an abelian category and $K(\mathcal{A})$ be the category of complexes in $\mathcal{A}$. There is a category $\mathcal{D}(\mathcal{A})$ and a functor $Q : K(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ (called the *derived category of* $\mathcal{A}$ )such that

a. If $f$ is a quasi-isomorphism, then $Q(f)$ is an isomorphism.

b. For any functor $F : K(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ satisfying (a), there is a unique functor $G : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ such that $F = G \circ Q$.

Let $\mathcal{B}$ be a category and $S$ be a class of morphisms in $\mathcal{B}$. We can construct a functor $Q : \mathcal{B} \to \mathcal{B}[S^{-1}]$ such that for any $f \in S$, $Q(f)$ is an isomorphism, where $B[S^{-1}]$ has the same object as $\mathcal{B}$ with morphisms in $S$ formally inverted. However, this construction loses the additive structure. To solve this problem, we shall only do this construction for $S$ a localizing system.

**Definition 21** Let $S$ be a class of morphisms in $\mathcal{B}$. We say that $\mathcal{B}$ is a *localizing system* (or *multiplicative system*) for the category $\mathcal{B}$ if

a. $S$ is closed under composition: $\mathrm{Id}_X \in S$ for any $X \in \mathcal{B}$ and for any $s, t \in S$, we have $st \in S$ (if they are composable).

b. Extension condition: for any $f \in \mathrm{Mor}(\mathcal{B})$, $s \in S$, we have the following commutative diagram

$$\begin{tikzcd} W \arrow[r, dashed, "\exists g"] \arrow[d, dashed, "t\in S"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

and a similar commutative diagram with arrows inverted.

c. For $f, g : X \to Y$, there exists $s \in S$ such that $sf = sg$ if and only if there exists $t \in S$ such that $ft = gt$.

Unfortunately, the class of quasi-isomorphisms does not form a localizing system for $K(\mathcal{A})$. In order to construct the derived category, we need to pass to the homotopy category $\mathcal{K}(\mathcal{A})$. It turns out that the quasi-isomorphisms form a localizing system $Q$ for $\mathcal{K}(\mathcal{A})$ and the category $\mathcal{K}(\mathcal{A})(Q^{-1})$ is the desired derived category $\mathcal{D}(\mathcal{A})$.

**Remark 5** The construction and properties of homotopy category, derived category and derived functor are discussed in class, which are not reproduced here. For this part of material, see Haiman's notes .

## Aside on spectral sequences

**Definition 22** Let $A \in \mathcal{A}$. A *decreasing filtration* $F^{\cdot} A$ is a sequence

$$\cdots \hookrightarrow F^p A \hookrightarrow F^{p-1} A \hookrightarrow \cdots.$$

Let $A^{\cdot} \in K(\mathcal{A})$ be a complex, a *decreasing filtration* $F^p A^{\cdot}$ is a double sequence $F^p A^k$ such that $F^p A^k \subseteq A^k$ and $d(F^p A^k) \subseteq F^p A^{k+1}$. The filtration on $A^{\cdot}$ induces a filtration on $H^i(A^{\cdot})$ given by $F^p H^i(A^{\cdot}) = \mathrm{Im}(H^i(F^p A^{\cdot}) \to H^i(A^{\cdot}))$.

**Theorem 8** (Spectral sequence of a filtered complex) Let $(A^{\cdot}, F^p A^{\cdot})$ be a filtered complex as above such that for any $k$, there exists $l \gg 0$, $F^l A^k = 0$. Then there exists complexes $(E_r^{p,q}, d_r)$, where $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that

a. $E_0^{p,q} = \mathrm{Gr}_p^F A^{p+q} = F^p A^{p+q} / F^{p+1} A^{p+q}$ and $d_0$ is the map induced from $d$.

b. $E_{r+1}^{p,q} = \ker(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}) / \mathrm{Im}(d_r : E_r^{p-r,q+r-1} \to E_r^{p,q})$.

c. For a fixed $n = p + q$ and $r \gg 0$, $E_r^{p,q} = \mathrm{Gr}_p^F H^{p+q}(A^{\cdot})$.

**Proof** Let $Z_r^{p,q} := \{x \in F^p A^{p+q} : dx \in F^{p+r} A^{p+q+1}\}$. Note that $Z_{r-1}^{p+1,q-1} \subseteq Z_r^{p,q}$ and $dZ_{r-1}^{p-r+1,q+r-2} \subseteq Z_r^{p,q}$. We can define $B_r^{p,q} := Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2}$. Then we define $E_r^{p,q} := Z_r^{p,q} / B_r^{p,q}$ and $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$. One can check that $d_r^2 = 0$.

a. $Z_0^{p,q} = F^p A^{p+q}$, $B_0^{p,q} = F^{p+1} A^{p+q}$. So $E_0^{p,q} = \mathrm{Gr}_p^F A^{p+q}$.

b. By construction.

c. For $k \gg 0$, $F^k A^n = 0$. So $Z_{k+1}^{p,q} = \ker(d : F^p A^n \to F^p A^{n+1})$ and $Z_k^{p+1,q-1} = \ker(d : F^{p+1} A^n \to F^{p+1} A^{n+1})$. Therefore $dZ_k^{p-k,p+q-1} = \mathrm{Im}\, d \cap F^p A^n$ and

$$\begin{aligned} E_{k+1}^{p,q} &= \ker(d : F^p A^n \to F^p A^{n+1}) / \ker(d : F^{p+1} A^n \to F^{p+1} A^{n+1}) + \mathrm{Im}\, d \cap F^p A' \\ &\cong \mathrm{Im}(H^n(F^p A^{\cdot}) \to H^n(A^{\cdot})) / \mathrm{Im}(H^n(F^{p+1} A^{\cdot}) \to H^n(A^{\cdot})) \\ &= \mathrm{Gr}_p^F H^n(A^{\cdot}). \end{aligned}$$

This completes the proof. □

**Example 4** Let $(K^{\cdot,\cdot}, D_1, D_2)$ be a double complex and $\mathrm{Tot}(K^{\cdot,\cdot})$ be the total complex. Define $F^p \mathrm{Tot}^n = \bigoplus_{r \geq p, r+s=n} K^{r,s}$. The associated spectral sequence for this filtration is $E_0^{p,q} = F^p \mathrm{Tot}^{p+q} / F^{p-1} \mathrm{Tot}^{p+q} = K^{p,q}$ and $E_1^{p,q} = H^q(K^{p,\cdot})$.

## Ordinary cohomology = Sheaf cohomology of the constant sheaf

**Lemma 4** If $X$ is a contractible space, then $R\Gamma(\underline{k}_X) = \underline{k}_X$.

**Proof** Cf. Corollary 2.7.7 (iii) of Kashiwara M., Schapira P, *Sheaves on manifolds*. □

**Proposition 2** Let $\mathcal{F}$ be a sheaf on $X$, $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of $X$, such that for any $J \subseteq I$, $q > 0$, $H^q(U_J; \mathcal{F}) = 0$. Then $R^i\Gamma(\mathcal{F}) \cong \check{H}^i(\mathcal{U}; \mathcal{F})$.

**Remark 6** This justifies writing $H^i(X; \mathcal{F}) = R\Gamma^i(\mathcal{F})$.

**Proof** To define the Cech cohomology $\check{H}$, we form $\check{C}^k := \bigoplus_{|J|=k+1} \mathcal{F}(U_J)$ with

$$(d\phi)(U_{i_0,\ldots,i_{k+1}}) = \sum_{j=0}^{k+1} (-1)^j \, \mathrm{Res}_{U_{i_0,\ldots,u_{k+1}}} \phi(U_{i_0,\ldots,\hat{i}_j,\ldots,i_{k+1}}).$$

So we get a complex of sheaves $\check{\mathcal{C}}^{\cdot}(\mathcal{U};\mathcal{F})$ on $X$ by restrictions. The sequence

$$\mathcal{F} \to \check{\mathcal{C}}^{\cdot}(\mathcal{U};\mathcal{F})$$

is a resolution, hence $\mathcal{C}$ is quasi-isomorphic to $\check{\mathcal{C}}^{\cdot}(\mathcal{U};\mathcal{F})$. Let $\mathcal{F} \to K^0 \to K^1 \to \cdots$ be a flabby resolution of $\mathcal{F}$. Then we can construct with the Cech complex of the $K^i$ 's a double complex $\check{\mathcal{C}}^q(\mathcal{U};K^p)$. Consider the spectral sequence from the filtration $F^p\mathrm{Tot}^n = \bigoplus_{r+s=n, r\geq p} A^{r,s}$ of the global sections. As $\check{\mathcal{C}}^q(\mathcal{U};K^p)$ is flabby and the columns are resolutions, we get

$$E_1^{p,q} = \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\ \Gamma(K^0) \arrow[u] \arrow[r] & \Gamma(K^1) \arrow[u] \arrow[r] & \Gamma(K^2) \arrow[u] \arrow[r] & \cdots \end{tikzcd}$$

And

$$E_2^{p,q} = \begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \\ R^0\Gamma(\mathcal{F}) \arrow[u] \arrow[r] & R^1\Gamma(\mathcal{F}) \arrow[u] \arrow[r] & R^2\Gamma(\mathcal{F}) \arrow[u] \arrow[r] & \cdots \end{tikzcd}$$

Using the other filtration we know that $R^q\Gamma(\mathcal{F}) = \check{H}^q(\mathcal{U},\mathcal{F})$. □

**Remark 7** If $K$ is flabby, then so is $\check{\mathcal{C}}^q(\mathcal{U};K)$.

**Corollary 6** If $X$ is a CW-complex, then $H^i(X;k) \cong R^i\Gamma(\underline{k}_X)$.

**Proof** Choose a cover of $X$ such that $U_J$ is contractible for any $J \subseteq I$ and apply the previous proposition. □

**Proposition 3** Let $f: X \to Y$ be a continuous map, then $R^q f_*\mathcal{F}$ is the sheafification of the presheaf $U \mapsto H^q(\mathcal{F}|_{f^{-1}(U)})$

**Proof** Let $\mathcal{F} \to I^{\cdot}$ be an injective resolution. Then $R^q f_*\mathcal{F} = H^q(f_*I^{\cdot})$ is the sheafification of $U \mapsto H^q(I^{\cdot}|_{f^{-1}(U)}) = H^q(\mathcal{F}|_{f^{-1}(U)})$. □

So $Rf_*$ gives a good notion of cohomology in *families*. For example, to compute the cohomology of $X$, we can compute $H^{\cdot}(X, \underline{k}_X) = H^{\cdot}(Y, Rf_*(\underline{k}_X))$ for any map $f: X \to Y$.

## Degeneracy of the Leray spectral sequences

Let $\mathcal{A}, \mathcal{B}$ be abelian categories and $F: \mathcal{A} \to \mathcal{B}$ be a left exact functor. Suppose $A$ has a class of $F$ -acyclic objects.

**Theorem 9** For any $M^{\cdot} \in \mathcal{D}^+(\mathcal{A})$, there exists a spectral sequence $E_2^{p,q} = R^pF(H^q(M^{\cdot})) \Rightarrow E_\infty^{p,q} = R^{p+q}F(M^{\cdot})$.

More generally, let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$, where $F$, $G$ are two left exact functors. Suppose $A$ has a class of $F$ -acyclic objects and $j$ has a class of $G$ -acyclic objects such that $F(F - \text{acyclic})$ is $G$ -acyclic.

**Theorem 10** (Grothendieck spectral sequence) For any $M^{\cdot} \in \mathcal{D}^+(\mathcal{A})$, there exists a spectral sequence $E_2^{p,q} = R^pG(R^qF(M^{\cdot})) \Rightarrow E_\infty^{p,q} = R^{p+q}(GF)(M^{\cdot})$.

Let $f: X \to Y$ be a smooth fiber bundle with smooth compact fibers $F$. Then the Grothendieck spectral sequence associated to $R\Gamma Rf_*$ applied to the constant sheaf $\underline{k}_X$ becomes the classical *Leray spectral sequence*

$$E_2^{p,q} = H^p(Y, \underline{H}^q(F)) \Rightarrow H^{p+q}(X),$$

where $\underline{H}^q(F)$ is the local system on $Y$ with fiber $H^q(F)$.

**Example 5** For the Hopf fibration $S^3 \to S^2$, $\underline{H}^0(F) = R^0f_*(\underline{k}_{S^3}) = \underline{k}_{S^2}$ and $\underline{H}^1(F) = R^1f_*(\underline{k}_{S^3}) = \underline{k}_{S^2}$ are constant sheaves as $\pi_1(S^2) = 0$.

**Definition 23** A *family of projective manifolds* is a proper, holomorphic submersion of smooth varieties $f: X \to Y$ factoring through $X \hookrightarrow Y \times \mathbb{P}^N$ with fibers smooth projective varieties.

**Remark 8** Ehresmann's theorem implies this is a smooth fibration.

Deligne proved the degeneracy of the Leray spectral sequence for a family of projective manifolds.

**Theorem 11** (Deligne) Let $f: X \to Y$ be a family of projective manifolds.

a. Version I: the Leray spectral sequence $E_2^{p,q} = H^q(Y; \underline{H}^q(F;\mathbb{C})) \Rightarrow E_\infty^{p,q} = H^{p+q}(X;\mathbb{C})$ degenerates at the $E_2$ -page. In particular, $H^i(X;\mathbb{C}) \cong \bigoplus_{p+q=i} H^p(Y; \underline{H}^q(F;\mathbb{C}))$ ; Version II: the derived pushforward $Rf_*\underline{\mathbb{C}}_X \cong \bigoplus_{i=0}^{2n} R^if_*\underline{\mathbb{C}}_X[-i]$ and $R^if_*\underline{\mathbb{C}}_X[-i] = \underline{H}^i(F;\mathbb{C})$.

b. The monodromy representation $\pi_1(Y, y_0) \to GL(H^q(F_{y_0};\mathbb{C}))$ for any $y_0 \in Y$ is *semisimple*.

**Remark 9** $H^i(X;\mathbb{C}) \cong \bigoplus_{p+q=i} H^p(Y; \underline{H}^q(F;\mathbb{C}))$ is NOT true for the Hopf fibration as one can check directly. It is a real algebraic proper submersion, but not holomorphic.

**Remark 10** We need smooth *projective* condition on the fibers, since the usage of the Hard Lefschetz Theorem is crucial.

**Remark 11** It is important that the coefficients of the cohomologies are $\mathbb{C}$ (at least characteristic 0). For example, in characteristic 2, the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not semisimple (consider $f: \mathbb{C}^\times \to \mathbb{C}^\times, z \mapsto z^2$, $f_*\underline{k}_X$ is a rank two local system and $0 \to \underline{k}_X \to f_*\underline{k}_X \to \underline{k}_X$ does not split).

**Proof** (Version II implies Version I) Let $K^\cdot \in \mathcal{D}^+(\mathrm{Sh}(Y))$. The Leray spectral sequence is the spectral sequence $E_2^{p,q} = R^p\Gamma(H^q(K^\cdot)) \Rightarrow E_\infty^{p+q} = R^{p,q}\Gamma(K^\cdot)$ applied to $K^\cdot = Rf_*\underline{\mathbb{C}}_X$ and it is degenerate if $K^\cdot$ is concentrated in a single degree. If $Rf_*\underline{\mathbb{C}}_X = \bigoplus_i R^if_*\underline{\mathbb{C}}_X[-i]$, then the Leray sequence is the direct sum of the spectral sequence of $R^if_*\underline{\mathbb{C}}_X[-i]$. Hence the Leray spectral sequence degenerates. □

**Remark 12** Since $\Gamma(\mathcal{F}) = \mathrm{Hom}(\underline{k}_X, \mathcal{F})$, we know that $H^i(X;k) = R^i\,\mathrm{Hom}(\underline{k}_X, \underline{k}_X) = \mathrm{Hom}_{\mathcal{D}}(\underline{k}_X, \underline{k}_X[i])$. Moreover, for $\alpha \in H^i(X;k)$, $\beta \in H^j(X;k)$, the cup product $\alpha \cup \beta$ is given by $\underline{k}_X \xrightarrow{\alpha} \underline{k}_X[i] \xrightarrow{\beta} \underline{k}_X[i+j]$.

**Proof** (Version II) Suppose $X \subseteq Y \times \mathbb{P}^N$. Let $\eta \in H^2(X;\mathbb{C})$ be the first Chern class of the hyperplane section. By the previous remark, we have $\eta: \underline{\mathbb{C}}_X \to \underline{\mathbb{C}}_X[2]$ and induces $\eta: R^if_*\underline{\mathbb{C}}_X \to R^{i+2}f_*\underline{\mathbb{C}}_X$. On each fiber, $\eta_{y_0}^i : H^{n-i}(F;\mathbb{C}) \to H^{n+i}(F;\mathbb{C})$ is an isomorphism by the Hard Lefschetz. Therefore $\eta^i : R^{n-i}f_*\underline{\mathbb{C}}_X \to R^{n+i}f_*\underline{\mathbb{C}}_X$ is an isomorphism on each stalk, thus $\eta^i$ is an isomorphism. Now applying the following Key Lemma to $\mathcal{A} = \mathrm{Sh}(Y;\mathbb{C})$, $X = Rf_*\underline{\mathbb{C}}_X[n]$ and $\eta$ the Lefschetz operator, we know that $Rf_*\underline{\mathbb{C}}_X \cong \bigoplus_i R^if_*\underline{\mathbb{C}}_X[-i]$. □

**Lemma 5** (Key Lemma) Let $\mathcal{A}$ be an abelian category. Let $X \in \mathcal{D}(\mathcal{A})$ and $\eta: X \to X[2]$ such that the induced maps $\eta^i : H^{-i}(X) \to H^i(X)$ are isomorphisms. Then $X \cong \bigoplus_i H^i(X)[-i]$.

**Proof** (van den Bergh) By induction downward on $d$, where $d$ is an integer such that $H^j(X) = 0$ for any $|j| > d$. Then

$$\alpha: H^{-d}(X) \cong \tau_{\le 0}(X[-d]) \to X[-d] \xrightarrow{\eta^d} X[d] \to \tau_{\ge 0}(X[d]) = H^d(X)$$

is the isomorphism induced by $\eta^d$ (by taking $H^0$). Using

$$\begin{tikzcd} H^{-d}(X) \arrow[r, "\alpha"] \arrow[d, "i"] & H^d(X) \arrow[d, "p"'] \\ X[-d] \arrow[r, "\eta^d"'] & X[d], \end{tikzcd}$$

We get a map $\Psi: X \to H^{-d}(X)[d] \oplus H^d(X)[-d]$ and $\Phi: H^{-d}(X)[d] \oplus H^d(X)[-d] \to X$. And $\Psi \circ \Phi = (\alpha^{-1} \circ \alpha, \alpha \circ \alpha^{-1}) = (\mathrm{Id}, \mathrm{Id})$. Using properties of triangles, we get $X = H^{-d}(X)[d] \oplus X' \oplus H^d(X)[-d]$. To finish the proof, we check that $H^j(X') = 0$ whenever $|j| \ge d$. □

Next part of this course is to define more operations on sheaves and establish the Poincare duality of sheaves for singular varieties.

## Operations on sheaves

Let $k$ be a commutative ring and $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X;k)$. We define $\mathcal{H}om(\mathcal{F},\mathcal{G}) \in \mathrm{Sh}(X;k)$ be the sheaf $U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. Then $\mathrm{Hom}(\mathcal{F},\mathcal{G}) = \Gamma(X, \mathcal{H}om(\mathcal{F},\mathcal{G}))$. Because $\mathcal{H}om$ is a left exact functor, we obtain a right derived functor $R\mathcal{H}om : D^-(\mathrm{Sh})^{\mathrm{op}} \times D^+(\mathrm{Sh}) \to D^+(\mathrm{Sh})$.

Similarly, we define $\mathcal{F} \otimes \mathcal{G} \in \mathrm{Sh}(X;k)$ by $U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U)$. Then $(\mathcal{F} \otimes \mathcal{G})_x = \mathcal{F}_x \otimes \mathcal{G}_x$. Because $\otimes$ is right exact, we obtain a left derived functor $\otimes^L : D^-(\mathrm{Sh}) \times D^-(\mathrm{Sh}) \to D^-(\mathrm{Sh})$. $\mathcal{F}$ is called *flat* if $\mathcal{F} \otimes -$ is exact. Note that $\mathcal{F}$ is flat if and only if $\mathcal{F}_x$ is a flat $k$ -module for any $x \in X$. Flat sheaves form an acyclic class for $\otimes$.

**Proposition 4** $\mathcal{H}om(\mathcal{H} \otimes \mathcal{F}, \mathcal{G}) \cong \mathcal{H}om(\mathcal{H}, \mathcal{H}om(\mathcal{F},\mathcal{G}))$.

**Corollary 7** $R\mathcal{H}om(\mathcal{H} \otimes^L \mathcal{F}, \mathcal{G}) \cong R\mathcal{H}om(\mathcal{H}, R\mathcal{H}om(\mathcal{F}, \mathcal{G}))$.

**Proof** If $\mathcal{F}$ is locally free and $\mathcal{G}$ injective, then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is injective. So

$$R\mathcal{H}om(\mathcal{F} \otimes^L \mathcal{H}, \mathcal{G}) = R\mathcal{H}om(\mathcal{F} \otimes \mathcal{H}, \mathcal{G}) = R\mathcal{H}om(\mathcal{H}, \mathcal{H}om(\mathcal{F}, \mathcal{G})) = R\mathcal{H}om(\mathcal{H}, R\mathcal{H}$$

The general case follows by taking a locally free resolution of $\mathcal{F}$ and an injective resolution of $\mathcal{G}$. □

**Definition 24** Let $f : X \to Y$ be a map of topological spaces. For any open subset $U \subseteq Y$, let $f_!(\mathcal{F})(U) = \{s \in \Gamma(f^{-1}(U); \mathcal{F}) : f : \mathrm{supp}(s) \to U \text{ is proper}\}$. Then $f_!(\mathcal{F})$ forms a subsheaf of $f_*(\mathcal{F})$, called the *direct image with compact support*. Note that $f_!$ is a left exact functor.

**Definition 25** Define $\Gamma_c(X; \mathcal{F}) = \{s \in \Gamma(X; \mathcal{F}) : \mathrm{supp}(s) \text{ is compact Hausdorff}\}$. Then $\Gamma_c(X; \mathcal{F}) = a_! \mathcal{F}$ for the morphism $a : X \to \mathrm{point}$.

**Proposition 5** Let $f : X \to Y$, where $X, Y$ locally compact. Then for any $y \in Y$, $\alpha : f_!(\mathcal{F})_y \to \Gamma_c(f^{-1}(y); \mathcal{F}|_{f^{-1}(y)})$ is an isomorphism.

**Proof** For any $s \in f_!(\mathcal{F})_y$, there exists $U$ an open neighborhood of $y$ and $t \in \Gamma(f^{-1}(U); \mathcal{F})$ such that $\mathrm{supp}(t) \to U$ is proper. It follows that $t|_{f^{-1}(y)}$ has compact support $f^{-1}(y) \cap \mathrm{supp}(t)$. We define $\alpha(s) = t|_{f^{-1}(y)}$. One can check that $\alpha$ is injective. For surjectivity, we use the following lemma. Hence $\alpha$ is an isomorphism. □

**Lemma 6** If $X$ is Hausdorff (resp. paracompact), $Z \subseteq X$ is compact (resp. closed). Then $\psi : \lim_{Z \subseteq U} \Gamma(U; \mathcal{F}) \to \Gamma(Z; \mathcal{F}) := \Gamma(Z; \mathcal{F}|_Z)$ is an isomorphism (i.e. we do not need to sheafify).

**Example 6** Consider $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$. Then $(j_! \underline{k}_{\mathbb{C}^\times})_0 = 0$ as the embedding $U^\times \hookrightarrow U$ is never proper.

**Definition 26** A sheaf $\mathcal{F}$ is *soft* if for any $K \subseteq X$ compact, the restriction map $\Gamma(X; \mathcal{F}) \to \Gamma(K; \mathcal{F})$ is surjective.

**Remark 13** Flabby sheaves are soft.

**Lemma 7** $\mathcal{F}$ is soft if and only if for any closed subset $Z \subseteq X$, the restriction map $\Gamma_c(X; \mathcal{F}) \to \Gamma_c(Z; \mathcal{F})$ is surjective.

**Corollary 8** If $\mathcal{F}$ is soft, then for any locally closed embedding $i : Z \hookrightarrow X$, $i^* \mathcal{F}$ is also soft.

**Proof** For any $Z' \subseteq Z \subseteq X$ closed, we have the surjection $\Gamma_c(Z; \mathcal{F}) \to \Gamma_c(Z'; \mathcal{F})$ by the softness of $\mathcal{F}$. □

**Proposition 6** If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact with $\mathcal{F}'$ soft. Then $0 \to f_! \mathcal{F}' \to f_! \mathcal{F} \to f_! \mathcal{F}'' \to 0$ is also exact. In particular, $\Gamma_c(X; -)$ is exact.

**Proof** By Corollary 8, $\mathcal{F}'|_{f^{-1}(y)}$ is soft for any $y \in Y$. Since it is enough to check the exactness on stalks, we reduce to the exactness of $\Gamma_c(X; -)$ by Proposition 5. By the left exactness of $f_!$, we only need to check the surjectivity of $g : \Gamma_c(X; \mathcal{F}) \to \Gamma_c(X; \mathcal{F}'')$. Let $s'' \in \Gamma_c(X; \mathcal{F}'')$. Choose a compact open subset $U \supseteq \mathrm{supp}(s'')$. Replace $\mathcal{F}$ by $\mathcal{F}|_U$ and $X$ by $U$, we may assume that $X$ is compact. Giving $s''$ is the same as giving a finite compact cover $\{K_i\}$ of $X$ and $s_i \in \Gamma(K_i; \mathcal{F})$ such that $g(s_i) = s''|_{K_i}$. One can check $(s_i - s_j)|_{K_i \cap K_j} = f(s')$ for some $s' \in \Gamma(K_1 \cap K_2, \mathcal{F}')$. By the softness of $\mathcal{F}'$, we get a global section $\tilde{s}$ maps to $s'$. Replace $s_2$ by $s_2 + \tilde{s}|_{K_2}$, then $s_1|_{K_1 \cap K_2} = s_2|_{K_1 \cap K_2}$. Now an induction shows that $K^i$ can be glued to be a section of $\mathcal{F}$. □

**Proposition 7** If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is exact and $\mathcal{F}', \mathcal{F}$ are soft, then $\mathcal{F}''$ is soft too.

**Proof** For any $Z \subseteq X$ closed, we have the following diagram

$$\begin{tikzcd}
\Gamma_c(X; \mathcal{F}) \arrow[r] \arrow[d, two heads] & \Gamma_c(X; \mathcal{F}'') \arrow[d, dashed, two heads] \\
\Gamma_c(Z; \mathcal{F}) \arrow[r, two heads] & \Gamma_c(Z; \mathcal{F}'').
\end{tikzcd}$$

Hence $\mathcal{F}''$ is soft. □

The above two propositions together imply the following theorem.

**Theorem 12** Soft sheaves form an acyclic class for $f_!$.

In fact more is true:

**Proposition 8** If $X$ is locally compact and countable at $\infty$ (its one-point compactification is Hausdorff). Then soft sheaves form an acyclic class for $\Gamma(X; -)$.

**Example 7** The de Rham resolution is a soft resolution, hence $R^i\Gamma(X, \underline{\mathbb{R}}_X) = H^i(X; \mathbb{R})$.

By Remark 13, we know there exists enough soft sheaves. So we have the right derived functor $Rf_! : \mathcal{D}^+(\mathrm{Sh}(X)) \to \mathcal{D}^+(\mathrm{Sh}(X))$. In particular, we have the derived functor $R\Gamma_c = R(a_x)_!$. Define the *cohomology with compact support* $H^i_c(X; \mathcal{F}) := R^i\Gamma_c(X; \mathcal{F})$. Note that $H^i_c(X; \underline{k}_X) = H^i_c(X; k)$.

**Theorem 13** (Proper Base change) If we have a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Then there exists a canonical isomorphism $g^* f_! \cong f'_!(g')^*$.

**Proof** There exists a canonical map $f_!(g')_* \to g_* f'_!$. By adjunction of $(g^*, g_*)$, have $\mathrm{Hom}(g^* f_! \mathcal{F}, f'_!(g')^* \mathcal{F}) = \mathrm{Hom}(f_! \mathcal{F}, g_* f'_!(g')^* \mathcal{F})$, which corresponds to $f_! \to f_!(g')_*(g')^* \to g_* f'_!(g')^*$. This induces an isomorphism. □

**Proposition 9** (Projection formula) There exists a natural map $f_! \mathcal{G} \otimes \mathcal{F} \to f_!(\mathcal{G} \otimes f^* \mathcal{F})$. It is an isomorphism if $\mathcal{F}$ is flat.

**Corollary 9**

a. $g^* R f_! \cong R f'_!(g')^*$.
b. $R f_!(\mathcal{G} \otimes^L f^* \mathcal{F}) \cong R f_! \mathcal{G} \otimes^L \mathcal{F}$.

**Example 8** (Stalks of $Rf_*$) $H^l(Rf_* \mathcal{F})$ is naturally isomorphic to the sheafification of the presheaf $U \mapsto H^l(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})$. Denote $i : \{y\} \hookrightarrow Y$. As $i^*$ is exact, we know that $H^l(i^* R f_* \mathcal{F}) = i^* H^l(R f_* \mathcal{F}) = \lim_{y \in U} H^l(f^{-1}(U); \mathcal{F}|_{f^{-1}(U)})$. For example, let $j : \mathbb{C}^\times \hookrightarrow \mathbb{C}$ and $i : \{0\} \hookrightarrow \mathbb{C}$, then $H^l(i^* j_* \underline{k}_{\mathbb{C}^\times}) = \lim_{y \in U} H^l(U \setminus \{0\}) = H^l(S^1; k)$.

**Example 9** (Stalks of $Rf_!$) Using the base change

$$\begin{tikzcd} f^{-1}(y) \arrow[r] \arrow[d] & X \arrow[d] \\ \{y\} \arrow[r] & Y, \end{tikzcd}$$

we know that $i^* R f_! \mathcal{F} = R\Gamma_c(\mathcal{F}|_{f^{-1}(y)})$.

We have seen in the exercise that if $i$ is a closed embedding, then $i_!$ has a right adjoint $i^! := i^* \circ \Gamma_Z$, where $\Gamma_Z$ is functor of taking the sheaf of sections with support inside $Z$. On the contrary, suppose $a : \mathbb{A}^1 \to \mathrm{pt}$, then $a_!$ does not admit a right adjoint. Does $Ra_!$ admit a right adjoint $a^!$ in the bounded below derived category? Or more generally, does $Rf_!$ admit a right adjoint functor for $f : X \to Y$ a continuous map between locally compact spaces? The answer is YES.

**Definition 27** There exists a right adjoint $f^! : \mathcal{D}^+(\mathrm{Sh}(Y; k)) \to \mathcal{D}^+(\mathrm{Sh}(X; k))$ to $f_!$. We call $f^!$ the *exceptional inverse image functor*. (See the next section for a brief discussion of $f^!$.)

**Definition 28** We define the *dualizing sheaf* $\omega_X := a^!_X \underline{k}_{\mathrm{pt}}$, where $a_X : X \to \mathrm{pt}$.

For a oriented manifold, $\omega_X \cong \underline{k}_X[\dim_{\mathbb{R}} X]$. In particular, by adjunction we have

$$\begin{aligned} R^l \,\mathrm{Hom}(R(a_X)_! \underline{k}_X, \underline{k}_{\mathrm{pt}}) &= R^l \,\mathrm{Hom}(\underline{k}_X, a^!_X \underline{k}_{\mathrm{pt}}) \\ &= R^l \,\mathrm{Hom}(\underline{k}_X, \underline{k}_X[\dim_{\mathbb{R}} X]) \\ &= H^{\dim_{\mathbb{R}} X + l}(X; k). \end{aligned}$$

On the other hand, $R^l \,\mathrm{Hom}(R(a_X)_! \underline{k}_X, \underline{k}_{\mathrm{pt}}) = H_c^{-l}(X; k)^\vee$. So in this way we recover the Poincare duality. (In general, for unoriented manifolds, the dualizing sheaf is the *orientation sheaf* shifted by the dimension.)

More generally, for any $\mathcal{F} \in \mathcal{D}^+(\mathrm{Sh}(X; k))$. We have

$$H_c^{-l}(X; \mathcal{F})^\vee \cong R^l \,\mathrm{Hom}(R(a_X)_! \mathcal{F}, \underline{k}_{\mathrm{pt}}) \cong H^l(X; R\mathcal{H}om(\mathcal{F}, \omega_X)).$$

**Definition 29** We define the *dualizing functor*

$$\mathbb{D} : \mathcal{D}^+(\mathrm{Sh}(X; k))^{\mathrm{op}} \to \mathcal{D}^+(\mathrm{Sh}(X; k)), \quad \mathcal{F} \mapsto R\mathcal{H}om(\mathcal{F}, \omega_X).$$

**Question** Given a "nice" singular space $X$, can we associate to $X$ some canonical object in $\mathcal{D}(\mathrm{Sh}(X; k))$ that is self-dual, i.e., $\mathbb{D}\mathcal{F} = \mathcal{F}$? If it is the case, then we will obtain a desired analog of the Poincare duality for singular spaces.

## Verdier duality

The original proof existence of $f^!$ is due to Verdier, using that we already know that $i^!$ exists for $A$ a locally closed embedding and then gluing them together. It is difficult since the derived category does not have good gluing property. Instead of Verdier's approach, we will give a proof due to A. Neeman.

**Definition 30** A *triangulated category* is an additive category $\mathcal{D}$ with an additive automorphism $[1] : \mathcal{D} \to \mathcal{D}$ and a set of distinguished triangles satisfying the following condition:

a. If a triangle is isomorphic to a distinguished triangle, then it is distinguished.

b. $X \xrightarrow{\mathrm{Id}} X \to 0 \to X[1]$ is distinguished.

c. For any map $X \xrightarrow{f} Y$ , there exists an distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ .

d. The following diagram commutes:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed, "\exists"] & X[1] \arrow[d] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1].
\end{tikzcd}$$

e. Rotations of distinguished triangles are distinguished.

f. The octahedral axiom:

$$\begin{tikzcd}
 & & A \arrow[dr, dashed] & & \\
 & Y \arrow[ur] \arrow[dr] & & C \arrow[dr, dashed] & \\
X \arrow[ur] \arrow[rr, bend right] & & Z \arrow[ur] \arrow[rr, bend right] & & B
\end{tikzcd}$$

**Theorem 14** For any abelian category $\mathcal{A}$ , the categories $\mathcal{K}(A)$ and $\mathcal{D}(A)$ are triangulated categories.

**Definition 31** A triangulated category $\mathcal{D}$ (with arbitrary coproducts) is called *well generated* if there exists a set $S$ of objects of $\mathcal{D}$ such that

a. For any $x \in \mathcal{D}$ , $x = 0$ if and only if for any $s \in X$ , $\mathrm{Hom}(s, x) = 0$ .

b. For any set of maps $X_i \to Y_i$ in $\mathcal{D}$ , if $\mathrm{Hom}(s, X_i) \to \mathrm{Hom}(s, Y_i)$ is surjective for any $s \in X$ and $i$ , then $\mathrm{Hom}(s, \coprod X_i) \to \mathrm{Hom}(s, \coprod Y_i)$ is also surjective.

c. There is a cardinal $\alpha$ such that all objects in $S$ are $\alpha$ -small, namely any map $s \to \coprod_{i \in I} X_i$ factors through $s \to \coprod_{i \in J} X_i$ , where $J \subseteq I$ and $J < \alpha$ .

**Theorem 15** (Alonso-Jeremias-Souto, Neeman) The unbounded derived theory of a Grothendieck abelian category is well generated. (A *Grothendieck abelian category* is an abelian category with generators such that small colimits and filtered colimits are exact.)

**Remark 14** $\mathrm{Sh}(X; k)$ is a Grothendieck abelian category, hence the unbounded derived category $\mathcal{D}(\mathrm{Sh}(X; k))$ is well generated.

**Theorem 16** (Brown representability) Let $\mathcal{D}_1, \mathcal{D}_2$ be two triangulated categories and $\mathcal{D}_1$ be well generated and with arbitrary coproducts. Then a functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ admits a right adjoint if and only if $F$ commutes with coproducts.

**Theorem 17** (Spaltenstein) For any $X, Y$ locally compact spaces, $Rf_!$ is defined on *all* of $\mathcal{D}(\mathrm{Sh}(X; k))$ .

**Lemma 8** $Rf_!$ commutes with arbitrary direct sums.

**Remark 15** This is not true for $f_*$ . For example, consider $X = \mathbb{R}$ and $\mathcal{F}_i$ be the skyscraper sheaf concentrated on $i$ . Then $\Gamma_c(\oplus(\mathcal{F}_i)) = \oplus\Gamma_c(\mathcal{F}_i)$ but $\Gamma(\oplus\mathcal{F}_9) = \prod \Gamma(\mathcal{F}_i) \neq \oplus\Gamma(\mathcal{F}_i)$ .

By the lemma and the Brown representability, we conclude that $Rf_!$ has a right adjoint functor $f^! : \mathcal{D}(\mathrm{Sh}(X; k)) \to \mathcal{D}(\mathrm{Sh}(Y; k))$ . To get a bounded functor $f^! : \mathcal{D}^+(\mathrm{Sh}(X; k)) \to \mathcal{D}^+(\mathrm{Sh}(Y; k))$ , we need further boundedness of $Rf_!$ .

**Definition 32** The *dimension with compact support* $\dim_c X$ for $X$ locally compact is the smallest $n$ such that $H^i_c(X; \mathcal{F}) = 0$ for any $\mathcal{F} \in \mathrm{Sh}(X; \mathbb{Z})$ and any $i > n$ .

**Proposition 10**

a. $\dim_c \mathbb{R}^n = n$.

b. If $Y \subseteq X$ is closed, then $\dim_c Y \leq \dim_c X$.

c. $\dim_c X$ is local, namely if for any $x \in X$, there exists a neighborhood $U$ of $x$ such that $\dim_c U \leq n$, then $\dim_c X \leq n$. In particular, $\dim_c M = n$ for any $n$-dimensional manifold.

d. For $f : X \to Y$ and $\dim_c X \leq n$, then $R^p f_! \mathcal{F} = 0$ for any $\mathcal{F}$ and $p > n$. Moreover, $Rf_!(\mathcal{D}^b) \subseteq \mathcal{D}^b$.

Now assume that $\dim_c X < \infty$. Let $\mathcal{F} \in \mathcal{D}^+(Y)$, then by adjunction

$$\mathrm{Hom}_{\mathcal{D}(X)}(\tau_{\leq i} f^! \mathcal{F}, f^! \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{D}(Y)}(Rf_! \tau_{\leq i} f^! \mathcal{F}, \mathcal{F}).$$

Suppose $\dim_c X = d < \infty$, then we know that $Rf_! \tau_{\leq i} f^! \mathcal{F} \subseteq \mathcal{D}^{\leq i+d}(Y)$. So for $i \ll 0$, we have $\mathrm{Hom}_{\mathcal{D}(Y)}(\tau_{\leq i} f^! \mathcal{F}, f^! \mathcal{F}) = 0$, hence $f^! \mathcal{F} \in \mathcal{D}^+(X)$. It follows that $f^!(\mathcal{D}^+(Y)) \subseteq D^+(X)$, hence the adjunction can be defined on $\mathcal{D}^+$. This adjoint pair $(Rf_!, f^!)$ is called the *(global) Verdier duality*.

Let $\mathcal{F}, \mathcal{G} \in \mathrm{Sh}(X)$. Then there is a canonical map

$$f_* \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(f_! \mathcal{F}, f_! \mathcal{G}).$$

Deriving this, for any $\mathcal{F}, \mathcal{G} \in \mathcal{D}^b(\mathrm{Sh}(X))$, we get

$$Rf_* R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \to R\mathcal{H}om(Rf_! \mathcal{F}, Rf_! \mathcal{G}).$$

Replacing $G$ by $f^! \mathcal{G}$ for some $\mathcal{G} \in \mathcal{D}^b(\mathrm{Sh}(Y))$, we obtain

$$Rf_* R\mathcal{H}om(\mathcal{F}, f^! \mathcal{G}) \to R\mathcal{H}om(Rf_! \mathcal{F}, \mathcal{G}).$$

**Proposition 11** (Local Verdier duality) The map

$$\phi : Rf_* R\mathcal{H}om(\mathcal{F}, f^! \mathcal{G}) \to R\mathcal{H}om(Rf_! \mathcal{F}, \mathcal{G})$$

is an isomorphism.

**Proof** We check that $R\Gamma(U, \phi)$ is an isomorphism on each open $U \subseteq Y$:

$$H^i(R\Gamma(U; Rf_* R\mathcal{H}om(\mathcal{F}, f^! \mathcal{G}))) \cong \mathrm{Hom}_{\mathcal{D}(f^{-1}(U))}(\mathcal{F}|_{f^{-1}(U)}, f^! \mathcal{G}[i]|_{f^{-1}(U)}),$$

by global Verdier duality, the right-hand-side is isomorphic to

$$\mathrm{Hom}_{\mathcal{D}(U)}(R(f|_U)_! \mathcal{F}|_U, \mathcal{G}[i]|_U) \cong H^i(R\Gamma(U; R\mathcal{H}om(Rf_! \mathcal{F}, \mathcal{G}))).$$

The proposition follows. □

We also have the following similar useful identity and base change.

**Proposition 12**

$$f^! R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \cong R\mathcal{H}om(f^* \mathcal{F}, f^! \mathcal{G}).$$

**Proposition 13** If

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

and assume that $f$ has fibers of finite dimensions (hence so does $f'$), then $f^! Rg_* \cong Rg'_*(f')^!$.

**Proof** The idea is to use the adjunction $(g^* Rf_!, f^! Rg_*)$. □

## Contraction of curves on complex surfaces

Throughout this section, we assume that the coefficient ring $k$ is a field. Let $f : (X, E) \to (Y, v)$ be a quotient map such that $Y = X/E$ and $f(E) = v$, namely a contraction of the union of curves $E = \cup E_i$ on a complex surface $X$ to a single point $v$.

**Theorem 18** (Grauert) $f$ is holomorphic if and only if the intersection matrix $[E_j \cdot E_k]$ is negative definite.

**Example 10** The contraction $(\mathbb{A}^1 \times \mathbb{P}^1, E = \{0\} \times \mathbb{P}^1)$ has $I = [0]$, hence is not holomorphic.

**Example 11** Let $\mathcal{L}$ be a line bundle over $\mathbb{P}^1$ such that $c_1(\mathcal{L}) < 0$, then the contraction of its zero-section on the total space of $\mathcal{L}$ has negative definite intersection pairing $I = [c_1(\mathcal{L})]$, hence is holomorphic.

**Example 12** Consider the blowup $\mathrm{Bl}_x(\mathbb{A}^1 \times \mathbb{P}^1, E = \{0\} \times \mathbb{P}^1)$, the contraction of $\hat{E} \cup \mathcal{E}$ ( $\hat{E}$ is the strict transform of $m$. $\mathcal{E}$ is the exceptional divisor) has $I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, hence is not holomorphic.

**Example 13** Consider the blowup $(\mathrm{Bl}_x(\mathcal{L}), \hat{E} \cup \mathcal{E})$ has $I = \begin{bmatrix} c_1(\mathcal{L}) - 1 & 1 \\ 1 & -1 \end{bmatrix}$, hence is holomorphic.

Now restrict $X$ to one of our four examples. Let $U = X \setminus E = Y \setminus v$ and $j : U \hookrightarrow X$, $i : E \hookrightarrow X$. The following is a fact from basic algebraic topology.

**Theorem 19** (Lefschetz duality) $H^i(X, U) \cong H_{4-i}(E)$.

Since $X$ retracts onto $E$, we have $i^* : H^*(X) \cong H^*(E)$. Let $\mathrm{cl}_E : H_2(E) \to H^2(X)$ be the cycle class map given by $i_* : H_2(E) \cong H_2(X)$ and $H_2(X) \to H_2(X)^\vee, \alpha \mapsto I(\alpha, -)$ and $H_2(X)^\vee \cong H^2(X)$. There is a long exact sequence associated to $(X, U)$, by the Lefschetz duality we have the following identification:

$$\begin{tikzcd} H^1(X,U) \arrow[r] \arrow[d, equal] & H^1(X) \arrow[r] & H^1(U) \arrow[r] & H^2(X,U) \arrow[r] \arrow[d, equal] & H^2(X) \arrow[r] \arrow[d, equal] & H^2(U) \arrow[r] & H^3(X,U) \arrow[d, equal] \cdot \\ H_3(E) = 0 & & & H_2(E) \arrow[r] \arrow[ur, "\mathrm{cl}_E"] & H^2(E) & & H_1(E) \end{tikzcd}$$

Note that $\dim H_2(E) = \dim H^2(X) = \dim H^2(E)$, so $\mathrm{cl}_E$ is an isomorphism if and only if it is surjective if and only it is injective, which is also equivalent to say that $j_1^*$ is an isomorphism, or $b_1 = 0$, or $j_2^* = 0$, or $b_2$ is an isomorphism, if and only if $I$ is non-degenerate (e.g., when $\mathrm{char}(k) = 0$ and $f$ is holomorphic.)

**Question** What does this mean in $\mathcal{D}(\mathrm{Sh}(Y; k))$?

Consider $Rf_*\underline{k}_X$. Then

$$R^i f_* \underline{k}_X = \begin{cases} \underline{k}_Y & i = 0, \\ \underline{H^1(E)}_v = 0 & i = 1, \\ \underline{H^2(E)}_v & i = 2, \\ 0 & \text{else.} \end{cases}.$$

Consider the truncation triangle

$$\tau_{\leq 1} Rf_*\underline{k}_X \to Rf_*\underline{k}_X \to R^2 f_*\underline{k}_X[-2] \to .$$

**Question** Does this split? Namely, does there exist $\sigma : R^2 f_*\underline{k}_X[-2] \to Rf_*\underline{k}_X$ such that $H^2\sigma$ is an isomorphism?

By the exact sequence of sheaves

$$0 \to i_! i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F},$$

We know that for any $\mathcal{F} \in \mathcal{D}(X)$, we have a distinguished triangle

$$i_! i^! \mathcal{F} \to \mathcal{F} \to Rj_* j^* \mathcal{F}.$$

Taking its cohomology, we have a long exact sequence

$$\to H^*(i_! i^! \underline{k}_X) \to H^*(X, \underline{k}_X) \to H^*(U, \underline{k}_X).$$

Hence $H^*(i_! i^! \underline{k}_X)$ is actually the relative cohomology.

$R^2 f_* \underline{k}_x[-2]$ has support on a single point $v$. By applying $\mathrm{Hom}(R^2 f_*\underline{k}_X[-2], -)$ to the above distinguished triangle,we get

$$\mathrm{Hom}(R^2 f_*\underline{k}_X[-2], \mathcal{F}) = \mathrm{Hom}(R^2 f_*\underline{k}_X[-2], i_! i^! \mathcal{F}).$$

Hence $\sigma$ must factor through $\sigma' : Rf_*\underline{k}_X[-2] \to i_! i^! Rf_*\underline{k}_X$. Taking $H^2$, we get $H^2(E) \xrightarrow{H^2(\sigma')} H^2(E)$. By base change, $H^2(i_! i^! Rf_*) = H^2(i_! R\Gamma \omega_E) = H_2(E) \xrightarrow{\mathrm{cl}_E} H^2(E)$. We conclude that the sequence splits if and only if $\mathrm{cl}_E$ is an isomorphism.

**Question** When does the complex $Rf_*\underline{k}_X$ split?

**Proposition 14** If the intersection pairing $I$ is invertible, then $Rf_*\underline{k}_X \cong R^2 f_*\underline{k}_X[-2] (= \underline{H^2(E)}_v) \oplus \tau_{\leq 1} Rf_*\underline{k}_X \cong \underline{H^2(E)}_v \oplus \tau_{\leq 1} Rj_*\underline{k}_U$. (Note $i^* Rf_*\underline{k}_X = H^*(X)$ and $i^* Rj_*\underline{k}_U$.)

**Proof** When $I$ is invertible, we want a splitting $\sigma : \underline{H^2(E)}_v \to Rf_*\underline{k}_X$. Since $\underline{H^2(E)}_v$ is supported on $v$. Then $\sigma$ exists if and only if there exists a map $\sigma' : \underline{H^2(E)}_v \to i_*i^!Rf_*\underline{k}_X$, which live in $\mathcal{D}(\mathrm{Sh}(v;k)) = \mathcal{D}(k\text{-v.s})$. Thus it is equivalent to giving a map $H^2(\sigma') : H^2(E) \to H^2(i_*i^!Rf_*\underline{k}_X) = H^2(X,U) = H_2(E)$ such that $\mathrm{cl}_E \circ H^2(\sigma') : H^2(E) = H^2(E)$ is the identity map, which is equivalent to saying that $\mathrm{cl}_E$ is an isomorphism, $I$ is invertible. □

**Remark 16** When $f$ is holomorphic and $k = \mathbb{Q}$, then by Grauert, $I$ is negative definite, hence invertible.

Truncating the adjunction map $Rf_*\underline{k}_X \to Rj_*j^*Rf_*\underline{k}_X = Rj_*\underline{k}_U$, we get $\tau_{\leq 1}Rf_*\underline{k}_X \to \tau_{\leq 1}Rj_*\underline{k}_U$, which is an isomorphism in $\mathcal{D}(\mathrm{Sh}(\mathrm{Y};))$. We can check this on $H^*$ locally, as $H^0 : \underline{k}_Y \xrightarrow{\mathrm{Id}} \underline{k}_y$ and $H^1 : H^1(X) = H^1(E) \xrightarrow{j_1^*} H^1(U)$, which an isomorphism if and only if $\mathrm{cl}_E$ is an isomorphism.

**Definition 33** We denote $IC_X = \underline{k}_X[2]$, $IC_v = \underline{k}_v$, $IC_Y = \tau_{\leq 1}(Rj_*\underline{k}_U[2])$. More generally, for any smooth $X$, we denote $IC_X = \underline{k}_X[\dim X]$. Then if $f$ is holomorphic and $k = \mathbb{Q}$, then $Rf_*IC_X \cong IC_v \oplus IC_Y$.

**Example 14** In the above non-holomorphic examples, $R^2f_*[-2]$ does not split. However, one can always split off a skyscraper sheaf of rank $\mathrm{rk}I$.

Here is another approach. Consider the adjunction map $a : \underline{k}_Y \to Rf_*f^*\underline{k}_Y = Rf_*\underline{k}_X$.

**Question** When does it split?

Again, truncating the adjunction map $Rf_*\underline{k}_X \to Rj_*j^*Rf_*\underline{k}_X = Rj_*\underline{k}_U$, we get $J : Rf_*\underline{k}_X \to \tau_{\leq 2}Rj_*\underline{k}_U$ as $R^if_*\underline{k}_X = 0$ for $i > 2$. Note that $R^0j_*\underline{k}_U = j_*\underline{k}_U = \underline{k}_Y$ and $R^kj_*\underline{k}_U = \underline{H^k(U)}_v$ for $k > 0$. We obtain that

$$\begin{tikzcd}
\underline{k}_Y \arrow[rr] \arrow[d, equal] & & Rf_*\underline{k}_X \arrow[dl, "\phi"'] \arrow[d] \\
\tau_{\leq 0}Rj_*\underline{k}_U \arrow[r] & \tau_{\leq 1}Rj_*\underline{k}_U \arrow[r] & \tau_{\leq 2}Rj_*\underline{k}_U \arrow[d] \\
& & R^2j_*\underline{k}_U[-2] = \underline{H^2(U)}_v[-2].
\end{tikzcd}$$

We get $J^2 : Rf_*\underline{k}_X \to R^2j_*\underline{k}_U[-2]$ given by $H^2(X) \xrightarrow{j_2^*} H^2(U)$. Applying $\mathrm{Hom}(Rf_*\underline{k}_X, -)$ to the triangle, we get

$$\mathrm{Hom}(Rf_*, \tau_{\leq 1}Rj_*) \to \mathrm{Hom}(Rf_*, \tau_{\leq 2}Rj_*) \to \mathrm{Hom}(Rf_*, R^2j_*[-2]) \to$$

which sends $J$ to $J^2$. Then $J$ lifts if and only if $j_2^* = 0$, if and only if $\mathrm{cl}_E$ is an isomorphism.

Thus $\phi$ exists if and only $I$ is invertible. Similarly form the triangle $\phi' : Rf_*\underline{k}_X \to \tau_{\leq 1}Rj_*\underline{k}_U \to Rj_*\underline{k}_U[-1] = \underline{H^1(U)}_v[-1]$, we know that $\phi'$ is an isomorphism. Therefore $\phi$ exists (and is unique) if and only if $\mathrm{cl}_E$ is an isomorphism and $H_1(E) = 0$ (in this case $IC_Y = \underline{k}_X[2]$).

## Borel-Moore homology and dualizing functor

Last time we studied the pushforward of constant sheaves and how they decompose. Now let us step back to duality.

We have seen that cohomology can be naturally expressed in terms of sheaves.

**Question** What about homology?

Let $C_\cdot^{\mathrm{BM}}(X)$ be the chain complex of possibly infinite singular (simplicial) chains $\sum a_i\sigma_i$ on $X$ together with the usual differential such that for any compact $D \subseteq X$, there exists at most finitely many $A$ such that $a_i \neq 0$ with $\mathrm{supp}(\sigma_i) \cap D \neq \varnothing$. Its homology is called the *Borel-Moore homology*.

**Example 15**

a. In $\mathbb{A}^1_{\mathbb{C}}$, a ray is a 1-chain. Its boundary is a single point and itself is a boundary. Hence $H_0^{\mathrm{BM}} = 0$ and $H_1^{\mathrm{BM}} = 0$. Also, $H_2^{\mathrm{BM}} = \mathbb{Z}$.

b. Consider a three rays $\gamma_1, \gamma_2, \gamma_3$ in plane branched at one point. Then its Borel-Moore $H_0^{\mathrm{BM}} = 0$ and $H_1^{BM} = \mathbb{Z}^3/(\gamma_1+\gamma_2+\gamma_3) \cong \mathbb{Z}^2$.

**Theorem 20** Let $C_{\mathrm{BM}}^{\cdot} := C_{-\cdot}^{\mathrm{BM}}$. $C_{\mathrm{BM}}^{\cdot}$ is in fact a complex of flabby sheaves, so $H^{-i}(U; C_{\mathrm{BM}}^{\cdot}) = H_\cdot^{\mathrm{BM}}(U)$.

**Remark 17** If $X$ is compact, then $H_i^{\mathrm{BM}}(X) = H_i(X)$.

The Poincare duality $H_i(M;k) \cong H_c^{m-i}(M;k)$ for smooth oriented manifolds $M$ can be also stated as $H_i^{\mathrm{BM}}(M;k) = H^{m-i}(M;k)$.

The universal coefficients theorem says that
$$0 \to \mathrm{Ext}(H_{i-1}(M;\mathbb{Z}), \mathbb{Z}) \to H^i(M;\mathbb{Z}) \to \mathrm{Hom}(H_i(M;\mathbb{Z}), \mathbb{Z}) \to 0.$$
By the Poincare duality, we obtain
$$0 \to \mathrm{Ext}(H_c^{n-i+1}(M;\mathbb{Z}), \mathbb{Z}) \to H^i(M;\mathbb{Z}) \to \mathrm{Hom}(H_c^{n-i}(M;\mathbb{Z}), \mathbb{Z}),$$
where $H^i(X;\mathbb{Z})$ can be also identified as $H_{n-i}^{\mathrm{BM}}(M;\mathbb{Z})$.

We want a general notion of a *dual complex* $B^\cdot$ for any $A^\cdot \in \mathcal{D}^b(\mathrm{Sh}(X;\mathbb{Z}))$ such that
$$0 \to \mathrm{Ext}(H_c^{i+1}(A^\cdot;, \mathbb{Z}) \to H^{-i}(B^\cdot) \to \mathrm{Hom}(H_c^i(A^\cdot), \mathbb{Z}) \to 0.$$

**Definition 34** Given a complex $A^\cdot$, we let $\tilde{B}^\cdot$ be the complex of presheaves $\tilde{B}^\cdot(U) := \mathrm{Hom}(\Gamma_c(U, A^i), \mathbb{Z})$ and define $B^\cdot$ to be its sheafification. In fact, $B^\cdot = R\mathcal{H}om(A^\cdot, a_x^! \underline{\mathbb{Z}}_{\mathrm{pt}} = \omega_X)$.

Let $A^\cdot = \underline{\mathbb{Z}}_X$, then the Borel-Moore homology is the dual of its cohomology with compact support. Hence $\omega_X \cong C^\cdot_{\mathrm{BM}}$. It follows from the calculation of basic Borel-Moore homology calculation that

**Corollary 10** Let $M$ be a smooth manifold of dimension $\dim_{\mathbb{R}} M = n$. Then $\omega_M \cong \mathrm{or}_M[n]$, where $\mathrm{or}_M$ is the orientation sheaf.

We would like to say that $\omega_X$ gives us a dualizing functor $\mathbb{D}_X : \mathcal{D}^b(X)^{\mathrm{op}} \to \mathcal{D}^b(X)$, with $\mathbb{D}^2 = \mathrm{Id}$. Unfortunately, $\mathcal{D}^b(X)$ is a bit too wild for this to be true. The first problem is that the image of the functor $R\mathcal{H}om : (\mathcal{D}^b)^{\mathrm{op}} \times \mathcal{D}^b \to \mathcal{D}$ may not lie in $\mathcal{D}^b$. This is not a major problem (e.g. for $\dim X < \infty$, which works out). The second problem is that there may exist bad sheaves on nice spaces. For example, let $j : X = \mathbb{R}^3 \setminus \{\text{Alexander horned sphere}\} \hookrightarrow \mathbb{R}^3$ and $\mathcal{L}$ be locally constant sheaf on $X$ and consider the sheaf $j_!\mathcal{L}$ on $\mathbb{R}^3$. We would like to eliminate those problems.

**Definition 35** An *analytic space* $X$ is a subset of an analytic manifold of $M$ cut out by analytic functions. A *subanalytic space* is one cut out by analytic equalities and inequalities.

**Definition 36** A sheaf $\mathcal{F}$ on a subanalytic space $M$ is called *constructible* if there exists $X = \coprod X_i$ a subanalytic locally finite stratification (i.e., $X_i \subseteq X$ is a subanalytic subspace which is also a manifold) such that $\mathcal{F}_{X_i}$ is a local system, i.e., a locally constant sheaf.

**Definition 37** Let $\mathcal{F}^\cdot \in \mathcal{D}^b(\mathrm{Sh}(X))$. We say $\mathcal{F}^\cdot$ is *(cohomologically) constructible* if $H^i(\mathcal{F}^\cdot)$ is constructible.

**Definition 38** The *constructible derived category* $\mathcal{D}_c^b(X;k)$ is defined to be the full subcategory of $D^b(\mathrm{Sh}(X;k))$ consisting of cohomologically constructible complexes.

**Theorem 21**

a. Let $f$ be an analytic map, then $Rf_!, Rf_*, R\mathcal{H}om, \otimes, f^*, f^!$ all preserve $\mathcal{D}_c^b$.

b. In $\mathcal{D}_c^b$, $\mathbb{D}_X$ is a dualizing functor, i.e., $\mathbb{D}^2(A^\cdot) \cong A^\cdot$.

Given $f : X \to Y$. We have
$$Rf_* \cong \mathbb{D}_Y Rf_! \mathbb{D}_X$$
as $Rf_* R\mathcal{H}om(\mathcal{F}, \omega_X) = Rf_* R\mathcal{H}om(\mathcal{F}, f^! \omega_Y) \cong R\mathcal{H}om(Rf_! \mathcal{F}, \omega_Y)$, hence $Rf_* \mathbb{D}_X \mathcal{F} \cong \mathbb{D}_Y(Rf_! \mathcal{F})$. Similarly, we have
$$f^! \cong \mathbb{D}_X f^* \mathbb{D}_Y.$$

## Stratification

Suppose we have a locally finite covering of smooth subanalytic subsets $X = \coprod X_\alpha$, we say that it is a (Whitney) *stratification* if it satisfies $\overline{X_\alpha} = \coprod_{\beta \leq \alpha} X_\beta$ and the Whitney conditions A and B. The condition A says that $T_y Y$ is contained the limit of the tangent spaces $T_{x_i} X$ for any $X = X_\alpha$ and $Y = X_\beta$ in the strata. It follows under these conditions that there exists a neighborhood $W_x$ of $x$ such that $W_x$ is strata-preserving homeomorphic to $\mathbb{R}^k \times \mathrm{Cone}_{\mathbb{R}}(L_x)$, where $L_x$ is the *link* of $x$. Let $Z$ be a transverse slice to the strata $X_\alpha$ containing $x$, then $L_x = \partial B_\varepsilon(x) \cap (Z \cap X)$. The following are the basic facts about stratification.

**Theorem 22**

a. Any locally finite covering by subanalytic subsets can be refined to a stratification.
b. Any algebraic variety admits a stratification by locally closed subvarieties.
c. Any map $f : X \to Y$ of varieties can be stratified (namely, there exists stratifications of $X$ and $Y$ such that the preiamge of a strata is a union of strata such that $f|_{X_\alpha}$ is a submersion and $f$ is locally constant over $Y_\beta$ .)

We have seen that $\mathbb{D}(\underline{k}_X) = \omega_X$,

$$\begin{aligned} H^l(X;k) &= H^l(X;\underline{k}_X) = Ra_* a^* \underline{k}_{\mathrm{pt}}, \\ H^l_c(X;k) &= H^l_c(X;\underline{k}_X) = Ra_! a^* \underline{k}_{\mathrm{pt}}, \\ H^{\mathrm{BM}}_l(X;k) &= H^{-l}(X;\omega_X) = Ra_* a^! \underline{k}_{\mathrm{pt}}, \\ H_l(X;k) &= H^{-l}_c(X;\omega_X) = Ra_! a^! \underline{k}_{\mathrm{pt}}. \end{aligned}$$

When $X$ is smooth and oriented, we have $\omega_X \cong \underline{k}_X[\dim_{\mathbb{R}} X]$, in fact $f^! \cong f^*[\dim_{\mathbb{R}} F] \otimes \mathrm{or}_{X/Y}$ for any smooth morphism $f : X \to Y$. Let $n = \dim_{\mathbb{R}} X/2$, we obtain the Poincare duality

$$H^{n+l}(X;k) \cong H^{\mathrm{BM}}_{n-l}(X;k), \quad H_{n+l}(X;k) \cong H^{n-l}_c(X;k).$$

This can be interpreted as the statement that $\underline{k}_X[n]$ is *self dual*.

An analog of Poincare duality on stratified space of even real dimension should then be $I_X \in \mathcal{D}^b_c(X)$ such that $\mathbb{D}I_X \cong I_X$ and $U \subseteq X$ an open smooth (strata) such that $I_X|_U \cong \underline{k}_U[\dim_{\mathbb{R}} U/2]$.

We have seen that for a smooth projective map $f : X \to Y$, $Rf_*\underline{\mathbb{Q}}_X \cong \bigoplus R^i f_* \underline{\mathbb{Q}}_X[-i] (= \underline{H^i(F)}_Y)$ and $H^i(F) \cong H^{2n-i}(F)$ by the Hard Lefschetz. Also, we have seen that for a contraction of curves $f : (X, E) \to (Y, v)$, $Rf_*\underline{\mathbb{Q}}_X \cong \underline{H^2(E)}_v \oplus \tau_{\leq 1} Rj_*\underline{\mathbb{Q}}_U$. As $\mathbb{Q}_X[2]$ is self dual, $Rf_*$ is self dual (as $f$ is proper) and $\underline{H^2(E)}_v$ is self dual, we know that $\tau_{\leq 1} Rj_*\underline{\mathbb{Q}}_U[2]$ is self dual.

In general for $f : X \to Y$ proper, we cannot hope $Rf_* = \bigoplus H^i(Rf_*)[-i]$, but instead we would hope that $Rf_* \cong {}^pH^i(Rf_*)[-i]$, where ${}^pH^i(Rf_*)$ is the form of $I_X$.

## Poincare duality for singular spaces

For $X$ singular, usually $\underline{k}_X[n] \ncong \omega_X[-n] = \mathbb{D}(\underline{k}_X[n])$. In order to obtain the Poincare duality for singular spaces, we need to find some $F \in \mathcal{D}^b_c(X;k) =: \mathcal{D}(X)$ such that $\mathbb{D}\mathcal{F} \cong \mathcal{F}$ and for some open $U \subseteq X$ such that $\mathcal{F}|_U \cong \underline{k}_U[\dim U]$. This is the goal for today.

We will go by induction. Let $X = U \coprod Z$, where $Z$ is smooth closed (but $U$ is not necessarily smooth). Let $j : U \hookrightarrow X$ is open and $i : Z \hookrightarrow X$. Assume there exists a stratification of $X$ such that $Z$ is smooth closed stratum and $\mathcal{F}_U$ is constructible with respect to the stratification.

**Definition 39** An *extension* of $\mathcal{F}_U \in \mathcal{D}(U)$ is a pair $(\mathcal{F}, \alpha)$, where $\mathcal{F} \in \mathcal{D}(X)$ and $\alpha : j^*\mathcal{F} \to \mathcal{F}_U$ an isomorphism.

**Remark 18** Extensions of $\mathcal{F}_U$ form a category.

**Lemma 9** Fix $\mathcal{F}_U \in \mathcal{D}(U)$, then there exists a natual bijection

$$\{\text{extensions of } \mathcal{F}_U\}/\text{iso.} \longleftrightarrow \begin{array}{c}\{\text{distinguished triangles in } \mathcal{D}(Z): \\ A \to i^*j_*\mathcal{F}_U \to B\}/\text{iso.}\end{array}$$

given by sending $\mathcal{F}$ to $i^*\mathcal{F} \to i^*j_*\mathcal{F}_U \to i^!\mathcal{F}[1]$.

**Proof** Associated to $\mathcal{F}$ we have an adjunction triangle

$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \to .$$

Applying $i^*$, we get

$$i^!\mathcal{F} \to i^*\mathcal{F} \to i^*j_*j^*\mathcal{F} \to .$$

Rotating the triangle we obtain the desired map for one direction.

For the other direction, the adjunction triangle for $j_*\mathcal{F}_U$ is

$$j_!j^*(j_*\mathcal{F}_U) = j_!\mathcal{F}_U \to j_*\mathcal{F}_U \to i_*i^*(j_*\mathcal{F}_U) \to .$$

Starting from $A \to i^*j_*\mathcal{F}_U \to B$, we get

$$\begin{tikzcd}
 & & i_*A \arrow[drr] & & & & j_!\mathcal{F}_U[1] \arrow[dr] & \\
 & \mathcal{F} \arrow[ur] \arrow[dr] & & & i_*i^*j_*\mathcal{F}_U \arrow[urr] \arrow[drr] & & & \mathcal{F}[1] \\
j_!\mathcal{F}_U \arrow[ur] \arrow[rr, bend right] & & j_*\mathcal{F}_U \arrow[urr] & & & & i_*B \arrow[ur] &
\end{tikzcd}$$

where $\mathcal{F} = \mathrm{Cone}(i_*A[-1] \to j_!\mathcal{F}_U) = \mathrm{Cone}(j_*\mathcal{F}_U \to i_*B)[-1]$. Let $\alpha = j^*(\mathcal{F} \to j_*\mathcal{F})$. □

**Example 16** The sheaf $j_!\mathcal{F}_U$ correpsonds to $i^*j_!\mathcal{F}_U = 0 \to i^*j_*\mathcal{F}_U \to i^!j_!\mathcal{F}_U[1] \to$. Namley, $i^*j_*\mathcal{F}_U \cong i^!j_!\mathcal{F}_U[1]$.

The sheaf $j_*\mathcal{F}_U$ correpsonds to $i^*j_*\mathcal{F}_U \to i^*j_*\mathcal{F}_U \to i^!j_*\mathcal{F}_U[1] = 0 \to$.

From $A \to i^*j_*\mathcal{F}_U \to B \to$, we get $\mathbb{D}\mathcal{F}$ corresponding to $\mathbb{D}B \to i^!j_!(\mathbb{D}\mathcal{F}_U) \to \mathbb{D}A \to$, hence we get $\mathbb{D}B[1] \to i^*j_*\mathbb{D}\mathcal{F}_U \to \mathbb{D}A[1] \to$. So for $\mathcal{F}$ to be self-dual, we need $\mathbb{D}\mathcal{F}_U \cong \mathcal{F}_U$ and $A \cong \mathbb{D}B[1]$. So we want to find $A \to i^*j_*\mathcal{F}_U \to B \to$ such that $A \cong \mathbb{D}B[1]$. The (only) way to find a splitting is by trucation:

$$\tau_{\leq k}(i^*j_*\mathcal{F}_U) \to i^*j_*\mathcal{F}_U \to \tau_{\geq k+1}(i^*j_*\mathcal{F}_U) \to .$$

As $H^i(A)$ is locally constant on $Z$, we know that

$$i_x^*(H^i(\mathbb{D}(A))) = H^i(i_x^*(\mathbb{D}_Z A)) = H^i(\mathbb{D}_x(i_x^! A)) = H^i(R\operatorname{Hom}(i_x^! A, k[0])) = H^{-i}(i_x^!$$

Decompose $i_x$ as $\alpha : x \to C$ and $\beta : C \hookrightarrow Z$, where $C$ is contractible and open. Then $i_x^! A = \alpha^! \beta^! A = \alpha^!(\beta^* A) = \alpha^!(\oplus H^i(\beta^* A)[-i])$, where $H^i(\beta^* A)$'s are all constant sheaves.

**Question** What is the "costalk", namely $i_x^!$ of a constant sheaf on a smooth $C$ of dimension $d$?

**Answer** As the dualizing sheaf of $x$ is $\underline{k}_{\mathrm{pt}}$, which is also equal to $i_x^! a^! \underline{k}_{\mathrm{pt}} = i_x^!(\underline{k}_C[d])$. Also $i_x^* \underline{k}_C[d] = \underline{k}_{\mathrm{pt}}[d]$. Hence $i_x^! = i_x^*[-d]$.

Therefore we $H^{-i}(i_x^! A)^\vee = H^{-i}(i_x^* A[-d])^\vee = H^{-d-i}(i_x^* A)^\vee$. Applying to $\tau_{\geq k} A$, we obtain that

$$i_x^* H^i(\mathbb{D}\tau_{\geq k} A) = \begin{cases} H^{-d-i}(i_x^* A) & -d-i \geq k, \\ 0 & \text{else.} \end{cases}$$

Hence

$$\mathbb{D}(\tau_{\leq k} A) = \tau_{\geq -d-k}(\mathbb{D}A).$$

So

$$\mathbb{D}B[1] = \tau_{\leq -d-k-1}(i^! j_! \mathcal{F}_U)[1] = \tau_{\leq -d-k-1}(i^* j_* \mathcal{F}_U)[1] = \tau_{\leq -d-k-2}(i^* j_* \mathcal{F}_U).$$

So we should take $k = -d-k-2$, namely $k = -d/2 - 1$.

To summarize, the above procudure works for even $d$ and $k = -d/2 - 1$. Starting with a self-dual sheaf $\mathcal{F}_U$, we get sheaf $j_{!*}\mathcal{F}_U \in \mathcal{D}(X)$ given by the extension corresponding to the distinguished triangle

$$\tau_{\leq -d-1}(i^*j_*\mathcal{F}_U) \to i^*j_*\mathcal{F}_U \to \tau_{\geq -d}(i^*j_*\mathcal{F}_U) \to .$$

In fact, we will see that $j_{!*}$ defines a functor $\mathcal{D}(U) \to \mathcal{D}(X)$.

**Proposition 15** Suppose we have the following diagram of distinguished triangles

$$\begin{tikzcd}
X \arrow[r, "u"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "d"] & {} \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "d'"] & {} .
\end{tikzcd}$$

Then the following are equivalent:

a. $v'gu = 0$.

b. There exists $f : X \to X'$ such that the first square commutes.

c. There exists $h : Z \to Z'$ such that the second square commutes.

d. There exists a morphism of triangle $(f, g, h)$.

Moreover, if these condition holds and $\operatorname{Hom}^{-1}(X, Z') = 0$, then $f$ and $h$ are unique.

**Proof** Consider the long exact sequence of $\operatorname{Hom}^{\cdot}(X, -)$, we know (a) is equivalent to (b) and the uniqueness. From the triangulated category axiom, we know that (b) is equivalent to (d). A dual argument implies that (c) is equivalent to (d). □

**Corollary 11** Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{d}$ be a distinguished triangulated and $\operatorname{Hom}^{-1}(X, Z) = 0$, then

a. The cone of it is unique up to unique isomorphism.

b. $d$ is the unique map $Z \xrightarrow{x} X[1]$ such that $X \to Y \to Z \xrightarrow{x}$ is distinguished.

**Proposition 16** $f_{!*}$ is a functor.

**Proof** Since $j_{!*}\mathcal{F} = \operatorname{Cone}(i_* A[-1] \to j_! \mathcal{F}_U)$ and

$$\operatorname{Hom}^{-1}(i_* A[-1], j_{!*}\mathcal{F}_U) = \operatorname{Hom}(A[-1], i^! j_{!*}\mathcal{F}_U[-1]) = \operatorname{Hom}(A, B[-1]) = 0$$

as $A \in \mathcal{D}^{\leq -d/2-1}$ and $B[-1] \in \mathcal{D}^{\geq -d/2+1}$. By Corollary 11, we know $j_{!*}$ is functorial. □

**Remark 19** $\mathbb{D} j_{!*} \cong j_{!*}\mathbb{D}$.

**Lemma 10** $\mathcal{F} \in \mathcal{D}(X)$ has a nontrivial summand with support in $Z$ ifa nd only if $i^! \mathcal{F} \to i^* \mathcal{F}$ can be expresseed as $\operatorname{Id}_Q \oplus h$, where $i^! \mathcal{F} \cong Q \oplus \mathcal{G}$, $i^* \mathcal{F} \cong Q \oplus \mathcal{G}'$ and $h : \mathcal{G} \to \mathcal{G}'$.

**Proof** If $\mathcal{F} \cong i_* Q \oplus \mathcal{F}'$, then $i^! \mathcal{F} \to i^* \mathcal{F}^*$ is the same as $i^!_i * Q \cong i^! \mathcal{F}' \to i^* i_* \oplus i^* \mathcal{F}'$ and $i^! i_* Q \cong i^* i_* Q \cong Q$.

For the other direction, we know $i_Q$ is a direct summand of $\mathcal{F}$ as the map $i_* Q \to i_* i^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to i_* Q$ is the identity. □

**Corollary 12** If $H^k(\phi : i^! \mathcal{F} \to i^* \mathcal{F}) = 0$ for any $k$, then $\mathcal{F}$ has no summands with support in $Z$.

**Proof** If not, then $H^k(\phi) = H^k(\operatorname{Id}_Q) \oplus H^k(h)$. But $H^k(\operatorname{Id}_Q) \cong \operatorname{Id}_{H^k(Q)}$ is nonzero for some $k$. □

**Remark 20** $H^k(\phi) = 0$ for any $k$ does not imply that $\phi = 0$. For example, $\operatorname{Ext}^1_{\mathbf{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \operatorname{Hom}_{\mathcal{D}(\mathbf{Ab})}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}[1])$ has a nonzero element corresponding to $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. The corresponding map $\phi \in \operatorname{Hom}_{\mathcal{D}(\mathbf{Ab})}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}[1])$ has trival cohomology but itself is not zero.

**Corollary 13** $j_{!*}\mathcal{F}_U$ has no summands with support in $Z$.

**Proof** Since $H^i(\tau_{-d-1} j_* \mathcal{F}_U) = 0$ for any $i \geq -d$ and $H^i(\tau_{\geq -d} i^* j_* \mathcal{F}_U)[-1] = 0$ for any $i \leq -d$. Hence $H^i(i^! j_{!*} \to i^* j_{!*}) = 0$ for any $A$. □

**Definition 40** Let $X$ be stratified by decreasing dimensions (so $X_0$ is open in $X$). Let $\mathcal{L}$ is locally constant on $X_0$. Define the *intersection cohomology complex* $IC(X; \mathcal{L}) = (j_n)_{!*} \cdots (j_0)_{!*} \mathcal{L}[\dim X/2]$, where $j_k : \coprod_{l=0}^{k-1} X_l \hookrightarrow \coprod_{l=0}^{k} X_l$. Write $IC(X) = IC(X; \underline{k}_{X_0})$.

We saw that $\mathbb{D}(IC(X; \mathcal{L})) \cong IC(X; \mathcal{L}^\vee)$. Also, $IC(X; \mathcal{L})$ is indecomposable if and only if $\mathcal{L}$ is indecomposable.

**Definition 41** Define the *intersection cohomology* $IH^i(X; k) = R^i\Gamma(IC(X; k))$ and $IH^i_c(X; k) = R^i\Gamma_c(IC(X; k))$.

As $IC(X; k)$ is self-dual, we get the Poincare-Verdier-Goresky-MacPherson duality

$$\begin{aligned} IH^k(X; k) &= H^k(R\operatorname{Hom}(IC(X; k), \omega_X = a^! \underline{k}_X)) \\ &= H^k(R\operatorname{Hom}(R\Gamma_c(IC(X; k), k))) \\ &= IH^{-k}_c(X; k)^\vee. \end{aligned}$$

Now many results can be extended to singular varieties using intesection cohomology.

**Theorem 23** The Lefschetz hyperplane thoerem is true for any projective variety with $H^{\cdot}$ replaced by $IH^{\cdot}$.

**Theorem 24** The Hard Lefschetz theorem is true for any projective variety with $H^{\cdot}$ replaced by $IH^{\cdot}$.

Recall that for a projective smooth morphism $f : X \to Y$, we have $f_* \underline{\mathbb{Q}}_X \cong \oplus H^i(f_* \underline{\mathbb{Q}}_X)[-i]$, where $H^i(f_* \underline{\mathbb{Q}}_X)$ is a semisimple local system by Deligne's theorem. Moreover, $\eta^i : H^{n-i}(f_* \underline{\mathbb{Q}}_X) \cong H^{n+i}(f_* \underline{\mathbb{Q}}_X)$.

**Theorem 25** (Decomposition theorem, BBD) If $f : X \to Y$ is proper and $X$ is smooth, then

$$f_*(\underline{\mathbb{Q}}_X) \cong \bigoplus_{\lambda\in\Lambda, k\in\mathbb{Z}, \mathcal{L}\in \mathrm{Loc}(Y_\lambda)} (i_{\lambda *}(IC(\overline{Y}_\lambda, \mathcal{L}))[-k]^{\oplus m_{\lambda,\mathcal{L},k}}.$$

Our next goal is to "filter" $\mathcal{D}(Y)$ in such a way that the $k$-th "associated graded piece" is ${}^pH^k f_*(\underline{\mathbb{Q}}_X)$. Namely, the following relative Hard Lefschetz holds: $f_*(\underline{\mathbb{Q}}_X) \cong \bigoplus_k {}^pH^k(f_*(\underline{\mathbb{Q}}_X))[-k]$ and $\eta : {}^pH^{\dim X - i}(f_*(\underline{\mathbb{Q}}_X)) \cong {}^pH^{\dim X + i}(f_*(\underline{\mathbb{Q}}_X))$.

## $t$-structures

Let $\mathcal{A}$ be a triangulated category. Let $\mathcal{A}^{\leq 0}$ and $\mathcal{A}^{\geq 0}$ be full categories of $\mathcal{A}$. Let $\mathcal{A}^{\leq n} := \mathcal{A}^{\leq 0}[-n]$ and $\mathcal{A}^{\geq n} := \mathcal{A}^{\geq 0}[n]$.

**Definition 42** $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ is a $t$-*structure* on $\mathcal{A}$ if

a. $\mathcal{A}^{\leq -1} \subseteq \mathcal{A}^{\leq 0}$ and $\mathcal{A}^{\geq 0} \supseteq \mathcal{A}^{\geq 1}$.

b. $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$ if $X \in \mathcal{A}^{\leq 0}$ and $Y \in \mathcal{A}^{\geq 1}$.

c. For $X \in \mathcal{A}$, there exists a distinguished triangle $X_0 \to X \to X_1 \to$, where $X_0 \in \mathcal{A}^{\leq 0}$ and $X_1 \in \mathcal{A}^{\geq 1}$. (Uniqueness also follows.)

**Definition 43** $\mathcal{C} = \mathcal{A}^{\geq 0} \cap \mathcal{A}^{\leq 0}$ is called the *heart* or *core* of the $t$-structure.

**Example 17** Let $Q$ be an abelian categroy, then $\mathcal{D}^{\geq 0}(Q)$ and $\mathcal{D}^{\leq 0}(Q)$ gives a $t$-structure on $\mathcal{D}(Q)$ with heart equivalent to $Q$.

**Proposition 17**

a. There exists a functor $\tau_{\leq n}\mathcal{A} \to A^{\leq n}$ which is right adjoint to the inclusion $\mathcal{A}^{\leq n} \hookrightarrow \mathcal{A}$. Similarly for a functor $\tau_{\geq n} : \mathcal{A} \to \mathcal{A}^{\geq n}$.

b. There exists a unique $d : \tau_{\geq n+1}(X) \to \tau_{\leq n}(X)[1]$ such that $\tau_{\leq n} X \to X \tau_{\geq n+1} \to X \xrightarrow{d} \tau_{\leq n}(X)[1]$ is a distinguished triangle.

**Proof** Define $\tau_{\leq 0} X = X_0$ and $\tau_{\geq 1} = X_1$. We want to show that given $f : X \to Y$, there exists a canonical map $X_0 \to Y_0$. Applying $\mathrm{Hom}(X_0, -)$ to $Y_0 \to Y \to Y_1 \to$, we get $\mathrm{Hom}(X_0, Y_1[-1]) \to \mathrm{Hom}(X_0, Y_0) \to \mathrm{Hom}(X_0, Y) \to \mathrm{Hom}(X_0, Y_1)$. Since $\mathrm{Hom}(X_0, Y_1[-1]) = 0$ and $\mathrm{Hom}(X_0, Y_1) = 0$, we know $\mathrm{Hom}(X_0, Y_0) \cong \mathrm{Hom}(X_0, Y)$, hence $\tau_{\leq 0}$ is a functor.

The same argument shows that $\mathrm{Hom}_{\mathcal{A}^{\leq 0}}(X, \tau_{\leq 0} Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, Y)$. Since $\mathrm{Hom}^{-1}(\tau_{\leq 0} X, \tau_{\geq 1} Y) = 0$, we know that there exists a unique $d : \tau_{\geq 1} X \to \tau_{\leq 0} X[1]$. □

**Corollary 14**

a. If $X \in \mathcal{A}^{\leq n}$, then $\tau_{\leq n} X \cong X$. Similarly for $\tau_{\geq n}$.

b. Let $X \in \mathcal{A}$. Then $X \in \mathcal{A}^{\leq n}$ if and only if $\tau_{> n} X = 0$. Similarly for $\tau_{< n}$.

c. If $A \to B \to C \to$ is a distinguished triangle in $\mathcal{A}$ and $A, C \in \mathcal{A}^{\geq n}$, then $B \in \mathcal{A}^{\geq n}$.

**Proof** For (a), use the adjunction from the last proposition. For (b), use (a). For (c), $\mathrm{Hom}(\tau_{< n} B, A) = \mathrm{Hom}(\tau_{< n} B, C) = 0$ and the long exact sequence implies that $\tau_{< n} B = 0$. □

**Proposition 18**

a. If $b \geq a$, then $\tau_{\geq b}\tau_{\geq a} \cong \tau_{\geq a}\tau_{\geq b} \cong \tau_{\geq b}$. Similarly for $\tau_{\leq a}$.

b. If $a > b$, then $\tau_{\leq b}\tau_{\geq a} \cong \tau_{\geq a}\tau_{\leq b} = 0$.

c. More generally, $\tau_{\leq b}\tau_{\geq a} \cong \tau_{\geq a}\tau_{\leq b}$.

**Definition 44** Let $\mathcal{C}$ be the heart. We define $\mathbb{H}^{\cdot}\mathcal{A} \to \mathcal{C}$, where $\mathbb{H}^0 := \tau_{\leq 0}\tau_{\geq 0}$ and $\mathbb{H}^n(X) := \mathbb{H}^0(X[n]) \cong (\tau_{\geq n}\tau_{\leq n} X)[n]$. So $X \in \mathcal{A}^{\geq n}$ if and only if $\mathbb{H}^k(X) = 0$ for every $k < n$.

**Theorem 26** The heart $\mathcal{C}$ is an abelian categroy.

**Proof** Note that for any $X, Y \in \mathcal{C}$, the distinguished triangle

$$X \to X \oplus Y \to Y \xrightarrow{0}$$

shows that $X \oplus Y \in \mathcal{C}$. Thus $\mathcal{C}$ is additive. For $f : X \to Y$ in $\mathcal{C}$, the distinguished triangle

$$X \xrightarrow{f} Y \to Z \to$$

shows that $Z \in \mathcal{A}^{\leq 0} \cap \mathcal{A}^{\geq -1}$. We claim that $\mathbb{H}^0(Z) = \tau_{\geq 0} Z \cong \operatorname{coker} f$ and $\mathbb{H}^{-1} Z = \tau_{\leq 0}(Z[-1]) \cong \ker f$. This claim can be checked using $\operatorname{Hom}(, -)$ and $\operatorname{Hom}(W, -)$. It remains to check that $\operatorname{coim} f \cong \operatorname{im} f$. Define $I$ such that

$$I \to Y \to \tau_{\geq 0} Z \to$$

is a distinguished triangle. Then $I \in \mathcal{A}^{\geq 0}$ and $I \cong \operatorname{Im} f$. By completing into a tetrahedron, we get an triangle

$$X \to I \to \tau_{\geq 0} Z \to \to .$$

Hence $I \in \mathcal{A}^{\leq 0}$ and $I \cong \operatorname{coim} f$. Hence $I \in \mathcal{C}$ and $\operatorname{coim} f \cong \operatorname{im} f$. □

**Proposition 19** The functor $\mathbb{H}^0 : \mathcal{A} \to \mathcal{C}$ is cohomological, i.e., for every triangle $X \to Y \to Z \to$ in $\mathcal{A}$, we obtain a long exact sequence of

$$\cdots \to \mathbb{H}^1(X) \to \mathbb{H}^1(Y) \to \mathbb{H}^1(Z) \to \mathbb{H}^2(X) \to \cdots .$$

**Proof** By rotation, it suffices to show that $\mathbb{H}^0(X) \to \mathbb{H}^0(Y) \to \mathbb{H}^0(Z)$ is exact.

a. Assume $X, Y, Z \in \mathcal{A}^{\geq 0}$, we shall show that $0 \to \mathbb{H}^0(X) \to \mathbb{H}^0(Y) \to \mathbb{H}^0(Z) \to 0$ is exact. In this case, we have $\operatorname{Hom}_{\mathcal{C}}(W, \mathbb{H}^0(X)) \cong \operatorname{Hom}_{\mathcal{A}}(W, \tau_{\geq 0} X = X)$ and $\operatorname{Hom}_{\mathcal{A}}(W, Z[-1]) = \operatorname{Hom}_{\mathcal{C}}(W, \mathbb{H}^0(Z[-1])) = 0$ for any $W \in \mathcal{C}$. Applying $\operatorname{Hom}_{\mathcal{A}}(W, -)$, we know the short exact sequence.

b. Assume that only $Z \in \mathcal{A}^{\geq 0}$. We shall show the short exact sequence $0 \to \mathbb{H}^0(X) \to \mathbb{H}^0(Y) \to \mathbb{H}^0(Z)$. By applying $\operatorname{Hom}_{\mathcal{A}}(W, -)$ for any $W \in \mathcal{A}^{<0}$, we know $\tau_{<0} X \cong \tau_{<0} Y$. By completing into a tetrahedron, we have the triangle $\tau_{\geq 0} X \to \tau_{\geq 0} Y \to Z \to$. By the first step, we know $0 \to \mathbb{H}^0(X) \to \mathbb{H}^0(Y) \to \mathbb{H}^0(Z)$ is exact.

c. Run the same argument, we know that if $X \in \mathcal{A}^{\leq 0}$, then $\mathbb{H}^0(X) \to \mathbb{H}^0(Y) \to \mathbb{H}^0(Z) \to 0$ is exact.

d. Let $X, Y, Z$ be arbitrary. Let $W$ such that $\tau_{\leq 0} X \to Y \to W \to$ is a distinguished triangle. Then $\mathbb{H}^0(X) \to \mathbb{H}^0(Y) \to \mathbb{H}^0(W)$ is exact by the third step. We also have the triangle $\tau_{>0} X \to W \to Z \to$, hence $0 \to \mathbb{H}^0(W) \to \mathbb{H}^0(Z)$ is exact by the second step. It follows that $\mathbb{H}^0(X) \to \mathbb{H}^0(Y) \to \mathbb{H}^0(Z)$ is exact. □

## Perverse $t$-structures

**Definition 45** Let $\mathcal{F} \in \mathcal{D}_c^b(X; k)$. We say $\mathcal{F}$ satisfies the *support condition* if $\dim(\operatorname{supp} H^i(\mathcal{F})) \leq -i$ for any $i \in \mathbb{Z}$. We define ${}^p\mathcal{D}^{\leq 0}(X) \subseteq \mathcal{D}_c^b(X; k)$ to be the full subcategory of objects satisfying the support condition.

**Definition 46** We say $\mathcal{F}$ satisfies the *cosupport condition* if $\mathbb{D}\mathcal{F} \in {}^p\mathcal{D}^{\leq 0}(X)$. We define ${}^p\mathcal{D}^{\geq 0}(X) \subseteq \mathcal{D}_c^b(X; k)$ to be the full subcategory of objects satisfying the cosupport condition.

**Theorem 27** $({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 0})$ is a $t$-structure on $\mathcal{D}_c^b(X; k)$ (called the *perverse $t$-structure*).

**Remark 21** $(i_\lambda)_* IC(X_\lambda; \mathcal{L})$ is in the heart of this perverse $t$-structure.

**Definition 47** We define $\operatorname{Perv}(X; k)$ to be the categroy of perverse sheaves in the heart of the perverse $t$-structure.

**Question** Why do we define the perverse $t$-structure in this way?

Here is another approach. Let $X = \coprod_{\lambda \in \Lambda} X_\lambda$ be a fixed stratification and $\mathcal{D}(X)$ be the complexes constructible with respect with this stratification. We would like to construct a self-dual $t$-structure on $X$ individually using descending induction on the strata.

On the top strata, we define $\mathcal{D}(X_{\text{top}})$ to be the complex of sheaves on $X_{\text{top}}$ with $H^i$ local systems on $X_{\text{top}}$. Let $X = X_{\text{top}}$, for $\mathcal{L}$ a local system on $X$, we have $\mathbb{D}\mathcal{L} = R\operatorname{Hom}(\mathcal{L}, \omega_X)$, where $\omega_X = \underline{k}_X[2\dim_{\mathbb{C}} X]$. So $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is not self-dual, but instead $(\mathcal{D}^{\leq 2\dim_{\mathbb{C}}(X)}, \mathcal{D}^{\geq 2\dim_{\mathbb{C}}(X)})$ is self-dual.

Now let $U \hookrightarrow X$ and $Z \hookrightarrow X$ be the open and closed embeddings obtained from the strata. Suppose we already have $({}^pD_U^{\leq 0}, {}^pD_U^{\geq 0})$ a self-dual $t$-structure on $U$ and also a self-dual $t$-structure on $\mathcal{D}(Z)$ given by $(\mathcal{D}^{\leq -d}, \mathcal{D}^{\geq -d})$, where $d = \dim Z$. We want a self-dual $t$-structure on $\mathcal{D}(X)$ such that

$${}^pD^{\leq 0} = \{\mathcal{F} \in \mathcal{D}(X) : j^*\mathcal{F} \in {}^p\mathcal{D}_U^{\leq 0}, i^*\mathcal{F} \in \mathcal{D}^{\leq -d}(Z)\}$$

and

$${}^pD^{\geq 0} = \{\mathcal{F} \in \mathcal{D}(X) : j^*\mathcal{F} \in {}^p\mathcal{D}_U^{\geq 0}, i^!\mathcal{F} \in \mathcal{D}^{\geq -d}(Z)\}.$$

This is self-dual by definition, so we only need to check it is actually a $t$-structure.

**Theorem 28** $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ is a $t$-structure on $\mathcal{D}(X)$.

**Proof**

a. Let $\mathcal{F} \in {}^pD^{\leq 0}$ and $\mathcal{G} \in {}^pD^{\geq 0}$. Applying $\mathrm{Hom}(-, \mathcal{G})$ to the adjunction distinguished triangle of $\mathcal{F}$, we obtain
$$\mathrm{Hom}(i_*i^*\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(j_!j^*\mathcal{F}, \mathcal{G}) \to .$$
Since $\mathrm{Hom}(i_*i^*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(i^*\mathcal{F}, i^!\mathcal{G}) = 0$ and $\mathrm{Hom}(j_!j^*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{G}) = 0$ by construction, we know that $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$.

b. Since shifts commute with restrtion, we know that $\mathcal{D}^{\leq -1} \subseteq \mathcal{D}^{\leq 0}$.

c. For $\mathcal{F} \in \mathcal{D}(X)$, we construct $\mathcal{G} \to \mathcal{F} \to j_* {}^p\tau_{>0,U} j^*\mathcal{F}$. We then construct $A$ such that
$$A \to \mathcal{G} \to i_*\tau_{>-d} i^*\mathcal{G} \to,$$
and $B$ such that
$$A \to \mathcal{F} \to B \to .$$
By completing the tetrahedron, we check that $A \in {}^pD^{\leq 0}$ and $B \in {}^pD^{>0}$. In fact, we have $j^*A \cong j^*\mathcal{G} = {}^p\tau_{\leq 0,U} j^*\mathcal{F} \subseteq {}^pD^{\leq 0}_U$ and $i^*\mathcal{A} \cong \tau_{\leq -d} i^*\mathcal{G} \in \mathcal{D}^{\leq -d}$, hence $A \in {}^pD^{\leq 0}$. Also, $j^B \cong j^*j_* {}^p\tau_{>0} j^*\mathcal{F} \in {}^p\mathcal{D}^{>0}$ and $i^!B \cong \tau_{>-d} i^*\mathcal{G} \in {}^pD^{>-d}$, hence $B \in {}^pD^{>0}$. □

It follows from the inductive construction that
$${}^pD^{\leq 0} = \{\mathcal{F} \in \mathcal{D}(X) : \forall X_\lambda \hookrightarrow X, i_\lambda^*\mathcal{F} \in \mathcal{D}^{\leq -\dim X_\lambda}\},$$
and
$${}^pD^{\geq 0} = \{\mathcal{F} \in \mathcal{D}(X) : \forall X_\lambda \hookrightarrow X, i_\lambda^!\mathcal{F} \in \mathcal{D}^{\geq -\dim X_\lambda}\}.$$
This explains why we defined perverse $t$-structures in such a way.

**Remark 22** Actually the above proof gives us a general procedure to glue any $t$-structures.

Notice that when $X_\lambda$ is smooth, $\mathcal{D}(X_\lambda)$ consists of complexes where $H^i$ are local systems. Then $\mathcal{D}^{\leq -\dim X_\lambda}, \mathcal{D}^{\geq -\dim X_\lambda})$ is th self-dual $t$-structure on $\mathcal{D}(X_\lambda)$.

**Remark 23** We are making an implicit assumption that $k$ is a field.

Let $(\mathcal{D}(U), 0)$ be the *degenerate* $t$-structure on $\mathcal{D}(U)$. By gluing with $({}^p\mathcal{D}_Z^{\leq 0}, {}^p\mathcal{D}_z^{\geq 0})$, we obtain a $t$-structure on $\mathcal{D}(X)$. Let $\tau_{\leq k}^Z$ be the corresponding truncating functor. Then $\tau_{\leq k}^Z$ is the right adjoint of the inclusion of objects $\mathcal{F}$ such that $i^*\mathcal{F} \in {}^p\mathcal{D}_Z^{\leq k}$ and
$$\tau_{\leq k}^Z \mathcal{F} \to \mathcal{F} \to i_* {}^p\tau_{>k} i^*\mathcal{F} \to .$$
Dually, define $\tau_{\geq k}^Z$ by using the $t$-structure glued from $(0, \mathcal{D}(U))$. We have
$$i_* {}^p\tau_{<k} i^!\mathcal{F} \to \mathcal{F} \to \tau_{\geq k}^Z \mathcal{F} \to .$$

**Lemma 11** Let $\mathcal{F}_U \in \mathcal{D}(U)$ and $k \in \mathbb{Z}$. Then there exists a unique up to a unique isomorphism extension $\mathcal{F}_k$ of $\mathcal{F}_U$ such that $i^*\mathcal{F} \in {}^p\mathcal{D}_Z^{\leq k-1}$ and $i^!\mathcal{F} \in {}^p\mathcal{D}_Z^{\geq k+1}$. Namely
$$\mathcal{F}_k = \tau_{\leq k-1}^Z j_*\mathcal{F}_U = \tau_{\geq k+1}^Z j_!\mathcal{F}_U.$$

**Proof** Use Lemma 9 and notice that $i^*j_*\mathcal{F}_U = i^!j_!\mathcal{F}_U[\pm 1]$. □

**Remark 24**

a. Suppose $\mathcal{F}(U) \in \mathrm{Perv}(U)$. When $k > 1$, we have $i^*\tau_{\leq k-1}^Z j_*\mathcal{F}_U = {}^p\tau_{\leq k-1} i^*j_*\mathcal{F}_U \in {}^p\mathcal{D}^{\leq k-1} \not\subseteq {}^p\mathcal{D}^{\leq 0}$, hence we do not expect $\mathcal{F}_k$ to be perverse. Similarly, if $k < -1$, we do not expect $\mathcal{F}_k$ is perverse. If $k = -1, 0, 1$, then $\mathcal{F}_k \in \mathrm{Perv}(X)$.

b. $j_{!*}\mathcal{F}_U = \tau_{\leq -1}^Z j_*\mathcal{F}_U = \tau_{\geq 1}^Z j_!\mathcal{F}_U$. In fact, by construction $i_\lambda^* j_{!*}\mathcal{F}_U \in \mathcal{D}^{\leq -\dim X_\lambda - 1}$ and $i_\lambda^! j_{!*}\mathcal{F}_U \in \mathcal{D}^{\geq -\dim X_\lambda - 1}$. This corresponds to the case $k = 0$.

c. For $k = -1$. On the open part $j^*j_!\mathcal{F}_u = \mathcal{F}_U \in \mathrm{Perv}(U)$, so $\tau_{\geq 0}^Z j_!\mathcal{F})_U = {}^p\tau_{\geq 0} j_!\mathcal{F}_U$. Since $j_!\mathcal{F}_U \in {}^p\mathcal{D}^{\leq 0}$ (as $i^*j_! = 0 \in {}^p\mathcal{D}^{\leq 0}$, we know it is equal to ${}^pH^0 j_!\mathcal{F}_U =: {}^pj_!\mathcal{F}_U$.

d. For $k = 1$, we have $\tau_{\leq 0}^Z j_*\mathcal{F}_u = {}^pH^0 j_*\mathcal{F}_u =: {}^pj_*\mathcal{F}_U$.

e. We obtain functors ${}^pj_!, {}^pj_* : \mathrm{Perv}(U) \to \mathrm{Perv}(X)$ and $({}^pj_!, j^*)$ and $(j^*, {}^pj_*)$ are adjoint pairs.

From ${}^p j_! j^! \to \mathrm{Id}$ and $\mathrm{Id} \to {}^p j_* j^*$, we have a morphism of functors ${}^p j_! \to {}^p j_*$.

**Proposition 20** The image functor $\mathrm{im}({}^p j_! \to {}^p j_*)$ is $j_{!*}$.

**Proof** Using the triangle for $\tau^Z_{\geq k}$, we have a short exact sequence

$$0 \to i_* {}^p H^0 i^! j_! \mathcal{F}_U \to {}^p j_! \mathcal{F}_U \to j_{!*} \mathcal{F}_U \to 0.$$

Similarly we have

$$0 \to j_{!*} \mathcal{F}_U \to {}^p j_* \mathcal{F}_u \to i_* {}^p H^0 i^* j_* \mathcal{F}_U \to 0.$$

It follows that $\mathrm{im}({}^p j_! \to {}^p j_*) = j_{!*}$ by identifying $i_* {}^p H^0 i^! j_! \mathcal{F}_U$ with $i_* {}^p H^{-1} i^* j_* \mathcal{F}_U$. □

## Simple objects

**Theorem 29** The simple objects in $\mathrm{Perv}(X) = P(X)$ are the $IC$-sheaves.

From the strata $i : Z \hookrightarrow X$ and $j : U \hookrightarrow X$, we have $i_* : P(Z) \to P(X)$ and $j^* : P(U) \to P(X)$. Let $\overline{P(Z)}$ be the essential image of $i_*$ in $P(X)$, in other words, the full subcategory with objects $\mathcal{F} \in P(Z)$ such that $j^* \mathcal{F} = 0$. Consider $A \in P(X)$ and the triangle $j_! j^* A \to A \to I_* i^* A \to$, applying ${}^p H^*$, we have a long exact sequence

$$0 \to i_* {}^p H^{-1} i^* A \to {}^p j_! j^* A \to A \to i_* {}^p i^* A \to 0.$$

We know that $i_* {}^p i^* A$ is the maximal quotient of $A$ with support in $Z$. Similarly, $i_* {}^p i^! A$ is the maximal subobject of $A$ with support in $Z$.

**Proposition 21** The functor $j^* : P(U) \to P(X)$ factors through the Serre quotient categroy $Q : P(X) \to P(X)/P(Z)$. Moreover, $T : P(X)/P(Z) \to P(U)$ is an equivalence of categories.

**Proof** $T$ is faithful: Let $f \in \mathrm{Hom}_{P(X)/P(Z)}$ and $T(f) = 0$. Let $f_1 \in \mathrm{Hom}_{P(X)}$ be a lift of $f$. Since $T(f) = 0$, we know that $j^*(f_!) = 0$, so $\mathrm{Im}(f_1) \in P(Z)$, therefore $Q(f_1) = 0$.

$T$ is essentially surjective and full: As $\mathrm{Id} \cong j^* {}^p j_! \cong T(Q({}^p j_!))$. □

**Proposition 22**

a. For $B \in P(U)$, $j_{!*} B$ is the unique extension of $j$ which has no nontrivial subquotients with support in $Z$.

b. The simple objects of $P(X)$ are

   a. $i_* S$ for $S \in P(Z)$ simple,

   b. $j_{!*} T$ for $T \in P(U)$ simple.

**Proof**

a. Recall that ${}^p i^* A$ is the maximal quotient ojbect of $A$ with support in $Z$ and ${}^p i^! A$ is the maximal subojet of $A$ with support in $Z$. So if $A$ has no subquotients with support in $Z$, then ${}^p i^* A = {}^p i^! A = 0$. Hence $i^* A \in {}^p \mathcal{D}^{<0}$ and $i^! A \in {}^p \mathcal{D}^{>0}$, which is equivalent to $A = j_{!*}(j^* A)$. Now use the triangle $i^* A \to i^* j_* B \to i^! A[1] \to$.

b. Any simple object in $P(X)$ is either

   a. the image of a simple object in $P(Z)$,

   b. an extension of a simple object in $P(X)/P(Z) = P(U)$ which has no nontrivial subobjects with support in $Z$, i.e. $j_{!*} T$ for $T \in P(U)$ simple. □

## Operations on perverse sheaves

Suppose $\mathcal{F} \in {}^p \mathcal{D}^{\leq 0}(Y)$ and $\dim(f^{-1}(y) \leq d$ for any $y \in Y$. Then

$$\dim(\mathrm{supp} H^i(f^* \mathcal{F})) \leq \dim(\mathrm{supp} H^i(\mathcal{F})) + d \leq -i + d.$$

Hence $f^* \mathcal{F} \in {}^p \mathcal{D}^{\leq d}(X)$. By duality, we know that for $\mathcal{G} \in {}^p \mathcal{D}^{\geq 0}(Y)$, $f^! \mathcal{G} \in {}^p \mathcal{D}^{\geq -d}(X)$.

**Lemma 12** Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{A}$ be an adjoint pair of triangulated functors. Then $f(\mathcal{A}^{\leq 0}) \subseteq \mathcal{B}^{\leq 0}$ (*right* $t$*-exact*) if and only if $g(\mathcal{B}^{\leq 0}) \subseteq \mathcal{A}^{\leq 0}$ (*left* $t$*-exact*).

**Corollary 15** $f_*({}^p\mathcal{D}^{\geq 0}(X)) \subseteq {}^p\mathcal{D}^{\geq -d}(Y)$ and $f_!({}^p\mathcal{D}^{\leq 0}(X)) \subseteq {}^p\mathcal{D}^{\leq d}(Y)$.

If $f$ is proper, then $f_* = f_!$, hence $f_*(\mathcal{C}) \subseteq {}^p\mathcal{D}^{[-d,d]}$. Recall that if $f$ is smooth of relative dimension $d$, then $f^! = f^*[2d]$. Hence $f^![-d] = f^*[d]$ and they take perverse sheaves to perverse sheaves ( $t$ *-exact*). In particular, when $f$ is etale, we know that $f^* = f^!$ is $t$ -exact.

Recall that if $X$ is smooth and affine, then $H^m(X;k) = 0$ for any $m > \dim_{\mathbb{C}} X$. The following is a generalization of this fact. The proof can be given by generalizing the original proof using Morse theory.

**Theorem 30** If $f : X \to Y$ is affine, then $f_*({}^p\mathcal{D}^{\leq 0}(X)) \subseteq {}^p\mathcal{D}^{\leq 0}(Y)$ and $f_!({}^p\mathcal{D}^{\geq 0}(X)) \subseteq {}^p\mathcal{D}^{\geq 0}(Y)$.

**Definition 48** Let $f : X \to Y$ be proper. $Y^n = \{y \in Y : \dim f^{-1}(y) = n\}$ is locally closed subvarieties and $Y = \coprod Y_n$. We say $f$ is *semismall* (resp. *small*) if $\dim Y^n + 2n \leq \dim X$ for any $n$ (resp. $<$ holds for any $n \neq 0$).

**Remark 25** If $f$ is semismall, then it is generically finite ( $\dim Y = \dim Y^0 = \dim X$ ).

Now suppose $f : X \to Y$ is semismall.

**Proposition 23** $f_*(P(X)) \subseteq P(Y)$.

**Proposition 24** Let $X$ be smooth. Then $f_*\underline{k}_X[\dim X] \in P(Y)$.

**Proof** $H^i(f_*\underline{k}_X[\dim X])$ has stalks $H^{i+\dim X}(f^{-1}(y))$. So it is zero if $i + \dim X > 2\dim f^{-1}(y)$. So

$$\dim \mathrm{supp} H^i(f_*\underline{k}_X[\dim X]) \leq \dim \coprod_{n > (i+\dim X)/2} Y^n \leq -i$$

as $2n \geq i + \dim X$. Namely $f_*\underline{k}_X[\dim X] \in {}^p\mathcal{D}^{\leq 0}$. One can similarly check that $\mathbb{D} f_*\underline{k}_X[\dim X] \in {}^p\mathcal{D}^{\leq 0}$. □

We showed that $f_*\underline{k}_X[\dim X] \in P(Y)$. Let $S_n = \coprod_{\dim Y_\lambda = n} Y_\lambda$. If $S_n \cap Y^k \neq 0$, then $n \leq \dim Y_n \leq \dim X - 2k$, hence $2k le \dim -n$. In other words, the fibers over $S_n$ have dimension $\leq \frac{1}{2}(\dim X - n)$. $H^j(i_\lambda^*(f_*\underline{k}_X[\dim X]))$ is a local system on $Y_n$, hence $f_*\underline{k}_X[\dim X] \in {}^p\mathcal{D}^{\geq 0}(Y)$ if and only if $H^j(i_\lambda^*\underline{k}_X[\dim X]) = 0$ for all $j > -\dim Y_\lambda$. This explaines why we defined "semismall" in such a way.

**Example 18** Let $\mathcal{N} = \{X \in gL_3(\mathbb{C}) : X^3 = 0\}$ and $\tilde{\mathcal{N}} = \{(x, \mathcal{F}^\cdot \in gL_3(\mathbb{C}) \times \mathrm{Fl}_3\} \cong T^*\mathrm{Fl}_3$. Then $\mathcal{N} = 0 \cup \mathcal{O}_{\min} \cup \mathcal{O}_{\mathrm{reg}}$, where $\mathcal{O}_{\min}$ consists of $\begin{bmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{bmatrix}$ and $\mathcal{O}_{\mathrm{reg}}$ consists of $\begin{bmatrix} 0 & 1 & \\ 0 & 0 & 1 \\ & 0 & 0 \end{bmatrix}$. Let $\pi : \tilde{\mathcal{N}} \to \mathcal{N}$ be the projection. Then $\pi^{-1}(0) = \mathrm{Fl}_3$, $\pi^{-1}\begin{bmatrix} 0 & 1 & \\ 0 & 0 & \\ & & 0 \end{bmatrix}$ is the glueing of two spheres along a point.

$\pi^{-1}\begin{bmatrix} 0 & 1 & \\ 0 & 0 & 1 \\ & 0 & 0 \end{bmatrix}$ is a single point. We have the following table

| | $-6$ | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|---|---|---|---|
| $0$ | $\mathbb{Q}$ | | $\mathbb{Q}^2$ | | $\mathbb{Q}^2$ | | $\mathbb{Q}$ |
| $\mathcal{O}_{\min}$ | $\mathbb{Q}$ | | $\mathbb{Q}^2$ | | | | |
| $\mathcal{O}_{\mathrm{reg}}$ | $\mathbb{Q}$ | | | | | | |

We know that $\pi$ is semismall and $\pi_*\underline{k}[\dim] \in P(\mathcal{N})$. By decomposition theorem, we know that $\pi_*\mathbb{Q}_{\tilde{\mathcal{N}}}[6] \cong IC^a_{\mathrm{reg}} \oplus IC^b_{\min} \oplus IC^c_0$. Comparing the stalk complexes of $IC_0$, $IC_{\min}$ and $IC_{\mathrm{reg}}$

| | $0$ |
|---|---|
| $0$ | $\mathbb{Q}$ |

$IC_0$

| | $-4$ | $-3$ | $-2$ | $-1$ | $0$ |
|---|---|---|---|---|---|
| $0$ | $*$ | $*$ | $*$ | $*$ | $0$ |
| $\mathcal{O}_{\min}$ | $\mathbb{Q}$ | | | | |

$IC_{\min}$

| | -6 | -5 | -4 | -3 | -2 | -1 | 0 |
|---|---|---|---|---|---|---|---|
| 0 | * | * | * | * | * | * | 0 |
| $\mathcal{O}_{\min}$ | * | * | 0 | | | | |
| $\mathcal{O}_{\mathrm{reg}}$ | $\mathbb{Q}$ | | | | | | |

$IC_{\mathrm{reg}}$

with the above table, we know that $a = 1$, $b = 2$, $c = 1$. We have $\mathcal{N} \subseteq \mathfrak{g}$, $\tilde{\mathcal{N}} \subseteq \mathfrak{g} \times G/B$. The Weyl group $W$ acts on $\pi_* \underline{k}_{\tilde{\mathcal{N}}}$.

## Kazhdan-Lusztig conjecture

Fix a connected reductive algebraic group $G$ and $T \subseteq B \subseteq G$. Let $W$ be the associated Weyl group.

**Definition 49** The *Hecke algebra* is defined to be

$$\mathcal{H} = \bigoplus_{x \in W} \mathbb{Z}[v^{\pm}] T_x,$$

where

$$T_x T_s = \begin{cases} T_{xs} & xs > x, \\ (v^{-1} - v) T_x + T_{xs} & xs < x. \end{cases}$$

**Remark 26** $\mathcal{H}$ is a deformation of the group algebra of $W$.

**Theorem 31** (Iwahori) Let $G_q = G(\mathbb{F}_q)$, $B_q = B(\mathbb{F}_q)$. Then $\mathbb{Z}[B_q \backslash G_q / B_q] \cong \mathcal{H}_{\frac{1}{q}}$. ??

**Remark 27** $T_s(T_s + (v - v^{-1})) = 1$, hence $T_s$ is invertible.

**Definition 50** The endomorphism of $\mathcal{H}$ given by $\bar{v} = v^{-1}$ and $\overline{T_w} = T_{w^{-1}}^{-1}$ is an involution, denoted by $h \mapsto \bar{h}$. We call $h \in \mathcal{H}$ is self-dual if $h = \bar{h}$.

**Theorem 32** (Kazhdan-Lusztig) There exists a unique baissi $\{C_w : w \in W\}$ of $\mathcal{H}$ such that

a. Each $C_w$ is self-dual.
b. $C_y = T_y + \sum_{x<y} h_{x,y} T_x$, where $h_{x,y} \in v\mathbb{Z}[v]$.

**Conjecture 1** If

a. $h_{x,y} \in v\mathbb{N}[v]$.
b. $C_x C_y = \sum_{z \in W} h_{xyz} C_z$, where $h_{xyz} \in \mathbb{Z}[v, v^{-1}]$ self-dual.

Then $h_{xyz} \in \mathbb{N}[v, v^{-1}]$.

The idea of the solution relates the $h_{x,y}$ with the geometry of $G/B$. By the Bruhat decomposition, $G/B \cong \coprod_{w \in W} BwB/B$, where each $BwB/B \cong \mathbb{C}^{\ell(w)}$. Write $i_w : BwB/B \hookrightarrow G/B$. Let $\mathcal{D}(G/B)$ be the constructible derived category with respect to the Bruhat stratification. For $\mathcal{F} \in \mathcal{D}(G/B)$, by the decomposition theorem, we know that $i_w^* \mathcal{F} \cong \bigoplus_k \underline{\mathbb{Q}}_{BwB/B}[\ell(w) + k]^{p_{w,k}}$.

**Definition 51** We define $\mathrm{ch} : \mathcal{D}(G/B) \to \mathcal{H}$ by $\mathcal{F} \mapsto \sum_w \sum_k p_{w,k} v^k T_w$.

**Example 19** $\Delta_w = (i_w)_! \underline{\mathbb{Q}}_{BwB/B}[\ell(w)]$, then $\mathrm{ch}(\Delta_w) = T_w$.

**Theorem 33** $\mathrm{ch}(IC_{BwB/B}) = C_w$.

Then the Kazhdan-Lusztig conjecture follows from this geometric characterization.

**Example 20** $C_e = T_e$, $C_s = T_s + v$. Then $\overline{C_s} = \overline{T_s} + v^{-1} = T_s + (v - v^{-1}) + v^{-1} = T_s + v$.

$$T_x C_s = T_x(T_s + v) = \begin{cases} T_{xs} + vT_x & xs > x \\ T_{xs} + v^{-1} T_x & xs < x \end{cases}$$

**Example 21** For $G = GL_2$, we have $G/B \cong \mathbb{P}^1 \cong B\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}B/B \cup B/B = \mathbb{A}^1 \cup \mathrm{pt}$. Then $\mathrm{ch}(IC_e) = T_e = C_e$, $\mathrm{ch}(IC_s) = T_s + vT_e = C_s$.

<definition> For $\mathcal{F} \in \mathcal{D}(G/B)$ is called $*$-even (resp. odd) if $H^i(\mathcal{F}) = 0$ for all $A$ odd (resp. even) (Equivalently, $H^i(i_x^*\mathcal{F}) = 0$ for all $x \in W$ and $A$ odd (resp. even). $\mathcal{F}$ is called $*$-parity if $\mathcal{F} = \mathcal{F}_{\mathrm{even}} \oplus \mathcal{F}_{\mathrm{odd}}$.

**Lemma 13** Let $\mathcal{F} \to \mathcal{G} \to \mathcal{H}$ be a distinguished triangle in $\mathcal{D}(G/B)$. If $\mathcal{F}, \mathcal{H}$ are $*$-even, then so is $\mathcal{G}$. Moreover, $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$.

**Proof** Applying $i_x^*$ and using the long exact sequence in cohomology, we know that $H^{2i-1}(i_x^*\mathcal{G}) = 0$ and $H^{2i}(i_x^*\mathcal{G}) \cong H^{2i}(i_x^*\mathcal{F}) \oplus H^{2i}(i_x^*\mathcal{H})$. □

**Remark 28** In general, it is not true that $\mathrm{ch}$ is additive on distinguished triangle.

Let $P_s = \overline{BsB}$ be the parabolic subgroup (e.g., block upper triangular matrices for $G = GL_n$). Let $\pi_s : G/B \to G/P_s$.

**Lemma 14** (Push-Pull (Springer, Brylinski, MacPherson)) If $\mathcal{F}$ is $*$-parity, then $\mathrm{ch}(\pi_s^*\pi_{s*})\mathcal{F}[1] = \mathrm{ch}(\mathcal{F})C_s$.

**Proof** Let $N(\mathcal{F}) = \{x \in W : i_x^* \neq 0$. We will induct on $|N(\mathcal{F})|$.

When $|N(\mathcal{F})| = 1$. We have $\mathcal{F} \cong \bigoplus \Delta_w[k]^{\oplus m_k}$. So it suffices to check on $\Delta_w = i_{w!}\underline{\mathbb{Q}}_{BwB/B}[\ell(w)]$. $G/B \to G/P$ reduces to the case of $\mathbb{P}^1 \to \mathrm{pt}$. For $G = GL_2$:

a. $w = e$, $\pi_s^*\pi_{s*}\underline{\mathbb{Q}}_{B/B}[1] \cong \underline{\mathbb{Q}}_{\mathbb{P}^1}[1]$. The corresponding $T_s + v = C_s$.

b. $w = s$, $\pi_s^*\pi_{s*}j_!\underline{\mathbb{Q}}_{\mathbb{A}^1}[1][1] = \pi_s^* H_c(\underline{\mathbb{Q}}_{\mathbb{A}^1})[2] = \pi_s^*(\underline{\mathbb{Q}}[-2]_{\mathrm{pt}})[2] = \underline{\mathbb{Q}}_{\mathbb{P}^1}$. The corresponding $v^{-1}T_s + 1 = T_sC_s$.

Now suppose $|N(\mathcal{F})| > 1$. Pick $w \in N(\mathcal{F})$ maximal. Using the adjunction triangle

$$i_{x!}i_x^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F},$$

where $i : \mathrm{supp}\mathcal{F} \setminus BwB/B \hookrightarrow \mathrm{supp}\mathcal{F}$. $i_{x!}i_x^*\mathcal{F} \cong \oplus\Delta_x[k]^{\oplus m_k}$ is $*$-even. And $i_*i^*\mathcal{F}$ is $*$-even since $\mathcal{F}$ is. Applying $\pi_s^*\pi_{s*}$, we know that $\mathrm{ch}(\pi_s^*\pi_{s*}\mathcal{F}) = \mathrm{ch}(\pi_s^*\pi_{s*}\Delta_x[k]^{\oplus m_k}) + \mathrm{ch}(\pi_s^*\pi_{s*}(i_*i^*\mathcal{F})) = \mathrm{ch}(\mathcal{F})C_s$ by induction. □

Now we will use the Push-Pull Lemma to prove the main theorem, namely $\mathrm{ch}(IC_{\overline{BwB/B}}) = C_w$.

**Proof** First we will inductively show that $\mathrm{ch}(IC_{\overline{BwB/B}})$ is self-dual. Choose a reduced expression for $w = s_1 \cdots s_m$. By the by Push-Pull Lemma, $\mathrm{ch}(\pi_{s_1}^*\pi_{s_1*}\cdots\pi_{s_m}^*\pi_{s_m*}\underline{\mathbb{Q}}_{B/B}[m]) = C_{s_m}\cdots C_{s_1}$.

$$\begin{tikzcd}
P_{s_1} \times_B P_{s_2}/B \arrow[r] \arrow[d] & P_{s_2}/B \arrow[r] \arrow[d] & G/B \arrow[r, "\pi_{s_3}"] \arrow[d, "\pi_{s_2}"] & G/P_{s_3} \\
P_{s_1}/B \arrow[r] \arrow[d] & G/B \arrow[r, "\pi_{s_2}"] \arrow[d, "\pi_{s_1}"] & G/P_{s_2} & \\
B/B \arrow[r, "\pi_{s_1}"] & G/P_{s_1} & &
\end{tikzcd}$$

The net effect of this push-pull sequence is pulling back the constant sheaf all the way up and pushing forward through the Bott-Samelson resolution,

$$\pi : P_{s_1} \times_B P_{s_2} \cdots \times_B P_{s_m}/B \to G/B, \quad (p_1, \ldots, p_m) \mapsto p_1 \cdots p_m B.$$

Its image is $\overline{BwB/B}$ and is in fact a resolution of singularities. By base change, we know

$$\mathrm{ch}(\pi_{s_1}^*\pi_{s_1*}\cdots\pi_{s_m}^*\pi_{s_m*}\underline{\mathbb{Q}}_{B/B}[m]) = \mathrm{ch}(\pi_*\underline{\mathbb{Q}}_{P_{s_1}\times_B P_{s_2}\cdots\times_B P_{s_m}/B}[m]),$$

which is isomorphic to

$$IC_{\overline{BwB/B}} \oplus \bigoplus_{y<w} V_y \otimes IC_{\overline{ByB/B}}$$

by the decomposition theorem, where $V_y$ is a self-dual vector space. Now the self-duality of $\mathrm{ch}(IC_{\overline{BwB/B}})$ follows from the induction hypothesis and the self-duality of $C_{s_m}\cdots C_{s_1}$.

Let $x \in W$. If $y > x$, then $H^k(i_y^*IC_x) = 0$. If $y = x$, then $i_x^*IC_x = \mathbb{Q}[\ell(x)]$. If $y < x$, then $H^k(i_y^*IC_x) = 0$ for any $k \geq -\ell(y)$. Then the second part follows from the definition of $\mathrm{ch}$. □

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