1. April 12

1.1.

Plan of the lectures:

1. Hilbert spaces, projections, and physics
2. Von Neumann algebras, states, and representations
3. Subfactors and index
4. The Temperley–Lieb algebra

"quantum mechanics $\rightsquigarrow$ functional analysis $\rightsquigarrow$ knot theory"

1.2.

Df A Hermitian space is a $\mathbb{C}$-vector space $\mathbb{H}$ with a pairing

$$\langle -, - \rangle : \mathbb{H} \times \mathbb{H} \to \mathbb{C}$$

that is positive-definite, sesquilinear, and s.t. $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

For any $v \in \mathbb{H}$ we set $\|v\| = \sqrt{\langle v, v \rangle}$.

$\mathbb{H}$ is a Hilbert space iff it is complete wrt the norm $\|-\|$.

1.3.

For any linear $a : \mathbb{H} \to \mathbb{H}'$, we set $\|a\| = \sup_{\|v\|=1} \|a(v)\|$.

By homogeneity + $\triangle$-inequality, the set of bounded maps

$$\mathcal{B}(\mathbb{H}, \mathbb{H}') = \{a \mid \|a\| < \infty\}$$

forms a vector space. Write $\check{\mathbb{H}} := \mathcal{B}(\mathbb{H}, \mathbb{C})$.

Thm (Riesz) If $\mathbb{H}$ is Hilbert, then the map $\mathbb{H} \to \check{\mathbb{H}}$ that sends $v \mapsto \langle -, v \rangle$ is an isomorphism. [Why not $\langle v, - \rangle$?]

1.4.

Df $\quad a \in \mathcal{B}(\mathbb{H}, \mathbb{H}') \quad \rightsquigarrow \quad$ adjoint $a^* = (-) \circ a \in \mathcal{B}(\check{\mathbb{H}}', \check{\mathbb{H}})$.

Cor $\quad$ If $\mathbb{H}$ is Hilbert, then we can regard $a^* \in \mathcal{B}(\mathbb{H}', \mathbb{H})$ and

$$\langle v, a^*(w) \rangle = \langle a(v), w \rangle$$

for $v \in \mathbb{H}$ and $w \in \mathbb{H}'$.

The map $a \mapsto a^*$ is anti-linear and $a^{**} = a$.

1.5.

Example $\quad$ For a measure space X, let

$$L^2(X) = \{\text{functions } f \mid \|f\| = \sqrt{\langle f, f \rangle} < \infty\}/\sim$$

where $\langle f, g \rangle = \int_X f(x)\overline{g(x)}\,dx$ and $f \sim 0$ iff $\|f\| = 0$.

For measurable $Y \subseteq X$, the composition

$$L^2(X) \twoheadrightarrow L^2(Y) \hookrightarrow L^2(X)$$

is a self-adjoint operator on $L^2(X)$ called the projection onto Y.

1.6.

Df $\quad \mathcal{B}(\mathbb{H}) := \mathcal{B}(\mathbb{H}, \mathbb{H})$. $\quad$ We say $a \in \mathcal{B}(\mathbb{H})$ is

- self-adjoint iff $a^* = a$
- positive iff $\langle v, a(v) \rangle \geq 0$ for all $v \in \mathbb{H}$.
- a projection iff $a^* = a = a^2$.

Lem $\quad$ projection $\implies$ positive $\implies$ self-adjoint.

1.7.

There is a bijection:

$$\begin{aligned} \{\text{closed subspaces of } \mathbb{H}\} &\leftrightarrow \{\text{projections in } \mathcal{B}(\mathbb{H})\} \\ \mathbb{K} &\mapsto \underbrace{\mathbb{H} = \mathbb{K} \oplus \mathbb{K}^\perp \twoheadrightarrow \mathbb{K} \hookrightarrow \mathbb{H}}_{e_{\mathbb{K}}} \\ \operatorname{im}(e) &\leftarrow\!\shortmid \; e \\ (\cap \text{ of subspaces}) &\leftrightarrow (\cdot \text{ of projections}) \end{aligned}$$

<u>Rem</u> Each subspace of X defines a closed subspace of $L^2(X)$, but not vice versa. [Example?]

1.8.

<u>Lem</u> a^*a is positive for any $a \in \mathcal{B}(\mathbb{H})$.

More strongly, a is a <u>partial isometry</u> iff a^*a is a projection. Then it looks like $\mathbb{H} \twoheadrightarrow \operatorname{im}(a^*) \xrightarrow{\sim} \operatorname{im}(a) \hookrightarrow \mathbb{H}$.

<u>Polar Decomposition Thm</u> Any $a \in \mathcal{B}(\mathbb{H})$ factors as

$$a = u|a|$$

where u is a partial isometry and $|a|$ is a positive sqrt of a^*a.

1.9.

<u>Example</u> We can embed $L^\infty(X) \hookrightarrow \mathcal{B}(L^2(X))$ via

$$[a(f)](x) = a(x)f(x) \quad \text{for } a \in L^\infty \text{ and } f \in L^2.$$

We have $a^*(x) = \overline{a(x)}$ and $|a|(x) = |a(x)|$.

If a is nonzero a.e., then $u = \frac{a}{|a|}$ is <u>unitary</u>: $u^*u = uu^* = 1$.

1.10.

Application to physics:

	classical	quantum
state state	measure space	projectivized Hilbert space
observables	functions	bounded operators
measurement	commutative	noncommutative

1.11.

Let Λ be a measure space of possible observations, e.g., $\{0, 1\}$ in the Boolean case.

<u>Classical</u> The state space is a measure space X.

$$\begin{array}{rl} \text{observable} & \text{function } X \xrightarrow{q} \Lambda \\ \text{observation in } E \subseteq \Lambda & \text{state in } q^{-1}(E) \end{array}$$

Boolean observables: $\mathbf{1}_Y(x) = \begin{cases} 1 & x \in Y \\ 0 & x \notin Y \end{cases}$ for $Y \subseteq X$.

1.12.

<u>Quantum</u> The state space is $\mathbb{PH} = \{v \in \mathbb{H} \mid \|v\| = 1\}/S^1$.

$$\begin{array}{rl} \text{observable} & \text{“measure” } q : \{E \subseteq \Lambda\} \to \{\mathbb{K} \subseteq \mathbb{H}\} \text{ s.t.} \\ & \begin{cases} q(\Lambda) = \mathbb{H} \\ \langle q(-)v, w\rangle \text{ is a measure for all } v, w \end{cases} \end{array}$$

$$\text{observation in } E \subseteq \Lambda \quad \text{probability } \langle a(\mathbf{1}_E)v, v\rangle\text{, given state } v$$

Boolean observables: $q_{\mathbb{K}}(a) = e_{\mathbb{K}}a(1) + e_{\mathbb{K}^\perp}a(0)$ for $\mathbb{K} \subseteq \mathbb{H}$.

1.13.

Classical logic is distributive:

$$\begin{aligned} (Y_1 \cup Y_2) \ \cap Z &= (Y_1 \cap Z) \cup (Y_2 \cap Z) \\ (\mathbf{1}_{Y_1} + \mathbf{1}_{Y_2}) \cdot \mathbf{1}_Z &= \mathbf{1}_{Y_1} \cdot \mathbf{1}_Z + \mathbf{1}_{Y_2} \cdot \mathbf{1}_Z \end{aligned}$$

Quantum logic is not in general:

$$\begin{aligned} (\mathbb{K}_1 + \mathbb{K}_2) \ \cap \mathbb{L} &\neq (\mathbb{K}_1 \cap \mathbb{L}) + (\mathbb{K}_2 \cap \mathbb{L}) \\ (e_{\mathbb{K}_1} + e_{\mathbb{K}_2}) \cdot e_{\mathbb{L}} &\neq (e_{\mathbb{K}_1} \cdot e_{\mathbb{L}}) + (e_{\mathbb{K}_2} \cdot e_{\mathbb{L}}) \end{aligned}$$

<u>Example</u> Double-slit experiment.

$\mathbb{K}_i$ is “pass through slit i” and $\mathbb{L}$ is “arrive at screen”.

1.14.

<u>Spectral Thm</u> Suppose $\Lambda \subseteq \mathbb{R}$ is bounded. Then

$$\left\{ \begin{array}{c} \text{quantum observables} \\ L^\infty(\Lambda) \to \mathcal{B}(\mathbb{H}) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{self-adjoint ops in } \mathcal{B}(\mathbb{H}) \\ \text{with spectrum in } \Lambda \end{array} \right\}$$

$$q \mapsto a_q = \int_\Lambda x \, dq(x)$$

The expectation of q in state v equals $\langle a_q(v), v \rangle$.

1.15.

Quantum observables “are” self-adjoint operators $a \in \mathcal{B}(\mathbb{H})$.

<u>Schrödinger</u>: Study $v \mapsto \langle av, v \rangle$ as the state v evolves.

<u>Heisenberg</u>: Study $a \mapsto \langle av, v \rangle$ as the observable a varies.

A state “is” a (positive) linear map $\mathcal{B}(\mathbb{H}) \to \mathbb{C}$ sending id $\mapsto 1$.

2. April 14

2.1.

Last time, we saw that:

- Quantum observables are bounded self-adjoint operators on Hilbert spaces: $a \in \mathcal{B}(\mathbb{H})$.
- Quantum states v induce linear functionals $\mathcal{B}(\mathbb{H}) \to \mathbb{C}$: namely, $a \mapsto \langle av, v\rangle$.

Sometimes, we use smaller subalgebras like $L^\infty \subseteq \mathcal{B}(L^2)$.

Today: algebraic properties of such subalgebras and functionals.

2.2.

Three topologies on $\mathcal{B}(\mathbb{H})$:

topology	basis of open nbds at 0
norm	$\{a \mid \|a\| < \epsilon\}$ for $\epsilon > 0$
strong operator (SOT)	$\{a \mid \|a(v_i)\| < \epsilon$ for all $i\}$ for $\epsilon > 0$ and finite sets $\{v_i\}_i$
weak operator (WOT)	$\{a \mid \lvert\langle a(v_i), w_i\rangle\rvert < \epsilon$ for all $i\}$ for $\epsilon > 0$ and finite sets $\{v_i, w_i\}_i$

Lem (weak) is coarser than (strong) is coarser than (norm).

2.3.

Df The commutant of a subalgebra $M \subseteq \mathcal{B}(\mathbb{H})$ is

$$M' := \mathrm{End}_M(\mathbb{H}) = \{b \in \mathcal{B}(\mathbb{H}) \mid ab = ba \text{ for all } a \in M\}.$$

Thm (von Neumann) If M is $*$-closed and contains 1, then

$$\text{SOT closure} = \text{WOT closure} = \text{bicommutant } M''$$

2.4.

<u>Proof</u> Want: (1) $\overline{M}_{\text{WOT}} \subseteq M''$, (2) $M'' \subseteq \overline{M}_{\text{SOT}}$.

(1) Claim that commutants are always WOT closed.
Then $M \subseteq M'' \implies \overline{M}_{\text{WOT}} \subseteq \overline{(M'')}_{\text{WOT}} = M''$.

To prove claim: For any $a \in \mathcal{B}(\mathbb{H})$ and $v, w \in \mathbb{H}$, the map

$$\lambda_{a,v,w}(b) = \langle (ab - ba)(v), w \rangle : \mathcal{B}(\mathbb{H}) \to \mathbb{C}$$

is WOT continuous. Observe $N' = \bigcap_{a \in N} \bigcap_{v,w} \lambda^{-1}_{a,v,w}(0)$.

2.5.

(2) Let $c \in M''$ and $\{v_i\}_{i=1}^n \subseteq \mathbb{H}$.
Let $\mathcal{B}(\mathbb{H}) \curvearrowright \mathbb{H}^{\oplus n}$ diagonally and $v = (v_i)_{i=1}^n \in \mathbb{H}^{\oplus n}$.
Want to show $c(v) \in \overline{Mv}$. [Closed subspace of $\mathbb{H}^{\oplus n}$]

Let $e \in \mathcal{B}(\mathbb{H}^{\oplus n})$ be the projection on $\overline{Mv}$. Check

$$e \in \mathrm{End}_M(\mathbb{H}^{\oplus n}), \quad \text{whence} \quad c \in \mathrm{End}_{\mathbb{C}e}(\mathbb{H}^{\oplus n}).$$

So $c\overline{Mv} \subseteq \overline{Mv}$. So $c(v) = c1_M(v) \in \overline{Mv}$.

2.6.

<u>Df</u> A subalgebra $M \subseteq \mathcal{B}(\mathbb{H})$ is <u>von Neumann</u> iff M is WOT closed, $*$-closed, and contains 1.

<u>Example</u> $\mathcal{B}(\mathbb{H})$
<u>Example</u> $L^\infty(X) \subseteq \mathcal{B}(L^2(X))$ for any ("σ-finite") X.
E.g., X countable or $X = [0, 1]$.

2.7.

Thm Commutative VNA's on separable $\mathbb{H}$ look like $L^\infty(X)$,
where X is a finite union of $[0, 1]$'s and countable sets.
["VNA's are noncommutative measure spaces."]

Df M is a factor iff its center $M \cap M'$ consists of scalars.

"Cor" General VNA's look like "$M = \int_X^\oplus M_x\,dx$",
where the M_x are factors and $L^\infty(X) = M \cap M'$.

2.8.

Classify factors using projections. Recall that $a \in M$ is

- positive iff $(\langle v, a(v)\rangle$ for all $v)$ iff $(a = b^*b$ for some $b)$
- a projection iff $a^* = a = a^2$.
- a partial isometry iff a^*a is a projection, not just positive.

Df For projections $e, f \in M$, we write $e \sim f$ iff there is a partial isometry $u \in M$ such that $u^*u = e$ and $uu^* = f$.

2.9.

Df $e \geq f$ iff $e = e_{\mathbb{K}}$ and $f = e_{\mathbb{L}}$ with $\mathbb{K} \supseteq \mathbb{L}$.
We set $e \succeq f$ iff $e \geq f_0 \sim f$ for some projection $f_0 \in M$.

Thm $\succeq$ induces a partial order on $\sim$-classes of projections.
If M is a factor, then $\succeq$ is a total order.

Example $M = \mathcal{B}(\mathbb{C}^n) = \mathrm{Mat}_n(\mathbb{C})$ is a factor.
$e \sim f$ iff $\mathrm{rank}(e) = \mathrm{rank}(f)$, $e \succeq e$ iff $\mathrm{rank}(e) \geq \mathrm{rank}(f)$.

2.10.

Df e is infinite iff $e > f$ and $e \sim f$ can hold simultaneously. Otherwise e is finite.

Nomenclature for factors:

type I	M contains a minimal nonzero projection
type II_1	M is not type I, but every projection is finite
type II_∞	M is not type I or II_1, but contains finite projections
type III	all else

2.11.

Thm Type I's look like $\mathcal{B}(\mathbb{H})$ for separable $\mathbb{H}$.

Thm Type II_∞'s look like $M \otimes \mathcal{B}(\ell^2(\mathbb{N}))$ for M of type II_1.

Study type II_1's via traces.

Df A $*$-preserving linear functional $\tau : M \to \mathbb{C}$ is

- positive iff $\tau(a) \geq 0$ for positive a.
- a state iff it is positive and $\tau(1) = 1$.
- a trace iff it is a state and $\tau(ab) = \tau(ba)$.

2.12.

For traces, get $(\{\text{projections}\}/\sim, \succeq) \xrightarrow{\tilde{\tau}} ([0, 1], \geq)$. [Why?]

Thm An ∞-dim. factor M admits a trace if and only if it is type II_1. In this case, there's a unique trace $\tau : M \to \mathbb{C}$ s.t.

1. τ is norm-continuous.
2. $\tilde{\tau}$ is bijective.

Cor If $M \subseteq \mathcal{B}(\mathbb{H})$ is type II_1, then $eMe \subseteq \mathcal{B}(e\mathbb{H})$ is type II_1 for nonzero projections e.

2.13.

<u>Pf sketch</u> Suppose type II_1. If τ exists, it's determined by $\tilde{\tau}$:

$$\text{(projections)} \xrightarrow{\text{spectral thm}} \text{(self-adjoint ops)} \xrightarrow{\text{Jones Ex. 2.1.7}} \text{(all ops)}$$

Construct $\tilde{\tau}$ dyadically: $\forall\, m\ \exists\, e_1, \ldots, e_{2^m}$ s.t. $\begin{cases} e_i \sim e_j \\ e_i \perp e_j \\ \sum_i e_i = 1 \end{cases}$

If ∞-dim. but not type II_1, then cannot have $\tau(1) = 1$.

2.14.

<u>Example</u> Let Γ be a discrete group with identity e, and let

$$\ell^2(\Gamma) = \left\{\textstyle\sum_{g \in G} c_g g \mid \sum_g |c_g|^2 < \infty\right\}.$$

Two embeddings $\Gamma \xrightarrow{\rho_\ell} \mathcal{B}(\ell^2(\Gamma)) \xleftarrow{\rho_r} \Gamma^{\mathrm{op}}$. Have factors

$$M_\Gamma := \rho_\ell(\mathbb{C}\Gamma)'', \qquad M'_\Gamma = \rho_r(\mathbb{C}\Gamma^{\mathrm{op}})''.$$

The map $\tau(\sum_g a_g g) = a_e$ defines traces on M and M'.
M is type II_1 iff every non-id conjugacy class is infinite.

2.15.

<u>Df</u> A <u>representation</u> of M on a separable Hilbert space $\mathbb{L}$ is a $*$-preserving, id-preserving algebra map $M \to \mathcal{B}(\mathbb{L})$.

<u>Example</u> The <u>standard rep</u> is

$$L^2(M) := \text{Cauchy completion of } M \text{ w.r.t. } |-|_\tau,$$

where $|a|_\tau = \tau(a^*a)$. Special case of the "GNS construction".

2.16.

Let $\Omega \in L^2(M)$ be the image of $1 \in M$.

There is an anti-linear unitary involution $J : L^2(M) \to L^2(M)$ determined by $J(a\Omega) = a^*\Omega$ for all $a \in M$.

Lem $M' = JMJ$. Thus M' is a type II_1 factor with trace τ.

2.17.

Now, two ways to build other reps from $L^2(M)$:

1. If I is countable, then
$$\ell^2(I) \otimes L^2(M) := \{(f_i)_i \in L^2(M)^I \mid \textstyle\sum_i \|f_i\|^2 < \infty\}$$
is "$|I|$ times as large".
2. If $e' \in M'$ is a projection, then $L^2(M)e'$ is "$\tau(e') \in [0, 1]$ times as large".

$\Longrightarrow$ For all $t \in [0, \infty]$, can build a rep "t times as large".

3. April 19

3.1.

Recap:

1. Factors are VNA's with trivial center $\mathbb{C}$.
2. M is a type II_1 factor iff there's a trace $\tau : M \to \mathbb{C}$ such that $\{\tau(e) : \text{projections } e\} = [0, 1]$.
3. For such M, there's an explicit rep "t times larger than $L^2(M)$" for all $t \in [0, \infty]$.

Thm Every separable rep of a type II_1 factor arises from (3).

$\Longrightarrow$ It's a subrep of $\ell^2(\mathbb{N}) \otimes L^2(M)$.

3.2.

Let the trace $\mathrm{Tr} : \ell^2(\mathbb{N}) \to \mathbb{C}$ send every rk-1 projection to 1.

Df The coupling constant of $M \curvearrowright \mathbb{L}$ is

$$\dim_M \mathbb{L} := (\mathrm{Tr} \otimes \tau)(uu^*)$$

for any(!) M-equivariant embedding $u : \mathbb{L} \to \ell^2(\mathbb{N}) \otimes L^2(M)$.

Behaves like the relative dimension of $\mathbb{L}$ over $L^2(M)$.

3.3.

- $\mathbb{L}$ is "$\dim_M \mathbb{L}$ times larger than $L^2(M)$".
- $\dim_M(\bigoplus_i \mathbb{L}_i) = \sum_i \dim_M(\mathbb{L}_i)$.
- If $e \in M$ is a projection, then $\dim_{eMe} e\mathbb{L} = \frac{1}{\tau(e)} \dim_M \mathbb{L}$.
- If $e' \in M'$ is a projection, then $\dim_M \mathbb{L}e' = \tau(e') \dim_M \mathbb{L}$.
- $(\dim_M \mathbb{L})(\dim_{M'} \mathbb{L}) = 1$.

3.4.

M type II_1 factor, $N \subseteq M$ subfactor

Df The index of N in M is

$$[M : N] = \dim_N L^2(M)$$

where $N \curvearrowright L^2(M)$ via M. Note $[M : N] \geq [N : N] = 1$.

Example If $M = \mathrm{Mat}_n(\mathbb{C}) \otimes N$, then $[M : N] = n^2$.

Example If $M = M_{\Gamma_1}$ and $N = M_{\Gamma_2}$ for groups $\Gamma_1 \supseteq \Gamma_2$, then $[M : N] = [\Gamma_1 : \Gamma_2]$.

3.5.

- $\dim_N \mathbb{L} = [M : N] \dim_M \mathbb{L}$.
- If $0 \neq e \in N$ is a proj., then $[eMe : eNe] = [M : N]$.
- If $0 \neq e' \in M'$ is a proj., then $[Me' : Ne'] = [M : N]$.
- If $(e_i)_i$ is a countable family of nonzero proj's in $N' \cap M$ such that $\sum_i e_i = 1$, then

  $$[M : N] = \sum_i \frac{1}{\tau_M(e_i)} [e_i M e_i : N e_i] \geq \sum_i \frac{1}{\tau_M(e_i)}.$$

  Thus $[M : N] < 4 \implies N' \cap M = \mathbb{C}$.

3.6.

Thm (Jones 1983) If $[M : N] < 4$, then

$$[M : N] = 4 \cos^2\left(\frac{\pi}{n+2}\right)$$

for some integer $n \geq 1$.

By contrast, every index ≥ 4 can occur.

Hard to construct $1 < [M : N] < 4$. [Why not $\Gamma_1 \supseteq \Gamma_2$?]

3.7.

Key idea: Build a tower of factors.

Df Let $e_1 \in \mathcal{B}(L^2(M))$ be the $\perp$-projection on $L^2(N)$ and

$$M_1 := (M + Me_1M)'' \subseteq \mathcal{B}(L^2(M)).$$

Lem $M_1 = JN'J$ and $e_1 = Je_1J \in N' \cap JN'J$.

Here, J is the involution on $L^2(M)$ s.t. $M' = JMJ$.

3.8.

<u>Lem</u> If $[M:N]<\infty$, then M_1 is type II_1 and

$$[M_1:M]=\tfrac{1}{\dim_{JN'J}L^2(M)}=\tfrac{1}{\dim_{N'}L^2(M)}=[M:N].$$

<u>Lem</u> If $[M:N]<\infty$, then:

1. $\tau_{M_1}(e_1)=\frac{1}{[M:N]}$.
2. $\tau_{M_1}(ae_1)=\frac{1}{[M:N]}\tau_M(a)$ for general $a\in M$.

3.9.

<u>Pf of (1)</u> We have $\tau_{M_1}(e_1)=\tau_{JN'J}(Je_1J)=\tau_{N'}(e_1)$.

But $\dim_N\mathbb{L}e_1=\tau_{N'}(e_1)\dim_N\mathbb{L}$ implies

$$\tau_{N'}(e_1)\underbrace{\dim_N L^2(M)}_{[M:N]}=\dim_N L^2(M)e_1=\underbrace{\dim_N L^2(N)}_{1}.$$

3.10.

<u>Cor</u> Warm-up to Jones: If $[M:N]>1$, then $[M:N]\geq 2$.

<u>Pf of Cor</u> Set $\lambda=\frac{1}{[M:N]}<1$. Since $e_1\in N'\cap M_1$, get

$$[M_1:N]\geq\tfrac{1}{\tau_{M_1}(e_1)}+\tfrac{1}{\tau_{M_1}(1-e_1)}=\tfrac{1}{\lambda}+\tfrac{1}{1-\lambda}.$$

At the same time,

$$[M_1:N]=[M_1:M][M:N]=[M:N]^2=\tfrac{1}{\lambda^2}.$$

Therefore $\lambda^{-2}\geq\lambda^{-1}+(1-\lambda)^{-1}$. Rearranging, $\lambda^{-1}\geq 2$.

3.11.

Df Set $M_{-1} = N$ and $M_0 = M$. For all $i \geq 0$,

1. Let $e_{i+1} \in \mathcal{B}(L^2(M_i))$ be the $\perp$-projection onto $L^2(M_{i-1})$.
2. Let $M_{i+1} = (M_i + M_i e_{i+1} M_i)'' \subseteq \mathcal{B}(L^2(M_i))$.

This is the Jones tower

$$N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \dots$$

If $[M : N] < \infty$, then all M_i are type II_1 factors.

3.12.

Lem Let $\lambda = \frac{1}{[M:N]}$. Then the e_i satisfy

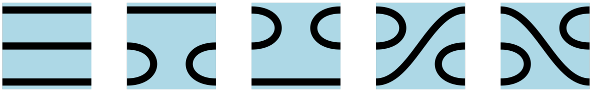
1. $e_i^2 = e_i = e_i^*$.
2. $e_i e_j = e_j e_i$ for $|i - j| \geq 2$.
3. $e_i e_{i\pm 1} e_i = \lambda e_i$.
4. $\tau(a e_{i+1}) = \lambda \tau(a)$ for any word a on $e_1, \dots, e_i$.

Df The Temperley–Lieb algebra is the $\mathbb{Z}[\lambda]$-algebra generated by the e_i modulo relations (1)-(3).

3.13.

Diagrammatics

1, λe_1, λe_2, $\lambda^2 e_2 e_1$, $\lambda^2 e_1 e_2$ respectively given by:



Impose $\mathbb{Z}[\lambda]$-linearity and the relation $\bigcirc = \lambda$.

3.14.

<u>Jones–Wenzl projectors</u>: $f_i = (1-e_1)\cdots(1-e_i)$

<u>Lem (Jones)</u> Set $P_0(q) = P_1(q) = 1$ and

$$P_{i+1}(q) = P_i(q) - qP_{i-1}(q).$$

If $\lambda = \frac{1}{[M:N]}$ and $P_i(\lambda) \neq 0$ for all $i \leq n$, then

$$f_n = f_{n-1} - \frac{P_{n-1}(\lambda)}{P_n(\lambda)} f_{n-1} e_n f_{n-1}.$$

3.15.

<u>Example</u> $P_2 = 1-q$, $P_3 = 1-2q$, $P_4 = 1-3q+q^2$

<u>Cor</u> If $P_i(\lambda) \neq 0$ for all $i \leq n$, then $\tau(f_n) = P_{n+1}(\lambda)$.

<u>Example</u> $f_1 = 1-e_1$ and $f_2 = 1-e_1-e_2+e_1e_2$.

$f_2 = f_1 - \frac{1}{1-\lambda} f_1 e_2 f_1$.

$\tau(f_1e_2f_1) = \lambda\tau(f_1) \implies \tau(f_2) = \frac{1-2\lambda}{1-\lambda}\tau(f_1) = P_3(\lambda)$.

3.16.

<u>Pf of the index thm</u> Check that

$$\frac{1}{4\cos^2(\frac{\pi}{n+2})} < \lambda < \frac{1}{4\cos^2(\frac{\pi}{n+1})} \implies \begin{cases} P_i(\lambda) > 0 \text{ for } i \leq n, \\ P_{n+1}(\lambda) < 0 \end{cases}$$

But if $\lambda = \frac{1}{[M:N]}$, then $P_{n+1}(\lambda) = \tau(f_n) \geq 0$.

4. April 21

4.1.

Last time: finite-index type II_1 subfactor $N \subseteq M$ yields

$$N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq \dots$$

If $e_i \in \mathcal{B}(L^2(M_i))$ is $\perp$ projection onto $L^2(M_{i-1})$, then:

$$\begin{aligned} e_i^2 &= e_i = e_i^*, & \\ e_i e_j &= e_j e_i & \text{for } |i-j| \geq 2 \\ e_i e_{i\pm 1} e_i &= \tfrac{1}{[M:N]} e_i & \end{aligned}$$

Embeds the Temperley–Lieb algebra into $\varinjlim_i \mathcal{B}(L^2(M_i))$.

4.2.

History:

- 1971: statistical mechanics (Temperley–Lieb)
- 1983: von Neumann algebras (Jones)
- 1984-5: knot theory (Jones)
- 1987: algebraic geometry (Jones + Kazhdan–Lusztig)

Today, focus on the last two.

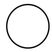
4.3.

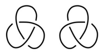
<u>Df</u> A <u>knot</u> is a smooth injective map $S^1 \hookrightarrow \mathbb{R}^3$.

An <u>isotopy</u> of knots u, v is a smooth map $\phi : S^1 \times [0, 1] \to \mathbb{R}^3$ such that:

- $\phi_t = \phi(-, t) : S^1 \to \mathbb{R}^3$ defines a knot for all $t \in [0, 1]$.
- $\phi_0 = u$ and $\phi_1 = v$.

4.4.

unknot 

trefoils 

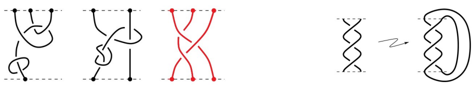
Are the trefoils isotopic to the unknot? To each other?

4.5.

Df A link is a finite disjoint union of knots.

Thm (Alexander 1923) Every link is the closure $\hat{\beta}$ of some braid β, generally not unique.

An n-strand braid connects n inputs at one end of a box to n outputs at the other, without trackbacks.



4.6.

Braids are easier than links, because isotopy classes of n-strand braids form a group.

Br_n is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is

Its relations are: $\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| \geq 2 \\ \sigma_i\sigma_{i\pm1}\sigma_i = \sigma_{i\pm1}\sigma_i\sigma_{i\pm1} & \end{cases}$

[Hatt–de la Harpe: braid relations look like Temperley–Lieb!]

4.7.

Let $|\beta|$ be the length of β w.r.t the $\sigma_i^{\pm 1}$'s.

<u>Thm (Jones 1985)</u> Let $TL_n = \mathbb{Z}[\lambda]\{e_1, \dots, e_n\}/(\text{relations})$.
There exist:

1. Homomorphisms $\mathrm{Br}_n \to TL_n^\times$.
2. $\mathbb{Z}[\lambda]$-linear traces $\tau_n^{TL} : TL_n \to \mathbb{Z}[\lambda]$.
3. An isotopy invariant $\mathbb{V} : \{\text{links}\} \to \mathbb{Z}[q^{\pm\frac{1}{2}}]$ s.t.

$$\mathbb{V}(\hat{\beta}) = (-q^{-1})^{|\beta|} \cdot \tau_n^{TL}(\beta)\Big|_{\lambda = q^{\frac{1}{2}} + q^{-\frac{1}{2}}} \quad \text{for all } \beta \in \mathrm{Br}_n$$

4.8.

<u>Df</u> The <u>Iwahori–Hecke algebra</u> is $H_n = \dfrac{\mathbb{Z}[z][\mathrm{Br}_n]}{\langle \sigma_i^2 - z\sigma_i - 1\rangle_{i=1}^{n-1}}$.

<u>Thm (Ocneanu 1986)</u> There exist:

1. Homomorphisms $\mathrm{Br}_n \to H_n^\times$.
2. $\mathbb{Z}[z]$-linear traces $\tau_n^H : H_n \to \mathbb{Z}[z, a^{\pm 1}]$.
3. An isotopy invariant $\mathbb{P} : \{\text{links}\} \to \mathbb{Z}[z^{\pm 1}, a^{\pm 1}]$ s.t.

$$\mathbb{P}(\hat{\beta}) = (-a)^{|\beta|}\tau_n^H(\beta) \quad \text{for all } \beta \in \mathrm{Br}_n$$

4.9.

$\mathbb{V} = \mathbb{P}|_{z = q^{\frac{1}{2}} - q^{-\frac{1}{2}},\, a = q^{-1}}$ through

$$\begin{tikzcd}
\mathrm{Br}_n \arrow[r] & H_n \arrow[r, "\tau_n^H"] \arrow[d] & \mathbb{Z}[z, a^{\pm 1}] \arrow[d] \\
& \mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes TL_n \arrow[r, "\tau_n^{TL}"] & \mathbb{Z}[q^{\pm\frac{1}{2}}]
\end{tikzcd}$$

4.10.

The explicit Jones–Ocneanu trace on H_n:

1. $\tau_n^H(1) = 1$.
2. $\tau_n^H(\beta) = \tau_{n-1}^H(\beta)$ for $\beta \in \mathrm{Br}_{n-1}$.
3. $\tau_n^H(\sigma_{n-1}^{\pm 1}\beta) = -\dfrac{a^{\mp 1}z}{a - a^{-1}}\,\tau_{n-1}^H(\beta)$ for $\beta \in \mathrm{Br}_{n-1}$.

<u>Example</u> $\mathbb{P}(\bigcirc) = 1, \quad \mathbb{P}(\text{trefoil}_+) = (2+z)a^2 - a^4$.

4.11.

<u>Mystery</u> The ring $\mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes H_n$ and the map

$$\mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes H_n \to \mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes TL_n$$

can be described via the geometry of flag varieties.

Over any field k, the <u>flag variety</u> X_n is the algebraic variety whose E-points are flags

$$0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = E^n$$

for any field extension $E \supseteq k$.

4.12.

<u>Lem</u> There is a well-defined map $X_n \times X_n \to S_n$ s.t.

$$(\vec{V}, \vec{W}) \mapsto x \quad \iff \quad \begin{array}{l} \exists \text{ set of lines } L_1, \ldots, L_n \text{ s.t.} \\ V_i = L_1 + \cdots + L_i, \\ W_i = L_{x^{-1}(1)} + \cdots L_{x^{-1}(i)} \end{array}$$

Fibers are precisely the diagonal GL_n-orbits on $X_n \times X_n$.

<u>Example</u> $X_2 = \mathbb{P}^1_k$. Orbits are diagonal and complement.

4.13.

Henceforth k is a finite field. For each $x \in S_n$, let

$$T_x \curvearrowright \mathbb{C}\langle X_n(k)\rangle \quad \text{by} \quad T_x \cdot \mathbf{1}_{\vec{V}} = \sum_{(\vec{V},\vec{W})\mapsto x} \mathbf{1}_{\vec{W}}.$$

<u>Thm (Iwahori 1964)</u> Set $q = |k|$. Then

$\mathrm{End}_{\mathrm{GL}_n}(\mathbb{C}\langle X_n(k)\rangle) \xrightarrow{\sim} \mathbb{C} \otimes H_n$ via $T_x \mapsto q^{\frac{1}{2}}\sigma_x$.

4.14.

Let $\mathsf{D} = \mathsf{D}^b_{\mathrm{GL}_n}(X_n \times X_n)$ be the constructible derived category.

Each GL_n-orbit $O_x \hookrightarrow X_n \times X_n$ defines an IC sheaf $\mathrm{IC}_x \in \mathsf{D}$. Via the function-sheaf dictionary, IC_x defines an element

$$C'_x \in \mathbb{C}\langle X_n(k) \times X_n(k)\rangle^{\mathrm{GL}_n} = \mathrm{End}_{\mathrm{GL}_n}(X_n(k)) = \mathbb{C} \otimes H_n.$$

<u>Thm (Kazhdan–Lusztig '79)</u> $\{C'_x\}_x$ = basis of $\mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes H_n$.

The change-of-basis $T_x \mapsto C'_x$ has coefficients in $\mathbb{Z}_{\geq 0}[q^{\pm\frac{1}{2}}]$.

4.15.

<u>Example</u> $X_2 = \mathbb{P}^1$. $X_2 \times X_2 = \underbrace{O_1}_{\text{diagonal}} \sqcup O_s$.

The Hecke algebra is

$$\begin{aligned}\mathbb{C} \otimes H_2 &= \mathbb{C}[\sigma]/\langle \sigma^2 - (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\sigma - 1\rangle \\ &= \mathbb{C}[T]/\langle T^2 - (q-1)T - q\rangle.\end{aligned}$$

$C'_1 = 1$ and $C'_s = \sigma + q^{-\frac{1}{2}} = q^{-\frac{1}{2}}(T+1)$.

4.16.

Thm (Jones 1987) Form the two-sided ideal

$$I = \langle C'_{s_i s_{i\pm 1} s_i} \rangle_i \subseteq \mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes H_n \quad \text{where } s_i = (i, i+1).$$

1. $\Theta : \mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes H_n \to \mathbb{Z}[q^{\pm\frac{1}{2}}] \otimes TL_n$ is quotienting by I.
2. $\Theta(C'_{s_i}) = (q^{\frac{1}{2}} + q^{-\frac{1}{2}})e_i$.

Thm (Fan–Green 1997) I is spanned by the C'_x such that x is 321-avoiding. [No $i < j < k$ with $x(i) > x(j) > x(k)$]

Why does H_n simultaneously relate to operator algebras, braids/knots/links, and algebraic geometry?