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# *A purity result for fixed point varieties in flag manifolds*

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## Introduction

Let $G$ be a connected reductive linear algebraic group and $X$ the corresponding flag variety, which we can view as the variety of Borel subgroups of $G$. If $x$ is an element of the Lie algebra of $G$ we denote by $X_x$ the subvariety of $X$ of the Borel subgroups whose Lie algebra contains $x$. Then $X_x$ is a projective variety, which in general is not smooth. Assume everything to be defined over a finite field $\boldsymbol{F}_q$.

The main result of this note is that—in spite of the non-smoothness—the $X_x$ "satisfy the Weil conjectures", i.e. that the eigenvalues of the Frobenius on the $l$-adic cohomology groups $H^i(X_x, \boldsymbol{Q}_l)$ have absolute values $q^{i/2}$ (under some mild restrictions on the characteristic).

It was shown by Slodowy [17] that over $\boldsymbol{C}$, with $x$ nilpotent, the complex space $X_x$ has the homotopy type of a smooth affine variety. The proof of the main result consists in making a cohomological adaptation of Slodowy's result which is suitable for use in $l$-adic cohomology. We use, as in [17], a transversal slice to the $G$-orbit of $x$. Its existence, in the situation discussed here, is established by Spaltenstein in [20].

We also give some applications. Another application is made by Beynon and Spaltenstein in [1, No. 5].

I am grateful to T. Shoji for pointing out an improvement in the proof of Theorem 1 and to N. Spaltenstein for making available the result about transversal slices of [20].

**1.** We denote by $G$ a connected reductive linear algebraic group over the algebraically closed field $k$ of characteristic $p$. We fix a Borel subgroup $B$ of $G$ and a maximal torus $T$ contained in it. Let $N$ be the normalizer of $T$ in $G$ and $W=N/T$ the Weyl group of $(G, T)$. We denote by $X=G/B$ the flag variety. If necessary we write $X=X^G$.

Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$, be the Lie algebras of $G, B, T$, respectively, and denote by Ad the adjoint representation of $G$ in $\mathfrak{g}$.

The quotient map $G \rightarrow X$ is a principal fibre bundle. Let $\tilde{\mathfrak{g}}$ be the associated fibre bundle $G\times^B\mathfrak{b}\to X$, i.e. the quotient of $G\times\mathfrak{b}$ by the $B$-action such that $b(g,x)=(gb^{-1}, \mathrm{Ad}(b)x)$, if $g\in G$, $x\in\mathfrak{b}$ (see [16, p. 25]). Denote by $g*x$ the image of $(g,x)\in G\times B$ in $\tilde{\mathfrak{g}}$ and let $\varphi:\tilde{\mathfrak{g}}\to\mathfrak{g}$ be the surjective morphism with $\varphi(g*x)=\mathrm{Ad}(g)x$.

The Weyl group $W$ acts on $\mathfrak{t}$, let $\psi:\mathfrak{t}\to\mathfrak{t}/W$ be the quotient morphism. If $G$ is adjoint the coordinate algebra of $\mathfrak{t}/W$ is isomorphic to the subalgebra of $\mathrm{Ad}(G)$-invariant polynomial functions on $\mathfrak{g}$ (see [3, p. 199–200]), from which one infers that we have in this case a morphism $\chi:\mathfrak{g}\to\mathfrak{t}/W$. The map defined by $\chi$ can be described as follows: Let $x\in\mathfrak{g}$ have the Jordan decomposition $x=x_s+x_n$. The intersection $\mathrm{Ad}(G)x_s\cap\mathfrak{t}$ is a $W$-orbit and the corresponding point of $\mathfrak{t}/W$ is $\chi(x)$.

We also introduce the morphism $\theta:\tilde{\mathfrak{g}}\to\mathfrak{t}$ which maps $g*x$ onto the canonical image of $x\in\mathfrak{b}$ in $\mathfrak{t}$ (this is well-defined). We then have a commutative diagram of morphisms

$$\begin{array}{ccc} \tilde{\mathfrak{g}} & \xrightarrow{\varphi} & \mathfrak{g} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\chi} \\ \mathfrak{t} & \xrightarrow{\psi} & \mathfrak{t}/W \end{array} \tag{1}$$

If $G$ is semi-simple and the characteristic $p$ of $k$ is very good (see [16, p. 38]) the diagram (1) is a "simultaneous resolution" of the morphism $\chi$, see [loc. cit., p. 58–59]. We notice, in particular, that $\varphi$ is proper and $\theta$ is smooth. Moreover, if $p$ is very good, the morphism $\chi$ is flat [loc. cit., p. 65].

If $x\in\mathfrak{g}$, the fiber $\varphi^{-1}(x)$ is isomorphic to the subvariety $X_x$ of $X$ defined by

$$X_x=\{gB\in X\,|\,\mathrm{Ad}(g)^{-1}x\in\mathfrak{b}\},$$

which we can view as the variety of the Borel subalgebras of $\mathfrak{g}$ which contain $x$. It is a projective variety and it is known that all its components have the same dimension [18, p. 47].

Now let $k$ be the algebraic closure of the finite field $\boldsymbol{F}_q$ and suppose that $G, B$ and $T$ are defined over $\boldsymbol{F}_q$. If $x$ is rational over $\boldsymbol{F}_q$ then $X_x$ is also defined over $\boldsymbol{F}_q$. Let $l$ be a prime number different from $p$. The $l$-adic cohomology groups $H^i(X_x, \boldsymbol{Q}_l)$ are finite dimensional vector spaces over the field $\boldsymbol{Q}_l$, with a Frobenius endomorphism denoted by $F$. The following theorem is the main result of this paper.

THEOREM 1. *If the characteristic $p$ is good then $H^i(X_x, \boldsymbol{Q}_l)$ is pure of weight $i$.*

This means that the eigenvalues of $F$ on $H^i(X_x, \boldsymbol{Q}_l)$ are algebraic numbers, all whose complex conjugates have absolute value $q^{i/2}$ (see [7, p. 153]). For the definition of good characteristic see [16, p. 38].

The proof of the theorem can be reduced quickly to the case that $x$ is nilpotent. In that case, Slodowy has proved [17, p. 50] in the case that $k=\boldsymbol{C}$, that the space $X_x$ has the same homotopy as a smooth affine algebraic variety. Some consequences of this fact, on the level of cohomology, can also be proved in $l$-adic cohomology, and the theorem will be a consequence of these, using Deligne's results proved in [7].

REMARK. Theorem 1 has a conjectural counterpart over $\boldsymbol{C}$. We assume now $x$ to be nilpotent and $k=\boldsymbol{C}$. According to Deligne [6] there is a mixed Hodge structure on the cohomology groups $H^i(X_x, \boldsymbol{Z})$. According to a well-known translation principle the counterpart would be: this mixed Hodge structure is pure of weight $i$.

This is only of interest for $i$ even since $H^i(X_x, \boldsymbol{C})=0$ if $i$ is odd, as a consequence of Corollary 1 to Theorem 2.

**2.** The reduction to the case of a nilpotent element $x$ proceeds as follows. Let $x=x_s+x_n$ be the Jordan decomposition of $x$ in $\mathfrak{g}$ and denote by $H=Z_G(x_s)^0$ the connected centralizer of $x_s$ in $G$. This is a reductive linear algebraic group. We assume that $x_s \in \mathfrak{t}$. We then have (with obvious notations):

LEMMA 1. *We have $X_x^G \simeq \coprod X_x^{H_i}$, where $H_i$ runs through the finite set of conjugates of $H$ whose Lie algebras contain $x$.*

See [21, p. 182] (the result is true in any characteristic). If $x$ is rational over $\boldsymbol{F}_q$, then by extending the ground field (which is harmless) we may, in proving Theorem 1, assume that all $H_i$ are defined on $\boldsymbol{F}_q$. If $H_i'$ is the derived group of $H_i$ (which is semi-simple), we have $X_x^{H_i} \simeq X_{x_n}^{H_i'}$. It is now clear that it suffices to prove Theorem 1 for $x$ nilpotent.

Also, for $x=0$ the theorem is easy to prove (using the Bruhat decomposition of $X$). So from now on we assume $x \in \mathfrak{g}$ to be nilpotent and nonzero. We perform a further reduction. First, it is easy to see that if $G \to \bar{G}$ is either a central isogeny of semi-simple groups or a quotient map for a quotient $\bar{G}=G/A$ by a central torus $A$, the result is true for $G$ if and only if it is true for $\bar{G}$. This implies that we can make the following assumption in proving Theorem 1:

(*) *either $G=GL_n$ or $G$ is simple and $p$ is good for $G$.*

It then follows from the results in [16, p. 58] that the diagram (1) is a simultaneous resolution (the statement in [loc. cit.] is slightly different, but the proof in our situation goes along the same lines).

We can now invoke the result of Spaltenstein, proved in the paper [20] following this one. It is shown there that there exists a multiplicative one parameter subgroup $\lambda$ of $G$ and a linear subspace $\Sigma$ of $\mathfrak{g}$ such that

(a) $\mathrm{Ad}(\lambda(t))x=t^{-a}x$, with $a>0$,
(b) $\mathrm{Ad}(\lambda(t))$ stabilizes $\Sigma$ and the weights $\xi$ in $\Sigma$ are of the form $\xi(t)=t^b$ with $b\geqq 0$,
(c) $\dim\Sigma$ equals the dimension of the centralizer $Z_G(x)$.

Then $S=x+\Sigma$ is a transversal slice to the $\mathrm{Ad}(G)$-orbit of $x$ at $x$ in the sense of [16, p. 60]. In fact, the arguments of [loc. cit., 7.4] can be applied to our one parameter group $\lambda$, and give the required properties of $S$. In particular, the morphism $G\times X\to\mathfrak{g}$ with $(g,s)\mapsto\mathrm{Ad}(g)s$ is smooth. One can now "localize" the diagram (1) at $x$. Put $\tilde{S}=\varphi^{-1}(S)$. We then have a commutative diagram of morphisms

$$(2)\qquad \begin{array}{ccc} \tilde{S} & \xrightarrow{\varphi} & S \\ {\scriptstyle\theta}\downarrow & \psi & \downarrow{\scriptstyle\chi} \\ \mathfrak{t} & \xrightarrow{\hphantom{\varphi}} & \mathfrak{t}/W\,. \end{array}$$

We have denoted restrictions of the morphisms $\varphi,\theta,\chi$, of (1) by the same symbols. It is proved in [16, p. 64] that in (2) the morphism $\chi$ is again surjective (and flat) and that (2) is a simultaneous resolution of $\chi$. In particular, $\theta$ is smooth.

Now define an action $\mu$ of $\boldsymbol{G}_m$ on $S$ by

$$\mu(t,s)=t^a\,\mathrm{Ad}(\lambda(t)s)\,.$$

Then $\mu$ is a contraction of $S$ to $x$, i.e. $\mu$ can be extended to a morphism $\mu:\boldsymbol{A}^1\times S\to S$ with $\mu(0,s)=x$. Define an action $\tilde{\mu}$ of $\boldsymbol{G}_m$ on $\tilde{S}$ by

$$\tilde{\mu}(t,g*y)=(\lambda(t)g*t^a y)\,,$$

if $t\in\boldsymbol{G}_m$, $g\in G$, $y\in\mathfrak{b}$. Then $\varphi$ is a $\boldsymbol{G}_m$-equivariant morphism $\tilde{S}\to S$. We have $\theta(\tilde{\mu}(t,g*y))=t^a\theta(g*y)$. So $\theta$ is also $\boldsymbol{G}_m$-equivariant, with respect to the action $(t,\xi)\mapsto t^a\xi$ of $\boldsymbol{G}_m$ on $\mathfrak{t}$. Henceforth we write $\mu(t,s)=t\cdot s$, similarly for $\tilde{\mu}$. Clearly the $\boldsymbol{G}_m$-action on $\mathfrak{t}$ is a contraction of $\mathfrak{t}$ to 0.

**3.** For the proof of Theorem 1 we need a homotopy result in $l$-adic cohomology. Similar results (for constant sheaves) can be found in the literature ([14, exp. XV, 2.1], [8, p. 136], [10, p. 181]).

Let $V$ be an algebraic variety over $k$, with a $\boldsymbol{G}_m$-action $\mu:\boldsymbol{G}_m\times V\to V$ which contracts $V$ to $a$. We also denote by $\mu$ the extension $\boldsymbol{A}^1\times V\to V$. Let $l$ be a prime different from $p$ and assume that $S$ is a constructible $l$-adic sheaf on $V$ which is a $\boldsymbol{G}_m$-sheaf, for the action $\mu$.

PROPOSITION 1. *We have* $H^i(V,S)=0$ *if* $i>0$ *and the canonical map* $H^0(V,S)\to S_a$ *is bijective.*

Let $j:\{a\}\to V$ be the injection morphism and put $S'=\mathrm{Ker}(S\to j_*j^*S)$. It is easy to see that it suffices to prove the assertion for $S'$. In other words, we may assume that the stalk $S_a$ is 0. It also suffices to prove the corresponding statement in étale cohomology, $S$ being a constructible étale sheaf of abelian $l$-groups.

Let $\pi_1:\boldsymbol{A}^1\times V\to\boldsymbol{A}^1$, $\pi_2:\boldsymbol{A}^1\times V\to V$ be the projections. Define the morphism $\tau:\boldsymbol{A}^1\times V\to\boldsymbol{A}^1\times V$ by $\tau(t,v)=(t,\mu(t,v))$. Then $\pi_1\circ\tau=\pi_1$, so $\tau$ is a morphism over $\boldsymbol{A}^1$.

The restrictions to $\boldsymbol{G}_m\times V$ of $\pi_2^*S$ and $\tau^*\pi_2^*S$ are isomorphic, since $S$ is a $\boldsymbol{G}_m$-sheaf. On the other hand, the restriction of $\tau^*\pi_2^*S$ to $\{0\}\times V$ is 0, since the stalk $S_a$ is 0. Let $i:\boldsymbol{G}_m\times V\to\boldsymbol{A}^1\times V$ be the inclusion. It follows that $\tau^*\pi_2^*S$ is isomorphic to $i_!i^*\pi_2^*S$, whence a homomorphism $\tau^*\pi_2^*S\to\pi_2^*S$ and a homomorphism of sheaves

$$\alpha: R^{\cdot}\pi_{1,*}\tau^*\pi^*S \longrightarrow R^{\cdot}\pi_{1,*}\pi_2^*S\,.$$

Now recall that if we have a commutative triangle of morphisms of varieties

$$\begin{tikzcd} V \arrow[rr, "f"] \arrow[dr, "a"'] & & W \arrow[dl, "b"] \\ & D & \end{tikzcd}$$

and an étale abelian sheaf $T$ on $W$ there is a canonical homomorphism of sheaves

$$R^{\cdot}b_*T \longrightarrow R^{\cdot}a_*(f^*T)\,,$$

coming from an edge homomorphism $R^{\cdot}b_*(f_*f^*T)\to R^{\cdot}a_*(f^*T)$ and the canonical homomorphism $T\to f_*f^*T$. Applying this with $f=\tau$, $a=b=\pi_1$, $T=\pi_2^*S$, we obtain a homomorphism

$$\beta: R^{\cdot}\pi_{1,*}\pi_2^*S \longrightarrow R^{\cdot}\pi_{1,*}\tau^*\pi_2^*S\,,$$

whence a homomorphism

$$\alpha\circ\beta: R^{\cdot}\pi_{1,*}\pi_2^*S \longrightarrow R^{\cdot}\pi_{1,*}\pi_2^*S\,.$$

Now $R^{\cdot}\pi_{1,*}\pi_2^*S$ is a constant sheaf on $\boldsymbol{A}^1$ (by smooth base change). In the stalk at the point 1, the homomorphism induced by $\alpha\circ\beta$ is bijective, and in the stalk at 0 it is 0. Since $\boldsymbol{A}^1$ is connected the sheaf $R^{\cdot}\pi_{1,*}\pi_2^*S$ must be 0, which means that $H^i(V,S)=0$ for all $i$. The argument of this proof is similar to the one given in [8, p. 136].

COROLLARY 1. *Let $K$ be a complex in the derived category $D_c^b(V, \boldsymbol{Q}_l)$, whose cohomology sheaf $H^{\cdot}(K)$ is a $\boldsymbol{G}_m$-sheaf. Then the hypercohomology $\mathcal{H}^{\cdot}(K)$ is isomorphic to the stalk $H^{\cdot}(K)_a$.*

Using a spectral sequence for hypercohomology this follows from the proposition. See [7, p. 146–150] for $D_c^b(V, \boldsymbol{Q}_l)$.

COROLLARY 2. *In the situation of the proposition there is an isomorphism $H^{\cdot}_{\{a\}}(V,S)\rightarrow H^{\cdot}_c(V,S)$.*

Let $j:\{a\}\rightarrow V$. The preceding corollary can be formulated as follows: the map $K\rightarrow j_*j^*K$ induces an isomorphism in hypercohomology. By Verdier duality this implies: the map $j^!j_*K\rightarrow K$ induces an isomorphism in hypercohomology with proper support, whence the corollary.

**4.** We shall apply Proposition 1 in the situation of the diagram (2). We use the notations of that diagram. Observe that $X_x=\varphi^{-1}(x)\subset\theta^{-1}(0)$.

LEMMA 2. (i) *The canonical maps $H^{\cdot}(\tilde{S}, \boldsymbol{Q}_l)\rightarrow H^{\cdot}(\theta^{-1}(0), \boldsymbol{Q}_l)\rightarrow H^{\cdot}(X_x, \boldsymbol{Q}_l)$ are bijective;*

(ii) *The canonical map $R^{\cdot}\theta_*(\boldsymbol{Q}_l)_0\rightarrow H^{\cdot}(\theta^{-1}(0), \boldsymbol{Q}_l)$ is bijective.*

Consider the Leray spectral sequence for $\varphi$:

$$E_2^{i,j}=H^i(S, R^j\varphi_*(\boldsymbol{Q}_l))\Longrightarrow H^{i+j}(\tilde{S}, \boldsymbol{Q}_l).$$

Since $\varphi$ is $\boldsymbol{G}_m$-equivariant for the actions $\mu$ and $\tilde{\mu}$ introduced in no. 2, the higher direct image sheaves $R^j\varphi_*(\boldsymbol{Q}_l)$ are $\boldsymbol{G}_m$-sheaves on $S$. Since $\mu$ is a contraction of $S$ to $x$, an application of Proposition 1 shows that $E_2^{i,j}=0$ if $i\neq 0$ and that

$$E_2^{0,j}\simeq R^j\varphi_*(\boldsymbol{Q}_l)_x.$$

Since $\varphi$ is proper, this is isomorphic to $H^j(X_x, \boldsymbol{Q}_l)$. These facts show that the spectral sequence collapses, and that we have an isomorphism

$$H^{\cdot}(S, \boldsymbol{Q}_l)\xrightarrow{\sim}H^{\cdot}(X_x, \boldsymbol{Q}_l).$$

There is a morphism of the Leray spectral sequence for $\varphi$ to the (trivial) spectral sequence for the morphism $\varphi^{-1}(x)\rightarrow\{x\}$. Using this morphism one readily obtains the connection with the canonical map

$$H^{\cdot}(\tilde{S}, \boldsymbol{Q}_l) \longrightarrow H^{\cdot}(X_x, \boldsymbol{Q}_l) .$$

Put $S_0=\varphi(\theta^{-1}(0))$, this is the set of nilpotent elements of $S$. Then $\varphi$ induces a morphism $\theta^{-1}(0)\to S_0$, which is $\boldsymbol{G}_m$-equivariant. Using the same argument as before, one proves that the canonical map

$$H^{\cdot}(\theta^{-1}(0), \boldsymbol{Q}_l) \longrightarrow H^{\cdot}(X_x, \boldsymbol{Q}_l)$$

is bijective[1]. This implies (i).

To prove (ii), we again apply the same argument to $\theta$ (which is $\boldsymbol{G}_m$-equivariant, see no. 2). We conclude that the canonical map

$$H^{\cdot}(\tilde{S}, \boldsymbol{Q}_l) \longrightarrow R^{\cdot}\theta(\boldsymbol{Q}_l)_0 \tag{3}$$

is bijective. Now the map

$$H^{\cdot}(\tilde{S}, \boldsymbol{Q}_l) \longrightarrow H^{\cdot}(\theta^{-1}(0), \boldsymbol{Q}_l) ,$$

which was proved to be bijective in (i), can be factored:

$$H^{\cdot}(\tilde{S}, \boldsymbol{Q}_l) \longrightarrow R^{\cdot}\theta_*(\boldsymbol{Q}_l)_0 \longrightarrow H^{\cdot}(\theta^{-1}(0), \boldsymbol{Q}_l) ,$$

the first map being the bijection (3), and the second one being the one of (ii). It follows that this map is also bijective, proving (ii).

We can now prove Theorem 1. As we saw above, we can assume that $x$ is nilpotent and that $G$ satisfies $(*)$. We then use the diagram (2). Since $X_x$ is projective, we know from Deligne's results [7, p. 206] that $H^i(X_x, \boldsymbol{Q}_l)$ has weight $\leqq i$. Similarly, $H_c^i(\theta^{-1}(0), \boldsymbol{Q}_l)$ has weight $\leqq i$. But $\theta^{-1}(0)$ is smooth, and Poincaré duality now shows that $H^i(\theta^{-1}(0), \boldsymbol{Q}_l)$ has weight $\geqq i$. Lemma 2(i) gives an isomorphism

$$H^i(\theta^{-1}(0), \boldsymbol{Q}_l) \xrightarrow{\sim} H^i(X_x, \boldsymbol{Q}_l) ,$$

which is compatible with the action of Frobenius. Hence $H^i(X_x, \boldsymbol{Q}_l)$ has also weight $\geqq i$, and the theorem follows.

Slodowy has proved over $\boldsymbol{C}$ that the projective variety $X_x$ is homotopy equivalent to the generic fiber of $\chi$, which is a smooth affine variety (see [17], p. 50). We can also establish the cohomological reflection of this fact in $l$-adic cohomology. This is the following result.

PROPOSITION 2. *Let $\xi \in \mathfrak{g}(\boldsymbol{F}_q)$. There is an isomorphism $H^{\cdot}(\theta^{-1}(\xi), \boldsymbol{Q}_l) \simeq H^{\cdot}(\theta^{-1}(0), \boldsymbol{Q}_l)$ which is compatible with the action of Frobenius.*

We may assume that $\xi \neq 0$. Let $l_\xi = k\xi$ be the 1-dimensional subspace of $\mathfrak{t}$ spanned by $\xi$ and put $\tilde{S}_\xi=\theta^{-1}(l_\xi)$, $S_\xi=\varphi(\tilde{S}_\xi)=\chi^{-1}(\pi\xi)$. Then $\tilde{S}_\xi$ and $S_\xi$

1) This argument was suggested by T. Shoji.

are subvarieties of $\tilde{S}$ and $S$, which are $\boldsymbol{G}_m$-stable and $\theta^{-1}(0)\subset\tilde{S}_\xi$, $x\in S_\xi$. Let $\theta_\xi:\tilde{S}_\xi\to l_\xi$ be the morphism induced by $\theta$. It is $\boldsymbol{G}_m$-equivariant. We shall prove that the sheaf $R^{\cdot}\theta_{\xi,*}(\boldsymbol{Q}_l)$ is locally constant, from which the proposition follows, by using the correspondence between locally constant sheaves and representations of the fundamental group.

To prove this we may work in étale cohomology and replace the coefficient field $\boldsymbol{Q}_l$ by a finite group $A=\boldsymbol{Z}/l^n\boldsymbol{Z}$. Now the morphism of $\boldsymbol{G}_m\times\theta^{-1}(\xi)$ to $\theta^{-1}(l_\xi-\{0\})$ sending $(t, x)$ to $t\cdot x$ is an étale covering. It follows that the restriction of $R^{\cdot}\theta_{\xi,*}(A)$ to $l_\xi-\{0\}$ is locally constant. It remains to be proved is that this sheaf is locally constant in a neighbourhood of 0. The argument used several times before shows that the canonical map

$$R^{\cdot}\theta_{\xi,*}(A)_0\longrightarrow H^{\cdot}(\theta^{-1}(0), A)$$

is bijective. Let $D$ be the strict localization of $l_\xi$ at the geometric point 0. One knows that

$$R^{\cdot}\theta_{\xi,*}(A)_0\simeq H^{\cdot}(\tilde{S}_\xi\times_{l_\xi}D, A)\,.$$

Let the geometric point $t$ of $l_\xi$ be a generisation of 0 [12, exp. VIII, p. 29]. As $D$ is the spectrum of a discrete valuation ring, there is essentially one such $t\neq 0$. Let $\tilde{S}_{\xi,t}$ be the corresponding fiber of $\theta_\xi$. The smoothness of $\theta$ implies that

$$H^{\cdot}(\tilde{S}_{\xi,t}, A)\simeq H^{\cdot}(S_\xi\times_{l_\xi}D, A)$$

(see [13, p. 58, th. 2.1 and p. 56, variante]). The local constancy of $R^{\cdot}\theta_{\xi,*}(A)$ on $l_\xi-\{0\}$ implies that

$$H^{\cdot}(\tilde{S}_{\xi,t}, A)\simeq R^{\cdot}\theta_{\xi,*}(A)_t\,.$$

We conclude from the preceding facts that the specialization map [12, exp. VIII, p. 31]

$$R^{\cdot}\theta_{\xi,*}(A)_0\longrightarrow R\,\theta_{\xi,*}(A)_t$$

is bijective. Hence [12, exp. IX, p. 21] $R^{\cdot}\theta_{\xi,*}(A)$ is locally constant around 0, which is what we wanted to prove.

REMARK. The isomorphism of the proposition holds over any algebraically closed field $k$, whose characteristic is 0 or a good prime, as the proof shows.

Next let $\mathfrak{t}_{\text{reg}}$ be the set of regular elements of $\mathfrak{t}$. This is a $W$-stable affine open subset of $\mathfrak{t}$ which is defined over $\boldsymbol{F}_q$. We put $\tilde{S}_{\text{reg}}=\theta^{-1}(\mathfrak{t}_{\text{reg}})$, $S_{\text{reg}}=\varphi(\tilde{S}_{\text{reg}})=\chi^{-1}(\mathfrak{t}_{\text{reg}}(W))$.

LEMMA 3. *The diagram*

$$\begin{array}{ccc} \tilde{S}_{\mathrm{reg}} & \xrightarrow{\varphi} & S_{\mathrm{reg}} \\ {\scriptstyle\theta}\downarrow & \psi & \downarrow{\scriptstyle\chi} \\ \mathfrak{t}_{\mathrm{reg}} & \longrightarrow & \mathfrak{t}_{\mathrm{reg}}/W \end{array}$$

*is cartesian.*

Let $\mathfrak{g}_{\mathrm{reg}}$ be the set of regular semi-simple elements of $\mathfrak{g}$. It follows readily that the map $(gT, x)\mapsto g*x$ (notation of no. 1) defines an isomorphism of $G/T\times\mathfrak{t}_{\mathrm{reg}}$ onto $\tilde{\mathfrak{g}}_{\mathrm{reg}}=\varphi^{-1}(\mathfrak{g}_{\mathrm{reg}})$. This means that Lemma 3 holds with $S$ replaced by $\mathfrak{g}$ (and $\tilde{S}$ by $\tilde{\mathfrak{g}}$). The assertion of the lemma follows easily. A closely related result is already in [17, p. 48].

PROPOSITION 3. *Let $\xi\in\mathfrak{t}_{\mathrm{reg}}(\boldsymbol{F}_q)$. There is an isomorphism $\alpha_\xi$: $H^\cdot(\chi^{-1}\psi(\xi), \boldsymbol{Q}_l)\simeq H^\cdot(X_x, \boldsymbol{Q}_l)$ which is compatible with the action of Frobenius.*

By Lemma 3 we have that $\theta^{-1}(\xi)$ is isomorphic to $\chi^{-1}\pi(\xi)$. The proposition now follows from Lemma 2 (i) and Proposition 2.

## 5. Applications to Green functions

We keep the same notations. The Weyl group $W$ acts on $H^\cdot(X_x, \boldsymbol{Q}_l)$, via the following composition of isomorphisms

$$H^\cdot(X_x, \boldsymbol{Q}_l) \xrightarrow{\alpha_\xi^{-1}} H^\cdot(\chi^{-1}\psi(\xi), \boldsymbol{Q}_l) \longrightarrow H^\cdot(\chi^{-1}\psi(w\xi), \boldsymbol{Q}_l) \xrightarrow{\alpha_{w\xi}} H^\cdot(X_x, \boldsymbol{Q}_l)\,.$$

This action can also be viewed as coming from a monodromy action of the fundamental group $\pi_1(\mathfrak{t}_{\mathrm{reg}}/W, \psi(\xi))$ on the stalk $H^\cdot(\theta^{-1}(\xi), \boldsymbol{Q}_l)$ of $R^\cdot\theta_*(\boldsymbol{Q}_l)$ (see [9, p. 868] where a similar situation occurs). It also follows from the results of Hotta in [loc. cit.] that the representation of $W$ on $H^\cdot(X_x, \boldsymbol{Q}_l)$ concides with the representation defined in [21, no. 4], tensored with the sign representation.

Put

$$Q_w(x)=\sum_{i\geq 0}(-1)^i\,\mathrm{Tr}(Fw, H^i(X_x, \boldsymbol{Q}_l))\,.$$

It follows from the preceding remarks, using [21, 5.10] that this is a Green function in the sense of [21], provided $q$ is sufficiently large. In fact, let $\eta\in\mathfrak{t}_{\mathrm{reg}}$ be such that $F\eta=w\cdot\eta$. Take $n\in N$ representing $w$ and let $g\in G$ be such that $n=(Fg)^{-1}g$. Then $F(\mathrm{Ad}(g)\eta)=\mathrm{Ad}(g)\eta$ and the connected centralizer of $\mathrm{Ad}(g)\eta$ is an $F$-stable maximal torus $T_1$ of $G$ with $Q_{T_1,G}(x)=Q_w(x)$ (notations of [loc. cit.]).

The following theorem has been established recently in all generality.

THEOREM 2. *Assume that $p$ is good. There is a polynomial $\tilde{Q}_w(x)\in$*

*$\boldsymbol{Z}(T)$, depending only on the isogeny class of $G$ over $\boldsymbol{F}_q$ such that*

$$Q_w(x)=\tilde{Q}_w(x)(q),$$

*if $q$ is sufficiently large.*

The proof is via a case by case check of the simple groups. The case of classical groups was already known (see [21, p. 201], [22] and [19] for the twisted $D_4$). The case of $G_2$ is dealt with in [21, p. 205-206] and $F_4$ is studied in [15]. Finally the most complicated cases $E_6, E_7, E_8$ were dealt with recently by Beynon and Spaltenstein [1, 2], using a computer. In fact, they have computed explicitly the $\tilde{Q}_w(x)$. The following is a consequence of Theorems 1 and 2. For the proof see [1, no. 5].

COROLLARY 1. *$H^i(X_x, \boldsymbol{Q}_l)=0$ if $i$ is odd.*

It would be desirable to have a geometric explanation of this result.

The next lemma also follows from the explicit results of [1, 2].

LEMMA 4. *The nonzero eigenvalues of $F$ in $H^{2i}(X_x, \boldsymbol{Q}_l)$ are of the form $\varepsilon q^i$ where $\varepsilon$ is a root of unity. If $\varepsilon q^i$ occurs as an eigenvalue so does $\varepsilon^{-1}q^i$.*

We shall discuss now another application of Theorem 2, which gives an alternative description of Green functions. Let $\eta$ be as before. Then $S_\eta=\chi^{-1}\pi(\eta)$ is a variety which is defined over $\boldsymbol{F}_q$. We also recall that $X_x$ is a connected projective variety, of pure dimension $e(x)=\frac{1}{2}(\dim Z_G(x)-\operatorname{rank} G)$, where $Z_G(x)$ is the centralizer of $x$ in $G$ (see [18, p. 163]).

LEMMA 5. *$S_\eta$ is a smooth, irreducible affine variety, of dimension $2e(x)$.*

$S_\eta$ is a closed subvariety of the affine variety $S$, hence is affine. Since $\chi$ is flat, all fibers of $\chi$ have dimension equal to $\dim S-\operatorname{rank} G=2\dim e(x)$. By Lemma 3, the fiber $S_\eta$ of $\chi$ is isomorphic to a fiber of the smooth morphism $\theta$, so $S_\eta$ is smooth. Finally, Proposition 2 and Lemma 2 imply that $H^0(S_\eta, \boldsymbol{Q}_l)\simeq H^0(X_x, \boldsymbol{Q}_l)$. Since $X_x$ is connected, the same is true for $S_\eta$.

REMARK. Lemma 5 is also true in characteristic 0.

THEOREM 3. *If $q$ is sufficiently large, the number of $\boldsymbol{F}_q$-rational points of $S_\eta$ equals $q^{2e(x)}\tilde{Q}_w(x)(q^{-1})$.*

By Grothendieck's formula this number of rational points equals

$$\sum_{i\geq 0}(-1)^i \operatorname{Tr}(F, H_c^i(S_\eta, \boldsymbol{Q}_l)).$$

By Poincaré duality, using Lemma 5, this also equals

$$q^{3e(x)}\sum_{i\geq 0}(-1)^i \operatorname{Tr}(F^{-1}, H^i(S_\eta, \boldsymbol{Q}_l)).$$

The theorem follows from Lemma 4, taking into account the results of no. 4.

If $G=GL_n$ there is a combinatorial description of the polynomials $T^{2e(x)}\tilde{\boldsymbol{Q}}_w(x)(T^{-1})$. It is given in [11, Ch. III, no. 7], where they are denoted by $X^\lambda_\rho$.

**6.** We finally give another application of Theorem 1. We keep the same notations and assumptions. The characteristic $p$ is assumed to be a good prime. Let $\mathcal{O}$ be the $G$-orbit of $x$. Its closure $\overline{\mathcal{O}}$ is an affine subvariety of $\mathfrak{g}$, which is defined over $\boldsymbol{F}_q$. Let $\boldsymbol{IC}^\cdot(\overline{\mathcal{O}})$ be the intersection cohomology complex of $\overline{\mathcal{O}}$. It is a complex in the derived category $D^b_c(\overline{\mathcal{O}}, \boldsymbol{Q}_l)$ (see [5]). We assume it to be normalized such that its restriction to $\mathcal{O}$ is (quasi-isomorphic to) the complex with the sheaf $\boldsymbol{Q}_l$ in degree 0 and the zero sheaf in the other dimensions.

Borho and MacPherson have determined $\boldsymbol{IC}^\cdot(\overline{\mathcal{O}})$ in "classical" cohomology over $\boldsymbol{C}$ [4]. The description of the stalks $H^\cdot(\boldsymbol{IC}(\overline{\mathcal{O}}))_y$ of the cohomology sheaves in a point $y\in\overline{\mathcal{O}}$ are in terms of the action of the Weyl group $W$ on the cohomology groups $H^\cdot(X_y)$ (see no. 5). The arguments of [loc. cit.] carry over to characteristic $p>0$ (at least if $p$ is good: in that case one knows that the number of nilpotent $G$-orbits in $\mathfrak{g}$ is finite, a fact which is used in the proof).

Now let the assumptions be as in Theorem 1, let $\mathcal{O}$ be as above.

THEOREM 4. *The stalks of the complex* $\boldsymbol{IC}^\cdot(\overline{\mathcal{O}})$ *are pure of weight* 0.

Theorem 4 follows readily from the results of Borho-MacPherson and Theorem 1, taking into account the self-duality of $\boldsymbol{IC}^\cdot(\overline{\mathcal{O}})$.

## References

[1] Beynon, W. M. and N. Spaltenstein, Green functions of finite Chevalley groups of type $E_n$ $(n=6, 7, 8)$, preprint.

[2] Beynon, W. M. and N. Spaltenstein, The computation of Green functions of finite Chevalley groups of type $E_n$ $(n=6, 7, 8)$, Univ. of Warwick, Computer Centre Report no. 23, 1982.

[3] Borel, A. et al., Seminar in algebraic groups and related finite groups, Lecture Notes in Math. No. 131, Springer-Verlag, Berlin-Heidelberg-New York, 1970.

[4] Borho, W. and R. MacPherson, Représentations des groupes de Weyl et homologie d'intersection pour les variétés de nilpotents, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 707-710.

[5] Brylinski, J.-L., (Co)-homologie d'intersection et faisceaux pervers, Sém. Bourbaki no. 585, Astérisque 92-93, p. 129-157, Soc. Math. France, 1982.

[6] Deligne, P., Théorie de Hodge III, Publ. Math. IHES **44** (1974), 5-77.

[7] Deligne, P., La conjecture de Weil II, Publ. Math. IHES **52** (1980), 137-252.

[8] Deligne, P. and G. Lusztig, Representations of reductive groups over finite fields, Ann. of Math. **103** (1976), 103-161.

[9] Hotta, R., On Springer's representation, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **18** (1982), 863-876.

[10] Kazhdan, D. and G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. **53** (1979), 165-184.

[11] Macdonald, I. G., Symmetric functions and Hall polynomials, Clarendon Press, Oxford, 1979.

[12] SGA 4, Théorie des topos et cohomologie étale des schémas (séminaire dirigé par M. Artin, A. Grothendieck et J. L. Verdier), Lecture Notes in Math. Nos. 269, 270 and 305, Springer-Verlag, Berlin-Heidelberg-New York, 1972/73.

[13] SGA 4 ½, Cohomologie étale (par P. Deligne avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier), Lecture Notes in Math. No. 569, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[14] SGA 7 II, Groupes de monodromie en géometrie algébrique (I, séminaire par P. Deligne et N. Katz), Lecture Notes in Math. No. 340, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[15] Shoji, T., On the Green polynomials of a Chevalley group of type $F_4$, Comm. Algebra **10** (1982), 505-543.

[16] Slodowy, P., Simple singularities and simple algebraic groups, Lecture Notes in Math. No. 815, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[17] Slodowy, P., Four lectures on simple groups and singularities, Comm. Math. Inst. Univ. Utrecht no. 11, 1980.

[18] Spaltenstein, N., Classes unipotents et sous-groupes de Borel, Lecture Notes in Math. No. 946, Springer-Verlag, Berlin-Heidelberg-NewYork, 1982.

[19] Spaltenstein, N., Caractères unipotents de ${}^3D_4(\boldsymbol{F}_q)$, Comment. Math. Helv. **57** (1982), 676-691.

[20] Spaltenstein, N., Existence of good transversal slices to nilpotent orbits in good characteristics, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1984), 283-286.

[21] Springer, T. A., Trigonometric sums, Green functions of finite groups and representations of Weyl groups, Invent. Math. **36** (1976), 173-207.

[22] Srinivasan, B., Green polynomials of finite classical groups, Comm. Algebra **5** (1977), 1241-1258.

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