What is the Tate conjecture?

The ℓ -adic etale cohomology of algebraic varieties is much richer than their classical cohomology in the sense that it admits the action of Galois groups. In the 1960's, motivated by its relation to poles of zeta functions and a geometric analogue of the BSD conjecture, Tate was led to formulate his famous conjecture saying roughly that the ℓ -adic cohomology classes fixed by the Galois action should arise from algebraic cycles. Besides making the Tate conjecture precise, we will spend most of the time discussing Tate's original motivation, various evidence supporting it and a few surprising implications.

This is an expanded note prepared for a STAGE talk at MIT, Spring 2013. Our main references are [1], [2], [3] and [4].

[-] **Contents**

The Tate conjecture

Let k be a field and let X be a smooth geometrically irreducible projective variety over k of dimension d. We denote by $\overline{X}=X\times_k\bar k$ the base change of X to the algebraic closure $\bar k$. The Galois group $G=\mathrm{Gal}(\bar k/k)$ then acts on $\overline{X}$ via the second factor.

Definition 1 Let $Z^r(\overline{X})$ be the free abelian group generated by the irreducible closed subvarieties of $\overline{X}$ of codimension r ($1\le r\le d$). An element of $Z^r(\overline{X})$ is called an *algebraic cycle* of codimension r on $\overline{X}$. Inside $Z^r(\overline{X})$, we have a subgroup $Z^r(X)\subseteq Z^r(\overline{X})$ consisting of algebraic cycles that are *defined over* k.

Let ℓ be a prime different from $p=\mathrm{char}(k)\ge0$. Recall from John's talk that there is *cycle class map* (of G -modules)

$$c^r: Z^r(\overline{X})\to H^{2r}(\overline{X},\mathbb{Q}_\ell(r)),$$

which associates to every algebraic cycle an ℓ -adic etale cohomology class.

Definition 2 We define $Z_h^r(\overline{X})\subseteq Z^r(\overline{X})$ to be the kernel of c^r. These are the algebraic cycles *ℓ -adically homologically equivalent to zero*. Denote the quotient group $A^r(\overline{X})=Z^r(\overline{X})/Z_h^r(\overline{X})$ and its subgroup $A^r(X)$ the image of $Z^r(X)$. Then c^r induces a map

$$A^r(\overline{X})\otimes_{\mathbb{Z}}\mathbb{Q}_\ell\xrightarrow{c^r\otimes1}H^{2r}(\overline{X},\mathbb{Q}_\ell(r)).\qquad(1)$$

Notice that the image of $A^r(X)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell$ is fixed by the G -action, so we have an induced map

$$A^r(X)\otimes_{\mathbb{Z}}\mathbb{Q}_\ell\to H^{2r}(\overline{X},\mathbb{Q}_\ell(r))^G.\qquad(2)$$

Our expectation will be that this is a bijection: the G -fixed cohomology classes are exactly the *$\mathbb{Q}_\ell$ -span* of the algebraic classes.

Example 1 When $r=1$, the cycle class map c^1 on divisors is relatively easier to describe. Taking the cohomology of the exact sequence of etale sheaves on $\overline{X}$

$$0\to\mu_{\ell^n}\to\mathbb{G}_m\xrightarrow{\ell^n}\mathbb{G}_m\to0$$

gives an injective map $\mathrm{Pic}(\overline{X})/\ell^n\,\mathrm{Pic}(\overline{X})\hookrightarrow H^2(\overline{X},\mu_{\ell^n})$, which by passing to the limit induces the cycle class map for $r=1$,

$$c^1: Z^1(\overline{X})\to\mathrm{Pic}(\overline{X})\to\mathrm{Pic}(\overline{X})\otimes\mathbb{Q}_\ell\hookrightarrow H^2(\overline{X},\mathbb{Q}_\ell(1)).$$

The kernel of c^1 therefore consists of the torsion elements and elements that are ℓ^∞ -divisible.

Notice the inclusions

$$\{\text{linearly equiv. to }0\}\subseteq\{\text{algebraically equiv. to }0\}\subseteq\{\text{numerically equiv. to }0\}$$

induces surjections

$$\mathrm{Pic}(\overline{X}) \twoheadrightarrow \mathrm{NS}(\overline{X}) \twoheadrightarrow \mathrm{Num}(\overline{X}).$$

By the theorem of the base, $\mathrm{NS}(\overline{X})$ is a finitely generated abelian group. The kernel of the first surjection is a divisible group $\mathrm{Pic}^0(\overline{X}) \subseteq \mathrm{Pic}(\overline{X})$ (points of the Picard variety) and the kernel of the second surjection is the torsion subgroup of $\mathrm{NS}(\overline{X})$. So we have $A^1(\overline{X}) = \mathrm{Num}(\overline{X})$, a free abelian group of rank equal to the *Picard number* $\rho(\overline{X}) = \mathrm{rank}\,\mathrm{NS}(\overline{X})$. In particular, $A^1(\overline{X})$ does not depend on the choice of ℓ. It also follows that the map (2) is at least injective when $r = 1$.

With this example in mind, we are now ready to state the famous Tate conjecture.

Conjecture 1 (Tate) Suppose k is finitely generated over its prime field.

- ($T^r(X)$) The map (2) is bijective.
- ($E^r(X)$) The ℓ-adically homological equivalence is the same as the numerical equivalence on $Z^r(\overline{X})$. In particular, $A^r(\overline{X})$ does not depend on the choice of ℓ.

Remark 1 Finitely generated fields are of "arithmetic nature": finite fields and global fields are finitely generated. Fields like $\mathbb{C}$ (or any algebraically closed field), $\mathbb{R}$ and p-adic fields are not finitely generated. The Tate conjecture is not expected to hold for the latter.

Remark 2 In characteristic 0, we can embed k into $\mathbb{C}$ and appeal to the comparison theorem between ℓ-adic etale cohomology and singular cohomology. Then the cycle class map c^r indeed factors through the finitely generated abelian group $H^2(X^{\mathrm{an}}, \mathbb{Z})$. It follows that $A^r(X)$ does not depend on the choice of ℓ and the map (1) is injective, hence the injectivity of the map (2) follows immediately.

Remark 3 In positive characteristic, the ℓ-independence and injectivity are less known for $r > 1$.

Remark 4 When k is a finite field, Tate proves that one can identify $H^{2r}(\overline{X}, \mathbb{Q}_\ell)^G$ with $H^{2r}(X, \mathbb{Q}_\ell)$ for any X. The Kummer sequence on X then gives an exact sequence

$$0 \to \mathrm{Pic}(X) \otimes \mathbb{Z}_\ell \to H^2(X, \mathbb{Z}_\ell(1)) \to T_\ell \,\mathrm{Br}(X) \to 0.$$

One sees that $T^1(X)$ is true if and only if $T_\ell \,\mathrm{Br}(X) = 0$, i.e., the ℓ-primary part $\mathrm{Br}(X)[\ell^\infty]$ is finite.

Remark 5 Since c^d maps a 0-cycle to its degree, the Tate conjecture is trivial for $r = d$. In particular, for X a curve, then Tate conjecture is trivially true; for X a surface, the only relevant conjecture is $T^1(X)$.

Remark 6 It is easy to see that $T^r(X)$ is insensitive to base field extension: if k' is a field extension, then $T^r(X_{k'}) \Longrightarrow T^r(X)$.

Evidence

Why should one believe the Tate conjecture? One should because it is a conjecture of Tate (proof by authority, QED). We are going to discuss two of Tate's major original motivations: one is the relation between T^1 and the geometric analogue of the remarkable conjecture of Birch and Swinnerton-Dyer concerning the rational points of elliptic curves; one has to do with the following conjecture (now a theorem) on homomorphisms between abelian varieties.

Conjecture 2 (Tate's $\mathrm{Hom}(A, B)$) Suppose k is finitely generated over its prime field. Let A, B be two abelian varieties over k. Then the natural map

$$\mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to \mathrm{Hom}_G(V_\ell(A), V_\ell(B))$$

is bijective, where $V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is the rational ℓ-adic Tate module of A.

Remark 7 This is very convenient for studying abelian varieties (geometric objects) via studying the linear algebraic objects (ℓ-adic Galois representations) on the right hand side! It follows that two abelian varieties are isogenous if and only their Tate modules are isomorphic as G-modules. For this reason the conjecture $\mathrm{Hom}(A, B)$ is usually called *Tate's isogeny conjecture*.

Remark 8 To rephrase, the conjecture $\mathrm{Hom}(A, B)$ says that taking the Tate module is a *fully faithful* functor from an isogenous category of abelian varieties where the Hom sets are $\mathrm{Hom}(A, B) \otimes \mathbb{Q}_\ell$ to the category of ℓ-adic G-representations. In the same spirit, the Tate conjecture implies that the functor from the category of pure motives over k (defined using the numerical equivalence) to the category of ℓ-adic G-representations given by taking ℓ-adic etale cohomology is fully faithful.

Remark 9 Fix a polarization on an abelian variety A, one can show using the properties of the Weil pairing that $A^1(A) \otimes \mathbb{Q}_\ell$ can be identified with the subset of elements of $\mathrm{End}(A) \otimes \mathbb{Q}_\ell$ fixed under the Rosati involution ([5, I 14.2]). Similarly, $H^2(\overline{A}, \mathbb{Q}_\ell) = \bigwedge^2 H^1(\overline{A}, \mathbb{Q}_\ell)$ can be identified with the subset of elements of $\mathrm{End}(V_\ell A)$ fixed under the Rosati involution. So $\mathrm{Hom}(A, A) \Longrightarrow T^1(A)$.

Theorem 1 $\mathrm{Hom}(A, B)$ is true for any finitely generated field k. In particular, T^1 is true for abelian varieties.

Remark 10 For finite fields $\mathrm{Hom}(A, B)$ was proved by Tate himself ([6]) soon after its formulation, now known as Tate's isogeny theorem. Zarhin [7] proved the case of function fields of positive characteristic. The number field case was proved by Faltings ([8]) as one of the steps in proving the Mordell conjecture. Faltings' method can be extended to any finitely generated fields.

The truth of $\mathrm{Hom}(A, B)$ allows us to prove the following

Proposition 1 $T^1(X\times Y) \Longleftrightarrow T^1(X)\times T^1(Y)$.

Proof Consider T^1 for the product $X\times Y$. On the one hand, the Kunneth formula gives decomposition

$$\begin{aligned} H^2(\overline{X}\times\overline{Y},\mathbb{Q}_\ell) &\cong \bigoplus_{i+j=2} H^i(\overline{X},\mathbb{Q}_\ell)\otimes H^j(\overline{Y},\mathbb{Q}_\ell) \\ &= H^2(\overline{X},\mathbb{Q}_\ell)\oplus H^2(\overline{Y},\mathbb{Q}_\ell)\oplus(H^1(\overline{X},\mathbb{Q}_\ell)\otimes H^1(\overline{Y},\mathbb{Q}_\ell))\end{aligned}$$

Notice that $H^1(\overline{X},\mathbb{Q}_\ell)=H^1(\mathrm{Alb}(\overline{X}),\mathbb{Q}_\ell)=V_\ell\,\mathrm{Alb}(\overline{X})^\vee$, where $\mathrm{Alb}(\overline{X})$ is the Albanese variety of $\overline{X}$. So using the duality between Pic^0 and Alb, we know that $V_\ell\,\mathrm{Pic}^0(\overline{Y})=V_\ell\,\mathrm{Alb}(\overline{Y})^\vee(1)$, hence

$$\begin{aligned} H^1(\overline{X},\mathbb{Q}_\ell)\otimes H^1(\overline{Y},\mathbb{Q}_\ell)(1) &\cong V_\ell\,\mathrm{Alb}(\overline{X})^\vee\otimes V_\ell\,\mathrm{Pic}^0(\overline{Y}) \\ &\cong \mathrm{Hom}(V_\ell\,\mathrm{Alb}(\overline{X}),V_\ell\,\mathrm{Pic}^0(\overline{Y})).\end{aligned}$$

On the other hand, extending the base field if necessary (Remark 6), one can choose $P\in X(k)$ and $Q\in Y(k)$ and obtain

$$\mathrm{Pic}(X\times Y)\cong\mathrm{Pic}(X)\oplus\mathrm{Pic}(Y)\oplus\mathrm{Hom}(\mathrm{Alb}(X),\mathrm{Pic}^0(Y)),$$

where $\mathrm{Hom}(\mathrm{Alb}(X),\mathrm{Pic}^0(Y))$ can be identified with the divisorial correspondence between (X,P) and (Y,Q) ([5, III 6.3]). This implies an isomorphism of free abelian groups

$$A^1(X\times Y)\cong A^1(X)\oplus A^1(X)\oplus\mathrm{Hom}(\mathrm{Alb}(X),\mathrm{Pic}^0(Y)).$$

Now the assertion follows by comparing the two decompositions and the truth of $\mathrm{Hom}(A,B)$. □

Remark 11 One can immediately enlarge the classes of varieties for which T^1 holds by taking products. In particular, T^1 is true for the product of any number of curves and abelian varieties.

Even better, Tate proved that

Proposition 2 T^1 is birationally invariant. More generally, if there is a dominant rational map $X\to Y$, then $T^1(X)\Longrightarrow T^1(Y)$.

Together with the previous proposition, we see that T^1 is true for any X that is dominated by products of varieties for which T^1 are known to be true.

Example 2 For example, the Fermat surface

$$S_n: z_0^n+z_1^n+z_2^n+z_3^n=0$$

is dominated by the product of two curves

$$C_1: x_0^n+x_1^n=x_2^n,\quad C_2: y_0^n+y_1^n=-y_2^n.$$

Explicitly,

$$C_1\times C_2\to S,\quad ([x_0,x_1,x_2],[y_0,y_1,y_2])\mapsto[x_0y_2,x_1y_2,y_0x_2,y_1x_2].$$

So T^1 is true for every Fermat surface S_n.

The Hodge conjecture is known for codimension 1 by Lefschetz 1-1 theorem. In contrast, T^1 is still not known in general (and as you expected, even less is known when $r>1$). We list some known case about K3 surfaces (the theme of this seminar).

Theorem 2 T^1 holds for

- (Folklore) K3 surfaces in characteristic 0;
- Over a finite field of characteristic $p\geq 5$:
    - (Artin, Swinnerton-Dyer [9]) Elliptic K3 surfaces;
    - (Nygaard, Ogus [10]) Non-supersingular K3 surfaces;
    - (Maulik [11], Charles [12]) Supersingular K3 surfaces;
- (Madapusi Pera [13]) K3 surfaces in odd characteristic.

Due to the work of many authors, T^1 is also known to be true for Hilbert modular surfaces, quaternionic Shimura surfaces, Picard modular surfaces, Siegel modular threefolds... See the references in [2].

Implications

There are many implications of the Tate conjecture. One surprising implication among them is the BSD conjecture for elliptic curves over global function fields. As we mentioned, this is one of Tate's original motivations for formulating T^1.

Assume now that $k=\mathbb{F}_q$ is a finite field of characteristic p. Recall that the zeta function of X/k is defined to be the generating function of $\#X(\mathbb{F}_{q^n})$,

$$Z(X,T)=\exp\sum_{n\ge1}\#X(\mathbb{F}_{q^n})\frac{T^n}{n}.$$

By the Lefschetz trace formula, this is in fact a rational function in T,

$$Z(X,T)=\frac{P_1(T)P_3(T)\cdots P_{2d-1}(T)}{P_0(T)P_2(T)\cdots P_{2d}(T)},$$

where here $P_j(T)=\det(1-\varphi T\mid H^j(\overline{X},\mathbb{Q}_\ell))$ is the characteristic polynomial of the geometric Frobenius φ on $H^j(\overline{X},\mathbb{Q}_\ell)$. By Deligne's proof of the Weil conjecture, this is a polynomial in $1+T\mathbb{Z}[T]$ independent of the choice of ℓ. The inverse roots of $P_i(T)$ have all absolute values $q^{j/2}$. For example, $P_0(T)=1-T$ and $P_{2d}(T)=1-q^dT$.

It follows that the poles of the zeta function $\zeta(X,s)=Z(X,q^{-s})$ lie on the lines $\mathrm{Re}\, s=0,1,\dots,d$. In particular, the order of pole at $s=j$ is exactly the multiplicity of q^j occurring as the eigenvalue of φ on $H^j(\overline{X},\mathbb{Q}_\ell)$, or the multiplicity of 1 occurring as the eigenvalue of φ on $H^{2j}(\overline{X},\mathbb{Q}_\ell(j))$. If φ acts semisimply, this is exactly the dimension of $H^j(\overline{X},\mathbb{Q}_\ell(j))^G$. In fact, by carefully doing linear algebra Tate showed that

Theorem 3 Assume k is a finite field. Then $T^r(X)$ and $E^r(X)$ is true if and only if the order of pole of $\zeta(X,s)$ at $s=r$ is equal to the rank of the group of codimension r cycles on X up to numerical equivalence. In particular, $T^1(X)$ is true if and only the order of the pole of $\zeta(X,s)$ at $s=1$ is equal to the the Picard number $\rho(X)$.

Example 3 Since $\varphi_{q^m}=(\varphi_q)^m$, the Tate conjecture implies that the rank of codimension r cycles on $\overline{X}$ *defined over* $\bar{k}$ up to numerical equivalence is the number of roots of unity occurring as eigenvalues of φ on $H^{2r}(\overline{X},\mathbb{Q}_\ell(r))$. Suppose d is even. By Poincare duality, α is an eigenvalue of φ on the middle cohomology $H^d(\overline{X},\mathbb{Q}_\ell(d/2))$ if and only if $1/\alpha$ is. In other words, the number of such roots of of unity has the same parity as $\dim H^d(\overline{X},\mathbb{Q}_\ell)$. In particular, when X is a K3 surface, $\dim H^2(\overline{X},\mathbb{Q}_\ell)=22$ and $T^1(X)$ implies that the geometric Picard number $\rho(\overline{X})$ is always even!

This may ring some bells if you have seen the remarkable conjecture of Birch and Swinnerton-Dyer, which predicts that the order of zero of the L-function $L(E,S)$ at $s=1$ of an elliptic curve E over a global field K (more generally, any finitely generated field) is equal to the rank of the group of K-rational points $E(K)$. When $K=k(C)$ is the function field of a smooth projective curve k defined over k, by resolving singularities one can always find a regular projective elliptic surface $\mathcal{E}\to C$ such that the generic fiber of $\mathcal{E}$ is E/K. If $\mathcal{E}\to C$ is smooth, the connection is extremely simple: on the one hand, unraveling the definition one finds that

$$L(E,s)=\frac{\zeta(C,s)\zeta(C,s-1)}{\zeta(\mathcal{E},s)}.$$

Since $\zeta(C,s)$ has simple poles at $s=0,1$, it follows that at $s=1$,

$$\text{order of zero of } L(E,s)=\text{order of pole of } \zeta(\mathcal{E},s)-2.$$

On the other hand,

$$\mathrm{NS}(\mathcal{E})\cong E(K)\oplus\mathbb{Z}O\oplus\mathbb{Z}f,$$

where f is the class of fibers and O is the class of the zero section. Hence

$$\mathrm{rank}\, E(K)=\rho(\mathcal{E})-2.$$

Therefore

$$\text{order of zero of } L(E,s)=\mathrm{rank}\, E(K)\iff\text{order of pole of } \zeta(\mathcal{E},s)=\rho(\mathcal{E}).$$

The left hand side is the BSD conjecture for E/K and by the previous theorem the right hand side is equivalent to the Tate conjecture for $\mathcal{E}/k$! In general $\mathcal{E}\to C$ has singular fibers and the theorem of Shioda-Tate says that

$$\operatorname{rank} E(K)=\rho(\mathcal{E})-2-\sum_v (m_v-1),$$

where m_v is the number of different components of the fiber over the closed point v of k. When taking into account of the modification to $L(E,s)$ from bad places, the change of the order of zero at $s=1$ miraculously matches up with $\sum_v (m_v-1)$. Hence we know that

Theorem 4 $T^1(\mathcal{E})$ is equivalent to the BSD conjecture for E/K.

In view of Remark 4, one naturally wonders if the finiteness of the Brauer group of the elliptic surface $\mathcal{E}$ has anything to do the finiteness of the Shafarevich-Tate group $Ш(E/K)$. Amazingly, Grothendieck [14] proved that

Theorem 5 There is a canonical isomorphism $\mathrm{Br}(\mathcal{E})\cong Ш(E/K)$.

Remark 12 It follows immediately that the BSD conjecture over global function fields is equivalent to the finiteness of $Ш(E/K)[\ell^\infty]$. This is what Artin and Swinnerton-Dyer actually proved in [9] for elliptic K3 surfaces.

Remark 13 Everyone believes $Ш$ is finite. As Tate commented, otherwise "the Galois cohomology of an abelian variety should be a mess and the determination of the group of rational points by descent would be ineffective". In view of the above equivalence, why shouldn't one also believe the Tate conjecture (at least T^1)?

Remark 14 Deligne proved that the Tate conjecture (for all codimensions) implies that all absolutely Hodge classes are algebraic ([4, 6.2]). Thus assuming Deligne's conjecture that all Hodge classes are absolutely Hodge (which is known for abelian varieties and K3 surfaces), the Tate conjecture implies the Hodge conjecture. There is no similar statement going the other way (e.g., the Hodge conjecture for K3 surfaces is easily true by Lefschetz 1-1, but the Tate conjecture for K3 surfaces is much harder).

Finally it is worth mentioning another interesting implication of the Tate conjecture for K3 surfaces recently proved in [15].

Theorem 6 (Lieblich-Maulik-Snowden) Assume k is a finite field of characteristic $p\ge 5$. Then T^1 is true for all K3 surfaces over $\bar{k}$ if and only there are only finitely many K3 surfaces defined over each finite extension of k.

References

[1] Tate, John T., *Algebraic cycles and poles of zeta functions,* Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), Harper \& Row, 1965, 93--110.

[2] Tate, John, *Conjectures on algebraic cycles in l-adic cohomology,* Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., **55** Amer. Math. Soc., 1994, 71--83.

[3] Ulmer, Douglas, *Elliptic curves over function fields,* Arithmetic of L-functions, IAS/Park City Math. Ser., **18** Amer. Math. Soc., 2011, 211--280.

[4] Milne, J.~S., *The Tate conjecture over finite fields (AIM talk),* ArXiv e-prints (2007).

[5] Milne, James S., *Abelian Varieties (v2.00),* Available at www.jmilne.org/math/.

[6] Tate, John, *Endomorphisms of abelian varieties over finite fields,* Invent. Math. **2** (1966), 134--144.

[7] Zarhin, Ju. G., *Endomorphisms of Abelian varieties over fields of finite characteristic,* Izv. Akad. Nauk SSSR Ser. Mat. **39** (1975), no.2, 272--277, 471.

[8] Faltings, G., *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern,* Invent. Math. **73** (1983), no.3, 349--366.

[9] Artin, M. and Swinnerton-Dyer, H. P. F., *The Shafarevich-Tate conjecture for pencils of elliptic curves on $K3$ surfaces,* Invent. Math. **20** (1973), 249--266.

[10] Nygaard, Niels and Ogus, Arthur, *Tate's conjecture for $K3$ surfaces of finite height,* Ann. of Math. (2) **122** (1985), no.3, 461--507.

[11] Maulik, D., *Supersingular K3 surfaces for large primes,* ArXiv e-prints (2012).

[12] Charles, F., *The Tate conjecture for K3 surfaces over finite fields,* ArXiv e-prints (2012).

[13] Madapusi Pera, K., *The Tate conjecture for K3 surfaces in odd characteristic,* ArXiv e-prints (2013).

[14] Grothendieck, Alexander, *Le groupe de Brauer. III. Exemples et compléments,* Dix Exposés sur la Cohomologie des Schémas, North-Holland, 1968, 88--188.

[15] Lieblich, M. and Maulik, D. and Snowden, A., *Finiteness of K3 surfaces and the Tate conjecture,* ArXiv e-prints (2011).

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