2.

Conference Board of the Mathematical Sciences
REGIONAL CONFERENCE SERIES IN MATHEMATICS

supported by the
National Science Foundation

Number 39

REPRESENTATIONS OF FINITE CHEVALLEY GROUPS

by

G. LUSZTIG

Published for the
Conference Board of the Mathematical Sciences

by the
American Mathematical Society
Providence, Rhode Island

$39521

Contents

Preface

These notes have arisen from a series of lectures given by the author at Madison, Wisconsin, from 8 August to 12 August, 1977, at a Reginal Conference sponsored by the Conference Board of the Mathematical Sciences and supported by the National Science Foundation.

Their main purpose was to show how l-adic cohomology of algebraic varieties over fields of characteristic $p > 1$ can be used to get information on the representations of finite Chevalley groups.

It is a great pleasure to thank Louis Solomon for organizing the lectures in Madison and for his hospitality. During the lectures, a set of notes was prepared by Bhama Srinivasan and these were of great help in writing the present notes (especially Parts 1 and 2); I wish to thank her for this.

Introduction

These notes are concerned with the representation theory of finite Chevalley groups (or finite groups of Lie type, or reductive algebraic groups over a finite field $\mathbf{F}_q$). Examples of such groups are the general linear group $\mathrm{GL}_n(\mathbf{F}_q)$ and the symplectic group $\mathrm{Sp}_{2n}(\mathbf{F}_q)$. The representations considered here will be always over an algebraically closed field K of characteristic zero. (The usual choice is to take K to be the complex numbers $\mathbf{C}$. We shall, however, take K to be an algebraic closure $\overline{\mathbf{Q}_l}$ of the l-adic numbers, where l is a prime number; this gives the same result, since $\overline{\mathbf{Q}_l}$ is abstractly isomorphic to $\mathbf{C}$.)

This subject started in 1896 with the work of Frobenius on the irreducible characters of $\mathrm{PSL}_2(\mathbf{F}_p)$ (Gesammelte Abhandlungen, Band III, Springer-Verlag, 1968, pp. 29–37).

A highlight in the subsequent development was Green's paper [13] on the irreducible characters of $\mathrm{GL}_n(\mathbf{F}_q)$. In this paper, Green combined ingeniously the Frobenius method of inducing characters of subgroups, with Brauer's theory of modular representations and with some rather deep combinatorics, to construct the irreducible characters. A further step was taken, for groups of rank 2, by Srinivasan [25], for $\mathrm{Sp}_4(\mathbf{F}_q)$, and by Chang-Ree [6] for $G_2(\mathbf{F}_q)$.

In the general case, almost all irreducible representations have been constructed. (Here, "almost all" means that the number of representations which have not been constructed divided by the total number of irreducible representations is $\leqslant C \cdot q^{-1}$, where C is independent of q.) There are two different methods to do this: they associate an irreducible representation of the group to each one dimensional representation, in general position, of a maximal torus (the difficult case being the case where the maximal torus is "anisotropic"). One method, which can be found in the author's joint paper with Deligne [11] realizes the representations in the l-adic (= étale) cohomology of certain algebraic varieties (over $\overline{\mathbf{F}}_q$) on which our group acts. This is similar, in spirit, to the Borel-Weil construction of the irreducible representation of a compact connected Lie group and to its extension to noncompact semisimple Lie groups. The other method (which applies only in sufficiently large characteristic) deals with characters rather than representations. One starts with an explicit class function on the group (which Springer conjectured is the character of the required representation). One has to prove that (a) it is a generalized character (this has been proved by Kazhdan) and (b) that it is irreducible (this has been proved by Springer). The proofs of both (a) and (b) involve again the use of l-adic cohomology. Thus l-adic cohomology seems to be essential for understanding the representations of finite Chevalley groups. However, for many purposes, it is already enough to consider not individual cohomology groups, but only alternating sums (e.g., Lefschetz numbers). We have included a preliminary part on Lefschetz numbers; their properties are stated (some are proved) in Part 1. Part 2 is essentially an exposition of [11]. In this part, only Lefschetz numbers rather than individual cohomology groups are used (except in the last section 2.20). There are some improvements on [11]: for instance, there is a more direct proof of the orthogonality theorem (2.3) which unlike the proof in [11] applies equally well to Ree and Suzuki groups over "the field with $\sqrt{2}$ or $\sqrt{3}$ elements". In the last section (2.20), we discuss the special case of $SL_2(F_q)$. The irreducible representations constructed in Part 2 provide only almost all of the representations of our Chevalley group. They are in some sense similar to the regular semisimple conjugacy classes among all conjugacy classes of a finite Chevalley group. It seems that to understand all irreducible representations, without exception, the essential thing is to understand a rather small class of representations, which play the same role as the unipotent conjugacy classes among all conjugacy classes of a finite Chevalley group. These representations were called "unipotent representations" in [11]. For large q, they could be characterized as those irreducible representations of our Chevalley group whose character takes a constant value on the set of regular elements inside any given maximal torus. Part 3 is concerned with the problem of classifying the "unipotent representations" of a finite Chevalley group. This aim is achieved (at least for sufficiently large q) when there are no factors of type E_8. This part is rather less self-contained than Part 2 in the sense that, generally, no proofs are included for results which are already in the literature. Moreover, in this part, Lefschetz numbers are no longer sufficient, and individual cohomology groups must be used. Part 4 contains some unsolved problems on unipotent representations.

The reader is assumed to have some familiarity with the structure theory and classification of reductive groups, as explained in [7], [15] (over algebraically closed fields) and [5], [24], [23] (over finite fields). He is also asked to accept, without proof, some fundamental properties of l-adic cohomology.

Part 1. Preliminaries

1.1. Throughout these notes, k will denote an algebraic closure of the field $\mathbf{F}_p$ with p elements, where p is a prime number. If q is a power of p, we denote by $\mathbf{F}_q$ the subfield of k which consists of q elements.

A separated scheme of finite type over k will be referred to as a variety. It will always be identified with its set of points over k, and it will often be affine. An $\mathbf{F}_q$-rational structure on a variety X gives rise to a Frobenius map $F: X \longrightarrow X$, and is completely determined by it. Here are some simple properties of F.

(a) It is bijective, its fixed point set X^F is finite and all its orbits on X are finite.

(b) If g is an automorphism of finite order of X commuting with F, then $g \cdot F$ is the Frobenius map for some $\mathbf{F}_q$-rational structure on X.

(c) If F, F' are Frobenius maps for an $\mathbf{F}_q$- (resp $\mathbf{F}_{q'}$-) rational structure on X, then there exist integers n, $n' \geqslant 1$ such that $F^n = (F')^{n'}$. For any $n \geqslant 1$, F^n is the Frobenius map for an $\mathbf{F}_{q^n}$-rational structure on X.

1.2. It is a very deep result that to any variety X one can attach canonically l-adic cohomology groups with compact support $H_c^i(X, \mathbf{Q}_l)$, with properties analogous to those familiar from algebraic topology. (Here l is a fixed prime $\neq p$ and $\mathbf{Q}_l$ denotes the l-adic numbers.) These are finite dimensional $\mathbf{Q}_l$-vector spaces, zero for $i < 0$ and for large i. They are functorial, in the sense that a finite map $f: X \longrightarrow X'$ induces a linear map $f^*: H_c^i(X', \mathbf{Q}_l) \longrightarrow H_c^i(X, \mathbf{Q}_l)$,

For the definition and properties of l-adic cohomology, the reader can consult S.G.A 4, S.G.A. 5 (but he will need much courage to do so) or S.G.A. 4½ (see [20]).

l-adic cohomology can be used to construct representations of finite groups as follows. Assume that G is a finite group acting on a variety X. Then $g \in G$ acts on $H_c^i(X, \mathbf{Q}_l)$ by $(g^*)^{-1}$ and this is a representation of G. We can also form the alternating sum $\Sigma_i(-1)^i H_c^i(X, \mathbf{Q}_l)$; this is a virtual representation of G (over $\mathbf{Q}_l$); its character at $g \in G$ is the Lefschetz number $\mathfrak{L}(g, X)$ of g.

Now let $F: X \longrightarrow X$ be the Frobenius map for an $\mathbf{F}_q$-rational structure on X. Grothendieck's trace formula (S.G.A. 4½, pp. 76—109) asserts that

$$|X^F| = \sum_i (-1)^i \operatorname{Tr}(F^*, H_c^i(X, \mathbf{Q}_l)).$$

Assume now that F commutes with g (this can be always achieved by replacing, if necessary, F by a power). Replacing F by $F^n g^{-1}$ (cf. 1.1(b), (c)) in the previous formula, we see that

$$|X^{F^n g^{-1}}| = \sum_i (-1)^i \operatorname{Tr}((F^*)^n (g^*)^{-1}, H_c^i(X, \mathbf{Q}_l))$$

for all integers $n \geqslant 1$. It follows easily that the formal power series $-\Sigma_{n=1}^{\infty} |X^{F^n g^{-1}}| t^n$ is a rational function in t with only simple poles and no pole at ∞; moreover, its value at ∞ is just $\mathfrak{L}(g, X)$. This can be used to compute $\mathfrak{L}(g, X)$ in certain examples and to check directly some simple properties of $\mathfrak{L}(g, X)$, like multiplicativity:

$$\mathfrak{L}(g, X)\mathfrak{L}(g', X') = \mathfrak{L}(g \times g', X \times X') \tag{1.2.1}$$

(g, g' automorphisms of finite order of X, X') or additivity:

$$\mathfrak{L}(g, X) = \sum_i \mathfrak{L}(g, X_i) \tag{1.2.2}$$

(where X_i, $1 \leqslant i \leqslant n$, form a partition of X into locally closed subvarieties stable under g: $X \longrightarrow X$). We also see that $\mathfrak{L}(g, X)$ is a rational number, independent of l (since $|X^{F^n g^{-1}}|$ are so) and, being the value of a character of a finite group, it is also an algebraic integer, hence an ordinary integer. In particular, we have $\mathfrak{L}(g, X) = \mathfrak{L}(g^{-1}, X)$. We now list some further properties of $\mathfrak{L}(g, X)$ in the form of lemmas.

LEMMA 1.3. *Let H be a finite group acting on X and let g: $X \longrightarrow X$ be an automorphism of finite order commuting with each element of H. Assume that the quotient $H\backslash X$ exists (this is always the case when X is affine). Then*

$$\mathfrak{L}(g, H\backslash X) = |H|^{-1} \sum_{h \in H} \mathfrak{L}(gh, X).$$

Indeed, if F: $X \longrightarrow X$ is the Frobenius map for an $\mathbf{F}_q$-rational structure on X commuting with g and with all elements of H, we have clearly

$$|(H\backslash X)^{F^n g^{-1}}| = |H|^{-1} \sum_{h \in H} |X^{F^n h^{-1} g^{-1}}|$$

for all $n \geqslant 1$.

LEMMA 1.4. *Let g: $X \longrightarrow X$ be an automorphism of finite order. Write $g = su = us$ where s: $X \longrightarrow X$ is of order prime to p and u: $X \longrightarrow X$ is of order a power of p. Then $\mathfrak{L}(g, X) = \mathfrak{L}(u, X^s)$ where X^s is the fixed point set of s.*

For the proof, which uses results in S.G.A. 4 and S.G.A. 5, see [11, Chapter 3].

LEMMA 1.5. *Let T be a torus acting on X. We have, for all $t \in T$: $\mathfrak{L}(t, X) = \mathfrak{L}(1, X)$.*

We can choose Frobenius maps F: $T \longrightarrow T$ and F: $X \longrightarrow X$ relative to $\mathbf{F}_q$-rational structures such that $F(tx) = F(t)F(x)$ for all $t \in T$, $x \in X$. We can also assume that $F(t) = t^q$ for all $t \in T$. It is then enough to show that $|X^{F^n t}| = |X^{F^n}|$ for any $n \geqslant 1$ and any $t \in T$. As T is isomorphic to $(k^*)^m$, we can write $t \in T$ in the form $t = t_1^{q^n - 1}$ for some $t_1 \in T$. Then $x \longrightarrow t_1 x$ gives a bijection $X^{F^n t} \xrightarrow{\sim} X^{F^n}$.

LEMMA 1.6. *With the notations of the previous lemma, let $s: X \longrightarrow X$ be an automorphism of finite order prime to p commuting with each element of T. Then $\mathfrak{L}(s, X) = \mathfrak{L}(s, X^T)$ where X^T is the fixed point set of T on X.*

Using Lemma 1.4 we see that $\mathfrak{L}(s, X) = \mathfrak{L}(1, X^s)$, $\mathfrak{L}(s, X^T) = \mathfrak{L}(1, (X^T)^s) = \mathfrak{L}(1, (X^s)^T)$ so that we can assume that s is the identity. By Lemmas 1.5 and 1.4 we have for any $t \in T$: $\mathfrak{L}(1, X) = \mathfrak{L}(t, X) = \mathfrak{L}(1, X^t)$ and it remains to observe that if $t \in T$ is sufficiently general, then $X^t = X^T$.

LEMMA 1.7. *Let $X = \amalg_{i \in I} X_i$ be a finite partition of X into disjoint, closed subsets X_i, and let G be a finite group acting on X, permuting the subsets X_i among them in such a way that the action of G on I is transitive. Let $H = \{g \in G | gX_{i_0} = X_{i_0}\}$ for a fixed $i_0 \in I$. Then the generalized character $g \longrightarrow \mathfrak{L}(g, X)$ of G is induced by the generalized character $h \longrightarrow \mathfrak{L}(h, X_{i_0})$ of H.*

We can choose a Frobenius map $F: X \longrightarrow X$ relative to an $\mathbf{F}_q$-rational structure such that $FX_{i_0} = X_{i_0}$ and F commutes with all elements of G. It is enough to observe that the identity

$$|X^{F^n g^{-1}}| = |H|^{-1} \sum_{\substack{z \in G \\ zgz^{-1} \in H}} |X_{i_0}^{F^n z g^{-1} z^{-1}}|$$

holds for all $g \in G$ and all $n \geqslant 1$.

LEMMA 1.8. *If X is isomorphic to affine m-space and F is the Frobenius map on X relative to an $\mathbf{F}_q$-rational structure, then $|X^F| = q^m$.*

We omit the proof.

LEMMA 1.9. *Let $f: X \longrightarrow Y$ be a morphism of varieties such that for some $m \geqslant 0$, $f^{-1}(y)$ is isomorphic to affine m-space, for all $y \in Y$. Let g, g' be automorphisms of finite order of X, Y such that $fg = g'f$. Then $\mathfrak{L}(g, X) = \mathfrak{L}(g', Y)$.*

Choose Frobenius maps $F: X \longrightarrow X$, $F: Y \longrightarrow Y$ relative to $\mathbf{F}_q$-rational structures such that $Ff = fF$, $Fg = gF$, $Fg' = g'F$. Using the previous lemma, we see that

$$|X^{F^n}| = \sum_{y \in Y^{F^n}} |f^{-1}(y)^{F^n}| = |Y^{F^n}| q^{mn}$$

for all $n \geqslant 1$. Hence

$$-\sum_{n=1}^{\infty} |X^{F^n}| t^n = -\sum_{n=1}^{\infty} |Y^{F^n}| (tq^m)^n.$$

We now set $t = \infty$ and the lemma follows.

LEMMA 1.10. *Assume that X is 0-dimensional (a finite set). Let $g\colon X \longrightarrow X$ be an automorphism. Then $\mathfrak{L}(g, X) = |X^g|$.*

We can take $F\colon X \longrightarrow X$ to be identity. We have

$$\mathfrak{L}(g, X) = -|X^{g^{-1}}| \sum_{n=1}^{\infty} t^n \Big|_{t=\infty} = |X^{g^{-1}}| = |X^g|.$$

Part 2. The Characters $R_T^G(\theta)$

2.1. Let G be a connected algebraic group (always over k), and let $F\colon G \longrightarrow G$ be the Frobenius map for an $\mathbf{F}_q$-rational structure, compatible with the group structure i.e. such that F is a group homomorphism.

According to a theorem of Lang, the map $L\colon G \longrightarrow G$ defined by $L(g) = g^{-1}F(g)$ is surjective. Note also that $L(g) = L(g')$ if and only if $g' = g_0 g$ for some $g_0 \in G^F$ and that L defines an isomorphism of varieties $G^F \backslash G \longrightarrow G$ (the finite group G^F acts on G, by left multiplication).

$Y \subset G$ normalized by G'

Now let Y be a locally closed subset of G. Then

$$L^{-1}(Y) = \{g \in G \mid g^{-1}F(g) \in Y\}$$

is a locally closed subset of G stable under left multiplication by elements in G^F. Assume that Y is normalized by an F-stable subgroup G' of G. Then $L^{-1}(Y)$ is stable under right multiplication by elements in G'^F. Thus $G^F \times G'^F$ acts on $L^{-1}(Y)$ by $(g_0, g_0')\colon g \longrightarrow g_0 g g_0'^{-1}$ $(g \in L^{-1}(Y),\ g_0 \in G^F,\ g_0' \in G'^F)$.

It follows that $H_c^i(L^{-1}(Y), \mathbf{Q}_l)$ can be regarded as a $G^F \times G'^F$-module (under $(g_0, g_0')^{*-1}$). If ρ is a finite dimensional representation of G'^F over $\overline{\mathbf{Q}}_l$ (an algebraic closure of $\mathbf{Q}_l$) we can consider the G'^F-invariant part

$$(H_c^i(L^{-1}(Y), \mathbf{Q}_l) \otimes \rho)^{G'^F}$$

of $H_c^i(L^{-1}(Y), \mathbf{Q}_l) \otimes \rho$. This is still a G^F-module (over $\overline{\mathbf{Q}}_l$). Let

$$\rho^* = \sum_i (-1)^i (H_c^i(L^{-1}(Y), \mathbf{Q}_l) \otimes \rho)^{G'^F}.$$

This is a virtual G^F-module (over $\overline{\mathbf{Q}}_l$). Its character at $g \in G^F$ is given by

$$\mathrm{Tr}(g, \rho^*) = |G'^F|^{-1} \sum_{g' \in G'^F} \mathfrak{L}((g, g'), L^{-1}(Y))\,\mathrm{Tr}(g', \rho).$$

($\mathfrak{L}$ = Lefschetz number). The correspondence $\rho \longrightarrow \rho^*$ defines a linear map from virtual G'^F-modules to virtual G^F-modules (depending on Y).

2.2. From now on, G will be assumed to be reductive. Let T be a maximal torus in G, defined over $\mathbf{F}_q$ (i.e. stable under F). Let B be a Borel subgroup of G containing T, with unipotent radical U. We can specialize the discussion in 2.1 to the case where $Y = U$, $G' = T$. Thus, the action of $G^F \times T^F$ on $L^{-1}(U)$ gives rise to a correspondence $\theta \longrightarrow \theta^*$ from one dimensional representation $\theta\colon T^F \longrightarrow \overline{\mathbf{Q}}_l^*$ to virtual representations of G^F. We shall denote θ^* as $R^G_{T,U}(\theta)$. (Later, we shall show that this does not depend on U and then we shall denote it simply as $R^G_T(\theta)$.) The character of the virtual G^F-module $R^G_{T,U}(\theta)$ at $g \in G^F$ is given by

$$(2.2.1) \qquad \operatorname{Tr}(g, R^G_{T,U}(\theta)) = |T^F|^{-1} \sum_{t \in T^F} \mathfrak{L}((g, t), L^{-1}(U))\theta(t).$$

The first main result on $R^G_{T,U}(\theta)$ is an orthogonality theorem. Let T' be another F-stable maximal torus in G and let B' be a Borel subgroup of G containing T' with unipotent radical U'. Let $\theta'\colon T'^F \longrightarrow \overline{\mathbf{Q}}_l^*$ be a homomorphism.

Let $N(T, T') = \{n \in G | n^{-1}Tn = T'\}$. Note that T acts on the left and T' acts on the right on $N(T, T')$ and $T\backslash N(T, T') \cong N(T, T')/T'$; this orbit space is denoted $W(T, T')$. It is a finite set. Now $N(T, T')$ is F-stable, therefore F acts naturally on $W(T, T')$. Using Lang's theorem for T and T', we see that $T^F\backslash N(T, T')^F = N(T, T')^F/T'^F = W(T, T')^F$ (this set is empty unless T, T' are conjugate under G^F). Choose representatives $\dot{w} \in N(T, T')$ for the elements $w \in W(T, T')$. Each element $w \in W(T, T')$ defines an isomorphism $\operatorname{ad}(\dot{w})$: $T' \longrightarrow T$, $\operatorname{ad}(\dot{w})(t') = \dot{w}t'\dot{w}^{-1}$. If $w \in W(T, T')^F$, then $\operatorname{ad}(\dot{w})$ defines an isomorphism $T'^F \longrightarrow T^F$; for such w, we define a character ${}^w\theta$ of T'^F by ${}^w\theta(t') = \theta(\dot{w}t'\dot{w}^{-1})$. We can now state

THEOREM 2.3. *With the notations above, we have*

$$(2.3.1) \qquad \langle R^G_{T,U}(\theta), R^G_{T',U'}(\theta')\rangle_{G^F} = \#\{w \in W(T, T')^F | {}^w\theta = \theta'\}.$$

where $\langle\ ,\ \rangle = \langle\ ,\ \rangle_{G^F}$ denotes the standard inner product of virtual representations (or characters) of G^F.

The proof is based on a study of the variety $S = G^F\backslash(L^{-1}(U) \times L^{-1}(U'))$ (G^F acts on $L^{-1}(U) \times L^{-1}(U')$ by left multiplication on both factors). The map $(g, g') \longrightarrow (x, x', y)$, $x = g^{-1}F(g)$, $x' = g'^{-1}F(g')$, $y = g^{-1}g'$ defines an isomorphism between S and the variety

$$\overline{S} = \{(x, x', y) \in U \times U' \times G | xF(y) = yx'\}.$$

The action of $T^F \times T'^F$ on S given by right multiplication by t^{-1} on the first factor and by t'^{-1} on the second factor becomes an action of $T^F \times T'^F$ on $\overline{S}$ given by $(x, x', y) \longrightarrow (txt^{-1}, t'x't'^{-1}, tyt'^{-1})$.

Let A be the left-hand side of (2.3.1). Using (2.2.1) and the fact that Lefschetz numbers are integers, we see that

$$A = |G^F|^{-1} \sum_{g_0 \in G^F} |T^F|^{-1}|T'^F|^{-1}$$

$$\sum_{\substack{t \in T^F \\ t' \in T'^F}} \mathfrak{L}((g_0, t), L^{-1}(U))\mathfrak{L}((g_0, t'), L^{-1}(U'))\theta(t)\theta'^{-1}(t').$$

Using (1.2.1) and Lemma 1.3, this can be rewritten as

$$A = |T^F|^{-1}|T'^F|^{-1} \sum_{\substack{t \in T^F \\ t' \in T'^F}} \mathfrak{L}((t, t'), G^F \backslash (L^{-1}(U) \times L^{-1}(U')))\theta(t)\theta'^{-1}(t')$$

$$= |T^F|^{-1}|T'^F|^{-1} \sum_{\substack{t \in T^F \\ t' \in T'^F}} \mathfrak{L}((t, t'), \bar{S})\theta(t)\theta'^{-1}(t').$$

We now define a partition of $\bar{S}$ into locally closed subvarieties $\bar{S}_w$ $(w \in W(T, T'))$ as follows.

Using Bruhat's lemma, we see that any element $g \in G$ can be written uniquely in the form $g = u \cdot n \cdot u'$ where $u \in F^{-1}(U) \cap \dot{w}F^{-1}(U'^-)\dot{w}^{-1}$, $n \in T\dot{w}$, $u' \in F^{-1}(U')$, $w \in W(T, T')$, where U'^- is the unipotent radical of the Borel subgroup containing T' and opposed to B'. The set of $g \in G$ corresponding to a given $w \in W(T, T')$ is denoted G_w; it is a locally closed subvariety of G, and we thus have a partition $G = \bigcup_{w \in W(T,T')} G_w$ of G. We then define

$$\bar{S}_w = \{(x, x', y) \in \bar{S} | y \in G_w\}.$$

This is the required partition of $\bar{S}$. Note that each $\bar{S}_w$ is stable under the action of $T^F \times T'^F$. Using now (1.2.2) we see that $A = \Sigma_{w \in W(T,T')} A_w$ where

$$A_w = |T^F|^{-1}|T'^F|^{-1} \sum_{\substack{t \in T^F \\ t' \in T'^F}} \mathfrak{L}((t, t'), \bar{S}_w)\theta(t)\theta'^{-1}(t').$$

Now $\bar{S}_w$ is just the set of all 5-tuples

(2.3.2) $(x, x', u, n, u') \in U \times U' \times (F^{-1}(U) \cap \dot{w}F^{-1}(U'^-)\dot{w}^{-1}) \times T\dot{w} \times F^{-1}(U')$

satisfying the equation

$$xF(u)F(n)F(u') = unu'x'.$$

The action of $T^F \times T'^F$ is given by

$$(x, x', u, n, u') \longrightarrow (txt^{-1}, t'x't'^{-1}, tut^{-1}, tnt'^{-1}, t'u't'^{-1}).$$

We now consider the algebraic group

$$H_w = \{(t, t') \in T \times T' | t'^{-1}F(t') = F(\dot{w})^{-1}t^{-1}F(t)F(\dot{w})\}.$$

As in **[11**, 6.6**]** we define an action of H_w on $\bar{S}_w$. As Deligne pointed out to the author, the formulae given in [loc. cit.] for the action of H_w are more transparent if we first make a change of coordinates $xF(u) \longrightarrow x$, $x'F(u')^{-1} \longrightarrow x'$ in $\bar{S}_w$. Thus, we now view $\bar{S}_w$ as the set of all 5-tuples (x, x', u, n, u') as in (2.3.2) satisfying the equation

(2.3.3) $$xF(n) = unu'x'.$$

(The action of $T^F \times T'^F$ is given by the same formula as in the old coordinates.) In these new coordinates the action of $(t, t') \in H_w$ on $\bar{S}_w$ is given by exactly the same formula as the action of $(t, t') \in T^F \times T'^F$. One checks easily that the equation (2.3.3) is preserved. Thus, we have extended the action of $T^F \times T'^F$ on $\bar{S}_w$ to an action of H_w.

Let H_w^0 be the identity component of H_w. It is a torus.

Suppose that $(x, x', u, n, u') \in \overline{S}_w$ is fixed by all elements of H_w^0. Using Lang's Theorem for T and T', we see that the maps $H_w \longrightarrow T$, $H_w \longrightarrow T'$ given by first and second projection, are surjective. It follows that the restrictions $H_w^0 \longrightarrow T$, $H_w^0 \longrightarrow T'$ are also surjective. Thus, we have $txt^{-1} = x$, $tut^{-1} = u$, for all $t \in T$ and $t'x't'^{-1} = x'$, $t'u't'^{-1} = u$, for all $t' \in T'$.

As $U^T = (U')^{T'} = \{1\}$, where T, T' act by conjugation on U, U', we must have $x = u = x' = u' = 1$. Thus the fixed point set $\overline{S}_w^{H_w^0}$ is empty if $Fw \neq w$ and it is the finite set

$$\{(1, 1, 1, n, 1) | F(n) = n\} \cong (T\dot{w})^F \quad \text{if } Fw = w.$$

By Lemmas 1.6 and 1.10, we have for all $(t, t') \in T^F \times T'^F$, and $w \in W(T, T')^F$:

$$\begin{aligned} \mathfrak{L}((t, t'), \overline{S}_w) &= \mathfrak{L}((t, t'), \overline{S}_w^{H_w^0}) = \mathfrak{L}((t, t'), (T\dot{w})^F) \\ &= \#\{n \in (T\dot{w})^F | tnt'^{-1} = n\}. \end{aligned}$$

It follows that $A_w = 0$ if $Fw \neq w$, while for $Fw = w$, we have

$$\begin{aligned} A_w &= |T^F|^{-1} |T'^F|^{-1} \sum_{t' \in T'^F} |T^F| \, {}^w\theta(t')\theta'^{-1}(t') \\ &= \begin{cases} 1 & \text{if } \theta' = {}^w\theta, \\ 0 & \text{if } \theta' \neq {}^w\theta. \end{cases} \end{aligned}$$

The theorem is proved.

REMARK. The previous proof is much simpler than the corresponding proof in [**11**, 6]. It also has the advantage that it applies equally to the Ree and Suzuki groups even for small q. (This case was left unsettled in [**11**].)

COROLLARY 2.4. *$R_{T,U}^G(\theta)$ is independent of U (hence it can be denoted $R_T^G(\theta)$).*

In [**11**] this was proved without using the orthogonality theorem. It is however simpler to use the following argument (pointed out to the author by Deligne).

If f, f' are two characters of G^F then the equalities $\langle f, f\rangle = \langle f, f'\rangle = \langle f', f'\rangle$ imply $\langle f - f', f - f'\rangle = 0$ hence $f = f'$. We apply this to $f = R_{T,U}^G(\theta)$, $f' = R_{T,U'}^G(\theta)$ where U, U' are the unipotent radicals of the Borel subgroup B, B' containing T. The orthogonality theorem shows that $\langle f, f\rangle = \langle f, f'\rangle = \langle f', f'\rangle$ and the corollary follows.

COROLLARY 2.5. *$\pm R_T^G(\theta)$ is irreducible if and only if θ satisfies the following condition*: $w \in W(T, T')^F$, ${}^w\theta = \theta \Rightarrow w = 1$ (*such θ is said to be in general position*).

Let T be an F-stable maximal torus in G. If $B \supset T$ can be chosen so that $FB = B$, then $R_T^G(\theta)$ is simply the representation $\mathrm{Ind}_{B^F}^{G^F}(\theta)$, where θ is regarded as a character of B^F, trivial on the unipotent radical. This is a special case of the following result.

PROPOSITION 2.6. *Let T be an F-stable maximal torus in G. Assume that T is contained in an F-stable parabolic subgroup $P \subset G$ with unipotent radical U_P. Let M be an*

F-stable Levi subgroup of P. If θ is a character of T^F, then $R_T^G(\theta) = \mathrm{Ind}_{P^F}^{G^F}(R_T^M(\theta))$ where $R_T^M(\theta)$ is regarded as a virtual representation of P^F, trivial on U_P^F. In particular, if $P = B$ is a Borel subgroup, we have $R_T^G(\theta) = \mathrm{Ind}_{B^F}^{G^F}(\theta)$.

Choose a Borel subgroup B such that $T \subset B \subset P$ and let U be its unipotent radical. Let $\mathcal{P}_0$ be the set of G^F-conjugates of P (a finite set). The fibres of the map $L^{-1}(U) \longrightarrow \mathcal{P}_0$ given by $g \longrightarrow gPg^{-1}$ define a partition of $L^{-1}(U)$ into finitely many closed subvarieties. One of these subvarieties is $Z = \{g \in P | g^{-1}Fg \in U\}$. Using Lemma 1.7 we see that $R_T^G(\theta)$ is the virtual character of G^F induced by the virtual character of P^F:

$$g \in P^F \longrightarrow |T^{F^{-1}}| \sum_{t \in T^F} \mathfrak{L}((g, t), Z)\theta(t).$$

But, by Lemma 1.9, applied to the map $Z \longrightarrow \{\bar{g} \in M | \bar{g}^{-1}F(\bar{g}) \in U \cap M\}$ (induced by the natural projection $g \longrightarrow \bar{g}$ of P onto M, with fibres $\approx U_P$) we see that:

$$\mathfrak{L}((g, t), Z) = \mathfrak{L}((\bar{g}, t), \{m \in M | m^{-1}F(m) \in U \cap M\}), \quad g \in P^F, \ t \in T^F,$$

and the proposition is proved.

We now prove

PROPOSITION 2.7. *The identity representation of G^F is equal to*

$$1 = |G^F|^{-1} \sum_T |T^F| R_T^G(1) \tag{2.7.1}$$

where Σ_T means sum over all F-stable maximal tori T in G.

We first note that, if T, U are as in 2.2, then $G^F \backslash L^{-1}(U) \simeq U$. It follows that, for any character θ of T^F, we have

$$\langle R_T^G(\theta), 1 \rangle = |T^F|^{-1} |G^F|^{-1} \sum_{\substack{g \in G^F \\ t \in T^F}} \mathfrak{L}((g, t), L^{-1}(U))\theta(t).$$

Using Lemma 1.3, this equals

$$|T^F|^{-1} \sum_{t \in T^F} \mathfrak{L}(t, G^F \backslash L^{-1}(U))\theta(t) = |T^F|^{-1} \sum_{t \in T^F} \mathfrak{L}(t, U)\theta(t),$$ (Here we use $G^F \backslash L^{-1}(U) \simeq U$)

where t acts on U by conjugation. By Lemma 1.9, $\mathfrak{L}(t, U) = 1$ for all $t \in T^F$ hence

$$\langle R_T^G(\theta), 1 \rangle = \begin{cases} 1 & \text{if } \theta = 1, \\ 0 & \text{if } \theta \neq 1. \end{cases} \tag{2.7.2}$$

Now let f be the character of the right-hand side of (2.7.1). Using (2.3.1), (2.7.2), we see that $\langle f, f \rangle = \langle f, 1 \rangle = \langle 1, 1 \rangle = 1$. (Note that $|G^F|^{-1} \Sigma_T |T^F| = 1$, by the classification of G^F-conjugacy classes of F-stable maximal tori, see [24].) It follows that $\langle f - 1, f - 1 \rangle = 0$ hence $f = 1$ and (2.7.1) is proved.

COROLLARY 2.8. *Let T be an F-stable maximal torus in G and let P be an F-stable parabolic subgroup of G containing no G^F-conjugate of T. Then*

$$\langle R_T^G(\theta), \operatorname{Ind}_{P^F}^{G^F}(1)\rangle = 0$$

for any character θ of T^F.

Let M be an F-stable Levi subgroup of P.

Using Propositions 2.7 and 2.6, we see that

$$\begin{aligned}\operatorname{Ind}_{P^F}^{G^F}(1) &= \operatorname{Ind}_{P^F}^{G^F}\left(|M^F|^{-1} \sum_{T' \text{ in } M} |T'^F| R_{T'}^M(1)\right)\\ &= |M^F|^{-1} \sum_{T' \text{ in } M} |T'^F| R_{T'}^G(1).\end{aligned}$$

By assumption, no T' in this sum is G^F-conjugate to T, hence the result is a consequence of the orthogonality Theorem 2.3.

Let us define $\epsilon_G = (1)^{\sigma(G)}$, where $\sigma(G)$ is the dimension of a maximal $\mathbf{F}_q$-split torus of G. (This depends on the $\mathbf{F}_q$-rational structure of G.)

THEOREM 2.9. *Let T be an F-stable maximal torus in G and let $\theta\colon T^F \longrightarrow \overline{\mathbf{Q}}_l^*$ be a character. Then*

$$\dim R_T^G(\theta) = \epsilon_G \epsilon_T \frac{|G^F|_{p'}}{|T^F|}$$

where, for any integer $m \geqslant 1$, m_p denotes the largest power of p dividing m and $m_{p'} = m/m_p$.

The proof will make use of the Steinberg representation St_G of G^F. We recall (cf. [23]) that this is an irreducible representation of degree $|G^F|_p$ whose character vanishes on nonsemisimple elements of G^F. It can be defined by the identity (due to Curtis):

$$\mathrm{St}_G = \sum \epsilon_{(P/U_P)_{\mathrm{der}}} \operatorname{Ind}_{P^F}^{G^F}(1) \tag{2.9.1}$$

(sum over all F-stable parabolic subgroups P containing a fixed F-stable Borel subgroup) where $(P/U_P)_{\mathrm{der}}$ denotes the derived group of P/U_P.

Assume now that T is not contained in any F-stable parabolic subgroup P of G other than G itself. Then (2.9.1), (2.7.2) and Corollary 2.8 show that

$$\begin{aligned}\langle R_T^G(1), \mathrm{St}_G\rangle &= \epsilon_{G_{\mathrm{der}}} \langle R_T^G(1), 1\rangle\\ &= \epsilon_{G_{\mathrm{der}}}\\ &= \epsilon_G \epsilon_T\end{aligned}$$

and

$$\langle R_T^G(\theta), \mathrm{St}_G\rangle = \epsilon_{G_{\mathrm{der}}}\langle R_T^G(\theta), 1\rangle = 0 \quad \text{if } \theta \neq 1.$$

Hence

$$\left\langle \sum_\theta R_T^G(\theta), \mathrm{St}_G \right\rangle = \epsilon_G\epsilon_T \tag{2.9.2}$$

(sum over all characters θ of T^F). But this sum over θ is just the virtual G^F-module $\Sigma_i(-1)^i H_c^i(L^{-1}(U))$ ($U=$ the unipotent radical of a Borel subgroup containing T), and the character of this virtual G^F-module vanishes on all semisimple elements $\neq 1$ of G^F (use Lemma 1.4 and the fact that G^F acts freely on $L^{-1}(U)$). On the other hand the character of St_G vanishes on all nonsemisimple elements of G^F, hence the left-hand side of (2.9.2) is just

$$|G^F|^{-1} \dim(\mathrm{St}_G) \dim\left(\sum_\theta R_T^G(\theta)\right).$$

We have

$$\begin{aligned} \dim R_T^G(\theta) &= |T^F|^{-1} \sum_{t\in T^F} \mathfrak{L}((1, t), L^{-1}(U))\theta(t) \\ &= |T^F|^{-1}\mathfrak{L}(1, L^{-1}(U)) \end{aligned}$$

(using Lemma 1.4 and the fact that T^F acts freely on $L^{-1}(U)$). In particular $\dim R_T^G(\theta) = \dim R_T^G(1)$ and we see that

$$|G^F|^{-1} \dim(\mathrm{St}_G)|T^F| \dim R_T^G(1) = \epsilon_T\epsilon_G$$

which gives the desired formula.

Next, we assume that T is contained in an F-stable parabolic $P \neq G$. Let $M \supset T$ be a Levi subgroup of P, stable under F. Using Proposition 2.6, we see that

$$\dim R_T^G(\theta) = |G^F||P^F|^{-1} \dim R_T^M(\theta).$$

By induction, we may assume known that

$$\dim R_T^M(\theta) = \epsilon_M\epsilon_T|M^F|_{p'}|T^F|^{-1}.$$

It follows that

$$\dim R_T^G(\theta) = |G^F|_{p'}|M^F|_{p'}\epsilon_M\epsilon_T|M^F|_{p'}|T^F|^{-1} = \epsilon_G\epsilon_T|G^F|_{p'}|T^F|^{-1}$$

since $\epsilon_M = \epsilon_G$.

COROLLARY 2.10. *If θ is in general position, then $\epsilon_G\epsilon_T R_T^G(\theta)$ is irreducible.*

Indeed, it has positive dimension.

COROLLARY 2.11. *The regular representation of G^F is equal to*

$$|G^F|_p^{-1} \sum_T \sum_\theta \epsilon_T \epsilon_G R_T^G(\theta) \tag{2.11.1}$$

(T runs through all F-stable maximal tori in G and θ runs through all characters of T^F).

Using the orthogonality Theorem 2.3, we see that the inner product of the character (2.11.1) with itself is

$$|G^F|_p^{-2} \sum_{T,T'} \sum_{\theta,\theta'} \epsilon_T \epsilon_{T'} |T^F|^{-1} \#\{g \in N(T, T')^F | g \text{ carries } \theta \text{ to } \theta'\}$$

$$= |G^F|_p^{-2} \sum_{T,\theta} \sum_{g \in G^F} |T^F|^{-1}$$

$$= |G^F||G^F|_p^{-2} \#(F\text{-stable maximal tori } T \text{ in } G)$$

$$= |G^F|$$

(the last step being a theorem of Steinberg [**26**, (14.16)]).

Using Theorem 2.3, we see that the inner product of the character (2.11.1) with the regular representation of G^F is

$$|G^F|_p^{-1} \sum_T \sum_\theta \dim(\epsilon_T \epsilon_G R_T^G(\theta)) = |G^F|_p^{-1} \sum_T |G^F|_{p'}$$

$$= |G^F||G^F|_p^{-2} \#(F\text{-stable maximal tori in } G)$$

$$= |G^F|.$$

Finally, the inner product of the regular representation with itself is again $|G^F|$. We then conclude as in the proof of Corollary 2.4 (or as in the proof of (2.7.1)).

COROLLARY 2.12. *For any irreducible representation ρ of G^F, there exist an F-stable maximal torus $T \subset G$ and a character θ of T^F such that $\langle \rho, R_T^G(\theta)\rangle \neq 0$.*

For otherwise, ρ would have inner product zero with the regular representation.

2.13. Let $T \subset G$ be an F-stable maximal torus and let θ be a character of T^F. If $u \in G^F$ is a unipotent element, the number $\mathrm{Tr}(u, R_T^G(\theta))$ is an integer independent of θ. Indeed, using Lemma 1.4 and the fact that the T^F acts freely on $L^{-1}(U)$ (U = unipotent radical for Borel subgroup containing T) we see that this number is equal to $|T^F|^{-1}\mathfrak{L}(u, L^{-1}(U))$. We denote it as $Q_T^G(u)$; Q_T^G is called the Green function corresponding to T. The character of $R_T^G(\theta)$ at an arbitrary element $g \in G^F$ can be expressed in terms of Green functions.

Let us write $g = su = us$ where $s \in G^F$ is semisimple and $u \in G^F$ is unipotent. Then $u \in Z^0(s)$, where $Z^0(s)$ is the connected centralizer of s in G (it is a reductive group defined over $\mathbf{F}_q$). We have the following character formula

$$\mathrm{Tr}(su, R_T^G(\theta)) = |Z^0(s)^F|^{-1} \sum_{\substack{g \in G^F \\ gsg^{-1} \in T}} Q_T^{Z^0(gsg^{-1})}(gug^{-1})\theta(gsg^{-1}). \tag{2.13.1}$$

A proof (using Lemma 1.4) can be found in **[11**, §4**]**.

If we take $g = s$ (so that $u = 1$), we have $Q_T^G(1) = \epsilon_G \epsilon_T |G^F|_{p'} |T^F|^{-1}$ by Theorem 2.9, and (2.13.1) gives an explicit formula for the character of $R_T^G(\theta)$ at g. This is best expressed in the following form:

$$\epsilon_T \epsilon_G R_T^G(\theta) \otimes \mathrm{St}_G = \mathrm{Ind}_{T^F}^{G^F}(\theta)$$

(note that the character of St_G is nonzero precisely at semisimple elements).

REMARK 2.14. Let $\mathcal{H}(T)$ be the set of all connected reductive subgroups H of G, defined over $\mathbf{F}_q$, such that H is the connected centralizer in G of some subset of T. Following Kazhdan **[16]**, we define a virtual representation $K_T^G(\theta)$ of G^F by the recurrent formula

$$R_T^G(\theta) = \sum_H \mathrm{Ind}_{H^F}^{G^F}(K_T^H(\theta)) \tag{2.14.1}$$

(sum over all $H \in \mathcal{H}(T)$). It is clear that giving a character formula for $R_T^G(\theta)$ is formally equivalent to giving a character formula for $K_T^G(\theta)$. In particular, the formula (2.13.1) is formally equivalent with the following statement (compare **[16]**):

$$\mathrm{Tr}(su, K_T^G(\theta)) = \begin{cases} \theta(s)\mathrm{Tr}(u, K_T^G(\theta)) & \text{if } s \text{ is in the centre of } G, \\ 0 & \text{otherwise.} \end{cases} \tag{2.14.2}$$

This can be proved directly as follows. We define a partition of U (U as above) into locally closed subsets $U^{(H)}$ ($H \in \mathcal{H}(T)$) as follows: $U^{(H)}$ is the set of points in $U \cap H$ which are not in H' for any $H' \in \mathcal{H}(T)$, $H' \subsetneqq H$. This gives rise to a partition of $L^{-1}(U)$ into locally closed subsets $L^{-1}(U^{(H)})$ stable under $G^F \times T^F$. This gives a geometrical interpretation of $K_T^G(\theta)$ and of (2.14.1): from this point of view, $K_T^G(\theta)$ appears as the virtual representation of G^F associated to the natural action of $G^F \times T^F$ on $L^{-1}(U^{(G)})$ and to θ (see 2.1). In particular, the character of $K_T^G(\theta)$ at su is given by

$$|T^F|^{-1} \sum_{t \in T^F} \mathfrak{L}((su, t), L^{-1}(U^{(G)}))\theta(t).$$

If s is in the centre of G, (2.14.2) is easy to prove.

Let us assume that s is not in the centre of G. To verify (2.14.2) it is enough, by Lemma 1.4, to show that the fixed point set of (s, t) on $L^{-1}(U^{(G)})$ is empty ($t \in T^F$). Assume that $x \in L^{-1}(U^{(G)})$ and $sx = xt$. Then $u = x^{-1}Fx \in U^{(G)}$ satisfies $t^{-1}ut = u$, so that $u \in Z^0(t) \cap U$. The definition of $U^{(G)}$ shows that t must be in the centre of G. Since $s = xtx^{-1}$, s must also be in the centre of G, a contradiction.

2.15. Just as the formula for the dimension of $R_T^G(\theta)$ leads to an expression of the regular representation of G^F as a $\mathbf{Q}$-linear combination of $R_T^G(\theta)$'s, the formula for the character of $R_T^G(\theta)$ on semisimple elements leads (cf. **[11**, 7.5**]**) to an expression for the characteristic function of a semisimple class (s) in G^F as

$$\mathrm{Tr}(s, \mathrm{St}_G)^{-1} \sum_{\substack{T \\ T \ni s}} \sum_{\theta} \theta(s)^{-1} \epsilon_G \epsilon_T R_T^G(\theta). \tag{2.15.1}$$

(Thus the character of (2.15.1) equals $|Z(s)^F|$ on elements conjugate to s in G^F and is zero on all other elements.)

Kilmoyer has proposed to call *uniform* those class functions on G^F which are linear combinations of the characters of $R_T^G(\theta)$'s. It is by no means true in general that the characteristic function of an arbitrary conjugacy class in G^F is uniform (although it is for the general linear and unitary groups). In general, a class function $f: G^F \longrightarrow \bar{\mathbf{Q}}_l^*$ is uniform if and only if

$$\langle f, f\rangle = \sum_{(T,\theta)} \langle f, R_T^G(\theta)\rangle\langle R_T^G(\theta), f\rangle\langle R_T^G(\theta), R_T^G(\theta)\rangle^{-1} \tag{2.15.2}$$

(sum over all G^F-conjugacy classes of pairs (T, θ)). This follows immediately from the orthogonality Theorem 2.3. We make the following

Conjecture 2.16. *Let C be a conjugacy class of G, stable under F. The function equal to* 1 *on C^F and to* 0 *on $G^F - C^F$ is uniform.*

In the special case where C is a unipotent class, this is equivalent (in terms of (2.15.2)) to the following identity for Green functions:

$$\sum_{(T)} |W(T)^F|^{-1}|T^F|Q_T^G(C)^2 = |G^F| \cdot |C^F|^{-1}$$

(sum over all G^F-conjugacy classes of maximal tori T in G) where $Q_T^G(C) = |C^F|^{-1}\sum_{g\in C^F} Q_T^G(g)$ and $W(T) = W(T, T)$.

More generally, given two unipotent conjugacy classes C, C' in G, stable under F, it is likely that the following identity holds:

$$\sum_{(T)} |W(T)^F|^{-1}|T^F|Q_T^G(C)Q_T^G(C') = \delta_{c'}^{c}|G^F| \cdot |C^F|^{-1} \tag{2.16.1}$$

where $\delta_{c'}^{c}$ equals 1 if $C = C'$ and equals 0 if $C \neq C'$. (This would imply that the conjecture is true and that the number of unipotent classes in G, stable under F, is less than or equal to the number of G^F-conjugacy classes of F-stable maximal tori in G.) Note that the somewhat related identity (see [**11**, 6.9.]):

$$|G^F|^{-1} \sum_{\substack{g\in G^F\\ \text{unipotent}}} Q_T^G(g)Q_{T'}^G(g) = |N(T, T')^F||T^F|^{-1}|T'^F|^{-1}$$

(T, T'-maximal tori in G, stable under F) is always true. It follows easily from Theorem 2.3. It implies that the number of G^F-conjugacy classes of unipotent elements in G^F is greater than or equal to the number of G^F-conjugacy classes of F-stable maximal tori in G.

2.17. Let ρ be an irreducible G^F-module (over $\bar{\mathbf{Q}}_l$). We say that ρ is *cuspidal* (cf. [**23**]), if the following condition is satisfied: for any F-stable parabolic subgroup $P \subsetneqq G$, the restriction of ρ to U_P^F (U_P = unipotent radical of P) does not contain the unit representation of U_P^F. Cuspidal representations can be characterized as follows:

PROPOSITION 2.18. *An irreducible G^F-module ρ is cuspidal if and only if $\langle \rho, R_T^G(\theta)\rangle = 0$ for any F-stable maximal torus T contained in some proper F-stable parabolic subgroup of G and any character θ of T^F.*

Let P be an F-stable parabolic subgroup of G, $P \neq G$, let T be an F-stable maximal torus in P, and let θ be a character of T^F. Assume that $\langle \rho, R_T^G(\theta)\rangle_{G^F} \neq 0$. Using Proposition 2.6 and Frobenius reciprocity, it follows that $\langle \rho, R_T^M(\theta)\rangle_{P^F} \neq 0$, where M is an F-stable Levi subgroup of P containing T. The restriction of ρ to P^F can be decomposed as $\rho = \rho' \oplus \rho''$ where ρ' is a P^F-module on which U_P^F acts trivially and ρ'' is a P^F-module which contains no trivial representation of U_P^F. As $R_T^M(\theta)$ comes from M^F, we have $\langle \rho'', R_T^M(\theta)\rangle_{P^F} = 0$, hence $\langle \rho', R_T^M(\theta)\rangle_{P^F} \neq 0$. In particular, $\rho' \neq 0$ hence ρ is not cuspidal.

Conversely, assume that ρ is not cuspidal. Then we can find a proper F-stable parabolic subgroup P of G such that $\langle \rho, \operatorname{Ind}_{U_P^F}^{G^F}(1)\rangle \neq 0$. Let M be an F-stable Levi subgroup of P, and let $\mathcal{R}_M$ be the regular representation of M^F. It can be computed using Corollary 2.11. Thus

$$\operatorname{Ind}_{U_P^F}^{G^F}(1) = \operatorname{Ind}_{P^F}^{G^F}(\mathcal{R}_M) = \operatorname{Ind}_{P^F}^{G^F}\left(|M_p^F|^{-1} \sum_{T \subset M} \sum_{\theta} \epsilon_T \epsilon_M R_T^M(\theta)\right)$$

$$= |M^F|_p^{-1} \sum_{T \subset M} \sum_{\theta} \epsilon_T \epsilon_M R_T^G(\theta).$$

As $\langle \rho, \operatorname{Ind}_{U_P^F}^{G^F}(1)\rangle \neq 0$, we can find $T \subset M$ and a character θ of T^F such that $\langle \rho, R_T^G(\theta)\rangle \neq 0$. This completes the proof.

Corollary 2.19. *Let T be an F-stable maximal torus of G and let θ be a character of T^F in general position. Then $\epsilon_T \epsilon_G R_T^G(\theta)$ (which is irreducible by Corollary 2.10) is cuspidal if and only if T is not contained in any proper F-stable parabolic subgroup of G.*

2.20. *Example.* Assume that $G = \mathrm{SL}_2(k)$ with its standard $\mathbf{F}_q$-rational structure. Then $G^F = \mathrm{SL}_2(\mathbf{F}_q)$. There is a unique G^F-conjugacy class of F-stable maximal tori in G which are not contained in any F-stable Borel subgroup. If T is such a torus there is an isomorphism $T \xrightarrow{\sim} k^*$ under which the action of F on T becomes the action $\lambda \longrightarrow \lambda^{-q}$ on k^*. Thus T^F is isomorphic to $H = \{\lambda \in k^* | \lambda^{q+1} = 1\}$. Let U be the unipotent radical of a Borel subgroup containing T. The variety $L^{-1}(U)$ with the action of $G^F \times T^F$ (see 2.2) can easily be identified with the following curve, first considered by Drinfeld:

$$\mathcal{C} = \{(x, y) \in k^2 | xy^q - x^q y = 1\},$$

on which $\mathrm{SL}_2(\mathbf{F}_q)$ acts by linear change of coordinates and H acts by homothety. $\mathcal{C}$ is an irreducible affine curve; it is known that this implies that $H_c^0(\mathcal{C}, \mathbf{Q}_l) = 0$ and $H_c^2(\mathcal{C}, \mathbf{Q}_l)$ is one dimensional with H acting trivially on it. It follows that for any nontrivial character $\theta\colon T^F \longrightarrow \overline{\mathbf{Q}}_l^*$ (or, equivalently, $\theta\colon H \longrightarrow \overline{\mathbf{Q}}_l^*$), the virtual $\mathrm{SL}_2(F_q)$-module $\epsilon_T \epsilon_G R_T^G(\theta) = -R_T^G(\theta)$ is an actual representation: it can be realized in the subspace $\mathcal{H}_{\theta^{-1}}$ of $H_c^1(\mathcal{C}, \mathbf{Q}_l) \otimes \overline{\mathbf{Q}}_l$ on which H acts according to θ^{-1}.

Corollary 2.19 shows that $\mathcal{H}_{\theta^{-1}}$ is an irreducible cuspidal $\mathrm{SL}_2(\mathbf{F}_q)$-module provided that θ is in general position, i.e. when $\theta^2 \neq 1$. Assume now that q is odd and that $\theta = \epsilon$ where $\epsilon^2 = 1$, $\epsilon \neq 1$. Then the orthogonality Theorem 2.3 shows that the inner product of the $\mathrm{SL}_2(\mathbf{F}_q)$-module $H_{\epsilon^{-1}} = H_\epsilon$ with itself equals 2. We now show that the decomposition of H_ϵ into two irreducible $\mathrm{SL}_2(\mathbf{F}_q)$-modules is achieved by taking the two eigenspaces of Frobenius F (for the obvious $\mathbf{F}_q$-rational structure of C). For this purpose we consider the projective nonsingular curve

$$\overline{C} = \{(x, y, z): xy^q - x^q y = z^{q+1}\}.$$

It contains C as the open subset defined by $z \neq 0$, and the action of $\mathrm{SL}_2(\mathbf{F}_q) \times H$ on C extends uniquely to $\overline{C}$; H will act trivially on $\overline{C} - C$. It follows that for $\theta \neq 1$, $H_{\theta^{-1}}$ can be identified with the subspace of $H^1(\overline{C}, \mathbf{Q}_l) \otimes \overline{\mathbf{Q}}_l$ on which H acts according to θ^{-1}.

Using Poincaré duality on $\overline{C}$, we see that H_ϵ has a canonical nondegenerate alternating form $(\ ,\ )$: $H_\epsilon \times H_\epsilon \longrightarrow H^2(\overline{C}, \overline{\mathbf{Q}}_l)$ compatible with the action of $\mathrm{SL}_2(\mathbf{F}_q)$ and such that $(Fx, Fy) = q(x, y)$ for $x, y \in H_\epsilon$. One can show that $F^2 = -q$ on $H^1(\overline{C}, \mathbf{Q}_l)$; indeed, over $\mathbf{F}_{q^2}$, the equation of $\overline{C}$ can be rewritten as $x^{q+1} + y^{q+1} + z^{q+1} = 0$, on which the unitary group $U_3(\mathbf{F}_q)$ acts naturally and we may then appeal to Theorem 3.23 for $U_3(\mathbf{F}_q)$ (or we may compute directly the eigenvalues of F^2 in terms of Jacobi sums). It follows that $F^2 = -q$ on H_ϵ. Let H'_ϵ, H''_ϵ be the two eigenspaces of F: $H_\epsilon \otimes \overline{\mathbf{Q}}_l \longrightarrow H_\epsilon \otimes \overline{\mathbf{Q}}_l$, corresponding to the two square roots λ', λ'' of $-q$.

If $x, y \in H'_\epsilon$, we have $q(x, y) = (Fx, Fy) = (\lambda' x, \lambda' y) = \lambda'^2(x, y) = -q(x, y)$ hence $(x, y) = 0$. Similarly, $(x, y) = 0$ if $x, y \in H''_\epsilon$. Thus, H'_ϵ, H''_ϵ are isotropic subspaces of $H_\epsilon \otimes \overline{\mathbf{Q}}_l$ with respect to $(\ ,\ )$. As $H'_\epsilon \oplus H''_\epsilon = H_\epsilon \otimes \overline{\mathbf{Q}}_l$, we see that H'_ϵ, H''_ϵ are maximal isotropic subspaces of $H_\epsilon \otimes \overline{\mathbf{Q}}_l$ so that they both have dimension equal to ½ dim H_ϵ = ½$(q - 1)$.

Thus we have proved that the decomposition of H_ϵ into irreducible $\mathrm{SL}_q(\mathbf{F}_q)$-modules is achieved by taking the two eigenspaces of Frobenius (corresponding to the two square roots of $(-q)$). It is easy to see, using 2.18, that the $\mathrm{SL}_2(\mathbf{F}_q)$-modules H'_ϵ, H''_ϵ are cuspidal. The previous argument shows that H'_ϵ, H''_ϵ are dual to each other. We note that H'_ϵ, H''_ϵ can be also realized in the cohomology of the curve obtained from C by taking the orbits of the group $\{\lambda \in H | \lambda^{(q+1)/2} = 1\}$. The map $(x, y) \longrightarrow (u, z)$ where $z = yx^{-1}$, $u = x^{-(q+1)/2}$, defines an isomorphism of this curve with

$$C' = \{(u, z) \in k^* \times k | u^2 = z^q - z\};$$

$H'_\epsilon \oplus H''_\epsilon$ appears as the (-1) eigenspace of the involution of $H^1_c(C', \mathbf{Q}_l)$ induced by $(u, z) \longrightarrow (-u, z)$.

Part 3. Unipotent Representations

An irreducible representation ρ of G^F (over $\bar{\mathbf{Q}}_l$) is said to be *unipotent* if $\langle \rho, R_T^G(1) \rangle \neq 0$ for some F-stable maximal torus T in G.

The following result is a special case of [**11**, Theorem 6.2]. Its proof is parallel to that of Theorem 2.3, and will not be given here.

THEOREM 3.1. *With the notations of* 2.3, T'^F *acts trivially on* $H_c^i((L^{-1}(U) \times L^{-1}(U'))/T^F, \mathbf{Q}_l)$.

The contragredient ρ' of a unipotent representation ρ of G^F is again unipotent (since the character of $R_T^G(1)$ is integer valued) hence we have the following

COROLLARY 3.2. *An irreducible G^F-submodule of $H_c^i(L^{-1}(U'), \mathbf{Q}_l)$ is unipotent if and only if it is pointwise fixed by T'^F. In particular, if θ' is a nontrivial character of T'^F and ρ is a unipotent representation of G^F, then $\langle \rho, R_{T'}^G(\theta') \rangle = 0$.*

3.3. Let B_0 be a fixed F-stable Borel subgroup of G with unipotent radical U_0 and let T_0 be an F-stable maximal torus of B_0. We denote by W the Weyl group of T_0 and, for each $w \in W$, we choose representatives $\dot{w}$ in the normalizer of T. F acts naturally on W and we shall assume, as we may, that the set of representatives $\{\dot{w} | w \in W\}$ is F-stable. Let S be the set of simple reflections in W corresponding to B_0 and let l be the length function on W defined by S. For each $w \in W$, let $\mathcal{O}_w$ be the set of pairs (B', B'') of Borel subgroups such that, for some $g \in G$, we have $B' = gB_0g^{-1}$, $B'' = g\dot{w}B_0\dot{w}^{-1}g^{-1}$. Then any pair of Borel subgroups lies in a well-defined $\mathcal{O}_w$. We shall also write $(B', B'') \in \mathcal{O}_w$ as $B' \xrightarrow{w} B''$. For each $w \in W$, we define X_w to be the set of all Borel subgroups $B \subset G$ such that $B \xrightarrow{w} F(B)$. This is a locally closed subvariety of the variety of all Borel subgroups, on which G^F acts by conjugation.

We also define

$$U_0^w = U_0 \cap \dot{w}U_0\dot{w}^{-1},$$

$$T_0^w = \{t \in T_0 | F(t) = \dot{w}^{-1}t\dot{w}\}.$$

For each $w \in W$, we choose $x \in G$ such that $x^{-1}F(x) = \dot{w}$ and we define $T_w = xT_0x^{-1}$. This is an F-stable maximal torus.

The map $g \longrightarrow gx$ defines an isomorphism $L^{-1}(F(x)U_0F(x)^{-1}) \xrightarrow{\sim} L^{-1}(\dot{w}U_0)$ commuting with the action of G^F (by left translation) and transforming the action of T_w^F (by right multiplication on $L^{-1}(F(x)U_0F(x)^{-1})$) into the action of T_0^w (by right multiplication on $L^{-1}(\dot{w}U_0)$).

This allows us to identify $R^G_{T_w}(1)$ with the virtual representation of G^F with character $g \longrightarrow \mathfrak{L}(g, L^{-1}(\dot{w}U_0)/T_0^w)$. Next, we note that the map $L^{-1}(\dot{w}U_0)/T_0^w \longrightarrow X_w$ given by $g \longrightarrow gB_0g^{-1}$ has as fibres affine spaces of dimension equal to dim U_0^w (the orbits of the action of U_0^w by right translation). It follows that $R^G_{T_w}(1)$ has character $g \longrightarrow \mathfrak{L}(g, X_w)$. We denote $R^G_{T_w}(1) = R_w$.

It is easy to see that any F-stable maximal torus in G is conjugate under G^F to some T_w and that, for $w, w' \in W$

$$|W(T_w, T_{w'})^F| = \#\{w_1 \in W | w_1 w' = wF(w_1)\}.$$

Thus, we have

PROPOSITION 3.4. *An irreducible representation ρ of G^F is unipotent if and only if $\langle \rho, R_w \rangle \neq 0$ for some $w \in W$, where R_w is the virtual representation of G^F with character $g \longrightarrow \mathfrak{L}(g, X_w)$. We have, for $w, w' \in W$:*

(3.4.1) $$\langle R_w, R_{w'} \rangle = \#\{w_1 \in W | w_1 w' = wF(w_1)\}.$$

Let w, w', w_1 be three elements of W. We consider the variety

$$Y_{w,w',w_1} = G^F \backslash ((X_w \times X_{w'}) \cap O_{w_1}),$$

where G^F acts on pairs of Borel subgroups by conjugating both factors. We also consider the variety Z_{w,w',w_1} consisting of all pairs (B_1, B_2) of Borel subgroups such that

$$B_0 \xrightarrow{w} B_1 \xrightarrow{F(w_1)} B_2 \xrightarrow{w'^{-1}} \dot{w}_1 B_0 \dot{w}_1^{-1}.$$

Let δ be the smallest integer $\delta \geqslant 1$ such that F^δ acts trivially on W. Then F^δ acts naturally on pairs of Borel subgroups, hence on Y_{w,w',w_1} and on Z_{w,w',w_1}. We shall now prove the following

LEMMA 3.5. *There exists a natural isomorphism $H^i_c(Y_{w,w',w_1}) \cong H^i_c(Z_{w,w',w_1})$ for each $i \geqslant 0$, commuting with the action of F^δ.*

The variety

$$Y^1 = \frac{G^F \backslash \{(g, g') \in G \times G | g^{-1}F(g) \in \dot{w}U_0, g'^{-1}F(g') \in \dot{w}'U_0, g^{-1}g' \in B_0 \dot{w}_1 B_0\}}{(T_0^w \times T_0^{w'})}$$

(where G^F acts on $G \times G$ by left multiplication on both factors and $T_0^w \times T_0^{w'}$ acts by right multiplication) maps to Y_{w,w',w_1} by $(g, g') \longrightarrow (gB_0g^{-1}, g'B_0g'^{-1})$ and all its fibres are isomorphic to the affine space of dimension equal to $\dim(U_0^w \times U_0^{w'})$.

The map $(g, g') \longrightarrow (x, x', y)$ defined by $x = \dot{w}^{-1}g^{-1}F(g)$, $x' = \dot{w}'^{-1}g'^{-1}F(g')$, $y = g^{-1}g'$ defines an isomorphism of Y^1 with

$$Y^2 = \{(x, x', y) \in U_0 \times U_0 \times (B_0 \dot{w}_1 B_0) | \dot{w}xF(y) = y\dot{w}'x'\}/(T_0^w \times T_0^{w'})$$

where the action of $(t, t') \in T_0^w \times T_0^{w'}$ is given by

$$(x, x', y) \longrightarrow (F(t)^{-1}xF(t), F(t')^{-1}x'F(t'), t^{-1}yt').$$

We now write y in the form $y = u\dot{w}_1 u'\tau$, $u \in U_0$, $u' \in U_0$, $\tau \in T_0$. This gives a map from the variety

$$Y^3 = \frac{\{(x, x', u, u', \tau) \in U_0 \times U_0 \times U_0 \times U_0 \times T_0 | \dot{w}xF(u)F(\dot{w}_1)F(u')F(\tau) = u\dot{w}_1 u'\tau\dot{w}'x'\}}{(T_0^w \times T_0^{w'})}$$

(where $T_0^w \times T_0^{w'}$ acts by

$$(x, x', u, u', \tau)$$
$$\longrightarrow (F(t)^{-1}xF(t), F(t')^{-1}x'F(t'), t^{-1}ut, (\dot{w}_1^{-1}t^{-1}\dot{w}_1^{-1})u'(\dot{w}_1^{-1}t\dot{w}_1), (\dot{w}_1^{-1}t^{-1}\dot{w}_1\tau t'))$$

to Y^2, with all the fibres isomorphic to the affine space of dimension $\dim(U_0^{w_1})$. We make the change of variable

$$(x, x', u, u', \tau) \longrightarrow (x_1, x_1', u, u', \tau)$$

where $x_1 = xF(u)$, $x_1' = F(\tau)x'F(\tau)^{-1}F(u')^{-1}$. This defines an isomorphism of Y^3 with

$$Y^4 = \frac{\{(x_1, x_1', u, u', \tau) \in U_0 \times U_0 \times U_0 \times U_0 \times T_0 | \dot{w}x_1F(\dot{w}_1) = u\dot{w}_1 u'\tau\dot{w}'F(\tau)^{-1}x_1'\}}{(T_0^w \times T_0^{w'})}$$

where $T_0^w \times T_0^{w'}$ acts by

$$(x_1, x_1', u, u', \tau) \longrightarrow (F(t)^{-1}x_1F(t), F(\dot{w}_1^{-1}t^{-1}\dot{w}_1)x_1'F(\dot{w}_1^{-1}t\dot{w}_1), t^{-1}ut,$$
$$(\dot{w}_1^{-1}t^{-1}\dot{w}_1)u'(\dot{w}_1^{-1}t\dot{w}_1), \dot{w}_1^{-1}t^{-1}\dot{w}_1\tau t').$$

By Lang's Theorem, the map $T_0 \longrightarrow T_0$ given by $\tau \longrightarrow \tilde{\tau} = \dot{w}'^{-1}\tau\dot{w}'F(\tau)^{-1}$ defines an isomorphism $T_0/T_0^{w'} \xrightarrow{\sim} T_0$. This gives an isomorphism of Y^4 with the variety

$$Y^5 = \frac{\{(x_1, x_1', u, u', \tilde{\tau}) \in U_0 \times U_0 \times U_0 \times U_0 \times T_0 | \dot{w}x_1F(\dot{w}_1) = u\dot{w}_1u'\dot{w}'\tilde{\tau}x_1'\}}{T_0^w}$$

where $t \in T_0^w$ acts by

$$(x_1, x_1', u, u', \tilde{\tau}) \longrightarrow (F(t)^{-1}x_1F(t), F(\dot{w}_1^{-1}t^{-1}\dot{w}_1)x_1'F(\dot{w}_1^{-1}t\dot{w}_1), t^{-1}ut,$$
$$(\dot{w}_1^{-1}t^{-1}\dot{w}_1)u'(\dot{w}_1^{-1}t\dot{w}_1), \dot{w}'^{-1}\dot{w}_1^{-1}t^{-1}\dot{w}_1\dot{w}'\tilde{\tau}F(\dot{w}_1)^{-1}F(t)F(\dot{w}_1)).$$

Setting $\tilde{\tau}x_1' = b_1$ we get an isomorphism of Y^5 with the variety Y^6/T_0^w where

$$Y^6 = \{(x_1, b_1, u, u') \in U_0 \times B_0 \times U_0 \times U_0 | \dot{w}x_1F(\dot{w}_1) = u\dot{w}_1u'\dot{w}'b_1\}$$

and $t \in T_0^w$ acts by

$$(x_1, b_1, u, u') \longrightarrow (F(t)^{-1}x_1F(t), \dot{w}'^{-1}\dot{w}_1^{-1}t^{-1}\dot{w}_1\dot{w}'b_1F(\dot{w}_1^{-1}t\dot{w}_1),$$
$$t^{-1}ut, (\dot{w}_1^{-1}t^{-1}\dot{w}_1)u'(\dot{w}_1^{-1}t\dot{w}_1)).$$

The action of $t \in T_0^w$ can be also written as

$$(x_1, b_1, u, u') \longrightarrow (\dot{w}^{-1}t^{-1}\dot{w}x_1\dot{w}^{-1}t\dot{w}, \dot{w}'^{-1}\dot{w}_1^{-1}t^{-1}\dot{w}_1\dot{w}'b_1F(\dot{w}_1)^{-1}\dot{w}^{-1}t\dot{w}F(\dot{w}_1),$$
$$t^{-1}ut, (\dot{w}_1^{-1}t^{-1}\dot{w}_1)u'(\dot{w}_1^{-1}t\dot{w}_1)).$$

But this formula defines also an action of the connected group T_0 on Y^6. It follows that T_0^w acts trivially on $H_c^i(Y^6, \mathbf{Q}_l)$ hence we have

$$H_c^i(Y^6/T_0^w, \mathbf{Q}_l) \xrightarrow{\sim} H_c^i(Y^6, \mathbf{Q}_l) \quad \text{for all } i.$$

Finally, the map $Y^6 \longrightarrow Z_{w,w',w_1}$ defined by $(x_1, b_1, u, u') \longrightarrow (B_1, B_2)$ where $B_1 = u^{-1}\dot{w}B_0\dot{w}^{-1}u$, $B_2 = \dot{w}_1u'\dot{w}'B_0\dot{w}'^{-1}u'^{-1}\dot{w}_1^{-1}$ is clearly well defined and all its fibres are isomorphic to the affine space of dimension equal to $\dim(U_0^w \times U_0^{w'} \times U_0^{w_1})$.

Now the lemma follows from consideration of the maps

$$Y_{w,w',w_1} \longleftarrow Y^1 \cong Y^2 \longleftarrow Y^3 \cong Y^4 \cong Y^5 \cong Y^6/T_0^w \longleftarrow Y^6 \longrightarrow Z_{w,w',w_1}.$$

3.6. Before stating our next lemma, we consider an abstract Weyl group $\mathcal{W}$ with simple reflections $\mathcal{S}$ and a function λ: $\mathcal{S} \longrightarrow \overline{\mathbf{Q}}_l$ such that $\lambda(s) = \lambda(s')$ whenever s, s' are conjugate in $\mathcal{W}$. Let $\mathcal{H}(\mathcal{W}, \lambda)$ be the abstract algebra over $\overline{\mathbf{Q}}_l$ with basis t_w ($w \in \mathcal{W}$) and multiplication defined by

$$t_wt_{w'} = t_{ww'} \quad \text{if } w, w' \in \mathcal{W}, l(ww') = l(w) + l(w'),$$
$$(t_s + 1)(t_s - \lambda(s)) = 0 \quad \text{if } s \in \mathcal{S}.$$

(Here l is the length function on $\mathcal{W}$ with respect to $\mathcal{S}$.) In the case where λ is a constant function we shall identify λ with its constant value. We have:

LEMMA 3.7. *Let $\underline{s} = (s_1, s_2, \ldots, s_n)$ be a sequence of simple reflections in W and let $w_1 \in W$. The eigenvalues of F^δ on the l-adic cohomology with compact supports of the variety $Z(\underline{s}, w_1)$ consisting of all sequences $(B_1, B_2, \ldots, B_n)$ of Borel subgroups in G such that*

$$B_0 \xrightarrow{s_1} B_1 \xrightarrow{s_2} B_2 \cdots \xrightarrow{s_n} B_n = \dot{w}_1B_0\dot{w}_1^{-1}$$

are integral powers of q^δ. The number $N(\underline{s}, w_1, q^{\delta e})$ of rational points of this variety over the finite field with $q^{\delta e}$ elements ($e = 1, 2, 3, \ldots$) is given by the following formula in the algebra $\mathcal{H}(W, q^{\delta e})$:

$$t_{s_1}t_{s_2}\cdots t_{s_n} = \sum_{w_1 \in W} N(\underline{s}, w_1, q^{\delta e})t_{w_1}.$$

When $l(s_1 s_2 \cdots s_n) = n$, the variety $Z(\underline{s}, w_1)$ consists of one point if $s_1 s_2 \cdots s_n = w_1$ and is empty if $s_1 s_2 \cdots s_n \neq w_1$, so that the result is clear in this case. Thus, we may assume that there exists $1 \leqslant i \leqslant n-1$ such that $l(s_1 s_2 \cdots s_i) = i$ and $l(s_1 s_2 \cdots s_i s_{i+1}) = i-1$. Let $s'_1, \ldots, s'_{i-1}$ be simple reflections such that $s_1 s_2 \cdots s_i = s'_1 s'_2 \cdots s'_{i-1} s_{i+1}$.

Let

$$\underline{s}' = (s'_1, s'_2, \ldots, s'_{i-1}, s_{i+1}, s_{i+2}, \ldots, s_n),$$

$$\underline{s}'' = (s'_1, s'_2, \ldots, s'_{i-1}, s_{i+2}, \ldots, s_n).$$

If $(B_1, B_2, \ldots, B_n)$, there is a unique sequence $B'_1, B'_2, \ldots, B'_{i-1}$ such that

$$B_0 \xrightarrow{s'_1} B'_1 \xrightarrow{s'_2} B'_2 \longrightarrow \cdots \xrightarrow{s_{i-1}} B'_{i-1} \xrightarrow{s_{i+1}} B_i.$$

If $B'_{i-1} \neq B_{i+1}$, then $(B'_1, B'_2, \ldots, B'_{i-1}, B_{i+1}, \ldots, B_n) \in Z(\underline{s}', w_1)$.

If $B'_{i-1} = B_{i+1}$, then $(B'_1, B'_2, \ldots, B'_{i-1}, B_{i+2}, \ldots, B_n) \in Z(\underline{s}'', w_1)$.

Thus we have a partition of $Z(\underline{s}, w_1)$ into two pieces: one open, which is the complement of the zero section of a line bundle over $Z(\underline{s}', w_1)$ and one closed which is a line bundle over $Z(\underline{s}'', w_1)$. We may assume that the lemma is true when $\underline{s}$ is replaced by $\underline{s}'$ or $\underline{s}''$. The statement on the eigenvalues of F^δ follows. We also see that

$$N(\underline{s}, w_1, q^{\delta e}) = N(\underline{s}', w_1, q^{\delta e})(q^{\delta e} - 1) + N(\underline{s}'', w_1, q^{\delta e})q^{\delta e}$$

from which the statement on $N(\underline{s}, w_1, q^{\delta e})$ follows.

We can now state the following:

THEOREM 3.8. *Let $w, w' \in W$. Consider the variety $G^F \backslash (X_w \times X_{w'})$ (where G^F acts by conjugation on both factors). F^δ acts naturally on this variety. All eigenvalues of F^δ on $H^i_c(G^F \backslash (X_w \times X_{w'}))$ are integral powers of q^δ. The number of rational points of $G^F \backslash (X_w \times X_{w'})$ over the finite field with $q^{\delta e}$ elements is equal to the trace of the linear transformation $x \longrightarrow t_w \cdot F(x) \cdot t_{w'^{-1}}$ of $\mathcal{H}(W, q^{\delta e})$ (where $x \longrightarrow F(x)$ is the linear transformation of $\mathcal{H}(W, q^{\delta e})$ given on the basis elements by $t_w \longrightarrow t_{F(w)}$).*

Consider the partition of $G^F \backslash (X_w \times X_{w'})$ given by the locally closed subvarieties Y_{x, w', w_1}. Using Lemmas 3.5 and 3.7 we see that the eigenvalues of F^δ on $H^i_c(Y_{w, w', w_1}, \mathbf{Q}_l)$ are integral powers of q^δ and that the number of rational points of Y_{w, w', w_1} over the finite field with $q^{\delta e}$ elements is the coefficient of t_{w_1} in the product $t_w t_{F(w_1)} t_{w'^{-1}}$ computed in $\mathcal{H}(W, q^{\delta e})$. The theorem follows.

COROLLARY 3.9. *Let ρ be an irreducible representation of G^F which is unipotent. Then, clearly, there exist $w \in W$, $i \geqslant 0$ and $\mu \in \overline{\mathbf{Q}}_l^*$ such that ρ is isomorphic to a G^F-submodule of $H^i_c(X_w, \mathbf{Q}_l)_\mu$ (the generalized μ-eigenspace of F^δ on $H^i_c(X_w, \mathbf{Q}_l) \otimes \overline{\mathbf{Q}}_l$). Assume that ρ is also isomorphic to a G^F-submodule of $H^{i'}_c(X_{w'}, \mathbf{Q}_l)_{\mu'}$. Then $\mu' \cdot \mu^{-1}$ is an integral power of q^δ. Thus, μ is uniquely determined by ρ up to a factor which is an integral power of q^δ.*

Let ρ' be the representation of G^F contragredient to ρ. As we have already noted, ρ' is again a unipotent representation. Hence there exist $w'' \in W$, $j \geqslant 0$ and $\mu'' \in \overline{\mathbf{Q}}_l^*$ such that ρ' is isomorphic to a G^F-submodule of $H_c^j(X_{w''}, \mathbf{Q}_l)_{\mu''}$. Then the G^F module $\rho \otimes \rho'$ is isomorphic to a G^F-submodule of $H_c^{i+j}(X_w \times X_{w''}, \mathbf{Q}_l)_{\mu\mu''}$. It follows that $\mu\mu''$ is an eigenvalue of F^δ on $H_c^{i+j}(G^F \backslash (X_w \times X_{w''}))$ so that $\mu\mu''$ is an integral power of q^δ. Similarly, $\mu'\mu''$ is an integral power of q^δ and the corollary follows.

3.10. *Examples.* (a) The variety X_1 is just the (finite) set of F_q-rational Borel subgroups of G. Clearly, F acts as the identity on its cohomology. Thus, all irreducible representations ρ of G^F such that $\langle \rho, \operatorname{Ind}_{B_0^F}^{G^F}(1) \rangle \neq 0$ are unipotent and, whenever such a representation occurs as a G^F-submodule of $H_c^i(X_w, \mathbf{Q}_l)_\mu$ (with the previous notation), μ must be an integral power of q^δ.

Applying Theorem 3.8 with $w' = 1$ and noting that $G^F \backslash (X_w \times X_1) = B_0^F \backslash X_w$, we see that the number of rational points of $B_0^F \backslash X_w$ over the field with $q^{\delta e}$ elements is equal to the trace of the linear transformation $x \longrightarrow t_w \cdot F(x)$ of $H(W, q^{\delta e})$. The same formula gives the number of rational points of $U_0^F \backslash X_w$. (Indeed, by Theorem 3.1, $U_0^F \backslash X_w \longrightarrow B_0^F \backslash X_w$ induces isomorphism in cohomology.)

If we assume that F acts trivially on W, we can identify the algebra of G^F endomorphisms of $\operatorname{Ind}_{B_0^F}^{G^F}(1)$ with the algebra $H(W, q)$, as in [3, Chapter 4], [9]. We see that, in this case, the irreducible representations of G^F occurring in $\operatorname{Ind}_{B_0^F}^{G^F}(1)$ are in 1-1 correspondence with the irreducible representations of $H(W, q)$.

(b) We shall denote by w_0 the longest element in W. Assume that $F(w) = w_0 w w_0^{-1}$ for each $w \in W$, so that $\delta = 1$ or 2. Using the definition of $H(W, q^{\delta e})$, we see easily that $t_{w_0} t_s = t_{F(s)} t_{w_0}$ for any simple reflection s, hence $t_{w_0} t_w = t_{F(w)} t_{w_0}$ for each $w \in W$. (It follows that $t_{w_0}^2$ is in the centre of $H(W, q^{\delta e})$.) Applying Theorem 3.8, we see that the number of rational points of $G^F \backslash (X_{w_0} \times X_{w_0})$ over the finite field with $q^{\delta e}$ elements is the trace of left multiplication by $t_{w_0}^2$ on $H(W, q^{\delta e})$.

One can prove that *all eigenvalues of F^2 on $H_c^i(X_{w_0}, \mathbf{Q}_l)$ are integral powers of* $(-q)$. Indeed, from the results of Benson and Curtis [1], [8] one can deduce easily that $t_{w_0}^2$ acts as an integral power of $(-q)$ times identity on any irreducible representation of $H(W, -q)$. Therefore, it is enough to construct a representation of $H(W, -q)$ on $H_c^i(X_{w_0}, \mathbf{Q}_l)$ such that $t_{w_0}^2$ acts as F^2. Let us define, for each $w \in W$, a map $t_w \colon X_{w_0} \longrightarrow X_{w_0}$ by the requirement that

$$B \xrightarrow{w} t_w(B) \xrightarrow{w^{-1} w_0} F(B), \qquad B \in X_{w_0}.$$

Since $l(w) + l(w^{-1} w_0) = l(w_0)$, $t_w(B)$ is well defined. Since $l(w^{-1} w_0) + l(F(w)) = l(w_0)$ and $w^{-1} w_0 F(w) = w_0$, we have $t_w(B) \in X_{w_0}$. One checks easily that $t_w t_{w'} = t_{w'w}$ (as maps of X_{w_0} into itself) if $l(w) + l(w') = l(ww')$. It is obvious that t_{w_0} acts on X_{w_0} as the Frobenius map. The maps $(-1)^{l(w)} t_w^*$ of $H_c^i(X_{w_0}, \mathbf{Q}_l)$ into itself will give the required representation of $H(W, -q)$ provided that we verify the identity $((-t_s^*)+1)((-t_s^*)+q)=0$, for any simple reflection s in W. Let P_s be the class of parabolic subgroups of G determined by s, and let P_s' be the open subset of P_s consisting of those $P \in P_s$ which are opposed to $F(P)$. If we associate to each $B \in X_{w_0}$ the unique $P \in P_s$ containing B, we get a map $X_{w_0} \longrightarrow P_s'$. This map is a fibre bundle with one-dimensional fibres and one verifies easily that $t_s\colon X_{w_0} \longrightarrow X_{w_0}$ maps each fibre into itself. Moreover each fibre can be identified with the projective line from which the $(q+1)$ rational points over F_q have been removed; the map t_s becomes simply the restriction of the Frobenius map on the projective line. It is then enough to observe that for this one dimensional variety, Frobenius acts trivially on H_c^1 and a multiplication by q on H_c^2; we have $H_c^i = 0$ for $i \neq 1$.

It is likely that the representation of $H(W, -q)$ on $\bigoplus_i H_c^i(X_{w_0}, \mathbf{Q}_l)$ which we have just defined here is the full endomorphism algebra of this G^F-module. (An equivalent statement would be that the G^F-modules $H_c^i(X_{w_0}, \mathbf{Q}_l)$, $H_c^j(X_{w_0}, \mathbf{Q}_l)$ are disjoint whenever $i \not\equiv j$ (mod 2).) It is also likely that $\mathrm{Trace}(t_w^*, \Sigma(-1)^i H_c^i(X_{w_0}, \mathbf{Q}_l)) = 0$ for all $w \neq 1$ in W. (Note that the fixed point set of $t_w\colon X_{w_0} \longrightarrow X_{w_0}$ is empty for $w \neq 1$.) If these statements were true, it would follow that the irreducible representations ρ of G^F such that $\langle \rho, R_{w_0}\rangle \neq 0$ would be in 1-1 correspondence with the irreducible representation of $H(W, -q)$ and that the dimension of such ρ would be formally computable from the structure of $H(W, -q)$.

(c) Assume that q is greater than the Coxeter number of G. Let ρ be an irreducible cuspidal G^F-submodule of $H_c^i(X_w, \mathbf{Q}_l)_\mu$ and let r be the semisimple $\mathbf{F}_q$-rank of G. Then *all complex conjugates of* μ *(which is known to be an algebraic number) have absolute value of the form* $q^{\delta m/2}$ *where* m *is an integer congruent to* r *modulo* 2. To prove this statement, let us define, for any sequence $\underline{s} = (s_1, s_2, \ldots, s_n)$ of simple reflections in W, a variety $X_{\underline{s}}$, consisting of all sequences $B_0, B_1, \ldots, B_n$ of Borel subgroups in G such that $B_0 \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} B_n$ and $B_n = F(B_0)$. Let $\underline{s}$ be a sequence with minimum possible n, such that ρ or ρ' is isomorphic to an irreducible G^F-submodule of $H_i^j(X_{\underline{s}}, \mathbf{Q}_l)$ for some j. Assume that there exists an integer i, $1 \leqslant i \leqslant n-1$, such that $l(s_1 s_2 \cdots s_i) = i$ and $l(s_1 s_2 \cdots s_i s_{i+1}) = i-1$. Let $s_1', \ldots, s_{i-1}'$ be simple reflections such that $s_1 s_2 \cdots s_i = s_1' s_2' \cdots s_{i-1}' s_{i+1}$ and let

$$\underline{s}' = (s_1', s_2', \ldots, s_{i-1}', s_{i+1}, s_{i+2}, \ldots, s_n),$$

$$\underline{s}'' = (s_1', s_2', \ldots, s_{i-1}', s_{i+2}, \ldots, s_n).$$

If $(B_0, B_1, \ldots, B_n) \in X_{\underline{s}}$, there is a unique sequence $B_1', B_2', \ldots, B_{i-1}'$ such that

$$B_0 \xrightarrow{s_1'} B_1' \xrightarrow{s_2'} B_2' \longrightarrow \cdots \xrightarrow{s_{i-1}'} B_{i-1}' \xrightarrow{s_{i+1}} B_i.$$

If $B_{i-1}' \neq B_{i+1}$, then $(B_0, B_1', B_2', \ldots, B_{i-1}', B_{i+1}, B_{i+2}, \ldots, B_n) \in X_{\underline{s}'}$; if $B_{i-1}' = B_{i+1}$ then $(B_0, B_1', B_2', \ldots, B_{i-1}', B_{i+2}, \ldots, B_n) \in X_{\underline{s}''}$. Thus, as in the proof of Lemma 3.7, we have a partition of $X_{\underline{s}}$ into two pieces: one open, which is the complement of the zero section of a line bundle over $X_{\underline{s}'}$ and one closed, which is a line bundle over $X_{\underline{s}''}$. This partition is G^F-equivariant. It follows that ρ or ρ' is isomorphic to an irreducible G^F-submodule of $H_c^{j'}(X_{\underline{s}'}, \mathbf{Q}_l)$ or of $H_c^{j''}(X_{\underline{s}''}, \mathbf{Q}_l)$ (for some j' or j'').

This contradicts the minimality of n. Thus, we see that $l(s_1 s_2 \cdots s_n) = n$, so that $X_{\underline{s}} = X_{w'}$, where $w' = s_1 s_2 \cdots s_n$. Let $\overline{X}_{w'}$ be the variety consisting of all sequences $(B_0, B_1, \ldots, B_n)$ of Borel subgroups such that $B_{i-1} \xrightarrow{s_i} B_i$ or $B_{i-1} = B_i$ (for $1 \leqslant i \leqslant n$) and such that $B_n = FB_0$. This is a projective, nonsingular variety, of pure dimension $l(w')$ (cf. [**11**, 9.11]). We may regard $X_{w'}$ as the open subset of $\overline{X}_{w'}$ defined by $B_0 \neq B_1 \neq \cdots \neq B_n$. The complement $\overline{X}_{w'} - X_{w'}$ has a natural (G^F-equivariant) partition into locally closed subvarieties of the form $X_{\underline{\tilde{s}}}$ where $\underline{\tilde{s}}$ runs through the subsequences of $\underline{s}$ other than $\underline{s}$ itself. Our choice of $\underline{s}$ shows that the cohomology with compact support of $\overline{X}_{w'} - X_{w'}$ contains no G^F-submodule isomorphic to ρ or ρ', and hence the natural map $H_c^i(X_{w'}, \mathbf{Q}_l) \longrightarrow H^i(\overline{X}_{w'}, \mathbf{Q}_l)$ induces isomorphism on the ρ-isotypic components and on the ρ'-isotypic components. By Poincaré duality, the ρ-isotypic component of $H^i(\overline{X}_{w'}, \mathbf{Q}_l)$ has the same dimension as the ρ'-isotypic component of $H^{2d-i}(\overline{X}_{w'}, \mathbf{Q}_l)$, where $d = \dim \overline{X}_{w'} = l(w')$.

It follows that the ρ-isotypic component of $H_c^i(X_{w'}, \mathbf{Q}_l)$ has the same dimension as the ρ'-isotypic component of $H_c^{2d-i}(X_{w'}, \mathbf{Q}_l)$. Using now our assumption on q, we see that $X_{w'}$ is affine (cf. [**11**, (9.7)]) hence $H_c^i(X_{w'}, \mathbf{Q}_l) = 0$ for $i < d$. It follows that both the ρ-isotypic component and the ρ'-isotypic component of $H_c^i(X_{w'}, \mathbf{Q}_l)$ are zero if $i \neq d$ and have the same dimension ($\neq 0$) if $i = d$. In particular, the exists μ' such that ρ is isomorphic to a G^F-submodule of $H_c^d(X_{w'}, \mathbf{Q}_l)_{\mu'}$ and to a G^F-submodule of $H_c^d(\overline{X}_{w'}, \mathbf{Q}_l)_{\mu'}$. Using Deligne's theorem [**10**] on the absolute values of eigenvalues of Frobenius (for $\overline{X}_w$), we see that all the complex conjugates of μ' have absolute value $q^{\delta d/2}$. Using Corollary 3.9, we see that $\mu' \cdot \mu^{-1}$ is an integral power of q^δ. It remains to prove that $l(w') \equiv r \pmod 2$. As ρ occurs in $H_c^i(X_{w'}, \mathbf{Q}_l)$ for exactly one value of i ($i = l(w')$), we see that $\langle \rho, R_{w'} \rangle \neq 0$. Using the assumption that ρ is cuspidal and Proposition 2.18, we deduce that the maximal torus $T_{w'}$ is not contained in any proper F-stable parabolic subgroup of G. It is well known that this implies that $l(w') \equiv r \pmod 2$.

(The proof above should be compared with the proof of Corollary (9.9) in [**11**] and with that of Proposition (4.3) in [**17**].)

In the case where ρ is known to be self-dual ($\rho \approx \rho'$), the above argument can be improved as follows. First, note that μ' above is unique, for another choice of μ' has the same absolute value as μ' and differs from μ' only by an integral power of q^δ, hence it is equal to μ'. Thus the ρ-isotypic component of $H^d(\overline{X}_{w'}, \mathbf{Q}_l)$ is contained in $H^d(\overline{X}_{w'}, \mathbf{Q}_l)_{\mu'}$. Poincaré duality gives a nondegenerate bilinear form on $H^d(\overline{X}_{w'}, \mathbf{Q}_l)$ with values in $H^{2d}(\overline{X}_{w'}, \mathbf{Q}_l)$ (which is 1-dimensional, by (d) below), compatible with F^δ. As $F^\delta = q^{d\delta}$ on $H^{2d}(\overline{X}_{w'}, \mathbf{Q}_l)$, it follows that $\mu'^2 = q^{d\delta}$, hence $\mu' = \pm q^{d\delta/2}$.

Thus, when ρ is self-dual, we have that $\mu = \pm q^{\delta m/2}$ where m is an integer congruent to r modulo 2.

(d) Let $w \in W$ be an element with the following property: $w = s_1 s_2 \cdots s_k$, $k = l(w)$, where s_i are simple reflections such that any simple reflection in W is in the F-orbit of s_i for some i, $1 \leqslant i \leqslant k$. We shall prove that for such w, *the variety X_w is irreducible.* (The converse is also true and easy to prove.) As a first step, we shall prove that the variety $U_0^F \backslash X_w$ is irreducible. We have seen in (a) that the number of rational points of $U_0^F \backslash X_w$ over the finite field $\mathbf{F}_{q^{\delta e}}$ is equal to the trace of the linear transformation $x \longrightarrow t_w F(x)$ of $H(W, q^{\delta e})$ into itself. But it is a purely algebraic fact (which is easily checked) that this trace is a polynomial in $q^{\delta e}$ with integral coefficients (independent of $q^{\delta e}$) of degree $l(w)$ and which is *monic* (here we use our assumption on w). As all components of $U_0^F \backslash X_w$ are known to have the same dimension ($= l(w)$), it follows that there is only one component. But the argument in the proof of [**17**, Proposition (4.8)] shows that X_w is irreducible if and only if $U_0^F \backslash X_w$ is irreducible, and our claim follows.

3.11. We shall now prove some general results on the dimension of unipotent representations. Let $Y_0 = \mathrm{Hom}(k^*, T_0)$ be the lattice of one parameter subgroups of T_0. We let F act on Y_0 by the formula

$$F(h(x)) = F(h)(x^q), \quad x \in k^*, h\colon k^* \longrightarrow T_0.$$

The Weyl group W acts also on Y_0, by $w(h(x)) = w(h)(x)$. We have $F(wh) = F(w)F(h)$ for $w \in W$, $h \in Y_0$. The order of the finite group $|T_w^F|$, $w \in W$, is given by the following formula:

$$|T_w^F| = \epsilon_{T_w} \det(1 - qF \circ w)$$

where $F \circ w$ is regarded as an automorphism of finite order of the free abelian group Y_0. It follows that

$$\text{(3.11.1)} \qquad \dim(R_w) = \epsilon_{T_w}\epsilon_G \frac{|G^F|_{p'}}{|T_w^F|} = \epsilon_G |G^F|_{p'} \det(1 - qF \circ w)^{-1}.$$

Let us denote by $(G^F)^\vee_u$ the set of (isomorphism classes of) unipotent representations of G^F (over $\overline{\mathbf{Q}}_l$).

PROPOSITION 3.12. *We have*

$$\text{(3.12.1)} \qquad \sum_\rho \dim(\rho)\rho = |W|^{-1} \sum_{w \in W} \dim(R_w) \cdot R_w$$

(*where ρ runs over the set $(G^F)^\vee_u$*).

We can rewrite the formula for the regular representations of G^F, given by Corollary 2.11, as follows:

$$\sum_\rho \dim(\rho)\rho - |G^F|_p^{-1} \sum_T \epsilon_T\epsilon_G R_T^G(1) = -\left(\sum_{\rho'} \dim(\rho')\rho' - |G^F|_p^{-1} \sum_T \sum_{\theta \neq 1} \epsilon_T\epsilon_G R_T^G(\theta)\right)$$

where ρ runs through the set $(G^F)^\vee_u$ and ρ' runs through the nonunipotent irreducible representations of G^F. Using Corollary 3.2, we see that the left-hand side of the previous equality is orthogonal to the right-hand side. It follows that both sides are zero. Thus, we have

$$\sum_\rho \dim(\rho)\rho = |G^F|_p^{-1} \sum_T \epsilon_T\epsilon_G R_T^G(1)$$

and it remains to use the classification of F-stable maximal tori in G in terms of elements of W.

Taking the dimensions of both sides in (3.12.1), and using (3.11.1), we find

COROLLARY 3.13.

$$\sum_{\rho} \dim(\rho)^2 = |W|^{-1} \sum_{w \in W} |G^F|^2_{p'} \cdot \det(1 - qF \circ w)^{-2} \tag{3.13.1}$$

(ρ runs over $(G^F)^{\vee}_u$).

Taking inner product of both sides in (3.12.1) with ρ we find

COROLLARY 3.14. *If $\rho \in (G^F)^{\vee}_u$, then*

$$\dim(\rho) = |W|^{-1} \sum_{w \in W} \langle \rho, R_w \rangle \epsilon_G |G^F|_{p'} \det(1 - qF \circ w)^{-1}.$$

Note also that:

(3.14.1) *The integer $\langle \rho, R_w \rangle$ satisfies $-|W|^{1/2} \leqslant \langle \rho, R_w \rangle \leqslant |W|^{1/2}$.*

Indeed, $\langle \rho, R_w \rangle^2 \leqslant \langle R_w, R_w \rangle \leqslant |W|$.

The study of unipotent representations of G^F can be reduced to the case where G is adjoint. Indeed, let G^{ad} be the adjoint group of G. The following result is proved in [**11**, 7.10].

PROPOSITION 3.15. *The restriction to G^F of a unipotent representation of $(G^{\mathrm{ad}})^F$ is irreducible. This defines a bijection $((G^{\mathrm{ad}})^F)^{\vee}_u \xrightarrow{\sim} (G^F)^{\vee}_u$.*

We can further reduce the general case to the case where G is adjoint, simple. Indeed, if G is a product $G_1 \times G_2$ where G_i are F-stable reductive subgroups of G then the map $(\rho_1, \rho_2) \longrightarrow \rho_1 \otimes \rho_2$ defines a bijection $(G_1^F)^{\vee}_u \times (G_2^F)^{\vee}_u \xrightarrow{\sim} (G^F)^{\vee}_u$. Moreover, if $G = G_1 \times G_2 \times \cdots \times G_m$ where G_i are reductive subgroups of G such that $FG_i = G_{i+1}$ $(1 \leqslant i \leqslant m-1)$ and $FG_m = G_1$, then $G^F \cong G_1^{F^m}$ and this gives a bijection $(G^F)^{\vee}_u \xrightarrow{\sim} (G_1^{F^m})^{\vee}_u$. (Compare [**17**, 1.18].)

From now on, G will be assumed to be adjoint, simple. (We shall also regard the group with one element as being simple.)

3.16. Let S be the algebra of polynomial functions $Y_0 \otimes \mathbf{Q} \longrightarrow \mathbf{Q}$. The Weyl group W acts naturally on S; we denote by I the ideal of S generated by W-invariant polynomials which vanish at 0. Then the algebra $\overline{S} = S/I$ is finite dimensional; it inherits a grading from S: $\overline{S} = \Sigma_{i=0}^{\nu} \overline{S}_i$ (where ν is the number of positive roots of G) and an action of W preserving this grading. The map F: $Y_0 \longrightarrow Y_0$ induces an algebra homomorphism F: $\overline{S} \longrightarrow \overline{S}$ (of finite order) preserving the grading. We have, for all $w \in W$:

$$\epsilon_G |G^F|_{p'} \cdot \det(1 - qF \circ w)^{-1} = \sum_{i=0}^{\nu} \mathrm{Tr}(F \circ w, \overline{S}_i) q^i.$$

Using this, we may rewrite (3.13.1) in the form

$$\sum_{\rho} \dim(\rho)^2 = |W|^{-1} \sum_{w \in W} \sum_{0 \leqslant i,j \leqslant \nu} \operatorname{Tr}(F \circ w, \bar{S}_i) \operatorname{Tr}(F \circ w, \bar{S}_j) q^{i+j} \tag{3.16.1}$$

$$= \sum_{0 \leqslant i,j \leqslant \nu} \operatorname{Tr}(F, (\bar{S}_i \times \bar{S}_j)^W) q^{i+j}.$$

3.17. Let $W^{\check{}}$ be the set of (isomorphism classes of) irreducible representations of W. The action of F on W induces an action of F on $W^{\check{}}$; we denote by $(W^{\check{}})^F$ its fixed point set. An irreducible representation E of W is in $(W^{\check{}})^F$ if and only if there exists an isomorphism $F\colon E \longrightarrow E$ such that $F(we) = F(w)F(e)$ for all $w \in W$, $e \in E$. This isomorphism, if it exists, is uniquely determined up to a constant factor and it can be chosen of order dividing δ (where δ is, as before, the order of $F\colon W \longrightarrow W$); in this case, the isomorphism is uniquely determined up to a δth root of 1.

Given $E \in (W^{\check{}})^F$, we define

$$R(E) = |W|^{-1} \sum_{w \in W} \operatorname{Tr}(F \circ w, E) R_w \tag{3.17.1}$$

(an element of the Grothendieck group of virtual representations of G^F, tensored by $\mathbf{Q}$). Here $F\colon E \longrightarrow E$ is an isomorphism as above (of order dividing δ) and $R(E)$ depends on the choice of F; thus, when $\delta = 2$, $R(E)$ is well defined only up to sign. When $\delta = 3$ (so that G is of type ${}^3D_4$), there is a canonical choice for F (for which $\operatorname{Tr}(F \circ w, E)$ is integral for all $w \in W$), so that, in this case, $R(E)$ is defined without ambiguity. This will be the content of the following

LEMMA 3.18. *Assume that G is of type ${}^3D_4$, and that E is an irreducible representation of W in $(W^{\check{}})^F$. Then there is a unique isomorphism $F\colon E \longrightarrow E$ of order dividing 3 such that $F(we) = F(w)F(e)$ ($w \in W$, $e \in E$) and such that $\operatorname{Tr}(F \circ w, E) \in \mathbf{Z}$ for all $w \in W$.*

The W-module E is defined over $\mathbf{Q}$. The three isomorphisms $F\colon E \longrightarrow E$ of order dividing 3 such that $F(we) = F(w)F(e)$ ($w \in W$, $e \in E$) can each be defined over $\mathbf{Q}(\sqrt[3]{1})$; they form a set stable under the Galois group of $\mathbf{Q}(\sqrt[3]{1})$ over $\mathbf{Q}$. Now a group of order 2 acting on a set with 3 elements must have at least one fixed point. So there is at least one F which is defined over $\mathbf{Q}$. The other F's will be obtained from it by multiplication with a primitive cubic root of 1, so they are not defined over $\mathbf{Q}$. Thus, there is a unique F which is defined over $\mathbf{Q}$. Then $\operatorname{Tr}(F \circ w, E)$ is clearly an integer and the lemma follows.

3.19. Thus, $R(E)$ is well defined, when $\delta = 1$ or 3 and is well defined up to sign, when $\delta = 2$. Its dimension is given by

$$\dim R(E) = |W|^{-1} \sum_{w \in W} \left(\operatorname{Tr}(F \circ w, E) \sum_{i=0}^{\nu} \operatorname{Tr}(F \circ w, \bar{S}_i) q^i\right) \tag{3.19.1}$$

$$= \sum_{i=0}^{\nu} \operatorname{Tr}(F, (E \otimes \bar{S}_i)^W) q^i.$$

(This implies that $\dim R(E) > 0$, when $\delta = 1$.)

We have the orthogonality relations:

$$\text{(3.19.2)} \qquad \langle R(E), R(E')\rangle = \begin{cases} 1 & \text{if } E = E', \\ 0 & \text{if } E \neq E', \end{cases}$$

where $E, E' \in (W^{\smile})^F$. (This follows immediately from the orthogonality relations (3.4.1) for the R_w and the following formula

$$|W|^{-1} \sum_{w \in W} \operatorname{Tr}(F \circ w, E) \operatorname{Tr}(F \circ w, E') = \begin{cases} 1 & \text{if } E = E', \\ 0 & \text{if } E \neq E'.) \end{cases}$$

Note also the formula

$$\sum_{E \in (W^{\smile})^F} \operatorname{Tr}(F \circ w, E) \operatorname{Tr}(F \circ w', E) = \#\{w_1 \in W | w_1 w = w' F(w_1)\}$$

($w, w' \in W$) from which it follows that

$$\text{(3.19.3)} \qquad R_w = \sum_{E \in (W^{\smile})^F} \operatorname{Tr}(F \circ w, E) R(E) \qquad (w \in W).$$

We see that $R(E)$ ($E \in (W^{\smile})^F$) form an orthonormal basis for the Q-vector space spanned by the virtual representations R_w ($w \in W$).

THEOREM 3.20 (SEE [13], [27], [19]). *Assume that G is of type A_l ($l \geqslant 1$) so that F acts on W as conjugation by w_0^e ($e = 0$ or 1), and $(W^{\smile})^F = W^{\smile}$. For each $E \in W^{\smile}$, $R(E)$ (when $e = 0$) and $\pm R(E)$ (when $e = 1$) is an irreducible unipotent representation of G^F. This gives a bijection $(W^{\smile})^F \xrightarrow{\sim} (G^F)_u^{\smile}$.*

COROLLARY 3.21. *With the same assumptions, any unipotent representation of G^F has nonzero inner product with $R_1 = \operatorname{Ind}_{B_0^F}^{G^F}(1)$ (when $e = 0$) or with R_{w_0} (when $e = 1$).*

For any $E \in (W^{\smile})^F$ and any $w \in W$, we have $\langle R(E), R_w\rangle = \operatorname{Tr}(F \circ w, E)$. When $e = 0$, we take $w = 1$ so that $\operatorname{Tr}(F \circ w, E) = \dim E \neq 0$. When $e = 1$, we note that $F \circ w = \pm w_0 w$ on E; taking $w = w_0$, we get $\operatorname{Tr}(F \circ w, E) = \pm \dim E \neq 0$, as required.

We now state a result on the unipotent cuspidal representations of the finite classical groups.

THEOREM 3.22 (SEE [18]). (i) *Let (G, F) be of type ${}^2A_l$ ($l \geqslant 2$). If $l + 1$ is of the form $\frac{1}{2}(s^2 + s)$, for some integer $s \geqslant 1$, then G^F has exactly one unipotent cuspidal representation. (It is $\pm R(E)$ where E is the unique irreducible representation of W whose character vanishes on all elements of even order of W.) If $l + 1$ is not of the form $\frac{1}{2}(s^2 + s)$, G^F has no unipotent cuspidal representation.*

(ii) *Let (G, F) be of type B_l or C_l ($l \geqslant 2$). If l is of the form $(s^2 + s)$ for some integer $s \geqslant 1$, then G^F has exactly one unipotent cuspidal representation; otherwise, it has none. (For $l = 2$ this is Srinivasan's θ_{10} [25].)*

(iii) *Let (G, F) be of type D_l ($l \geqslant 4$). If l is an even square, then G^F has exactly one unipotent cuspidal representation; otherwise, it has none.*

(iv) *Let (G, F) be of type ${}^2D_l$ ($l \geqslant 4$). If l is an odd square, then G^F has exactly one unipotent cuspidal representation; otherwise, it has none.*

(v) *These unipotent cuspidal representations have the following dimension*:

$$^2A_l(l+1 = \tfrac{1}{2}(s^2+s) \geqslant 3): \frac{(q+1)(q^2-1)\cdots(q^{l+1}-(-1)^{l+1})q^{\binom{s}{2}+\binom{s-1}{2}+\cdots}}{(q+1)^s(q^3+1)^{s-1}\cdots(q^{2s-3}+1)^2(q^{2s-1}+1)}.$$

$$B_l,\ C_l(l = s^2+s \geqslant 2): \frac{|G^F|_{p'}q^{\binom{2s}{2}+\binom{2s-2}{2}+\cdots}}{2^s(q+1)^{2s}(q^2+1)^{2s-1}\cdots(q^{2s}+1)}.$$

$$\left.\begin{array}{l} D_l(l = s^2,\ even \geqslant 4) \\ ^2D_l(l = s^2,\ odd \geqslant 9)\end{array}\right\}: \frac{|G^F|_{p'}q^{\binom{2s-1}{2}+\binom{2s-3}{2}+\cdots}}{2^{s-1}(q+1)^{2s-1}(q^2+1)^{2s-2}\cdots(q^{2s-1}+1)}.$$

In contrast with the classical groups, each exceptional group always has at least two unipotent cuspidal representations, as the following result shows (at least when (G, F) is not of type $^3D_4$).

THEOREM 3.23 (SEE [17]). *Let r be the $\mathbf{F}_q$-rank of G and let $w = s_1 s_2 \cdots s_r \in W$ be the product of r simple reflections, in some order, such that s_i, s_j are not in the same orbit of F, for $i \neq j$. Then F acts semisimply on $H_c^r(X_w, \mathbf{Q}_l)$ and its eigenspaces $H_c^r(X_w, \mathbf{Q}_l)_\mu$ are irreducible, mutually nonisomorphic G^F-modules. The μ-eigenspace is an irreducible, cuspidal unipotent representation of G^F precisely in the cases listed in Table* I.

(Here θ, i, ρ) denote a fixed primitive root of 1 in $\overline{\mathbf{Q}}_l$ of order 3, 4, 5 respectively.)
We have

THEOREM 3.24. *If (G, F) is of type E_6 or E_7 then G^F has exactly two unipotent cuspidal representations: the ones described in Theorem* 3.23.

The proof will be given in 3.27. For the other exceptional groups there exist unipotent cuspidal representations other than those described in Theorem 3.23.

3.25. We now show how the problem of classifying the unipotent representations of G^F can be reduced to classifying unipotent cuspidal representations. (Recall that G is assumed to be adjoint, simple.)

Let ρ be a unipotent representation of G^F. Assume that ρ is not cuspidal. We can find a proper F-stable parabolic subgroup P of G containing B_0, with an F-stable Levi subgroup M containing T_0, and an irreducible cuspidal representation ρ_1 of M^F such that $\langle \rho, \mathrm{Ind}_{P^F}^{G^F}(\rho_1)\rangle \neq 0$, where ρ_1 is regarded as a P^F-module. Using (3.2), we see easily that ρ_1 must be unipotent.

We show that the adjoint group M^{ad} of M is necessarily simple (possibly reduced to one element). Assume that this is not so. Now, given a connected Dynkin graph with an automorphism ϕ and given a disconnected subgraph invariant under ϕ, one can always write this subgraph as a disjoint union of two subgraphs, one of which is of the following kind: (a) a graph of type A_l $(l \geqslant 1)$, with ϕ acting trivially, or (b) a graph of type $A_l \times A_l$ $(l \geqslant 1)$ with ϕ permuting the two factors and with $\phi^2 = 1$, or (c) a graph of type $A_1 \times A_1 \times A_1$ with ϕ permuting cyclically the three factors. It follows that M^{ad} decomposes into a product of two F-stable semisimple subgroups $G' \times G''$, and G' decomposes as $G' = G_1' \times G_2' \times \cdots \times G_s'$, where G_i' are semisimple subgroups with $F(G_i') = G_{i+1}'$ $(1 \leqslant i \leqslant s-1)$,

TABLE I

(G, F)	μ	$\dim H_c^r(X_w, \mathbf{Q}_l)_\mu$
${}^2A_2$:	$-q$	$\dfrac{q\lvert G^F\rvert_{p'}}{(q+1)(q^3+1)}$
B_2:	$-q$	$\dfrac{q\lvert G^F\rvert_{p'}}{2(q+1)^2(q^2+1)}$
G_2:	$-q$	$\dfrac{q\lvert G^F\rvert_{p'}}{2(q+1)(q^3+1)}$
	$\theta q, \theta^2 q$	$\dfrac{q\lvert G^F\rvert_{p'}}{3(q^4+q^2+1)}$
D_4:	$-q^2$	$\dfrac{q^3\lvert G^F\rvert_{p'}}{q(q+1)^3(q^2+1)^2(q^3+1)}$
${}^3D_4$:	$-q^3$	$\dfrac{q^3\lvert G^F\rvert_{p'}}{q(q^3+1)^2(q^4-q^2+1)}$
F_4:	$iq^2, -iq^2$	$\dfrac{q^4\lvert G^F\rvert_{p'}}{4(q^2+1)(q^4+1)(q^6+1)}$
	$\theta q^2, -\theta q^2$	$\dfrac{q^4\lvert G^F\rvert_{p'}}{3(q^4+q^2+1)(q^8+q^4+1)}$
${}^2A_5$:	$-q^3$	$\dfrac{q^4\lvert G^F\rvert_{p'}}{(q+1)^2(q^3+1)^2(q^5+1)}$
E_6:	$\theta q^3, \theta^2 q^3$	$\dfrac{q^7\lvert G^F\rvert_{p'}}{3(q^2+q+1)(q^4+q^2+1)(q^6+q^3+1)(q^8+q^4+1)}$
${}^2E_6$:	$\theta q^4, \theta^2 q^4$	$\dfrac{q^7\lvert G^F\rvert_{p'}}{3(q^2-q+1)(q^4+q^2+1)(q^6-q^3+1)(q^8+q^4+1)}$
E_7:	$iq^{7/2}, -iq^{7/2}$	$\dfrac{q^{11}\lvert G^F\rvert_{p'}}{2(q+1)^2(q^3+1)^2(q^5+1)(q^7+1)(q^9+1)}$
E_8:	$-\theta q^4, -\theta^2 q^4$	$\dfrac{q^{16}\lvert G^F\rvert_{p'}}{6(q+1)(q^3+1)(q^4+q^2+1)(q^6-q^3+1)(q^{10}-q^5+1)(q^{16}+q^8+1)}$
	$\rho q^4, \rho^2 q^4, \rho^3 q^4, \rho^4 q^4$	$\dfrac{q^{16}\lvert G^F\rvert_{p'}}{5(q^{16}+q^{12}+q^8+q^4+1)(q^{24}+q^{18}+q^{12}+q^6+1)}$

$F(G'_s) = G'_1$; moreover G'_1 is of type A_l $(l \geqslant 1)$ with F^s acting trivially on the Weyl group of G'_1.

By (3.15), ρ_1 comes from a unipotent cuspidal representation of $(M^{\mathrm{ad}})^F$. It follows easily that G'^F and G''^F must have some unipotent cuspidal representation and then that $G_1'^{F^s}$ must have some unipotent cuspidal representation. But this contradicts Corollary 3.21.

We see not only that M^{ad} is simple, but (using Theorem 3.22) that its Dynkin graph is of one of the following types: A_0 (empty), ${}^2A_l$ $(l+1 = \frac{1}{2}(s^2+s) \geqslant 3)$, B_l $(l = s^2 + s \geqslant 2)$, C_l $(l = s^2 + s \geqslant 2)$, D_l $(l = s^2$, even $\geqslant 4)$, ${}^2D_l$ $(l = s^2$, odd $\geqslant 9)$, E_6, E_7.

Each of these Dynkin graphs (with automorphism) has the property that it is the unique F-stable subgraph of its type of the Dynkin graph of (G, F). It follows that P is uniquely determined by ρ. (If P' is an F-stable parabolic subgroup containing B_0, with an F-stable Levi subgroup M' containing T_0, such that M, M' are conjugate under G^F, then P, P' correspond to F-stable subgraphs of the same type of the Dynkin graph of (G, F); these two subgraphs must coincide, by the previous remark, hence $P = P'$.) Then M is also uniquely determined by ρ.

Moreover, the M^F-module ρ_1 is uniquely determined (up to isomorphism) by ρ. This is clear if M is not of type E_6, E_7 since then M^F has a unique unipotent cuspidal representation. In the cases where M is of type E_6 or E_7 we appeal to [17, Lemma (6.6)] which shows that ρ_1 extends to an $N(M)^F$-module, where $N(M)$ is the normalizer of M in G; it follows again that ρ_1 is uniquely determined by ρ.

Thus, we can define a partition of the set of noncuspidal unipotent representations, two such representations being regarded as equivalent if they give rise to the same (P, M, ρ_1), as above. Conversely, if P is a proper, F-stable parabolic subgroup of G containing B_0, corresponding to an F-stable subgraph Γ' of the Dynkin graph Γ of (G, F), and if ρ_1 is a unipotent cuspidal representation of M^F (M as above), then all irreducible representations of G^F occurring in $\mathrm{Ind}_{P^F}^{G^F}(\rho_1)$ are unipotent.

We shall now describe the endomorphism algebra of the G^F-module $E = \mathrm{Ind}_{P^F}^{G^F}(\rho_1)$. We associate to the Dynkin graph Γ and its subgraph Γ' a new graph $\overline{\Gamma}$. The vertices of $\overline{\Gamma}$ will be in 1-1 correspondence with the orbits of F on the set of vertices of Γ which are not in Γ'. If $\gamma \neq \gamma'$ are two such orbits, we define

$$m(\gamma, \gamma') = \frac{2(|\Phi_{\Gamma' \cup \gamma \cup \gamma'}| - |\Phi_{\Gamma'}|)}{|\Phi_{\Gamma' \cup \gamma}| + |\Phi_{\Gamma' \cup \gamma'}| - 2|\Phi_{\Gamma'}|}$$

where $\Phi_{\Gamma'}, \Phi_{\Gamma' \cup \gamma}$, etc., denote the set of roots of a root system of type Γ', $\Gamma' \cup \gamma$, etc. It follows from [17, Theorem 5.9] that $m(\gamma, \gamma')$ is one of the integers 2, 3, 4, 6. The vertices of $\overline{\Gamma}$ corresponding to γ, γ' will be joined by 0, 1, 2 or 3 bonds, according as $m(\gamma, \gamma') =$ 2, 3, 4 or 6. The resulting graph $\overline{\Gamma}$ is the Coxeter graph of a Weyl group W with set of simple reflections S (cf. [loc. cit.]); we shall identify the set of vertices of $\overline{\Gamma}$ with S. We define a function λ on the set of vertices of $\overline{\Gamma}$ as follows. Given an orbit γ of F on $\Gamma - \Gamma'$, we consider the parabolic subgroup $P(\gamma)$ of G corresponding to $\Gamma' \cup \gamma$. The induced representation $\mathrm{Ind}_{P^F}^{P(\gamma)^F}(\rho_1)$ splits into two irreducible $P(\gamma)^F$-modules of dimensions say, $d_\gamma \geqslant d'_\gamma$.

We then define the value of λ on the vertex corresponding to γ to be the ratio $d_\gamma \cdot d_\gamma'^{-1}$. We can also regard γ as a function $S \longrightarrow \overline{\mathbf{Q}}_l$.

We can now state

THEOREM 3.26. *The function $\lambda: S \longrightarrow \overline{\mathbf{Q}}_l$ takes equal values on elements in S which are conjugate in W. Hence the $\overline{\mathbf{Q}}_l$-algebra $H(W, \lambda)$ is well defined. There exists an algebra isomorphism (preserving* 1)

$$H(W, \lambda) \cong \mathrm{End}_{G^F}(E),$$

under which the basis elements t_w $(w \in W, w \neq 1)$ correspond to endomorphisms with trace zero of E.

The proof is given in [17, §5] and [18, §5]. We shall now list (in Table II) following [17, (7.7)] and [18, §8] the precise form of $(\overline{\Gamma}, \lambda)$ for each pair $(\Gamma \supset \Gamma')$, $\Gamma \neq \Gamma'$, as above. We shall describe λ by attaching the values of λ to the various vertices of $\overline{\Gamma}$. We also make the following conventions: ${}^2A_0, {}^2A_{-1}, B_0, C_0$ denote the empty graph;

$$(\overline{\Gamma}, \lambda) = (B_n, \overset{q^a}{\circ}\!-\!\overset{q^a}{\circ}\!-\!\overset{q^a}{\circ} \cdots \overset{q^a}{\circ}\!-\!\overset{q^a}{\circ}\!-\!\overset{q^b}{\circ})$$

is interpreted for $n = 1$ as (A_1, q^b).

This reduces completely the problem of classification of noncuspidal unipotent representations of G^F to that of classifying the irreducible representations of the algebra $H(W, \lambda)$ where (W, λ) is given by Table II: there is a 1-1 correspondence $\rho \longrightarrow \chi_\lambda$ between the (isomorphism classes of) unipotent representations ρ of G^F which give rise to (P, M, ρ_1) and the (isomorphism classes of) irreducible representations χ_λ of $H(W, \lambda)$ (where (W, λ) is given by Table II) with the following property:

$$\dim(\rho) = \dim(\rho_1)|G^F| \cdot |P^F|^{-1} d(\chi_\lambda) P(\lambda)^{-1}. \tag{3.26.1}$$

Here $d(\chi_\lambda)$ denote the "generic degrees" [9] of the algebra $H(W, \lambda)$ and $P(\lambda)$ denotes its Poincaré polynomial; they are defined by the condition that the linear combination of $H(W, \lambda)$-modules: $\Sigma_{\chi_\lambda} d(\chi_\lambda)\chi_\lambda$ should have trace zero on all t_w $(w \neq 1)$ and dimension equal to $P(\lambda)$, together with the normalization $d(1_\lambda) = 1$ where 1_λ is the one dimensional representation of $H(W, \lambda)$ on which t_s acts as $\lambda(s) \cdot 1$ $(s \in S)$.

The classification of the irreducible representations of $H(W, \lambda)$ is the same as that of the irreducible representations of W (see [3, Chapter 4], [9]). The Poincaré polynomials $P(\lambda)$ are known explicitly in each case; moreover the generic degrees $d(\chi_\lambda)$ are known explicitly in each case listed in Table II except when (Γ, Γ') is $(E_8$, empty) or (E_8, D_4) (see Hoefsmit [14] for the case of classical groups and Surowski [29], Benson-Grove-Surowski [2] for the case of exceptional groups).

3.27. We shall now indicate how the proof of Theorem 3.24 can be reduced to mechanical computation.

Assume first that (G, F) is of type E_6. We can make use of the results in 3.25 and 3.26 to describe the noncuspidal unipotent representations of G^F. We see that there are 25 unipotent representations corresponding to the pair $(E_6$, empty), one for each irreducible representation of $H(W, q)$, W of type E_6 (their dimensions are known explicitly from the

TABLE II

Γ	Γ'	$\overline{\Gamma}$	λ
Γ arbitrary	empty	Γ	identically q.
${}^2A_{2n+(s^2+s)/2-1}$ $(n \geqslant 1, s \geqslant 0)$	${}^2A_{(s^2+s)/2-1}$	B_n	$q^2\ q^2\ q^2 \cdots q^2\ q^2\ q^{2s+1}$
$B_{n+(s^2+s)}$ $(n \geqslant 1, s \geqslant 1)$	B_{s^2+s}	B_n	$q\ q\ q \cdots q\ q\ q^{2s+1}$
$C_{n+(s^2+s)}$ $(n \geqslant 1, s \geqslant 1)$	C_{s^2+s}		
D_{n+s^2} $(n \geqslant 1, s \geqslant 2, s$ even)	D_{s^2}	B_n	$q\ q\ q \cdots q\ q\ q^{2s}$
${}^2D_{n+s^2}$ $(n \geqslant 1, s \geqslant 3, s$ odd)	${}^2D_{s^2}$		
${}^2D_n$ $(n \geqslant 4)$	empty	B_{n-1}	$q\ q\ q \cdots q\ q\ q^2$
${}^3D_4$	empty	G_2	$q\ q^3$
F_4	B_2	B_2	$q^3\ q^3$
E_6	D_4	A_2	$q^4\ q^4$
E_7	D_4	B_3	$q\ q^4\ q^4$
E_8	D_4	F_4	$q\ q\ q^4\ q^4$
${}^2E_6$	empty	F_4	$q\ q\ q^2\ q^2$
${}^2E_6$	${}^2A_5$	A_1	q^9
E_7	E_6	A_1	q^9
E_8	E_6	G_2	$q\ q^9$
E_8	E_7	A_1	q^{15}

work of Surowski [29]); in addition, there are 3 unipotent representations corresponding to the pair (E_6, D_4), one for each irreducible representation of $\mathcal{H}(\mathcal{W}, q^4)$, $\mathcal{W}$ of type A_2 (their dimensions are determined by (3.26.1) since the generic degrees are in this case $1, q^8 + q^4, q^{12}$). We compute the sum of squares of the dimensions of these 28 representations plus the sum of squares of the dimensions of the two cuspidal representations of G^F described in Theorem 3.23. We compare the result with the expression obtained by computing the right-hand side of (3.13.1), using the explicit formulae [4] for the characteristic polynomials of the elements in the 25 conjugacy classes of W; we find the same result. Using (3.13.1), we see that there are no other unipotent representations than the ones considered. (Thus, G^F has $25 + 3 + 2 = 30$ unipotent representations altogether.)

Assume now that G is of the type E_7. Since we now know precisely the unipotent cuspidal representations in the case E_6, we can make use of the results in 3.25, 3.26 to describe the noncuspidal unipotent representations of G^F. We see that there are:

(a) 60 unipotent representations corresponding to the pair (E_7, empty), one for each irreducible representation of $\mathcal{H}(\mathcal{W}, q)$, $\mathcal{W}$ of type E_7 (their dimensions are known explicitly from the work of Surowski [29]).

(b) 10 unipotent representations corresponding to the pair (E_7, D_4), one for each irreducible representation of $\mathcal{H}(\mathcal{W}, \lambda)$, $\mathcal{W}$ of type B_3, λ as in Table II. (Their dimensions are given by (3.26.1) together with Hoefsmit's formulae [14] for the generic degrees for B_3.)

(c) 4 unipotent representations corresponding to the pair (E_7, E_6), two for each of the two unipotent cuspidal representations of E_6. (Their dimensions can be computed explicitly using (3.26.1).)

We compute the sum of squares of the dimensions of these 74 representations plus the sum of squares of the dimensions of the two unipotent cuspidal representations of G^F, described in Theorem 3.23. We also compute explicitly the right-hand side of (3.13.1) using again the information on characteristic polynomials for the 60 conjugacy classes in W [4], and we find the same result as before. Using (3.13.1), we conclude again that there are no unipotent representations other than the ones considered. Thus, G^F has $60 + 10 + 4 + 2 = 76$ unipotent representations altogether.

This proof, although very simple in principle, involves a great deal of rather tedious (mechanical) computation. A less computational proof will be given in (3.32).

We now consider the case when G is of type G_2.

THEOREM 3.28. *If G is of type G_2, G^F has exactly four unipotent cuspidal representations: three of them described in Theorem 3.23 and a fourth one of dimension:*

$$\frac{q|G^F|_{p'}}{6(q+1)^2(q^2+q+1)}. \tag{3.28.1}$$

(This result is known when q is not a power of 2, see Chang-Ree [6] and Enomoto [12].)

The degrees of the 6 noncuspidal unipotent representations are known explicitly (see Chang-Ree [6]). Using (3.13.1) we can compute the sum of squares of the dimensions of all unipotent cuspidal representations of G^F other than those described in Theorem 3.23; we find that this equals the square of the expression (3.28.1).

Now let ρ be a unipotent cuspidal representation of G^F other than those described in Theorem 3.23; we consider the inner products $\langle \rho, R_c \rangle$, $\langle \rho, R_{c^2} \rangle$, $\langle \rho, R_{c^3} \rangle$, where c is a Coxeter element in W. Note that, up to conjugacy, c, c^2, c^3 are the only elements of W without eigenvalue 1 on Y_0, i.e. such that the corresponding F-stable maximal torus in G is not contained in a proper F-stable parabolic subgroup; thus for all $w \in W$ which are not conjugate to c, c^2 or c^3 we have $\langle \rho, R_w \rangle = 0$. By assumption, ρ does not occur in the G^F module $H_c^2(X_c, \mathbf{Q}_l)$. It is known [17, (4.3)] that this implies that $\langle \rho, R_c \rangle = 0$. We can write (2.7.1) in the form $1 = |W|^{-1} \Sigma_{w \in W} R_w$. We have $\langle \rho, 1 \rangle = 0$, hence $2\langle \rho, R_{c^2} \rangle + \langle \rho, R_{c^3} \rangle = 0$.

Using now Corollary 3.14, we see that

$$\begin{aligned}\dim(\rho) &= \frac{1}{12}|G^F|_{p'}(2\langle \rho, R_{c^2}\rangle \det(1 - q \cdot c^2)^{-1} + \langle \rho, R_{c^3}\rangle \det(1 - q \cdot c^3)^{-1}) \\ &= \frac{\langle \rho, R_{c^2}\rangle}{6}|G^F|_{p'}(\det(1 - qc^2)^{-1} - \det(1 - qc^3)^{-1}) \\ &= \frac{\langle \rho, R_{c^2}\rangle}{6}|G^F|_{p'}\frac{q}{(q+1)^2(q^2+q+1)}.\end{aligned}$$

Now $\langle \rho, R_{c^2} \rangle$ must be an integer $\geqslant 1$. It follows that ρ is the unique unipotent cuspidal representation of G^F such that $\langle \rho, R_c \rangle = 0$ and that we have $\langle \rho, R_{c^2} \rangle = 1$, $\langle \rho, R_{c^3} \rangle = -2$.

For the groups of type ${}^3D_4$, ${}^2E_6$, F_4 one has the following result.

THEOREM 3.29. (i) *Assume that (G, F) is of type ${}^3D_4$. If q is sufficiently large, then G^F has exactly two unipotent cuspidal representations: the one given by Theorem 3.23, and one of dimension*

$$\frac{q^3|G^F|_{p'}}{2(q^3+1)^2(q^2+q+1)^2}.$$

(ii) *Assume that (G, F) is of type ${}^2E_6$. If q is sufficiently large, then G^F has exactly three unipotent cuspidal representations: the two given by Theorem 3.23 and one of dimension*

$$\frac{q^7|G^F|_{p'}}{6(q+1)^5(q^2+1)^2(q^3+1)(q^4+q^2+1)^2}.$$

(iii) *Assume that (G, F) is of type F_4. If q is sufficiently large, then G^F has exactly seven unipotent cuspidal representations: the four representations given by Theorem 3.23 and three representations of dimension given respectively by*:

$$\frac{q^4|G^F|_{p'}}{8(q+1)^2(q^2+1)^2(q^3+1)^2}, \tag{3.29.1}$$

$$\frac{q^4|G^F|_{p'}}{24(q+1)^2(q^2+q+1)^2(q^3+q^2+q+1)^2}, \tag{3.29.2}$$

$$\frac{q^4|G^F|_{p'}}{4(q+1)^2(q^3+1)^2(q^4+1)}. \tag{3.29.3}$$

We shall only give the proof of (iii); the cases (i), (ii) are simpler. We shall need the following.

LEMMA 3.30. *Assume that F acts trivially on W. Let w be a regular element of order d of W (in the sense of Springer [22]) i.e. such that there exists a nonzero vector $v \in Y_0 \otimes C$ which is an eigenvector of w and whose stabilizer in W is trivial. We denote G as $G[q]$ to show that it depends on q (there is one such group for each q, a power of a prime).*

Assume that for any sufficiently large prime power q we are given a unipotent representation $\rho[q]$ of $G[q]^F$ such that the dimension of $\rho[q]$ is a polynomial $P(q)$ in q with rational coefficients (independent of q). Then, we have $\langle \rho[q], R_w \rangle = P(\zeta)$ for sufficiently large q, where R_w refers to $G[q]$ and ζ is a fixed primitive dth root of 1.

For each $w' \in W$, the integer $\langle \rho[q], R_w \rangle$ can only take finitely many values when q varies (see (3.14.1)). It follows that we can find an integer $m \geq 1$ and disjoint, infinite sets of integers $A_1, A_2, \ldots, A_m$ consisting of prime powers, with the following properties:

(a) $\rho[q]$ is defined for all $q \in A_1 \cup \cdots \cup A_m$.

(b) For each i, $1 \leq i \leq m$, and each $w' \in W$, the function $q \longrightarrow \langle \rho[q], R_{w'} \rangle$ is constant on A_i.

(c) Any sufficiently large prime power q is in $A_1 \cup \cdots \cup A_m$.

We fix i, $1 \leq i \leq m$. We have the identity (see (3.14))

$$P(q) = |W|^{-1} \sum_{w' \in W} \langle \rho[q], R_{w'} \rangle \, |G[q]^F|_{p'} \cdot \det(1 - qw')^{-1}$$

for all $q \in A_i$. Since A_i is infinite and $\langle \rho[q], R_{w'} \rangle$ are constants, this must be true as an identity between polynomials in q. We set $q = \zeta$ in this identity. Using a result of Springer [22, Theorem 4.2], we see easily that

$$\epsilon_G |G[q]^F|_{p'} \det(1 - qw')^{-1} |_{q=\zeta} = \begin{cases} 0 & \text{if } w' \text{ is not conjugate to } w, \\ |Z_W(w)| & \text{if } w' \text{ is conjugate to } w. \end{cases}$$

It follows that $P(\zeta) = \langle \rho[q], R_{w'} \rangle$ for all $q \in A_i$. It remains to use (c) above.

3.31. We shall now give a proof of Theorem 3.29(iii). Let (G, F) be of type F_4. Then G^F has 25 unipotent representations corresponding to the empty subgraph of F_4, one for each irreducible representation of $H(W, q)$, W of type F_4 (these dimensions are explicitly known from the work of Benson, Grove, Surowski [2]) and 5 unipotent representations corresponding to the subgraph $B_2 \subset F_4$, one for each irreducible representation of $H(W, q^3)$, W of type B_2 (their dimensions are given by (3.26.1) together with the well-known formulae for the generic degrees of B_2). These exhaust the noncuspidal unipotent representations of G^F. We also know explicitly the dimension of four unipotent cuspidal representations of G^F, from Theorem 3.23. Using Lemma 3.30, we can compute, for each of these 34 representations, the inner product with R_w, where w is the unique regular element (up to conjugacy) of order 8 in W (for large q); this we achieve by setting $q = \sqrt[8]{1}$ in the dimension of these representations, which are known to be polynomials in q. We find that for 27 of these we get zero and for 7 of these we get the value ± 1. As $\langle R_w, R_w \rangle = 8$, there exists a unique representation ρ distinct from the known 34 ones and such that $\langle \rho, R_w \rangle = \pm 1$; ρ is necessarily cuspidal. Its dimension can be determined from the formula

$$\dim R_w = |G^F|_{p'} \det(1 - qw)^{-1} = \sum_{\rho'} \langle \rho', R_w \rangle \dim(\rho'),$$

and turns out to be given by (3.29.3).

Next we compute for each of the 35 representations known already, the inner product with $R_{w'}$, where w' is the unique regular element (up to conjugacy) of order 6 in W (for large q); this we achieve, using Lemma 3.30, by setting $q = \sqrt[6]{1}$ in their dimension polynomial. We find, in each case, one of the numbers $0, \pm 1, \pm 2, \pm 3$ and the sum of squares of these 35 numbers equals 71. We have $\langle R_{w'}, R_{w'} \rangle = |Z_W(w')| = 72$. It follows that there exists a unique unipotent representation ρ' distinct from the known 35 ones and such that $\langle \rho', R_{w'} \rangle = \pm 1$; ρ' is necessarily cuspidal. Its dimension can be computed in the same way as the dimension of ρ; we find that it is given by (3.29.2).

Next we compute for each of the 36 representations known already, the inner product with $R_{w''}$, where w'' is the unique regular element (up to conjugacy) of order 4 in W (for large q); this we achieve by setting $q = \sqrt[4]{1}$ in their dimension polynomial. We find, in each case, one of the numbers $0, \pm 1, \pm 2, \pm 3, \pm 4$ and the sum of squares of these 36 numbers equals 95. We have $\langle R_{w''}, R_{w''} \rangle = |Z_W(w'')| = 96$. It follows that there exists a unique unipotent representation ρ'' distinct from the known 36 ones and such that $\langle \rho'', R_{w''} \rangle = \pm 1$; ρ'' is necessarily cuspidal. Its dimension can be computed as before and is given by (3.29.1).

We must now show that the 37 representations we have obtained so far exhaust the set of unipotent representations of G^F. In principle, this could be done by showing that the sum of squares of the dimensions of these 37 representations is equal to the right-hand side of (3.13.1). This would, however, involve rather long computations. We shall instead argue as follows. First, we notice that the right-hand side of (3.13.1) is invariant under $q \longrightarrow -q$, since $-1 \in W$. We can also check directly that our 37 dimension polynomials are permuted among them when q is replaced by $-q$ (with possible sign changes), hence the sum of their squares is invariant under $q \longrightarrow -q$. Assume that there exist infinitely many q for which $G[q]^F$ has strictly more than 37 unipotent representations. We can find an infinite subset A of this set of q's such that, for $q \in A$, $G[q]^F$ has, in addition to the known 37 unipotent representations, exactly s unipotent cuspidal representations $\rho_1[q], \ldots, \rho_s[q]$ ($s \geqslant 1$ independent of q) such that for each i, $1 \leqslant i \leqslant s$, and each $w \in W$, the function $q \longrightarrow \langle \rho_i[q], R_w \rangle$ is constant on A. As $\rho_i[q]$ is cuspidal, $\langle \rho_i[q], R_w \rangle = 0$ unless w has no eigenvalue 1. There are exactly 9 conjugacy classes of such elements in W, and 6 of them are regular in the sense of Springer. For each regular $w \in W$, we can compute the inner product of each of our 37 representations with R_w and notice that the sum of squares of these 37 numbers equals $|Z_W(w)|$.

It follows that $\langle \rho_i[q], R_w \rangle = 0$ unless w has no eigenvalue 1 and is not regular ($q \in A$). Let w_1, w_2, w_3 be representatives for the three nonregular classes in W, without eigenvalue 1; their characteristic polynomials are (cf. [4]): $(q+1)(q^3+1)$, $(q+1)(q^3+1)$, $(q+1)^2(q^2+1)$. Let $c_{ij} = \langle \rho_i[q], R_{w_j} \rangle$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant 3$. We have (cf. (3.14)):

$$\begin{aligned}\dim \rho_i[q] &= \sum_{j=1}^{3} |Z_W(w_j)|^{-1} c_{ij} |G[q]^F|_{p'} \cdot \det(1 - qw_j)^{-1} \\ &= |G[q]^F|_{p'} (c_i'(q+1)^{-1}(q^3+1)^{-1} + c_i''(q+1)^{-2}(q^2+1)^{-1})\end{aligned}$$

where $c_i' = |Z_W(w_1)|^{-1}c_{i1} + |Z_W(w_2)|^{-1}c_{i2}$, $c_i'' = |Z_W(w_3)|^{-1}c_{i3}$ are constants. It follows that, for $q \in A$, we have

$$\sum_{i=1}^{s} \dim(\rho_i[q])^2 = |G[q]^F|_{p'}^2\left(\left(\sum_{i=1}^{s} c_i'^2\right)(q+1)^{-2}(q^3+1)^{-2}\right.$$
$$+ 2\left(\sum_{i=1}^{s} c_i'c_i''\right)(q+1)^{-3}(q^2+1)^{-1}(q^3+1)^{-1}$$
$$\left.+\left(\sum_{i=1}^{s} c_i''^2\right)(q+1)^{-4}(q^2+1)^{-2}\right).$$

Using (3.13.1) and an earlier part of the proof, we see that the previous expression is a polynomial in q invariant under $q \longrightarrow -q$. Clearly, $|G[q]^F|_{p'}$ is also invariant under $q \longrightarrow -q$. We obtain the identity

$$\left(\sum_{i=1}^{s} c_i'^2\right)\left(\frac{1}{(q+1)^2(q^3+1)^2} - \frac{1}{(1-q)^2(1-q^3)^2}\right)$$
$$+ 2\left(\sum_{i=1}^{s} c_i'c_i''\right)\left(\frac{1}{(q+1)^3(q^2+1)(q^3+1)} - \frac{1}{(1-q)^3(q^2+1)(1-q^3)}\right)$$
$$+\left(\sum_{i=1}^{s} c_i''^2\right)\left(\frac{1}{(q+1)^4(q^2+1)^2} - \frac{1}{(1-q)^4(q^2+1)}\right) = 0.$$

We multiply this by $(q^3+1)^2$ and then set $q = \sqrt[6]{1}$; we deduce that $\Sigma_{i=1}^{s} c_i'^2 = 0$, hence $c_i' = 0$ $(1 \leqslant i \leqslant s)$. Next, we multiply by $(q^2+1)^2$ and then set $q = \sqrt[4]{1}$; we deduce that $\Sigma_{i=1}^{s} c_i''^2 = 0$ hence $c_i'' = 0$ $(1 \leqslant i \leqslant s)$.

It follows that $\dim \rho_i[q] = 0$, a contradiction. This completes the proof.

3.32. We now return to the proof of Theorem 3.24 (see (3.27)); we shall show how to avoid the very long computations which were necessary in (3.27). Assume first that (G, F) is of type E_6. We have 30 unipotent representations of G^F and we want to show that they exhaust the set of unipotent representations of G^F. We can compute, using Lemma 3.30, the inner product of each of these thirty representations with R_w (for q large) where w is a regular element of W without eigenvalue 1 (there are four such w, up to conjugacy). We compute the sum of squares of these 30 numbers and we find $|Z_W(w)|$ in each case. It follows that, if G^F had a unipotent cuspidal representation other than the 30 known ones, we would have $\langle \rho, R_{w'}\rangle = 0$ unless w' is a nonregular element of W without eigenvalue 1. But there is a unique such w', up to conjugacy. We have $1 = |W|^{-1}\Sigma_{w\in W} R_w$, and $\langle \rho, 1\rangle = 0$, hence $|Z_W(w')|^{-1}\langle \rho, R_{w'}\rangle = 0$. Thus $\langle \rho, R_w\rangle = 0$ for all $w \in W$, contradicting (3.14). It follows that the sum of squares of the dimensions of our 30 representations is equal to the right-hand side of (3.13.1), for large q, and hence for all q.

Next, we assume that (G, F) is of type E_7. We want to prove that the 76 known unipotent representations of G^F exhaust the set of all unipotent representations of G^F. Assume that there exist infinitely many values of q for which $G[q]^F$ has strictly more than 76 unipotent representations. We can find an infinite subset A of this set of q's such that, for

$q \in A$, $G[q]^F$ has, in addition to the known 76 unipotent representations, exactly s unipotent cuspidal representations $\rho_1[q], \ldots, \rho_s[q]$ ($s \geqslant 1$ independent of q) such that, for each i, $1 \leqslant i \leqslant s$, and each $w \in W$, the function $q \longrightarrow \langle \rho_i[q], R_w \rangle$ is constant on A. As $\rho_i[q]$ is cuspidal, we have $\langle \rho_i[q], R_w \rangle = 0$ unless w has no eigenvalue 1. There are exactly 12 conjugacy classes of such elements w, with representatives $w_1, w_2, \ldots, w_{12}$. Their characteristic polynomials, listed in [4], are respectively: $p_1(q) = \phi_2^7, p_2(q) = \phi_4^2\phi_3^2, p_3(q) = \phi_6\phi_3^2\phi_2, p_4(q) = \phi_8\phi_4\phi_2, p_5(q) = \phi_6\phi_2^5, p_6(q) = \phi_{10}\phi_2^3, p_7(q) = \phi_6^2\phi_2^3, p_8(q) = \phi_{18}\phi_2, p_9(q) = \phi_{14}\phi_2, p_{10}(q) = \phi_{12}\phi_6\phi_2, p_{11}(q) = \phi_{10}\phi_6\phi_2, p_{12}(q) = \phi_6^3\phi_2$, where $\phi_d = \phi_d(q)$ is the dth cyclotomic polynomial.

Next, we observe that the right-hand side of (3.13.1) and $|G[q]^F|_{p'}$ are both invariant under $q \longrightarrow -q$, since $-1 \in W$. Moreover, one can check that the dimension polynomials of our 76 representations are permuted among them (up to sign) when q is changed into $-q$, hence the sum of their squares is invariant under $q \longrightarrow -q$. Using (3.13.1) it follows that $\Sigma_{i=1}^s \dim(\rho_i[q])^2$, which may be regarded as a polynomial in q, is invariant under $q \longrightarrow -q$. Let $c_{ij} = \langle \rho_i[q], R_{w_j} \rangle$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant 12$ and let $c'_{ij} = |Z_W(w_j)|^{-1} c_{ij}$. We have (cf. (3.14))

$$\dim \rho_i[q] = -|G[q]^F|_{p'} \sum_{j=1}^{12} c'_{ij} \det(1 - qw_j)^{-1}$$

hence

$$\sum_{i=1}^{s} \dim(\rho_i[q])^2 = |G[q]^F|_{p'}^2 \sum_{i=1}^{s} \left(\sum_{j=1}^{12} c'_{ij} \cdot p_j(q)^{-1} \right)^2$$

($q \in A$) where c'_{ij} are constants. It follows that we have the identity:

(3.32.1) $$\sum_{i=1}^{s} \left(\sum_{j=1}^{12} c'_{ij} p_j(q)^{-1} \right)^2 = \sum_{i=1}^{s} \left(\sum c'_{ij} p_j(-q)^{-1} \right)^2.$$

We now show that this identity implies $c'_{ij} = 0$ for all i, j. First, we note that if $p_{j_0}(q)$ is divisible by $\phi_d^n(q)$ and all other $p_i(q)$ and all $p_i(-q)$ are not divisible by $\phi_d^n(q)$, then multiplying (3.32.1) by $\phi_d^{2n}(q)$ and setting $q = \sqrt[n]{1}$, we get: $\Sigma_{i=1}^s c_{ij_0}^2 = 0$, hence c_{ij_0} for all i, $1 \leqslant i \leqslant s$. We apply this first to $j = 1$, using ϕ_2^7, then to $j = 5$, using ϕ_2^5, then to $j = 12$, using ϕ_6^3, then to $j = 8$, using ϕ_{18} and then to $j = 9$, using ϕ_{14}. Thus $c_{ij} = 0$ for $1 \leqslant i \leqslant s$, $j = 1, 5, 12, 8, 9$. Next, we multiply (3.32.1) with ϕ_4^4 and set $q = \sqrt[4]{1}$; we get $\Sigma_{i=1}^s c_{i2}^2 = -\Sigma_{i=1}^s c_{i2}^2$, hence $c_{i2} = 0$ for $1 \leqslant i \leqslant s$.

We now multiply (3.32.1) with ϕ_8^2 and set $q = \sqrt[8]{1} = \zeta$; we get

$$\sum_{i=1}^{s} c_{i4}^2 \cdot (\zeta + 1)^{-2} = \sum_{i=1}^{s} c_{i4}^2 (\zeta - 1)^{-2}.$$

As $(\zeta + 1)^{-2} \neq (\zeta - 1)^{-2}$, we deduce $\Sigma_{i=1}^s c_{i4}^2 = 0$ hence $c_{i4} = 0$ for $1 \leqslant i \leqslant s$. We now multiply (3.32.1) with ϕ_6^4 and we set $q = \sqrt[6]{1}$; we get $-(1/3)\Sigma_{i=1}^s c_{i7}^2 = (1/2)\Sigma_{i=1}^s c_{i3}^2$ hence $c_{i7} = c_{i3} = 0$ for $1 \leqslant i \leqslant s$. Using ϕ_2^3, we see that $c_{i6} = 0$ $(1 \leqslant i \leqslant s)$; using ϕ_{10}, we see that $c_{i,11} = 0$ $(1 \leqslant i \leqslant s)$ and finally, using ϕ_2, we see that $c_{i,10} = 0$ $(1 \leqslant i \leqslant s)$.

Thus, we have shown that all $c_{ij} = 0$ hence $\dim \rho_i[q] = 0$ $(1 \leqslant i \leqslant s)$, a contradiction. Thus, for all q, except possibly finitely many, $G[q]^F$ has exactly 76 unipotent representations.

Hence the sum of squares of the dimensions of our 76 representations is equal to the right-hand side of (3.13.1). But this is a polynomial identity, hence it must be true for all q, hence $G[q]^F$ has exactly 76 unipotent representations, for all q. This completes the proof.

REMARKS. (1) Theorem 3.29 is valid for q large, but its proof does not give any information on how large q should be. It is extremely likely that it remains true for arbitrary q.

(2) If (G, F) is of type ${}^2E_6$, and q is large, G^F has exactly 30 unipotent representations: 25 corresponding to the empty subgraph of ${}^2E_6$, one for each representation of $H(W, \lambda)$ (W of type F_4, λ as in Table II), 2 corresponding to the subgraph ${}^2A_5$ of ${}^2E_6$ and 3 cuspidal ones. One can check that the polynomials giving the dimensions of these 30 representations are transformed under $q \longrightarrow -q$ into the polynomials giving the dimensions (up to sign) of the 30 unipotent representations of a split group of type E_6.

(3) If (G, F) is of type ${}^3D_4$, and q is large, G^F has exactly 8 unipotent representations: 6 corresponding to the empty subgraph and 2 cuspidal ones. The 8 polynomials in q giving their dimension are permuted among them (up to sign) when q is changed into $-q$.

(4) When (G, F) is of type ${}^2E_6$, and q is large, the third unipotent cuspidal representation ρ of G^F described in 3.29(ii) has the property that $\langle \rho, R_w \rangle = 1$ where w is an element of W such that $\det(1 - qFw) = (q^2 - q + 1)^3$ (for all q).

When (G, F) is of type ${}^3D_4$, and q is large, the second unipotent cuspidal representation ρ of G^F described in 3.29(i) has the property that $\langle \rho, R_w \rangle = 1$ where w is an element of W such that $\det(1 - qFw) = (q^2 - q + 1)^2$ (for all q).

3.33. We shall now discuss the possibilities for the eigenvalues of F^δ on $H_c^i(X_w, \mathbf{Q}_l)$ for w an arbitrary element of W. In 3.9 we have attached to each unipotent representation ρ of G^F an eigenvalue $\mu \in \overline{\mathbf{Q}}_l$, well defined up to an integral power of q^δ. Assume that ρ gives rise to a triple (P, M, ρ_1) as in 3.25 (with ρ_1 a unipotent cuspidal M^F-module). Now to ρ_1 there corresponds $\mu_1 \in \overline{\mathbf{Q}}_l$ (describing the eigenvalue of F^δ on an M^F-submodule isomorphic to ρ_1 of the cohomology of some X_{w_1} (relative to M)). Note that G and M have the same δ, except when M is a torus. It is easy to see that μ and μ_1 are equal up to a power of q^δ (when M is not a torus); when M is a torus, μ is a power of q^δ. Thus, in order to determine the eigenvalues of F^δ on $H_c^i(X_w, \mathbf{Q}_l)$ up to powers of q^δ it is enough to determine the numbers μ associated to unipotent cuspidal representations of the various M. Let S be the set of all eigenvalues of F^δ on $H_c^i(X_w, \mathbf{Q}_l)$, for various $i \geqslant 0$, $w \in W$, taken up to integral powers of q^δ. We have the following

THEOREM 3.34. (i) *If (G, F) is of type A_l $(l \geqslant 1)$, then all eigenvalues of F^δ on $H_c^i(X_w, \mathbf{Q}_l)$ $(w \in W)$ are integral powers of q, so that $S = \{1\}$.*

(ii) *If (G, F) is of type ${}^2A_l$ $(l \geqslant 2)$, then all eigenvalues of F^2 on $H_c^i(X_w, \mathbf{Q}_l)$ $(w \in W)$ are integral powers of $(-q)$.*

(iii) *If (G, F) is of type G_2 or E_6, then S consists of $1, -1$ and the primitive cubic roots of 1.*

(iv) *If (G, F) is of type E_7, then S consists of $1, -1$, the primitive cubic roots of 1 and of $\pm\sqrt{-q}$.*

(v) *If (G, F) is of type B_l, C_l $(l \geqslant 2)$ (resp., D_l $(l \geqslant 4)$) and if $q \geqslant 2l$ (resp. $q \geqslant 2l - 2$) then $S = \{\pm 1\}$.*

(vi) *If (G, F) is of type ${}^2D_l$ $(l \geqslant 4)$ and if $q \geqslant 2l-2$ then $S \subset \{\pm 1\}$.*

(vii) *If (G, F) is of type ${}^4D_3$ and if q is large, then $S = \{\pm 1\}$.*

(viii) *If (G, F) is of type ${}^2E_6$ and if q is large, then S consists of $1, -1$ the primitive cubic roots of 1 and of $-q$.*

(ix) *If (G, F) is of type F_4 and if q is large, then S consists of $1, -1$, the primitive cubic roots of 1 and the primitive fourth roots of 1.*

Part (i) follows from Corollary 3.21 (with $e = 0$); part (ii) follows from Corollary 3.21 (with $e = 1$) and from 3.10(b). Part (iii) for (G, F) of type E_6 follows from Theorem 3.24 and Table I; when (G, F) is of type G_2 we must also check that the fourth unipotent cuspidal representation ρ of G^F defined in Theorem 3.28 has associated with it the eigenvalue $\mu = \pm 1$. As $\langle \rho, R_{w_0} \rangle \neq 0$, we see from the discussion in 3.10(b) that it is enough to prove the following statement: t_{w_0} acts on any irreducible representation of $\mathcal{H}(\mathcal{W}, -q)$ ($\mathcal{W}$ of type G_2), as scalar multiplication by $\pm q^n$ (for an integer $n \geqslant 0$). This can be easily checked. Part (iv) follows from Theorem 3.24 and Table I. For parts (v) and (vi) we notice that a unipotent cuspidal representation of G^F is necessarily self-dual (since it is unique if it exists), therefore we may apply to it the discussion in 3.10(c). (Note that the $\mathbf{F}_q$-rank of G is always even in these cases, whenever G^F has a unipotent cuspidal representation, see Theorem 3.22.)

In the cases (vii), (viii), (ix) we must only check that the unipotent cuspidal representations of G^F described in Theorem 3.29 (and which are not described in Theorem 3.23) are self-dual, for then we may use the discussion in 3.10(c). But they are the only unipotent cuspidal representations of G^F which have inner product zero with R_w (w as in Theorem 3.23), and the character of R_w has integral values. This is already enough for the cases (vii), (viii); in the case (ix), we observe (using Lemma 3.30) that the three representations of G^F under consideration have distinct inner products with R_{w_0} (4, 12, 16 respectively) hence one cannot be the dual of another one, which proves that they are all self-dual. This completes the proof.

REMARKS. (a) We see that in all cases considered in Theorem 3.34 (i.e. for G not of type E_8) the eigenvalues of F^δ are roots of 1 times integral powers of q^δ except when (G, F) is of type E_7, ${}^2E_6$ or ${}^2A_l$. Let us consider more closely the case ${}^2A_l$, with $l + 1 = \frac{1}{2}(s^2 + s)$ for some integer $s \geqslant 2$. Then G^F has a unique unipotent cuspidal representation ρ which is necessarily self-dual. Let $\mu \in \overline{\mathbf{Q}}_l$ be the number associated to ρ (see (3.9)). We know from part (ii) of Theorem 3.34 that $\mu = (-q)^n$ for some integer $n \geqslant 1$. If we assume that $q \geqslant l + 1$, we may use the discussion in 3.10(c) and we see that n must have the same parity as the $\mathbf{F}_q$-rank of G^F, which equals $(l + 1)/2$ if $l + 1$ is even and equals $l/2$ if l is even.

It follows that μ is an integral power of q^2, if s is congruent to 6, 7, 8 or 9 modulo 8 and that μ is of the form $(-q)$ times an integral power of q^2, if s is congruent to 2, 3, 4 or 5 modulo 8.

(b) It is extremely likely that the assumptions on q in (v)–(ix) are unnecessary.

Part 4. Some Open Problems

4.1. We wish to describe a conjectural relationship, in the case of classical groups, between the unipotent representations of G^F and the class functions $R(E)$ on G^F defined by (3.17.1). These are essentially equivalent with giving formulae for the multiplicities $\langle \rho, R_w \rangle$, where ρ is a unipotent representation of G^F, and $w \in W$.

We shall use the formalism of symbols [18, §3]. A *symbol* is an unordered pair $\Lambda = (S, T)$ of finite subsets of $\{0, 1, 2, \ldots\}$. The *rank* of Λ is defined by

$$\mathrm{rk}(\Lambda) = \sum_{\lambda \in S} \lambda + \sum_{\mu \in T} \mu - \left[\left(\frac{a+b-1}{2}\right)^2\right]$$

where $a = |S|$, $b = |T|$ and, for any real number z, we denote by $[z]$ the largest integer m such that $m \leqslant z$.

The defect of Λ is defined by $\mathrm{def}(\Lambda)$ = absolute value of $|S| - |T|$.

One can check easily that $\mathrm{rk}(\Lambda) \geqslant |(\mathrm{def}(\Lambda)/2)^2|$.

A symbol $\Lambda = (S, T)$ is said to be *reduced* if $0 \notin S \cap T$. The following result was proved in [18, §8].

THEOREM 4.2. *Assume that G is of type B_l or C_l $(l \geqslant 1)$. There is a natural* 1-1 *correspondence $\Lambda \longrightarrow \rho_\Lambda$ between the set of reduced symbols of rank l and odd defect and the set of (isomorphism classes of) unipotent representations of G^F.*

On the other hand, by [18, (2.7(i))] there is a natural 1-1 correspondence $\Lambda \longrightarrow E_\Lambda$ between the set of reduced symbols of rank l and defect one and the set of (isomorphism classes of) irreducible representations of the Weyl group W of G. Hence we may consider the element $R(E_\Lambda)$ of the Grothendieck group of virtual representations of G^F tensored by $\mathbf{Q}$ (see 3.17).

To describe the relationship between ρ_Λ and $R(E_\Lambda)$ we need some further remarks. Note that a reduced symbol of rank l and odd defect can be written uniquely as an ordered pair $(X \cup (Y - I), X \cup I)$ where X, Y are disjoint subsets of $\{0, 1, 2, \ldots\}$ such that $0 \notin X$, $|Y| \equiv 1 \pmod 2$ and where I is a subset of Y such that $|Y| \equiv 2|I| + 1 \pmod 4$.

Define a partition $Y = Y^0 \cup Y^1$ as follows: if $Y = \{\lambda_0 < \lambda_1 < \cdots < \lambda_{2s}\}$, we set

$$Y^0 = \{\lambda_0 < \lambda_2 < \cdots < \lambda_{2s-2} < \lambda_{2s}\}, \quad Y^1 = \{\lambda_1 < \lambda_3 < \cdots < \lambda_{2s-1}\}.$$

We now define $\hat{\rho}_{(X \cup (Y-I), X \cup I)}$ by the formula

$$(-1)^{|I\cap Y^0|}\hat{\rho}_{(X\cup(Y-I),X\cup I)} = 2^{-s} \sum_{\substack{J\subset Y\\ |J|\equiv s \,(\mathrm{mod}\ 2)}} (-1)^{|I\cap J|}(-1)^{|J\cap Y^1|}\rho_{(X\cup(Y-J),X\cup J)}, \tag{4.2.1}$$

where $s = \frac{1}{2}(|Y| - 1)$. (This is essentially a Fourier transform over a vector space of dimension $2s$ over F_2.)

Conjecture 4.3. *If Λ is a symbol of defect one then $\hat{\rho}_\Lambda = R(E_\Lambda)$. If Λ has defect >1 then $\hat{\rho}_\Lambda$ is orthogonal to all R_w $(w \in W)$.*

Using the Fourier inversion formula we deduce that

$$(-1)^{|I\cap Y^1|}\rho_{(X\cup(Y-I),X\cup I)} - 2^{-s}\sum_{\substack{J\subset Y\\ |J|=s}} (-1)^{|I\cap J|}(-1)^{|J\cap Y^0|}R(E_{(X\cup(Y-J),X\cup J)}) \tag{4.3.1}$$

is orthogonal to all R_w $(w \in W)$, hence

$$(-1)^{|I\cap Y^1|}\langle\rho_{(X\cup(Y-I),X\cup I)}, R_w\rangle = 2^{-s}\sum_{\substack{J\subset Y\\ |J|=s}} (-1)^{|I\cup J|}(-1)^{|J\cap Y^0|}\mathrm{Tr}(w, E_{(X\cup(Y-J),X\cup J)}. \tag{4.3.2}$$

The fact that the dimension of (4.3.1) is zero has been checked in [18, §1.2]. An analogous conjecture can be made for the even orthogonal groups. The conjecture implies that ρ_Λ is uniform (see 2.15) if and only if $s = 0$, or equivalently if $\Lambda = (S, T)$ with $|(S \cup T) - (S \cap T)| = 1$. In this case, Λ must have defect one and $\rho_\Lambda = R(E_\Lambda)$.

4.4. Let us consider the 37 polynomials in q giving the degrees of the unipotent representations of G^F for G of type F_4, for q large (see 3.31). Eight of them are polynomials with integral coefficients. It is likely that the corresponding representations are equal to $R(E)$ (for some $E \in \hat{W}$) and, in particular, are uniform. Next, there are four representations of degree $\frac{1}{2}q + \cdots + \frac{1}{2}q^{11}$ and four representations of degree $\frac{1}{2}q^{13} + \cdots + \frac{1}{2}q^{23}$. These will probably form the same pattern as the four nonuniform unipotent representations of a group of type B_2.

There remain 21 representations; their degrees are of the form $q^4/n + \cdots + q^{20}/n$, where n is an integer in the following list

$$(4, 4, 4, 4; 3, 3, 3; 8, 8, 8, 8, 4; 4, 4, 4, 4; 24, 24, 8, 8, 12). \tag{4.4.1}$$

To find an interpretation of this list, we consider the symmetric group $\mathfrak{S}_4$ and we construct the set $M(\mathfrak{S}_4)$ consisting of all pairs (x, σ) where x is an element of $\mathfrak{S}_4$ (up to conjugacy) and σ is an irreducible representation of the centralizer $Z(x)$ of x in $\mathfrak{S}_4$. It is easy to see that $M(\mathfrak{S}_4)$ has exactly 21 elements. Moreover the 21 numbers $|Z(x)|/\dim\sigma$ (for $(x, \sigma) \in M(\mathfrak{S}_4)$) form a list identical to (4.4.1).

It seems natural to expect that there is a canonical 1-1 correspondence between $M(\mathfrak{S}_4)$ and the set of 21 representations of G^F considered above.

This correspondence should have the following property: if ρ is the representation of G^F corresponding to $(x, \sigma) \in M(\mathfrak{S}_4)$, then the eigenvalue μ of Frobenius attached to ρ (see 3.9) is equal to an integral power of q times the scalar by which x acts on the $Z(x)$-module σ.

The reason for which $\mathfrak{S}_4$ plays a role here is probably the following one: G has a unique unipotent class C such that the dimension of the variety of Borel subgroups containing u ($u \in C$ fixed) equals 4 (at least for large characteristic). It is known [21] that the group of components of the centralizer of u ($u \in C$ fixed) is isomorphic to $\mathfrak{S}_4$. There is considerable evidence to suggest that this is a general phenomenon.

We shall illustrate this in the case where G is of type E_8. The 6 unipotent cuspidal representations of G^F described in Theorem 3.23 have degree of the form $q^{16}/n + \cdots + q^{104}/n$ where $n = 5$ or 6.

It may be conjectured that G^F has exactly 39 unipotent representations with degree given by a polynomial in q of the form $q^{16}/n + \cdots + q^{104}/n$, $(n \geqslant 2)$. They should be in a natural 1-1 correspondence with the set $M(\mathfrak{S}_5)$ defined in analogy with $M(\mathfrak{S}_4)$. The eigenvalues of Frobenius attached to these 39 representations will be determined as before. (Note that, at least for large characteristic, there exists a unipotent class C in G such that if $u \in C$, the variety of Borel subgroups containing u has dimension 16 and the group of components of the centralizer of u is isomorphic to $\mathfrak{S}_5$; this was shown by Steinberg (unpublished).) Presumably, all unipotent cuspidal representations of G^F will be among these 39 representations.

Added in proof. The case of E_8 is discussed in the author's forthcoming paper entitled "Unipotent representations of a finite Chevalley group of type E_8."

Bibliography

1. C. T. Benson and C. W. Curtis, *On the degrees and rationality of certain characters of finite Chevalley groups,* Trans. Amer. Math. Soc. **165** (1972), 251–273.

2. C. T. Benson, L. C. Grove and D. B. Surowski, *Semilinear automorphisms and dimension functions for certain characters of finite Chevalley groups,* Math. Z. **144** (1975), 149–159.

3. N. Bourbaki, *Groupes et algèbres de Lie,* Chapitres 4, 5, 6, Hermann, Paris, 1968.

4. R. W. Carter, *Conjugacy classes in the Weyl group,* Compositio Math. **251** (1972), 1–59.

5. ———, *Simple groups of Lie type,* John Wiley & Sons, London, 1972.

6. B. Chang and R. Ree, *The characters of $G_2(q)$,* Ist. Naz. di Alta Mat. Sympos. Math., vol. XIII, 1974, pp. 385–413.

7. C. Chevalley, *Classification des groupes de Lie algébriques,* Paris, 1956/57.

8. ———, Corrections and additions to [1], Trans. Amer. Math. Soc. **202** (1972), 405–406.

9. C. W. Curtis, N. Iwahori and R. Kilmoyer, *Hecke algebras and characters of parabolic type of finite groups with (B, N) pairs,* Inst. Hautes Études Sci. Publ. Math. **40** (1972), 81–116.

10. P. Deligne, *La conjecture de Weil.* I, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307.

11. P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields,* Ann. of Math. (2) **103** (1976), 103–161.

12. H. Enomoto, *The characters of the finite Chevalley group $G_2(q)$, $q = 3^f$,* Japan J. Math. (N.S.) **2** (1976), no. 2, 191–248.

13. J. A. Green, *The characters of the finite general linear groups,* Trans. Amer. Math. Soc. **80** (1973), 402–447.

14. P. N. Hoefsmit, *Representations of Hecke algebras of finite groups with BN-pairs of classical type,* Ph. D. dissertation, Univ. British Columbia, Vancouver, 1974.

15. J. E. Humphreys, *Linear algebraic groups,* Springer-Verlag, New York, 1975.

16. D. Kazhdan, *Proof of Springer's hypothesis,* Israel J. Math **28** (1977), 272–286.

17. G. Lusztig, *Coxeter orbits and eigenspaces of Frobenius,* Invent. Math. **38** (1976), 101–159.

18. ———, *Irreducible representations of finite classical groups,* Invent. Math. **43** (1977), 125–175.

19. G. Lusztig and B. Srinivasan, *The characters of the finite unitary groups,* J. Algebra **49** (1977), 167–171.

20. Séminaire de Géometrie Algébrique du Bois Marie (S.G.A. 4, 4½, 5), Lecture Notes in Math., vols. 269, 270, 305, 569, 589, Springer-Verlag, Berlin.

21. T. Shoji, *The conjugacy classes of Chevalley groups of type* (F_4) *over finite fields of characteristic* $p \neq 2$, J. Fac. Sci. Univ. Tokyo, Sect. I–A Math. **21** (1970), no. 1, 1–17.

22. T. A. Springer, *Regular elements of finite reflection groups,* Invent. Math., **25** (1974), 159–193.

23. ———, *Caractères de groupes de Chevalley finis,* Séminaire Bourbaki no. 429, 1972/73, Lecture Notes in Math., vol. 383, Springer-Verlag, Berlin.

24. T. A. Springer and R. Steinberg, *Conjugacy classes.* Part E in Seminar on Algebraic Groups and Related Finite Groups by A. Borel et al., Lecture Notes in Math., vol. 131, Springer, Berlin.

25. B. Srinivasan, *The characters of the finite symplectic group* Sp(4, q), Trans. Amer. Math. Soc. **131** (1968), 488–525.

26. R. Steinberg, *Endomorphisms of linear algebraic groups,* Mem. Amer. Math. Soc. No. 80 (1968).

27. ———, *A geometric approach to the representations of the full linear group over a Galois field,* Trans. Amer. Math. Soc. **71** (1951), 274–282.

28. ———, *Lectures on Chevalley groups,* Yale Univ. Press, New Haven, Conn., 1967.

29. D. B. Surowski, Ph. D. dissertation, Univ. of Arizona, 1976.