The algebraic fundamental group

Frans Oort

An apology. - On 12-IV-1996 a "Grothendieck day" was organized in Utrecht. For non-specialists we tried to explain some of the basic ideas of Grothendieck and their impact on modern mathematics. The text below is the contents of one of the talks. It is written for a general mathematical audience. Simplifications were made in an attempt to make these ideas accessible for a general audience, with the effect that it certainly dilutes the ideas discussed. - The reader should not expect anything more than just an informal exposé, which perhaps would suit better the coffee table than any serious publication.

§1. Introduction.

In 1984 *Alexandre Grothendieck* wrote:

..aujourd'hui je ne suis plus, comme naguère,
le prisonnier volontaire de tâches interminables,
qui si souvent m'avaient interdit de m'élancer dans l'inconnu,
mathématique ou non.

See [2], page 51.

(1.1) [EGA] J. Dieudonné and A. Grothendieck published in 1960 - 1967 eight volumes:

Elements de la géométrie algébrique.

There are 4 chapters, in 8 volumes, together more than 1800 pp. All published as volumes in the series Publ. Math. at the IHES.

In 1960 Grothendieck had planned 12 chapters, at that moment he already had a rather clear picture of what should be contained in the various volumes.

(1.2) [SGA] In 1960 - 1969 Grothendieck, together with many collaborators, had seminars:

Séminaire de géométrie algébrique du Bois Marie.

There are 12 volumes , together more than 6200 pp. Eleven volumes are published as Lecture Notes Math., Springer - Verlag; one volume is published by the North - Holland Publ. Cy.

(1.3) [FGA] In 1957 - 1962 Grothendieck gave 8 talks in the Séminaire Bourbaki:

Fondements de la géométrie algébrique.

There are 8 exposés plus comments, together more than 200 pp. They appeared in one volume: Notes, secr. math. Paris, 1962.

(1.4) How can we understand the citation of 1984 above? Does it really imply that the many pages, the impressive results obtained by Grothendieck, say between 1958 and 1970, belonged to "known territory" for him? Amazing!

(1.5) Considering work by Grothendieck we see his insight, we see the influence of his ideas on present day mathematics. Some of the aspects which we see:

- On several occasions Grothendieck revised existing theory in a superb way.
- Notions previously well known but seemingly unconnected were unified by Grothendieck with the effect that totally new insight became available.
- New, mostly revolutionary, ideas were launched by Grothendieck. We are still reaping the fruits, and trying to see where it all leads to.

Just to give a few examples:

Schemes, methods of sheaves and similar concepts were developed. But Chevalley, Serre and many others stayed in line with classical algebraic geometry over a field; Grothendieck generalized the notion of scheme (locally) to arbitrary rings, and many applications became available.

The fundamental idea of A. Weil to prove the Riemann hypothesis over finite fields via an analogue of the Lefschetz fixed point formula had a decisive influence on the research by Grothendieck. As we see in [1] this was a starting point for Grothendieck to set up algebraic geometry on new foundations.

Serre had the idea of "isotrivial fibre spaces" meeting the desire to describe certain maps in the Zariski topology, see [1], page 104; it took the insight of Grothendieck to see that quite generally "Grothendieck topologies" give the natural framework... Etale cohomology filled in the need for a cohomology theory which could lead to a proof of the Weil conjectures, see [1], page 104.

Fundamental groups and Galois groups existed, Grothendieck realized that arithmetic applications could follow once you see these as two aspects of one theory, to exploit the arithmetic in geometric situations. Below we give one very particular example of this.

Around 1964 Grothendieck had the idea of studying "motives", e.g. see [20].

Does his idea have traceable roots in earlier mathematics? To my feeling this was a completely new insight at that time. Some people however argue that Galois representations were studied already at the time Grothendieck started in algebraic geometry, and that the idea of considering one piece of (co)homology was present at that time; still I feel it is a big step from previous theory to the basic and functorial way Grothendieck approached this concept.

Many more examples can be given. It is an illuminating exercise to study notions introduced and developed by Grothendieck, and to (try to) see where they came from. In some cases one can see the influence of earlier mathematical ideas; in such cases I am impressed by the way Grothendieck gave a new direction to existing concepts. In other cases I do not see where the new ideas had their origin but in the mind of Grothendieck. It seems a beautiful task for us to unravel his ideas together with the historical roots. It is time to start!

(1.6) We try to explain some ideas by Grothendieck (and we will be very far from anything like a complete survey). We have tried to show his work and his ideas between two extremes in time. Some of his ideas around the fundamental group mainly stem from 1960-1964; for Grothendieck this was known territory.... Can we imagine where his ideas, "l'inconnu", in his "longue marche" (1981) might lead us?

(1.7) Remark: In [23] we find a fierce defense by Francesco Severi of "Italian algebraic geometry". It would be nice to have a close historical analysis of the arguments raised by Severi. Certainly one could encounter the following (and much more).

1) By phrasing questions in a certain way, by laying foundations in a certain manner, some aspects are focussed upon, but other do not show up simply because of the way the questions are asked. It would be nice to see what is the influence of the various ways algebraic geometry was approached in this century.

2) As Schappacher and Schoof warn us, see footnote (2) on page 59 of [22], modern schools of algebraic geometry do not always place algebraic geometry performed in the first half of this century in the correct context. Should we talk about "Italian algebraic geometry" (as Severi does)? What do we mean, and how do we place this in the correct historical context?

3) Let us be more concrete about one aspect. In the first page and his footnote (1) of [23] Severi comments on algebraic geometry over the complex numbers and in the "abstract case". We have seen in the development of algebraic geometry in which way the geometric ideas triggered by questions coming from number theory lead to fundamentally new ideas. It would be nice to describe these historical lines, not in the sense of a discussion about "rigor", but in the spectrum of various questions with different backgrounds about basically the same kind of objects, of the same kind of theory. And we would conclude how much geometry and the abstract case have profited

from the insight which Grothendieck has taught us. Applications in arithmetic geometry (such as Weil conjectures, Ramanujan conjecture, Mordell conjecture, Shafarevich conjecture, Tate conjectures) are unthinkable in the classical style, these really need Grothendieck's foundations of algebraic geometry. It would be nice to have a good historical view on these developments and their background.

§2. Schemes

In [7] we find the word "scheme". Chevalley introduced this name in 1955. Note that in that context only rings contained in a field were considered. It seems natural from a classical point of view. Grothendieck generalized the notion, saw that a theory with less conditions on the objects to be studied gives much more possibilities. Here is the central idea.

Convention. *All rings considered are supposed to be commutative, and are supposed to have an identity element.* NB. the ring $A = \{0\}$ is not excluded. The term "prime ideal" (an ideal such that $x, y \in R$ and $x{\cdot}y \in I \Rightarrow x \in I$ or $y \in I$) will be used only for proper ideals.

Example. Let V be an affine variety, with coordinate ring A, and let $W \subset V$ be a subvariety. Then $\mathcal{I}(W) = I \subset R$, the set of "polynomial" functions which are zero on W is a prime ideal in A. (Note: in the theory of function algebras this is the usual way to recover points by considering maximal ideals in the ring of functions. By his pre-1958 interests Grothendieck knew that very well.)

Example. Let A and B be commutative rings, $\varphi : A \to B$ a ring homomorphism, and let $J \subset B$ be a prime ideal. Then

$$\varphi^{-1}(J) \subset A$$

is a prime ideal.

From a classical point of view it seems much more natural to consider maximal ideals (corresponding with true geometric points in the classical sense), but note that for a maximal ideal $J \subset B$, the ideal $\varphi^{-1}(J) \subset A$ need not be maximal, e.g. $A = \mathbb{Z} \to \mathbb{Q} = B$, and $J = (0)$. Hence we have to consider all prime ideals as "points", which was counter-intuitive for many people.

(2.1) $\pm$ Definition: Let A be a ring (commutative, with $1 \in A$). We define the *affine scheme* $X = \mathrm{Spec}(A)$ as follows: the points of X are the prime ideals of A, the topology on X is the Zariski topology (for $S \subset A$ we define $\mathcal{Z}(S) \subset X$ as the set of ideals containing S, these are the closed sets of

X), and for every point $x \in A$ corresponding with a prime ideal $I \subset A$ we define the local ring $\mathcal{O}_{X,x} := A_I$, the localization of A in I. (See EGA I for more extensive definitions, also see [11], Chapter II; we should talk about sheaves, and so on, to obtain precise definitions).

As is usual in geometry one can define geometric objects (manifolds, abstract varieties, schemes) by atlasses, once charts are given. This can be done: using affine schemes, one defines schemes, which are given locally as affine schemes. Certainly this aspect deserves a longer introduction.

(2.2) Exercise. Place yourself in the situation that you know algebraic geometry à la André Weil very well, and that you know the theory of sheaves (as was studied in the Séminaire Cartan, as in Serre's FAC). Try to find good foundations for algebraic geometry. Would you arrive at FAC, or something like [8], or closer to FGA ?

§3. Galois groups of field extensions, fundamental groups of topological spaces.

(3.1) Galois groups. Let K be a field, and let $K \subset L$ be a finite extension. Let $G = \mathrm{Aut}(L/K)$ be the group of K-automorphisms of L (elements of K are left fixed). One can show that $\#(G) \leq [L : K]$, and in fact:

$$\#(G) = [L : K] \quad \Longleftrightarrow \quad L/K \quad \text{is normal and separable.}$$

Such a finite extension is called a *finite Galois extension.* For such an extension the Galois correspondence gives a bijective correspondence between intermediate fields $K \subset E \subset L$ and subgroups $G \supset H \supset \{1\}$ by:

$$H \mapsto L^H, \quad E \mapsto \mathrm{Aut}(L/E).$$

An algebraic extension (not necessarily finite) $K \subset L$ is called a Galois extension if it is normal and separable.

(3.2) Fundamental groups. Let X be a topological space, which we suppose to be arcwise connected. Suppose we are given a base point $x \in X$. Loops in X starting and ending at x can be composed; up to homotopy they form a group, denoted by $\pi_1(X,x)$, called the *fundamental group.* In order to avoid confusion below here we shall write $\pi_1^{top}(X,x)$, and say *the topological fundamental group.*

Covering spaces. Suppose moreover that X is locally arcwise connected and locally simply connected. There exists a "universal covering space" $\tilde{X} \to X$ such that $G := \pi_1^{top}(X,x)$ acts on $\tilde{X}/X$, such that for every connected covering $Y \to X$ there exists a subgroup $H \subset G$ such that the covering Y/X fits uniquely in a diagram

$$\tilde{X} \longrightarrow H\backslash\tilde{X} \cong Y \longrightarrow G\backslash\tilde{X} \cong X.$$

This gives a bijection between such coverings and subgroups of $\pi_1^{top}(X, x)$.

Here is a basic example: Let $X = S^1$, the circle, e.g. $X = \{z \in \mathbb{C} \mid |z| = 1\}$. Then

$$\pi_1^{top}(X, x) \cong \mathbb{Z},$$

and the universal covering space can be given by:

$$\tilde{X} \approx \mathbb{R} \quad \longrightarrow \quad X = S^1, \quad \text{by} \quad x \quad \mapsto \quad e^{2\pi\sqrt{-1}\cdot x}$$

Any finite connected covering $Y \to X = S^1$ can be given by a subgroup of finite index of $\pi_1^{top}(X, x)$:

$$Y \cong S^1 \longrightarrow X = S^1 \quad \text{with} \quad s \mapsto s^n = z \in X.$$

(3.3) We summarize: if $K \subset L$ is a finite Galois extension with $G = \mathrm{Gal}(L/K)$, then

$$G \supset H = \mathrm{Aut}_K(E) \supset \{1\} \leftrightarrows K \subset L^H = E \subset L,$$

and

$$\tilde{X} \to H\backslash\tilde{X} \cong Y \to G\backslash\tilde{X} \cong X \quad \leftrightarrows \quad \{1\} \subset H \subset G = \pi_1^{top}(X, x).$$

(3.4) Exercise. Place yourself in the situation that you know these correspondences in Galois theory and in the topological theory of coverings. Try to find a general theory of which these two are special cases.

§4. Profinite completions. Actions defined by an extension.

In this section we gather some easy algebraic tools.

(4.1) Let G be a group. Consider all surjective homomorphisms $G \to I$, where I is a *finite group*, i.e. consider all normal subgroups $H \subset G$ of finite index (and write $G \to H\backslash G \cong I$). The system of such finite quotients forms a *projective system* (sometimes called an *inverse system*). The limit is written as

$$\varprojlim I = \hat{G}.$$

Note that for normal subgroups of finite index $H_I \subset H_J \subset G$ we have a natural homomorphism $\varphi_I^J : J \to I$, and we write:

$$\hat{G} := \{\{x_{H_I}\} \in \prod (H_I\backslash G) \,|I = H_I\backslash G, \text{ and } \\ \varphi_I^J(x_J) = x_J \bmod H_I = x_I \;\; \forall H_I \subset H_J.\}$$

It is called the *profinite* completion of G. There is a canonical homomorphism

$$G \longrightarrow \hat{G}.$$

On $\hat{G}$ we have the profinite topology: every finite quotient $H\backslash G \cong I$ is given the discrete topology (every subset is open), and the projective limit is given the limit topology. This can be characterized by: it is the coarsest topology which makes all projections $\varphi_I : G \to I$ continuous. Or: for any I, for any $a \in I$ take $(\varphi_I)^{-1}(a)$, and use these as a basis for all open sets in the topology on $\hat{G}$. Note that the image of $G \to \hat{G}$ is dense in $\hat{G}$.

(4.2) Examples. 1) Consider $G = \mathbb{Z}$. The finite quotients are $\mathbb{Z} \to \mathbb{Z}/n$ for $n \in \mathbb{Z}_{\geq 0}$. The projective limit is isomorphic to:

$$\hat{\mathbb{Z}} \cong \prod_{\ell} \mathbb{Z}_\ell,$$

product taken over all prime numbers ℓ; here we write $\mathbb{Z}_\ell$ for the (additive group of) ℓ-adic integers. Note that the natural map $\mathbb{Z} \to \hat{\mathbb{Z}}$ is injective.
2) It can happen that the map $G \to \hat{G}$ is not injective, e.g. $\hat{\mathbb{Q}} = \{0\}$. Below we shall see more interesting examples.
3) Let p be a prime number, let $\mathbb{F}_p$ be the field having p elements, let $k = \overline{\mathbb{F}_p}$ be its algebraic closure (which is an algebraic, separable, normal extension). Every finite extension of $\mathbb{F}_p$ of degree n has a cyclic group $\mathbb{Z}/n$ of automorphisms, and in fact

$$\mathrm{Gal}(k/\mathbb{F}_p) \cong \hat{\mathbb{Z}}.$$

(4.3) Actions defined by an extension. Suppose we have a group π, and a normal subgroup $N \subset \pi$, i.e. an exact sequence

$$1 \to N \longrightarrow \pi \longrightarrow \pi/N =: G \to 1.$$

This extension defines a representation

$$\rho : G \longrightarrow \mathrm{Out}(N);$$

here $\mathrm{Out}(N) := \mathrm{Aut}(N)/\mathrm{Inn}(N)$ is the group of "outer automorphisms", i.e. the factor group of automorphisms of N by the normal subgroup consisting of inner automorphisms. The representation is obtained as follows: for $\sigma \in G$, choose a lifting $\sigma' \in \pi$; it acts by inner conjugation in π on the normal subgroup $N \subset \pi$. In this way we obtain an automorphism

$$\varphi_{\sigma'} \mid_N : N \longrightarrow N.$$

Taking another lifting of σ the new action differs from the old one by an inner automorphism of N:

$$\rho(\sigma) := (\varphi_{\sigma'} \mid_N) \bmod \mathrm{Inn}(N) \in \mathrm{Aut}(N)/\mathrm{Inn}(N)$$

is well-defined; this defines a homomorphism ρ.

Such actions play an important role if one considers "arithmetic fundamental groups", i.e. a combination of a geometric fundamental group N and a Galois group G. This is an algebraic way to see how in a family the fundamental group of a base acts on the geometric fundamental group of a fibre.

Remark. A section of $\pi \to \pi/N =: G$ defines a homomorphism $G \to \mathrm{Aut}(N)$.

Suppose H is commutative, and

$$1 \to H \longrightarrow \pi \longrightarrow \pi/H =: G \to 1$$

exact. The we obtain a representation

$$G \longrightarrow \mathrm{Aut}(H).$$

This plays an important role in the theory of algebraic monodromy. For example we can obtain such a situation by choosing $H = N_{Ab}$, the abelianized N, starting with an extension $\pi'/N = G$, and pushing this out, achieving the above exact sequence $\pi/H = G$.

§5. The algebraic fundamental group

Seeing the definition of a Galois group, with the property $K = L^G$ (as above) on the one hand, and the definition of the fundamental group, with the property $X = G\backslash(\tilde{X})$ on the other hand, we can follow Grothendieck in his idea that these are two aspects of one more general concept.

In order to define this concept one needs the notion of "unramified" both in an algebraic setting and in a topological context. An inseparable extension looks like a ramified extension (in the sense that "roots coming together" is the geometric notion of being ramified). This can be combined in one notion.

To this end Grothendieck defined the notion of an *étale morphism* which combines these two aspects;

La mer était étale, mais le reflux commençait à se faire sentir.
V. Hugo, *Les travailleurs de la mer.*

(see [14], page v.)

An étale morphism is a smooth morphism of relative dimension zero. This implies it is flat and unramified (see [11], II.10, and Ex. III.10.3). If a morphism $f : Y \to X$, of finite type, is étale then it is flat, and for every $y \in Y$, with $f(y) =: x \in X$, the extension

$$\mathcal{O}_{X,x}/m_{X,x} \subset \mathcal{O}_{Y,y}/m_{X,x}\mathcal{O}_{Y,y} \quad \text{is a finite separable extension of fields.}$$

Here are some very easy examples which could explain this concept:
1) The morphism $\mathrm{Spec}(\mathbb{C}[S]) \to \mathrm{Spec}(\mathbb{C}[T])$ via $T \mapsto S^2$ is ramified, because

$$\mathbb{C}[S] \otimes_{\mathbb{C}[T]} \mathbb{C}[T]/(T) \cong \mathbb{C}[S]/(S^2),$$

which is not a field. Geometrically: the map $z \mapsto z^2$ on the complex plane is two-to-one outside $z = 0$, but one-to-one at $z = 0$, i.e. ramified at that point.
2) The morphism $\mathrm{Spec}(\mathbb{Z}[\sqrt{-1}]) \to \mathrm{Spec}(\mathbb{Z}) = X$ is ramified at $(2) = x \in X$, and unramified elsewhere. For $p = 5$ we have a factorization $5 = (2+i)\cdot(2-i)$, the prime $p = 3$ is also prime in $\mathbb{Z}[i]$, and above $\mathbb{Z}/(3)$ there are two geometric points in $\mathrm{Spec}(\mathbb{Z}[\sqrt{-1}])$. We see that $\mathbb{Z}[\sqrt{-1}]/(p)$ is a field or a product of fields for prime numbers $p > 2$, and for $p = 2$ this ring has nilpotents:

$$\mathbb{Z}[\sqrt{-1}] \otimes (\mathbb{Z}/2) \quad \cong \quad \mathbb{F}_2[\varepsilon]/(\varepsilon^2).$$

Note that $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}, \sqrt{-1}]) \to \mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$ is an étale morphism.
3) The extension $\mathbb{F}_p(t) = K \subset L = \mathbb{F}_p(\sqrt[p]{t})$ is inseparable, and $\mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is not étale.

(5.1) Construction/Properties: Suppose given a connected scheme X, an algebraically closed field Ω, and a (geometric) point $x \in X(\Omega)$. Consider all étale coverings $f : Y \to X$. For a given X we want to classify such coverings. There exists a profinite group $\pi = \pi_1(X, x)$, called the (algebraic) fundamental group of X with base point x, such that there is an equivalence:

$$\text{finite, continuous} \quad \pi - \text{sets} \quad \leftrightarrows \quad \text{étale coverings of } X;$$

in this correspondence to a covering f we associate the set $S := f^{-1}(x)$. We refer to SGA I, or to [15] for details.

Here are two special cases:
If K is a field, with K^s its separable closure, there is an isomorphism:

$$\mathrm{Gal}(K^s/K) \quad \cong \quad \pi_1(X)$$

(one has to specify a base point, etc.).
Let X be an (irreducible) algebraic variety defined over $\mathbb{C}$, then the profinite completion of the topological fundamental group is canonically the algebraic fundamental group:

$$\pi_1^{top}(X(\mathbb{C})) \to \hat{\pi}_1^{top}(X(\mathbb{C})) \quad \cong \quad \pi_1(X).$$

Conclusion. We see that the theory of the algebraic fundamental group combines Galois theory and the theory of (profinitely completed) fundamental groups of algebraic varieties.

(5.2) Properties of the arithmetic fundamental group. Let K be a field (e.g. think of $K = \mathbb{Q}$), and let X be an algebraic variety defined over K. Then we have morphisms

$$X \otimes \overline{K} =: \overline{X} \to X \to \mathrm{Spec}(K).$$

Suppose we have geometric points

$$\overline{a} \in \overline{X}(\Omega), \quad a \in X(\Omega), \quad b : K \to \Omega$$

mapping to each other. Grothendieck proves that there exists an exact sequence:

$$1 \to \pi_1(\overline{X}, \overline{a}) \to \pi_1(X, a) \to \pi_1(K, b) \cong \mathrm{Gal}(K^s/K) \to 1.$$

See SGA I, pp. 251, 268. (Intuitively this is clear: a function field has "two kind of extensions": one can extend the base field, and one can extend the function field by taking coverings! Combination of these two yields all coverings, that is what the theorem says.)

(5.3) Anabelian algebraic geometry. The Galois group of the ground field K operates (via inner conjugation in $\pi_1(X)$) as outer automorphisms on the (profinite completion of the) geometric fundamental group $\pi_1(\overline{X})$. And Grothendieck writes:
"C'est ainsi que mon attention s'est portée vers ce que j'ai appelé depuis la **"géométrie algébrique anabélienne"** *...des groupes fondamentaux qui*

sont très éloignés des groupes abéliens (et que pour cette raison je nomme **"anabéliens"***)..."*
see [2], page 14. Strictly speaking there is no precise definition for the notion "anabelian", but it indicates a type of questions.

(5.4) Example. Let $X = \mathbb{P}^1_{\mathbb{Q}} - \{0, \infty\}$. What is $= \pi_1(X)$? Note that $\mathbb{P}^1_{\mathbb{Q}} - \{0, \infty\}$ =Spec($\mathbb{Q}[X, \frac{1}{X}]$). Further note that $X(\mathbb{C}) = \mathbb{C}^*$; from this we easily see that

$$\pi_1^{top}(X(\mathbb{C})) \cong \mathbb{Z}, \quad \text{and} \quad \pi_1(X \otimes \overline{\mathbb{Q}}) \cong \hat{\mathbb{Z}}.$$

We need to know how the Galois group of $\mathbb{Q}$ acts on this. In order to study this we first consider
An easy example (Kummer theory). Consider the field $M := \mathbb{Q}(t)$, a purely transcendental extension of $\mathbb{Q}$, let $n \in \mathbb{Z}_{>0}$, and let N be the splitting field of $X^n - t \in M[X]$. What is Gal(N/M)? Note that we have a tower

$$M := \mathbb{Q}(t) \quad \subset \quad \mathbb{Q}(\zeta_n)(t) \quad \subset \quad N,$$

and

$$\mathrm{Gal}(\mathbb{Q}(\zeta_n)(t)/M) = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \cong (\mathbb{Z}/n)^*, \text{ and } \mathrm{Gal}(N/\mathbb{Q}(\zeta_n)(t)) \cong \mathbb{Z}/n.$$

In the exact sequence

$$1 \to \mathrm{Gal}(N/\mathbb{Q}(\zeta_n)(t)) \longrightarrow \mathrm{Gal}(N/M) \longrightarrow \mathrm{Gal}((\mathbb{Q}(\zeta_n)(t)/M) \to 1$$

the cokernel Gal$(\mathbb{Q}(\zeta_n)(t)/M) \cong (\mathbb{Z}/n)^*$ acts by inner conjugation inside Gal(N/M) on the abelian kernel Gal$(N/\mathbb{Q}(\zeta_n)(t)) \cong \mathbb{Z}/n$. What is this action? An easy exercise shows it is the natural way $(\mathbb{Z}/n)^*$ acts on $\mathbb{Z}/n$ (check !).

Conclusion. The fundamental group $\pi_1(\mathbb{P}^1_{\mathbb{Q}} - \{0, \infty\})$ fits into an exact sequence:

$$0 \to \hat{\mathbb{Z}} \longrightarrow \pi_1(\mathbb{P}^1_{\mathbb{Q}} - \{0, \infty\}) \longrightarrow \mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q}) \to 0,$$

and the representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(\hat{\mathbb{Z}})$$

defined by this extension is the **cyclotomic character**.

(5.5) Example. Let $X = \mathbb{P}^1_{\mathbb{Q}} - \{0, 1, \infty\}$. Then the topological fundamental group of $X(\mathbb{C})$ is a free group on two generators (on three generators with one relation). The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the profinite completion of this group via the exact sequence above. This is a central theme of research, see [10], see [12]. On many occasions Grothendieck emphasized that this is an important object to study.

(5.6) Remark. There was a striking analogy between Galois extensions of a number field and (ramified) covering of an algebraic curve (Galois extension of a function field in one variable), e.g. see [26]. The theory of the algebraic fundamental group by Grothendieck does more than just hinting at an analogy: it unifies both theories as aspects of one theory. The exact sequence above shows how the arithmetic and geometric aspects combine into an action of Galois groups on (the profinite completion of the geometric) fundamental group.

Once you feel that Galois theory and the theory of the fundamental group should be combined, this will come out. It took the insight of Grothendieck to do so, and to predict what kind of applications will follow.

(5.7) Geometric versus algebraic fundamental group: *In general the algebraic fundamental group does not determine the geometric fundamental group.*

1) Serre constructed an example of an algebraic variety X over a number field K plus two embeddings of K into $\mathbb{C}$ such that the geometric fundamental groups of $X_1(\mathbb{C})$ and $X_2(\mathbb{C})$ are not isomorphic, while the algebraic fundamental groups (i.e. their profinite completions) clearly are isomorphic, see [Serre].

2) In general the canonical homomorphism $G \to \hat{G}$ is not injective; in fact this can also happen for a geometric fundamental group, see an example by Catanese and Tovena, [9].

§6. The punctured disc in an algebraic setting.

(6.1) Inertia. Consider $K = \mathbb{C}((t))$ the field of Laurent power series over the field $\mathbb{C}$ of complex numbers (or more generally any local field with an algebraically closed residue field, say, with a residue class field of characteristic zero). Clearly for every $n \in \mathbb{Z}_{>0}$ we have a finite extension

$$K = \mathbb{C}((t)) \subset \mathbb{C}((\sqrt[n]{t})) = L_n$$

which is finite cyclic with $\mathrm{Gal}(L_n/K) \cong \mathbb{Z}/n$. This is "Kummer theory" which applies because the root of unity $\zeta_n \in \mathbb{C} \subset K$ is in the base field.

Moreover:

$$\bigcup_n L_n \quad \text{is the algebraic closure} \quad \overline{\mathbb{C}((t))}.$$

This is a classical topic; for a proof e.g. see [25], IV.3. We conclude:

$$\pi_1(\mathrm{Spec}(\mathbb{C}((t))) \cong \hat{\mathbb{Z}}.$$

By the way, we should indicate a base point before we can use the notation "π_1"; here we choose an algebraically closed field Ω, fix $x : K = \mathbb{C}((t)) \to \Omega$, which we write as $x : \mathrm{Spec}(\Omega) \to \mathrm{Spec}(K)$, and in fact we were considering $\pi_1(\mathrm{Spec}(K), x)$.

(6.2) Let us denote the disc and the punctured disc by:

$$D := \{z \in \mathbb{C} \mid \quad |z| \leq 1\}, \quad D - \{0\} = D^*.$$

An algebraic extension

$$\mathbb{C}((\sqrt[n]{t})) \supset K = \mathbb{C}((t))$$

gives rise to a map

$$\mathbb{C}^* \stackrel{n\,exp}{\longrightarrow} \mathbb{C}^*, \quad s \mapsto s^n,$$

by restriction this gives

$$D^* \stackrel{n\,exp}{\longrightarrow} D^*,$$

hence

$$S^1 \stackrel{n\,exp}{\longrightarrow} S^1,$$

here $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.

Conclusion.

$$\pi_1^{top}(S^1) \cong \mathbb{Z} \quad \longrightarrow \quad \hat{\mathbb{Z}} \cong \pi_1(\mathrm{Spec}(\mathbb{C}((t))) = \mathrm{Gal}(\overline{K}/K).$$

This is a particular case of the comparison between topological and algebraic fundamental groups.

Remarks. Instead of D^* we can also consider $\mathbb{C}^*$.
We can consider $\mathrm{Gal}(\overline{\mathbb{Q}((t))}/\mathbb{Q}((t)))$. After what has been said it will be clear what the structure of this group is.

§7. The monodromy theorem.

(7.1) In various settings monodromy has been studied. One of the central results is:

"*the eigenvalues of monodromy are roots of unity,*"

i.e. the action is quasi-unipotent. Here we mean that the action of the fundamental group of a parameter space (say around a singular fibre) acts via matrices on a commutative algebraic object such as a (co)homology group, and the eigenvalues of these are roots of unity. Usually this is called the "algebraic monodromy" when the action is on homology classes of cycles, in contrast with "geometric monodromy" when the action on e.g. loops is considered.

The monodromy theorem in the analytic context is about "Picard-Lefschetz" transformations. It was proved by Landman in his Berkeley PhD-thesis, see [13], page 90, (1.6), Theorem 1, and by Steenbrink in his Amsterdam PhD-thesis, see [24], page 50. Also see [17], page 245, (6.1). Monodromy locally on the fibre around one singularity was studied by Brieskorn, [6], page 113, Satz 4.

We sketch the proof by Grothendieck of the monodromy theorem; details can be found in [21], Appendix, pp. 514-516, also see [16], Section 1, especially Coroll. 1.3.

(7.2) An apology. Quite often when giving a talk, Grothendieck would start by writing X, vertical arrow, S, saying "let X be a scheme over S", and in the rest of the talk results would be given in the greatest possible generality. For the audience not always easy to follow. But then, much later only, you discover the enormous quality of avoiding unnecessary assumptions.

In this talk I try to highlight at least one idea by Grothendieck, but, I will not do this in *his* style. I hope you will see what the basic idea is. Certainly many mathematicians in the audience will see the general form in which this should be put.

(7.3) The monodromy theorem (in simplified form). *Let $K = \mathbb{C}(t)$, and let C_K be an algebraic curve over K, absolutely irreducible, complete, and smooth over K. Let $L := \mathbb{C}((t))$, and $C := C_K \otimes_K L$, and let*

$$\rho_I : I \longrightarrow \mathrm{Aut}(H)$$

be the monodromy representation. Here $I = \mathrm{Gal}(\overline{L}/L)$, and H is a cohomology group attached to C, say $H = \mathrm{H}^1_{et}(C, \mathbb{Z}_\ell)$, where we have chosen a prime number ℓ. Let $\gamma \in I(\ell) \cong \mathbb{Z}_\ell$ be a topological generator, and let

$$\rho(\gamma) \in \mathrm{Aut}(H) \cong \mathrm{GL}(2g, \mathbb{Z}_\ell),$$

where g is the genus of C. Then the eigenvalues of the matrix $\rho(\gamma) =: A$ are roots of unity.

Instead of fixing ℓ and considering $\mathrm{H}^1_{et}(C, \mathbb{Z}_\ell)$, we could have started with (ordinary) (co)homology.

In the proof there are two ingredients. Grothendieck remarks that all data can be defined over a much smaller field, so that an arithmetic Galois group acts on the geometric situation. Once you see this the Kummer theory discussed in §5 gives extra structure on the monodromy representation, and we shall see that is all we need for this case.

(7.4) Sketch of the proof. Note that C_K is defined over $K = \mathbb{C}(t)$ by a finite number of polynomial equations. There is an extension of finite type $\mathbb{Q} \subset P$ and a curve $D = D_{P((t))}$ such that $D_{P((t))} \otimes \mathbb{C}((t)) \cong C$. We write $G = \mathrm{Gal}(\overline{P((t))}/P((t)))$. We obtain a representation

$$\rho : G \longrightarrow \mathrm{Aut}(H).$$

Note that the restriction of this to $I \cong \mathrm{Gal}((\overline{P((t))}/\overline{P}((t)))$ is ρ_I. We indicate how to obtain this representation. The fundamental group $\pi_1(D)$ gives an exact sequence $\pi_1(D)/\pi_1(D \otimes \overline{P((t))}) = G$. This defines a representation of G on $(\pi_1(D \otimes \overline{P((t))}))_{Ab}$; note that its ℓ-adic part is:

$$(\pi_1(D \otimes \overline{P((t))}))_{Ab}(\ell) \cong H.$$

Thus we derive the ρ indicated above. Further note that the fact that $P/\mathbb{Q}$ is of finite type implies that for any given prime ℓ the field P does not contain all roots of unity of a power of ℓ. Hence $\bigcup_n P(\zeta_{\ell^n}) \supset P$ is an infinite extension. Hence the image of the cyclotomic character in $\mathrm{Aut}(H)$ is infinite. Using the ideas exposed in (5.4), using that $\mathbb{Z} \to \hat{\mathbb{Z}}$ has a dense image we see: there exists an element

$$\sigma \in \mathrm{Gal}(\overline{P}((t))/P((t)))$$

an integer $q \in \mathbb{Z}_{>1}$ and a topological generator $\gamma \in I(\ell)$ such that

$$\sigma' \cdot \gamma \cdot (\sigma')^{-1} \quad = \quad \gamma^q$$

in $\mathrm{Gal}(\overline{P((t))}/P((t)))$ for any lift σ' of σ.

Applying ρ, and using $\rho(\gamma) = A$ we obtain:

$$S \cdot A \cdot S^{-1} = A^q, \quad \text{with} \quad S := \rho(\sigma').$$

This proves that A and A^q have the same eigenvalues, and raising to the q-th power is a permutation of these eigenvalues. Hence for any such eigenvalue λ it follows that

$$\lambda^{q^{(2g)!}} \quad = \quad \lambda.$$

$\diamondsuit$

References

[1] A. Grothendieck, The cohomology theory of abstract algebraic varieties. Proceed. ICM Edinburgh 1968; Cambridge Univ. Press, 1960; pp. 103-118.

[2] A. Grothendieck, Esquisse d'un programme. Ms. 57 pp., 1984.

[3] A. Grothendieck, Récoltes et semailles: réflexions et témoignages sur un passé de mathématicien. Manuscript, Montpellier, 1985.

[4] "La longue marche à travers la théorie de Galois." Tome 1 (§§ 1-37). Transcr. manuscr. by A. Grothendieck by J. Malgoire (with P. Lochak & L. Schneps), Univ. de Montpellier II, 1995; 253 pp.

[5] "Grothendieck", article in: Science & Vie **935**, August 1995, pp. 52-57.

[6] E. Brieskorn, Die Mononodromie der isolierten Singularitäten von Hyperflächen. Manuscr. Math. **2** (1970), 103-161.

[7] C. Chevalley, Les schémas I, II. Sém. Cartan - Chevalley, **8** (1955/56), Exp. 5, 6. Notes, Secr. math. Paris 1956.

[8] C. Chevalley, Fondements de la géométrie algébrique. Notes, Secr. Math., Paris, 1958.

[9] Classifications of irregular surfaces, Proceed. Trento 1990 (Ed. E. Ballico, F. Catanese & C. Ciliberto), Lect. Notes Math. 1515, Springer - Verlag 1992, "Trento Example 2": Fundamental groups of algebraic varieties; pp. 136-139.

[10] P. Deligne, Le groupe fondamental de la droite projective moins trois points. In: Galois groups over $\mathbb{Q}$ (Ed. Y. Ihara, K. Ribet, J-P. Serre). Math. Sc. Res. Inst. Publ. 16. Springer - Verlag, 1989; pp. 79 - 297.

[11] R. Hartshorne, Algebraic geometry. Grad. Texts Math. 52. Springer - Verlag, first printing: 1977.

[12] Y. Ihara, Braids, Galois groups, and some arithmetic functions. Proceed. ICM Kyoto 1990. Math. Soc. Japan, Springer - Verlag, 1991. Vol. I, pp. 99-120.

[13] A. Landman, On the Picard - Lefschetz transformation for algebraic manifolds acquiring general singularities. Transact. Amer. Math. Soc. **181** (1973), 89 - 126.

[14] J. S. Milne, Etale cohomology. Princeton Math. Series 33, Princeton Univ. Press, 1980.

[15] J. P. Murre, Lectures on an introduction to Grothendieck's theory of the fundamental group. Notes, Tata Inst. Fund. Res., Bombay, 1967.

[16] F. Oort, Good and stable reduction of abelian varieties. Manuscr. Math. **11** (1974), 171 - 197.

[17] W. Schmid, Variation of Hodge structures: the singularities of the period map. Invent. Math. **22** (1973), 211-320.

[18] J-P. Serre, Faisceaux algébriques cohérents. Ann. Math. **61** (1955), 197-278, cited as [FAC], [Œl, 29].

[19] J-P. Serre, Exemples de variétés projectives conjuguées non homéomorphes. C. R. Acad. Sc. Paris **258** (1964), 4194 - 4196 [ŒII, 63].

[20] J-P. Serre, Motifs. In: Journ. arithm. de Luminy, 1989. Astérisque 198-199-200, Soc. Math. France 1991; pp. 333-341.

[21] J-P. Serre & J. Tate, Good reduction of abelian varieties. Ann. Math. **88** (1968), 492-517 [Serre ŒII, 79].

[22] N. Schappacher & R. Schoof, Beppo Levi and the arithmetic of elliptic curves. Math. Intelligencer **18** (1996), 57-68.

[23] F. Severi, La géométrie algébrique Italienne. Sa rigueur, ses méthodes, ses problèmes. Centre Belge Rech. math., Colloq. géom. algébrique. Liège 19-21/XII/1949, pp. 9-55 (1950).

[24] J. H. M. Steenbrink, Limits of Hodge structures. PhD-thesis, Amsterdam, 1974.

[25] R. J. Walker, Algebraic curves. Dover Publ., 1950.

[26] A. Weil, Sur l'analogie entre les corps de nombres algébriques et les corps de fonctions algébriques. Revue Scient. **77** (1939), 104-106 [Collected Papers I, 1939a].

Frans Oort
Mathematisch Instituut
Budapestlaan 6
NL - 3508 TA Utrecht
The Netherlands
email: oort@math.ruu.nl