Intersection theory in algebraic geometry

These are my live-TeXed notes for the course *Math 266: Intersection theory in algebraic geometry* taught by Joe Harris at Harvard, Spring 2015.

Any mistakes are the fault of the notetaker. Let me know if you notice any mistakes or have any comments!

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Introduction

This is a course not only about intersection theory but intended to introduce modern language of algebraic geometry and build up tools for solving concrete problems in algebraic geometry. The textbook is Eisenbud-Harris, *3264 & All That, Intersection Theory in Algebraic Geometry*. It is at the last stage of revision and will be published later this year.

We will fix a base field K, an algebraically closed field of characteristic 0. The starting point of an enormous amount of mathematics (including all of cohomology) is the classical theorem of Bezout:

Theorem 1 Let $C, D \subsetneq \mathbb{P}^2$ be two curves of degree d, c intersecting transversely (i.e., with linearly independent differentials at intersection points), then $\#(C \cap D) = dc$.

Remark 1 Thinking of all polynomials of a fixed dgree on $\mathbb{P}^2$ as a projective space with the coefficients of the polynomials as coordinates. We can deform C, D by moving around their coefficients. We notice intersection number of them is *invariant under deformation* (*preservation of numbers*) and this number is dc. Notice this common cardinality is easily calculated by taking the intersection of d lines and c lines and thus the really crucial thing is the invariance. This observation leads to the notion of *Chow groups*.

Definition 1 For X any variety, a *cycle* is a formal linear combination $\sum n_i Z_i$ of subvarieties $Z_i \subsetneq X$. We say two cycles $Z, Z' \subsetneq X$ are *rationally equivalent* if there exists a family of subvarieties parametrized by $\mathbb{P}^1$ interpolated between them. Namely, there exists $\Phi \subsetneq \mathbb{P}^1 \times X$ a subvariety, not contained in one fiber (better way to say this: Φ is flat over $\mathbb{P}^1$) such that $\Phi_0 = Z$ and $\Phi_\infty = Z'$.

Remark 2 By varieties, we mean irreducible (usually quasi-projective) varieties. By subvarieties, we always mean closed subvarieties.

Definition 2 We define the *Chow group* $A(X)$ of X to be the group of cycles modulo rational equivalence (the equivalence relation generated by $Z' \sim Z$). Notice the rational equivalence always preserves the dimension, so we can write

$$A(X) = \bigoplus_{k=0}^{\dim X} A_k(X).$$

Remark 3 One crucial aspect is that there is a well-defined pushforward on Chow groups: given a *proper* map $f : X \to Y$ and $A \subseteq X$ a subvariety, then $f(A)$ is closed in Y and we define $f_*[A] = [f(A)]$ if $\dim f(A) = \dim A$ and $f|_A$ is generically one-to-one.

Remark 4 The *key* fact is that for X smooth and projective, one can define a product on $A(X)$ such that for any $A, B \subseteq X$ intersecting generically transversely (i.e., the intersection is transverse at a generic point of every component of $A \cap B$) such that $[A][B] = [A \cap B]$. In other words, the intersection behaves well under rational equivalence: Bezout's theorem is simply a tiny instance of this phenomenon. The reason for this key fact is the *moving lemma*, which says that, firstly, for any classes $\alpha, \beta \in A(X)$, there exist cycles A, B representing them and intersecting generically transversely; secondly, $[A \cap B]$ is independent of this choice. Severi had a wonderful idea of the proof for the first part. Everybody believed that the second part follows easily from the first part, which is however not the case. Fulton wrote an influential book in 1984 and figured out a way to prove the key fact without proving the moving lemma (which implies the moving lemma) and put the theory on a solid basis. For us, it does no real harm to take the moving lemma for granted and use it as a tool to do algebraic geometry.

Remark 5 In general, the fiber Φ_∞ might be non-reduced. We will enhance the construction $Z \mapsto [Z]$ to associate any *subscheme* of X to a cycle in X , with multiplicity. The seek for the right definition of multiplicity is one of the major driving force for developing commutative algebra.

Remark 6 In general, if f is a proper map which is not necessarily generically one-to-one, one defines $f_*[A] = \deg f|_A \cdot [f(A)]$ if $\dim f(A) = \dim A$, and 0 if $\dim f(A) < \dim A$.

Remark 7 When X is projective, we have a well-defined degree map $A_0(X) \to \mathbb{Z}$ by sending the class of a point to 1.

Remark 8 Given $f : X \to Y$, where X, Y are *smooth*. Then the moving lemma implies that there exists a well-defined pullback map $f^* : A(Y) \to A(X)$, given by

$$f^*[A] = [f^{-1}(A)]$$

when $\operatorname{codim} f^{-1}(A) = \operatorname{codim} A$ and $f^{-1}(A)$ is generically reduced. It also follows the *push-pull formula*: for $\alpha \in A(Y)$, $\beta \in A(X)$, then

$$f_*((f^*\alpha).\beta) = (f_*\beta).\alpha.$$

Since pullback preserves codimension but not dimension, we will also use the grading by the codimension of cycles (only when X is smooth)

$$A(X) = \bigoplus_{k=0}^{\dim X} A^k(X).$$

An analogy: the intersection pairing on homology is only well-defined on nice topological spaces (= oriented smooth manifolds) since one needs a good notion of index (which can be negative) for components in the intersection. On the other hand, the cup product on *cohomology* doesn't need such restrictive assumption and recovers the intersection pairing on homology via Poincare duality for oriented smooth manifolds.

In contrast to homology/cohomology, the Chow groups are not computed for 99 percent of algebraic varieties: even for simple cases like surfaces in $\mathbb{P}^3$ of degree ≥ 4 (for $d \leq 3$, the Chow ring is $\mathbb{Z}$: every two points are connected by rational curves; we don't know if this is true or not for $d \geq 4$). Nevertheless this is not the end of the world: a lot of calculation are performed on spaces whose Chow rings are known (e.g., projective spaces; product of projective spaces; Grassmannians); we also know many subrings of Chow rings (of certain special degrees).

Chow rings of projective spaces

We will compute that $A(\mathbb{P}^n) = \mathbb{Z}[t]/t^{n+1}$. Namely, if $X \subseteq \mathbb{P}^n$ is of codimension k , degree d , then $[X] = d \cdot t^k$. The key to proving this is that every subvariety $X \subseteq \mathbb{P}^n$ is rationally equivalent to a multiple of a linear subspace. We observe that $\mathbb{P}^n$ has an affine stratification

$$\mathbb{P}^n = \mathbb{A}^n \coprod \mathbb{A}^{n-1} \cdots \coprod \mathbb{A}^0.$$

Definition 3 A *stratification* of X is the expression $X = \coprod U_i$ as a disjoint union of locally closed subvarieties $U_i \subseteq X$, such that for any i, the closure of U_i is a disjoint union of certain strata. We say it is an *affine stratification* if U_i is isomorphic to an affine space for each i.

The above assertion about $A(\mathbb{P}^n)$ follows from the following more general

Lemma 1 If X has an affine stratification, then $A(X)$ is generated by the classes of $\overline{U_i}$.

Remark 9 In fact, it is a theorem that these classes $\overline{U_i}$ generate $A(X)$ freely.

Remark 10 This is very special circumstance that we can actually find the generators of the Chow ring: it only applies to a special class of rational varieties. For example, consider the blow-up of $\mathbb{P}^n$ along some subvariety Y, unless Y is linear, the blow-up will not have an affine strata (even more: unless Y belongs to some special classes of varieties, we will lose the finite generation of the Chow ring!)

Proof The proof is based on the following claim. If $A \subseteq X$ is a subvariety such that $A \subsetneq \overline{U_i}$ and $A \cap U_i \neq \varnothing$, i.e., $A = \overline{A \cap U_i}$. Then A is rationally equivalent to A' such that A' is supported on ∂U_i. This process will allow us to arrive at $A = \overline{U_i}$, or finally reduce the case of dimension 0.

To prove the claim, choose an isomorphism $\mathbb{A}^k \cong U_i$ such that the origin of $\mathbb{A}^k$ maps to some point $\notin A$. Take Φ to be the closure in $\mathbb{P}^1 \times X$ of the subset

$$\{(t,p) : t \in \mathbb{A}^1, t \neq 0, p \in U_i : tp \in A\}.$$

Any fiber of Φ over t is a translation of A (in particular $\Phi_1 = A$) and its limit Φ_0 when $t \to 0$ is contained in ∂U_i. □

Remark 11 The existence in the moving lemma is easy when there is a group acting on X, due to the following theorem of Kleiman (which again only applies to special classes of rational varieties).

Theorem 2 let G be an algebraic group acting transitively on X. Let $A, B \subseteq X$ be two subvarieties. Then for a general $g \in G$, gA and B intersect generically transversely. Moreover, when G is affine, gA is rationally equivalent to A.

Examples

Example 1 (Product of projective spaces) A product of projective spaces also has an affine stratification. It follows that

$$A(\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}) = \mathbb{Z}[t_1, \ldots, t_k]/(t_1^{n_1+1}, \ldots, t_k^{n_k+1}).$$

One can think of this formula as the Kunneth formula for Chow rings. But be careful that there is no such a thing in general: e.g., if $X = C \times D$ is a product of two curves of genus ≥ 2, then $A(X)$ is very different from $A(C) \otimes A(D)$ (see Example 5).

Example 2 (Veronese varieties) Let

$$v_{n,d} : \mathbb{P}^n \to \mathbb{P}^N, \quad [x_0, \ldots x_n] \mapsto [\ldots, X^I, \ldots]$$

be the Veronese embedding, where $N = \binom{d+n}{n} - 1$ and X^I runs over all monomials of degree d in $n+1$ variable $x_0, \ldots, x_n$. Let $\Phi_{n,d}$ be its image. What is $\deg \Phi_{n,d}$? The degree of $\Phi_{n,d}$ is the size of its intersection with a general linear subspace $H_1 \cap \cdots \cap H_n$ in $\mathbb{P}^N$ of dimension $N - n$. Since the preimage of a general hyperplane under $v_{n,d}$ is a general hypersurface of degree d, it is equal to the intersection number of n hypersurfaces of degree d in $\mathbb{P}^n$, which is d^n by Bezout.

Example 3 (degree of duals) Let $X \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d. Let $\{H_\lambda\}$ be a general pencil of hyperplanes, that is, choose a general codimension 2 linear subspace $\lambda = \mathbb{P}^{n-2} \subseteq \mathbb{P}^n$ and take all hyperplanes H_λ containing λ. How many of these H_λ's are tangent to X?

We observe there is a regular map, called the *Gauss map* $G : X \to (\mathbb{P}^n)^*$, sending $p \mapsto T_pX$. Explicitly, if $X = V(F)$, then this map sends p to $[\frac{\partial F}{\partial z_0}(p), \ldots, \frac{\partial F}{\partial z_n}(p)]$. Notice this map can be viewed as a composition of $v_{n,d-1}$ followed by a projection map. Since a pencil H_λ corresponds to a line L in $(\mathbb{P}^n)^*$, we are asking nothing but the intersection number $\#G(X) \cap L$ for a general line L.

Write $L = H_1 \cap \cdots \cap H_{n-1}$. Then $\#G(X) \cap L$ is equal to (see the remark below) $\#(X \cap W_1 \cdots \cap W_{n-1})$, where W_i is hypersurface of degree $d-1$ cut out by a general linear combination of $\frac{\partial F}{\partial z_j}$. This is $d \cdot (d-1)^{n-1}$, by Bezout.

Remark 12 Notice We are implicitly using the fact that G is *generically one-to-one*: this is is true in characteristic 0 (somewhat tricky to prove), but false in characteristic p (there are plane curves such that each tangent line is bi-tangent).

Example 4 (Segre varieties) Let

$$\sigma : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N, \quad ([X_i], [Y_j]) \mapsto [X_i Y_j]$$

be the Segre map, where $N = (r+1)(s+1) - 1$. Let $\Sigma_{r,s}$ be its image. What is $\deg \Sigma_{r,s}$? Write

$$A(\mathbb{P}^N) = \mathbb{Z}[t]/t^{N+1}, \quad A(\mathbb{P}^r \times \mathbb{P}^s) = \mathbb{Z}[\alpha, \beta]/(\alpha^{r+1}, \beta^{s+1})$$

Then

$$\deg \Sigma_{r,s} = \deg(\sigma_*[\mathbb{P}^r \times \mathbb{P}^s]).t^{r+s}) = \deg(\sigma^*(t)^{r+s})$$

by the push-pull formula. Since $\sigma^* t = \alpha + \beta$, we know that $\deg(\sigma^*(t)^{r+s}) = \deg(\alpha+\beta)^{r+s} = \binom{r+s}{s}$ (only the $\alpha^r \beta^s$ -term is nonzero).

Today's motivating questions:

Question If F_0, F_1, F_2 are 3 general homogeneous quadratic polynomials in 3 variables, how many (nonzero) solutions are there to the system of equations $(X_0, X_1, X_2) = (F_0(X), F_1(X), F_2(X))$?

Question If F_0, F_1, F_2 are 3 general homogeneous cubic polynomials in 3 variables, for how many triples $[t_0, t_1, t_2]$ does $t_0 F_0(X) + t_1 F_1(X) + t_2 F_2(X)$ factor?

Example 5 (Kunneth does not hold for Chow rings) Let C be a smooth curve of genus $g > 0$. Let $\Delta \subseteq C \times C$ be the diagonal. We claim its class $[\Delta]$ is not the linear combination of fibers (hence $A(C \times C) \neq A(C) \otimes A(C)$). To see this, we use the usual Kunneth formula in topology: the cohomology class of Δ has nontrivial component in $H^1(C) \otimes H^1(C)$, but $H^1(C)$ is not seen in Chow rings: Chow rings only see even degree cohomology.

Example 6 Let $\Delta \subseteq \mathbb{P}^r \times \mathbb{P}^r$ be the diagonal. What is the class $\delta = [\Delta] \in A^r(\mathbb{P}^r \times \mathbb{P}^r)$?

We will discuss two methods. The first method is *undetermined coefficients*, which applies many other settings. We can write

$$\delta = c_0 \alpha^r + c_1 \alpha^{r-1} \beta + \cdots + c_r \beta^r$$

for $c_i \in \mathbb{Z}$. To determine c_i, we take the intersection of δ with $\alpha^i \beta^{r-i}$ and observe that, by the shape of the Chow ring of $\mathbb{P}^r \times \mathbb{P}^r$,

$$c_i = \deg(\delta \cdot \alpha^i \beta^{r-i}).$$

This is equal to $\#(\Delta \cap (\Lambda \times \Phi))$, where $\Lambda \cong \mathbb{P}^{r-i} \subseteq \mathbb{P}^r$ and $\Phi \cong \mathbb{P}^i \subseteq \mathbb{P}^r$ are general linear subspaces such that the intersection is transverse. This is the same as the intersection number $\#(\Lambda \cap \Phi)$, which is equal to 1. Hence

$$[\Delta] = \alpha^r + \alpha^{r-1}\beta + \cdots + \beta^r.$$

This is reminiscence of the cohomology class of the diagonal in topology. (The secret property we are using is that the cohomology ring and Chow ring agree for products of projective spaces. A tiny proportion of varieties for which we can compute the Chow rings actually has this secret property).

The second method is less general: *specialization*. Let $\Phi \subseteq \mathbb{P}^1 \times \mathbb{P}^r \times \mathbb{P}^r$ be the closure of the triples

$$\{(t, X, Y) : t \neq 0, \infty, [Y_0, \dots, Y_r] = [X_0, tX_1, \dots, t^r X_r]\}.$$

Then the fiber $\Phi_1 = \Delta$ and Φ_t is isomorphic to $\mathbb{P}^r$ (as the graph of an automorphism of $\mathbb{P}^r$). The formula for $[\Delta]$ then follows from the fact that

$$\Phi_0 = \bigcup_{i=0}^r V(X_0, \dots, X_{i-1}) \times V(Y_{i+1}, \dots, Y_r).$$

One subtle point is that we need to actually write down local equations of Φ_0 to check that it is reducible, i.e., multiplicity-free.

Example 7 More generally, let $f : \mathbb{P}^r \to \mathbb{P}^r$ be a map given by $[F_0, \dots, F_r]$, where F_i are homogeneous of degree d. Let $\Gamma = \Gamma_f$ be its graph. What is the class $\gamma = [\Gamma] \in A^r(\mathbb{P}^r \times \mathbb{P}^r)$?

The method of undetermined coefficients reduces to the computation of $\#(\Gamma \cap (\Lambda \times \Phi))$. This is given by $\#(\Lambda \cap f^{-1}(\Phi))$, which by Bezout, is d^{i-1}. Hence

$$\gamma = d^r \alpha^r + d^{r-1} \alpha^{r-1} \beta + \cdots + \beta^r.$$

In particular, the number of fixed point of f is equal to

$$\deg(\gamma \cdot \delta) = d^r + d^{r-1} + \cdots + 1.$$

This answers our first question in the beginning: take $r=2$, $d=2$, so there are 7 fixed points, i.e., 7 nonzero solutions to the system of equations $(X_0,X_1,X_2)=(F_0(X),F_1(X),F_2(X))$.

Plane cubics

Example 8 An importance aspect of algebraic geometry is that many algebro-geometric objects are naturally parametrized by another variety. It is exactly these parameter spaces we will apply intersection theory to solve enumerative problem in algebraic geometry. For example, the collection of plane cubics can be identified as nonzero homogeneous cubic polynomials in three variables (up to scalars), which forms a $\mathbb{P}^9$.

A subset of objects are then usually a locally closed subset of the parameter space. For example, the space of singular cubics, reducible cubics (denoted by Γ), triangles (Σ), asterisks (Φ) are cut out from $\mathbb{P}^9$ by algebraic conditions and have dimension 8,7,6,5 respectively.

Question What are the degrees of Γ,Σ,Φ? (one can also ask the degree of the locus of singular cubics, but we will wait until we talk about Chern classes.)

Example 9 Notice $\deg\Gamma$ is simply the answer to the second question in the beginning (the intersection of Γ with a generic codimension 2 subspace). To compute it, look at the regular map defined by multiplying a linear form and a quadratic form in 3 variables

$$\tau:\mathbb{P}^2\times\mathbb{P}^5\to\mathbb{P}^9,\quad (F,G)=F\cdot G.$$

If we write $A(\mathbb{P}^9)=\mathbb{Z}[\zeta]/\zeta^{10}$, then $\tau^*\zeta=\alpha+\beta$, $\Gamma=\tau_*([\mathbb{P}^2\times\mathbb{P}^5])$ and thus

$$\deg\Gamma=\deg(\Gamma\cdot\zeta^7)=\deg((\tau^*\zeta)^7)=\deg(\alpha+\beta)^7=\binom{7}{2}=21.$$

Example 10 To compute $\deg\Sigma$, look at the map defined by multiplying three linear forms

$$\tau:\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2\to\mathbb{P}^9,\quad (L,M,N)\mapsto LMN.$$

If $A(\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2)=\mathbb{Z}[\alpha,\beta,\gamma]/(\alpha^3,\beta^3,\gamma^3)$, then we have

$$\deg\Sigma=\deg(\frac{1}{6}\tau_*[\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2].\zeta^6)$$

Be careful about the extra factor $\frac{1}{6}$ arising from the fact that τ is not generically one-to-one: there are six ways of writing a product three linear forms as a triple of linear forms. By the push-pull formula, this is equal to

$$\frac{1}{6}\deg\tau^*(t)^6=\frac{1}{6}(\alpha+\beta+\gamma)^6=\frac{1}{6}\cdot\frac{6!}{2!2!2!}=15.$$

Example 11 To compute $\deg\Phi$, we need to compute the class $[A]$ of the subspace A consisting of concurrent triples of lines in $\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$. If the three lines are given by $a_iX+b_iY+c_iZ=0$, for $i=1,2,3$. Then A is given by the vanishing locus of the determinant of the matrix

$$\begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix},$$

which is again a homogeneous trilinear form. Hence $[A]=\alpha+\beta+\gamma$. Thus

$$\deg\Phi=\deg(\frac{1}{6}[A].\zeta^5)=\frac{1}{6}\deg([A].(\tau^*\zeta)^5)=\frac{1}{6}(\alpha+\beta+\gamma)^6=15.$$

This agrees with $\deg\Sigma$ and one naturally wonders if Φ is a hyperplane section of Σ. The answer is no, however. Notice the image of i is a hyperplane section under Segre embedding

$$\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2\to\mathbb{P}^{26}$$

but the projection $\mathbb{P}^{26}\to\mathbb{P}^9$ is a regular map of degree 6 on the image of $\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$ and does not send this hyperplane section in $\mathbb{P}^{26}$ to a hyperplane section in $\mathbb{P}^9$.

Example 12 Here is another way of computing $\deg\Sigma$. For any point $p\in\mathbb{P}^2$, we have a hyperplane $H_p\subseteq\mathbb{P}^9$ of cubics passing through p. Let $p_1,\ldots,p_6\in\mathbb{P}^2$ be general points, then

$$\deg\Gamma=\#(\Sigma\cap H_{p_1}\cap\cdots\cap H_{p_6}).$$

Be careful that one needs to verify that this intersection is transverse since H_{p_i} are no longer general hyperplanes and we cannot invoke Bertini's theorem directly. On the other hand, this intersection can be compute directly (without push-pull formula), it is the number of ways of grouping six points into three pairs of points, which is $\frac{1}{6}\cdot\frac{6!}{2!2!2!}=15$.

Curves on surfaces

Let X be a quasi-projective variety variety. Any line bundle L on X has a rational section σ. The vanishing locus $V(\sigma)$ and $V(\sigma')$ of two rational sections σ, σ' of L are rationally equivalent since they can be connected by the family $V(t\sigma + (1-t)\sigma')$. Hence taking the vanishing locus of σ gives a well-defined *Chern class* map

$$c_1 : \mathrm{Pic}(X) \to A^1(X).$$

If X is smooth, then c_1 is indeed an isomorphism since every codimension one cycle can be represented by a Cartier divisor. If X is not smooth, c_1 can be neither injective nor surjective.

Now assume X is smooth. We have the *canonical bundle* $K_X = \wedge^n T_X^*$. By abuse of notation, we denote the *canonical class* $c_1(K_X) \in A^1(X)$ again by K_X. If $D \subseteq X$ is a smooth divisor, then we have an exact sequence

$$0 \to T_D \to T_X|_D \to N_{D/X} \to 0.$$

The (I-would-call) adjunction formula says that the normal bundle $N_{D/X} = \mathcal{O}_X(D)|_D$. Therefore

$$\wedge^n T_X^*|_D = \wedge^{n-1} T_D^* \otimes \mathcal{O}_X(-D)|_D,$$

which is the usual *adjunction formula*

$$K_D = (K_X + D)|_D.$$

Example 13 Consider $\mathbb{P}^n$. Then $dx_1 \wedge \cdots \wedge dx_n$ is a rational differential form, everywhere regular and nonzero on $\mathbb{A}^n \subseteq \mathbb{P}^n$ and has a pole of order $n+1$ on the boundary. Hence $K_{\mathbb{P}^n} = -(n+1)\zeta$, where ζ is the hyperplane section. For $X \subseteq \mathbb{P}^n$ a smooth hypersurface of degree d, the adjunction formula then implies that $K_X = (d-n-1)\zeta$.

Now let X be a smooth surface. Let C and D be curves on X. If k and D intersect transversely, then $\deg([C].[D])$ is the number of intersection points. We will write this as $C.D$ for short.

Suppose k is a smooth curve of genus g. Then by the adjunction formula,

$$2g - 2 = \deg K_C = C^2 + K_X.C.$$

Hence we deduce the *genus formula*

$$g = \frac{C^2 + K_X.C}{2} + 1.$$

Example 14 Let $X = \mathbb{P}^2$ and C is a smooth curve of degree d. Then $C^2 = d^2$ and $C.K_{\mathbb{P}^2} = -3d$. Hence

$$g = \frac{d^2 - 3d}{2} + 1 = \binom{d-1}{2}.$$

Example 15 Let X be a quadric surface in $\mathbb{P}^3$. Then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $A(X) = \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$, where α, β are the classes of the two rulings. We say $C \subseteq \mathbb{P}^1 \times \mathbb{P}^2$ has bidegree a, b if $[C] = a\alpha + b\beta$, or equivalently, C is the locus of bi-homogeneous polynomial of degree (a, b). By the adjunction formula, we have

$$K_X = (-4\zeta + 2\zeta)|_X = -2(\alpha + \beta).$$

Suppose C has bidegree (a, b), then $C^2 = 2ab$, $K_X \cdot C = -2(a+b)$ and

$$g = \frac{2ab - 2(a+b)}{2} + 1 = (a-1)(b-1).$$

Example 16 Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree d. Suppose X contains a line L. This is the first case that the curve cannot be visibly moved. To compute L^2, the idea is to reverse the genus formula since we know that $g(L) = 0$. Since $K_X = (d-4)\zeta$, we have $K_X.L = d - 4$. By the genus formula, $0 = \frac{L^2 + d - 4}{2} + 1$, hence

$$L^2 = 2 - d.$$

Notice when $d \geq 3$, L has *negative* self intersection so we can never move L to another rationally equivalent *curve* that meets L transversely.

The moving lemma however does not fail: there is a rationally equivalent *cycle* that meets L transversely. In fact, let H be a general plane containing L. Then $H \cap X = L \cup D$, where $D \subseteq H$ is a plane curve of degree $d - 1$. Therefore the two cycles L and $H - D$ are rationally equivalent and they intersect transversely. Notice $H.L = 1$, $D.L = d - 1$, we again have

$$L^2 = L.(H - D) = 1 - (d-1) = 2 - d.$$

Example 17 Let $C \subseteq \mathbb{P}^3$ be a smooth curve. In general, C may not always be defined by exactly two equations. Suppose $C \subseteq S \cap T$, where S, T are smooth surfaces of degree s, t. We write $S \cap T = C \cup D$ and say that C and D are *linked*. We define *liaison* to be the equivalence relation generated by linkage. The classification of liaison has been done by Hartshorne and Rao.

If C has degree c, genus g, what can we say about $d = \deg D$ and $h = g(D)$? By Bezout, one easily sees that $d = st - c$. To compute h, we proceed as follows by using the genus formula on S twice.

a. Since $K_S = (s-1)\zeta$, hence $K_S.C = (s-1)c$. By the genus formula, we find that
$$C.C = 2g - 2 - (s-4)c.$$
b. Since $C + D = t\zeta \in A^1(S)$, we have
$$C.D = C.(t\zeta - C) = tc + (s-4)c - (2g-2) = (s+t-4)c - (2g-2).$$
c. We compute that
$$D.D = D.(C - t\zeta) = td - (s+t-4)c + (2g-2).$$
d. By the adjunction formula, we get
$$2h - 2 = K_S.D + D.D = (s-4)d + td - (s+t-4)c + (2g-2).$$
Namely,

So the genus of D is indeed determined and is very easy to remember: the difference of the genera is proportional to the difference of degrees.

Grassmannians

Today's motivating question is the first nontrivial question in enumerative geometry:

Question Given four general lines in $\mathbb{P}^3$, how many lines meet all these four?

To answer an enumerative problem like this, we are going to proceed as follows.

a. Introduce a parameter space for the objects we are studying (in this case: the Grassmannian of lines in $\mathbb{P}^3$, which is 4 dimensional);
b. Describe the Chow ring of this space;
c. Find the classes of the loci of objects satisfying desired conditions (in this case: the locus of lines meeting a given line, which is codimension 1 in);
d. Take the product of these classes and evaluate its degree (in this case: intersecting four such codimension 1 classed);
e. Verify the transversality of the intersection!

We will begin with some generality on Grassmannians and come back to later. There are three stages of for understanding Grassmannians:

a. Definition via Plucker embedding; local coordinates;
b. Description of the tangent spaces. This is useful for verifying transversality);
c. Functorial description. This also helps us get familiar with Hilbert schemes.

Definition 4 Let be an n-dimensional vector space. We denote by the set of k-planes in . This is also denoted by when viewing everything projectively.

Given , then we have an inclusion of the line . This gives an embedding

The claim is that the image is closed, which gives the set the structure of an algebraic variety. Namely, we will exhibit polynomials of degree that cut out the image (however, this is the *wrong* way of doing things since it does not define the image scheme-theoretically: the homogeneous ideal is in fact generated by quadratic polynomials — the Plucker relations).

Definition 5 Let . We say that if for some . Then if and only if in .

Notice the map givens a linear map

It follows that is totally decomposable (i.e.,) if and only if , and the last condition is cut out by polynomials of degree on (vanishing of -minors).

In terms of matrices, can be viewed by the spaces of -matrices of rank k (up to -action), since any k-plane can be viewed as the row space of such matrix. The Plucker embedding is simply taking the matrix to the set of its -minors. This gives us a way of writing down the local coordinates concretely.

Remark 13 Let be an -plane. Let be the set of such that . Then is the complement of a hyperplane section: if with basis and , then consists of matrices with nonzero *first* -minor (up to -action). Modifying the first -minor to be the identity matrix by the -action, we get

Notice this particularly gives an affine covering of (by varying Γ).

Now let us come back to . We have a stratification of G by closed stratas. Choose a flag . We have the following subset of G:

These are again algebraic (with dimension 3,2,1,0 respectively), known as *Schubert cycles*. To be rigorous: depends on the choice of p and better denoted by ; similar remarks apply to other Schubert cycles.

Notice that each open stratum (the complement in a closed stratum of all its substrata) is an affine space by the argument in Remark 13. We will denote the classes of these cycles by the with lower case symbols σ. By Lemma 1, these classes generate . We will compute the intersection product on case by case.

a. First take two Schubert cycles of complementary dimension and compute their intersections:

   Notice by Kleiman

   for two generic points p, . This intersection number is 1 since there is a unique line passing through . Similarly

   since two generic planes intersect at a unique line, and

   since two generic lines don't intersect. It follows the and are linearly independent (with their intersect matrix the identity). Finally,

   since there unique line passing through and the intersection point .

b. By Kleinman,

   for a generic point p and a generic line . This is the Schubert cycle

   Similarly,

c. The only remaining intersection product is . It is given by the number of lines intersecting both general lines t, . This is not any Schubert cycle, but we can use the method of undetermined coefficients: write

   Then

   Similarly

   Hence we conclude that

Now we can answer the earlier motivating question:

So there are exactly two lines intersecting all four given general lines.

Remark 14 One can also visualize this directly: any three general lines lie on a unique smooth irreducible quadric surface (3 points on each line gives 9 points and there is 9 dimension of quadrics in $\mathbb{P}^3$). The 2 lines simply correspond to the 2 intersection points of with the fourth general line.

More generally,

Example 18 (lines intersecting four curves) Given four curves with degree . How many lines meet all these four?

We only need to figure out the classes of the cycles we would like to intersect. Let be the set of all lines meeting a given curve C. First we need to show that is closed. There is a universal family of lines over the Grassmannian

Notice that Φ is simply the zero locus of in , where and , hence is a closed subset. Let and be the projections. Now , and hence is also closed and of dimension 3.

Write for some . To determine , we take a cycle of complementary dimension, , and intersect. We obtain

for a general point p and a general plane p. This is the same as , which is the by Bezout.

Therefore the answer to the question is .

Example 19 (Common chords of curves) Given two general twisted cubics . How many common chords do they have?

When C is smooth, a chord on k is either a secant line or a tangent line. The set of chords of C is then the image of the symmetric square of C into , hence is closed and of dimension 2. We can also describe as .

Let C be a curve of degree d and genus g. Write . We again compute a, b by intersecting with cycles of complementary dimension. We have

This is the number of pair of points in (check that no three points are collinear!), hence . Similarly,

This is the number of points where the projection of C away from p is not one-to-one, which is the number of nodes of the projection of C in $\mathbb{P}^2$, i.e., the difference of the arithmetic genus ( ) and the geometric genus g. Hence .

Now we can answer the question about common chords of two twisted cubics: and .

Remark 15 There is a cute way to get the answer 10 directly by choosing two specific twisted cubics (but still such that is transverse). This leads to our next topic on the criterion for transversality.

Tangent spaces of Grassmannians and specialization

Question Given four (not necessarily general) skew lines and be a line meeting all four. When is L a point of transverse intersection for ?

This question has a simple answer.

Answer Let and . Then the intersection is not transverse if and only if the cross ratio of the four points is equal the cross ratio of .

In general, to test the transversality, we need to describe the tangent space of the cycles, which lie inside the tangent space of the Grassmannian.

Question Let . For , how to describe the tangent space ?

Answer If we want to describe how can move in G, we can use all one-parameters families passing through . Let be a one-parameter family of k-planes such that . Let and choose a one-parameter family such that and for any t. One can check that the tangent direction is well-defined modulo . So gives us a map . This association provides a natural isomorphism

Remark 16 Another way to see this: recall that given a subspace of dimension . Then is the complement of a Schubert cycle and is isomorphic to (Remark 13). For , we have . So any point in can be identified with the graph of a linear map . Hence .

Remark 17 The point now is that the identification is *natural* in the following sense. On G, there is a universal (aka. tautological) sub-bundle S of rank k

whose fiber . There is also a universal quotient bundle sitting in the exact sequence

whose fiber . The word "natural" here means that the isomorphism patches nicely when varying and gives an isomorphism of vector bundles

Example 20 Let $C \subseteq \mathbb{P}^3$ be a curve. Let (Example 18). Let be a line that meets C in just one point p and . Then is smooth at if and only if , and . Using the previous description of in terms of one-parameter families, one can see that

In particular, when is a line, this gives the tangent space of the Schubert cycle .

Example 21 Let us calculate again using the fundamental idea of *specialization*. We would like to compute . But this time we take to be special: . Let . Then we find that

This gives the right answer but we need to check the transversality without Kleinman. To check this we simply compute the tangent space of and and see in fact they are *distinct* in the tangent space of . For example, suppose and , . Then

and

are distinct.

Remark 18 To see that the class does not depend on the choice of L (even for special L), we can use the fact that is rationally connected.

Example 22 For $C \subseteq \mathbb{P}^3$, let us calculate again using the idea of specialization.

Slightly more generally, a one-parameter subgroup of has the form

under some coordinates system. The simplest and most useful case is that all is either 0 or 1:

Suppose is cut out by and is cut out by . Then when every point outside Γ and Σ is moving toward Γ (Γ is called the *attracting* subspace).

Now come back to the case $C \subseteq \mathbb{P}^3$. Take a general and . Assume Γ is cut out by and H is cut out by . Take . What is the flat limit of as ? Namely, let

and let Φ be the closure of in . Then the flat limit of is defined to be Φ_0. Notice that every point of C that does not interesting H gets sucked into but the points in stay fixed. Thus

It follows that as in Example 18. This is correct set-theoretically but not scheme-theoretically: there must be extra embedded points to make the arithmetic genus match: for example, if $C \subseteq \mathbb{P}^3$ has degree four then k has arithmetic genus 3, but four lines intersecting at one point has arithmetic genus 1 (as the intersection of two singular quadrics), hence there must be embedded points in the limit . The embedded points however do not affect the computation of , since no entire component of is non-reduced.

Example 23 Let us calculate the number of common chords of two twisted cubics again using specialization. Every twisted cubic lies on a smooth quadric and has type . We take special twisted cubics C, D on the same smooth quadric of type and . Then there is no line on that is a common chord. All common chords are then given by the lines through any two intersection points in . There are 5 such intersection points (since in). Hence we can conclude the number of common chords is (Example 19), after checking the transversality. Notice if we chose the types to be both , then the intersection consisting of points and a one dimensional component coming from one ruling. In fact, Fulton has a formula for computing the components even when the intersection is not transverse and we will come back to this later.

Digression on multiplicities

There are three levels of intersection theory. Suppose X is smooth of dimension n and A, B are subvarieties of codimension C, D.

a. Suppose A, B intersect transversely. Then , where $A \cap B$ is the scheme-theoretical intersection.
b. Suppose $A \cap B$ has the correct codimension but does not necessarily intersect transversely. Then we can look at the the components of the intersection and assign a multiplicity to each component such that

   This applies to much broader range than the first level.
c. Fulton was able to drop all assumptions except a small one: assume A is locally Cohen-Macaulay, then

   where is is the pushforward of a class under . The point is that there is a *formula* for the class . This puts the intersection theory on solid basis: one takes the formula as the definition of the intersection pairing. After one checks this intersection pairing is well-defined, the independence of choice of the cycles in first level follows automatically.

Suppose we are in the second level: if are locally Cohen-Macaulay. Then is simply the multiplicity of the component of $A \cap B$ supported at (and this was the reason for introducing the notion of Cohen-Macaulay in the first place). However, when the Cohen-Macaulay assumption is not satisfied, the intersection formula in terms of multiplicity no longer holds in general. Serre eventually found a general formula (without any assumptions) in terms of alternating sums of Tor groups (we are not going into that but it is good to know it is there).

Example 24 If , then is simply the degree of $A \cap B$ supported at the point . This implies the stronger version of Bezout: if are plane curves without common component, then .

General Schubert cycles

Let . We introduce the flag ,

Suppose is a general k -dimensional subspace. We look at the intersection of with the flag ,

Notice the intersection will be empty for the first terms and jumps by one for each of the last k terms. For an arbitrary k -dimensional subspace , we should specify how the jumps occur.

Definition 6 For any sequence , we define the *Schubert cycle*

In other words, this consists of k -dimensional subspaces whose i -th jump occurs at least steps earlier. In particular, consists of all general k -dimensional subspaces.

Definition 7 We define . Since the dimension is nondecreasing, we will adopt the convention that unless is nondecreasing. We also drop the ending zeros from the notation (so).

Remark 19 Notice that the codimension of is equal to . In particular there is always a unique class in codimension one.

Remark 20 The notation behaves well with respect to the standard inclusions and : we have .

Remark 21 The key fact is that form the closed strata of an affine stratification (Remark 13). Therefore by Lemma 1, the classes generate the Chow ring (and as we will see, freely).

The role played by the Schubert cycles for Grassmannians is like the role played by linear subspaces for projective spaces. But there are differences. For example, the Schubert cycles may be singular: for , is singular at the line : can be viewed as a hyperplane section of under the Plucker embedding by the hyperplane tangent to). It is known (but not easy) which Schubert cycles are singular. But in general it is not known which Schubert cycles are rationally equivalent to smooth cycles.

Also, there are algorithms for computing the intersection product between 's but we don't know in general which coefficient in the intersection product is zero or not. The situation in complimentary dimension is much better as in the following theorem.

Theorem 3 If , then , unless , in which case .

Proof Notice . Suppose is in the intersection. Then

So

This gives exactly . Adding this inequality for all i we obtain that . Hence the equality holds and for each i. The previous intersection is exactly 1-dimensional for each i and uniquely determines , hence . □

Example 25 Let be general quadrics and . How many lines does S contain?

Let be the set of all lines on . We would like to compute . By Kleinman, this is the class , which only involves intersection in complimentary dimension since and . Write . Let be a complete flag. Then

Then one computes that and

which is zero since a general point p does not lie on the quadric . Similarly,

This intersection contains the lines passing through any of the two points and lying on the quadric surface . There are 2 lines on the quadric surface passing through each point, so . Hence and .

We end by discussing the method of *dynamic specialization*.

Example 26 Our goal is calculate . Here

We would like to compute . If we take two general lines we would not be able to describe the intersection easily. If take two special lines lying in a plane with . Then

This looks nice but the problem is that not only the intersection is not transverse, the component has the wrong dimension! We are stuck now since there is no more options between "general" and "lying on a plane".

The idea of dynamic specialization is to choose a one-parameter family of lines such that for but . Let

Take Φ to be the closure of in and let Φ_0 be the fiber of Φ at (key point: this is the limit of the intersection, rather than the intersection of the limits). We have , but Φ_0 now has the correct dimension 2. In other words, must satisfies some additional assumption to be in the limit Φ_0.

Let for and take be the limit of as . Then the additional assumption is . So

The final claim is that this is indeed an equality and thus .

We can show geometrically that coincides with the support of Φ_0. So it remains to show that Φ_0 is generically reduced. We are going to write down local equations to prove this claim. We can take , . Then , , . Let us choose the nice (= the limit does not go outside) affine open of

Then the equation of is given by , and is given by and . So the ideal of Φ_t is given by . When , Φ_t has dimension 2. But when , the ideal contains the hyperplane (this is what goes wrong when taking the intersection of the limit). Notice that

we know that for any , so we should throw in as well. This is the limit of the intersection and gives the correct Φ_0. From the equations we see that Φ_0 is reduced.

Remark 22 Notice, however, if we arrived at the formula by a different method (e.g., undetermined coefficients), then it already implies that the fiber Φ_0 is generically reduced. This is the power of abstraction in intersection theory: it allows you to say useful things about polynomial equations without actually writing down the polynomial equations!

Example 27 Let be a curve. Let . Then X is a closed subvariety of dimension . This is easily seen by incidence correspondence

Then , where α, β are the projections of Φ to $\mathbb{P}^n$ and respectively. What is ? Let be a general subspace of complimentary dimension, then , which is equal to . This is exactly the degree of C under the Plucker embedding .

Chern classes

Today's motivating questions:

Question If is a general cubic surface, how many lines does S contain?

Question If is a general pencil of quartic surfaces, how many contain lines?

Before we see how Chern classes help answer these two questions, we should first ask the following simpler question. If $X \subseteq \mathbb{P}^n$ is a general hypersurface of degree d, we introduce its *Fano scheme*,

When is ? And what is ? This can again be answered using the incidence correspondence

The projection is easier to understand: its fiber above consists of all hypersurfaces of degree d containing L, which is the the projectivization of the kernel of the surjection

(An intrinsic description is that if , then). It follows that Φ is irreducible and has dimension

since . So if , a general X is expected to contain no line and when , we expect that .

Example 28 In the first question, we have and , so there should be finitely many lines. In the second question, and , so a general quartic surface should contain no line and there should be a hypersurface of special quartic surfaces which contain lines. The second question is simply asking the degree of this hypersurface. Ultimately, we would like a polynomial in 35 variables (coefficients of homogeneous quartic polynomials in four variables) which exactly determines whether a quartic surface contains lines or not. It turns out the degree is 320 (and one shouldn't try to write it down!)

The key new idea is *linearization*: we replace such a complicated polynomial by a family of of system of *linear* equations.

Example 29 For the first question, instead of asking which line the cubic surface contains, we ask: given a line, which cubic surfaces contain this line? And when we vary the line, how many times the corresponding condition is satisfied?

Let us introduce a vector bundle over of rank 4: its fiber

is a 4-dimensional vector space. Let be a cubic surface, for . Then we obtain a section of the vector bundle given by . The question of number of lines on S can be rephrased as: how many *zeros* does the section have?

Locally, the section can be viewed as a 4-tuple of functions on a 4-dimensional space, so there should be finitely many common zeros. Notice that if are two sections of . Then

gives a linear equivalence between $V(\sigma)$ and (assuming they have the correct dimension 0). So the rational equivalence class of the zeros of does not depend on the choice of .

Example 30 For the second question, similarly we introduce vector bundle over of rank 5 such that

If are general quartic polynomials on $\mathbb{P}^3$, we again obtain two sections of . Then we are asking for how many points , are *linearly dependent*. Now a section locally is a 5-tuple of functions on a 4-dimensional space, so two sections pin down finitely many such L's.

Let X be a smooth variety. Let be a vector bundle of rank . Notice that is a trivial bundle is equivalent to saying that there are -linearly independent sections. More generally, let be sections of . What is the locus where they becomes linearly dependent? If , the zero locus of the section should be of codimension . In general, these sections locally gives a matrix of functions and the locus where the matrix has rank is of codimension k.

Definition 8 We define the *degeneracy locus* to be the subscheme of X cut out by the maximal minors of this matrix. If has the expected codimension k, we then define the *Chern class*

The Chern class does not depend on the choice of the sections (as long as the codimension is as expected). One issue with this definition is that it takes effort to extend the definition to general vector bundles without global sections satisfying the assumption (okay when is generated by global sections). Define

What have we achieved? We gave (nice) *names* for the answers of the questions: the answer should be and . Apart from that we hardly accomplished anything. The point of the abstract construction of Chern classes is that it allows us to carry the information about vector bundles around: if a vector bundle is built up (via multilinear operations) from simpler vector bundles whose Chern classes we can calculate, then we can also calculate the Chern classes of itself.

Example 31 Let be a rank 2 vector bundle on such that

consists of linear functions on L. This is simply the dual of the universal subbundle over . Then Chern class of should reflect the number of lines on planes (instead of cubic surfaces), which is an easy linear problem. Given a linear form H on $\mathbb{P}^3$, one obtains a section of by . The zero locus of is simply the Schubert cycle . Hence . Similarly, given two linear forms on $\mathbb{P}^3$, then are linearly dependent if and only if . So . Our remaining task is to relate and using .

The general tools for calculating Chern classes:

a. *Whitney formula*. If , then . The same formula applies more generally we have an exact sequence

   This more general formula easily follows from the split case: since the extension group is a vector space, the class of does not depend on its extension class.

b. *Splitting principle*. We say that is *totally filtered* if there exists a sequence of subbundles

   such that is a line bundle. In this case,

Theorem 4 (Splitting principle) Given a vector bundle . There exists a morphism such that

a. is injective.
b. is totally filtered.

It follows from the splitting principle that any formula for Chern classes that holds for direct sum of line bundles, holds in general.

Example 32 Let be a vector bundle of rank 2. Let us calculate . Suppose first that is a direct sum of two line bundles. Write and . Then . In particular, , . Notice that

it follows that

which is expressible in terms of and since it is symmetric in α, β:

This formula makes sense and in fact is true for any (not necessarily split) rank 2 vector bundle. Similarly,

gives

Applying this to in Example 31, we obtain that

In this way we obtain the well-known number of lines on general cubic surfaces!

Remark 23 Can one actually write down the equations of the 27 lines in terms of the cubic polynomial? The answer is no and the reason has to do with that the symmetry group of the 27 lines is non-solvable.

Example 33 Let be general polynomials of degree d in $\mathbb{P}^n$. How many of are singular? Let $\mathbb{P}^N$ be the space of hypersurfaces of degree d in $\mathbb{P}^n$ and let be the singular locus. We are simply asking the degree of Δ. To linearize, instead of asking if a given hypersurface is singular, we ask if a given hypersurface is singular at *a given point*. Let be the vector bundle over $\mathbb{P}^n$ such that

where consists of polynomials vanishing to degree 2 at p. Given a degree d polynomial, we obtain a section of . The question now becomes how many points in $\mathbb{P}^n$ where and are linearly dependent. By definition of Chern classes, this is since (determined by the value and n partial derivatives).

Example 34 Let be a vector bundle on X. What is the Chern class of its dual bundle ? If with . Then

Similarly, , we have

since . It follows that .

Example 35 Let be a line bundle with and be a vector bundle on X. What is ? (Notice twisting by ample line bundles can help create enough sections). Again we may assume that . So and it follows that

In particular,

and

More generally, let be vector bundles of rank . Again assuming they are direct sum of line bundles, we have . Notice the terms of a given degree is symmetric in and , hence is expressible in the Chern classes of and . For example,

In general it seems a mess to obtain an explicit closed formula in higher degrees.

Example 36 Let . We have the universal vector bundles over G with and . Let us first compute the Chern class of (it has many sections since it is a quotient of a trivial bundle). Let , its image in gives rise to a section of . Let be linearly independent vectors and . Where are linearly dependent? It happens exactly when . In other words,

Similarly, we can obtain directly that

Hence .

On the other hand, we have the exact sequence

Whitney's formula then implies that

Comparing the quadratic terms recovers the intersection product .

Example 37 Let us compute the Chern class of the tangent bundle of $\mathbb{P}^n$. The easy way is to look at the Euler sequence

where the second map is simply . (Notice that is not a well-defined element of but multiplying it by a linear form gives a well-defined element.) By Whitney's formula, we obtain that

Alternatively, we can use the general fact for , . Applying to we know that

We can compute

The formula in Example 35 then gives the same answer for .

Remark 24 For general , both S and have higher rank, and it is not easy to compute .

Now let come back to the fundamental question:

Question Where does the Whitney formula come from?

Example 38 Let be vector bundles of rank . The Whitney formula says that the top Chern class

This is complete straightforward: let be sections of , let be the corresponding section of , then the zero locus of is simply the intersection of the zero loci of σ and τ.

Example 39 The next graded piece of the Whitney formula says that

Suppose and be two general sections of . Let

We have a map

so that and . The degeneracy locus is the preimage of the diagonal . The Whitney formula in this case boils down to the class of Δ in . Making this argument work in general requires lots of technical work.

The preimage of the second factor gives . Similarly, the pullbuck And

I owe you the proof of the following theorem.

Theorem 5 There exists an association to any vector bundle on a quasi-projective scheme X of a class such that

- For any morphism , .
- For a line bundle, .
- The Whitney formula holds.
- If is globally generated, for general sections .

The last item is how Chern classes were originally defined and is what many people use to do computation. But unlike in the differentiable or continuous category, there may not be enough global sections in the algebraic category. I will defer the proof as it is slightly more subtle than one may think. As we have seen in examples, the last item characterizes Chern classes and is useful for solving enumerative problems: its existence will be shown in due course.

Let us return to Example 33 .

Let $\mathbb{P}^N$ be the space of all hypersurfaces of degree d in $\mathbb{P}^n$. We would like to compute the degree the singular locus . To see Δ is in fact of codimension one, we look at the incidence correspondence

The fiber at along the second projection is a *linear* subspace (defined by the vanishing of the value and n partial derivatives), hence Φ is irreducible of dimension . It remains to show that the first projection is generically finite (in fact generically one-to-one). For this it suffices to show that there is an isolated point in the fiber of the first projection, which is clear by Bertini, or by taking a singular cone (whose nearby points are all smooth).

Now we introduce a vector bundle on $\mathbb{P}^n$ such that

Let be a general pencil of hypersurface of degree d . Then

- there are no elements singular at the same point (since by Bertini, the common zero locus of two general hypersurfaces is smooth).
- any element of is singular at at most one point (since one can show that is generally one-to-one).

Therefore the number of singular is equal to

- the number of points p such that is singular at p , which is,
- the number of points p such that viewing as general sections of , which is,
- by definition, the Chern class .

To calculate , we look at the exact sequence

where

In other words, we filter the vector bundle by the order of vanishing. Notice that is simply the the line bundle . The fiber of at p is and hence

Now we win because the Chern classes of and can be easily computed.

Example 40 Suppose . Write . We know that

Suppose , then

Hence

Reality check: when , we get the answer 0 (no line is singular); when $d = 2$, we get the answer 2 (a general pencil of conics has a base consisting of four points, there are exactly 3 pairs of lines passing through 4 points); when , we get the answer 12 (the modular discriminant is of degree 12).

For a general n , one can compute that .

Remark 25 In fact one can count singular fibers for any pencil of hypersurfaces given by two sections of any line bundle on any smooth variety X as long as we can calculate the tangent bundle of X . But in this generality the singular locus Δ may not be codimension one and additional verification is needed.

Space of complete conics and 3264

Today's motivating question:

Question Given five general conics in $\mathbb{P}^2$, how many conics are tangent to all 5?

We would like to ask this question only for *smooth* conics since double lines are trivial solutions. The key problem is that the parameter space for smooth conics is *non-compact*. We need to introduce a compactification for the space of smooth conics for doing intersection theory and moreover the intersection on the compactification does not contain extraneous points coming from the boundary.

To see how the boundary may cause trouble, let us consider a simpler problem.

Example 41 Given two conics , how many triangles are inscribed in k and circumscribed about ?

Here we define a *triangle* to be a triple of non-colinear points . The parameter space for triangles is an open subspace of . A natural compactification is simply $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. But the boundary (colinear points) adds lots of trivial solutions of the problem. To deal with this issue, we also need to keep the extra information of lines passing through pairs of points. We redefine a triangle to be a triple of points *and* a triple of lines such that , and . This gives the correct compactified subspace in . The intersection number turns out to be zero: for general conics, there is no such triangle, but when there is one, then there is always a 1-dimensional family of such triangles (Poncelet's theorem), meaning that the intersection is in fact 1-dimensional.

Let us come back to the motivating question.

Definition 9 We define

where C, D are smooth conics dual to each other: , . We simply take the closure X of this locally closed subspace, and call it the space of *complete conics*.

Remark 26 Let be a conic in , it gives a linear map . Then the smoothness of implies that is an isomorphism. The inverse then corresponds to the dual conic . The closure X is simply the projective space of symmetric 3 by 3 matrices. We remark that the space of complete plane curves of higher degree is much harder to study (even for cubics, it is a huge mess).

What happens for the dual conic when the conic becomes singular? The dual of two lines is a double line dual to the intersection point. And vice versa. Therefore , where consists of pairs , where k is union of two distinct lines ( ) and D is the double line , consists of and , and consists of the common specialization: , , where .
Notice that and are of codimension 1 and is of codimension 2.

One can check that:

a. X is smooth.
b. If is smooth and is the closure in X of the locus of smooth conics tangent to . Then and the intersection is transverse.

After checking these, we get a desired nice parameter space. The next step is to calculate the Chow ring. For our purpose we only need to calculate and the degree 5 map .

We claim that . In fact,

a. . Since is the complement of a cubic hypersurface, any divisor on extends to a divisor on , which is of the form . If , the there exists a section of which vanishes to order along and nowhere zero in . Hence in . In general, the Picard group of the complement of a hypersurface in a projective space is always *torsion*.
b. Let be any line bundle on X. Then is trivial on . Find a section of nowhere vanishing on . Extend it to a section σ of on X. Then . Hence has rank at most 2.
c. To see the rank is equal to 2, we introduce

We claim that and are linearly independent in . Let Γ be a general pencil of conics in $\mathbb{P}^2$. Let Φ be a general pencil of conics in . Let be their classes in . Then we can compute the intersection product

since a general pencil has one member passing through a given point and has two members tangent to a given line. The intersection matrix is nonsingular, so α, β are linearly independent.

Now we calculate . We see that

by counting the number of conics passing through 5, 4, 3 points and tangent to 0,1,2 lines. The remaining three products are then given by symmetry.

The last step is to calculate the class . We compute that , hence that . Using the above results we see that

Projective bundles

Today's motivating question:

Question Given 8 general lines , how many (plane) conics in $\mathbb{P}^3$ meet all 8?

This question is significant for us because the parameter space in question will be a projective bundle and calculating the Chow ring of projective bundles is a useful tool and in fact can also be used to give a rigorous definition of the Chern classes. Moreover, the parameter space in question will be a simple example of *Hilbert schemes*.

As usual, we need to first construct a parameter space for conics in $\mathbb{P}^3$. We should have a map , sending a conic to the unique plane containing it. The fiber of this map consists of conics contained in a given plane and has a natural compacitifaction . So we define

where is any homogeneous quadratic polynomial on H.

Remark 27 This is the Hilbert scheme , classifying subschemes of $\mathbb{P}^3$ with Hilbert polynomial . Hilbert schemes are in general horrendously behaved. This is one of the few examples that the Hilbert scheme is a nice space.

The map has fiber and hence is irreducible of dimension 8. Let be the universal family

We have two projection maps , For a given line . Define

Then has codimension 1 in . Hence we expect that the above question has a finite number as its answer.

Next we would like to describe and and compute . is an example of an important class of varieties appearing in enumerative geometry: projective bundles. Let us first discuss projective bundles in general.

Definition 10 Let X be a smooth variety. Let be a vector bundle of rank . Define

One can verify directly that it is well-defined by looking at trivializations and transition functions. More intrinsically,

since homogeneous polynomial of degree d on is . Notice on we have an exact sequence of universal sub and quotient bundles

The hyperplane class is then the first Chern class of . Similarly, we have an exact sequence in families

where . We then define and let . Its restriction to each fiber of is the hyperplane class. We remark that there may not exist a divisor on such that its restriction to each fiber is an actual hyperplane (rather than in the hyperplane class).

The following theorem provides a way of computing .

Theorem 6

a. is injective.

b. Additively,

This can be viewed as an additive Kunneth formula for Chow ring.

c. In the case of a single vector space, , but there is no reason for this to be true in . In fact, as a ring,

Remark 28 By b), there is a unique relation among the powers in ζ. So c) serves as a rigorous definition for the Chern classes of any vector bundle (even without enough sections).

Proof To prove c), we use the exact sequence of bundles on ,

Then . The Whitney formula gives (we suppress from the notation using a). So

In particular, . □

Remark 29 Is any morphism with all fibers projective spaces and locally trivial always a projective bundle? This is true for the complex analytic topology but not true for Zariski topology. This subtlety leads to the notion of Brauer-Severi varieties.

Remark 30 We can also use the Kontsevich space as the parameter space (which, interestingly, differs from the Hilbert scheme) and the similar calculation will give the same answer.

Recall that , a projective bundle over . To compute , we need to compute . Write

Then

Another way: the restriction map from the trivial bundle of rank 10 gives an exact sequence

where the kernel is of rank 4 (its fiber at consists of quadratic forms vanishing on the plane corresponding to , which can be identified with the space of linear forms). We know that .

It follows that

Now it remains to find δ and compute . We do this by undetermined coefficients. Let . We need two curves in : fix a plane H and let Γ be a general pencil of conics in H; fix a quadric and take Φ to be the intersection of with a general pencil of planes. Then one can compute the following intersection matrix

It follows that . After checking that the intersection is transverse (one needs to know more about the tangent space of Hilbert schemes, which we will do next time), the answer to the motivating question would be

Since the pullback of is the class of the fiber and ζ restricts to the hyperplane class in the fiber, we know that . Using the relation in , we know that

Similarly, we obtain that , and . Finally we can conclude that

Segre classes and trisecants

Question If is a general rational curve of degree d, how many trisecant lines does it have?

Instead of looking at lines in and imposing condition that they intersect C at three points, we look at triples of points on the curve and impose the condition that they are colinear, which becomes a linear problem. The compact parameter space we need is then the space of effect divisors on $\mathbb{P}^1$ of degree 3, which is the same as (projectivization of homogeneous polynomials of degree 3 on $\mathbb{P}^1$). To impose the condition that the three points are colinear (i.e. the failure of the three points to impose independent conditions), we introduce the vector bundle on $\mathbb{P}^1$ given by

In other words, the homogeneous degree d polynomials modulo those vanishing on the divisor D. More precisely, we introduce the universal divisor of degree 3 on $\mathbb{P}^1$,

Notice is the vanishing locus of , hence is a closed subvariety of . Let α, β be the projection maps to $\mathbb{P}^3$ and $\mathbb{P}^1$ respectively. Then

We have an evaluation map from the trivial bundle of rank 5

The colinear locus is the locus where fails to be surjective, whose class is by definition the *Segre class* . Let . We have an exact sequence

To compute the Segre class of it suffices to compute that of G.

Let

which is given by a pair of homogeneous polynomials of degree 3 and d such that . It follows that

Under the map , , we have three hyperplane classes on Φ,

a. σ the pullback of ,
b. τ the pullback of (linear forms on a the 1-dimensional space of degree d polynomials vanishing on).
c. the pullback of .

In particular, . Also we know that , so , which gives a degree monic polynomial in σ satisfied by τ. Using Theorem 6, we know that

Hence . Therefore the class of the locus of colinear degree 3 divisor is

Contact problems

Question If is a general quintic surface, how many lines meet S in only one point?

To linearize the problem, we look at the the space of lines *together* with a point

Then the condition that the quintic polynomial restricted to L vanishes to order 5 at the *given* point becomes a linear condition. We introduce the vector bundle of rank 5

the space of quintic polynomials on L modulo the 1-dimensional space of quintic polynomials on L vanishing to order 5 at p (5-th power of a linear form). Now a quintic polynomials on $\mathbb{P}^3$ gives a section of . The locus we want is simply the zero locus of . So the answer is the Chern class .

Remark 31 can be formally defined as , here are the two projections , α, β are the two projections and is the diagonal. This is a teaser: all the vector bundles we have encountered are direct images and the Grothendieck-Riemann-Roch theorem gives a (theoretic) formula for calculating the Chern classes of direct image bundles.

We define an increasing filtration

Then and for , , where is the relative tangent bundle, which is the same as the kernel of the differential . To find , we use the exact sequence on Φ,

where is the universal subbundle on and . Then

Since Φ is a $\mathbb{P}^1$-bundle over , by Theorem 6, we obtain

where (see Example 36). Then and hence . Thus

Using the filtration on , it follows that

in particular,

Porteous' formula

Question Let be a smooth surface and be the projection from a general -plane . At how many points does fail to be injective?

Remark 32 This was an important question in 19th century: before abstract varieties were invented, what people studied were curves in plane, surfaces in space, etc. For a curve in $\mathbb{P}^n$, a general projection to the plane is a plane curve with at most nodal singularities. Since curves are birational if and only if they are isomorphic, it suffices to study plane curves with nodes for curve theory. For a smooth surface in $\mathbb{P}^n$, a general projection to $\mathbb{P}^3$ locally has the following equations:

a. (smooth),
b. (two sheets intersecting transversely),
c. (three sheets intersecting transversely) or,
d. (two sheets winding around at the point). The singular point at is known as a *pinch point*. So the above question is simply asking for the number of pinch points for a general projection of a smooth surface.

Let be a vector bundle of rank . Recall that the Chern class is the class of the locus where

fails to be *injective* and the Segre class is the class of the locus where

fails to be *surjective*. Given a general map between two vector bundles , we can ask for the class of the locus

This class again does not depend on Φ (any two maps can be interpolated) and only depends on the vector bundle and . The key fact is that these classes are all *expressible* in terms of and , which is the content of Porteous' formula.

Remark 33 Notice that the set is indeed a scheme: locally we can trivialize and and represent as a matrix of functions on X. Then is naturally defined as the the -minors of this matrix. We observe that in the space of matrices , the locus of matrices of rank has codimension . Hence the expected codimension of is . Our goal is to find a formula for when the codimension is as expected.

The three steps to obtain Porteous' formula:

a. Linearize;
b. Calculate;
c. Get incredibly lucky.

Linearize. The vanishing of minors is not a linear condition. To linearize, we specify a subspace of the source and require that this *specific subspace* lies in the kernel. So we introduce a parameter space

It is a vector bundle of rank . We denote the universal subbundle of by S and , and we have an exact sequence of vector bundles over Φ ,

Composing with we obtain . Then the locus

is exactly , where is viewed as a section of the vector bundle over Φ . In particular, has codimension

This is equal to the rank of . Therefore is the top Chern class and hence

Calculate. Our next goal is to calculate this Chern class. Suppose are vector bundles of rank . Then . The splitting principle ensures (in theory) that the Chern classes of can be expressible in terms of the Chern classes of and : if , , then

One case this can be calculated explicitly is the top Chern class:

Suppose is the space of degree polynomials on $\mathbb{P}^1$. Similarly for $\mathbb{P}^n$. Look at

The fiber of Φ over a point of $\mathbb{P}^n$ is a disjoint union of n hyperplanes. Hence Φ is a hypersurface of bidegree .

What is this bidegree polynomial? Notice that have a common factor if and only if there exists a relation with , , if and only if the linear map between two dimension spaces

has a kernel. So the bidegree polynomial is simply famous determinant of the *Sylvester matrix* of .

Let . Multiplying the Sylvester matrix by

we obtain the matrix

where

and . Hence have a common root if and only if .

Now let and . It follows that

Hence

Therefore we get the formula

Get lucky. Notice . So we can write , where . So the determinant is an -matrix with entries a linear combination of with coefficients Chern classes of a *pullback*. We observe that if is any monomial in , then if for dimension reason. Hence the only nonzero contribution comes from

By the push-pull formula, we finally obtain *Porteous' formula*

Example 42 Now let us apply Porteous' formula to the question in the beginning. Since , , the number of pinch points is simply the class

Let . Let be the Chern classes of . Then we get the answer (see Example 37)

For example, let be the Veronese surface. Suppose . Then , , . We obtain pinch points. To see this directly, we observe that the secant variety of S is a cubic hypersurface (rather than the whole like every other surface in). So projecting from a line of S we obtain 6 pinch points (2 branch points of a conic projecting onto a line, corresponding to each of the three intersection points).

Excess intersection

What happens when the intersection is not of expected codimension?

Question Let be surfaces of degree such that , where C is a smooth curve of degree d , genus g and Γ is a collection of isolated points. What is $\deg \Gamma$?

Question Let be smooth surfaces such that , where C is a smooth curve of degree d , genus g and Γ is a collection of isolated points. What is $\deg \Gamma$?

Let us assume S is smooth in the first question (though it is necessarily to do so: adding a generic linear combination of the ideals of can make S smooth while keeping the intersection the same). Suppose $S \cap T = C \cup D$, and . Let be the hyperplane class. Then

Hence

By adjunction,

Hence

We recognize that the term as the expected number of intersection points, and the remaining term comes from the excess intersection. We also observe the formula not only depends on the class of C (i.e., its degree d) but also depends on its genus g. We can use this method to compute the excess intersection whenever the objects in question are hypersurfaces (the intersection is reducible one step before the intersection has the wrong dimension), but not in general.

To answer the second question, we use deformations. Suppose there exist deformations that intersect transversely: i.e., families over a disk Δ such that , and intersect transversely when . Therefore , where Σ is finite over Δ of degree . The question becomes: how many of the points of Σ in the generic fiber specialize to C ? To calculate this , we characterize the points of as the singular points of . If , then we would have

In general, we only have the map

The locus is precisely the locus where this map fails to be an isomorphism. Hence

The first two bundles indeed does not depend on the choice of and gives

The third term is equal to since

Using

and

We obtain the general formula

Remark 34 In the deduction of the formula, we used the existence of deformations and realized the excess intersection as the isolated singularities of their intersection. These deformations may not exist on general varieties. But the point is that it doesn't matter and the final formula is still valid!

Theorem 7 (Excess intersection formula) Let X be a smooth variety. Let be smooth varieties. Suppose , where 's are smooth. Let d be the expected dimension of intersection. Let . Then

where is a dimension d class in , given by the dimension d component of

Remark 35 The class measures the failure of the intersection being transverse. When the intersection is transverse, becomes the direct sum of and), hence is simply the fundamental class .

From the beginning of our course, we assumed the ambient variety X is smooth. This is absolutely essential for defining the intersection product.

Example 43 Suppose is a cone over a quadric surface . Let be a 2-plane. Let be a line in X. We can rotate L in the same ruling as to get a line . Then , and . However, if we rotate L in the other ruling to get another line , then but intersect transversely at one point.

This examples destroys the hope to define a general intersection product on singular varieties. But the excess intersection formula helps: though the Chern class does not make sense (coherent sheaf may not have a resolution by vector bundles on a singular variety) and Segre class does make sense (Fulton's brilliant observation). We saw an alternative definition of the Segre class: for any coherent sheaf on X (not necessarily smooth), let be it projectivization with the projection map . Let . Then

Notice one cannot invert the Segre class to define the Chern class : as we already have seen in Example 43, the Chow group does not have a ring structure on the singular varieties.

Now under an extra assumption we can take the excess intersection formula as the definition of intersection product when the intersection is not transverse.

Definition 11 Let such that is locally complete intersection (which implies that is a vector bundle, whose Chern class still makes sense). Suppose , where is *not* necessarily smooth. The we define where is the dimension d class of .

Remark 36 There is a price one must pay: it takes effort to prove the commutativity and associativity of the intersection product thus defined.

Remark 37 One can think of the class as a *homology* class while the class (coming from a locally complete intersection cycle) as a *cohomology* class. As in topology, one can take the cup product of a homology class and a cohomology class to obtain another homology class.

Example 44 Let us come back to the situation where X is smooth. Suppose is smooth of codimension . Let be subvarieties (of codimension a, b in) such that they intersect generically transversely in (so $A \cap B$ has codimension in). Let . We would like to relate and . Let and . Then it follows from the excess intersection formula that

In particular,

is simply the multiplication map by . For example, if has dimension and X has dimension , then . Slightly more generally, suppose X is smooth of dimension , is smooth of dimension of . Then a generic map is an embedding except at a finite number of double points. The number of double points is then given by , half of the different between the self-intersection of and . One recover the genus formula for plane curves from this easily.

Chow rings of blow-ups

One main application of the excess intersection formula is the computation of Chow rings of blow-ups. Suppose are both smooth. Consider the blow-up diagram,

The key fact is that the is generated by and . The remaining problem is to calculate the intersection product between them.

Notice that

By the push-pull formula, we also have

Let . Since `(the universal subbundle), we have $c_1(N_{E/W})-`\zeta$. By the excess intersection formula, we get

As an application, we answer the following earlier question using blow-ups.

Example 45 Given five general conics , how many smooth conics are tangent to all five?

Let be the space of conics in $\mathbb{P}^2$. Then and . Let be the set of conics tangent to . We would like to intersection all five . To avoid the intersection corresponding to double lines, we need to blow-up X along the locus of double lines.

Let be the locus of double lines (= the Veronese surface). Then , . The pullback map is given by . We compute that

Hence by Theorem 6,

Moreover,

Let be the proper transform of D in . One can check that and the intersection is transverse. Hence the number of conics tangent to all five is given by

where . By push-pull formula, the answer is then

which is again!

Grothendieck-Riemann-Roch

The Hirzebruch-Riemann-Roch formula relates the Euler characteristic of some sheaf and the degree of certain dimension zero class in the Chow group. There are three flavors of this formula: the basic package, the standard package and the deluxe package.

Example 46 Let X be a smooth projective curve. In this case the basic package says

Let L be a line bundle . The standard package the says

which can be deduced from the basic package using the exact sequence

The deluxe package says

which can be deduced from the standard package for any coherent sheaf by induction on its rank.

Example 47 Now let X a smooth projective surface. The basic package says that

Similarly, for any line bundle , one deduce the standard package by combining the basic package and the formula for curves:

For any coherent sheaf , we obtain

The formula looks more complicated and you may start to worry what to write down when . But it becomes simpler once we introduce some new language.

Definition 12 Let be a vector bundle over X. Write . We define the *Chern character*

For example, , , and

Remark 38 Notice the Chern character has better formal properties than Chern classes,

and

Definition 13 Let be a vector bundle over X. Write . We define the *Todd class* by

For example, , and .

Remark 39 These definitions can be extended to any coherent sheaf on a smooth projective variety X using resolutions by vector bundles.

Now let X be a smooth projective variety of dimension n. Then the basic package the simply says

This formula in fact originally motivates the weird looking definition of the Todd class: Todd discovered his definition simply by *reverse engineering* using the case when X is a dimension n product of projective spaces. The deluxe formula is also simple: for any coherent sheaf ,

We summarize the three flavors of Hirzebruch-Riemann-Roch formulas in the following table.

X	Basic	Standard	Deluxe
dim 1			
dim 2			
dim n			

If you know Grothendieck, anything that can be done for a single variety should also be done for a family of varieties. Given a family of coherent sheaves on a family of projective varieties , we should be able to fit together the cohomology on the fibers into a sheaf on B. Moreover we should also be able to see the "twisting". Associated to this situation we have a sheaf on B which is a best approximation of the i-th cohomology of the fibers, i.e., for a *general* ,

Theorem 8 (Grothendieck-Riemann-Roch)

Remark 40 The degree 0 piece of the left hand side recovers . The classical Hirzebruch-Riemann-Roch is simply the case when the base B is a point.

Remark 41 The Grothendieck-Riemann-Roch can be used to compute the Chern classes of virtually any vector bundles we encountered before. However, the bad news is that in 99% situation it is more difficult to carry out than the more ad-hoc methods we discussed.