FAMILIES OF IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{S}_2 \wr \mathfrak{S}_3$

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We would like to consider the representation theory of the Weyl group of type B_3, which is isomorphic to the wreath product $\mathfrak{S}_2 \wr \mathfrak{S}_3 = (\mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_2) \rtimes \mathfrak{S}_3$. We will consider the elements of $\mathfrak{S}_2$ to be $\{1, x\}$ throughout. Also we let $C^1 = \{1\}$ and $C^2 = \{x\}$ be the two conjugacy classes of $\mathfrak{S}_2$. We will denote an element of the wreath product by $(x_1, x_2, x_3 : \pi)$ where $x_i \in \mathfrak{S}_2$ and $\pi \in \mathfrak{S}_3$.

1. Conjugacy Classes

We recall from [JK09, Theorem 4.2.8] that the conjugacy classes in $\mathfrak{S}_2 \wr \mathfrak{S}_n$ are parameterised by their **type**. We now define what we mean by type using an example. First note that it is clear that for two entries in $\mathfrak{S}_2 \wr \mathfrak{S}_n$ to be conjugate we must have that the entries in the $\mathfrak{S}_n$ component are conjugate, i.e. of the same cycle type. The notion of **type** for the wreath product is a generalisation of cycle type for $\mathfrak{S}_n$. Consider our case $\mathfrak{S}_2 \wr \mathfrak{S}_3$ and think of the 2×3 matrix

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ x \end{array} & \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \end{array}.$$

Down the left hand side we give conjugacy class representatives for $\mathfrak{S}_2$ and along the top row we give the lengths of possible cycles in a cycle decomposition in $\mathfrak{S}_3$. The entry a_{ik} in the matrix corresponds to the number of **cycle products**, (see [JK09, 4.2.1]), associated to a cycle of length k which are in the conjugacy class C^i of $\mathfrak{S}_2$. These matrices satisfy certain conditions [JK09, 4.2.3] and a complete list of such matrices for $\mathfrak{S}_2 \wr \mathfrak{S}_3$ is given by

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From these matrices it is now possible to write down corresponding entries in the wreath product that are conjugacy class representatives.

$$\begin{array}{llll}
(1,1,1:1) & (x,1,1:1) & (x,x,1:1) & (x,x,x:1), \\
(1,1,1:(12)) & (1,1,x:(12)) & (x,1,1:(12)) & (x,1,x:(12)), \\
(1,1,1:(123)) & (x,1,1:(123)). & &
\end{array}$$

From left to right we label these conjugacy classes $K^1, \ldots, K^{10}$. By [JK09, Lemma 4.2.10] we have a simple way to calculate the order of the conjugacy class in the wreath product and hence the order of the centraliser. We give this information below

	K^1	K^2	K^3	K^4	K^5	K^6	K^7	K^8	K^9	K^{10}
$\lvert K^i \rvert$	1	3	3	1	6	6	6	6	8	8
$\lvert C_{\mathfrak{S}_2 \wr \mathfrak{S}_3}(K^i) \rvert$	48	16	16	48	8	8	8	8	6	6

From this it is clear that the conjugacy class K^4 must contain the longest element of the Weyl group as this is always in its own conjugacy class.

2. Obtaining Representations

We let $1, \varepsilon$ denote the two irreducible representations of $\mathfrak{S}_2$ where ε denotes the sign representation. Recall from the theory of wreath products that $N = \mathfrak{S}_2 \wr \{1\} \lhd \mathfrak{S}_2 \wr \mathfrak{S}_3$ is a normal subgroup isomorphic to $\mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_2$, (see [JK09, 4.1.15]). A complete list of irreducible representations for this normal subgroup is given by

$$\begin{array}{lll}
1 \times 1 \times 1 & & \varepsilon \times \varepsilon \times \varepsilon, \\
\varepsilon \times 1 \times 1 & 1 \times \varepsilon \times 1 & 1 \times 1 \times \varepsilon, \\
1 \times \varepsilon \times \varepsilon & \varepsilon \times 1 \times \varepsilon & \varepsilon \times \varepsilon \times 1,
\end{array}$$

As N is a normal subgroup of $\mathfrak{S}_2 \wr \mathfrak{S}_3$ we can consider the action of this group on N by conjugation. We then also have that $\mathfrak{S}_2 \wr \mathfrak{S}_3$ will act on the irreducible characters of N by composition with conjugation. We consider the stabiliser of this action in $\mathfrak{S}_2 \wr \mathfrak{S}_3$. Clearly N is in the stabiliser and in fact we will have that the stabiliser is of the form $\mathfrak{S}_2 \wr H$ for some subgroup $H \leqslant \mathfrak{S}_3$. Let n_1 be the number of copies of the trivial character in the representation of $\mathfrak{S}_2^3$ and n_2 the number of copies of the sign character. Then by [JK09, Theorem 4.3.27] the stabiliser is isomorphic to the group $\mathfrak{S}_2 \wr (\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2})$.

We call the stabiliser of this action the **inertia group** of the irreducible representation. The component $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ is called the **inertia factor**. By [JK09, Theorem 4.4.3] we have that two irreducible representations χ, ψ of $\mathfrak{S}_2^3$ are in the same equivalence class under the conjugation action by $\mathfrak{S}_2 \wr \mathfrak{S}_3$ if and only if the representations contain the same number of copies of the trivial and sign representation. Hence representatives of these equivalence classes are given by

$$1 \times 1 \times 1 \qquad \varepsilon \times 1 \times 1 \qquad 1 \times \varepsilon \times \varepsilon \qquad \varepsilon \times \varepsilon \times \varepsilon.$$

By [JK09, 4.3.28] we can extend these representations of N to its inertia group and we denote this extension $\chi^{\sim}$. Let $[3], [2,1], [1^3]$ be the irreducible representations of $\mathfrak{S}_3$, $[2], [1^2]$ be the irreducible representations of $\mathfrak{S}_2$ and $[1]$ be the irreducible representation of $\mathfrak{S}_1$. Then every irreducible representation of $\mathfrak{S}_2 \wr \mathfrak{S}_3$ is given by tensoring the extension of an irreducible representation of N with the inflation of an irreducible representation of its inertia factor and inducing to the whole group. For example, letting $[\lambda]'$ denote the inflation of $[\lambda]$, we have the complete list of irreducible representations of this wreath product is given by

(λ, μ)	Representation	$\chi(1)$	Inertia Factor	Inertia Group
$(3, -)$	$(1 \times 1 \times 1)^{\sim} \otimes [3]'$	1	$\mathfrak{S}_3$	$\mathfrak{S}_2 \wr \mathfrak{S}_3$
$(21, -)$	$(1 \times 1 \times 1)^{\sim} \otimes [2,1]'$	2	$\mathfrak{S}_3$	$\mathfrak{S}_2 \wr \mathfrak{S}_3$
$(1^3, -)$	$(1 \times 1 \times 1)^{\sim} \otimes [1^3]'$	1	$\mathfrak{S}_3$	$\mathfrak{S}_2 \wr \mathfrak{S}_3$
$(2, 1)$	$(\varepsilon \times 1 \times 1)^{\sim} \otimes ([2]' \times [1]') \uparrow \mathfrak{S}_2 \wr \mathfrak{S}_3$	3	$\mathfrak{S}_2 \times \mathfrak{S}_1$	$\mathfrak{S}_2 \wr \mathfrak{S}_2$
$(1^2, 1)$	$(\varepsilon \times 1 \times 1)^{\sim} \otimes ([1^2]' \times [1]') \uparrow \mathfrak{S}_2 \wr \mathfrak{S}_3$	3	$\mathfrak{S}_2 \times \mathfrak{S}_1$	$\mathfrak{S}_2 \wr \mathfrak{S}_2$
$(1, 2)$	$(1 \times \varepsilon \times \varepsilon)^{\sim} \otimes ([1]' \times [2]') \uparrow \mathfrak{S}_2 \wr \mathfrak{S}_3$	3	$\mathfrak{S}_1 \times \mathfrak{S}_2$	$\mathfrak{S}_2 \wr \mathfrak{S}_2$
$(1, 1^2)$	$(1 \times \varepsilon \times \varepsilon)^{\sim} \otimes ([1]' \times [1^2]') \uparrow \mathfrak{S}_2 \wr \mathfrak{S}_3$	3	$\mathfrak{S}_1 \times \mathfrak{S}_2$	$\mathfrak{S}_2 \wr \mathfrak{S}_2$
$(-, 3)$	$(\varepsilon \times \varepsilon \times \varepsilon)^{\sim} \otimes [3]'$	1	$\mathfrak{S}_3$	$\mathfrak{S}_2 \wr \mathfrak{S}_3$
$(-, 21)$	$(\varepsilon \times \varepsilon \times \varepsilon)^{\sim} \otimes [2,1]'$	2	$\mathfrak{S}_3$	$\mathfrak{S}_2 \wr \mathfrak{S}_3$
$(-, 1^3)$	$(\varepsilon \times \varepsilon \times \varepsilon)^{\sim} \otimes [1^3]'$	1	$\mathfrak{S}_3$	$\mathfrak{S}_2 \wr \mathfrak{S}_3$

Note that when the inertia factor is $\mathfrak{S}_3$ we don't need to induce to the whole group because we already have an irreducible representation of the wreath product. Also we recall that the index of $\mathfrak{S}_2 \wr \mathfrak{S}_2$ is given by

$$[\mathfrak{S}_2 \wr \mathfrak{S}_3 : \mathfrak{S}_2 \wr \mathfrak{S}_2] = \frac{|\mathfrak{S}_2|^3 |\mathfrak{S}_3|}{|\mathfrak{S}_2|^3 |\mathfrak{S}_2|} = \frac{|\mathfrak{S}_3|}{|\mathfrak{S}_2|} = \frac{6}{2} = 3.$$

Hence this explains the dimensions of the representations. We have listed 10 irreducible representations of the wreath product. By a specialisation of [JK09, 4.3.6] we have that the number of irreducible representations of $\mathfrak{S}_2 \wr \mathfrak{S}_n$ is given by

$$\sum_{(n_1, n_2)} p(n_1) p(n_2),$$

such that $n_1, n_2 \in \mathbb{N}^0$ and $n_1+n_2 = n$ and $p(n_i)$ is the number of partitions of n_i. Therefore applying this to our case we have the number of irreducible representations of $\mathfrak{S}_2 \wr \mathfrak{S}_3$ is

$$2p(3)p(0) + 2p(2)p(1) = 2 \times 3 + 2 \times 2 = 6 + 4 = 10.$$

Hence we are likely correct. Indeed recalling the fact that $|\mathfrak{S}_2 \wr \mathfrak{S}_3| = |\mathfrak{S}_2|^3|\mathfrak{S}_3| = 2^3 \cdot 3! = 48$ we have from the dimensions of the irreducible representations that

$$1^2 + 2^2 + 1^2 + 3^2 + 3^2 + 3^2 + 3^2 + 1^2 + 2^2 + 1^2 = 48$$

and so we are almost certainly correct.

3. The Character Table

We now go about calculating the character table of $\mathfrak{S}_2 \wr \mathfrak{S}_3$. We note that in the table above we have indexed the characters in the standard way by bi-partitions. A bi-partition is a pair (λ, μ) such that $\lambda, \mu \vdash n$ are partitions of n and $|\lambda| + |\mu| = n$.

	K^1	K^2	K^3	K^4	K^5	K^6	K^7	K^8	K^9	K^{10}
$\lvert K^i\rvert$	1	3	3	1	6	6	6	6	8	8
$\lvert C_{\mathfrak{S}_2\wr\mathfrak{S}_3}(K^i)\rvert$	48	16	16	48	8	8	8	8	6	6
$\chi_{(3,-)}$	1	1	1	1	1	1	1	1	1	1
$\chi_{(21,-)}$	2	2	2	2	0	0	0	0	-1	-1
$\chi_{(1^3,-)}$	1	1	1	1	-1	-1	-1	-1	1	1
$\chi_{(2,1)}$	3	1	-1	-3	1	-1	1	-1	0	0
$\chi_{(1^2,1)}$	3	1	-1	-3	-1	1	-1	1	0	0
$\chi_{(1,2)}$	3	-1	-1	3	1	1	-1	-1	0	0
$\chi_{(1,1^2)}$	3	-1	-1	3	-1	-1	1	1	0	0
$\chi_{(-,3)}$	1	-1	1	-1	1	-1	-1	1	1	-1
$\chi_{(-,21)}$	2	-2	2	-2	0	0	0	0	-1	1
$\chi_{(-,1^3)}$	1	-1	1	-1	-1	1	1	-1	1	-1

To calculate the first and last three characters in this table it is simply a case of multiplying the character values together as we already obtain an irreducible representation of the whole wreath product by tensoring. In order to calculate the middle four characters we will need to induce the representations from the inertia group. We note that the inertia group is $\mathfrak{S}_2 \wr \langle(23)\rangle \cong \mathfrak{S}_2 \wr \mathfrak{S}_2$ for all of these representations. Therefore we will need to know how the conjugacy classes split upon restriction to this subgroup. By sheer calculation we see that

$$
\begin{aligned}
K^1 \downarrow \mathfrak{S}_2 \wr \langle (23) \rangle &= \{(1,1,1:1)\},\\
K^2 \downarrow \mathfrak{S}_2 \wr \langle (23) \rangle &= \{(x,1,1:1)\} \sqcup \{(1,x,1:1),(1,1,x:1)\},\\
K^3 \downarrow \mathfrak{S}_2 \wr \langle (23) \rangle &= \{(x,x,1:1),(x,1,x:1)\} \sqcup \{(1,x,x:1)\},\\
K^4 \downarrow \mathfrak{S}_2 \wr \langle (23) \rangle &= \{(x,x,x:1)\},\\
K^5 \downarrow \mathfrak{S}_2 \wr \langle (23) \rangle &= \{(1,1,1:(23)),(1,x,x:(23))\},\\
K^6 \downarrow \mathfrak{S}_2 \wr \langle (23) \rangle &= \{(x,1,1:(23)),(x,x,x:(23))\},\\
K^7 \downarrow \mathfrak{S}_2 \wr \langle (23) \rangle &= \{(1,x,1:(23)),(1,1,x:(23))\},\\
K^8 \downarrow \mathfrak{S}_2 \wr \langle (23) \rangle &= \{(x,x,1:(23)),(x,1,x:(23))\}.
\end{aligned}
$$

This information gives us the structure of the conjugacy classes in $\mathfrak{S}_2 \wr \langle (23) \rangle$. We can now use this information to calculate the uninduced characters of this group

	K^1	K^2	$K^{2'}$	K^3	$K^{3'}$	K^4	K^5	K^6	K^7	K^8
$\lvert K^i \downarrow \mathfrak{S}_2 \wr \langle (23) \rangle \rvert$	1	2	1	2	1	1	2	2	2	2
$\lvert C_{\mathfrak{S}_2 \wr \mathfrak{S}_3}(K^i) \rvert$	48	16	16	16	16	48	8	8	8	8
$\lvert C_{\mathfrak{S}_2 \wr \langle (23) \rangle}(K^i) \rvert$	16	8	16	8	16	16	8	8	8	8
$(\varepsilon \times 1 \times 1)^{\sim} \otimes ([2]' \times [1]')$	1	1	-1	-1	1	-1	1	-1	1	-1
$(\varepsilon \times 1 \times 1)^{\sim} \otimes ([1^2]' \times [1]')$	1	1	-1	-1	1	-1	-1	1	-1	1
$(1 \times \varepsilon \times \varepsilon)^{\sim} \otimes ([1]' \times [2]')$	1	-1	1	-1	1	1	1	1	-1	-1
$(1 \times \varepsilon \times \varepsilon)^{\sim} \otimes ([1]' \times [1^2]')$	1	-1	1	-1	1	1	-1	-1	1	1

This then makes it very easy to construct the induced characters in the table of $\mathfrak{S}_2 \wr \mathfrak{S}_3$. Alternatively one could use [GHL+96] to produce the character table.

4. Lusztig's Symbols

We are interested in how the irreducible representations of $\mathfrak{S}_2 \wr \mathfrak{S}_3$ fall into **families** in the sense of [Lus84, 4.2]. Lusztig gives a criterion [Lus84, 4.5.6] for two irreducible representations to be in the same family in a Weyl group of type B_n. This criterion is based upon certain combinatorial symbols that he associates to the representations. Recall that to each representation we associated a bi-partition (λ, μ) of n.

The current way we have expressed the partitions λ, μ are as weakly decreasing partitions. However it would make no different to consider them as weakly increasing. This allows us to add as many 0's as we like to the front of our partitions without affecting their meaning. By doing this we can always arrange that λ has exactly one more entry than μ by adding a suitable number of zeroes to the front of the partitions. We let this new partition be denoted (λ', μ').

To each bipartition (λ, μ) we can associate a combinatorial symbol $[\Lambda]$ using the bipartition (λ', μ'). Let λ'_i, μ'_i denote the entries of λ', μ' then we define a new pair $(\bar{\lambda}, \bar{\mu})$ by $\bar{\lambda} = (\lambda'_1, \lambda'_2 + 1, \lambda'_3 + 2, \dots, \lambda'_{m+1} + m)$ and $\bar{\mu} = (\mu'_1, \mu'_2 + 1, \mu'_3 + 2, \dots, \mu'_m + m - 1)$. Then Lusztig's combinatorial symbol, denoted $[\Lambda]$, associated to (λ, μ) is given by

$$[\Lambda] = \begin{bmatrix} \bar{\lambda}_1 & & \bar{\lambda}_2 & & \dots & & \bar{\lambda}_m & & \bar{\lambda}_{m+1} \\ & \bar{\mu}_1 & & \bar{\mu}_2 & & \dots & & \bar{\mu}_m & \end{bmatrix}.$$

In the following table we give the partitions (λ', μ') and the Lusztig symbols $[\Lambda]$ for each of the partitions (λ, μ).

(λ, μ)	(λ', μ')	$(\bar{\lambda}, \bar{\mu})$	$[\Lambda]$	$[\Lambda] \otimes \chi_{(-,1^3)}$
$(3, -)$	$(3, -)$	$(3, -)$	$\begin{bmatrix} 3 \\ - \end{bmatrix}$	$\begin{bmatrix} 0 & & 1 & & 2 & & 3 \\ & 1 & & 2 & & 3 & \end{bmatrix}$
$(21, -)$	$(12, 0)$	$(13, 0)$	$\begin{bmatrix} 1 & & 3 \\ & 0 & \end{bmatrix}$	$\begin{bmatrix} 0 & & 1 & & 2 \\ & 1 & & 3 & \end{bmatrix}$
$(1^3, -)$	$(1^3, 0^2)$	$(123, 01)$	$\begin{bmatrix} 1 & & 2 & & 3 \\ & 0 & & 1 & \end{bmatrix}$	$\begin{bmatrix} 0 & & 1 \\ & 3 & \end{bmatrix}$
$(2, 1)$	$(02, 1)$	$(03, 1)$	$\begin{bmatrix} 0 & & 3 \\ & 1 & \end{bmatrix}$	$\begin{bmatrix} 0 & & 1 & & 3 \\ & 1 & & 2 & \end{bmatrix}$
$(1^2, 1)$	$(1^2, 1)$	$(12, 1)$	$\begin{bmatrix} 1 & & 2 \\ & 1 & \end{bmatrix}$	$\begin{bmatrix} 0 & & 2 \\ & 2 & \end{bmatrix}$
$(1, 2)$	$(01, 2)$	$(02, 2)$	$\begin{bmatrix} 0 & & 2 \\ & 2 & \end{bmatrix}$	$\begin{bmatrix} 1 & & 2 \\ & 1 & \end{bmatrix}$
$(1, 1^2)$	$(0^2 1, 1^2)$	$(013, 12)$	$\begin{bmatrix} 0 & & 1 & & 3 \\ & 1 & & 2 & \end{bmatrix}$	$\begin{bmatrix} 0 & & 3 \\ & 1 & \end{bmatrix}$
$(-, 3)$	$(0^2, 3)$	$(01, 3)$	$\begin{bmatrix} 0 & & 1 \\ & 3 & \end{bmatrix}$	$\begin{bmatrix} 1 & & 2 & & 3 \\ & 0 & & 1 & \end{bmatrix}$
$(-, 21)$	$(0^3, 12)$	$(012, 13)$	$\begin{bmatrix} 0 & & 1 & & 2 \\ & 1 & & 3 & \end{bmatrix}$	$\begin{bmatrix} 1 & & 3 \\ & 0 & \end{bmatrix}$
$(-, 1^3)$	$(0^4, 1^3)$	$(0123, 123)$	$\begin{bmatrix} 0 & & 1 & & 2 & & 3 \\ & 1 & & 2 & & 3 & \end{bmatrix}$	$\begin{bmatrix} 3 \\ - \end{bmatrix}$

Now Lusztig's criterion, (see [Lus84, 4.5.6]), states that two irreducible representations of the Weyl group of type B_n are in the same family if and only if their symbols $[\Lambda]$ contain the same entries. For example $\{\chi_{(21,-)}, \chi_{(2,1)}, \chi_{(-,3)}\}$ is a family of irreducible characters in $\mathfrak{S}_2 \wr \mathfrak{S}_3$ because their associated symbols $[\Lambda]$ are

$$\begin{bmatrix} 1 & & 3 \\ & 0 & \end{bmatrix} \qquad \begin{bmatrix} 0 & & 3 \\ & 1 & \end{bmatrix} \qquad \begin{bmatrix} 0 & & 1 \\ & 3 & \end{bmatrix}.$$

Therefore using this criterion we have the families of irreducible characters of $\mathfrak{S}_2 \wr \mathfrak{S}_3$ are

$$\{\chi_{(3,-)}\} \quad \{\chi_{(1^2,1)}\} \quad \{\chi_{(1,2)}\} \quad \{\chi_{(-,1^3)}\} \quad \{\chi_{(21,-)}, \chi_{(2,1)}, \chi_{(-,3)}\} \quad \{\chi_{(1^3,-)}, \chi_{(1,1^2)}, \chi_{(-,21)}\}.$$

Indeed using [GHL$^+$96] to calculate the truncated induction tables from parabolic subgroups one can verify that these are the families of irreducible characters in $\mathfrak{S}_2 \wr \mathfrak{S}_3$.

Recall that $\chi_{(-,1^3)}$ is the sign representation of $\mathfrak{S}_2 \wr \mathfrak{S}_3$. Now Lusztig prescribes a way to determine $[\Lambda] \otimes \mathrm{sgn}$ in the case of a Weyl group of type B_n. Let $[\Lambda]$ be a symbol representing an irreducible representation of $W(\mathrm{B}_n)$ with associated partitions (λ', μ'). Then if $t = \max\{\lambda'_i, \mu'_j \mid 1 \leqslant i \leqslant m+1 \text{ and } 1 \leqslant j \leqslant m\}$ we define the symbol for $[\Lambda] \otimes \mathrm{sgn}$ in the following way. The top row is given by $(t - j \mid 0 \leqslant j \leqslant t \text{ and } j \neq \mu_1, \dots, \mu_m)$ and the bottom row is given by $(t - i \mid 0 \leqslant i \leqslant t \text{ and } i \neq \lambda_1, \dots, \lambda_{m+1})$. In the table we have calculated these symbols in our case.

5. The Fourier Transform Matrix

We wish to describe how, given a family $\mathcal{F}$ of irreducible characters of $\mathfrak{S}_2 \wr \mathfrak{S}_3$, we can obtain the associated Fourier transform matrix. Let $\mathcal{F}$ be the family $\{\chi_{(1^3,-)}, \chi_{(1,1^2)}, \chi_{(-,21)}\}$ which have associated symbols

$$\begin{bmatrix} 0 & & 1 & & 3 \\ & 1 & & 2 & \end{bmatrix} \qquad \begin{bmatrix} 0 & & 1 & & 2 \\ & 1 & & 3 & \end{bmatrix} \qquad \begin{bmatrix} 1 & & 2 & & 3 \\ & 0 & & 1 & \end{bmatrix}.$$

Now in each family there is a unique **special** symbol. In the case of type B_n the unique special symbol is the symbol satisfying the condition that

$$\bar{\lambda}_1 \leqslant \bar{\mu}_1 \leqslant \bar{\lambda}_2 \leqslant \bar{\mu}_2 \leqslant \cdots \leqslant \bar{\mu}_m \leqslant \bar{\lambda}_{m+1}.$$

Therefore the unique special symbol in the family $\mathcal{F}$ is

$$[Z] = \begin{bmatrix} 0 & & 1 & & 3 \\ & 1 & & 2 & \end{bmatrix}.$$

To the special symbol $[Z]$ we wish to define a set of symbols $\mathcal{M}_{\mathcal{F}}$. A symbol in $\mathcal{M}_{\mathcal{F}}$ will be of the form

$$\begin{bmatrix} \lambda_1 & & \lambda_2 & & \dots & & \dots & & \dots & \lambda_b \\ & \mu_1 & & \mu_2 & & \dots & & \mu_{b'} & & \end{bmatrix},$$

such that $b' < b$ and $b + b' = 2m+1$, (in our case $2m+1 = 5$), and $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_b$, $0 \leqslant \mu_1 < \mu_2 < \cdots < \mu_{b'}$, where $\{\lambda_1, \dots, \lambda_b, \mu_1, \dots, \mu_{n'}\}$ is precisely the set of entries in $[Z]$. For example, in our case we would have the set of symbols $\mathcal{M}_{\mathcal{F}}$ is

$$\begin{bmatrix} 0 & & 1 & & 3 \\ & 1 & & 2 & \end{bmatrix} \quad \begin{bmatrix} 0 & & 1 & & 2 \\ & 1 & & 3 & \end{bmatrix} \quad \begin{bmatrix} 1 & & 2 & & 3 \\ & 0 & & 1 & \end{bmatrix} \quad \begin{bmatrix} 0 & & 1 & & 2 & & 3 \\ & 1 & & & & & \end{bmatrix}.$$

It's clear to see that we can identify the symbols corresponding to $\mathcal{F}$ in $\mathcal{M}_{\mathcal{F}}$ as the set of all symbols such that $b = b' + 1$. Hence we have an embedding $\mathcal{F} \hookrightarrow \mathcal{M}_{\mathcal{F}}$.

We have any element of $\mathcal{M}_{\mathcal{F}}$ is of the form

$$[\Lambda_M] = \begin{cases} \begin{bmatrix} Z_2 \sqcup (Z_1 - M) \\ Z_2 \sqcup M \end{bmatrix} & \text{if } |Z_2 \sqcup (Z_1 - M)| > |Z_2 \sqcup M|, \\ \begin{bmatrix} Z_2 \sqcup M \\ Z_2 \sqcup (Z_1 - M) \end{bmatrix} & \text{if } |Z_2 \sqcup M| > |Z_2 \sqcup (Z_1 - M)|, \end{cases}$$

where we define the sets Z_1, Z_2 and M in the following way. We let Z_2 be the set of elements which appear in both rows of the special symbol $[Z]$. We let Z_1 be the set of entries that appear only once in $[Z]$ and M is a subset of Z_1 such that $|M| \equiv d \pmod 2$ where d is such that $|Z_1| = 2d + 1$. For example, in our case we have $Z_2 = \{1\}$, $Z_1 = \{0, 2, 3\}$ and the respective sets M are $\{2\}$, $\{3\}$, $\{0\}$ and $\{0, 2, 3\}$. Note that we can identify the special symbol $[Z]$ as $[\Lambda_{M_0}]$ where M_0 is the set of elements of Z_1 appearing in the second row of $[Z]$. Hence in our case $M_0 = \{2\}$.

Now we associate to each set M a subset $M^\# \subset Z_1$ where $M^\# = (M \cup M_0) - (M \cap M_0)$. This gives us a bijection $[\Lambda_M] \leftrightarrow M^\#$ between the set of symbols $\mathcal{M}_{\mathcal{F}}$ and the set V_{Z_1} of all subsets of Z_1 of even cardinality. We consider the sets $M^\#$ in our example

$[\Lambda_M]$	M	$M^\#$
$\begin{bmatrix} 1 & & 2 & & 3 \\ & 0 & & 1 & \end{bmatrix}$	$\{0\}$	$\{0, 2\}$
$\begin{bmatrix} 0 & & 1 & & 3 \\ & 1 & & 2 & \end{bmatrix}$	$\{2\}$	$\varnothing$
$\begin{bmatrix} 0 & & 1 & & 2 \\ & 1 & & 3 & \end{bmatrix}$	$\{3\}$	$\{2, 3\}$
$\begin{bmatrix} 0 & & 1 & & 2 & & 3 \\ & 1 & & & & & \end{bmatrix}$	$\{0, 2, 3\}$	$\{0, 3\}$

It's easy to verify that $V_{Z_1} = \{\varnothing, \{0, 2\}, \{2, 3\}, \{0, 3\}\}$ is the set of all subsets of Z_1 of even cardinality.

We can turn V_{Z_1} into a vector space over the field $\mathbb{F}_2$ by defining addition to be the **symmetric difference** of sets. Let A, B be two sets then we define their symmetric difference to be the set $A \ominus B = (A \cup B) - (A \cap B)$. For example $\{0, 2\} \ominus \{0, 3\} = \{2, 3\}$. This vector space is, in general, of dimension $2d$ and hence is of dimension 2 in our case.

Let $M_1^\#, M_2^\#$ be two elements of V_{Z_1} then we define a nonsingular symplectic form $\langle -, - \rangle : V_{Z_1} \times V_{Z_1} \to \mathbb{F}_2$ on V_{Z_1} by $\langle M_1^\#, M_2^\# \rangle = |M_1^\# \cap M_2^\#| \pmod 2$.

We wish to define a basis for V_{Z_1}. Let $Z_1 = \{z_1, \dots, z_{2d+1}\}$ be such that $z_1 \leqslant z_2 \leqslant \cdots \leqslant z_{2d+1}$, then for each $1 \leqslant i \leqslant 2d$ we define an element $e_i = \{z_i, z_{i+1}\} \in V_{Z_1}$. For example in our case we will have two elements $e_1 = \{0, 2\}$ and $e_2 = \{2, 3\}$. The elements e_i will form a basis for the vector space V_{Z_1}. Let Γ_1 be the span over $\mathbb{F}_2$ of the basis elements $e_1, e_3, e_5, \dots$ and Γ_2 be the span over $\mathbb{F}_2$ of the basis elements $e_2, e_4, e_6, \dots$, then we have $V_{Z_1} = \Gamma_1 \oplus \Gamma_2$. For example in our case we have $\Gamma_1 = \{\varnothing, e_1\} = \{\varnothing, \{0, 2\}\}$ and $\Gamma_2 = \{\varnothing, e_2\} = \{\varnothing, \{2, 3\}\}$ and it's clear that $V_{Z_1} = \Gamma_1 \oplus \Gamma_2$.

We now associate to the family $\mathcal{F}$ a finite group $\mathcal{G}(\mathcal{F})$ in the following way. Let $\mathcal{G}(\mathcal{F})$ be the abelian group Γ_1. In general this will be isomorphic to $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ but in our case it is just isomorphic to $\mathbb{Z}_2$. Recall that $\mathcal{M}(\mathcal{G}(\mathcal{F}))$ is the collection of all pairs (x, σ) such that $x \in \mathcal{G}(\mathcal{F})$ and σ is an irreducible character of $C_{\mathcal{G}(\mathcal{F})}(x)$. Note that in the case of B_n we have $\mathcal{G}(\mathcal{F})$ is always abelian and so $C_{\mathcal{G}(\mathcal{F})}(x) = \mathcal{G}(\mathcal{F})$, for all $x \in \mathcal{G}(\mathcal{F})$. Using the pairing $\langle -, - \rangle$ we have $\Gamma_2 \cong \mathrm{Hom}(\Gamma_1, \mathbb{F}_2)$ via the map $\gamma \mapsto \varphi_\gamma$, where $\varphi_\gamma(\delta) = (-1)^{\langle \delta, \gamma \rangle}$ for each $\delta \in \Gamma_1$ and $\gamma \in \Gamma_2$. Therefore we canonically have $V_{Z_1} = \Gamma_1 \oplus \Gamma_2 \cong \mathcal{M}(\mathcal{G}(\mathcal{F}))$.

Now for each two pairs (x, σ), $(y, \tau) \in \mathcal{M}(\mathcal{G}(\mathcal{F}))$ we have their corresponding entry in the Fourier transform matrix will be $1/2^k \sigma(y)\tau(x)$, where k is the number of copies of $\mathbb{Z}_2$ that appear in $\mathcal{G}(\mathcal{F})$. Therefore in our case we will have $k = 1$ and hence the entry of the Fourier transform matrix will be $1/2\sigma(y)\tau(x)$. This means the Fourier transform matrix will be

$$
\begin{array}{c|c}
 & \begin{array}{cccc} (\varnothing, \varphi_\varnothing) & (\varnothing, \varphi_{e_2}) & (e_1, \varphi_\varnothing) & (e_1, \varphi_{e_2}) \end{array} \\
\begin{array}{c} (\varnothing, \varphi_\varnothing) \\ (\varnothing, \varphi_{e_2}) \\ (e_1, \varphi_\varnothing) \\ (e_1, \varphi_{e_2}) \end{array} &
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\end{array}.
$$

Finally we comment that we have an embedding $\mathcal{F} \hookrightarrow \mathcal{M}(\mathcal{G}(\mathcal{F}))$ given by the embedding $\mathcal{F} \hookrightarrow \mathcal{M}_\mathcal{F}$ composed with the bijection $\mathcal{M}_\mathcal{F} \to V_{Z_1}$ and the isomorphism $V_{Z_1} \cong \mathcal{M}(\mathcal{G}(\mathcal{F}))$.

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