

# Chromatic and Temperley-Lieb algebras

## A homomorphism

Lawrence Hook

University of Virginia

MATH 4840 - 2016

# Overview

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

- Introduce
  - Chromatic algebra
  - Temperley-Lieb algebra
- Define a map  $\Phi$ : Chromatic  $\rightarrow$  Temperley-Lieb
- Prove  $\Phi$  is well-defined & preserves trace
- Further applications - knotted graph invariant

# Terminology

Chromatic  
and  
Temperley-  
Lieb  
algebras

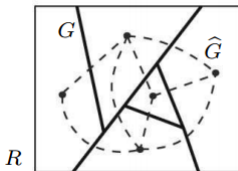
Lawrence  
Hook

## Definition

A **trivalent** graph is a graph in which all vertices have three associated edges

## Definition

Given a planar graph  $G$ , the **dual** graph  $\hat{G}$ , is obtained as shown below.



# Free Algebra

## Definition

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

### Definition

The **free algebra** in degree  $n$ ,  $\mathcal{F}_n$  over  $\mathbb{C}[Q]$   
Elements are linear combinations of trivalent graphs contained  
in a rectangle, with  $n$  endpoints on both the top and the  
bottom.

# Chromatic Algebra

## Definition

### Definition

The **chromatic algebra** in degree  $n$ ,  $C_n$  is the algebra over  $\mathbb{C}[Q]$  given by the quotient of the free algebra  $\mathcal{F}_n$  by  $I_n$

Sometimes denoted  $C_n^Q$ . Set  $C = \cup_n C_n$

The ideal  $I_n$  is generated by the “H-I relation” and the fact that “tadpoles” vanish.

The diagram shows an equation between two sets of circular diagrams. On the left, a circle with three lines meeting at a central point (a 'Y' shape) is added to a circle with two horizontal arcs (one at the top, one at the bottom). This is equal to a circle with three lines meeting at a central point (an 'X' shape) added to a circle with two vertical arcs (one on the left, one on the right). To the right of this, a circle with a small circle inside it (a 'tadpole') is shown to be equal to zero.

Also note, the value of a simple, closed curve is set to  $Q - 1$

# Chromatic Polynomial

## Intuitive definition

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

### Definition

The **chromatic polynomial**  $\chi_\Gamma(Q)$  of a graph  $\Gamma$ ,  $Q \in \mathbb{Z}^+$  is the number of colorings of the vertices of  $\Gamma$  with  $Q$  colors where no adjacent vertices have the same color.

# Chromatic Polynomial

## definition

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

### Properties:

- 1  $\chi_{\Gamma}(\mathcal{Q}) = \chi_{\Gamma \setminus e}(\mathcal{Q}) - \chi_{\Gamma/e}(\mathcal{Q})$
- 2 if  $\Gamma$  has no edges and  $V$  vertices, then  $\chi_{\Gamma}(\mathcal{Q}) = \mathcal{Q}^V$
- 3 if  $\Gamma$  contains a loop, then  $\chi_{\Gamma}(\mathcal{Q}) = 0$

These properties allow us to generalize the chromatic polynomial from  $\mathbb{Z}^+$  to  $\mathbb{C}$

# Chromatic Polynomial

## State space formula

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

An explicit formula can be derived from the preceding properties. This formula is called the **state space formula** and is given by:

$$\chi_G(Q) = \sum_{s \subseteq E(G)} (-1)^{|s|} Q^{k(s)}$$

where  $k(s)$  is the number of connected components of  $G$ .



# Chromatic Algebra

## Trace

### Definition

**Trace**  $tr_\chi : \mathcal{C}^Q \rightarrow \mathbb{C}$

Defined on the additive generators, i.e. a graph  $G$

Connect the endpoints of the rectangle and denote the result  $\bar{G}$

Now, evaluate

$$tr_\chi(G) = Q^{-1} \cdot \chi_{\hat{G}}(Q)$$

$$Tr \left( \begin{array}{|c|} \hline \text{Y-shape} \\ \hline \end{array} \right) = \left( \begin{array}{|c|} \hline \text{Y-shape} \\ \hline \end{array} \right) \text{ with boundary} = \left( \begin{array}{|c|} \hline \text{Y-shape} \\ \hline \end{array} \right) \text{ with boundary and internal structure} = (Q-1)^2(Q-2).$$

# Temperley-Lieb

## Algebraic definition

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

### Definition

The Temperley-Lieb algebra in degree  $n$ ,  $TL_n$  is an algebra over  $\mathbb{C}[d]$  ( $d$  is some complex number) generated by  $\{1, E_1, \dots, E_{n-1}\}$  with relations:

$$E_i^2 = E_i$$

$$E_i E_{i \pm 1} E_i = \frac{1}{d^3} E_i$$

$$E_i E_j = E_j E_i \text{ for } |i - j| \geq 1$$

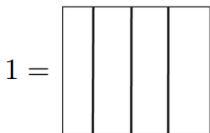
# Temperley-Lieb

## Geometric

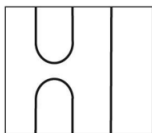
Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

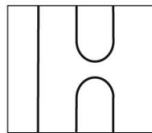
The generators of  $TL_3$



$E_1 = \frac{1}{d}$



$E_2 = \frac{1}{d}$



# Temperley-Lieb

## Trace

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

### Definition

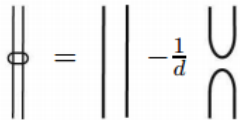
**Trace**  $tr_d : TL_n^d \rightarrow \mathbb{C}$

Connect the endpoints and evaluate  $d^{\#circles}$

# Homomorphism

## Definition

Define a homomorphism  $\Phi : \mathcal{F}_n \rightarrow TL_n^d$



# Theorem 1

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

## Theorem

$\Phi$  induces a well-defined algebra homomorphism

$$C_n^Q \rightarrow TL_{2n}^d$$

where  $Q = d^2$

# Theorem 1

## Proof

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

Suffices to show that the relations in  $C_n^Q$  hold in  $TL_{2n}^d$   
Namely, check the following

- H-I relation
- Tadpoles vanish
- Simple closed curves go to  $Q - 1 (= d^2 - 1)$

# Theorem 2

## Theorem

*Let  $G$  be a trivalent planar graph. Then*

$$Q^{-1}\chi_{\hat{G}}(Q) = \Phi(G)$$

*where  $Q = d^2$ .*

*It follows that the following diagram commutes*

$$\begin{array}{ccc} C_n^Q & \xrightarrow{\Phi} & TL_{2n}^d \\ \downarrow tr_\chi & & \downarrow tr_d \\ \mathbb{C} & \xrightarrow{=} & \mathbb{C} \end{array}$$

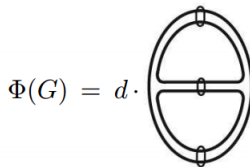
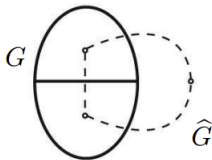


# Theorem 2

## Example

### Example

$$Q^{-1}\chi_Q(\hat{G}) = (Q-1)(Q-2) = d^4 - 3d^2 + 2 = \Phi(G).$$



# Theorem 2

## Example

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

$$\begin{aligned}
 d \left( \text{Diagram 1} \right) &= \left( \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \right) \\
 &+ \frac{1}{d} \left( \text{Diagram 5} + \text{Diagram 6} + \text{Diagram 7} \right) - \frac{1}{d^2} \left( \text{Diagram 8} \right)
 \end{aligned}$$

The diagrams are oriented graphs with 4 vertices arranged in a square. Solid lines represent edges, and dashed lines represent edges. The diagrams are as follows:

- Diagram 1:** A cycle of length 4 with a dashed edge between the top and bottom vertices.
- Diagram 2:** A cycle of length 4 with a dashed edge between the left and right vertices.
- Diagram 3:** A cycle of length 4 with a dashed edge between the top and bottom vertices.
- Diagram 4:** A cycle of length 4 with a dashed edge between the left and right vertices.
- Diagram 5:** A cycle of length 4 with a dashed edge between the top and bottom vertices.
- Diagram 6:** A cycle of length 4 with a dashed edge between the left and right vertices.
- Diagram 7:** A cycle of length 4 with a dashed edge between the top and bottom vertices.
- Diagram 8:** A cycle of length 4 with a dashed edge between the left and right vertices.

# Theorem 2

## Proof

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

First assume  $G$  connected, and recall the state sum formula of the chromatic polynomial, here applied to the dual graph  $\hat{G}$

$$\chi_{\hat{G}}(\mathcal{Q}) = \sum_{s \subseteq E(\hat{G})} (-1)^{|s|} \mathcal{Q}^{k(s)}$$

Now, since  $\Phi(G)$  maps the edges of  $G$  to a binary sum, the total number of summands of  $\Phi(G)$  is  $2^{|E(G)|}$ .

This power of two is precisely the number of terms in the state sum formula above, and hints toward a potential correspondence.

# Theorem 2

## Proof

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

Indeed!  $\Phi(G)$  can be parameterized by the subsets of the edge set of  $\hat{G}$ . (i.e. the same indexing of chromatic state-sum)

### State sum of $\Phi$ (v1)

$$\Phi(G) = d^{V(G)/2} \sum_{s \subseteq E(\hat{G})} (-1)^{|s|} \frac{1}{d^{|s|}} d^{k(s)+n(s)}$$

$k(s)$  is the number of connected components.

$n(s)$  is “the rank of the first homology of  $\hat{G}_s$ ”

The number of edges remaining after the removal of a spanning tree. Also, called “circuit rank”

# Quick aside

## Euler characteristic

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

The **Euler characteristic** is a topological invariant.

In particular, for planar connected graphs it is defined to be  $V - E + F$  and always equals 2.

# Theorem 2

## Proof

Since we restricted ourselves to trivalent graphs,  $\hat{G}$  is a triangulation.

$$\begin{aligned}\implies 2V(\hat{G}) &= F(\hat{G}) + 4, \text{ derived from Euler characteristic} \\ &= V(G) + 4, \text{ by construction of } \hat{G}\end{aligned}$$

$$\implies V(G)/2 = V(\hat{G}) - 2$$

### State sum of $\Phi$ (v2)

$$\Phi(G) = \sum_{s \subseteq E(\hat{G})} (-1)^{|s|} d^{V(\hat{G})-2+k(s)+n(s)-|s|}$$

# Theorem 2

## Proof

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

**Claim:**  $V(\hat{G}) = k(s) - n(s) + |s|$

**Proof:** True for  $s = 0$ , and the addition of one edge either decrements  $k(s)$  or increments  $n(s)$ .

State sum of  $\Phi$  (v3)

$$\Phi(G) = \sum_{s \subseteq E(\hat{G})} (-1)^{|s|} d^{2k(s)-2}$$

# Theorem 2

## Proof

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

Substituting  $Q$  for  $d^2$ , we get

$$\begin{aligned}\Phi(G) &= Q^{-1} \sum_{s \subseteq E(\hat{G})} (-1)^{|s|} Q^{k(s)} \\ &= Q^{-1} \chi_{\hat{G}}(Q)\end{aligned}$$

This concludes the proof for connected graphs.

The proof for unconnected graphs is not difficult.

If  $G = G_1 \sqcup G_2$  then

$\phi(G) = \phi(G_1)\phi(G_2)$  and

$\chi_{\hat{G}}(Q) = Q^{-1} \chi_{\hat{G}_1}(Q) \chi_{\hat{G}_2}(Q)$



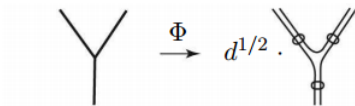


# Knotted Graph Invariant

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook

Using the  $\Phi$  constructed above,  
we can now map knotted graphs to the Jones polynomial.



$$d = -A^2 - A^{-2}$$

This turns out to be an invariant of knotted graphs!  
So, with  $\Phi$ , we can extend the application of the Jones  
polynomial beyond knots.

# For Further Reading I

Chromatic  
and  
Temperley-  
Lieb  
algebras

Lawrence  
Hook



P. Fendley, V. Krushkal.

Tutte Chromatic Identities from the Temperley-Lieb Algebra

19 July 2008



L. Kauffman, S. Lins

Temperley-Lieb Recoupling Theory and Invariants of 3-Manifolds

1994