# An Algebraic Integration

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## Introduction

Let $\mathbf{G}$ be a reductive group over $\mathbb{Z}$. For any field $F$ we can consider the group $\mathbf{G}(F)$ of $F$-points of $\mathbf{G}$. If $F$ is a finite or a local field, then there exists a rich theory of representation $\rho : \mathbf{G}(F) \to \mathrm{Aut}(V_\rho)$ of the group $\mathbf{G}(F)$. If one can choose a realization of $V_\rho$ as the space of functions on the set of $F$-points $\mathbf{X}_\rho(F)$ of a $\mathbb{Z}$-variety $\mathbf{X}_\rho$, then the operators $\rho(g)$ are given by kernels $K_\rho(g; x', x'')$, $g \in \mathbf{G}(F)$, $x', x'' \in \mathbf{X}_\rho(F)$. Often one can find *algebraic* constructions for the kernels $K_\rho(g; x', x'')$ over $\mathbb{Z}$. The goal of this paper is to introduce a notion of *algebro-geometric numbers*, *functions* and *operators*, and to give some examples which are useful for the theory of representations.

We start with the case of *algebro-geometric numbers.* Our *algebro-geometric numbers* are functions on the set $\mathcal{E}$ of finite and local fields. But first we consider such functions on the subset $\mathcal{E}_F \subset \mathcal{E}$ of finite extensions of a fixed field $F$.

Let $\mathbf{X}$ be an algebraic variety over a finite field $F$. One is often interested in the function $a_{\mathbf{X}}$ on $\mathcal{E}_F$ given by $a_{\mathbf{X}}(E) := \#\mathbf{X}(E)$. As is well known, the function $a_{\mathbf{X}}$ has the form $a_{\mathbf{X}}(E) = \mathrm{Trace}\,\mathrm{Fr}_{\mathbf{X}}^{\deg E/F}$, where $\mathrm{Fr}_{\mathbf{X}}$ is an endomorphism of a certain finite complex $\mathcal{C}^*_{\mathbf{X}}$ of finite-dimensional vector spaces over a field of characteristic zero. Moreover, the complex $\mathcal{C}^*_{\mathbf{X}}$ has a purely geometric origin. In other words, this complex [or more precisely its image in the derived category] depends only on the isomorphism class of $\mathbf{X} \otimes_F \bar{F}$, where $\bar{F}$ is an algebraic closure of $F$. Therefore we can often use results about the structure of $\mathbf{X} \otimes_F \bar{F}$ to obtain information about $a_{\mathbf{X}}$.

EXAMPLE 0.1. Let $\mathbf{X}$ be a variety such that $\mathbf{X} \otimes \bar{F}$ admits a fibration by affine lines. Then for any $E \in \mathcal{E}_F$, the number $a_{\mathbf{X}}(E)$ is divisible by $\#E$.

REMARK 0.1. Of course the claim of Example 0.1 is obvious in the case when $\mathbf{X}$ admits a fibration by affine lines. But the geometric interpretation of the function $a_{\mathbf{X}}(E)$ allows an extension of the "naive" arguments to the "nonsplit" case, where the fibration is not defined over $E$.

Assume now that $\mathbf{X}$ is a complete smooth algebraic variety over a local field $F$ and $\boldsymbol{\omega}$ is a *volume* form on $\mathbf{X}$. That is, $\boldsymbol{\omega} \neq 0$ is a regular section of the canonical bundle $\boldsymbol{\Omega}_{\mathbf{X}}$. As is well known, the form $\boldsymbol{\omega}$ defines a measure $|\boldsymbol{\omega}|_E$ on $\mathbf{X}(E)$ for any $E \in \mathcal{E}_F$, and we can define a function $a_{\mathbf{X}}$ on $\mathcal{E}_F$ by

$$a_{\mathbf{X}}(E) := \int_{\mathbf{X}(E)} |\boldsymbol{\omega}|_E.$$

QUESTION 0.1. Is it possible to find a geometric interpretation for the function $a_{\mathbf{X}}$?

Of course, in the case when $\mathbf{X}$ and $\boldsymbol{\omega}$ have good reduction mod $\mathfrak{P}$, where $\mathfrak{P}$ is the maximal ideal in the ring of integers of $F$, it is easy to reduce the computation of the function $a_{\mathbf{X}}$ to the computation of $a_{\bar{\mathbf{X}}}$, where $\bar{\mathbf{X}}$ is the reduction of $\mathbf{X}$ mod $\mathfrak{P}$. On the other hand, the only case in which I know a cohomological interpretation for $a_{\mathbf{X}}$ in the case of an arbitrary bad reduction is when $\mathbf{X}$ is an abelian variety. The computations are due to Vadim Vologodsky and are presented in the Appendix.

The main body of the paper deals with "Fourier type" *algebro-geometric numbers* which generalize the functions of the type $a_{\mathbf{X}}$. To explain the problem we return to the case where $F$ is a finite field. Fix a nontrivial additive character $\psi$ on $F$. For any regular function $\mathbf{f}$ on $\mathbf{X}$ we can define a function $a_{\mathbf{X},\mathbf{f}}$ on $\mathcal{E}_F$ by

$$a_{\mathbf{X},\mathbf{f}}(E) := \sum_{m \in \mathbf{X}(E)} \psi(\mathrm{Tr}_{E/F}[\mathbf{f}(m)]).$$

In this case one can also define geometrically a complex $\mathcal{C}^*_{\mathbf{X},\mathbf{f}}$ and its endomorphism $\mathrm{Fr}_{\mathbf{X}}$ in such a way that $a_{\mathbf{X}}(E) = \mathrm{Trace}\,\mathrm{Fr}_{\mathbf{X}}^{\deg E/F}$. One can often apply this algebraic interpretation of $a_{\mathbf{X},\mathbf{f}}$ for computations of Fourier type *numbers* over finite fields. To illustrate the situation we bring one example which is crucial for the theory of representations of reductive groups over finite fields.

EXAMPLE 0.2. Let $\mathcal{G}$ be a Lie algebra of a semi-simple algebraic group over $F$, $d := \dim(\mathcal{G})$, $r := \mathrm{rank}(\mathcal{G})$, $\mathcal{G}^\vee(F)$ the dual space, and $O \subset \mathcal{G}$ a regular conjugacy class. For any regular element $x \in \mathcal{G}^\vee(F)$ we denote by $f_x$ the restriction of $x$ to $O$. Using the algebro-geometric interpretation of the function $a_{O,f_x}$ one can show that

$$a_{O,f_x}(E) = (\pm) \sum_{y \in (Z(x) \bigcap O)(E)} \psi(\mathrm{Tr}_{E/F}[< x, y >]) \cdot (\#E)^{(d-r)/2},$$

where $Z(x)$ is the centralizer of $x$ in $\mathcal{G}$.

REMARK 0.2. The "Fundamental Lemma" in the theory of automorphic forms is a conjecture which can be considered to be a local field analogue of Example 0.2.

As in the case of Example 0.1 it is easy to prove the claim of Example 0.2 directly in the split case, when the conjugacy class $\mathcal{O}$ consists of elements diagonalizable over $E$. But there is no elementary proof in the general case. On the other hand, any regular conjugacy class $\mathcal{O}$ splits over $\bar{F}$, and the algebro-geometric interpretation of the function $a_{\mathcal{O},f_x}$ allows one to extend the "naive" arguments to the general [=nonsplit] case.

Roughly speaking an *algebro-geometric number* $\boldsymbol{\alpha}$ is a triple $\boldsymbol{\alpha}=(\mathbf{Z},\boldsymbol{\omega},\mathbf{f})$, where $\mathbf{Z}$ is a variety over $\mathbb{Z}$, $\boldsymbol{\omega}$ is a volume form on $\mathbf{Z}$, and $\mathbf{f}$ is a rational function on $\mathbf{Z}$. For any local field $F$ we define the $((F,\psi)-$ or) $F$-realization $\boldsymbol{\alpha}(F)=\boldsymbol{\alpha}_\psi(F)$ of $\boldsymbol{\alpha}$ as the integral

$$\boldsymbol{\alpha}(F) := \int_{z\in\mathbf{Z}(F)} \psi(\mathbf{f}(z))|\boldsymbol{\omega}|.$$

If one knows how to define the concept of *algebro-geometric numbers*, then it is clear how to define the notions of *algebro-geometric functions* and *algebro-geometric operators*. These notions will be defined in more detail in the paper.

REMARK 0.3. In the case when $F=\mathbb{R}$ one can use the technique of $D$-modules to study the "algebro-geometric" behaviour of integrals.

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## 1. Algebro-geometric integral operators

In this section, $F$ is an arbitrary field. If $\mathbf{X}$ is a smooth $F$-variety, we denote by $T_{\mathbf{X}}$ the tangent bundle and by $T^*_{\mathbf{X}}$ the cotangent bundle of $\mathbf{X}$, and by $\boldsymbol{\Omega}_{\mathbf{X}}=\Lambda^d T^*_{\mathbf{X}}, d=dim\mathbf{X}$ the canonical bundle on $\mathbf{X}$.

DEFINITION 1.1. We call a volume form $\boldsymbol{\omega}$ on $\mathbf{X}$ finite if it extends to a regular differential form on some smooth compactification of $\mathbf{X}$.

REMARK 1.1. a) In the case when $\mathbf{X}$ is an affine space, it is clear that a volume form $\boldsymbol{\omega}$ on $\mathbf{X}$ is uniquely defined up to multiplication by a nonzero scalar $c\in F^\times$.

b) If $\boldsymbol{\omega}$ is a finite form on $\mathbf{X}$, then it extends to a regular form on any smooth compactification of $\mathbf{X}$.

Let $\mathbf{X},\mathbf{Y}$ be smooth $F$-varieties, $\mathbf{L}$ a smooth algebraic subvariety of $\mathbf{Y}$ and $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ a smooth morphism. We define $\hat{\mathbf{L}}:=\mathbf{f}^{-1}\mathbf{L}$. Let $\boldsymbol{\Omega}_{\mathbf{X}}|_{\hat{\mathbf{L}}}$ be the restriction of the canonical line bundle $\boldsymbol{\Omega}_{\mathbf{X}}$ to $\hat{\mathbf{L}}$.

LEMMA 1.1. *There exists a natural isomorphism* $\iota:\boldsymbol{\Omega}_{\mathbf{X}}|_{\hat{\mathbf{L}}}\to\boldsymbol{\Omega}_{\hat{\mathbf{L}}}\otimes \mathbf{f}^*(\boldsymbol{\Omega}_{\mathbf{Y}}\otimes\boldsymbol{\Omega}_{\mathbf{L}}^{-1})$.

PROOF. The exact sequence $0\to T_{\hat{\mathbf{L}}}\to T_{\mathbf{X}}|\hat{\mathbf{L}}\to\mathbf{f}^*(T_{\mathbf{Y}}/T_{\mathbf{L}})\to 0$ yields $\Lambda^\bullet(T^*_{\mathbf{X}}|\hat{\mathbf{L}})=\Lambda^\bullet T^*_{\hat{\mathbf{L}}}\otimes\Lambda^\bullet\mathbf{f}^*(T^*_{\mathbf{Y}}/T^*_{\mathbf{L}})$. □

In the case when $\mathbf{L} = \{\mathbf{y}\}$ is a point and $\boldsymbol{\omega}$ is a volume form on $\mathbf{X}$, we denote $\iota(\boldsymbol{\omega}|\hat{\mathbf{L}})$ by $\boldsymbol{\omega}_{\{\mathbf{y}\}}$. By definition $\boldsymbol{\omega}_{\{\mathbf{y}\}}$ is a volume form on $\mathbf{f}^{-1}(\mathbf{y})$ which is well defined up to multiplication by a (nonzero) scalar.

DEFINITION 1.2. Let $\mathbf{f} : \mathbf{X} \to \mathbf{Z}$ be a smooth morphism and $\boldsymbol{\omega}_{\mathbf{X}}$ a volume form on $\mathbf{X}$. We say that $\boldsymbol{\omega}_{\mathbf{X}}$ is $\mathbf{f}$-finite if there exists an embedding $\mathbf{X} \hookrightarrow \hat{\mathbf{X}}$, a volume form $\boldsymbol{\omega}_{\hat{\mathbf{X}}}$ on $\hat{\mathbf{X}}$ extending $\boldsymbol{\omega}_{\mathbf{X}}$, and a proper smooth extension $\hat{\mathbf{f}} : \hat{\mathbf{X}} \to \mathbf{Z}$ of $\mathbf{f}$ all of whose critical points are in $\mathbf{X}$.

It is clear that in such a case the volume forms $\boldsymbol{\omega}_{\{\mathbf{z}\}}$ are finite for all $\{\mathbf{z}\} \in \mathrm{Im}(\mathbf{f})$ outside a subvariety of positive codimension.

Let $\mathbf{X}$, $\mathbf{Y}$ be smooth varieties and $\boldsymbol{\Omega}_{\mathbf{X}}^{1/2}, \boldsymbol{\Omega}_{\mathbf{Y}}^{1/2}$ square roots of the canonical bundles on $\mathbf{X}$ and $\mathbf{Y}$. For a morphism $p : \mathbf{R} \to \mathbf{X} \times \mathbf{Y}$, we write $\mathbf{p}_{\mathbf{X}}$, $\mathbf{p}_{\mathbf{Y}}$ for the compositions of $\mathbf{p}$ with the natural projections from $\mathbf{X} \times \mathbf{Y}$ to $\mathbf{X}$ and $\mathbf{Y}$.

DEFINITION 1.3. We denote by $\tilde{S}^*(\mathbf{X}, \mathbf{Y}, \boldsymbol{\Omega}_{\mathbf{X}}^{1/2}, \boldsymbol{\Omega}_{\mathbf{Y}}^{1/2})$ the category such that it's objects are quadruples

$$\boldsymbol{\eta} = (\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X} \times \mathbf{Y}, \boldsymbol{\omega}, \mathbf{f}).$$

Here $\mathbf{R}$ is a smooth $F$-variety, $\boldsymbol{\omega}$ a section of the line bundle $\boldsymbol{\Omega}_{\mathbf{R}} \otimes p_{\mathbf{X}}^*(\boldsymbol{\Omega}_{\mathbf{X}}^{-1/2}) \otimes p_{\mathbf{Y}}^*(\boldsymbol{\Omega}_{\mathbf{Y}}^{-1/2})$, $\mathbf{f}$ a rational function on $\mathbf{R}$, and the morphism $\mathbf{p}$ is required to have the propery that the restrictions of $\mathbf{p}_{\mathbf{X}} : \mathbf{R} \to \mathbf{X}$ and $\mathbf{p}_{\mathbf{Y}} : \mathbf{R} \to \mathbf{Y}$ to any irreducible component of $\mathbf{R}$ are dominant.

Given two quadruples $\boldsymbol{\eta} = (\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X} \times \mathbf{Y}, \boldsymbol{\omega}, \mathbf{f}) \in \tilde{S}^*(\mathbf{X}, \mathbf{Y}, \boldsymbol{\Omega}_{\mathbf{X}}^{1/2}, \boldsymbol{\Omega}_{\mathbf{Y}}^{1/2})$ and $\boldsymbol{\eta}' = (\mathbf{R}', \mathbf{p}' : \mathbf{R}' \to \mathbf{X}' \times \mathbf{Y}', \boldsymbol{\omega}', \mathbf{f}') \in \tilde{S}^*(\mathbf{X}', \mathbf{Y}', \boldsymbol{\Omega}_{\mathbf{X}'}^{1/2}, \boldsymbol{\Omega}_{\mathbf{Y}'}^{1/2})$ we define morphisms from $\boldsymbol{\eta}$ to $\boldsymbol{\eta}'$ as birational isomorphisms $\mathbf{R} \to \mathbf{R}'$, $\mathbf{X} \to \mathbf{X}'$, and $\mathbf{Y} \to \mathbf{Y}'$, compatible with the rest of the data.

b) Denote by $S^*(\mathbf{X}, \mathbf{Y})$ the set of equivalent classes of elements in $\tilde{S}^*(\mathbf{X}, \mathbf{Y}, \boldsymbol{\Omega}_{\mathbf{X}}^{1/2}, \boldsymbol{\Omega}_{\mathbf{Y}}^{1/2})$.

REMARK 1.2. Since any line bundle is locally trivial in the Zariski topology, the set $S^*(\mathbf{X}, \mathbf{Y})$ does not depend on the choice of the square roots $\boldsymbol{\Omega}_{\mathbf{X}}^{1/2}, \boldsymbol{\Omega}_{\mathbf{Y}}^{1/2}$.

We think about the elements of the set $S^*(\mathbf{X}, \mathbf{Y})$ as kernels of operators from "$\Gamma(\mathbf{Y}, \boldsymbol{\Omega}^{1/2})$" to "$\Gamma(\mathbf{X}, \boldsymbol{\Omega}^{1/2})$" and call them *algebro-geometric integral operators*. Analogously by *algebro-geometric* $1/2$*-forms* on $\mathbf{X}$ we understand the elements of $S^*(\mathbf{X}, \star)$, and by *algebro-geometric numbers* we mean the elements of $S^*(\star, \star)$, where $\star$ indicates a one point space.

We denote by $S(\mathbf{X}, \mathbf{Y}) \subset S^*(\mathbf{X}, \mathbf{Y})$ the subset of quadruples $\boldsymbol{\eta} = (\mathbf{R}, \mathbf{p}, \boldsymbol{\omega}, \mathbf{f})$ such that $\boldsymbol{\omega}$ is $\mathbf{p} \times \mathbf{f}$-finite. The algebro-geometric integral operators in $S(\mathbf{X}, \mathbf{Y})$ will be called *finite*.

In the next section we show that in the case when $F$ is a local field, finite algebro-geometric integral operators have realizations over finite extensions $E$ of $F$, which are operators from $1/2$-forms on $\mathbf{Y}(E)$ to $1/2$-forms on $\mathbf{X}(E)$.

We will also consider a variant of the notion of algebro-geometric integral operators which we call *algebro-geometric measures*. Let $\mathbf{X}$ be a smooth variety. We denote by $M^*(\mathbf{X})$ the set of quadruples $\boldsymbol{\mu} = (\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X}, \boldsymbol{\omega}, \mathbf{f})$, where $\boldsymbol{\omega}$ is a section of the line bundle $\boldsymbol{\Omega}_{\mathbf{R}}$, and $\mathbf{f}$ a rational function on $\mathbf{R}$. We denote by $M(\mathbf{X}) \subset M^*(\mathbf{X})$ the set of the quadruples $\boldsymbol{\mu} = (\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X}, \boldsymbol{\omega}, \mathbf{f})$ where $\omega$ is $\mathbf{p} \times \mathbf{f}$-finite. The elements of $M^*(\mathbf{X})$ will be called *algebro-geometric measures*, those of $M(\mathbf{X})$ will be called *finite algebro-geometric measures*.

Next we define a composition law $S^*(\mathbf{X}, \mathbf{Y}) \times S^*(\mathbf{Y}, \mathbf{Z}) \to S^*(\mathbf{X}, \mathbf{Z})$.

For any smooth varieties $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ we denote by $\Delta_{\mathbf{Y}} : \mathbf{Y} \to \mathbf{Y} \times \mathbf{Y}$ the diagonal embedding. Define $\mathbf{L} \subset \mathbf{X} \times \mathbf{Y} \times \mathbf{Y} \times \mathbf{Z}$ to be the image of $\mathbf{X} \times \mathbf{Y} \times \mathbf{Z}$ under the embedding $\mathbf{id}_{\mathbf{X}} \times \Delta_{\mathbf{Y}} \times \mathbf{id}_{\mathbf{Z}}$. Let $\mathbf{p}'_{\mathbf{X}} : \mathbf{L} \to \mathbf{X}$, $\mathbf{p}'_{\mathbf{Y}} : \mathbf{L} \to \mathbf{Y}$, $\mathbf{p}'_{\mathbf{Z}} : \mathbf{L} \to \mathbf{Z}$ be the natural projections. It is clear that $\boldsymbol{\Omega}_{\mathbf{X}\times\mathbf{Y}\times\mathbf{Y}\times\mathbf{Z}} \times \boldsymbol{\Omega}_{\mathbf{L}}^{-1} = \mathbf{p}'^*_{\mathbf{Y}}(\boldsymbol{\Omega}_{\mathbf{Y}})$. Given $\boldsymbol{\eta}' = (\mathbf{R}', \mathbf{p}' : \mathbf{R}' \to \mathbf{X} \times \mathbf{Y}, \boldsymbol{\omega}', \mathbf{f}') \in S^*(\mathbf{X}, \mathbf{Y})$ and $\boldsymbol{\eta}'' = (\mathbf{R}'', \mathbf{p}'' : \mathbf{R}'' \to \mathbf{Y} \times \mathbf{Z}, \boldsymbol{\omega}'', \mathbf{f}'') \in S^*(\mathbf{Y}, \mathbf{Z})$, we define $\tilde{\mathbf{R}} := \mathbf{R}' \times \mathbf{R}''$, put $\mathbf{r} = (\mathbf{p}', \mathbf{p}'') : \tilde{\mathbf{R}} \to \mathbf{X} \times \mathbf{Y} \times \mathbf{Y} \times \mathbf{Z}$, denote by $\mathbf{q}'$, $\mathbf{q}''$ the natural projections $\mathbf{q}' : \tilde{\mathbf{R}} \to \mathbf{R}'$ and $\mathbf{q}'' : \tilde{\mathbf{R}} \to R''$, and put $\mathbf{R} := \mathbf{r}^{-1}(\mathbf{L})$. Let $\omega = \boldsymbol{\omega}_{\mathbf{R}}$ be the restriction of $\mathbf{q}'^{-1}(\omega') \cdot \mathbf{q}''^{-1}(\omega'')$ to $\mathbf{R}$. It is easy to deduce from Lemma 1.1 that $\boldsymbol{\omega}$ can be considered to be a section of $\boldsymbol{\Omega}_{\mathbf{R}} \otimes \mathbf{p}^*_{\mathbf{X}}(\boldsymbol{\Omega}_{\mathbf{X}}^{-1/2}) \otimes \mathbf{p}^*_{\mathbf{Z}}(\boldsymbol{\Omega}_{\mathbf{Z}}^{-1/2})$, where $\mathbf{p}_{\mathbf{X}} = \mathbf{p}'_{\mathbf{X}} \circ \mathbf{r}$ and $\mathbf{p}_{\mathbf{Z}} = \mathbf{p}'_{\mathbf{Z}} \circ \mathbf{r}$.

DEFINITION 1.4. Given $\boldsymbol{\eta}' = (\mathbf{R}', \mathbf{p}' : \mathbf{R}' \to \mathbf{X} \times \mathbf{Y}, \boldsymbol{\omega}', \mathbf{f}') \in S^*(\mathbf{X}, \mathbf{Y})$ and $\boldsymbol{\eta}'' = (\mathbf{R}'', \mathbf{p}'' : \mathbf{R}'' \to \mathbf{Y} \times \mathbf{Z}, \boldsymbol{\omega}'', \mathbf{f}'') \in S^*(\mathbf{Y}, \mathbf{Z})$, we define $\boldsymbol{\eta} = \boldsymbol{\eta}' \circ \boldsymbol{\eta}'' \in S^*(\mathbf{X}, \mathbf{Z})$ by $\boldsymbol{\eta} := (\mathbf{R}, \mathbf{p}, \boldsymbol{\omega}, \mathbf{f})$, where $\mathbf{p} : \mathbf{R} \to \mathbf{X} \times \mathbf{Z}$ is $\mathbf{p}_{\mathbf{X}} \times \mathbf{p}_{\mathbf{Z}} = (\mathbf{p}'_{\mathbf{X}} \times \mathbf{p}'_{\mathbf{Z}}) \circ \mathbf{r}$, and $\mathbf{f}$ is the restriction of $\mathbf{f}' + \mathbf{f}''$ to $\mathbf{R}$.

REMARK 1.3. Unfortunately, one cannot guarantee that the composition $\boldsymbol{\eta}' \circ \boldsymbol{\eta}''$ will be in $S(\mathbf{X}, \mathbf{Z})$ for $\boldsymbol{\eta}' \in S(\mathbf{X}, \mathbf{Y})$ and $\boldsymbol{\eta}'' \in S(\mathbf{Y}, \mathbf{Z})$.

## 2. Local realizations of algebro-geometric measures

Let now $F$ be a local field and $|| : F \to \mathbb{R}$ the norm map. For simplicity we assume that the local field $F$ is nonarchimedean. All the definitions below could be extended to the cases of $F = \mathbb{R}$ or $F = \mathbb{C}$.

For any smooth $F$-variety $\mathbf{X}$ and finite extension $E$ of $F$ we denote by $\mathcal{S}(\mathbf{X}(E))$ the space of smooth [= locally constant since $F$ is nonarchimedean] compactly supported functions on the set $\mathbf{X}(E)$.

Let $\boldsymbol{\mu} = (\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X}, \omega, \mathbf{f}) \in M(\mathbf{X})$ be a finite algebro-geometric measure. It is easy to see that for any compact subsets $K \subset E$ and $C \subset \mathbf{X}(E)$ we have that $\int_{(\mathbf{p}\times\mathbf{f})^{-1}(K\times C)} |\boldsymbol{\omega}|$ is finite. Therefore for any function $\phi \in \mathcal{S}(\mathbf{X}(E))$ we can define a measure $l_E(\boldsymbol{\mu}, \phi)$ on $E$ by

$$l_E(\boldsymbol{\mu}, \phi)(K) := \int_{r\in\mathbf{f}^{-1}(K)} \psi(\mathbf{f}(r))\phi(\mathbf{p}(r))|\boldsymbol{\omega}|$$

for any compact $K \subset E$. For any $r \in \mathbb{R}_{>0}$ we denote by $K_r$ the open compact subgroup $\{a \in E; |a| < r\}$ of $E$.

The following result is proven by V. Vologodsky (unpublished)

PROPOSITION 2.1. *For any $r \in \mathbb{R}_{>0}$ there exists an $R \in \mathbb{R}_{>0}$ such that the restriction of $l_E(\boldsymbol{\mu}, \phi)$ to $E - K_R$ is $K_r$-invariant.*

As follows from Proposition 2.1, the sequence

$$I_E^r(\boldsymbol{\mu}) := \int_{a \in K_r} \psi(\mathrm{Tr}_{E/F}(a)) l_E(\boldsymbol{\mu})$$

stabilizes for $r \gg 0$ and we can define $I_E(\boldsymbol{\mu}) := \lim_{r \to \infty} I_E^r(\boldsymbol{\mu})$. We say that $I_E(\boldsymbol{\mu})$ is the *realization* of $\boldsymbol{\mu}$ over $E$.

Even in the case where the algebraic measure $\boldsymbol{\mu}$ is not finite we can often define a realization of $\boldsymbol{\mu}$ if $\boldsymbol{\mu}$ belongs to a family of algebraic measures.

DEFINITION 2.1. Let $\mathbf{X}$, $\mathbf{R}$, $\mathbf{L}$ be $F$-varieties, $\boldsymbol{\omega}$ a volume form on $\mathbf{R}$, $\mathbf{p}$ a morphism from $\mathbf{R}$ to $\mathbf{X}$ and $\mathbf{f}$ a rational function on $\mathbf{R} \times \mathbf{L}$. We say that an $\mathbf{L}$-family $\boldsymbol{\eta}$ of quadruples $(\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X}, \boldsymbol{\omega}, \mathbf{f})$ is integrable if for any finite extension $E$ of $F$, for any locally constant measure $\nu$ on $\mathbf{L}(E)$ with compact support, and for any compact open set $K \subset \mathbf{X}(E)$, the restriction to $\mathbf{p}^{-1}(K)$ of the complex-valued measure

$$\boldsymbol{\omega}_{\psi,E,\mathbf{f}}(\nu) := |\boldsymbol{\omega}| \cdot \int_{\mathbf{L}(E)} \psi(\mathrm{Tr}_{E/F}\, \mathbf{f})\nu$$

on $\mathbf{R}(E)$ is absolutely integrable. Denote by $\mathbf{p}_*(\boldsymbol{\omega}_{\psi,E,\mathbf{f}}(\nu))$ the push forward under $\mathbf{p}$ to $\mathbf{X}(E)$ of this measure.

REMARK 2.1. Let $\mathbf{A} = (\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X}, \omega, \mathbf{f})$ and $\mathbf{A}' = (\mathbf{R}', \mathbf{p}' : \mathbf{R}' \to \mathbf{X}', \boldsymbol{\omega}', \mathbf{f}')$ be two integrable $\mathbf{L}$- and $\mathbf{L}'$-families of quadruples. Then the product

$$\mathbf{A} \circ \mathbf{A}' := (\mathbf{R} \times \mathbf{R}', \mathbf{p} \times \mathbf{p}' : \mathbf{R} \times \mathbf{R}' \to \mathbf{X} \times \mathbf{X}', \boldsymbol{\omega} \times \boldsymbol{\omega}', \mathbf{f} \times \mathbf{f}')$$

is an integrable $\mathbf{L} \times \mathbf{L}'$-family.

If a $\mathbf{L}$-family of quadruples $(\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X}, \boldsymbol{\omega}, \mathbf{f})$ is integrable we can interpret $\mathbf{p}_*(\boldsymbol{\omega}_{\psi,E,\mathbf{f}})$ as a family of generalized functions on $\mathbf{L}(E)$ with values in the space of distributions on $\mathbf{X}(E)$. Often there exists an open subspace $\mathbf{L}_0 \subset \mathbf{L}$ such that these distributions can be represented by families of functions $a_E(t) = a_{E,\psi}(t)$ on $\mathbf{L}_0(E)$.

DEFINITION 2.2. Let $\mathbf{L}_0 \subset \mathbf{L}$ be an open subset. We say that a family of locally constant functions $a_E(t)$ on $\mathbf{L}_0(E)$ with values in the space of measures on $\mathbf{X}(E)$ represents $\mathbf{p}_*(\boldsymbol{\omega}_{\psi,E,\mathbf{f}})$, if for any $h \in \mathcal{S}(\mathbf{X})$ the function $t \mapsto (a_E(t))(h)$ on $\mathbf{L}_0(E)$ belongs to $L^1(\mathbf{L}(E))$, and for any locally constant measure $\nu$ on $\mathbf{L}(E)$ with compact support we have $\int_{\mathbf{L}(E)} (a_E(t))(h)\nu = \mathbf{p}_*(\boldsymbol{\omega}_{\psi,E,\mathbf{f}}(\nu))(h)$.

EXAMPLE 2.1. Assume that $\mathbf{X}$ is a subvariety of an affine space $\mathbf{V}$, $\mathbf{R} = \mathbf{X}$, $\mathbf{L} := \mathbf{V}^\vee$ the dual space and $\mathbf{f}$ the restriction of the natural pairing $\mathbf{V} \times \mathbf{V}^\vee \to \mathbb{A}^1$ to $\mathbf{X} \times \mathbf{V}^\vee$. Define a distribution $\delta_\omega(E)$ on $\mathbf{V}(E)$ by $\delta_\omega(E)(g) := \int_{\mathbf{X}(E)} g|\boldsymbol{\omega}|$. Then the distribution $\mathbf{p}_*(\boldsymbol{\omega}_{\psi,E,\mathbf{f}})$ on $\mathbf{V}^\vee(E)$ is equal to the Fourier transform $\mathcal{F}(\delta_{\boldsymbol{\omega}}(E))$.

REMARK 2.2. It is not difficult to show that there exists a nonempty open subset $\mathbf{L}_0$ in $\mathbf{V}^\vee$ such that the restriction of $\mathcal{F}(\delta_\omega(E))$ to $\mathbf{L}_0(E)$ is given by a smooth measure $h(\boldsymbol{\omega})$. But it is an interesting question to find when the measure $h(\boldsymbol{\omega})$ is locally integrable on $\mathbf{V}^\vee(E)$, and, moreover, defines a distribution on $\mathbf{V}^\vee(E)$ equal to $\mathcal{F}(\delta_\omega(E))$.

EXAMPLE 2.2. Let $\mathbf{V}$ be an affine space, $\mathbf{V}^\vee$ the dual space, $\mathbf{X} = \mathbf{V} \times \mathbf{V}^\vee$, $\mathbf{p}_V, \mathbf{p}_{V^\vee}$ the natural projections from $\mathbf{X}$ to $\mathbf{V}$ and $\mathbf{V}^\vee$. Let $\boldsymbol{\omega}$ be the preimage by $\mathbf{p}_V$ of a nonzero invariant form $\varpi \in \Lambda^d(\mathbf{V})$, $d = \dim \mathbf{V}$, and $\mathbf{L}$ the space of quadratic forms $\mathbf{Q}_t$ ($t \in \mathbf{L}$) on $\mathbf{V}$. Define $\mathbf{f}_t : \mathbf{X} \to \mathbb{A}^1$ by $\mathbf{f}_t(v, v^\vee) = \mathbf{Q}_t(v) + < v, v^\vee >$. It is easy to check that this family is weakly integrable.

The associated family $\mathbf{p}_*(\boldsymbol{\omega}_{\psi,E,\mathbf{f}})$ of generalized functions is represented by a family of locally $L^1$-functions

$$a_E(\mathbf{Q}) = \epsilon_\psi(\mathbf{Q}, E)|\operatorname{disc}(\mathbf{Q})|^{-1/2},$$

which is defined on the subset of nondegenerate quadratic forms $\mathbf{Q}$. Both the discriminant $\operatorname{disc}(\mathrm{Q})$ ($\in E^\times$) of the quadratic form $\mathbf{Q}$, and $\epsilon_\psi(\mathbf{Q}, E)$, which is a certain root of unity, are independent of the choice of a form $\varpi$.

Example 2.2 is really a special case of the following construction. Let $\mathbf{G}$ be a semi-simple group, $\boldsymbol{\rho} : \mathbf{G} \to \mathbf{V}$ an irreducible representation, $\mathbf{X}$ the orbit of a highest weight vector in $\mathbf{V}$, and $\boldsymbol{\omega}$ a $\mathbf{G}$-invariant volume form on $\mathbf{X}$.

CONJECTURE 2.1. *When $\mathbf{X} \neq \mathbf{V}$, the distribution $\mathcal{F}(\delta_\omega(E))$ is given by a locally $L^1$-function.*

Example 2.2 corresponds to the case when $\mathbf{V} = \operatorname{Sym}^2 \mathbf{V}_1$ and $\mathbf{X} \subset \mathbf{V}$ is the image of $\mathbf{V}_1$ under the map $v \mapsto v \otimes v$.

## 3. Realizations of algebro-geometric integral operators

As before, let $E$ be a local field. For any finite extension $E$ of $F$ the norm map $|| : E^\times \to \mathbb{R}_{>0}$ associates to the canonical line bundle $\boldsymbol{\Omega}_\mathbf{X}$ a principal $\mathbb{R}_{>0}$-bundle $|\boldsymbol{\Omega}_\mathbf{X}|$ over the set $\mathbf{X}(E)$. Since the group $\mathbb{R}_{>0}$ is uniquely divisible, we can define also a principal $\mathbb{R}_{>0}$-bundle $|\boldsymbol{\Omega}_\mathbf{X}|^{1/2}$. The group $\mathbb{R}_{>0}$ acts naturally on $\mathbb{R}$ and we denote the associated $\mathbb{R}$-bundle also by $|\boldsymbol{\Omega}_\mathbf{X}|^{1/2}$

We denote by $\mathcal{S}^{1/2}(\mathbf{X}(E))$ the space of smooth [= locally constant if $E$ is nonarchimedean] complex-valued sections $\phi$ of $|\boldsymbol{\Omega}_X|^{1/2}$ with compact support. Let $(,)_\mathbf{X}$ be the natural Hermitian form on $\mathcal{S}^{1/2}(\mathbf{X}(E))$:

$$(\phi_1, \phi_2)_\mathbf{X} := \int_{\mathbf{X}(E)} \phi_1 \bar{\phi_2}.$$

We denote by $L^2(\mathbf{X}(E))$ the completion of $\mathcal{S}^{1/2}(\mathbf{X}(E))$ with respect to $(,)_\mathbf{X}$, and by $\mathcal{D}^{1/2}(\mathbf{X}(E))$ the space of continuous linear functionals on $\mathcal{S}^{1/2}(\mathbf{X}(E))$. The form $(,)_\mathbf{X}$ defines an embedding from $\mathcal{S}^{1/2}(\mathbf{X}(E))$ into $\mathcal{D}^{1/2}(\mathbf{X}(E))$.

For any $\boldsymbol{\eta} \in S(\mathbf{X}, \mathbf{Y})$ and for any finite extension $E$ of $F$, we define a bilinear form $B_{\boldsymbol{\eta}}(E)$ on $\mathcal{S}^{1/2}(\mathbf{X}(E)) \times \mathcal{S}^{1/2}(\mathbf{Y}(E))$ with values in the space of complex-valued measures on $E$ by $B_{\boldsymbol{\eta}}(E)(\phi_{\mathbf{X}}, \phi_{\mathbf{Y}}) = \int_{\mathbf{R}(E)} (\phi_{\mathbf{X}} \times \phi_{\mathbf{Y}}) \circ \mathbf{p} \cdot \psi(\mathbf{f})|\boldsymbol{\omega}|$. The following generalizes Proposition 2.1.

CONJECTURE 3.1. *For any $\boldsymbol{\eta} \in S(\mathbf{X}, \mathbf{Y})$, any finite extension $E$ of $F$, any $\phi_{\mathbf{X}} \in \mathcal{S}^{1/2}(\mathbf{X}(E))$, $\phi_{\mathbf{Y}} \in \mathcal{S}^{1/2}(\mathbf{Y}(E))$, and any $r \in \mathbb{R}_{>0}$, there exists $R \in \mathbb{R}_{>0}$ such that the restriction of $B_{\boldsymbol{\eta}}(E)(\phi_{\mathbf{X}}, \phi_{\mathbf{Y}})$ to $E - K_R$ is $K_r$ invariant.*

We assume from now on that Conjecture 3.1 is true. In this case we can associate to any $\boldsymbol{\eta} \in S(\mathbf{X}, \mathbf{Y})$ a realization $A^{\psi}_{\boldsymbol{\eta}}(E)$ which is a continuous operator from $\mathcal{S}^{1/2}(\mathbf{Y}(E))$ to $\mathcal{D}^{1/2}(\mathbf{X}(E))$.

## 4. Algebro-geometric unitary operators

Let $F$ be a global field. For any smooth $F$-varieties $\mathbf{X}, \mathbf{Y}$ we denote by $S_{\mathrm{uni}}(\mathbf{X}, \mathbf{Y}) \subset S(\mathbf{X}, \mathbf{Y})$ the set of all $\boldsymbol{\eta}$ such that for any local field $E$ containing $F$ and for any nontrivial character $\psi$ of $E$ we have $A^{\psi}_{\boldsymbol{\eta}}(E)(\mathcal{S}^{1/2}(\mathbf{Y}(E))) \subset L^2(\mathbf{X}(E))$, and, moreover, there exists a real number $c_{\psi} > 0$ such that the operator $c_{\psi} A^{\psi}_{\boldsymbol{\eta}}(E)$ is unitary. Elements of $S_{\mathrm{uni}}(\mathbf{X}, \mathbf{Y}) \subset S(\mathbf{X}, \mathbf{Y})$ will be called *unitary algebro-geometric operators.* Elements of the set $S_{\mathrm{uni}}(\star, \star)$ will be called *unitary algebro-geometric numbers*, where $\star$ is the one point variety.

We will now give examples of unitary algebro-geometric operators over an arbitrary global field $F$. In some cases the proof of unitarity is simple and local. But in a number of cases one has to use global arguments to prove the unitarity.

EXAMPLE 4.1. Let $\mathbf{X} = \mathbf{Y}$ be a point. Let $\mathbf{R}$ be an affine space, $\boldsymbol{\omega}$ a volume form on $\mathbf{R}$, and $\mathbf{Q}$ a nondegenerate quadratic form on $\mathbf{R}$. Then $\epsilon_{\mathbf{Q}} := (\mathbf{R}, \boldsymbol{\omega}, \mathbf{Q})$, where $\mathbf{p} : \mathbf{R} \to \mathbf{X} \times \mathbf{Y}$ is omitted to simplify the notations, is a unitary algebro-geometric number.

It will be very interesting to find other examples of unitary algebro-geometric numbers.

EXAMPLE 4.2. Let $\mathbf{X}, \mathbf{Y}$ be smooth varieties over $F$, $\boldsymbol{\phi} : \mathbf{X} \to \mathbf{Y}$ a birational isomorphism, and $\mathbf{f}$ a rational function on $\mathbf{X}$. Let $\mathbf{X}_0 \subset \mathbf{X}$ be a dense open set such that $\boldsymbol{\phi}$ and $\mathbf{f}$ are well defined on $\mathbf{X}_0$, $\boldsymbol{\phi}$ defines an isomorphism from $\mathbf{X}_0$ to $\mathbf{Y}_0 := \boldsymbol{\phi}(\mathbf{X}_0)$, and the restriction of $\boldsymbol{\Omega}_{\mathbf{X}}$ to $\mathbf{X}_0$ is trivial. Denote by $\mathbf{R}_{\phi} \subset \mathbf{X}_0 \times \mathbf{Y}_0$ the graph of the restriction of $\boldsymbol{\phi}$ to $\mathbf{X}_0$, by $\mathbf{p} : \mathbf{R}_{\phi} \to \mathbf{X}_0 \times \mathbf{Y}_0$ the natural embedding, and by $\boldsymbol{\omega}$ the canonical section of $\boldsymbol{\Omega}_{\mathbf{R}_{\phi}} \otimes \mathbf{p}^*_{\mathbf{X}}(\boldsymbol{\Omega}_{\mathbf{X}}^{-1/2}) \otimes \mathbf{p}^*_{\mathbf{Y}}(\boldsymbol{\Omega}_{\mathbf{Y}}^{-1/2})$ (which is isomorphic to the structure sheaf of $\mathbf{R}_{\phi}$). Then $\boldsymbol{\eta}_{\phi,\mathbf{f}} := (\mathbf{R}_{\phi}, \mathrm{p}, \boldsymbol{\omega}, \mathbf{f} \circ \mathbf{p}_{\mathbf{X}}) \in S^*(\mathbf{X}, \mathbf{Y})$ does not depend on the choice of $\mathbf{X}_0$, and we have $\boldsymbol{\eta}_{\phi,\mathbf{f}} \in S_{\mathrm{uni}}(\mathbf{X}, \mathbf{Y})$.

If $\mathbf{X} = \mathbf{Y}$, $\mathbf{f} = 0$ and $\boldsymbol{\phi} = \mathbf{id}_{\mathbf{X}}$, then $\boldsymbol{\eta}_{\mathrm{id}_{\mathbf{X}}} \circ \boldsymbol{\eta} = \boldsymbol{\eta} = \boldsymbol{\eta} \circ \boldsymbol{\eta}_{\mathrm{id}_{\mathbf{X}}}$ for all $\boldsymbol{\eta} \in S_{\mathrm{uni}}(\mathbf{X}, \mathbf{X})$. We denote $\boldsymbol{\eta}_{\mathrm{id}_{\mathbf{X}}}$ by $\mathbf{id}_{\mathbf{X}}$.

EXAMPLE 4.3. Let $\mathbf{X} = \mathbf{V}$ be an affine space, $\mathbf{Y} = \mathbf{V}^\vee$ the dual affine space, $\mathbf{R} := \mathbf{V} \times \mathbf{V}^\vee$, $\boldsymbol{\omega}$ the canonical volume form on $\mathbf{R}$, and $\mathbf{f} : \mathbf{R} \to \mathbb{A}^1$ the canonical pairing. Then $\boldsymbol{\eta}_{\mathbf{V}} := (\mathbf{R}, \mathbf{id}, \boldsymbol{\omega}, \mathbf{f})$ belongs to $S_{\mathrm{uni}}(\mathbf{V}, \mathbf{V}^\vee)$, and, moreover, the corresponding operator $A^{\psi}_{\boldsymbol{\eta}_{\mathbf{V}}}$ coincides with the Fourier transform.

Let $\Gamma$ be a finite group.

DEFINITION 4.1. An algebro-geometric representation of $\Gamma$ on $\mathbf{X}$ is a map $\gamma \mapsto \boldsymbol{\eta}(\gamma)$ from $\Gamma$ into $S_{\mathrm{uni}}(\mathbf{X}, \mathbf{X})$, such that for any $\gamma_1, \gamma_2 \in \Gamma$ we have $A^{\psi}_{\boldsymbol{\eta}}(\gamma_1)(E) A^{\psi}_{\boldsymbol{\eta}}(\gamma_2)(E) = A^{\psi}_{\boldsymbol{\eta}}(\gamma_1\gamma_2)(E)$ for all local fields $E$ containing $F$.

EXAMPLE 4.4. Let $\boldsymbol{\eta}_{\mathbf{V}}$ be as in Example 4.3 and $\mathbf{B}$ a symplectic form on $\mathbf{V}$. Then $\mathbf{B}$ defines an isomorphism between $\mathbf{V}$ and $\mathbf{V}^\vee$. Define $\boldsymbol{\eta}_B$ to be the composition of $\boldsymbol{\eta}_V$ and the isomorphism $\hat{\mathbf{B}} : \mathbf{V}^\vee \to \mathbf{V}$ induced by $\mathbf{B}$. It is an element of $S_{\mathrm{uni}}(\mathbf{V}, \mathbf{V})$, and the map $0 \mapsto \mathbf{id}_{\mathbf{V}}$, $1 \mapsto \boldsymbol{\eta}_{\mathbf{B}}$ defines an algebraic representation of the group $\Gamma = \mathbb{Z}/2\mathbb{Z}$ on $\mathbf{V}$.

Let $\mathbf{G}$ be a split semi-simple group over $F$, $\mathcal{G}$ the Lie algebra of $\mathbf{G}$, $\mathbf{B} = \mathbf{TU} \subset \mathbf{G}$ a Borel subgroup, where $\mathbf{U} \subset \mathbf{B}$ is the unipotent radical of $\mathbf{B}$ and $\mathbf{T}$ a Cartan subgroup. Let $\Delta$ be the set of simple roots of $\mathbf{G}, \mathbf{B}$, and $W$ the Weyl group. We fix a splitting of $\mathbf{G}$. That is for any $i \in \Delta$ we fix a nonzero element $x_i \in \mathcal{G}$. Put $\mathbf{X} := \mathbf{U}\backslash\mathbf{G}$. As in [5], we can associate with any $w \in W$ and a shortest decomposition $w = s_1 \dots s_l$, $l = l(w)$, of $w$, an $F$-variety $\mathbf{R}(s_1, \dots, s_l)$, a morphism $\mathbf{p}_w : \mathbf{R}(s_1, \dots, s_l) \to \mathbf{X} \times \mathbf{X}$, and a function $\mathbf{f}_w$ on $\mathbf{R}(s_1, \dots, s_l)$. Since the splitting of $\mathbf{G}$ defines canonically $\mathbf{G}$-invariant volume forms on $\mathbf{X}$ and $\mathbf{R}(s_1, \dots, s_l)$, we can define

$$\boldsymbol{\eta}_{s_1,\dots,s_l} = (\mathbf{R}(s_1, \dots, s_l), \mathbf{p}_{(s_1,\dots,s_l)}, \boldsymbol{\omega}_{(s_1,\dots,s_l)}, \mathbf{f}_{(s_1,\dots,s_l)}) \in S(\mathbf{X}, \mathbf{X}).$$

It follows from Proposition 2.2.10 of [5] that the element $\boldsymbol{\eta}_{s_1,\dots,s_l}$ does not depend on the choice of a decomposition $w = s_1 \dots s_l$. We denote it by $\boldsymbol{\eta}_w$.

REMARK 4.1. Actually one has to be more careful in the definition of $\boldsymbol{\eta}_w$. Let $\mathcal{C}(w)$ be the groupoid whose objects are shortest decompositions of $w$ into a product of simple reflections and morphisms are compositions of elementary morphisms where elementary morphisms come from the braid relations in the Weyl group $W$. Then one can construct canonically a functor from the category $\mathcal{C}(w)$ into $S(\mathbf{X}, \mathbf{X})$.

EXAMPLE 4.5. a) For any $w \in W$ we have $\boldsymbol{\eta}_w \in S_{\mathrm{uni}}(\mathbf{X}, \mathbf{X})$.

b) If $w_1, w_2 \in W$ are such that $l(w_1 w_2) = l(w_1) + l(w_2)$ then $\boldsymbol{\eta}_w = \boldsymbol{\eta}_{w_1} \circ \boldsymbol{\eta}_{w_2}$.

c) The map $w \mapsto \boldsymbol{\eta}_w$ defines an algebro-geometric representation of the group $W$ on $\mathbf{X}$.

Let $\mathbf{G}$ be an algebraic group and $\mathbf{X}$ a smooth algebraic variety over $F$.

DEFINITION 4.2. A strongly algebro-geometric representation of $\mathbf{G}$ on $1/2$-*forms* on $\mathbf{X}$ is a quadruple

$$\boldsymbol{\eta} = (\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{G} \times \mathbf{X} \times \mathbf{X}, \boldsymbol{\omega}, \mathbf{f}).$$

Here $\mathbf{R}$ is a smooth equidimensional $F$-variety, $\boldsymbol{\omega}$ a section of the line bundle $\boldsymbol{\Omega}_{\mathbf{R}} \otimes \mathbf{p}_{\mathbf{G}}^*(\boldsymbol{\Omega}_{\mathbf{G}}^{-1}) \otimes \mathbf{p}_{\mathbf{X}\times\mathbf{X}}^*(\boldsymbol{\Omega}_{\mathbf{X}\times\mathbf{X}}^{-1/2})$, and $\mathbf{f}$ is a rational function on $\mathbf{R}$ which satisfies: for any $g \in \mathbf{G}(F)$ the restriction $\boldsymbol{\eta}(g) := \boldsymbol{\eta}|\mathbf{p}^{-1}(g\times\mathbf{X}\times\mathbf{X})$ belongs to $S_{\mathrm{uni}}(\mathbf{X},\mathbf{X})$, and for any local field $E$ containing $F$ the map $g \mapsto A^{\psi}_{\boldsymbol{\eta}(g)}(E)$ defines a representation $\boldsymbol{\rho}(E)$ of the group $\mathbf{G}(E)$ on $L^2(\mathbf{X}(E))$.

EXAMPLE 4.6. Let $\mathbf{G}$ be a unimodular algebraic group, $\mathbf{U} \subset \mathbf{G}$ a unipotent subgroup and $\mathbf{X} := \mathbf{U}\backslash\mathbf{G}$. For any $g \in \mathbf{G}(F)$ we denote by $\phi(g) : \mathbf{X} \to \mathbf{X}$ the right shift by $g$. We choose a rational section $\mathbf{s} : \mathbf{X} \to \mathbf{G}$ and define a rational map $\mathbf{r} : \mathbf{X} \times \mathbf{G} \to \mathbf{U}$ by $\mathbf{r}(x,g) = \mathbf{s}(x)g\mathbf{s}^{-1}(xg)$. Let $\boldsymbol{\psi} : \mathbf{U} \to \mathbb{A}^1$ be a group homomorphism. We define $\boldsymbol{\rho}_{\mathbf{s},\boldsymbol{\psi}}(g) := \boldsymbol{\eta}_{\phi(g),\mathbf{f}_{\mathbf{s}}}$ where $\mathbf{f}_{\mathbf{s}} := \boldsymbol{\psi} \circ \mathbf{r}$. Then $\boldsymbol{\rho}_{\mathbf{s},\boldsymbol{\psi}}$ is a strong algebro-geometric representation of $\mathbf{G}$, and the equivalence class of $\boldsymbol{\rho}_{\mathbf{s},\boldsymbol{\psi}}$ does not depend on a choice of a section $\mathbf{s}$. We denote this strong algebro-geometric representation of $\mathbf{G}$ on 1/2-forms on $\mathbf{X}$ by $\mathrm{ind}_{\mathbf{U}}^{\mathbf{G}}(\boldsymbol{\psi})$.

REMARK 4.2. For any smooth variety $\mathbf{Y}$, the sets $S_{\mathrm{uni}}(\mathrm{ind}_{\mathbf{U}}^{\mathbf{G}}(\boldsymbol{\psi}), \mathbf{Y})$ and $S_{\mathrm{uni}}(\mathbf{Y}, \mathrm{ind}_{\mathbf{U}}^{\mathbf{G}}(\boldsymbol{\psi}))$ have been defined in [**3**].

Often it is convenient to define a slightly weaker notion, of an *algebro-geometric representation* of a group $\mathbf{G}$. Let $\mathbf{Q}, \mathbf{H}$ be subgroups of $\mathbf{G}$. For any $n \geq 0$ we denote by $\mathbf{p}_n : \mathbf{Q}^n \times \mathbf{H}^n \to \mathbf{G}$ the map given by $\mathbf{p}_n(q_1,\dots,q_n;h_1,\dots,h_n) := q_1h_1q_2h_2\dots q_nh_n$. We say that $\mathbf{Q},\mathbf{H}$ are *rational generators* of $\mathbf{G}$ if for some $n \geq 0$ the map $\mathbf{p}_n$ admits a rational section.

DEFINITION 4.3. An algebro-geometric representation of a group $\mathbf{G}$ is a pair of strong algebro-geometric representation $\boldsymbol{\rho}^{\mathbf{Q}}, \boldsymbol{\rho}^{\mathbf{H}}$ on 1/2-forms on $\mathbf{Y}$ of rational generators $\mathbf{Q}$, $\mathbf{H} \subset \mathbf{G}$, such that for any local field $E$ containing $F$ the representations $\boldsymbol{\rho}^{\mathbf{Q}}(E), \boldsymbol{\rho}^{\mathbf{H}}(E)$ on $L^2(\mathbf{Y}(E))$ of the subgroups $\mathbf{Q}(E)$, $\mathbf{H}(E)$ of $\mathbf{G}(E)$ are the restriction of a representation $\boldsymbol{\rho}(E)$ of $\mathbf{G}(E)$.

Let $\mathbf{U} \subset \mathbf{GL}_n$ be the subgroup of upper-triangular matrices, $\boldsymbol{\psi} : \mathbf{U} \to \mathbb{A}^1$ a nondegenerate homomorphism, $\mathbf{T}$ the diagonal Cartan subgroup of $\mathbf{GL}_n$, $\sigma$ an outer automorphism of $\mathbf{GL}_n$ preserving $\mathbf{T}$, $\mathbf{U}$ and $\boldsymbol{\psi}$. Then $\sigma^2 = 1$ and we denote by $\mathbf{G}_n$ the semi-direct product $\{e,\sigma\} \ltimes \mathbf{GL}_n$. Let $\mathbf{X} := \mathbf{U}\backslash\mathbf{GL}_n$. The automorphism $\sigma$ defines an automorphism of $\mathbf{X}$ which we denote by the same letter.

Let $\tilde{\mathbf{Q}} \subset \mathbf{GL}_n$ be the stabilizer of the line of $\mathbf{U}$-invariant vectors; thus this is the group of matrices in $\mathbf{GL}_n$ with zero entries in the first column, below the first entry. Let $\mathbf{Q} := \tilde{\mathbf{Q}} \times \mathbf{T}/\mathbb{G}_m$ where $\mathbb{G}_m$ is embedded diagonally in the product $\tilde{\mathbf{Q}} \times \mathbf{T}$. Of course we can consider $\mathbf{U}$ as a subgroup of $\mathbf{Q}$.

In [**3**] we constructed elements

$$\boldsymbol{\beta} = \boldsymbol{\beta}_n \in S_{\mathrm{uni}}(\mathbf{X}, \mathrm{ind}_{\mathbf{U}}^{\mathbf{Q}}(\boldsymbol{\psi})) \quad \text{and} \quad \boldsymbol{\alpha} = \boldsymbol{\alpha}_n \in S_{\mathrm{uni}}(\mathrm{ind}_{\mathbf{U}}^{\mathbf{Q}}(\boldsymbol{\psi}), \mathbf{X})$$

such that for any local field $E$ containing $F$ we have $A^{\psi}_{\boldsymbol{\alpha}}(E) \circ A^{\psi}_{\boldsymbol{\beta}}(E) = \mathbf{id}$ and $A^{\psi}_{\boldsymbol{\beta}}(E) \circ A^{\psi}_{\boldsymbol{\alpha}}(E) = \mathbf{id}$. Put $\boldsymbol{\eta}'_{\sigma} := \boldsymbol{\beta} \circ \boldsymbol{\eta}_{\sigma} \circ \boldsymbol{\alpha}$. If we choose a rational section of the projection $\mathbf{Q} \to \mathbf{Y} := \mathbf{U}\backslash\mathbf{Q}$ we can write $\boldsymbol{\eta}'_\sigma$ in the form $(\mathbf{R}, \mathbf{p}, \boldsymbol{\omega}, \mathbf{f}) \in S^*(\mathbf{Y}, \mathbf{Y})$. The justification of the following example is contained in [**3**].

EXAMPLE 4.7. a) $\boldsymbol{\eta}'_\sigma \in S_{\text{uni}}(\text{ind}_{\mathbf{U}}^{\mathbf{Q}}(\psi), \text{ind}_{\mathbf{U}}^{\mathbf{Q}}(\psi))$.

b) There exists an *algebro-geometric representation* $\boldsymbol{\rho}$ of the group $\mathbf{G}_n$ on 1/2-forms on $\mathbf{X}$ such that $\boldsymbol{\rho}(\sigma) = \boldsymbol{\eta}'_\sigma$ and the restriction of $\boldsymbol{\rho}$ to $\mathbf{Q}$ is equal to $\text{ind}_{\mathbf{U}}^{\mathbf{Q}}(\psi)$.

The Weyl group $S_n$ of $\mathbf{GL}_n$ acts naturally on the torus $\mathbf{T}$ and therefore on $\mathbf{Q}$ and $\mathbf{Y} = \mathbf{U}\backslash\mathbf{Q}$. It follows from [**6**] that for any local field $E$ containing $F$ the induced action of $S_n$ on 1/2-forms on $\mathbf{Y}(E)$ commutes with the representation $\boldsymbol{\rho}(E)$ of the group $\mathbf{G}(E)$.

PROPOSITION 4.1. *There exists a natural action of $S_n$ on $\mathbf{R}$ commuting with $\mathbf{p}$ and preserving $\boldsymbol{\omega}$ and $\mathbf{f}$, where as before $\boldsymbol{\eta}'_\sigma = (\mathbf{R}, \mathbf{p}, \boldsymbol{\omega}, \mathbf{f})$.*

REMARK 4.3. This claim is trivial for $n = 3$ (since in this case $\mathbf{R} = \mathbf{Y} \times \mathbf{Y}$) and it is proven in [**3**] for $n = 4$.

## 5. Forms

Let $\mathbf{X}$ be an algebraic $F$-variety and $\Gamma$ a finite group of automorphisms of $\mathbf{X}$. As is well known, for any homomorphism $\alpha$ of the Galois group $\text{Gal}(\bar{F}/F)$ to $\Gamma$ one can define an $F$-variety $\mathbf{X}^\alpha$ such that $\mathbf{X}^\alpha(F) = \{x \in \mathbf{X}(\bar{F}); \gamma(x) = (\alpha(\gamma))(x) \forall \gamma \in \text{Gal}(\bar{F}/F)\}$.

Let $\boldsymbol{\eta} = (\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X} \times \mathbf{X}, \boldsymbol{\omega}, \mathbf{f})$ be an element on $S^*(\mathbf{X}, \mathbf{X})$, and $\Gamma$ a finite group of automorphisms of $\boldsymbol{\eta}$. Then for any homomorphism $\alpha$ of the Galois group $\text{Gal}(\bar{F}/F)$ to $\Gamma$ one can define an $\alpha$-twist $\boldsymbol{\eta}^\alpha = (\mathbf{R}^\alpha, \mathbf{p}^\alpha : \mathbf{R}^\alpha \to \mathbf{X}^\alpha \times \mathbf{X}^\alpha, \boldsymbol{\omega}^\alpha, \mathbf{f}^\alpha)$ of $\boldsymbol{\eta}$. Now we can formulate our main question.

QUESTION 5.1. *a) Given $\boldsymbol{\eta} \in S_{\text{uni}}(\mathbf{X}, \mathbf{X})$, a finite group $\Gamma$ of automorphisms of $\boldsymbol{\eta}$, and a homomorphism $\alpha$ of the Galois group* $\text{Gal}(\bar{F}/F)$ *to $\Gamma$, when can one guarantee that $\boldsymbol{\eta}^\alpha \in S_{\text{uni}}(\mathbf{X}^\alpha, \mathbf{X}^\alpha)$?*

*b) Given an algebro-geometric representation $\boldsymbol{\rho}$ of the group $\mathbf{G}$ on 1/2-forms on $\mathbf{X}$, a finite group $\Gamma$ of automorphisms of $\boldsymbol{\eta}$, and a homomorphism $\alpha$ of the Galois group* $\text{Gal}(\bar{F}/F)$ *to $\Gamma$, when can one guarantee that the $\alpha$-twist $\boldsymbol{\rho}^\alpha$ defines an algebro-geometric representation of the group $\mathbf{G}^\alpha$?*

I do not know any general criterion but will formulate some results and conjectures concerning the forms of examples 4.5 and 4.7.

In the situation of Example 4.5, let $\mathbf{G}$ be a semi-simple split $F$-group and $\Gamma$ the group of automorphisms of the Dynkin diagram of $\mathbf{G}$. Then the group $\Gamma$ acts naturally on the Weyl group $W$ and on the maximal-split torus $\mathbf{T}$. Let $\alpha$ be a homomorphism $\text{Gal}(\bar{F}/F) \to \Gamma$, and $\mathbf{G}^\alpha$ the corresponding quasi-split form of $\mathbf{G}$. We denote by $\mathbf{X}^\alpha := \mathbf{U}^\alpha\backslash\mathbf{G}^\alpha$ the corresponding form of $\mathbf{X}$. Then the group $\text{Gal}(\bar{F}/F)$ acts naturally on $W$ and we denote by $W' \subset W$ the subgroup of elements $w'$ invariant under this action. For any $w' \in W'$ we have a natural action of the group $\text{Gal}(\bar{F}/F)$ on $\mathbf{R}(w')$ (which is the first component of the quadruple $\boldsymbol{\eta}_{w'}$ of Example 4.5). It is easy to see that for any $w' \in W'$ the group $\Gamma$ acts naturally on $\boldsymbol{\eta}_{w'} \in S_{\mathrm{uni}}(\mathbf{X},\mathbf{X})$. Therefore for any $w' \in W'$ we can define an element $\boldsymbol{\eta}^{\alpha}_{w'} \in S^*(\mathbf{X}^{\alpha},\mathbf{X}^{\alpha})$.

CONJECTURE 5.1. *a) We have* $\boldsymbol{\eta}^{\alpha}_{w'} \in S_{\mathrm{uni}}(\mathbf{X}^{\alpha},\mathbf{X}^{\alpha})$.

*b) If* $w_1', w_2' \in W'$ *are such that* $l(w_1'w_2') = l(w_1') + l(w_2')$, *then* $\boldsymbol{\eta}^{\alpha}_{w'} = \boldsymbol{\eta}^{\alpha}_{w_1'} \circ \boldsymbol{\eta}^{\alpha}_{w_2'}$.

*c) There exist nondegenerate quadratic forms* $\mathbf{Q}^{w'}$, $w' \in W'$, *such that the map* $w' \mapsto \epsilon_{\mathbf{Q}^{w'}} \circ \boldsymbol{\eta}^{\alpha}_{w'}$ *(*$\epsilon_{\mathbf{Q}^{w'}}$ *is defined in Example 4.1) defines an algebraic representation of the group* $W'$ *on* $\mathbf{X}^{\alpha}$.

REMARK 5.1. a)Parts a) and b) of Conjecture 5.1 are easy to check. Using the formulae which define the Weil representation one can show the validity of Conjecture 5.1 in the case that $\mathbf{G}$ does not have any factor of type $D_4$.

Consider now Example 4.7. To any homomorphism $\alpha : \mathrm{Gal}(\bar{F}/F) \to S_n$ we can associate a maximal torus $\mathbf{T}^{\alpha} \subset \mathbf{GL}_n$. Let $\mathbf{Q}^{\alpha}$ be the quotient of the product $\tilde{\mathbf{Q}} \times \mathbf{T}^{\alpha}$ by the diagonally embedded $\mathbb{G}_m$. Assume now the validity of Conjecture 5.1. Then we have an action of the group $S_n$ on $\boldsymbol{\eta}'_{\sigma} \in S_{\mathrm{uni}}(\mathrm{ind}_{\mathbf{U}}^{\mathbf{Q}} \boldsymbol{\psi}, \mathrm{ind}_{\mathbf{U}}^{\mathbf{Q}} \boldsymbol{\psi})$ and therefore we can define

$$\boldsymbol{\eta}'^{\,\alpha}_{\sigma} \in S^*(\mathrm{ind}_{\mathbf{U}}^{\mathbf{Q}^{\alpha}} \boldsymbol{\psi}, \mathrm{ind}_{\mathbf{U}}^{\mathbf{Q}^{\alpha}} \boldsymbol{\psi}).$$

CONJECTURE 5.2. *a) We have* $\boldsymbol{\eta}'^{\,\alpha}_{\sigma} \in S_{\mathrm{uni}}(\mathrm{ind}_{\mathbf{U}}^{\mathbf{Q}^{\alpha}} \boldsymbol{\psi}, \mathrm{ind}_{\mathbf{U}}^{\mathbf{Q}^{\alpha}} \boldsymbol{\psi})$.

*b) Let* $A_n \subset S_n$ *be the alternating group. If* $\mathrm{Im}(\alpha) \subset A_n$, *then there exists an algebro-geometric representation* $\boldsymbol{\rho}^{\alpha}$ *of the group* $\mathbf{G}_n$ *on* $1/2$*-forms on* $\mathbf{X}^{\alpha}$ *such that* $\boldsymbol{\rho}^{\alpha}(\sigma) = \boldsymbol{\eta}'^{\,\alpha}_{\sigma}$ *and the restriction of* $\boldsymbol{\rho}^{\alpha}$ *to* $\mathbf{Q}^{\alpha}$ *is equal to* $\mathrm{ind}_{\mathbf{U}}^{\mathbf{Q}^{\alpha}}(\boldsymbol{\psi})$.

REMARK 5.2. a) One can formulate Conjecture 5.2 without the assumption that $\mathrm{Im}(\alpha) \subset A_n$. But this will demand the introduction of additional notations.

b) Conjecture 5.2 is proven in [**6**] in the case that $n = 3$, without the assumption that $\mathrm{Im}(\alpha) \subset A_n$.

## 6. Equivalence relation between algebro-geometric integrals

Let $\mathbf{A} = (\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X}, \boldsymbol{\omega}, \mathrm{f})$ and $\mathbf{A}' = (\mathbf{R}', \mathrm{p}' : \mathbf{R}' \to \mathbf{X}', \boldsymbol{\omega}', \mathrm{f}')$ be integrable quadruples over a global field $F$ and $\psi$ be a nontrivial complex valued character of the group $\mathbb{A}/F$ where $\mathbb{A}$ is the ring of adeles for $F$.

DEFINITION 6.1. We say that $\mathbf{A}$, $\mathbf{A}'$ are equivalent and write $\mathbf{A} \equiv \mathbf{A}'$ if for any local field $E$ containing $F$ the functions $\mathbf{p}_*(\boldsymbol{\omega}_{\psi,E,\mathbf{f}})$ and $\mathbf{p}'_*(\boldsymbol{\omega}'_{\psi,E,\mathbf{f}'})$ on $\mathbf{R}(E)$ coincide.

EXAMPLE 6.1. Let $\mathbf{V}$ be an affine space over $F$, $\mathbf{V}^{\vee}$ the dual space, $\boldsymbol{\omega}$ a volume form on $\mathbf{V}$ and $\boldsymbol{\omega}^{\vee}$ the dual volume form on $\mathbf{V}^{\vee}$. Take $\mathbf{A} = (\mathbf{R}, \mathbf{p} : \mathbf{R} \to \mathbf{X}, \boldsymbol{\omega}, \mathrm{f})$ and $\mathbf{A}' = (\mathbf{R}, \mathbf{p}' : \mathbf{R} \to \mathbf{X}', \boldsymbol{\omega}', \mathrm{f}')$, where $\mathbf{X} = \mathbf{X}' = \mathbf{V} \times \mathbf{V}$, $\mathbf{R} = \mathbf{V}$, $\mathbf{R}' = \mathbf{V} \times \mathbf{V}^{\vee} \times \mathbf{V}$, $\mathbf{p} : \mathbf{R} \to \mathbf{X}$ the diagonal embedding, $\mathbf{p}' : \mathbf{R}' \to$

$\mathbf{X}'$ the natural projection, $\boldsymbol{\omega}' = \boldsymbol{\omega} \times \boldsymbol{\omega}^\vee \times \boldsymbol{\omega}$, $\mathbf{f} = 0$ and $\mathbf{f}'(v', v^\vee, v'') = < v^\vee, v' - v'' >$. Then $\mathbf{A} \equiv \mathbf{A}'$.

REMARK 6.1. The statement "$\mathbf{A} \equiv \mathbf{A}'$" is equivalent to the formula for the inverse Fourier transform.

Let $W_F$ be the Witt group of $F$. So $W_F$ is a commutative ring generated by pairs $(\mathbf{Q}, \mathbf{V})$, where $\mathbf{V}$ is an $F$-affine space and $\mathbf{Q}$ is a nondegenerate quadratic form on the affine space $\mathbf{V}$. Addition in $W_F$ corresponds to direct sum of forms and multiplication to tensor product. Let $I \subset W_F$ be the ideal generated by pairs $(\mathbf{Q}, \mathbf{V})$ such that $\dim \mathbf{V}$ is even. As is well known, the ideal $I^2$ is generated by pairs $(\mathbf{Q}, \mathbf{V})$ such that $\dim \mathbf{V}$ is even and $\operatorname{disc}(\mathbf{Q}) \in F^{\times 2}$. Moreover, by **[9]** we can identify the quotient $I^2/I^3$ with the group $K_2(F)/2K_2(F)$.

EXAMPLE 6.2. Let $\mathbf{V}$, $\mathbf{V}'$ be affine spaces over $F$, and $\boldsymbol{\omega}$, $\boldsymbol{\omega}'$ invariant volume forms on $\mathbf{V}, \mathbf{V}'$. Let $\mathbf{Q}$, $\mathbf{Q}'$ be nondegenerate quadratic forms on $\mathbf{V}$, $\mathbf{V}'$ such that the difference $\{\mathbf{Q}\} - \{\mathbf{Q}'\} \in W(F)$ lies in $I^3$ and $\operatorname{disc}(\mathbf{Q})/\boldsymbol{\omega}^2 = \operatorname{disc}(\mathbf{Q}')/\boldsymbol{\omega}'^2$. Let $\mathbf{A} = (\mathbf{V}, \mathbf{p} : \mathbf{V} \to \star, \boldsymbol{\omega}, \mathbf{Q})$ and $\mathbf{A}' = (\mathbf{V}', \mathbf{p} : \mathbf{V}' \to \star, \boldsymbol{\omega}', \mathbf{Q}')$. Then $\mathbf{A} \equiv \mathbf{A}'$.

REMARK 6.2. This statement is clearly local. It is trivial for nonarchimedean local fields and in the case of the field $\mathbb{C}$ of complex numbers, since for such fields $I^3 = \{0\}$. For the field $\mathbb{R}$ it follows from the Fresnel formula for $\int_{-\infty<t<\infty} \exp(it^2)dt$. But I do not know a conceptual proof of this result.

DEFINITION 6.2. For any $\bar{\mathbf{Q}} \in I^2/I^3$ we choose a representative $(\mathbf{Q}, \mathbf{V})$ of $\bar{\mathbf{Q}}$ in $W_F$ and a top-form $\boldsymbol{\omega}$ on $\mathbf{V}$ such that $\operatorname{disc}(\mathbf{Q}) = \boldsymbol{\omega}^2$. As follows from Example 2.2, the equivalence class of the quadruple $(\star, \mathbf{p} : \star \to \mathbf{V}, \boldsymbol{\omega}, \mathbf{Q})$ does not depend on a choice of $\mathbf{Q}$ and $\boldsymbol{\omega}$. We denote it by $\eta(\bar{\mathbf{Q}})$.

We present now a series of pairs of equivalent algebro-geometric measures. In most of these cases we do not know how to prove locally the equality of the corresponding integrals. Since the proof depends on results from the theory of representations of reductive groups, we start with exposition of some concepts and results from this theory.

## 7. Representations of reductive groups

Let $\mathbf{G}$ be a connected, simply connected semi-simple linear group over a field $F$ of characteristic zero and $\boldsymbol{\rho} : \mathbf{G} \to \operatorname{Aut}(\mathbf{V})$ a representation. As is well known (see **[1]**), there exists a nonempty open subset $\mathbf{V}_0 \subset \mathbf{V}$ and a subgroup $\mathbf{St} = \mathbf{St}_\rho$ in $\mathbf{G}$ such that for all $v \in \mathbf{V}_0(F)$ the stabilizer $\mathbf{St}_v$ of $v$ in $\mathbf{G}$ is conjugate to $\mathbf{St}$ over $\bar{F}$.

Assume that the group $\mathbf{St}$ is reductive. Then it is well known (see **[10]**) that the $\mathbf{G}$-orbits $\mathcal{O}_v = \mathbf{G} \cdot v$ are closed for any $v \in \mathbf{V}_0$.

Let $\operatorname{Lie}(\mathbf{St})$ be the Lie algebra of $\mathbf{St}$ and $\mathbf{O}$ the group of linear transformations of $\operatorname{Lie}(\mathbf{St})$ preserving the Killing form. Let $\tilde{\mathbf{O}}$ be the spinor cover of $\mathbf{O}$ and $\tilde{\mathbf{St}}$ the induced cover of $\mathbf{St}$.

DEFINITION 7.1. We say that a representation $\boldsymbol{\rho}$ is nice if $\mathbf{St}$ is a connected reductive group and the extension $\tilde{\mathbf{St}} \to \mathbf{St}$ splits.

In this paper we will consider only nice representations. It is easy to see that for any nice representation $\boldsymbol{\rho} : \mathbf{G} \to \mathrm{Aut}(\mathbf{V})$ of $\mathbf{G}$ the dual representation $\boldsymbol{\rho}^\vee : \mathbf{G} \to \mathrm{Aut}(\mathbf{V}^\vee)$ is also nice.

For any nice representation $\boldsymbol{\rho} : \mathbf{G} \to \mathrm{Aut}(\mathbf{V})$ of $\mathbf{G}$ we put $\mathbf{B} := \mathrm{Spec}(F[\mathbf{V}]^{\mathbf{G}})$ and denote by $\mathbf{p} : \mathbf{V} \to \mathbf{B}$ the quotient map. We denote by $\mathbf{B}_0 \subset \mathbf{B}$ the image of $\mathbf{V}_0$ under $\mathbf{p}$. For any $y \in \mathbf{B}_0(F)$ we denote by $\mathcal{O}_y$ the preimage $\mathbf{p}^{-1}(y)$. It is easy to see that $\mathcal{O}_y$ is a homogeneous $\mathbf{G}$-variety. We fix invariant volume forms $dg$ and $ds$ on $\mathbf{G}$ and $\mathbf{St}$.

For any $y \in \mathbf{B}_0(F)$ the form $ds$ defines an invariant volume form $ds_y$ on $\mathbf{St}_y$, and we denote by $\boldsymbol{\omega}_y$ the $\mathbf{G}$-invariant volume form $dg/ds_y$ on $\mathcal{O}_y$. For any nice representation $\boldsymbol{\rho} : \mathbf{G} \to \mathrm{Aut}(\mathbf{V})$ of $\mathbf{G}$ we denote by $\mathbf{B}^\vee$ the quotient $\mathbf{V}^\vee/\mathbf{G}$ and by $\mathbf{B}_0^\vee \subset \mathbf{B}^\vee$ the image of $\mathbf{V}_0$.

For any $v^\vee \in \mathbf{V}_0^\vee(F)$ and $y \in \mathbf{B}_0(F)$ we denote by $\mathbf{R}(v^\vee, y)$ the set of critical points of the restriction of the linear function $v^\vee$ to $\mathcal{O}_y(F)$. For any $r \in \mathbf{R}(v^\vee, y)$ we denote by $T_r$ the tangent space to $\mathcal{O}_y$ at $r$ and by $\mathbf{Q}_{r,v^\vee}$ the quadratic form on $T_r$ defined by the restriction of the function $v^\vee$ to $\mathcal{O}_y$.

LEMMA 7.1. *a) The set $\mathbf{R}(v^\vee, y)$ is finite.*

*b) For any $r \in \mathbf{R}(v^\vee, y)$ the form $\mathbf{Q}_{r,v^\vee}$ is nondegenerate.*

*c) For any $r, r' \in \mathbf{R}(v^\vee, y)$ we have: $\mathrm{disc}(\mathbf{Q}_{r,v^\vee})/\boldsymbol{\omega}_y^2$ is equal to $\mathrm{disc}(\mathbf{Q}_{r',v^\vee})/\boldsymbol{\omega}_y^2$; and: $\{\mathbf{Q}_{r,v^\vee}\} - \{\mathbf{Q}_{r',v^\vee}\} \in W_F$ lies in $I^3$.*

*Denote by $\bar{\mathbf{Q}}_{y,v^\vee}$ the image of the form $\mathbf{Q}_{r,v^\vee}$ in the quotient $W_F/I^3$.*

*d) For any $y^\vee = \mathbf{p}(v^\vee) \in \mathbf{p}(\mathbf{V}_0^\vee(F)) \subset \mathbf{B}_0^\vee(F)$, the class $\bar{\mathbf{Q}}_{y,v^\vee}$ in $W_F/I^3$ does not depend on a choice of $v^\vee \in \mathcal{O}_y(F)$. Denote this class by $\bar{\mathbf{Q}}_{y,y^\vee}$.*

PROOF. Clear.

Let $\boldsymbol{\rho} : \mathbf{G} \to \mathrm{Aut}(\mathbf{V})$ be a nice representation,$v_0 \in \mathbf{V}_0(F)$. We denote by $\hat{Z}$ the center of the Langlands dual $\hat{\mathbf{St}}$ of $\mathbf{St}$. Let $\pi_1' := \pi_1(\mathbf{V}_0, v_0)$ be the fundamental group of $\mathbf{V}_0$ in the etale topology. The group $\pi_1'$ acts naturally on $\hat{Z}$. Since $\hat{\mathbf{St}}$ is connected and the image of $\pi_1(\hat{\mathbf{St}})$ in $\pi_1'$ acts trivially on $\hat{Z}$, we see that the action of $\pi_1'$ on $\hat{Z}$ comes from an action $\iota : \pi_1 \to \mathrm{Aut}(\hat{Z})$, where $\pi_1 = \pi_1(\mathbf{B}_0)$ is the fundamental group of $\mathbf{B}_0$. It is easy to see that the image $\bar{\pi}_1$ of $\pi_1$ in $\mathrm{Aut}(\hat{Z})$ is finite. Let $q : \tilde{\mathbf{B}}_0 \to \mathbf{B}_0$ be the covering corresponding to the kernel of the map $\pi_1 \to \bar{\pi}_1$.

For any extension $E$ of $F$ and any point $y \in \mathbf{B}_0(E)$ we denote by $\iota_y$ the corresponding representation $\iota_y : \mathrm{Gal}(\bar{E}/E) \to \mathrm{Aut}(\hat{Z})$, and we define $\Gamma_y := (\hat{Z})^{\mathrm{Im}(\iota_y)}$. We denote by $\Gamma_y^\vee$ the dual group.

For any $v^\vee \in \mathbf{V}_0^\vee(F)$ we denote by $V(v^\vee)$ the subset of $v \in \mathbf{V}_0(F)$ such that the $F$-group $\mathbf{St}_v$ is an inner form of the group $\mathbf{St}_{v^\vee}$. It is clear that the set $V(v^\vee)$ has the form $V(v^\vee) = \mathbf{p}^{-1}(B(v^\vee))$, where $B(v^\vee)$ is a subset of $\mathbf{B}_0(F)$.

Let $\boldsymbol{\rho}$ be a nice representation. For any $\gamma \in \bar{\pi}_1$ we denote by $Y_{\boldsymbol{\rho}}(\gamma)$ the connected component of the identity in the group $\hat{Z}^\gamma$ of $\gamma$-fixed points of $\hat{Z}$.

We denote by $Y_{\rho}$ the connected component of the identity in the group $\hat{Z}^{\bar{\pi}_1}$ of $\bar{\pi}_1$-invariants of $\hat{Z}$.

The paper [**1**] contains a classification of all nice representations of semi-simple groups such that $\mathbf{St} \neq \{e\}$. We will use later the following result which can be derived from that classification.

LEMMA 7.2. *For any nice representation* $\boldsymbol{\rho}$ *we have* $Y_{\boldsymbol{\rho}} = \bigcap_{\gamma \in \bar{\pi}_1} Y_{\boldsymbol{\rho}}(\gamma)$.

Assume now that $F$ is a local field. All the results we prove are true for any local field but for simplicity we assume that $F$ is nonarchimedean. In this case $H^1(F, \tilde{\mathbf{G}}(\bar{F}) = \{0\}$ for any semi-simple simply connected group $\tilde{\mathbf{G}}$. Therefore for any connected semi-simple group $\mathbf{G}$ one can identify the set $H^1(F, \mathbf{G}(\bar{F})$ with the abelian group $H^1(F, \mathbf{C}(\bar{F})$ where $\mathbf{C} := \pi_1(\underline{G})$. In particular for any nice representation the set $H^1(F, \mathbf{St}(\bar{F})$ has a natural structure of an abelian group.

Let $\mathcal{S}(\mathbf{V}(F))$ be the space of Schwartz-Bruhat functions on $\mathbf{V}(F)$ and $\mathcal{D}(\mathbf{V}(F))$ the dual space of distributions. For any $y \in \mathbf{B}_0(F)$ we denote by $\delta_y \in \mathcal{D}(\mathbf{V}(F))$ the distribution on $\mathbf{V}(F)$ such that $\delta_y(f) := \int_{a \in \mathcal{O}_y(F)} f(a)|\boldsymbol{\omega}_y|$. In particular $\delta_y = 0$ if $y$ does not belong to $\mathbf{p}(\mathbf{V}_0(F))$.

Let $\mathcal{S}_0(\mathbf{V}(F)) \subset \mathcal{S}(\mathbf{V}(F))$ be the subspace of functions $f$ such that $\delta_y(f) = 0$ for all $y \in \mathbf{B}_0(F))$. We denote by $\mathcal{S}_0(\mathbf{V}_0(F))$ the intersection $\mathcal{S}_0(\mathbf{V}(F)) \bigcap \mathcal{S}(\mathbf{V}_0(F))$. We define $\mathcal{D}(\mathbf{V}(F))^{st} := \{\alpha \in \mathcal{D}(\mathbf{V}(F)); \alpha(f) = 0\}$ for all $f \in \mathcal{S}_0(\mathbf{V}_0(F))$.

By our assumption, the group $\mathbf{G}$ is simply connected, the group $\mathbf{St}$ is connected and the local field $F$ is nonarchimedean. Therefore the set $H^1(F, \mathbf{St})$ has a natural structure of an abelian group and for any $y \in \tilde{\mathbf{B}}_0(F)$ the set $\bar{\mathcal{O}}_y$ of $\mathbf{G}(F)$-orbits on $\mathcal{O}_y(F)$ has a natural structure of a principal homogeneous $H^1(F, \mathbf{St})$-space. Moreover, the local Tate-Nakayama duality theory (see, e.g., [**7**]) shows that the group $H^1(F, \mathbf{St})$ is isomorphic to the group $\Gamma_y^{\vee}$.

For any function $f \in \mathcal{S}(\mathbf{V}(F))$ and point $y \in \mathbf{B}_0(F)$ we denote by $\bar{f}_y$ the $\mathbf{G}(F)$-invariant function on $\mathcal{O}_y(F)$ whose value at $v \in \mathcal{O}_y(F)$ is given by $\bar{f}_y(v) := \int_{a \in \mathbf{G}(F)v} f(a)|\boldsymbol{\omega}_y|$.

We can consider $\bar{f}_y$ as a function on the set $\bar{\mathcal{O}}_y$.

For any $y \in \tilde{\mathbf{B}}_0(F)$ we denote by $\mathcal{S}_y^{st} \subset \mathcal{S}(\mathcal{O}_y(F))$ the subspace of functions $f \in \mathcal{S}(\mathcal{O}_y(F))$ such that the function $v \mapsto \bar{f}_y(v)$ does not depend on a choice of $v \in \mathcal{O}_y(F)$.

For any $y^{\vee} \in \mathbf{B}_0^{\vee}(F)$ we denote by $\mathcal{S}^{st}(y^{\vee})$ the subspace of functions $f \in \mathcal{S}(V(v^{\vee}))$ such that for any $y \in \mathbf{B}_0(F)$ the restriction of $\mathbf{f}$ on $\mathcal{O}_y(F)$ lies in $\mathcal{S}_y^{st}$.

The following result is an easy generalization of a special case of Proposition 8.2 in [**11**].

PROPOSITION 7.1. *Let* $F$ *be a local nonarchimedean field, and* $\boldsymbol{\rho} : \mathbf{G} \to \operatorname{Aut}(\mathbf{V})$ *a nice representation of an* $F$*-group* $\mathbf{G}$*. Then for any* $y^{\vee} \in \mathbf{B}_0^{\vee}(F)$

*there exists a function $\phi_{y^\vee} \in \mathcal{S}^{st}(y^\vee)$ such that the restriction of the Fourier transform $\mathcal{F}(\phi)$ to $\mathcal{O}_{y^\vee}$ belongs to $\mathcal{S}^{st}_{y^\vee}$ and $\delta_{y^\vee}(\mathcal{F}(\phi)) \neq 0$.*

PROOF. Choose $v^\vee \in \mathbf{p}^{-1}(y^\vee)(F)$. Let $\mathbf{T}_{v^\vee} \subset \mathbf{V}^\vee$ be the tangent space to $\mathcal{O}_{y^\vee}$ at $v^\vee$. Since the group $\mathbf{St}_{v^\vee}$ is reductive there exists $v \in \mathbf{V}_0(F)$ such that $v$ is orthogonal to $\mathbf{T}_{v^\vee}$ and $\mathbf{St}_v = \mathbf{St}_{v^\vee}$. Let $L := \{l \in \mathbf{V}_0; \mathbf{St}_l = \mathbf{St}_v\}$, and let $W$ be a compact open neighborhood of $v$ in $L$. Since $\mathbf{G}$ is a simply connected group, we have $H^1(F, \mathbf{G}) = 0$, and hence for any $\alpha \in H^1(F, \mathbf{St})$ there exists $g_\alpha \in \mathbf{G}(\bar{F})$ such that $\alpha(\gamma) = \gamma(g_\alpha)g_\alpha^{-1}$ for all $\gamma \in F$. Let $W_\alpha := g_\alpha(W) \subset \mathbf{V}(\bar{F})$. It is clear then that the set $W_\alpha$ belongs to $\mathbf{V}(F)$ and does not depend on the choice of $g_\alpha(W)$. Let $K \subset \mathbf{G}(F)$ be an open compact subgroup, and put $U := \bigcup_{\alpha \in H^1(F,\mathbf{St})} KW_\alpha$. For any $t \in F$ we denote by $\phi_{y^\vee,t}$ the characteristic function of the set $tU$. It is clear that for all $t \in F^\times$ we have $\phi_{y^\vee,t} \in \mathcal{S}^{st}(y^\vee)$. On the other hand it is easy to see that, for any $t \in F^\times$ such that $|t| >> 1$, the restriction of the Fourier transform $\mathcal{F}(\phi_{y^\vee,t})$ to $\mathcal{O}^\vee_y$ is equal to its stationary phase approximation. That is

$$\mathcal{F}(\phi_{y^\vee,t})(v^\vee) = a_F(\mathbf{Q}_{y,v^\vee}) \sum_{r \in \mathbf{R}(v^\vee,y)} \psi(< v^\vee, r >)$$

where $a_F(\mathbf{Q})$ is defined in Example 2.2. It follows now from Lemma 7.1 that the function $\phi_t$ satisfies the conditions of Proposition 7.1. □

## 8. Stability

THEOREM 1. *Every nice representation $\boldsymbol{\rho} : \mathbf{G} \to \mathrm{Aut}(\mathbf{V})$ has the following stability property: for every $v^\vee$, $v'^\vee \in \mathbf{V}_0^\vee(F)$ such that $\mathbf{p}(v^\vee) = \mathbf{p}(v'^\vee)$, the quadruples $(\mathbf{V}_0, \mathbf{p} : \mathbf{V}_0 \to \mathbf{B}_0, \boldsymbol{\omega}, < *, v^\vee >)$ and $(\mathbf{V}_0, \mathbf{p} : \mathbf{V}_0 \to \mathbf{B}_0, \boldsymbol{\omega}, < *, v'^\vee >)$ are equivalent.*

Let $E$ be a local field, $\boldsymbol{\rho} : \mathbf{G} \to \mathrm{Aut}(\mathbf{V})$ a nice representation over $E$, and $\psi$ a nontrivial additive character of $E$. Let $\mathcal{F} : C^\infty(\mathbf{V}(F)) \to C^\infty(\mathbf{V}^\vee(F))$ be the Fourier transform. It defines the Fourier transform on the space of distributions which we will also denote by $\mathcal{F}$.

It is clear that Theorem 1 is equivalent to the following result.

THEOREM **1′**. *For any local field $E$ of characteristic zero, any nice representation $\boldsymbol{\rho} : \mathbf{G} \to \mathrm{Aut}(\mathbf{V})$ over $E$, and any $y \in \mathbf{B}_0(E)$, we have $\mathcal{F}(\delta_y) \in \mathcal{D}(V^\vee(E))^{st}$ where the space $\mathcal{D}(V^\vee(E))^{st}$ is defined after Lemma 7.2.*

It will be convenient to work with the following reformulation of Theorem 1′.

THEOREM **1″**. *For any nice representation $\boldsymbol{\rho} : \mathbf{G} \to \mathrm{Aut}(\mathbf{V})$ and any local field $E$, $F \subset E$, we have $\mathcal{F}(\mathcal{S}_0(\mathbf{V}_0(E))) \subset \mathcal{S}_0(\mathbf{V}^\vee(E))$.*

Of course the statement of Theorem 1″ is local [that is, we can fix a local field $E$]. But I do not know a local proof of this theorem. In the proof we assume that $E$ is nonarchimedean. But the analogous arguments are applicable in the case when $E = \mathbb{R}$. In the case when $E = \mathbb{C}$ the claim is obviously true.

Let now $F$ be a global field of characteristic zero. We denote by $\mathfrak{P} = \mathfrak{P}(F)$ the set of places of $F$ and by $\mathbb{A}$ the ring of adeles for $F$. For any finite subset $S \subset \mathfrak{P}$ we define $\mathbb{A}_S := \prod_{\mathfrak{p}\in S} F_{\mathfrak{p}}$ where $F_{\mathfrak{p}}$ is the completion of $F$ at $\mathfrak{p}$. We denote by $\mathbb{A}^S$ the ring of adeles outside $S$. So $\mathbb{A} = \mathbb{A}_S\mathbb{A}^S$.

As is well known, there exists a finite set $S_\rho \subset \mathfrak{P}(F)$ containing the set $\mathfrak{P}_\infty$ of archimedean places, such that $\boldsymbol{\rho}$ is defined over $\mathcal{O}_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathfrak{P} - S_\rho$. For any $\mathfrak{p} \in \mathfrak{P} - S\boldsymbol{\rho}$ we denote by $B_{00}(\mathcal{O}_{\mathfrak{p}})$ the set of points $y_{\mathfrak{p}} \in \mathbf{B}_0(\mathcal{O}_{\mathfrak{p}})$ such that the reduction $\bar{y}_{\mathfrak{p}}$ belongs to $\mathbf{B}_0(k_{\mathfrak{p}})$. It is clear that for $y_{\mathfrak{p}} \in B_{00}(\mathcal{O}_{\mathfrak{p}})$ the covering $q$ is unramified at $\bar{y}_{\mathfrak{p}}$. We denote by $\gamma(y_{\mathfrak{p}}) \subset \bar{\pi}_1$ the conjugacy class of the Frobenius at $\bar{y}_{\mathfrak{p}}$.

It is easy to see that for any $y \in \mathbf{B}_0(F)$ there exists a finite set $S_y \subset \mathfrak{P}$ containing $S_\rho$ such that $y_{\mathfrak{p}} \in B_{00}(\mathcal{O}_{\mathfrak{p}})$ for all $\mathfrak{p} \in \mathfrak{P} - S_y$.

As in [**7**] we define for any $y \in \mathbf{B}_0(F)$ a map $\mathrm{inv}_y : \mathcal{O}_y(\mathbb{A}) \to \Gamma_y^\vee$ such that $\mathrm{inv}_y^{-1}(e) = G(\mathbb{A})(\mathcal{O}_y(F))$.

Assume now that the $F$-group $\mathbf{G}$ is anisotropic. Let $\mathcal{S}, \mathcal{S}^\vee$ be the spaces of Schwartz-Bruhat functions on $\mathbf{V}(\mathbb{A})$ and $\mathbf{V}^\vee(\mathbb{A})$. We denote by $\theta, \theta^\vee$ the distributions on $\mathcal{S}, \mathcal{S}^\vee$ such that $\theta(f) := \sum_{v\in\mathbf{V}(F)} f(v)$, $\theta(f^\vee) := \sum_{v\in\mathbf{V}^\vee(F)} f^\vee(v)$ for $f \in \mathcal{S}$, $f^\vee \in \mathcal{S}^\vee$. Since $\mathbf{G}$ is anisotropic over $F$, the quotient space $\mathbf{G}(\mathbb{A})/\mathbf{G}(F)$ is compact. We define distributions $\Theta$, $\Theta^\vee$ on $\mathcal{S}$ and $\mathcal{S}^\vee$ by integrals: $\Theta := \int_{g\in\mathbf{G}(\mathbb{A})/\mathbf{G}(F)} \theta^g|dg|$ and $\Theta^\vee := \int_{g\in\mathbf{G}(\mathbb{A})/\mathbf{G}(F)} \theta^{\vee g}|dg|$; here $|dg|$ is the Tamagawa measure on $\mathbf{G}(\mathbb{A})$.

Let $\mathcal{S}_r \subset \mathcal{S}$ be the space of functions $f$ such that $\mathrm{supp}(f) \cap \mathbf{p}^{-1}(y)(\mathbb{A})$ is $\emptyset$ for any $y \in \mathbf{B}(F) - \mathbf{B}_0(F)$.

LEMMA 8.1. *For any $f \in \mathcal{S}_r$ we have*

$$\Theta(f) = \sum_{y\in\mathbf{B}_0(F)} \sum_{\kappa\in\Gamma_y} \int_{a\in\mathcal{O}_y(\mathbb{A})} \kappa(\mathrm{inv}_y(a))f(a)|\boldsymbol{\omega}_y|.$$

This lemma follows from results of Kottwitz (see [**7**] and [**8**]).

Let $\psi$ be a nontrivial additive character of $\mathbb{A}/F$ and $\mathcal{F} : \mathcal{S} \to \mathcal{S}^\vee$ the Fourier transform. By the Poisson summation formula we have $\mathcal{F}(\theta) = \theta^\vee$ and therefore $\mathcal{F}(\Theta) = \Theta^\vee$.

COROLLARY 8.1. *For any $f \in \mathcal{S}_r$ such that $\mathcal{F}(f) \in \mathcal{S}_r^\vee$ we have*

$$\sum_{y\in\mathbf{B}_0(F)} \sum_{\kappa\in\Gamma_y} \int_{a\in\mathcal{O}_y(\mathbb{A})} \kappa(\mathrm{inv}_y(a))f(a)|\boldsymbol{\omega}_y| =$$

$$\sum_{y^\vee\in\mathbf{B}_0^\vee(F)} \sum_{\kappa^\vee\in\Gamma_{y^\vee}} \int_{a^\vee\in\mathcal{O}_{y^\vee}(\mathbb{A})} \mathcal{F}(f)\kappa^\vee(\mathrm{inv}_{y^\vee}(a))f(a^\vee)|\boldsymbol{\omega}_y|.$$

Let $\mathfrak{p}_0$ be a place of $F$, $w_{\mathfrak{p}_0} \in \mathbf{B}_0(F_{\mathfrak{p}_0})$ and $U_{\mathfrak{p}_0} \subset \mathbf{B}_0(F_{\mathfrak{p}_0})$ a neighborhood of $w_{\mathfrak{p}_0}$ in the $F_{\mathfrak{p}_0}$-topology.

LEMMA 8.2. *There exists $v \in \mathbf{V}_0(F)$ and a finite set $R \subset \mathfrak{P}(F) - S_v$ such that $\mathbf{p}(v_{\mathfrak{p}_0}) \in U_{\mathfrak{p}_0}$ and $\bar{\pi}_1 = \bigcup_{\mathfrak{p}\in R} \gamma_{(y_\mathfrak{p})}(v)$.*

PROOF. Since the group **St** is connected, it follows from Lang's theorem that $p^{-1}(y_\mathfrak{p}) \neq \emptyset$ for any nonarchimedean $\mathfrak{p} \in \mathfrak{P}(F)$ and $y_\mathfrak{p} \in \mathbf{B}_0(\mathcal{O}_\mathfrak{p})$. The lemma follows now from Chebotarev's theorem and the weak approximation theorem. □

Let $E$ be a nonarchimedean local field of characteristic zero, and $\boldsymbol{\rho}_E : \mathbf{G}_E \to \mathrm{Aut}(\mathbf{V}_E)$ be a nice representation. We start the proof of theorem $1''$ by choosing a global field $F$ such that $\mathfrak{P}(F)$ contains a place $\mathfrak{p}_0$ such that $F_{\mathfrak{p}_0} = E$ and the set $\mathfrak{P}_\infty(F)$ contains a point $p_\infty$ such that $F_{p_\infty} = \mathbb{R}$.

It is easy to show that one can find an $F$-group $\mathbf{G}$ and a representation $\boldsymbol{\rho}$ of $\mathbf{G}$ such that $\mathbf{G}_{\mathfrak{p}_0} = \mathbf{G}$, $\boldsymbol{\rho}_{\mathfrak{p}_0} = \boldsymbol{\rho}_E$ and the group $\mathbf{G}(\mathfrak{p}_\infty)$ is compact.

To prove theorem $1''$ we have to show that for any function $f_{\mathfrak{p}_0} \in \mathcal{S}_0(\mathbf{V}_0(F_{\mathfrak{p}_0}))$ and for any $y_E{}^\vee \in \mathbf{B}_0^\vee(F_{\mathfrak{p}_0})$ we have $\delta_{y_E{}^\vee}(\mathcal{F}_{\mathfrak{p}_0}(f_{\mathfrak{p}_0})) = 0$. Fix a function $f_{\mathfrak{p}_0} \in \mathcal{S}_0(\mathbf{V}_0(F_{\mathfrak{p}_0}))$. Since $\mathcal{F}_{\mathfrak{p}_0}(f_{\mathfrak{p}_0}) \in \mathcal{S}(\mathbf{V}(F_{\mathfrak{p}_0}))$, the function $y'^\vee \mapsto \delta_{y'^\vee}(\mathcal{F}_{\mathfrak{p}_0}(f_{\mathfrak{p}_0}))$ on $\mathbf{B}_0^\vee(F_{\mathfrak{p}_0})$ is locally constant. Therefore it follows from Lemma 8.2 that there exists $y^\vee \in \mathbf{B}_0^\vee(F)$ and a finite set $R \subset \mathfrak{P}(F) - (S_v \cup \mathfrak{p}_0)$ such that

$$\delta_{y_E^\vee}(\mathcal{F}_{\mathfrak{p}_0}(f_{\mathfrak{p}_0})) = \delta_{y_{\mathfrak{p}_0}^\vee}(\mathcal{F}_{\mathfrak{p}_0}(f_{\mathfrak{p}_0}))$$

and $Y_\rho = \bigcap_{\mathfrak{p}\in R} Y_\rho(\gamma(y_\mathfrak{p}))$, where $Y_\rho$ is defined in Section 4.

As follows from Proposition 7.1, there exists a function $\phi_R \in \mathcal{S}^{st}(y_R^\vee)$ such that the restriction of the Fourier transform $\mathcal{F}_R(\phi_R)$ on $\mathcal{O}_{y_R^\vee}$ belongs to $\mathcal{S}^{st}_{y_R^\vee}$ and $\delta_{y_R^\vee}(\mathcal{F}_R(\phi_R)) \neq 0$.

Let $R' := R \cup \mathfrak{p}_0$.

LEMMA 8.3. *For any function $f^{R'} \in \mathcal{S}^{R'}$ we have $\Theta(f_{\mathfrak{p}_0} \otimes \phi_R \otimes f^{R'}) = 0$.*

PROOF. As follows from Lemma 8.1, it is sufficient to show that for any $y \in \mathbf{B}_0(F)$ and any $\kappa \in \Gamma_y$ we have $I_{y,\kappa} = 0$, where

$$I_{y,\kappa} := \int_{a\in\mathcal{O}_y(\mathbb{A})} \kappa(\mathrm{inv}_y(a)) f_{\mathfrak{p}_0}(a_{\mathfrak{p}_0}) \phi_R(a_R) f^{R'}(a^{R'}) |\omega_y|.$$

By definition

$$I_{y,\kappa} = I_{y_{\mathfrak{p}_0},\kappa_{\mathfrak{p}_0}} I_{y_R,\kappa_R} I_{y^{R'},\kappa^{R'}},$$

where

$$I_{y_{\mathfrak{p}_0},\kappa_{\mathfrak{p}_0}} := \int_{a_{\mathfrak{p}_0}\in\mathcal{O}_y(\mathfrak{p}_0)} \kappa_{\mathfrak{p}_0}(\mathrm{inv}_{y_{\mathfrak{p}_0}}(a_{\mathfrak{p}_0})) f_{\mathfrak{p}_0}(a_{\mathfrak{p}_0}) |(\omega_y)_{\mathfrak{p}_0}|,$$

$$I_{y_R,\kappa_R} := \int_{a_R\in\mathcal{O}_y(\mathbb{A}_R)} \kappa_R(\mathrm{inv}_y(a_R)) \phi_R(a_R) |(\omega_y)_R|,$$

and

$$I_{y^{R'},\kappa^{R'}} = \int_{a^{R'}\in\mathcal{O}_y(\mathbb{A}^{R'})} \kappa^{R'}(\mathrm{inv}_y(a^{R'})) f^{R'}(a^{R'}) |(\omega_y)^{R'}|.$$

By the construction $I_{y_{\mathfrak{p}_0},\kappa_{\mathfrak{p}_0}} = 0$ if $\kappa = 0$. On the other hand, it follows from the choice of the set $R$ that for $\kappa \neq 0$ we have $\kappa_R \neq 0$. But then the inclusion $\phi_R \in \mathcal{S}^{st}(y_R^\vee)$ implies that $I_{y_R,\kappa_R} = 0$. □

Now we finish the proof of Theorem $1''$. Choose a nonarchimedean place $\mathfrak{p}_1 \in \mathfrak{P} - R'$ and a Schwartz-Bruhat function $f_{\mathfrak{p}_1}$ on $\mathbf{V}(F_{\mathfrak{p}_1})$ such that $\mathrm{supp}(\mathcal{F}_{\mathfrak{p}_1}(f_{\mathfrak{p}_1})) \subset \mathbf{V}_0^\vee(F_{\mathfrak{p}_1})$ and $\delta_{y_{\mathfrak{p}_1}^\vee}(\mathcal{F}_{\mathfrak{p}_1}(f_{\mathfrak{p}_1})) \neq 0$. Let $R'' := R' \cup \mathfrak{p}_1$. For any Schwartz-Bruhat function $h \in \mathcal{S}(\mathbb{A}^{R''})$ define $f_h := f_{\mathfrak{p}_0} \otimes f_{\mathfrak{p}_1} \otimes \phi_R \otimes h$. By the construction, $f_h$ satisfies the conditions of Lemma 8.3 and of Corollary 8.1. So $\Theta(f_h) = 0$ and therefore

$$\Theta(\mathcal{F}(f_h)) = \Theta(\mathcal{F}_{\mathfrak{p}_0}(f_{\mathfrak{p}_0}) \otimes \mathcal{F}_{\mathfrak{p}_1}(f_{\mathfrak{p}_1}) \otimes \mathcal{F}_R(f_R) \otimes \mathcal{F}^{R''}(h)) = 0$$

for all $h \in \mathcal{S}(\mathbb{A}^{R''})$. Let

$$C := \mathbf{B}_0(F) \bigcap \mathbf{p}_{R''}^{-1}(\mathrm{supp}(\mathcal{F}_{\mathfrak{p}_0}(f_{\mathfrak{p}_0}) \otimes \mathcal{F}_{\mathfrak{p}_1}(f_{\mathfrak{p}_1}) \otimes \mathcal{F}_R(f_R))$$

and $M^{R''} := \bigcup_{y \in C} \mathbf{p}^{R''-1}(y^{R''}) \subset \mathbf{V}(\mathbb{A}^{R''})$. Then

$$M^{R''} = \mathcal{O}_{y_{R''}}(\mathbb{A}^{R''}) \bigcup M'^{R''},$$

where $\mathcal{O}_{y_{R''}}(\mathbb{A}^{R''})$ and $M'^{R''}$ are closed disjoint subsets of $\mathbf{V}(\mathbb{A}^{R''})$.

Therefore one can find a Schwartz-Bruhat function $h$ on $\mathbf{V}(\mathbb{A}^{R''})$ such that $\mathrm{supp}(\mathcal{F}^{R''}(h)) \bigcap M'^{R''} = \emptyset$ and $\delta_{y^\vee,R''}(\mathcal{F}^{R''}(h)) \neq 0$. Then

$$\Theta(\mathcal{F}(f_h)) = \delta_{y_{\mathfrak{p}_0}^\vee}(f_{\mathfrak{p}_0}) \delta_{y_{\mathfrak{p}_1}^\vee}(f_{\mathfrak{p}_1}) \delta_{y_{\mathfrak{p}_2}^\vee}(f_{\mathfrak{p}_2}) \delta_{y^\vee,R''}(\mathcal{F}^{R''}(h))$$

and the equality

$$\Theta(\mathcal{F}_{\mathfrak{p}_0}(f_{\mathfrak{p}_0}) \otimes \mathcal{F}_{\mathfrak{p}_1}(f_{\mathfrak{p}_1}) \otimes \mathcal{F}_{\mathfrak{p}_2}(f_{\mathfrak{p}_2}) \otimes \mathcal{F}^{R''}(f^{R''})) = 0$$

implies that $\delta_{y_{\mathfrak{p}_0}^\vee}(f_{\mathfrak{p}_0}) = 0$.

Theorem $1''$ [and therefore Theorem 1] is proved. □

## 9. Inner forms

Let $\mathbf{G}$ be a semi-simple connected, simply connected and quasi-split group over a field $F$ of characteristic zero, $\mathbf{Z}_0$ a subgroup of the center of $\mathbf{G}$, $\alpha \in H^1(F, \mathbf{G}/\mathbf{Z}_0)$ and $\mathbf{G}'$ the inner form of $\mathbf{G}$ corresponding to $\alpha$. Let $\mathbf{Q}$, $\mathbf{Q}'$ be the Killing forms on Lie algebras of $\mathbf{G}$ and $\mathbf{G}'$. It is easy to see that the difference $\kappa(\mathbf{G}, \mathbf{G}') := \{\mathbf{Q}\} - \{\mathbf{Q}'\} \in W_F$ belongs to $I^2$.

For any $\mathbf{G}/\mathbf{Z}_0$-variety $\mathbf{X}$ we denote by $\mathbf{X}'$ the $\mathbf{G}'/\mathbf{Z}_0$-variety obtained from $\mathbf{X}$ by twisting by $\alpha$. By definition, $\mathbf{X}'(F) = \{x \in \mathbf{X}(\bar{F}); \gamma(x) = \alpha(\gamma)(x) \forall \gamma \in \mathrm{Gal}(\bar{F}/F)\}$.

Let $\mathbf{X}/\mathbf{G}$, $\mathbf{X}'/\mathbf{G}'$ be the geometric quotients and $\mathbf{p} : \mathbf{X} \to \mathbf{X}/\mathbf{G}$, $\mathbf{p}' : \mathbf{X}' \to \mathbf{X}'/\mathbf{G}'$ the natural projections. It is clear that the natural isomorphism $\bar{i} : \mathbf{X}(\bar{F}) \to \mathbf{X}'(\bar{F})$ induces a bijection $i_F : (\mathbf{X}/\mathbf{G})(F) \to (\mathbf{X}'/\mathbf{G}')(F)$.

LEMMA 9.1. *There exists an isomorphism* $\mathbf{i} : \mathbf{X}/\mathbf{G} \to \mathbf{X}'/\mathbf{G}'$ *which induces the bijection* $i_F$.

PROOF. Clear.

Let $\mathbf{G}$ be a semi-simple simply connected group over $F, \mathbf{Z}_0$ a central subgroup of $\mathbf{G}, \alpha$ an element in $H^1(F, \mathbf{G}/\mathbf{Z}_0)$ and $\mathbf{G}'$ the $\alpha$-twist of $\mathbf{G}$. Let $\boldsymbol{\rho} : \mathbf{G} \to \mathrm{Aut}(\mathbf{V})$ be a nice representation such that the restriction of $\boldsymbol{\rho}$ to $\mathbf{Z}_0$ is trivial. As before we can define the twisted representation $\boldsymbol{\rho}' : \mathbf{G}' \to \mathrm{Aut}(\mathbf{V}')$. As follows from Lemma 9.1 we can identify the quotient $\mathbf{B}' := \mathbf{V}'/\mathbf{G}'$ with $\mathbf{B} := \mathbf{V}/\mathbf{G}$.

For any $v^\vee \in \mathbf{V}_0^\vee(F)$ we denote by $\mathbf{A}(v^\vee)$ the quadruple $(\mathbf{V}_0, \mathbf{p} : \mathbf{V}_0 \to \mathbf{B}_0, \boldsymbol{\omega}, < *, v^\vee >)$; here $\boldsymbol{\omega}$ is an invariant top-form on $\mathbf{V}$.

Analogously, for any $v'^\vee \in \mathbf{V}_0'^\vee(F)$ we denote by $\mathbf{A}(v'^\vee)$ the quadruple $(\mathbf{V}_0', \mathbf{p}' : \mathbf{V}_0' \to \mathbf{B}_0, \mathbf{p}', \boldsymbol{\omega}', < *, v'^\vee >)$; here $\boldsymbol{\omega}'$ is the twist of $\boldsymbol{\omega}$.

THEOREM 2. *For any $v^\vee \in \mathbf{V}_0{}^\vee(F)$ and $v'^\vee \in \mathbf{V}_0'^\vee(F)$ such that $\mathbf{p}^\vee(v^\vee) = \mathbf{p}'^\vee(v'^\vee)$ we have $\mathbf{A}(v^\vee) \equiv \mathbf{A}(v'^\vee)\eta(\kappa(G, G'))$ (see Def. 6.2).*

It is clear that it is sufficient to prove Theorem 2 in the case when the field $F$ is local. For convenience we denote this local field by $E$. As in the case of the proof of Theorem 1, it is convenient to reformulate the claim in terms of Schwartz-Bruhat functions.

Let $E$ be a local field of characteristic zero, $\mathbf{G}$ a semi-simple simply connected $E$-group, $\mathbf{Z}$ a central subgroup of $\mathbf{G}$, $\boldsymbol{\rho} : \mathbf{G} \to \mathrm{Aut}(\mathbf{V})$ a nice representation of $\mathbf{G}$ trivial on $\mathbf{Z}$ and $\alpha$ an element of $H^1(E, \mathbf{G}/\mathbf{Z})$. We denote by $\boldsymbol{\rho}' : \mathbf{G}' \to \mathrm{Aut}(\mathbf{V}')$ the twisted representation. As before we will identify the quotient $\mathbf{B}' = \mathbf{V}'/\mathbf{G}'$ with $\mathbf{B} = \mathbf{V}/\mathbf{G}$.

Given $f$, $f' \in \mathcal{S}(\mathbf{V}(E))$ we say that $f \sim f'$ if for any $y \in \mathbf{B}_0(E)$ we have $\delta_y(f) = \delta_y'(f')$, where $\delta_y, \delta_y'$ are the functionals on $\mathcal{S}(\mathbf{V}_0(E))$ and $\mathcal{S}(\mathbf{V}_0'(E))$ defined prior to Proposition 7.1. It is clear that the following result is equivalent to Theorem 2.

THEOREM $\mathbf{2}'$. *For any local field $E$ of characteristic zero and any $f \in \mathcal{S}(\mathbf{V}_0(E))$, $f' \in \mathcal{S}(\mathbf{V}_0'(E))$, such that $f \sim f'$, we have*

$$\mathcal{F}(f) \sim \kappa(\mathbf{G}, \mathbf{G}')\mathcal{F}'(f').$$

PROOF. Fix a global field $F$ and a place $\mathfrak{p}_0 \in \mathfrak{P}(F)$ such that $F_{\mathfrak{p}_0} = E$ and the set $\mathfrak{P}_\infty(F)$ of nonarchimedean places contains a point $\mathfrak{p}_\infty \neq \mathfrak{p}_0$. It is easy to show that one can find an $F$-group $\mathbf{G}$, a central subgroup $\mathbf{Z}$ of $\mathbf{G}, \alpha \in H^1(F, \mathbf{G}/\mathbf{Z})$ and a representation $\sigma$ of $\mathbf{G}$ such that $\mathbf{G}_{\mathfrak{p}_0} = \mathbf{G}_E, \mathbf{Z}_{\mathfrak{p}_0} = \mathbf{Z}_E, \boldsymbol{\rho}_{\mathfrak{p}_0} = \boldsymbol{\rho}_E$ and the restriction of $\alpha$ to $H^1(F_{\mathfrak{p}_0}, \mathbf{G}/\mathbf{Z})$ is equal to $\alpha_E$. Let $\mathbf{G}'$ be the $\alpha$-twist of $\mathbf{G}$. We can assume that the groups $\mathbf{G}(\mathfrak{p}_\infty)$ and $\mathbf{G}'(\mathfrak{p}_\infty)$ are compact.

Let $S \subset \mathfrak{P}(F)$ be the set of places $\mathfrak{p}$ such that the restriction of $\alpha$ to $F_\mathfrak{p}$ is not zero. As is well known, $S$ is a finite set. Given $f_S \in \mathcal{S}(\mathbf{V}(\mathbb{A}_S))$ and $f_S' \in \mathcal{S}(\mathbf{V}'(\mathbb{A}_S))$, we write $f_S \sim f_S'$ if for any $y_S \in \mathbf{B}_0(\mathbb{A}_S)$ we have $\delta_{y_S}(f_S) = \delta_{y_S}'(f_S')$.

PROPOSITION 9.1. *Under the conditions of Theorem 2, for any $f_S \in \mathcal{S}(\mathbf{V}_0(\mathbb{A}_S))$, $f_S' \in \mathcal{S}(\mathbf{V}_0'(\mathbb{A}_S))$, such that $f_S \sim f_S'$, we have $\mathcal{F}_S(f_S) \sim \mathcal{F}_S'(f_S')$.*

PROOF. Fix functions $f_S, f'_S$ which satisfy the conditions of the proposition. We have to show that for any $y_S^\vee \in \mathbf{B}_0(\mathbb{A}_S)$ we have $\delta_{y_S^\vee}(\mathcal{F}_S(f_S)) = \delta'_{y_S^\vee}(\mathcal{F}_S(f'_S))$. As in the proof of Theorem 1$''$, we can assume that there exists $y^\vee \in \mathbf{B}_0^\vee(F)$ such that $y_S^\vee$ is equal to the $S$-component of $y^\vee$. Let $R \subset \mathfrak{P}(F)$ and $\phi_R \in \mathcal{S}(\mathbb{A}_R)$ be as in the proof of Theorem 1$''$ and $R' := R \cup S$. Since the restriction of $\alpha$ to $F_\mathfrak{p}$ is zero for all $\mathfrak{p} \in \mathfrak{P} - S$, we can identify the group $\mathbf{G}'(\mathbb{A}^S)$ with $\mathbf{G}(\mathbb{A}^S)$.

As in the proof of Theorem 1$''$, we choose a nonarchimedean place $\mathfrak{p}_1 \in \mathfrak{P} - R'$ and a Schwartz-Bruhat function $f_{\mathfrak{p}_1}$ on $\mathbf{V}(F_{\mathfrak{p}_1})$ such that $\mathrm{supp}(\mathcal{F}_{\mathfrak{p}_1}(f_{\mathfrak{p}_1})) \subset \mathbf{V}_0^\vee(F_{\mathfrak{p}_1})$ and $\delta_{y^\vee_{\mathfrak{p}_1}}(\mathcal{F}_{\mathfrak{p}_1}(f_{\mathfrak{p}_1})) \neq 0$. Let $R'' := R' \cup \mathfrak{p}_1$. For any Schwartz-Bruhat function $h \in \mathcal{S}(\mathbb{A}^{R''})$ define $f_h := f_S \otimes f_{\mathfrak{p}_1} \otimes \phi_R \otimes h$ and $f'_h := f'_S \otimes f_{\mathfrak{p}_1} \otimes \phi_R \otimes h$. By their construction, $f_h, f'_h$ satisfy the conditions of Lemma 8.3 and of Corollary 8.1. So $\Theta(f_h) = \Theta(f'_h)$ and therefore $\Theta(\mathcal{F}(f_h)) = \Theta(\mathcal{F}'(f'_h))$. On the other hand

$$\Theta(\mathcal{F}(f_h)) = \Theta(\mathcal{F}_S(f_S) \otimes \mathcal{F}_{\mathfrak{p}_1}(f_{\mathfrak{p}_1}) \otimes \mathcal{F}_R(f_R) \otimes \mathcal{F}^{R''}(h)) = 0$$

and

$$\Theta(\mathcal{F}'(f'_h)) = \Theta(\mathcal{F}'_S(f'_S) \otimes \mathcal{F}_{\mathfrak{p}_1}(f_{\mathfrak{p}_1}) \otimes \mathcal{F}_R(f_R) \otimes \mathcal{F}^{R''}(h)) = 0$$

for all $h \in \mathcal{S}(\mathbb{A}^{R''})$. The same arguments as in the proof of Theorem 1$''$ show that $\delta_{y_S^\vee}(\mathcal{F}_S(f_S)) = \delta'_{y_S^\vee}(\mathcal{F}_S(f'_S))$. □

Now we can finish the proof of theorem 2$'$. It follows immediately from Proposition 9.1 that there exists $\kappa \in \mathbb{C}^\times$ such that for any pair $f \in \mathcal{S}(\mathbf{V}_0(E))$, $f' \in \mathcal{S}(\mathbf{V}'_0(E))$, such that $f \sim f'$, and for any $y \in \mathbf{B}_0(E)$, we have $\delta_y(\mathcal{F}(f)) = \kappa\delta'_y(\mathcal{F}'(f'))$. If we replace $y$ by $ty$, $t \in E^\times$, $|t| >> 1$, and apply the stationary phase approximation, we find that $\kappa = \kappa(\mathbf{G}_E, \mathbf{G}'_E)$. Theorem 2$'$ is proved. □

## 10. Appendix

Notations:

$K$ is a finite extension of $Q_p$;

$O_K \subset K$ is the ring of integers;

$m \subset O_K$ is the maximal ideal, $k = O_K/m$, $q = \sharp k$;

$A_K$ is an abelian variety over $K$, $n = \dim A$;

$A_0$ is the Neron model of $A_K$ over $O_K$;

$\boldsymbol{\omega}$ is an invariant differential form of degree $n$ on $A_0$, which does not vanish on the special fiber $A_0 \otimes \operatorname{Spec} k$;

$\boldsymbol{\mu}_{\boldsymbol{\omega}}$ denotes the corresponding measure on $A_K(K)$.

What follows is a computation of $\boldsymbol{\mu}_{\boldsymbol{\omega}}(A_K(K))$ in terms of Tate modules $T_l(A_K)$.

The first simple observation (due to A. Weil) is that $\boldsymbol{\mu}_{\boldsymbol{\omega}}(A_K(K)) = N_k q^{-n}$, where $N_k$ denotes the order of the group $A_0(k)$. Consider the exact sequence of group schemes over Spec $k$

$$0 \to A_0^0 \to A_0 \otimes \operatorname{Spec} k \to C \to 0$$

where $A_0^0$ is the connected component of the identity in $A_0 \otimes \operatorname{Spec} k$, and $C$ is a finite smooth group scheme over Spec $k$. Since $H^1(\mathrm{Gal}(\bar{k}/k); A_0^0(\bar{k})) = 0$, it follows that the sequence

$$0 \to A_0^0(k) \to A_0(k) \to C(k) \to 0$$

is exact. Hence $N_k = \sharp A_0^0(k) \cdot \sharp C(k)$.

Let $I \subset \mathrm{Gal}(\overline{K}/K)$ be the inertia subgroup and $J \subset Z[I]$ stand for the augmentation ideal.

Put $a := n - \frac{1}{2} dim(T_l(A_K) \otimes Q)^{J^2}$. We claim that $a$ is an integer. Moreover, one can prove the following.

PROPOSITION 10.1. *$H_c^i(A_0^0; Q_l) = (\bigwedge^{2n+2a-i}(T_l(A_K) \otimes Q)^I(-n-a)$ for $i \geq 2a$ and $H_c^i(A_0^0; Q_l) = 0$ otherwise.*

Therefore, by the Lefschetz formula

$$\sharp A_0^0(k) = q^{n+a} \sum_i (-1)^i \operatorname{Tr}\{\mathrm{Fr}^*; (\bigwedge^i T_l(A_K)^I)\}.$$

The following theorem is essentially due to Grothendieck [**4**].

THEOREM 3. *Let $C(\bar{k})_l$ denote the $l$-part of $C(\bar{k})$. For any $l \neq p$ there is a canonical isomorphism of* $\mathrm{Gal}(\bar{k}/k)$*-modules*

$$C(\bar{k})_l = \operatorname{coker}((T_l(A_K) \otimes Q_l)^I \to (T_l(A_K) \otimes Q_l/Z_l)^I).$$

The case of $l = p$ is more involved. First we suppose that $A_0$ is semi-abelian.

There is a weight filtration on $T_p(A_K)$:

$$W_{-2}T_p(A_K) \subset W_{-1}T_p(A_K) \subset W_0 T_p(A_K) = T_p(A_K),$$

where $W_{-1}T_p(A_K) = \lim \ker(p^n : A_0(O_{\overline{K}}) \to A_0(O_{\overline{K}}))$, $W_{-2}(T_p(A_K))$ is the orthogonal complement to $W_{-1}T_p(A_K^*)$,($A_K^*$ stands for the dual abelian variety). One can prove [**4**] that the restriction of the representation $\mathrm{Gal}(\overline{K}/K)$ of $T_p(A_K)/W_{-1}$ to $\mathrm{Gal}(\overline{K}/L)$, where $L$ is a finite, unramified extension of $K$, is trivial.

Let $B_{ss}$ denote the Fontaine ring [**2**]. Consider the space

$$H = (T_p(A_K) \otimes B_{ss})^{\mathrm{Gal}(\overline{K}/K)}.$$

It is a vector space over Fract $W(k)$ equipped with an action of a nilpotent operator (called the monodromy operator) $N : H \to H$, and Frobenius-linear operator $\phi : H \to H$, satisfying $N\phi = p\phi N$. The operators $N$ and $\phi$ come from the corresponding operators on $B_{ss}$. The weight filtration defines one on $H$. Moreover, its adjoint factors in weights 0 and $-2$ have canonical sublattices over $W(k)$ of maximal ranks (the lattice in weight 0 is isomorphic to $((T_p(A_K)/W_{-1}) \otimes O_{\overline{K}_{nr}})^{\mathrm{Gal}(\bar{K}_{nr}/K)}$, the definition of the other lattice involves the dual variety). We denote these lattices by $L_0$ and $L_{-2}$. One can show that $\phi$ and the monodromy operator $N$ induce corresponding operators on $L_i$: $\phi_i : L_i \to L_i$ $(i = 0; -2)$, $N : L_0 \to L_{-2}$ with $N\phi_0 = p\phi_{-2}N$.

THEOREM 4. *a) Suppose that $A_0$ is semi-abelian. Then there is an isomorphism*

$$C(k)_p \to (\ker[N : L_0 \otimes Q_p/Z_p \to L_{-2} \otimes Q_p/Z_p])^{\phi_0}.$$

*b) Let $L$ be a finite, normal, totally ramified extension of $K$, such that $A_K \otimes \operatorname{Spec} L$ admits a semi-abelian Neron model $A_{0,L}$. Then for any prime $l$, which does not divide degree of the extension $L$ over $K$, there is an isomorphism $C_l \to C_{L;l}$, where $C_{L;l}$ stands for the group of connected components of $A_{0,L}$.*

REMARK 10.1. One can show that for any $A_K$ of dimension $n$ there exists a normal extension $L$ of degree not greater then $\sharp GL_{2n}(\mathbb{Z}/12\mathbb{Z})$ such that $A_K \otimes \operatorname{Spec} L$ has semi-stable reduction.

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