LECTURE 1: REMINDER ON AFFINE HECKE ALGEBRAS

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Abstract. These are notes for a seminar talk at the MIT-Northeastern Spring 2017 Double Affine Hecke Algebras and Elliptic Hall Algebras (DAHAEHA) Seminar.

Contents

1. Goals

The purpose of this talk is to introduce affine and double affine Hecke algebras and certain structural results regarding these algebras. When possible, I will make all definitions and statements for general Cartan types. However, I will emphasize the type A case and use this case as an example throughout, as in the later parts of the seminar we will be primarily concerned with type A.

The main references are Macdonald's book [M] and Kirillov Jr.'s lecture notes [K]. Essentially everything in these notes can be found in those references in a more complete form.

2. Review of Coxeter Groups and Their Hecke Algebras

In this section we will quickly review Coxeter groups and their associated Hecke algebras. All statements and their proofs can be found in (one of) the references [GP] or [H]. We will largely omit proofs in this section, as these results and definitions are standard.

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2.1. **Coxeter Groups.** Let I be a finite set, and let $m : I \times I \to \mathbb{Z}^{\geq 1} \cup \{\infty\}$ be a function satisfying $m(i,j) = m(j,i) \geq 2$ for all $i \neq j \in I$ and $m(i,i) = 1$ for all $i \in I$. One may view this data equivalently as a finite $(\mathbb{Z} \cup \{\infty\})$-labeled undirected graph Γ with vertex set I and edge labels at least 3 (by convention, a missing edge between i and j indicates $m(i,j) = 2$); we refer to Γ as the *Coxeter graph*. Given this data, let W be the group generated by the set

$$\{s_i : i \in I\}$$

with the relations

$$(s_i s_j)^{m(i,j)} = 1$$

(whenever $m(i,j) < \infty$). The groups W appearing in this manner are called *Coxeter groups*, and such a pair (W, I) is called a *Coxeter system*. Note that the generators s_i satisfy $s_i^2 = 1$. This allows the defining relations to be replaced with the *braid relations*

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots$$

for all $i \neq j \in I$ whenever $m(i,j) < \infty$ (with $m(i,j)$ factors on each side) and the *quadratic relations*

$$s_i^2 = 1$$

for all $i \in I$. This presentation will be particularly convenient for us when we consider associated braid groups and Hecke algebras shortly.

Any Coxeter such group W admits a faithful real representation in which the generators s_i act by reflections; using this representation, one can show that for any $i \neq j \in I$ the order of the product $s_i s_j$ in W is precisely $m(i,j)$. In particular, the function $m(i,j)$, and hence the Coxeter graph Γ, describing the relations among the generators, is uniquely recovered from the Coxeter system (W, I). Note, however, that this data is not uniquely recovered from the Coxeter group W as an abstract group, and an abstract group may be a Coxeter group in many different ways. For example, when $n > 4$ is an odd integer, the Coxeter group B_n is isomorphic as an abstract group to the product $A_1 \times D_n$ (take the nontrivial element in A_1 to be $-1 \in B_n$).

Recall that a *finite real reflection group* is a finite subgroup $W \subset GL(V)$ of the general linear group $GL(V)$ of a finite-dimensional real vector space V that is generated by reflections, i.e. by elements $s \in W$ satisfying $\mathrm{rank}(s - 1) = 1$ and $s^2 = 1$. The following characterizes the finite Coxeter groups as the finite (real) reflection groups (with additional structure):

Theorem 2.1.1. *A finite group W is a real reflection group if and only if there is a generating subset $I \subset W$ such that (W, I) is a finite Coxeter system.*

Proof Sketch. This theorem is proved, for example, in Humphrey's book [H]. The idea of proof is as follows. Any Coxeter group admits a faithful representation generated by reflections in a finite-dimensional real vector space, and in particular finite Coxeter groups are finite real reflection groups. Conversely, given a finite real reflection group W with reflection representation V, choose a component $\mathcal{C}$ of the disconnected space

$$V^{reg} := V \setminus \cup_{s \in \mathrm{Ref}(W)} \ker(s - 1),$$

where $\mathrm{Ref}(W) \subset W$ denotes the set of reflections in W with respect to its action on V. We refer to $\mathcal{C}$ as an (open) fundamental chamber for W. Then $\mathcal{C}$ is a simplicial cone with boundary defined by hyperplanes $\ker(s)$ for a certain subset $I \subset \mathrm{Ref}(W)$ of the reflections (the *simple reflections*). Then I generates W, and (W, I) is a Coxeter system. The numbers $m(s, s')$ are obtained as the order of the products ss' for $s, s' \in I$ with $s \neq s'$. These orders

$m(s,s')$ themselves are easily read off by the angle $2\pi/m(s,s')$ between the hyperplanes $\ker(s)$ and $\ker(s')$. $\square$

In particular, it follows from the classification of finite reflection groups that the irreducible finite Coxeter groups coincide with the finite Weyl groups (which come in types A through G) along with the dihedral groups $I_2(m)$ for $m \geq 3$ and the exceptional non-crystallographic Coxeter groups H_3 and H_4.

Again let (W,I) be a Coxeter system, finite or infinite. Any $w \in W$ equals some product $s_1 \cdots s_q$ of simple reflections. Let the *length* $l(w)$ of w be the minimal length of such an expression, and refer to any such minimal expression $w = s_{i_1} \cdots s_{i_{l(w)}}$ as a *reduced expression.* A typical element w of W admits many distinct reduced expressions, and it will be important to us to understand the relationship between these expressions. This question is answered by Matsumoto's Theorem. In particular, let RE denote the set of reduced expressions $\mathbf{s} = (s_{i_1}, ..., s_{i_l})$. Let $\sim_=$ denote the equivalence relation on RE given by $\mathbf{s} \sim_= \mathbf{s}'$ if and only if $s_{i_1} \cdots s_{i_l} = s_{i'_1} \cdots s_{i'_l}$. Let $\sim_{br}$ denote the equivalence relation on RE generated by the braid relations, i.e. by replacing a sequence $(s_{i_1}, s_{i_2}, ...)$ of length $m(i_1,i_2)$ with the sequence $(s_{i_2}, s_{i_1}, ...)$ of length $m(i_1,i_2)$. We then have:

Theorem 2.1.2 (Matsumoto)**.** $\sim_= \, = \, \sim_{br}$.

In other words, any two reduced expressions for the same element $w \in W$ are connected by a sequence of braid relations. The proof can be found in [L].

2.2. **Braid Groups.** Let (W,I) be a Coxeter system. The *braid group* B_W associated to (W,I) is the group generated by the set $\{T_i : i \in I\}$ subject to the *braid relations*

$$T_i T_j \cdots = T_j T_i \cdots$$

($m(i,j)$ factors on each side) for $i \neq j \in I$ with $m(i,j) \neq \infty$. In other words, the braid group B_W has a description by generators and relations identical to that of W except that the quadratic relations $s_i^2 = 1$ are omitted.

For any $w \in W$ and reduced expression $w = s_{i_1} \cdots s_{l(w)}$, it follows from the definition of the braid group and Matsumoto's theorem that the product $T_{i_1} \cdots T_{i_{l(w)}}$ is independent of the choice of reduced expression for w. We denote any such product by the symbol T_w. As $T_{i_1} = T_{s_{i_1}}$, it follows that the set $\{T_w : w \in W\}$ generates B_W, and it is immediate that the relation

$$T_w T_{w'} = T_{ww'} \quad \text{whenever} \quad l(ww') = l(w) + l(w')$$

holds in B_W. It is easy to see that this gives another presentation for B_W. Similarly, we may give a presentation for B_W by specifying generators $\{T_w : w \in W\}$ with relations

$$T_{s_i} T_w = T_{s_i w} \quad \text{whenever} \quad l(s_i w) > l(w)$$

(or the analogous "right handed" relations, or both types of relations simultaneously).

Clearly, there is a surjection $B_W \to W$ sending T_i to s_i, with kernel generated by the elements T_i^2. The kernel P_W is the *pure braid group.*

Remark 2.2.1. *When W is finite, so W is a finite real reflection group with reflection representation V, the braid group has a standard topological interpretation. In particular, let $V_{\mathbb{C}}$ denote the complexification of V, and let $V_{\mathbb{C}}^{reg}$ denote the set of points in $V_{\mathbb{C}}$ with trivial stabilizer in W (i.e., the complement of the reflection hyperplanes). Then there are identifications $\pi_1(V_{\mathbb{C}}^{reg}/W) = B_W$ and $\pi_1(V_{\mathbb{C}}^{reg}) = P_W$ compatible with the obvious short exact sequences.*

Example : **Type A** The symmetric group S_n on n letters is a real reflection group with respect to its standard representation by coordinate permutations in $\mathbb{R}^n$ - the transposition (i, j) is given by reflection through the hyperplane $x_i = x_j$. This representation is faithful but not irreducible - the space $\{x : \sum_i x_i = 0\}$ is the *irreducible reflection representation* for S_n. A set of simple reflections can be given by the adjacent transpositions $(i, i+1)$ for $1 \leq i < n$, and the corresponding Coxeter graph is a line of $n-1$ connected dots (by convention an unlabeled connection indicates $m = 3$), and this type of reflection group is said to be of type A_{n-1}. When $W = S_n$, the braid group B_{S_n} is the familiar standard braid group B_n on n strands, and the pure braid group P_{S_n} is the standard pure braid group on n strands.

2.3. **Hecke Algebras.** In this section we will recall certain deformations of group algebras of Coxeter groups, the *Hecke algebras*. Let (W, I) be a Coxeter system. Let $\tau : I \to \mathbb{C}^\times$ be a function such that $\tau(i) = \tau(j)$ whenever s_i and s_j are conjugate in W, and write $\tau_i = \tau_{s_i} = \tau(i)$. The *Hecke algebra* $H_\tau(W, I)$ attached to the Coxeter system (W, I) and *parameter* τ is the $\mathbb{C}$-algebra with generators $\{T_i : i \in I\}$ and relations consisting of the braid relations seen above and the *Hecke relations* (or *quadratic relations*)

$$(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0$$

for all $i \in I$. We will write $H_\tau(W)$ rather than $H_\tau(W, I)$ when the meaning is clear.

It is immediate from the Hecke relations that the generators T_i are invertible and that the Hecke relations can be equivalently written

$$T_i - T_i^{-1} = \tau_i - \tau_i^{-1}.$$

In particular, there is a natural surjection

$$\mathbb{C}B_W \to H_\tau(W), \quad T_i \mapsto T_i.$$

It follows that the Hecke algebra $H_\tau(W)$ has an alternative description as the quotient of the complex group algebra $\mathbb{C}B_W$ by the Hecke relations. We use the notation T_w for $w \in W$ to denote both elements of the braid group and their images in $H_\tau(W)$ when the meaning is clear.

When τ is the constant function 1, the Hecke relations read

$$T_i^2 = 1,$$

and in particular the Hecke algebra $H_1(W)$ is identified with the group algebra $\mathbb{C}W$ of W. In this way, the family of algebras $H_\tau(W)$ form a deformation of $\mathbb{C}W$. As it happens, this deformation is flat:

Theorem 2.3.1. *The set $\{T_w : w \in W\}$ forms a $\mathbb{C}$-basis of $H_\tau(W)$.*

The proof is standard. In particular, it is clear that the elements T_w span $H_\tau(W)$ because the span of the T_w contains 1, is stable under multiplication by the T_i, and the T_i generate $H_\tau(W)$. So, what one needs to do is to prove linear independence. This is achieved by the standard trick of writing down a representation of $H_\tau(W)$ in a space in which the linear operators by which the T_w act are manifestly linearly independent. In this case, one uses the regular representation as a model. In particular, one considers the $\mathbb{C}$-vector space H' with basis $\{e_w : w \in W\}$ and tries to define a representation of $H_\tau(W)$ in this space by letting the generator T_i act by

$$T_i(e_w) = \begin{cases} e_{s_i w} & \text{if } l(s_i w) > l(w) \\ e_{s_i w} + (\tau_i - \tau_i^{-1})e_w & \text{if } l(s_i w) < l(w). \end{cases}$$

It is obvious that the T_i satisfy the Hecke relations, so to see that this defines a representation of $H_\tau(W)$ one need only check the braid relations. For this, one considers the operators on H' that should correspond to right multiplication by T_i, and checks that the "left multiplication" operators commute with the "right multiplication" operators, reducing the check of the braid relations to the check that the braid relations hold when applied to the element e_1, which is obviously true. Details can be found, for example, in Humphrey's book [H].

Remark 2.3.2. *The same proof shows that when the parameter τ is viewed as a formal invertible variable(s), the Hecke algebra $H_\tau(W)$ is a free $\mathbb{C}[\tau_i^{\pm 1}]$-module with basis $\{T_w : w \in W\}$. The case of numeric τ is then obtained by specialization to $\mathbb{C}$.*

Remark 2.3.3. *In some other contexts, the presentation/definition of Hecke algebras attached to (W, I) looks slightly different in that the Hecke relation seen above. Specifically, the Hecke relation may be of the form $(T+1)(T-q) = 0$ (as one sees, for example, in the context of Hecke algebras attached to BN pairs) or $(T-1)(T+q)$ (as one sees, for example, in the context of the KZ functor appearing for rational Cherednik algebras). These two forms involving q are easily reconciled by a rescaling of the generators T_i (notice that the braid relations are homogenous). The version seen above with τ (often the letter v is used instead) amounts to choosing a square root of q and rescaling the generators, which is important from some representation theoretic perspective that we won't discuss here.*

3. Affine Hecke Algebras

3.1. **Affine Root Systems.** We will assume the reader is familiar with finite root systems. For the rest of this talk, any finite Coxeter group appearing will be a Weyl group, i.e. a crystallographic real reflection group (i.e., types H and I are excluded). Similarly, all finite root systems appearing will also be crystallographic. All definitions and results in this section can be found in [M, Chapter 1]. Also, the notion of "affine root system" I will use here is in the sense appearing in [M]; the roots appearing in these affine root systems are the real roots of the affine root systems discussed in the context of Kac-Moody Lie algebras.

Affine root systems are related and similar in spirit to finite root systems. The essential differences are that in the affine case there are infinitely many affine roots and an affine root determines an affine reflection, i.e. a reflection through an affine hyperplane, while in the finite case there are finitely many roots and a root determines an honest reflection through a linear hyperplane. The standard theory of finite root systems has an analogue for affine root systems. Specifically, one has an axiomatic definition and concise classification involving diagrams closely related to Dynkin diagrams, notions of affine Weyl groups with length functions, convenient fundamental domains (now called *alcoves* rather than *Weyl chambers*), etc. For the sake of concreteness, simplifying the notation, and with the goals of this seminar in mind, we will not consider arbitrary affine root systems, but rather only certain affine root systems R^a that are easily associated to finite irreducible reduced root systems R. This is not a significant restriction, and in fact the classification of arbitrary affine root systems is easily stated in terms of these affine root systems R^a and some mild additional constructions (see [M, Chapter 1, Section 3]). For the full-fledged general case of all the material to follow in this talk, see Macdonald's book [M].

3.1.1. *Affine functions, affine reflections, and translations.* Fix a Euclidean vector space V with inner product $(\cdot,\cdot)$. Identify V with its dual V^* via the inner product. Let F be the set of affine-linear functions on V, i.e. the functions $f : V \to \mathbb{R}$ that are sums of linear functionals and constant functions. Then $F = V \oplus \mathbb{R}\delta$, where $\delta \in F$ is the constant function with value 1. For any $f \in F$, let $Df \in V$ denote the projection of f to V under the splitting $F = V \oplus \mathbb{R}\delta$ (then Df is the gradient of f in the usual sense of calculus, and $f = Df + f(0)$). Extend the inner product $(\cdot,\cdot)$ to F by defining

$$(v + c\delta, w + d\delta) = (v, w)$$

for all $v, w \in V$ and $c, d \in \mathbb{R}$. On F, the form $(\cdot,\cdot)$ is a degenerate symmetric bilinear form with kernel $\mathbb{R}\delta$.

Let $f \in F$ be non-constant. Then $(f, f) > 0$, and we define

$$f^\vee := \frac{2f}{(f,f)}.$$

The subset $f^{-1}(0) \subset V$ is an affine hyperplane, and we denote by s_f the orthogonal reflection in V through this affine hyperplane. The affine reflection s_f is given by a familiar formula:

$$s_f(x) = x - f^\vee(x)Df = x - f(x)Df^\vee.$$

Then s_f also acts on functions g on V by the usual formula $s_f.g = g \circ s_f^{-1} = g \circ s_f$, and clearly this action preserves the space F. This action of s_f is given by the familiar formula

$$s_f(g) = g - (g, f^\vee)f = f - (g, f)f^\vee.$$

Naturally, a translation $t : V \to V$ is an affine linear transformation of V of the form $t(x) = x + v$, for some $v \in V$; we denote this translation by $t(v)$. For a subset $X \subset V$, we define $t(X) := \{t(v) : v \in X\}$. When $L \subset V$ is a lattice, $t(L)$ is a lattice isomorphic to L. Any translation $t(v)$ also acts on the space F of affine-linear functions on V:

$$t(v)(f) = f - (v, f)\delta.$$

3.1.2. *The affine root systems R^a.* Fix a finite irreducible reduced root system $R \subset V$ spanning V (so R has rank $\dim V$). As usual, let $Q := \sum_{\alpha\in R}\mathbb{Z}\alpha$ denote the root lattice, let $Q^\vee := \sum_{\alpha\in R}\mathbb{Z}\alpha^\vee$ denote the coroot lattice, let $P \subset V$ be the weight lattice (i.e. those $\lambda \in V$ with integral pairing with all coroots), and let $P^\vee$ be the coroot lattice (i.e. those $\lambda \in V$ with integral pairing with all roots.

Define the associated *affine root system* R^a to be the subset of F given by:

$$R^a := \{\alpha + n\delta : \alpha \in R, n \in \mathbb{Z}\}.$$

We call elements $a \in R^a$ *affine roots*. Note that R is a subset of R^a. Let W denote the Weyl group attached to R, i.e. the subgroup of $GL(V)$ generated by the reflections $\{s_\alpha : \alpha \in R\}$. Similarly, let W^a, the *affine Weyl group*, be the group of invertible affine transformations of V generated by the s_a for $a \in R^a$. Clearly, $W \subset W^a$.

Proposition 3.1.1. *The lattice $t(Q^\vee)$ is a normal subgroup of W^a and $W^a = W \ltimes t(Q^\vee)$.*

Proof. Let $a = \alpha + n\delta$ be an affine root. Then we have

$$s_\alpha s_a(x) = s_\alpha(x - ((x,\alpha) + n)\alpha^\vee) = x - (x,\alpha)\alpha^\vee + ((x,\alpha) + n)\alpha^\vee = x + n\alpha^\vee$$

so $s_a s_\alpha = t(n\alpha^\vee)$. It follows that $t(Q^\vee)$ is a subgroup of W^a. We also see that $s_a = t(n\alpha^\vee)s_\alpha$, so W^a is generated by W and $t(Q^\vee)$. It's also clear that for any $w \in W$ and $\lambda \in Q^\vee$, we have $wt(\lambda)w^{-1} = t(w\lambda)$, so $t(Q^\vee)$ is normal in W^a and $W^a = W.t(Q^\vee)$. As W fixes $0 \in V$, it follows that $W \cap t(Q^\vee) = 1$ and the claim follows. □

It's now easy to see the following:

Proposition 3.1.2.
(1) R^a spans F.
(2) $s_a(b) \in R^a$ for all $a, b \in R^a$.
(3) $(a^\vee, b) \in \mathbb{Z}$ for all $a, b \in R^a$.
(4) the action of W^a on V is proper, i.e. for any compact subset $K \subset V$ the set of $w \in W^a$ such that $wK \cap K \neq \emptyset$ is finite.

Remark 3.1.3. *The preceding proposition says that R^a is indeed an affine root system in the axiomatic sense.*

3.1.3. *Alcoves and positive and simple roots.* Let $\alpha_1, ..., \alpha_n$ be a choice of simple positive roots for the finite root system R, determining a Weyl chamber

$$\mathcal{C} := \{x \in V : \alpha_i(x) \geq 0 \text{ for } 1 \leq i \leq n\}.$$

Recall that the Weyl group W of R is then a Coxeter group with respect to the corresponding simple reflections through the walls of $\mathcal{C}$. Recall also that the set of *positive roots* $R_+ \subset R$ is given by

$$R_+ := \{\alpha \in R : \alpha(x) \geq 0 \text{ for all } x \in \mathcal{C}\},$$

the negative roots are $R_- := -R_+$, and that every positive root is a linear combination of positive simple roots with nonnegative integer coefficients. There is a very similar story for the affine Weyl group W^a that we now explain.

The set of affine hyperplanes $\{a^{-1}(0) : a \in R^a\}$ is a locally finite arrangement of real hyperplanes in V, and it follows that the complement is open and has a natural W^a-action. A connected component of this complement is called an *alcove*, and we denote the set of alcoves by $\mathcal{A}$. Let A denote the closure of the unique alcove A° contained in $\mathcal{C}$ and such that $0 \in \overline{A^\circ}$. We call A an *affine Weyl chamber* for R^a. Clearly, A is a n-dimensional simplex

$$A = \{x \in V : a_i(x) \geq 0 \text{ for } 0 \leq i \leq n\}$$

with $n + 1$ walls given by affine hyperplanes $\{a_i^{-1}(0) : 0 \leq i \leq n\}$ for some uniquely determined affine roots $a_0, a_1, ..., a_n$. We call the a_i the *simple affine roots*, or just *simple roots* when the meaning is clear. Up to reordering, we have $a_i = \alpha_i$ for $1 \leq i \leq n$ (corresponding to the walls that A shares with $\mathcal{C}$) and a root a_0 with nonzero constant term that defines the remaining wall of A. Let I be the set $\{0, ..., n\}$ and let I_0 be the set $I \backslash \{0\}$. For $0 \leq i \leq n$, define the i^{th} *simple reflection* by $s_i := s_{a_i}$. Sometimes by abuse of notation I'll confuse $i \in I$ with s_i.

Let's fix some terminology and notation. Define the *positive (affine) roots* $R^a_+ \subset R^a$ by

$$R^a_+ := \{a \in S : a(x) \geq 0 \text{ for all } x \in A\}$$

and define the *negative (affine) roots* R^a_- by $R^a_- := -R^a_+$.

Let $\theta \in R^+$ denote the highest root of the finite root system R. For $i \in I_0$, let $m_i \in \mathbb{Z}^{>0}$ denote the unique positive integers such that

$$\theta = \sum_{i \in I_0} m_i \alpha_i.$$

Proposition 3.1.4. $a_0 = -\theta + \delta$.

Proof. Certainly $a_0 = \alpha + n\delta$ for some $\alpha \in R$ and $n \in \mathbb{Z}$. As $0 \in A \backslash a_0^{-1}(0)$, we have $a_0(0) > 0$, so $n > 0$. Certainly for any $n > 0$ the simplex A' defined by any $\alpha + n\delta$ along with $\alpha_1, ..., \alpha_n$ contains A, so we have $n = 1$. Any root α can be written $\alpha = \sum_i k_i \alpha_i$, and we have $k_i \geq -m_i$ for all i. As the 1-dimensional faces of $\mathcal{C}$ are given by $\mathbb{R}^{\geq 0}\lambda_i$, where the λ_i are the fundamental weights, it is clear that simplex determined by $-\theta + \delta$ and the α_i is contained in the simplex determined by any $\alpha + \delta$ and the α_i, and the claim follows. □

Corollary 3.1.5. *A has the alternative, slightly more concrete description:*

$$A = \{x \in V : (\alpha_i, x) \geq 0 \textit{ for } i \in I_0 \textit{ and } (x, \theta) \leq 1\}.$$

We can now give a convenient description of the positive affine roots and see that the simple positive affine roots give a basis for R^a in the same familiar way that the simple positive roots give a basis for R:

Corollary 3.1.6.
(1) R^a_+ has the following description:

$$R^a_+ = \{\alpha + r\delta : \alpha \in R, r \geq \chi(\alpha)\}$$

where χ is the indicator function on R of the subset $R^- \subset R$ of negative roots.
(2) $R^a = R^a_+ \coprod R^a_-$ and every positive affine root $a \in R^a_+$ is of the form

$$a = \sum_{i \in I} n_i a_i$$

for some non-negative integers n_i.

Proof. (1) follows easily from the previous corollary (in particular, note that any root $\alpha \in R$ is positive if and only if $\alpha(x) \in [0, 1]$ for all $x \in A$). (2) follows from (1) and the fact that for any root $\alpha \in R$ the difference $\theta - \alpha$ is a sum of positive simple roots α_i with nonnegative coefficients. □

3.1.4. *W^a as a Coxeter group, and its length function.* Proofs of the following two propositions can be found in [H, Chapter 4].

Proposition 3.1.7.
(1) The simple affine reflections $s_0, ..., s_n$ generate W^a, and in fact the pairs (W^a, I) and (W, I_0) are Coxeter systems. As usual, the entries $m(i, j)$ of the Coxeter matrix are read off from the relative angles of the affine hyperplanes $a_i^{-1}(0)$ (equivalently, from the pairings $(a_i^\vee, a_j)$).
(2) W^a acts simply transitively on the set $\mathcal{A}$ of alcoves, and A is a fundamental domain for the action of W^a on V.

Let $l : W^a \to \mathbb{Z}^{\geq 0}$ be the length function on W^a as a Coxeter group with generating simple reflections $s_0, ..., s_n$.

Proposition 3.1.8. *For any $w \in W^a$, the length $l(w)$ can be equivalently described as:*
(a) The length of a reduced expression for w.
(b) $l(w) = |R^a_+ \cap w^{-1}R^a_-|$
(c) The number of affine hyperplanes $a^{-1}(0)$ with $a \in R^a_+$ separating A and wA.

3.2. **Extended Affine Weyl Groups.** In light of the decomposition $W^a = W \ltimes t(Q^\vee)$, we can enlarge the group W^a by replacing the coroot lattice $Q^\vee$ with a larger lattice L' on which W acts.

Definition 3.2.1. *The* extended affine Weyl group W^{ae} *attached to R^a is the semidirect product*

$$W^{ae} := W \ltimes t(P^\vee).$$

Clearly, W^{ae} admits a natural action on V extending that of W^a.

Proposition 3.2.2. *The extended affine Weyl group W^{ae} acts on the affine roots R^a.*

Proof. For a coweight $\lambda \in P^\vee$, the translation $t(\lambda)$ acts on F by

$$t(\lambda)(a) = a - (\lambda, a)\delta.$$

Any $\lambda \in P^\vee$ has integral pairing with any $a \in R^a$, and the claim follows. □

So it follows as well that W^{ae} acts on the set of affine hyperplanes and the set of alcoves $\mathcal{A}$ etc., but the action on the alcoves is no longer faithful.

Now we want to relate the extended affine Weyl group W^{ae} to the affine Weyl group W^a:

Definition 3.2.3. *Extend the definition of the length function l from the affine Weyl group W^a to the extended affine Weyl group W^{ae} by either of the two equivalent definitions (b) or (c) appearing in Proposition 3.1.8*

Definition 3.2.4. *Let Ω be the finite group*

$$\Omega := \{w \in W^{ae} : l(w) = 0\} = \{w \in W^{ae} : wA = A\}.$$

Definition 3.2.5. *Recall that a weight $\lambda \in P$ is called* miniscule *if $0 \leq (\lambda, \alpha^\vee) \leq 1$ for* every *positive root $\alpha \in R^+$. Similarly, recall that a coweight $\lambda' \in P^\vee$ is called* miniscule *if $0 \leq (\lambda', \alpha) \leq 1$ for every positive root $\alpha \in R^+$.*

Recall that the miniscule weights form a system of representatives for P/Q, just as the miniscule coweights do for $P^\vee/Q^\vee$.

Proposition 3.2.6.
(1) $W^{ae} = \Omega \ltimes W^a$.
(2) $\Omega \cong P^\vee/Q^\vee$. In particular, every $\pi_r \in \Omega$ is of the form

$$\pi_r = t(b_r)w_r$$

for some miniscule coweight $b_r \in P^\vee$ and $w_r \in W^a$.

Proof. (1) follows immediately from the facts that W^a acts simply transitively on the set of alcoves and that Ω is the set-wise stabilizer of the alcove A. (2) follows from the semidirect product decomposition/definition of W^a and W^{ae} and the fact that the minscule coweights form a system of representatives for $P^\vee/Q^\vee$. □

Remark 3.2.7. *In fact, in (2) above one has $w_r \in W$ (see* [M, Chapter 2, Section 5]*).*

Remark 3.2.8. *The faithful action of Ω on the alcove A gives rise to a faithful action of Ω on the set of walls of A, and therefore on the set of simple roots. If $\pi_r(a_i) = a_j$, then $\pi_r s_i \pi_r^{-1} = s_j$. This describes the semidirect product appearing in (1) concretely, and we see that the action of Ω on W^a is by diagram automorphisms.*

Let P_+ denote the dominant weights (i.e. those $\lambda \in P$ with $(\lambda, \alpha_i^\vee) \geq 0$ for all $i \in I_0$) and let $P_+^\vee$ denote the dominant coweights (i.e. those $\lambda' \in P^\vee$ with $(\lambda', \alpha_i) \geq 0$ for all $i \in I_0$). As usual, let $\rho \in P_+$ denote the half sum of the positive roots.

We have the following facts about the length function l on the extended affine Weyl group W^{ae}:

Proposition 3.2.9.

(1) The restriction of the length function on W^{ae} to W^a coincides with the usual length function, and $l(\pi w) = l(w\pi) = l(w)$ for all $\pi \in \Omega$, $w \in W^{ae}$.

(2) $l(ws_i) = \begin{cases} l(w) + 1 & \text{if } w(a_i) \in R_+^a \\ l(w) - 1 & \text{if } w(a_i) \in R_-^a \end{cases}$

(3) If $w \in W^{ae}$ an $\lambda \in P^\vee$, then

$$l(wt(\lambda)) = \sum_{\alpha \in R^+} |(\lambda, \alpha) + \chi(w\alpha)|$$

where χ is the indicator function of the negative roots $R_- \subset R$.

Proof. (1) is immediate from the definition and (2) follows from (1), the definition of l, and the fact that s_i permutes the set $R_+^a \backslash \{a_i\}$. A proof of (3) can be found in [M, Chapter 2] - it is not difficult and it makes use of the description

$$R_+^a = \{\alpha + r\delta : \alpha \in R, r \geq \chi(\alpha)\}.$$

□

The translation elements $t(\lambda) \in W^{ae}$ will be of particular relevance in what follows, so we record some facts about these elements and the length function in the following corollary:

Corollary 3.2.10.

(1) If $\lambda \in P^\vee$, then $l(t(\lambda)) = 2(\lambda^+, \rho)$, where λ^+ is the dominant coweight lying in the W-orbit of λ.

(2) If $\lambda \in P_+^\vee$, then $l(wt(\lambda)) = l(w) + l(t(\lambda))$.

(3) If $(\lambda, \alpha_i) = 0$ for any $i \in I_0$, then $l(t(\lambda)s_i) = l(s_i t(\lambda)) = l(t(\lambda)) + 1$

(4) If $(\lambda, \alpha_i) = -1$, then $l(s_i t(\lambda)) = l(t(\lambda)) - 1$.

Proof. Follows immediately from Proposition 3.2.9(3). □

3.3. **Affine and Extended Affine Braid Groups.**

Definition 3.3.1. *The* affine braid group *B^a attached to the affine Weyl group W^a is the braid group attached to the Coxeter system (W^a, I). The* extended affine braid group *B^{ae} attached to the extended affine Weyl group W^{ae} has exactly the same description as B^a except with W^{ae} and its length function in place of W^a. In particular, B^{ae} has generators T_w for $w \in W^{ae}$ and relations $T_w T_w' = T_{ww'}$ whenever $l(ww') = l(w) + l(w')$.*

Remark 3.3.2. *Like for finite-type braid groups, the affine braid group B^a has a topological interpretation as the fundamental group $\pi_1(V_{\mathbb{C}}^{reg}/W^a)$ where*

$$V_{\mathbb{C}}^{reg} = (\mathbb{C} \otimes_{\mathbb{R}} V) \backslash \cup_{a \in R^a} (1 \otimes a)^{-1}(0)$$

(this quotient is sensible because the action of W^a on V is proper). In light of the decomposition $W^a = W \ltimes t(Q^\vee)$, it is easy to see that $V_{\mathbb{C}}^{reg}/W^a$ can also be described as T^{reg}/W, where $T = V_{\mathbb{C}}^{reg}/t(Q^\vee)$ is the complement of the corresponding hypersurfaces in the complex torus $V_{\mathbb{C}}/t(Q^\vee)$.

Theorem 3.3.3. *The elements T_π with $\pi \in \Omega$ form a subgroup of B^{ae} isomorphic to Ω, and we have*

$$B^{ae} = \Omega \ltimes B^a$$

where the action of Ω on B^a is by the same diagram automorphisms discussed earlier, i.e. if $\pi_r(a_i) = a_j$ then $\pi_r T_i \pi_r^{-1} = T_j$, where T_i denotes T_{s_i} and π_r denotes T_{π_r}.

Proof. This is easy to see from our previous discussion of braid groups and the fact that $\pi_r s_i = s_j \pi_r$ and that $l(\pi_r s_i) = 1 = l(s_i)$ etc. □

Along with the description of B^a by generators and relations from our discussion of Coxeter groups, this gives a more tractable description of B^{ae} by generators and relations.

Recall that the affine and extended affine Weyl groups had the semidirect product decompositions

$$W^a = W \ltimes t(Q^\vee) \qquad W^{ae} = W \ltimes t(P^\vee).$$

We now will upgrade this decomposition to the affine and extended affine braid groups, which will allow us to do the same for the affine Hecke algebras to come.

Definition 3.3.4. *For $\lambda \in P^\vee$, define elements $Y^\lambda \in B^{ae}$ by*
(1) $Y^\lambda = T_{t(\lambda)}$ if $\lambda \in P_+^\vee$
(2) $Y^\lambda = Y^\mu (Y^\nu)^{-1}$ if $\lambda = \mu - \nu$ with $\mu, \nu \in P_+^\vee$.

Theorem 3.3.5. *Y^λ is well-defined for all $\lambda \in P^\vee$, and we have $Y^\lambda Y^\mu = Y^{\lambda+\mu}$ for all $\lambda, \mu \in P^\vee$. The mapping $\lambda \mapsto Y^\lambda$ determines a lattice isomorphism between $P^\vee$ and the subgroup $\{Y^\lambda : \lambda \in P^\vee\}$ of B^{ae}.*

Proof. Clearly the Y^λ are well-defined for $\lambda \in P_+^\vee$ and satisfy $Y^\lambda Y^\mu = Y^{\lambda+\mu}$. It follows immediately that the Y^λ are well-defined for all $\lambda \in P^\vee$ and satisfy $Y^\lambda Y^\mu = Y^{\lambda+\mu}$. The restriction of the natural surjection $B^{ae} \to W^{ae}$ gives the inverse lattice isomorphism, completing the proof.

□

We will now describe the commutation relations between elements Y^λ and T_i. Note that as the lattice $\{Y^\lambda : \lambda \in P^\vee\}$ is generated by the elements Y^{ω_i}, where the $\omega_i \in P_+^\vee$ are the fundamental dominant coweights, it suffices to explain the commutation relations between Y^λ and T_i in the case that $(\alpha_i, \lambda) \in \{0, 1\}$.

Theorem 3.3.6.
(1) The elements Y^λ, $\lambda \in P^\vee$, and the elements $T_1, ..., T_n$ together generate B^{ae} as a group.
(2) If $(\lambda, \alpha_i) = 0$ for some $i \in I_0$, then $T_i Y^\lambda = Y^\lambda T_i$.
(3) If $(\lambda, \alpha_i) = 1$ for some $i \in I_0$, then $Y^\lambda = T_i Y^{s_i \lambda} T_i$.

Proof. For statement (1), in view of the decomposition $B^{ae} = \Omega \ltimes B^a$, it follows that the elements Y^λ together with the $T_0, ..., T_n$ generate B^{ae}, so we need only understand why generator T_0 is redundant in this collection. But this follows from the equality $s_0 s_\theta = t(\theta^\vee)$ in W^a. Indeed, note that

$$s_\theta(a_0) = s_\theta(-\theta + \delta) = \theta + \delta \in R_+^a$$

so the expression $s_0 \cdot s_\theta$ is reduced (i.e. $l(s_0 s_\theta) = l(s_0) + l(s_\theta)$). It follows that we have an equality

$$T_0 T_{s_\theta} = T_{t(\theta^\vee)} = Y^{\theta^\vee}$$

in the affine braid group.

In the case $\lambda \in P_+^\vee$, (2) follows from Corollary 3.2.10 which says that $l(t(\lambda)s_i) = l(s_i t(\lambda)) = l(t(\lambda)) + 1$. This then gives

$$T_i Y^\lambda = T_i T_{t(\lambda)} = T_{s_i t(\lambda)} = T_{t(\lambda)s_i} = Y^\lambda T_i$$

as needed. The general case then follows from the fact that any $\lambda \in P^\vee$ with $(\lambda, \alpha_i) = 0$ can be written as $\lambda = \mu - \nu$ with $\mu, \nu \in P_+^\vee$ and $(\mu, \alpha_i) = (\nu, \alpha_i) = 0$.

For case (3), we can again reduce to the case $\lambda \in P_+^\vee$ by noticing that any $\lambda \in P^\vee$ satisfying $(\lambda, \alpha_i) = 1$ can be written as $\lambda = \mu - \nu$ for some $\mu, \nu \in P_+^\vee$ satisfying $(\mu, \alpha_i) = 1$ and $(\nu, \alpha_i) = 0$. So take any $\lambda \in P_+^\vee$ satisfying $(\lambda, \alpha_i) = 1$. Define another element

$$\pi := \lambda + s_i \lambda = 2\lambda - \alpha_i^\vee.$$

Note that $\pi \in P_+^\vee$. From Corollary 3.2.10, we know that if $l(t(\lambda)) = 2(\lambda, \rho) = p$ then $l(t(\pi)) = 2p - 2$. From the same corollary (statement (3)) we also know that $l(s_i t(\pi)) = l(t(\pi)) + 1 = 2p - 1$, and from statement (4) we know that $l(t(\lambda)s_i) = l(t(\lambda)) - 1 = p - 1$. It follows that each side of the equality

$$s_i \cdot t(\pi) = (t(\lambda)s_i) \cdot t(\lambda)$$

is a reduced expression, and therefore that

$$T_i Y^\pi = T_{t(\lambda)s_i} Y^\lambda.$$

But $Y^\pi = Y^{s_i \lambda} Y^\lambda$ and $T_{t(\lambda)s_i} = T_{t(\lambda)} T_i^{-1} = Y^\lambda T_i^{-1}$. Rearranging the above equality, we get

$$T_i Y^{s_i \lambda} T_i = Y^\lambda$$

as needed. □

We can now state a presentation for B^{ae} analogous to the decomposition $W^{ae} = W \ltimes t(P^\vee)$ of the extended affine Weyl group W^{ae} seen earlier.

Theorem 3.3.7. *The extended affine braid group B^{ae} is generated by the finite-type braid group B (generated by $T_1, ..., T_n$) and the lattice $Y^{P^\vee}$ subject only to the relations appearing in (2) and (3) of the previous Theorem 3.3.6.*

The proof can be found in [M, Section 3.3]. The idea is to define elements T_0 and U_i in the group described by generators and relations in the theorem, and then to show by calculation that the relations of the extended affine braid group hold.

3.4. **Affine Hecke Algebras.** We now define affine Hecke algebras as quotients of group algebras of extended affine braid groups, in complete analogy with the Coxeter case seen earlier:

Definition 3.4.1. *Let $\{\tau_a\}_{a \in R^a}$ be a collection of complex numbers indexed by the affine roots such that $\tau_a = \tau_b$ whenever s_a and s_b are conjugate in W^{ae}. Write $\tau_i = \tau_{a_i}$ for the simple affine roots $a_0, ..., a_n$. The* affine Hecke algebra $H_\tau(W^{ae})$ *and parameter τ is the quotient of the group algebra $\mathbb{C}[B^{ae}]$ by the relations*

$$(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0, \quad i = 0, ..., n.$$

Remark 3.4.2. *One could just as well consider the τ_a to be formal invertible commuting variables, in which case the algebra defined above arises as a specialization.*

When the meaning is clear, we will use the notation T_w for $w \in W^{ae}$ and Y^λ for $\lambda \in P^\vee$ to denote both elements of the extended affine braid group B^{ae} and their images in $H_\tau(W^{ae})$.

There is a nice description of the interaction of generators T_i with elements Y^λ in $H_\tau(W^{ae})$ that follows from Theorem 3.3.6:

Lemma 3.4.3. *For any $i = 1, ..., n$ and $\lambda \in P^\vee$, the following relation holds:*

$$T_i Y^\lambda - Y^{s_i(\lambda)} T_i = (\tau_i - \tau_i^{-1}) \frac{s_i - 1}{Y^{-\alpha_i^\vee} - 1} Y^\lambda := (\tau_i - \tau_i^{-1}) \frac{Y^{s_i\lambda} - Y^\lambda}{Y^{-\alpha_i^\vee} - 1}.$$

Proof of Lemma 3.4.3. The key observation is that if the relation holds for Y^λ, Y^μ, then it also holds for $Y^{-\lambda}$ and $Y^{\lambda+\mu}$, which follows from a straightforward calculation. So, it suffices to establish the relation when $(\lambda, \alpha_i) = 1$ and when $(\lambda, \alpha_i) = 0$. In the case $(\lambda, \alpha_i) = 0$, the relation reads $Y^\lambda T_i = T_i Y^\lambda$, which is Theorem 3.3.6(2). When $(\lambda, \alpha_i) = 1$ we have $Y^{s_i(\lambda)} = Y^{\lambda - \alpha_i^\vee}$ so the relation reads

$$T_i Y^\lambda - Y^{s_i(\lambda)} T_i = (\tau_i - \tau_i^{-1}) Y^\lambda$$

which follows immediately from Theorem 3.3.6(3) and the identity

$$T_i = T_i^{-1} + \tau_i - \tau_i^{-1}$$

in $H_\tau(W^{ae})$. □

3.5. **Two Presentations.** Recall that we have seen two presentations of the extended affine braid group B^{ae}, one describing B^{ae} in terms of the affine braid group B^a and some group Ω of diagram automorphisms, and the other in terms of the finite-type braid group B and the lattice $P^\vee$. Each of these immediately gives a presentation for the Hecke algebra $H_\tau(W^{ae})$ in which one just adds the quadratic Hecke relations to the mix. We'll refer to the presentation for $H_\tau(W^{ae})$ arising from the presentation of the extended affine braid group appearing in Theorem 3.3.3 as the *Coxeter presentation*, and we'll refer to the the presentation of $H_\tau(W^{ae})$ arising from the presentation in Theorem 3.3.7 as the *Bernstein presentation*. In this section we elaborate on these presentations slightly, describing certain subalgebras of and bases for $H_\tau(W^{ae})$.

Theorem 3.5.1. *The subalgebra of $H_\tau(W^{ae})$ generated by the T_i for $i \in I$ is isomorphic to the usual Hecke algebra $H_\tau(W^a)$ attached to the Coxeter system (W^a, I), and similarly for the finite-type Hecke algebra $H_\tau(W)$. Furthermore, $H_\tau(W^{ae})$ has the following description:*

$$H_\tau(W^{ae}) \cong \Omega \ltimes H_\tau(W^a),$$

where the action of $\pi_r \in \Omega$ on T_i is the same as seen earlier for the extended affine braid group. In particular, the set $\{T_w : w \in W^{ae}\}$ forms a $\mathbb{C}$-basis for $H_\tau(W^{ae})$.

Proof. Clear, given our earlier discussion of Coxeter and braid groups. □

In view of Theorem 3.5.1, we will regard $H_\tau(W^a)$ and $H_\tau(W)$ as subalgebras of $H_\tau(W^{ae})$.

In addition to the "Coxeter presentation" above, we would also like to have a description of $H_\tau(W^{ae})$ involving the finite Weyl group W and the lattice $P^\vee$, as we did for B^{ae} and W^{ae}.

Theorem 3.5.2. *The natural map*

$$H_\tau(W) \otimes \mathbb{C}Y^{P^\vee} \to H_\tau(W^{ae})$$

given by multiplication is an isomorphism of vector spaces. In particular, the elements $\{T_w Y^\lambda : w \in W, \lambda \in P^\vee\}$ *form a* $\mathbb{C}$*-basis for* $H_\tau(W^{ae})$.

Proof. We've seen that the elements T_i and Y^λ generate the extended affine braid group B, and so they also generate $H_\tau(W^{ae})$. From our discussion of Hecke algebras attached to Coxeter groups, we know that any product of the T_i generators lies in the span of the T_w. It then follows from the relations in Lemma 3.4.3 that the set $\{T_w Y^\lambda : w \in W, \lambda \in P^\vee\}$ spans $H_\tau(W^{ae})$, so we need only to prove linear independence. Consider a linear relation of the form

$$\sum_{w,\lambda} c_{w,\lambda} T_w Y^\lambda = 0$$

wth almost all $c_{w,\lambda} = 0$. There is a dominant coweight $\mu \in P_+^\vee$ such that $\mu + \lambda$ is also dominant for any of the finitely many $\lambda \in P^\vee$ satisfying $c_{w,\lambda} \neq 0$. Multiplying on the right by Y^μ, we may assume that all (w, λ) with $c_{w,\lambda} \neq 0$ satisfy $\lambda \in P_+^\vee$. But for such λ we have

$$T_w Y^\lambda = T_w T_{t(\lambda)} = T_{wt(\lambda)}$$

(the first equality by the definition of Y^λ, and the second by Corollary 3.2.10(2)). The linear independence then follows from that of the basis $\{T_w : w \in W^{ae}\}$ and the description $W^{ae} = W \ltimes t(P^\vee)$ of W^{ae}. $\square$

Remark 3.5.3. *Obviously, we have right- and left-handed versions of the previous theorem.*

3.6. **Cherednik's Basic Representation.** To simplify the notation slightly, let's write $\mathbb{C}[Y]$ to denote $\mathbb{C}Y^{P^\vee}$. Recall that by Theorem 3.5.2 we have an isomorphism

$$\mathbb{C}[Y] \otimes H_\tau(W) \cong H_\tau(W^{ae})$$

of $\mathbb{C}$-vector spaces. In particular, for any representation E of the finite Hecke algebra $H_\tau(W)$, we can form the induced representation

$$\mathrm{Ind}E := H_\tau(W^{ae}) \otimes_{H_\tau(W)} E$$

that as a $\mathbb{C}[Y]$-module has the convenient description

$$\mathrm{Ind}E \cong_{\mathbb{C}[Y]} \mathbb{C}[Y] \otimes_{\mathbb{C}} E.$$

In particular, taking $E = \mathbb{C}$ to the the deformed trivial representation of $H_\tau(W)$ in which each generator T_i acts by the scalar τ_i, we obtain a representation of $H_\tau(W^{ae})$ in the space

$$\mathrm{Ind}\mathbb{C} = \mathbb{C}[Y] \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}[Y].$$

Lemma 3.6.1. *In the representation* $\mathbb{C}[Y]$, *the elements* T_i *for* $i \in I_0$ *(note we exclude* $i = 0$ *here) act by the formulas*

$$T_i \mapsto \tau_i s_i + (\tau_i - \tau_i^{-1}) \frac{s_i - 1}{Y^{-\alpha_i^\vee} - 1}.$$

Proof. This is an immediate consequence of the definition of the representation $\mathbb{C}[Y]$ and of Lemma 3.4.3. In particular, for any $f(Y) \in \mathbb{C}[Y]$ and $i \in I_0$, by Lemma 3.4.3 we have

$$T_i f(Y) = (s_i f(Y)) T_i + (\tau_i - \tau_i^{-1}) \frac{s_i - 1}{Y^{-\alpha_i^\vee} - 1} f(Y).$$

As T_i acts on $\mathbb{C}$ by the scalar τ_i, the claim follows. $\square$

We call this representation of $H_\tau(W^{ae})$ in $\mathbb{C}[Y]$ the *polynomial representation.* In order to define double affine Hecke algebras (DAHA), we will need to introduce a related representation of $H_\tau(W^{ae})$, Cherednik's *basic representation.* First, we need to discuss the affine weight lattice.

Let $\widehat{P} = P \oplus \mathbb{Z}\delta$ denote the *affine weight lattice* of the root system R. Let $\mathbb{C}[X]$ denote the group algebra of the weight lattice P, where the element of $\mathbb{C}[X]$ corresponding to $\lambda \in P$ will be denoted X^λ. Let $e \in \mathbb{Z}^{>0}$ be the positive integer satisfying

$$(P, P^\vee) = \frac{1}{e}\mathbb{Z}$$

(for example, when R is of type A_{n-1}, we have $e = n$, as you can compute). Fix a nonzero complex number q *that is NOT a root of unity,* and fix a primitive e^{th} root $q_0 = q^{1/e}$ of q. We can extend the definition of the $X^\lambda \in \mathbb{C}[X]$ to include affine functions of the form $f = \lambda + r\delta$ with $r \in \frac{1}{e}\mathbb{Z}$ by defining

$$X^f := q^r X^\lambda$$

(we've implicitly used our chosen e^{th} root q_0 of q in the above definition of X^f). Such X^f act on $\mathbb{C}[X]$ by multiplication.

Recall that any element $w \in W^{ae}$ of the extended affine Weyl group is (uniquely) of the form $w = t(\lambda')v$ for some $\lambda' \in P^\vee$ and $v \in W$, and that such w acts on any affine linear function f on V by

$$w \cdot f = vf - (\lambda', vf)\delta.$$

In particular, we can define an action of the affine Weyl group W on $\mathbb{C}[X]$ by setting, for such $w = t(\lambda')v$,

$$w(X^\mu) = X^{w\mu} = q^{-(\lambda', v\mu)} X^{v\mu}.$$

Obviously, this action is faithful, because q is not a root of unity.

We can now define Cherednik's basic representation, which is a deformation of the representation of W^{ae} in $\mathbb{C}[X]$ discussed above:

Theorem 3.6.2. *There is a representation β* (Cherednik's basic representation) *of $H_\tau(W^{ae})$ in the space $\mathbb{C}[X]$ in which the generators T_i for $i \in I$ (note we are now including $i = 0$!) and $\pi_r \in \Omega$ act as follows:*

$$\beta : T_i \mapsto \tau_i s_i + (\tau_i - \tau_i^{-1}) \frac{s_i - 1}{X^{a_i} - 1}$$
$$\pi_r \mapsto \pi_r.$$

Proof. Replacing the root system R with its dual $R^\vee$ does not change the finite Weyl group W or its Hecke algebra $H_\tau(W)$. It therefore follows from Lemma 3.6.1 that the operators $\beta(T_i)$ operators for $i \in I_0$ define a representation of the finite Hecke algebra $H_\tau(W)$, i.e. that these operators satisfy the Hecke and braid relations. Indeed, replacing R by its dual root system $R^\vee$ in Lemma 3.6.1 (and the basis α_i of R with the basis $-\alpha_i^\vee$ of $R^\vee$, $P^\vee$ with P, etc.), the operators by which the T_i act in the resulting induced representation $\mathbb{C}[X]$ coincide with the $\beta(T_i)$.

Furthermore, it is clear that for any $\pi_r \in \Omega$ and and $i \in I$ (including $i = 0$), the operators π_r and T_i satisfy

$$\pi_r T_i \pi_r^{-1} = T_j \quad \text{when } \pi_r(a_i) = a_j.$$

Therefore, the only relations among the operators $\beta(T_i)$ and $\beta(\pi_r)$ remaining to be checked are the Hecke and braid relations involving $\beta(T_0)$. For this, note that the fact that the operators $\beta(T_i)$ and $\beta(T_j)$ for $i, j \in I_0$, $i \neq j$, satisfy the Hecke and braid relations is a statement about the rank 2 root subsystem generated by a_i and a_j. When R is not of type A_1, the subsystem generated by a_0 and any a_i for $i \neq 0$ is a finite-type root system of rank 2, and the Hecke and braid relations follow in that case again by comparison with the polynomial representation. When R is of type A_1, there are no braid relations between T_0 and T_1 to check, and the Hecke relations follow again by comparison with the polynomial representation. It follows that the Hecke and braid relations involving $\beta(T_i)$ and $\beta(T_j)$ for *any* $i, j \in I$ (allowing $i = 0$ or $j = 0$) hold. □

Theorem 3.6.3. *The operators $X^\lambda \beta(T_w)$ with $\lambda \in P$ and $w \in W^{ae}$ on $\mathbb{C}[X]$ are linearly independent, as are the operators $\beta(T_w)X^\lambda$.*

Lemma 3.6.4. *Let F be a field, and let $\varphi_1, ..., \varphi_n$ be distinct automorphisms of F. Then $\varphi_1, ..., \varphi_n$ are linearly independent as elements of the F-vector space of functions $\varphi : F \to F$.*

Proof. This lemma is standard. A proof can be found in most introductory algebra textbooks. □

Proof of Theorem 3.6.3. We will prove that the operators $X^\lambda \beta(T_w)$ are linearly independent - the proof for the operators $\beta(T_w)X^\lambda$ is entirely analogous.

Let $w \in W^{ae}$ and let $w = \pi_r s_{i_1} \cdots s_{i_p}$ be a reduced expression with $\pi_r \in \Omega$ and all $i_j \in I_0$. Then we have

$$\beta(T_w) = \pi_r \beta(T_{i_1}) \cdots \beta(T_{i_p}).$$

By the definition of the operators $\beta(T_i)$ it follows that $\beta(T_w)$ is of the form

$$\beta(T_w) = \sum_{v \leq w} f_{vw}(X) v$$

where $f_{vw} \in \mathbb{C}(X)$, the field of fractions of the Laurent polynomial algebra $\mathbb{C}[X]$. Note that we have

$$f_{ww}(X) \neq 0$$

for all $w \in W^{ae}$.

For contradiction, now suppose there is a linear dependence among the $X^\lambda \beta(T_w)$. It follows that there is a relation of the form

$$\sum_{w \in W^{ae}} g_w(X) \beta(T_w)$$

with the $g_w(X) \in \mathbb{C}[X]$ not all 0 (but only finitely many nonzero). From the above expression for $\beta(T_w)$, this gives a relation

$$\sum_{v \leq w \in W^{ae}} g_w(X) f_{vw}(X) v = 0.$$

The group elements v act on $\mathbb{C}(X)$ by field automorphisms, and these automorphisms are distinct as the action of W^{ae} on $\mathbb{C}[X]$ is faithful. We therefore have

$$\sum_{w \geq v} g_w(X) f_{vw}(X) = 0$$

for all $v \in W^{ae}$. As $g_w = 0$ for all but finitely many w, choose $w \in W^{ae}$ maximal in the Bruhat ordering among those w with $g_w \neq 0$. The above equation then reads

$$g_w(X) f_{ww}(X) = 0.$$

As $f_{ww}(X) \neq 0$, this implies $g_w = 0$, a contradiction. ☐

Corollary 3.6.5. *The basic and polynomial representations of $H_\tau(W^{ae})$ are faithful.*

Proof. The set $\{T_w : w \in W\}$ is a basis for $H_\tau(W^{ae})$, so the claim for the basic representation follows immediately from Theorem 3.6.3. For the polynomial representation, it suffices to prove the claim with the root system R with its basis α_i replaced by the dual root system $R^\vee$ with its basis $-\alpha_i^\vee$. Denote the corresponding affine Hecke algebra by $H_\tau(W^\vee)$. It has a basis $X^\lambda T_w$ with $\lambda \in P$ and $w \in W$ (finite Weyl group); its polynomial representation is in the space $\mathbb{C}[X]$, and the action of any such element $X^\lambda T_w$ in the polynomial representation $\mathbb{C}[X]$ is precisely by the operator $X^\lambda \beta(T_w)$, and the claim follows. ☐

Corollary 3.6.6. *The center $Z(H_\tau(W^{ae}))$ coincides with the invariant Laurent polynomials $\mathbb{C}[Y]^W$.*

Proof. First note that it follows immediately from the relations in Lemma 3.4.3 and the fact that $H_\tau(W^{ae})$ is generated by $\mathbb{C}[Y]$ and the T_i for $i \in I_0$ that any $f \in \mathbb{C}[Y]^W$ is central in $H_\tau(W^{ae})$. By the same Lemma, for any $f \in \mathbb{C}[Y]$ and $i \in I_0$ we have the relation

$$T_i f(Y) - (s_i f)(Y) T_i = g(Y)$$

for some $g(Y) \in \mathbb{C}[Y]$. If f is central, this gives

$$(f(Y) - (s_i f)(Y)) T_i = g(Y)$$

which implies that $f(Y) - (s_i f)(Y) = 0$, so that $f = s_i f$. In particular, f is W-invariant.

It therefore suffices to check that any central element $z \in H_\tau(W^{ae})$ is a Laurent polynomial. But such z commutes with all Laurent polynomials on the faithful polynomial representation, so z itself must indeed be a Laurent polynomial. ☐

3.7. **DAHA Definition and PBW Theorem.** Cherednik's basic representation gives an easy definition of DAHA:

Definition 3.7.1. *The* double affine Hecke algebra (DAHA) $\mathcal{H}_{q,\tau} = \mathcal{H}_{q,\tau}(R)$ *attached to the finite root system R and parameters q, τ as in the previous section is the subalgebra of $End_{\mathbb{C}}(\mathbb{C}[X])$ generated by the operators $X^\lambda T_w$ for $\lambda \in P$ and $w \in W^{ae}$.*

The algebra $\mathcal{H}_{q,\tau}$ can be alternatively described by generators and relations as follows:

Theorem 3.7.2. *The algebra $\mathcal{H}_{q,\tau}$ is generated by the T_i with $i \in I$ (including $i = 0$), $\pi_r \in \Omega$, and X^λ with $\lambda \in P$ subject to the relations:*

(1) the relations of the affine Hecke algebra among the T_i and π_r
(2) $X^0 = 1$ and $X^\lambda X^\mu = X^{\lambda+\mu}$
(3) for any $i \in I$ (including $i = 0$) and $\mu \in P$,

$$T_i X^\mu = X^\mu T_i \quad \text{if } (\mu, a_i^\vee) = 0$$
$$T_i X^\mu = X^{s_i(\mu)} T_i^{-1} \quad \text{if } (\mu, a_i^\vee) = 1$$

(4) $\pi_r X^\mu \pi_r^{-1} = X^{\pi_r(\mu)}$.

Remark 3.7.3. *The parameter(s) τ appears in the Hecke relations in (1). The parameter q, slightly more hidden, appears in the second relation in (3). In particular, for $i = 0$, we have $a_0 = -\theta + \delta$, and if $(\mu, a_0^\vee) = 1$ so that $s_0(\mu) = \mu - a_0 = \mu + \theta - \delta$, the relation reads*

$$T_0 X^\mu = X^{s_i(\mu)} T_0^{-1} = X^{\mu - a_i} T_0^{-1} = q^{-1} X^{\mu+\theta} T_0^{-1}.$$

Proof of Theorem 3.7.2. That the elements generate $\mathcal{H}_{q,\tau}$ follows from the definition. Relations (1), (2), (4) and the $(\mu, a_i^\vee) = 0$ case of (3) are clear from the definition of the basic representation and the fact that it is a representation of $H_\tau(W^{ae})$. As $T_i^{-1} = T_i - (\tau_i - \tau_i^{-1})$, it's easy to verify the $(\mu, a_i^\vee) = 1$ case of (3) as well by a direct computation from the definition of the operators $\beta(T_i)$. In particular, all of the relations (1)-(4) hold in $\mathcal{H}_{q,\tau}$.

Let $\mathcal{H}'_{q,\tau}$ be the algebra given by generators and relations as in the theorem. Then we have a surjection $\mathcal{H}'_{q,\tau} \to \mathcal{H}_{q,\tau}$ arising from the basic representation. As the affine Hecke algebra relations hold among the T_i and π_r, we can define elements $T_w \in \mathcal{H}'_{q,\tau}$ for $w \in W^{ae}$ as for the affine Hecke algebra. It is then clear from the other relations that the elements of the form $X^\mu T_w$ (or just as well $T_w X^\mu$) with $\mu \in P$ and $w \in W^{ae}$ span $\mathcal{H}'_{q,\tau}$. But these elements are linearly independent in the basic representation, and it follows that the map $\mathcal{H}'_{q,\tau} \to \mathcal{H}_{q,\tau}$ is an isomorphism and that the relations (1)-(4) are a complete set of relations. □

Define elements $Y^\lambda \in \mathcal{H}_{q,\tau}$ as we did for the affine Hecke algebra $H_\tau(W^{ae})$.

Corollary 3.7.4. *(PBW Theorem for DAHA) The set* $\{X^\mu T_w Y^\lambda : \mu \in P, w \in W, \lambda \in P^\vee\}$ *(or* $\{Y^\lambda T_w X^\mu : \lambda \in P^\vee, w \in W, X^\mu\}$*) forms a basis for the DAHA* $\mathcal{H}_{q,\tau}$.

Proof. Immediate from Theorem 3.7.2, the basis $T_w Y^\lambda$ for $H_\tau(W^{ae})$, and the linear independence of the operators $X^\mu \beta(T_w)$. □

3.8. **Affine Hecke Algebras for GL_n.** So far we've discussed affine Hecke algebras in the generality of affine root systems of the form R^a, where R is a reduced irreducible finite root system. In most of this seminar, we will be concerned with type A structures. We'll now discuss how the results of this talk look in the GL_n case.

In type A_{n-1}, it is actually quite natural to consider a slight enlargement of the affine Hecke algebra, and this is what is often done. This is entirely analogous to considering the reducible reflection representation $\mathbb{R}^n$ of S_n rather than the irreducible reflection representation $\{x \in \mathbb{R}^n : \sum_i x_i = 0\}$. In particular, in this approach one defines the affine Hecke algebra $H_{n,\tau}^{aff}$ for GL_n via the Bernstein presentation, but replacing the coweight lattice $P^\vee$ (which coincides with the weight lattice P in type A_{n-1}) with the permutation lattice $L := \bigoplus_{i=1}^n \mathbb{Z}\epsilon_i = \mathbb{Z}^n$ and its obvious S_n-action. (see, e.g., [A, Chapter 13]).

Specifically, what we get by following this recipe is the following. Let $\epsilon_1, ..., \epsilon_n$ be the standard basis for $\mathbb{R}^n$, and choose simple roots $\alpha_i := \epsilon_{i+1} - \epsilon_i \in \mathbb{R}^n$, $i = 1, .., n-1$, for the root system R_n of type A_{n-1}. For $1 \leq i < n$, let s_i be the simple reflection $s_i = (i, i+1)$ attached to simple root α_i, and let T_i be the associated braid group/Hecke algebra generator. Identify $\mathbb{C}[Y^L]$ with the Laurent polynomial ring $\mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]$ by identifying $Y_i = Y^{\epsilon_i}$. The commutation identities $T_i Y^\lambda = Y^\lambda T_i$ when $(\lambda, \alpha_i) = 0$ and $Y^\lambda = T_i Y^{s_i \lambda} T_i$ when $(\lambda, \alpha_i) = 1$ for all $\lambda \in L$ and $1 \leq i < n$ are then equivalent to the commutation identities $Y_{i+1} = T_i Y_i T_i$ for $1 \leq i < n$ and $T_i Y_j = Y_j T_i$ when $j \neq i, i+1$. So we have the following definition:

Definition 3.8.1. *Let $\tau \in \mathbb{C}^\times$. The affine Hecke algebra $H_{n,\tau}^{aff}$ for GL_n is the $\mathbb{C}$-algebra given by generators $T_1, ..., T_{n-1}$ and $Y_1^{\pm 1}, \cdots, Y_n^{\pm 1}$ and relations:*

(1) $Y_i Y_i^{-1} = Y_i^{-1} Y_i = 1, \quad Y_i Y_j = Y_j Y_i \quad (1 \leq i, j \leq n)$
(2) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i < n)$
(3) $T_i T_j = T_j T_i \quad (|i - j| > 1)$
(4) $(T_i - \tau)(T_i + \tau^{-1}) = 0 \quad (1 \leq i < n)$
(5) $Y_{i+1} = T_i Y_i T_i \quad (1 \leq i < n), \qquad T_i Y_j = Y_j T_i \quad (j \neq i, i+1)$.

The affine Hecke algebra of type A_{n-1} and the affine Hecke algebra for GL_n are very related. Using the same methods as for the algebra $H_\tau(S_n^{ae})$, one can show that $H_{n,\tau}^{aff}$ has a basis of the form $\{Y_1^{k_1}\cdots Y_n^{k_n}w : w \in S_n, k_i \in \mathbb{Z}\}$ (and similarly of the form $\{wY_1^{k_1}\cdots Y_n^{k_n} : w \in S_n, k_i \in \mathbb{Z}\}$), that the center coincides with the invariant Laurent polynomials $\mathbb{C}[Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^{S_n}$, etc.

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LECTURE 2: DOUBLE AFFINE HECKE ALGEBRAS

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Abstract. These are notes for a talk given at the MIT-Northeastern Graduate Student Seminar on Double Affine Hecke Algebras and Elliptic Hall Algebras, Spring 2017.

Contents

1. Goals and structure of the talk

This talk introduces one of the main objects of study in our seminar: the double affine Hecke algebra (DAHA). We will make the definitions in great (but not complete) generality, and we will emphasize the $\mathfrak{gl}_n$ case. In the first part of the talk we will briefly recall from Seth's talk the main ingredients for the construction of DAHA: the affine Hecke algebras and Cherednik's basic representation. After this, we will give the definition of DAHA and exhibit an explicit basis of it. We will then spend some time looking at two explicit cases: the DAHA for A_1 and that for $\mathfrak{gl}_n$. In particular, we will give explicit presentations by generators and relations, present their trigonometric and rational degenerations, and exhibit a large group of automorphisms of the DAHA for $\mathfrak{gl}_n$. After that, we will study certain operators on the polynomial representation of DAHA. The importance of these operators is that, first, they can be used to form a big commuting family of difference operators on the group algebra of the weight lattice and, second, they are connected to Macdonald polynomials, which is the topic of a subsequent talk in this seminar. The study of these difference operators naturally leads to the definition of spherical DAHA. We finish the notes with a discussion of trigonometric and rational degenerations in the general setting, the description of (trigonometric, difference-rational and rational) Dunkl operators, and applications to the theory of quantum integrable systems.

2. Double Affine Hecke Algebras

2.1. **Reminders.**

2.1.1. *Root systems and Weyl groups.* We will work with affine root systems that are of the form R^a, where R is an irreducible finite root system (so, for example, we will ignore the affine root systems of the form $(C_n^\vee, C_n)$, etc.). Throughout these notes, we will use the following notation.

- $\{\alpha_1, \dots, \alpha_n\}$ denotes the set of simple roots and $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ the set of simple co-roots of R.
- $Q, Q^\vee$ denote the root and coroot lattice of R, respectively. Similarly, $P, P^\vee$ denote the weight and coweight lattice of R, respectively.
- $W := \langle s_1, \dots, s_n\rangle$ is the Weyl group of R, where s_i denotes the reflection s_{α_i}.
- $\alpha_0 := -\theta + \delta$, so that $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$ forms a set of simple roots for R^a.
- $W^a := \langle s_0, \dots, s_n\rangle$ is the Weyl group of R^a, aka the affine Weyl group. Recall that we have an isomorphism
$$W^a = W \ltimes t(Q^\vee)$$
- $W^{ae} := W \ltimes t(P^\vee)$ is the extended affine Weyl group.
- $\Omega \subseteq W^{ae}$ denotes the subgroup of all elements of length 0. This is a finite subgroup of W^{ae}, acting faithfully on the set of simple roots $\{\alpha_0, \dots, \alpha_n\}$, and it is actually isomorphic to $P^\vee/Q^\vee$.
- We have an isomorphism
$$W^{ae} = \Omega \ltimes W^a$$
where the action of Ω' on W^a is given as follows: if $\pi_r \in \Omega'$ is such that $\pi_r(\alpha_i) = \alpha_j$, then $\pi_r s_i \pi_r^{-1} = s_j$.

2.1.2. *Affine Hecke algebras.* Throughout this talk, $\tau := \{\tau_0, \dots, \tau_n\}$ will denote a collection of *formal variables* such that $\tau_i = \tau_j$ whenever the reflections s_i and s_j are conjugate in W^a, and let $\mathbb{C}_\tau := \mathbb{C}(\tau_0, \dots, \tau_n)$ denote the field of rational functions in these variables. Recall that we have the *affine Hecke algebra* $\mathcal{H}_\tau$ of W, which is a quotient of the group algebra $\mathbb{C}_\tau B^{ae}$, where B^{ae} is the extended affine braid group. We have two presentations of this algebra.

The Coxeter presentation. $\mathcal{H}_\tau = \mathbb{C}_\tau\langle T_0, \dots, T_n, \Omega\rangle$ with the following relations.

(a) $T_iT_j \cdots = T_jT_i \cdots$, where term has m_{ij} factors.
(b) $(T_i - \tau_i)(T_i + \tau_i^{-1}) = 0$.
(c) $\pi_r T_i \pi_r^{-1} = T_j$, if $\pi_r(\alpha_i) = \alpha_j$.

The Bernstein presentation. $\mathcal{H}_\tau = \mathbb{C}_\tau\langle T_1, \dots, T_n, Y^{P^\vee}\rangle$ with relations:

(a') Relations (1) and (2) above for the T_i.
(b') $Y^\lambda Y^\mu = Y^{\lambda+\mu}$.
(c') $T_iY^\lambda = Y^\lambda T_i$ if $\langle \lambda, \alpha_i\rangle = 0$.
(d') $T_iY^{s_i(\lambda)}T_i = Y^\lambda$ if $\langle\lambda, \alpha_i\rangle = 1$.

2.1.3. *Cherednik's basic representation.* Let us now recall Cherednik's basic representation of the affine Hecke algebra $\mathcal{H}_\tau$. We let q be a variable, and consider the affine Hecke algebra defined over the field $\mathbb{C}_{q,\tau} := \mathbb{C}_\tau(q^{1/e})$, where e is such that $\langle P, P^\vee\rangle = \frac{1}{e}\mathbb{Z}$. Now let $\mathbb{C}_{q,\tau}[X]$ denote the group algebra of P. Note that this contains $\mathbb{C}_\tau[\widehat{X}]$, the group algebra of the affine weight lattice $\widehat{P} := P \oplus \mathbb{Z}\delta$, by setting $X^{\lambda + r\delta} := q^rX^\lambda$. So the extended affine Weyl group W^{ae} acts on $\mathbb{C}_{q,\tau}[X]$ by setting, for $w = t(\lambda)v$, $\lambda \in P^\vee, v \in W$ and $\mu \in P$,

$$w(X^\mu) := X^{w(\mu)} = q^{-\langle\lambda, v(\mu)\rangle}X^{v(\mu)}$$

We have that $\mathbb{C}_{q,\tau}[X]$ becomes a $\mathcal{H}_\tau$-module via the formulas:

$$\pi_r \mapsto \pi_r, \pi_r \in \Omega$$
$$T_i \mapsto \tau_i s_i + (\tau_i - \tau_i^{-1})\frac{s_i - \mathrm{id}}{X^{\alpha_i} - 1}, i = 0, \ldots, n$$

Let us be more explicit on the action of T_0. Recall that $\alpha_0 = -\theta + \delta$, and that we are identifying $q = X^\delta$. So it follows that $s_0 X^\mu = X^{\mu - (-\theta+\delta)\langle\mu, -\theta^\vee\rangle} = X^\mu(q^{-1}X^\theta)^{\langle\mu,-\theta^\vee\rangle}$. Thus,

$$T_0 : X^\mu \mapsto \left(\tau_0(q^{-1}X^\theta)^{\langle\mu,-\theta^\vee\rangle} + (\tau_0 - \tau_0^{-1})\frac{(q^{-1}X^\theta)^{\langle\mu,-\theta^\vee\rangle} - 1}{qX^{-\theta} - 1}\right)X^\mu$$

In particular, if $\langle\mu, -\theta^\vee\rangle = 0$, then $T_0(X^\mu) = \tau_0 X^\mu$, while if $\langle\mu, -\theta^\vee\rangle = 1$, then $T_0(X^\mu) = \tau_0 q^{-1}X^{\theta+\mu} - (\tau_0 - \tau_0^{-1})q^{-1}X^{\theta+\mu}$.

2.1.4. *The induced representation.* Let us denote by $\mathcal{H}_\tau^X$ the affine Hecke algebra for the root system $(R^\vee)^a$. In particular, we have the Bernstein presentation for this Hecke algebra, which is completely analogous to the Bernstein presentation above. We have the induced representation of $\mathcal{H}_\tau^X$ on $\mathbb{C}_{q,\tau}[X]$, where the X^μ act by multiplication and the T_i act by

$$T_i \mapsto \tau_i s_i + (\tau_i - \tau_i^{-1})\frac{s_i - \mathrm{id}}{X^{\alpha_i} - 1}, i = 1, \ldots, n$$

Let us remark that the induced representation of $\mathcal{H}_\tau^X$ on $\mathbb{C}_{q,\tau}[X]$ is obtained by the eponymous representation on $\mathbb{C}_\tau[X]$ by base-change to the field $\mathbb{C}_{q,\tau}$.

2.2. **Double affine Hecke algebras.** We are now ready to define the double affine Hecke algebra for R. The idea here is to glue together the affine Hecke algebras $\mathcal{H}_\tau$ and $\mathcal{H}_\tau^X$ along their common representation $\mathbb{C}_{q,\tau}[X]$.

Definition 2.2.1. *The* double affine Hecke algebra $\mathbb{H} := \mathbb{H}(W)$ *is the $\mathbb{C}_{q,\tau}$-algebra generated by elements $T_0, \ldots, T_n, \Omega, X^P$ with relations.*

1. *The relations (a)-(c) above for the affine Hecke algebra between $T_0, \ldots, T_n$ and Ω.*
2. *Denote $\alpha_0^\vee := -\theta^\vee$. Then, for $i = 0, \ldots, n$:*

$$T_i X^\mu = X^\mu T_i \quad \text{if } \langle\mu, \alpha_i^\vee\rangle = 0$$
$$T_i X^\mu = X^{s_i(\mu)} T_i^{-1} \quad \text{if } \langle\mu, \alpha_i^\vee\rangle = 1$$

3. $\pi_r X^\mu \pi_r^{-1} = X^{\pi_r(\mu)}$.

Definition 2.2.2. *Note that, by its very definition, the DAHA $\mathbb{H}$ admits a representation on the space $\mathbb{C}_{q,t}[X]$, where X^μ acts by multiplication and both π_r and T_i act as in Cherednik's basic representation. We call this representation the* polynomial representation *of $\mathbb{H}$.*

Note that, by Matsumoto's theorem, if $w = \alpha_{i_1} \ldots \alpha_{i_k}$ is a reduced decomposition of $w \in W^a$, then we have a well-defined element $T_w \in \mathbb{H}$.

Theorem 2.2.3 (PBW theorem for DAHA)**.** *Every element $h \in \mathbb{H}$ can be uniquely written in the form*

$$h = \sum_{\substack{\mu\in P \\ \pi_r\in\Omega \\ w\in W^a}} a_{\mu,r,w} X^\mu \pi_r T_w, \quad a_{\mu,r,w} \in \mathbb{C}_{q,\tau}$$

The existence of such an expression for h is a standard exercise. The uniqueness is harder. We will use a standard trick that we have already seen in Seth's lecture: we will write down a representation of $\mathbb{H}$ in a space in which the operators $X^\mu \pi_r T_w$ are linearly independent. It turns out that we already know such a representation: the polynomial representation, cf. Definition 2.2.2.

Theorem 2.2.4. *Consider the polynomial representation $\mathbb{C}_{q,\tau}[X]$ of $\mathbb{H}$. Then, the operators $\{X^\mu \pi_r T_w : \mu \in P, \pi_r \in \Omega, w \in W^a\}$ are linearly independent over the field $\mathbb{C}_{q,\tau}$. In particular, the polynomial representation is faithful.*

Proof. Note that, even though the operators $\pi_r T_w$ are not $\mathbb{C}_{q,\tau}[X]$-linear, we still have an action of $\mathbb{C}_{q,\tau}[X]$ on $\mathrm{End}_{\mathbb{C}}(\mathbb{C}_{q,\tau}[X])$, $f : \varphi \mapsto (x \mapsto f\varphi(x))$. It clearly suffices to show that the operators $\{\pi_r T_w\}_{\mu \in P, w \in W^a}$ are linearly independent over $\mathbb{C}_{q,\tau}[X]$. In order to do this, we will relate this action to the action of the extended affine Weyl group W^{ae} on $\mathbb{C}_{q,\tau}[X]$, which we know from Seth's talk it is faithful.

Recall that for $i = 0, \ldots, n$, T_i acts via the operator:

$$T_i := \tau_i s_i + (\tau_i - \tau_i^{-1})\frac{s_i - \mathrm{id}}{X^{\alpha_i} - 1} = \left(\tau_i + \frac{\tau_i - \tau_i^{-1}}{X^{\alpha_i} - 1}\right) s_i + \left(\frac{\tau_i^{-1} - \tau_i}{X^{\alpha_i} - 1}\right) \mathrm{id}$$

It follows that, for $\pi_r w \in W^{ae}$, $w \in W^a$, we can write

$$\pi_r T_w = \sum_{w' \leq w} f_{w',w} \pi_r w'$$

where $f_{w',w} \in \mathbb{C}_{q,\tau}(X)$ are rational functions on X and the order on W^a is the usual Bruhat order. Note that $f_{w,w} \neq 0$.

Now assume that we have a linear combination of the form

$$\sum_{\substack{\pi_r \in \Omega \\ w \in W^a}} g_{r,w}(X) \pi_r T_w = 0$$

where $g_{r,w} \in \mathbb{C}_{q,\tau}[X]$ are not all 0. It follows from the above that we get

$$\sum_{\substack{w \in W^a, \pi_r \in \Omega \\ w' \leq w}} g_{r,w}(X) f_{r,w',w}(X) \pi_r w' = 0$$

The operators $\pi_r w$ are all distinct, since the representation of W^a on $\mathbb{C}_{q,\tau}[X]$ is faithful, and can be extended to automorphisms of the field $\mathbb{C}_{q,\tau}(X)$ (= the field of quotients of $\mathbb{C}_{q,\tau}[X]$). It follows that the operators $\pi_r w$ are linearly independent over the field $\mathbb{C}_{q,\tau}(X)$. So for every $\pi_r \in \Omega$, $w \in W^a$ we have

$$\sum_{w' \geq w} g_{r,w'} f_{r,w,w'} = 0$$

If we pick $w_0 \in W^a$ such that w_0 is maximal w.r.t. the Bruhat order in the set $\{w \in W^a : g_{r,w} \neq 0 \text{ for some } \pi_r \in \Omega\}$ then we get $g_{r,w_0} f_{r,w_0,w_0} = 0$. But since $f_{r,w_0,w_0} \neq 0$, this is a contradiction. We are done. $\square$

Corollary 2.2.5. *We define the following subalgebras of $\mathbb{H}$:*

1. $\mathcal{H}^X := \langle T_1, \ldots, T_n, X^\mu (\mu \in P)\rangle$.
2. $\mathcal{H}^Y := \langle T_0, \ldots, T_n, \Omega\rangle$.
3. $H := \langle T_1, \ldots, T_n\rangle$.

Then, $\mathcal{H}^X$ is (isomorphic to) the affine Hecke algebra for the root system $R^\vee$; $\mathcal{H}^Y$ is (isomorphic to) the affine Hecke algebra for the root system R; and H is (isomorphic to) the finite Hecke algebra of W.

Recall from Seth's talk the definition of the elements $Y^\lambda \in \mathcal{H}^Y$, $\lambda \in P^\vee$. Namely, $Y^\lambda := T_{t(\lambda)}$ if $\lambda \in P_+^\vee$, while $Y^\lambda := Y^\mu (Y^\nu)^{-1}$ if $\lambda = \mu - \nu$ with $\mu, \nu \in P_+^\vee$. Since $\mathcal{H}^Y$ is an affine Hecke algebra, the elements Y^λ are well-defined. The following result follows immediately from Seth's talk.

Theorem 2.2.6 (PBW theorem for DAHA, v2)**.** *Every element $h \in \mathbb{H}$ can be* <u>*uniquely*</u> *written in the form*

$$h = \sum_{\substack{\lambda \in P^\vee \\ \mu \in P \\ w \in W}} a_{\lambda,\mu,w} X^\mu Y^\lambda T_w, \quad a_{\lambda,\mu,w} \in \mathbb{C}_{q,\tau}$$

Let us remark that the weight and co-weight lattice play a symmetric role in the definition of DAHA. In order to state this precisely, let $\omega_1, \ldots, \omega_n$ be the fundamental weights of R, so $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, and denote by $\omega_1^\vee, \ldots, \omega_n^\vee$ the fundamental coweights. We will denote $X_i := X^{\omega_i}$, $Y_i := Y^{\omega_i^\vee}$.

Theorem 2.2.7. *The following assignment can be extended to a $\mathbb{C}$-automorphism of $\mathbb{H}$:*

$$X_i \mapsto Y_i, \quad Y_i \mapsto X_i, \quad T_j \mapsto T_j^{-1}, \quad \tau_j \mapsto \tau_j^{-1}, \quad q \mapsto q^{-1}$$

We will not prove Theorem 2.2.7 in full generality. We will show it for specific types of root systems below. Let us remark that a consequence of Theorem 2.2.7 is the following.

Corollary 2.2.8 (PBW theorem for DAHA, v3)**.** *Every element $h \in \mathbb{H}$ can be* <u>*uniquely*</u> *written in the form*

$$h = \sum_{\substack{\lambda \in P^\vee \\ \mu \in P \\ w \in W}} a_{\lambda,\mu,w} Y^\lambda X^\mu T_w, \quad a_{\lambda,\mu,w} \in \mathbb{C}_{q,\tau}$$

Note that we could have also defined $\mathbb{H}$ to be an algebra generated by $T_1, \ldots, T_{n-1}, X^P, Y^{P^\vee}$ with certain relations. Of course, the relations among (T_i, X^μ) or among (T_i, Y^λ) can be explicitly written - they are just the relations of the affine Hecke algebra. But it is not easy to write the relations among (X^μ, Y^λ). We will give a couple of examples where these relations can actually be written. As we will see, they are topological in nature.

2.3. **Example: DAHA for A_1.**

2.3.1. *Generators and relations.* We give explicit generators and relations for the DAHA of A_1. So we have that the (co-)root lattice is $Q = Q^\vee = \mathbb{Z}\alpha$ and the (co-)weight lattice is $P = P^\vee = \mathbb{Z}\rho$, with $\rho = \alpha/2$. Let us denote $s = s_\alpha$. We have that $\Omega = \{1, \pi_\rho\}$, we denote $\pi := \pi_\rho = t(\rho)s$. Setting now $X := X^\rho$, we have that the DAHA $\mathbb{H}$ is generated by $T_0, T_1, X^{\pm 1}$ and π. Note, however, that $T_0 = \pi T_1 \pi$, so we may ignore T_0 from our list of generators. Thus, we have

$$\mathbb{H} = \mathbb{C}_{q,\tau}\langle X, T, \pi \rangle \Big/ \left\{ \begin{array}{cc} TXT = X^{-1}, & \pi X \pi^{-1} = qX^{-1}, \\ \pi^2 = 1, & (T-\tau)(T+\tau^{-1}) = 0 \end{array} \right\}$$

Setting $Y := \pi T$, we have the following alternative presentation of $\mathbb{H}$:

$$\mathbb{H} = \mathbb{C}_{q,\tau}\langle X, T, Y \rangle \Big/ \left\{ \begin{array}{cc} TXT = X^{-1}, & Y^{-1}X^{-1}YX = q^{-1}T^{-2}, \\ TY^{-1}T = Y, & (T-\tau)(T+\tau^{-1}) = 0 \end{array} \right\}$$

Note that this presentation reveals a symmetry between X and Y. The following proposition is obvious, note that its second part is a special case of Theorem 2.2.7.

Lemma 2.3.1. *We have a $\mathbb{C}_{q,\tau}$ anti-involution $\phi : \mathbb{H} \to \mathbb{H}^{\mathrm{opp}}$, defined on generators by the following formulas*

$$\phi(X) = Y^{-1}, \quad \phi(Y) = X^{-1}, \quad \phi(T) = T$$

and a $\mathbb{C}$-involution $\varepsilon : \mathbb{H} \to \mathbb{H}$, defined by

$$\varepsilon(X) = Y, \quad \varepsilon(Y) = X, \quad \varepsilon(T) = T^{-1}, \quad \varepsilon(\tau) = \tau^{-1}, \quad \varepsilon(q) = q^{-1}$$

2.3.2. *The polynomial representation.* Let us give formulas for the action of the elements X, Y, T on the polynomial representation $\mathbb{C}_{q,\tau}[X]$. First of all, we have that the action of T is given by

$$\tau s + (\tau - \tau^{-1})\frac{s - \mathrm{id}}{X^2 - 1}$$

while the action of $t(\rho)$ is given by $t(\rho)(X) = qX$. Moreover, for a Laurent polynomial $f(X) \in \mathbb{C}_{q,\tau}[X]$, we have that $t(\rho)f(X) = f(qX)$, so that $\pi(X) = t(\rho)s(X) = q^{-1}X^{-1}$. Thus, we have

$$\begin{aligned} Y = {} & \pi\left(\tau s + (\tau - \tau^{-1})\frac{s - \mathrm{id}}{X^2 - 1}\right) \\ = {} & \tau t(\rho) + (\tau - \tau^{-1})\pi\frac{s - \mathrm{id}}{X^2 - 1} \\ = {} & \tau t(\rho) + (\tau - \tau^{-1})\frac{t(\rho) - \pi}{q^{-2}X^{-2} - 1} \\ = {} & \tau t(\rho) + (\tau - \tau^{-1})t(\rho)\frac{\mathrm{id} - s}{X^{-2} - 1} \\ = {} & t(\rho)\left(\tau + (\tau - \tau^{-1})\frac{\mathrm{id} - s}{X^{-2} - 1}\right) \end{aligned}$$

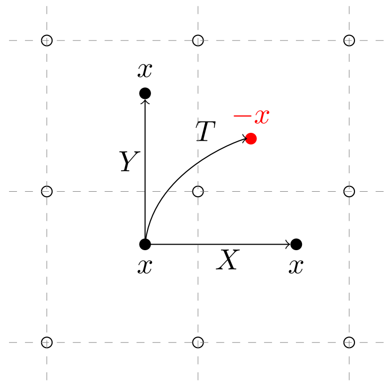
So, for example, we get $Y(X^n) = \tau^{-1}(q^n X^n + q^{n-2}X^{n-2} + \cdots + q^{2-n}X^{2-n})$. The operator Y is known as the *difference-trigonometric Dunkl operator.*

2.3.3. *Topological interpretation.* Let $E = \mathbb{C}/\Lambda$ be an elliptic curve, where we take the lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\iota$. Let $0 \in E$ be the zero point, and consider the automorphism $-1 : x \mapsto -x$ of E. Note that $\pi_1((E \setminus \{0\})/\mathbb{Z}_2)$ is trivial, as $(E \setminus \{0\})/\mathbb{Z}_2$ being a disk is contractible. We will consider the *orbifold fundamental group* $\pi_1^{\mathrm{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x)$, where $x \in E \setminus \{0\}$ is a generic point (i.e., not one of the three branching points of $E \setminus \{0\} \to (E \setminus \{0\})/\mathbb{Z}_2$).

Let us recall that the orbifold fundamental group is generated by homotopy classes of paths in $E \setminus \{0\}$ from x to $\pm x$, with multiplication defined by $\gamma_1 \circ \gamma_2$ is γ_2 followed by $-\gamma_1$, if γ_2 connects x to $-x$. So we have an exact sequence

$$1 \to \pi_1(E \setminus \{0\}, x) \to \pi_1^{\mathrm{orb}}((E \setminus \{0\})/\mathbb{Z}_2, x) \to \mathbb{Z}_2 \to 1,$$

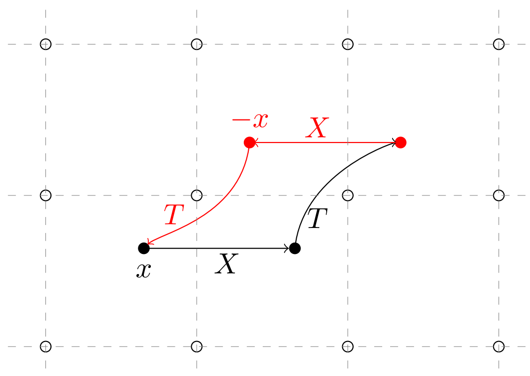
i.e., $\pi_1^{\mathrm{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x)$ is an extension by $\mathbb{Z}_2$ of the group $\pi_1(E \setminus \{0\}, x)$, the fundamental group of the punctured torus. The latter group has three generators, X (the "horizontal" cycle of the torus), Y (the "vertical" cycle of the torus) and C (a loop around the missing point 0). The orbifold fundamental group $\pi_1^{\mathrm{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x)$ is then generated by X, Y and an element T (a half-loop around 0) connecting x to $-x$ such that $T^2 = C$.



The elements X, Y, T satisfy the relations

$$TXT = X^{-1}, \quad TY^{-1}T = Y, \quad Y^{-1}X^{-1}YXT^2 = 1$$

Let us show, for example, the first relation, which says that $TXTX = 1$. So we have to first go through the loop X, then through the path T, and then through the path $-X$, since the endpoint of T is $-x$. Finally, we go through the path $-T$. It follows from the following picture that this path is null-homotopic.



So we see that $\mathbb{H}$ can be seen as a quotient of the group algebra $\mathbb{C}_{q,\tau}\pi_1^{\mathrm{orb}}((E \setminus \{0\})/\mathbb{Z}_2, x)$, as follows. First of all, recall that out base field $\mathbb{C}_{q,\tau}$ includes $q^{\pm 1/2}$, since $\langle \rho, \rho \rangle = 1/2$. Now set $\widetilde{T} := q^{-1/2}T, \widetilde{X} := q^{1/2}X, \widetilde{Y} := q^{-1/2}Y$, so that $\widetilde{X}, \widetilde{Y}, \widetilde{T}$ satisfy the relations of $\mathbb{H}$ with the exception of the quadratic relation for $\widetilde{T}$. Thus:

$$\mathbb{H} = \mathbb{C}_{q,\tau}\pi_1^{\mathrm{orb}}(E \setminus \{0\}/\mathbb{Z}_2, x)\Big/((T - q^{1/2}\tau)(T + q^{1/2}\tau^{-1}))$$

2.3.4. *Trigonometric degeneration.* Now let $\hbar$, c and t be variables. Set $Y := \exp(\hbar\hat{y})$, $q := \exp(t\hbar)$, $\tau := q^c = \exp(\hbar tc)$ and $T := s\exp(\hbar cs)$, where $s \in S_2$ is the non-trivial element. We can consider $\mathbb{H}$ as a $\mathbb{C}[[\hbar, c, t]]$-algebra, with the same generators and relations as above. Then, $\mathbb{H}/\hbar\mathbb{H}$ is generated by $s, \hat{y}$ and X, with relations

$$s^2 = 1, \quad sXs = X^{-1}, \quad s\hat{y} + \hat{y}s = 2c, \quad X^{-1}\hat{y}X - \hat{y} = t - 2cs$$

We call $\mathbb{H}^{\mathrm{trig}} := \mathbb{C}[c,t]\langle s, X, \hat{y}\rangle$ with the relations above the *trigonometric* DAHA of A_1.

Lemma 2.3.2. *Every element $h \in \mathbb{H}^{\mathrm{trig}}$ can be uniquely written as*

$$h = \sum_{\substack{m\in\mathbb{Z}\\ n\in\mathbb{Z}_{\geq 0}\\ i=0,1}} a_{m,i,n}X^m s^i \hat{y}^n, \quad a_{m,i,n} \in \mathbb{C}[c,t]$$

The lemma again can be proved using a faithful representation of $\mathbb{H}^{\mathrm{trig}}$. Here the space is $\mathbb{C}[c,t][X^{\pm 1}]$. The element s acts by $X \mapsto X^{-1}$ and X acts by multiplication. To give the action of $\hat{y}$, first define the *trigonometric derivative* by $\partial(X) = X$. Then, $\hat{y}$ acts by the *trigonometric Dunkl operator*

$$D^{\mathrm{trig}} := t\partial - 2c\frac{1}{1-X^{-2}}(\mathrm{id} - s) + c$$

This is known as the *differential polynomial representation* of $\mathbb{H}^{\mathrm{trig}}$. We also have a different polynomial representation of $\mathbb{H}^{\mathrm{trig}}$, which stems from the fact that the variables X and $\hat{y}$ are not symmetric. This is an action of $\mathbb{H}^{\mathrm{trig}}$ on $\mathbb{C}[c,t][\hat{y}]$. Note that we have an action of S_2 on $\mathbb{C}[c,t][\hat{y}]$, where s acts by $y \mapsto -y$. Then, $s \in \mathbb{H}^{\mathrm{trig}}$ acts on $\mathbb{C}[c,t][\hat{y}]$ via the operator

$$S := s - \frac{c}{\hat{y}}(s - \mathrm{id})$$

To define an action of X, let $\pi : \mathbb{C}[c,t][\hat{y}] \to \mathbb{C}[c,t][\hat{y}]$ be defined by $f(\hat{y}) \mapsto f(-\hat{y}+t)$. Then, X acts via the operator πS. This is known as the *difference-rational polynomial representation* of $\mathbb{H}^{\mathrm{trig}}$. The operator πS is known as the *difference-rational Dunkl operator.*

Corollary 2.3.3. *The following are subalgebras of* $\mathbb{H}^{\mathrm{trig}}$*:*

(1) *The group algebra of the extended affine Weyl group for A_1: it is isomorphic to the subalgebra of $\mathbb{H}^{\mathrm{trig}}$ generated by s, X.*
(2) *The degenerate affine Hecke algebra for A_1: it is isomorphic to the subalgebra of $\mathbb{H}^{\mathrm{trig}}$ generated by $s, \hat{y}$.*

2.3.5. *Rational degeneration.* Now in $\mathbb{H}^{\mathrm{trig}}$ set $X = \exp(\hbar x)$ and $y = \hbar\hat{y}$. Then, modulo $\hbar$, the elements s, x, y, satisfy the following relations

$$s^2 = 1, \quad sx = -xs \quad sy = -ys \quad yx - xy = t - 2cs$$

Define the algebra $\mathbb{H}^{\mathrm{rat}} := \mathbb{C}[c,t]\langle s, x, y\rangle$ with the relations above. This is known as the *rational* DAHA of A_1.

Lemma 2.3.4. *Every element $h \in \mathbb{H}^{\mathrm{rat}}$ can be uniquely written in the form*

$$h = \sum_{\substack{m,n\in\mathbb{Z}\\ i=0,1}} a_{m,i,n} x^m s^i y^n, \quad a_{m,i,n} \in \mathbb{C}[c,t]$$

Lemma 2.3.4 may be proven using the *polynomial representation* of $\mathbb{H}^{\mathrm{rat}}$. This is the representation on $\mathbb{C}[c,t][x]$, where s acts by $x \mapsto -x$, x acts by multiplication and y acts by the *rational Dunkl operator*

$$D^{\mathrm{rat}} := t\frac{d}{dx} + c\frac{1}{x}(s - \mathrm{id})$$

2.4. Example: DAHA for $\mathfrak{gl}_n$.

2.4.1. *The affine Hecke algebra for $\mathfrak{gl}_n$, revisited.* We will now define the DAHA for $\mathfrak{gl}_n$, which is different (but closely related to) from the DAHA of type A_n. So the first step is to study the affine Hecke algebra for $\mathfrak{gl}_n$, which has already appeared at the end of Seth's talk. Recall that we denote $\mathbb{C}_\tau := \mathbb{C}(\tau)$, the field of rational functions on the variable τ.

Definition 2.4.1. *Let $n > 0$. The* affine Hecke algebra of $\mathfrak{gl}_n$, *$\mathcal{H}_n$, is the $\mathbb{C}_\tau$-algebra with generators $T_1, \ldots, T_{n-1}, Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$ and relations*

(Quadratic relations) $(T_i - \tau)(T_i + \tau^{-1}) = 0$, *for* $i = 1, \dots, n-1$.
(Braid relations) $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ *for* $i = 1, \dots, n-2$; $T_iT_j = T_jT_i$ *if* $|i-j| > 1$.
(Action relations) $T_i^{-1}Y_iT_i^{-1} = Y_{i+1}$, $i = 1, \dots, n-1$; $T_jY_i = Y_iT_j$ *if* $j \neq i, i-1$.
(Laurent relations) $Y_iY_j = Y_jY_i$ *for* $i, j = 1, \dots, n$; $Y_iY_i^{-1} = 1$.

In order to define DAHA, we will need Cherednik's basic representation for $\mathcal{H}_n$. This is more easily given in the Coxeter presentation, so we will need an analogue of the Coxeter presentation of $\mathcal{H}_n$. Let us introduce the following element of $\mathcal{H}_n$:

$$\pi := T_1^{-1} \cdots T_{i-1}^{-1}Y_i^{-1}T_i \cdots T_{n-1}$$

Note that, thanks to the action relations, the element π is well-defined, i.e., it does not depend on $i = 1, \dots, n$. So, for example, $\pi = Y_1^{-1}T_1 \cdots T_{n-1} = T_1^{-1} \cdots T_{n-1}^{-1}Y_n$.

Lemma 2.4.2. *The element* π^n *is central in* $\mathcal{H}_n$.

Proof. We have

$$\begin{aligned} \pi^n = &\ (Y_1^{-1}T_1 \cdots T_{n-1})(Y_1^{-1}T_1 \cdots T_{n-1}) \cdots (Y_1^{-1}T_1 \cdots T_{n-1}) \\ = &\ Y_1^{-1}Y_2^{-1} \cdots Y_n^{-1}A_1A_2 \cdots A_n \end{aligned}$$

where $A_i := T_1^{-1}T_2^{-1} \cdots T_{n-i}^{-1}T_{n-i+1} \cdots T_{n-1}$, so, for example, $A_1 = T_1^{-1}T_2^{-1} \cdots T_{n-1}^{-1}$ and $A_n = T_1T_2 \cdots T_{n-1}$. We claim that $A_1A_2 \cdots A_n = 1$, note that this will finish the proof of the lemma. Indeed, this can already be seen in the braid group B_n: first of all, the associated permutation in S_n of every A_i's is the cycle $1 \mapsto n \mapsto n-1 \mapsto \cdots \mapsto 2 \mapsto 1$, and the n-th power of this cycle is the identity. So $A_1 \cdots A_n$ is, at least, an element of the pure braid group.

Now note that, in A_i, the strand starting at 1 passes *below* the strands starting at $2, \dots, n-i$ and *above* the strands starting at $n-i+1, \dots, n-1$. So, in the product $A_1 \cdots A_n$, the strand connecting 1 to 1 passes below all other strands; the strand connecting 2 to 2 passes above the strand connecting 1 to 1 and below all other strands and, in general, the strand connecting i to i passes above the strand connecting j to j if $j < i$, and below the strand connecting j to j if $j > i$. So $A_1 \cdots A_n = 1$. ☐

Lemma 2.4.3. *We have* $\pi T_i \pi^{-1} = T_{i+1}$, $i = 1, \dots, n-2$.

Proof. Here we use $\pi = T_1^{-1} \cdots T_{n-1}^{-1}Y_n$. So

$$\begin{aligned} \pi T_i \pi^{-1} = &\ (T_1^{-1} \cdots T_{n-1}^{-1}Y_n)T_i(Y_n^{-1}T_{n-1} \cdots T_1) \\ = &\ (T_1^{-1} \cdots T_i^{-1})(T_{i+1}^{-1}T_iT_{i+1})(T_i \cdots T_1) \end{aligned}$$

Now we use the identity $T_{i+1}^{-1}T_iT_{i+1} = T_iT_{i+1}T_i^{-1}$, which follows immediately form the braid relation involving i, $i+1$. From here, the result follows easily. ☐

Theorem 2.4.4. *The affine Hecke algebra* $\mathcal{H}_n$ *is generated by* $T_1, \dots, T_{n-1}, \pi^{\pm 1}$ *with relations:*

(1) *The braid and quadratic relations involving the* T_i.
(2) $\pi T_i \pi^{-1} = T_{i+1}$, $i = 1, \dots, n-2$.
(3) π^n *is central.*

Proof. Let $\mathcal{H}'_n$ denote the algebra defined in the statement of the theorem. Define

$$Y_i := T_i \cdots T_{n-1}\pi^{-1}T_1^{-1} \cdots T_{i-1}^{-1}.$$

We have to check that the Y_i's satisfy the action and commutativity relations. Let us check the action relations. First of all, it is clear that $T_i^{-1}Y_iT_i^{-1} = Y_{i+1}$, for $i = 1, \dots, n$. Now, if $j > i$ we have

$$\begin{aligned} T_jY_i = & \; T_jT_iT_{i+1}\cdots T_{n-1}\pi^{-1}T_1^{-1}\cdots T_{i-1}^{-1} \\ = & \; T_i\cdots T_jT_{j-1}T_j\cdots T_{n-1}\pi^{-1}T_1^{-1}\cdots T_{i-1}^{-1} \\ = & \; T_i\cdots T_{j-1}T_j\cdots T_{n-1}T_{j-1}\pi^{-1}T_1^{-1}\cdots T_{i-1}^{-1} \\ = & \; T_i\cdots T_{n-1}\pi^{-1}T_jT_1^{-1}\cdots T_{i-1}^{-1} \\ = & \; Y_iT_j \end{aligned}$$

and if $j < i-1$ we have

$$\begin{aligned} T_jY_i = & \; T_i\cdots T_{n-1}T_j\pi^{-1}T_1^{-1}\cdots T_{i-1}^{-1} \\ = & \; T_i\cdots T_{n-1}\pi^{-1}T_{j+1}T_1^{-1}\cdots T_{i-1}^{-1} \\ = & \; T_i\cdots T_{n-1}\pi^{-1}T_1^{-1}\cdots T_{j+1}T_j^{-1}T_{j+1}^{-1}\cdots T_{i-1}^{-1} \\ = & \; T_i\cdots T_{n-1}\pi^{-1}T_1^{-1}\cdots T_j^{-1}T_{j+1}^{-1}T_jT_{j+2}^{-1}\cdots T_{i-1}^{-1} \\ = & \; Y_iT_j \end{aligned}$$

Let us proceed to the commutation relations. We prove them in several steps.

Step 1: If $Y_1Y_j = Y_jY_1$ for every $j = 1, \ldots, n$, then $Y_iY_j = Y_jY_i$ for every $i, j = 1, \ldots, n$. Indeed, assume that $i < j$. Then, using the action relations that we have already shown:

$$\begin{aligned} Y_iY_j = & \; T_{i-1}^{-1}\cdots T_1^{-1}Y_1T_1^{-1}\cdots T_{i-1}^{-1}Y_j \\ = & \; T_{i-1}^{-1}\cdots T_1^{-1}Y_1Y_jT_1^{-1}\cdots T_{i-1}^{-1} \\ = & \; T_{i-1}^{-1}\cdots T_1^{-1}Y_jY_1T_1^{-1}\cdots T_{i-1}^{-1} \\ = & \; Y_jT_{i-1}^{-1}\cdots T_1^{-1}Y_1T_1^{-1}\cdots T_{i-1}^{-1} \\ = & \; Y_jY_i \end{aligned}$$

Step 2: If $Y_1Y_2 = Y_2Y_1$, then $Y_1Y_j = Y_jY_1$ for every $j = 1, \ldots, n$. This is done similarly to Step 1.

Step 3: $Y_1Y_2 = Y_2Y_1$. We need to show that $Y_1T_1^{-1}Y_1T_1^{-1} = T_1^{-1}Y_1T_1^{-1}Y_1$. The left-hand side of this equation becomes

$$\begin{aligned} Y_1T_1^{-1}Y_1T_1^{-1} = & \; T_1\cdots T_{n-1}\pi^{-1}T_1^{-1}T_1\cdots T_{n-1}\pi^{-1}T_1^{-1} \\ = & \; T_1\cdots T_{n-1}T_1\cdots T_{n-2}\pi^{-2}T_1^{-1} \end{aligned} \tag{2.4.1}$$

And the right-hand side becomes

$$\begin{aligned} T_1^{-1}Y_1T_1^{-1}Y_1 = & \; T_1^{-1}T_1\cdots T_{n-1}\pi^{-1}T_1^{-1}T_1\cdots T_n\pi^{-1} \\ = & \; T_2\cdots T_{n-1}T_1\cdots T_{n-2}\pi^{-2} \end{aligned} \tag{2.4.2}$$

Now we use that π^n is central in $\mathcal{H}_n'$. Indeed, we have $\pi^{-n}T_1\pi^n = T_1$, which implies that $\pi^{-2}T_1\pi^2 = T_{n-1}$, or $T_1\pi^2 = \pi^2T_{n-1}$, so $\pi^{-2}T_1^{-1} = T_{n-1}^{-1}\pi^{-2}$. We use this on the right-hand side of Equation (2.4.1). Now inductively use the identity $T_{i-1}T_i^{-1} = T_i^{-1}T_{i-1}^{-1}T_iT_{i-1}$, together with the braid relations, to get an equality with (2.4.2). □

We also need an analog of Cherednik's basic representation. This is given by the following.

Theorem 2.4.5. *The following assignment defines a representation of $\mathcal{H}_n$ on the space $\mathbb{C}_{q,\tau}[X_1^{\pm1}, \ldots, X_n^{\pm1}]$:*

$$T_i \mapsto \tau s_i + (\tau - \tau^{-1})\frac{s_i - \mathrm{id}}{1 - X_iX_{i+1}^{-1}}$$
$$\pi(X_1^{a_1}\cdots X_n^{a_n}) = q^{-a_n}X_1^{a_n}X_2^{a_1}\cdots X_n^{a_{n-1}}$$

Proof. We need to check that these operators satisfy the relations of $\mathcal{H}_n$. That the T_i satisfy the braid and quadratic relations is very similar to what Seth has already done. Note also that $\pi^n(X_1^{a_1}\cdots X_n^{a_n}) = q^{-\sum a_i}X_1^{a_1}\cdots X_n^{a_n}$. Since the operators T_i preserve the grading, it follows that they commute with π^n. The only relation we need to check now is that $\pi T_i = T_{i+1}\pi$ for $i = 1,\dots,n-2$. This is clear. □

Let us examine the relations between the operators X_i (multiplication) and π. First of all, it is clear that for $i = 1,\dots,n-1$, we have that $\pi X_i = X_{i+1}\pi$. For $i = n$, we get $\pi X_n = q^{-1}X_1\pi$. And since π^n is a grading operator, we get that $\pi^n X_i = q^{-1}X_i\pi^n$.

2.4.2. *Generators and relations.* The DAHA for $\mathfrak{gl}_n$ is the $\mathbb{C}_{q,\tau}$-algebra generated by the operators $T_j, j = 1,\dots,n-1$, $X_i, i = 1,\dots,n$ and π. Let us give a precise definition by generators and relations.

Definition 2.4.6. *The DAHA $\mathbb{H}_n$ is the $\mathbb{C}_{q,\tau}$-algebra generated by $T_1,\dots,T_{n-1}, X_1^{\pm 1},\dots,X_n^{\pm 1},\pi^{\pm 1}$ with relations:*

1. *The quadratic relations for $T_1,\dots,T_{n-1}$: $(T_i-\tau)(T_i+\tau^{-1}) = 0$.*
2. *The braid relations for $T_1,\dots,T_{n-1}$: $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, $T_iT_j = T_jT_i$ if $|i-j|>1$.*
3. *The Laurent relations for $X_1^{\pm},\dots,X_n^{\pm}$: $X_iX_j = X_jX_i$, $X_iX_i^{-1} = X_i^{-1}X_i = 1$.*
4. *The action relations involving T_i, X_j: $T_iX_iT_i = X_{i+1}$ if $i = 1,\dots,n-1$; $T_iX_j = X_jT_i$ if $i \neq j, j-1$.*
5. *$\pi X_i = X_{i+1}\pi, i = 1,\dots,n-1$; ; $\pi^n X_i = q^{-1}X_i\pi^n$.*
6. *$\pi T_i = T_{i+1}\pi$, $i = 1,\dots,n-2$; $\pi^n T_i = T_i\pi^n$, $i = 1,\dots,n-1$.*

Remark 2.4.7. *Let us remark that the relations $\pi X_n = q^{-1}X_1\pi$ and $\pi^2T_{n-1} = T_1\pi^2$ are formal corollaries of the relations (5), (6) in Definition 2.4.6.*

Now set

$$Y_i := T_i\cdots T_{n-1}\pi^{-1}T_1^{-1}\cdots T_{i-1}^{-1} \in \mathbb{H}_n$$

It is clear that the Y_i's satisfy the Laurent relations, as well as the relations

$$T_i^{-1}Y_iT_i^{-1} = Y_{i+1}, \quad T_iY_j = Y_jT_i \text{ if } i \neq j, j-1$$

Let us examine the relations of Y_i with X_j. First of all, since $Y_1\cdots Y_n = \pi^{-n}$, we get

$$\widetilde{Y}X_j = qX_j\widetilde{Y} \tag{2.4.3}$$

where $\widetilde{Y} := Y_1\cdots Y_n$. Now, setting $\widetilde{X} := X_1\cdots X_n$, we have that $\widetilde{X}$ commutes with all the T_i's while we have that $\pi\widetilde{X} = q^{-1}\widetilde{X}\pi$. This easily implies that

$$\widetilde{X}Y_j = q^{-1}Y_j\widetilde{X}. \tag{2.4.4}$$

Finally, we have the following relation.

$$\begin{aligned} Y_2^{-1}X_1Y_2X_1^{-1} = {} & (T_1\pi T_{n-1}^{-1}\cdots T_2^{-1})X_1(T_2\cdots T_{n-1}\pi^{-1}T_1^{-1})X_1^{-1} \\ = {} & T_1\pi T_{n-1}^{-1}\cdots T_2^{-1}T_2\cdots T_{n-1}X_1\pi^{-1}T_1^{-1}X_1^{-1} \\ = {} & T_1(\pi X_1\pi^{-1})T_1^{-1}X_1^{-1} \\ = {} & T_1X_2(T_1^{-1}X_1^{-1}T_1^{-1})T_1 \\ = {} & T_1(X_2X_2^{-1})T_1 \\ = {} & T_1^2. \end{aligned} \tag{2.4.5}$$

Theorem 2.4.8. *The DAHA $\mathbb{H}_n$ is isomorphic to the $\mathbb{C}_{q,\tau}$-algebra generated by $T_1, \ldots, T_{n-1}$, $X_1^{\pm 1}, \ldots, X_n^{\pm 1}$, $Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}$ subject to the following relations.*

1. *The quadratic and braid relations for $T_1, \ldots, T_{n-1}$.*
2. *The Laurent relations for $\{X_1^{\pm 1}, \ldots, X_n^{\pm 1}\}$ and for $\{Y_1^{\pm 1}, \ldots, Y_n^{\pm 1}\}$.*
3. *The action relations for (T_i, X_j) and for (T_i, Y_j).*
4. *Relations (2.4.3), (2.4.4) and (2.4.5).*

Proof. Let $\mathbb{H}'_n$ denote the algebra defined in the statement of the theorem. Define π by

$$\pi := T_1^{-1} \cdots T_{i-1}^{-1} Y_i^{-1} T_i \cdots T_{n-1}$$

Thanks to the action relations involving T and Y, π is independent of i. We need to check that $T_1, \ldots, T_{n-1}, X_1^{\pm 1}, \ldots, X_n^{\pm 1}, \pi$ satisfy the relations of $\mathbb{H}_n$. Note that we only need to check the relations involving π and X_i. Moreover, since $\pi^n = Y_1^{-1} \cdots Y_n^{-1}$, we only need to check that $\pi X_i = X_{i+1}\pi$ for $i = 1, \ldots, n-1$. Furthermore, note that because of the action relations involving T, X and the relations $\pi T_i = T_{i+1}\pi$, we only need to check the relation $\pi X_1 = X_2 \pi$. Using the relation (2.4.5) we have

$$\begin{aligned} \pi X_1 = &\; T_1^{-1} Y_2^{-1} T_2 \cdots T_{n-1} X_1 \\ = &\; T_1^{-1}(Y_2^{-1} X_1) T_2 \cdots T_{n-1} \\ = &\; T_1^{-1}(T_1^2 X_1 Y_2^{-1}) T_2 \cdots T_{n-1} \\ = &\; T_1 X_1 (T_1 T_1^{-1}) Y_2^{-1} T_2 \cdots T_{n-1} \\ = &\; (T_1 X_1 T_1) T_1^{-1} Y_2^{-1} T_2 \cdots T_{n-1} \\ = &\; X_2 \pi \end{aligned}$$

and the result follows. $\square$

Just as in the A_1 case, the T, X, Y presentation of the DAHA $\mathbb{H}_n$ has the advantage of revealing a symmetry between the X and Y parameters.

Lemma 2.4.9. *The following defines a $\mathbb{C}_{q,\tau}$-linear anti-involution of $\mathbb{H}_n$*

$$\phi(X_i) = Y_i^{-1}, \quad \phi(Y_i) = X_i^{-1}, \quad \phi(T_j) = T_j, \quad 1 \leq i \leq n, 1 \leq j \leq n-1$$

and the following defines a $\mathbb{C}$-linear involution of $\mathbb{H}_n$:

$$\varepsilon(X_i) = Y_i, \quad \varepsilon(Y_i) = X_i, \quad \varepsilon(T_j) = T_j^{-1}, \quad \varepsilon(\tau) = \tau^{-1}, \quad \varepsilon(q) = q^{-1}, \quad 1 \leq i \leq n, 1 \leq j \leq n-1$$

Proof. For the first statement, we only need to check that the relation (2.4.5) is self-dual with respect to ϕ. Note that we can write this relation as:

$$\begin{aligned} 1 = &\; T_1^{-1} Y_2^{-1} X_1 Y_2 X_1^{-1} T_1^{-1} \\ = &\; (T_1^{-1} Y_2^{-1} T_1^{-1})(T_1 X_1 T_1)(T_1^{-1} Y_2 T_1^{-1})(T_1 X_1^{-1} T_1) T_1^{-2} \\ = &\; Y_1^{-1} X_2 (T_1^{-2} Y_1 T_1^{-2})(T_1^2 X_2^{-1} T_1^2) T_1^{-2} \\ = &\; Y_1^{-1} X_2 T_1^{-2} Y_1 X_2^{-1} \end{aligned} \tag{2.4.6}$$

which we can rewrite as $T_1^2 = Y_1 X_2^{-1} Y_1^{-1} X_2$, and so (2.4.5) is self-dual with respect to ϕ. Note that this is also the equation required to prove that ε extends to a morphism $\mathbb{H} \to \mathbb{H}$. This finishes the proof. $\square$

Exercise 2.4.10. *The following relations hold in $\mathbb{H}_n$:*

$$Y_{i+1}^{-1} X_i Y_{i+1} X_i^{-1} = T_i^2, \quad Y_{j+1}^{-1} X_i Y_{j+1} X_i^{-1} = T_j \cdots T_{i+1} T_i^2 T_{i+1}^{-1} \cdots T_j^{-1}, j > i$$

2.4.3. *Topological interpretation.* Let E be a 2-torus. Consider the n-fold product E^n, and let $(E^n)^{reg} := \{(x_1, \dots, x_n) \in E^n : x_i \neq x_j \text{ if } i \neq j\}$, $C := (E^n)^{reg}/S_n$. The fundamental group $\pi_1(C)$ is known as the *elliptic braid group.*

Lemma 2.4.11. *We have $\pi_1(C) = \langle T_1, \dots, T_{n-1}, X_1, \dots, X_n, Y_1, \dots, Y_n\rangle$ with relations*

$$\begin{array}{ccc} T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} & T_iT_j = T_jT_i \text{ if } |i-j| > 1, & X_iX_j = X_jX_i \\ Y_iY_j = Y_jY_i & T_iX_iT_i = X_{i+1} & T_i^{-1}Y_iT_i^{-1} = Y_{i+1} \\ T_iX_j = X_jT_i, i \neq j, j-1 & T_iY_j = Y_jT_i, i \neq j, j-1 & Y_2^{-1}X_1Y_2X_1^{-1} = T_1^2 \\ (Y_1 \cdots Y_n)X_j = X_j(Y_1 \cdots Y_n) & & (X_1 \cdots X_n)Y_j = Y_j(X_1 \cdots X_n) \end{array}$$

In the previous lemma, the generator X_i corresponds to the i-th point going around a loop in the "horizontal" direction on E; Y_i corresponds to the i-th point going around in the "vertical" direction on E; while T_i corresponds to the transposition of the i-th and $(i+1)$-th points. Let us remark that, unlike the A_1 case, it is *not* possible to renormalize the generators so that the DAHA $\mathbb{H}_n$ becomes an honest quotient of the group algebra of $\pi_1(C)$. However, one may form a *twisted group algebra*, which is a deformation of the group algebra $\pi_1(C)$ arising from a central extension of $\pi_1(C)$ (so that the central element z becomes q in the twisted group algebra) and we indeed have

$$\mathbb{H}_n = \mathbb{C}^{tw}_{q,\tau}\pi_1(C)\big/((T_i - \tau)(T_i + \tau^{-1}))_{i=1,\dots,n-1}$$

2.4.4. *From $\mathfrak{gl}_n$ to $\mathfrak{sl}_n$.* Let us explain how to recover the DAHA for the root system A_{n-1} from $\mathbb{H}_n$. First of all, in the lattice generated by Y_i we must have $Y_1 \cdots Y_n = 1$. Thus, we pass to the algebra

$$\widetilde{\mathbb{H}}_n := \mathbb{H}_n/(\pi^n - 1)$$

In this algebra, we take the subalgebra generated by $T_1, \dots, T_{n-1}$, the elements $\overline{Y}_i := Y_1 \cdots Y_i$, $i = 1, \dots, n-1$ and their multiplicative inverses, and the elements $\overline{X}_i := X_1 \cdots X_i(\widetilde{X})^{-i/n}$. We remark that the element $\widetilde{X}$ does have an n-root in $\mathbb{H}_n$ (this can be seen, for example, using the automorphism ε defined in Lemma 2.4.9 and using the fact that $\widetilde{Y} = \pi^{-n}$) so this expression makes sense. We also take $\overline{X}_i^{-1}$. This subalgebra is isomorphic to $\mathbb{H}(A_{n-1})$.

2.4.5. *Trigonometric degeneration.* Let us introduce the trigonometric degeneration of the DAHA $\mathbb{H}_n$. This is done completely analogously to the A_1 case. So the first thing we need to do is to think of $\mathbb{H}_n$ as a $\mathbb{C}[t, c][[\hbar]]$-algebra. Set

$$Y_i := e^{\hbar \hat{y}_i}, \quad q := e^{t\hbar}, \quad \tau := e^{\hbar c}, \quad T_j := s_je^{\hbar c s_j}, \quad i = 1, \dots, n, \;\; j = 1, \dots, n-1$$

where $s_i \in S_n$ is the transposition $(i, i+1)$. We have $\mathbb{H}_n^{\text{trig}} := \mathbb{H}_n/\hbar\mathbb{H}_n$. So $\mathbb{H}^{\text{trig}}$ is generated by $s_i, i = 1, \dots, n-1$, $X_i^{\pm 1}$, $i = 1, \dots, n$, and $\hat{y}_i$, $i = 1, \dots, n$. We have $s_i^2 = 1$. From the identity $T_iX_iT_i = X_{i+1}$, we get $s_iX_is_i = X_{i+1}$. Let us now examine the identity $T_i^{-1}Y_iT_i^{-1} = Y_{i+1}$, we have

$$(s_i - \hbar cs_i^2 + \hbar^2\frac{c^2s_i^3}{2!} + \dots)(1 + \hbar\hat{y}_i + \hbar^2\frac{\hat{y}_i^2}{2!} + \dots)(s_i - \hbar cs_i^2 + \hbar^2\frac{c^2s_i^3}{2!} + \dots) = 1 + \hbar\hat{y}_{i+1} + \hbar^2\frac{\hat{y}_{i+1}^2}{2!} + \dots$$

looking at the coefficient of $\hbar$ we get the identity $s_i\hat{y}_is_i - 2cs_i = \hat{y}_{i+1}$ or, equivalently,

$$s_i\hat{y}_i - \hat{y}_{i+1}s_i = 2c.$$

Similarly, we have the following relations:

$$\begin{aligned}
&(\hat{y}_1 + \cdots + \hat{y}_n)X_j = X_j(t + \hat{y}_1 + \cdots + \hat{y}_n),\\
&X_1 \cdots X_n \hat{y}_j = (-t + \hat{y}_j)X_1 \cdots X_n,\\
&X_1 \hat{y}_2 X_1^{-1} - \hat{y}_2 = 2cs_1
\end{aligned}$$

Definition 2.4.12. *The* trigonometric double affine Hecke algebra *for $\mathfrak{gl}_n$, $\mathbb{H}_n^{\text{trig}}$, is the $\mathbb{C}[c,t]$-algebra with generators $s_1, \dots, s_{n-1}, \hat{y}_1, \dots, \hat{y}_n, X_1^{\pm}, \dots, X_n^{\pm}$ subject to the relations*

(1) $s_i^2 = 1, i = 1, \dots, n-1$; $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$; $s_i s_j = s_j s_i$ *if* $|i-j| > 1$.
(2) $s_i X_i s_i = X_{i+1}$; $s_i X_j = X_j s_i$ *if* $i \neq j, j-1$.
(3) $s_i \hat{y}_i - \hat{y}_{i+1} s_i = 2c$, $s_i \hat{y}_j = \hat{y}_j s_i$ *if* $i \neq j, j-1$.
(4) $(\hat{y}_1 + \cdots + \hat{y}_n)X_j = X_j(t + \hat{y}_1 + \cdots + \hat{y}_n)$.
(5) $X_1 \cdots X_n \hat{y}_j = (\hat{y}_j - t)X_1 \cdots X_n$.
(6) $X_1 \hat{y}_2 X_1^{-1} - \hat{y}_2 = 2cs_1$

Lemma 2.4.13. *Every element $h \in \mathbb{H}_n^{\text{trig}}$ can be uniquely written as a sum*

$$h = \sum_{\substack{P \in \mathbb{C}[X_i^{\pm 1}] \\ w \in S_n \\ f \in \mathbb{C}[\widehat{y}_i]}} a_{P,w,f} P(X) w f(\hat{y}), \quad a_{P,w,f} \in \mathbb{C}[c,t]$$

Of course, the lemma is proven by means of an action of $\mathbb{H}_n^{\text{trig}}$ on its *polynomial representation* $\mathbb{C}[c,t][X_1^{\pm 1}, \dots, X_n^{\pm n}]$, where s_i acts by transposing the i-th and $(i+1)$-th variables, X_i acts by multiplication. Now, for $i = 1, \dots, n$, define the trigonometric derivative by $\partial_i(X_j) = \delta_{ij} X_j$. So $\hat{y}_i$ acts by the *trigonometric Dunkl operator*

$$D_i^{\text{trig}} := t\partial_i + 2c \sum_{i \neq j} \frac{1}{1 - X_i X_j^{-1}} (\operatorname{id} - s_{ij}) - 2cis_i$$

Corollary 2.4.14. *The following are subalgebras of $\mathbb{H}_n^{\text{trig}}$:*

(1) *The group algebra of the extended affine Weyl group of $\mathfrak{gl}_n$, which is isomorphic to the subalgebra of $\mathbb{H}_n^{\text{trig}}$ generated by $s_1, \dots, s_{n-1}$, $X_1^{\pm 1}, \dots, X_n^{\pm 1}$.*
(2) *The degenerate affine Hecke algebra of $\mathfrak{gl}_n$, which is isomorphic to the subalgebra of $\mathbb{H}_n^{\text{trig}}$ generated by $s_1, \dots, s_{n-1}, \hat{y}_1, \dots, \hat{y}_n$.*

Let us remark that we also have a *difference-rational polynomial representation* of $\mathbb{H}_n^{\text{trig}}$. This is the representation of $\mathbb{H}_n^{\text{trig}}$ on the space $\mathbb{C}[c,t][\hat{y}_1, \dots, \hat{y}_n]$ which is defined as follows. The element $\hat{y}_i$ just acts by multiplication, the element s_i acts by the *Demazure-Lusztig operator*:

$$S_i := \tilde{s}_i - 2c \frac{1}{\hat{y}_i - \hat{y}_{i+1}} (\tilde{s}_i - \operatorname{id})$$

where $\tilde{s}_i$ is the operator on $\mathbb{C}[c,t][\hat{y}_1, \dots, \hat{y}_n]$ that transposes the variables $\hat{y}_i$ and $\hat{y}_{i+1}$. To state the action of X_i, first define the operator $\pi : \mathbb{C}[c,t][\hat{y}_1, \dots, \hat{y}_n] \to \mathbb{C}[c,t][\hat{y}_1, \dots, \hat{y}_n]$ by

$$\pi(f(\hat{y}_1, \dots, \hat{y}_n)) = f(\hat{y}_2, \dots, \hat{y}_n, \hat{y}_1 - t)$$

And now define the action of X_i to be by the operator $S_{i-1} \cdots S_1 \pi S_{n-1} \cdots S_i$. This is known as the *difference-rational Dunkl operator*.

2.4.6. *Rational degeneration.* Now we define the rational degeneration for the DAHA $\mathbb{H}_n$. Similarly to what was done for the case of A_1, let $X_i = \exp(\hbar x_i)$, $y_i := \hbar \hat{y}_i$. Then $s_1, \dots, s_{n-1}, y_1, \dots, y_n, x_1, \dots, x_n$ satisfy the following relations modulo $\hbar$.

$$s_i x_i s_i = x_{i+1}, \quad s_i y_i s_i = y_{i+1}, \quad s_i x_j = x_j s_i (i \neq j, j-1), \quad s_i y_j = y_j s_i (i \neq j, j-1),$$
$$y_i x_j - x_j y_i = 2c s_{ij}, i \neq j \quad (y_1 + \cdots + y_n) x_j = t + x_j (y_1 + \cdots + y_n)$$

Note that, in view of all the other relations, the last relation is equivalent to $y_i x_i - x_i y_i = t - 2c \sum_{j \neq i} s_{ij}$. These are the defining relations for the *rational* DAHA of $\mathfrak{gl}_n$, $\mathbb{H}_n^{\mathrm{rat}}$.

Lemma 2.4.15. *Every element $h \in \mathbb{H}_n^{\mathrm{rat}}$ can be written as*

$$h = \sum_{\substack{f \in \mathbb{C}[x] \\ w \in S_n \\ g \in \mathbb{C}[y]}} a_{f,w,g} f(x) w g(y), \quad a_{f,w,g} \in \mathbb{C}[c,t]$$

We have the polynomial representation $\mathbb{C}[c,t][x_1, \dots, x_n]$ of $\mathbb{H}_n^{\mathrm{rat}}$. Here, S_n acts by permutation of the indices, x_i acts by multiplication, and y_i acts via the rational Dunkl operator

$$D_i^{\mathrm{rat}} := t \frac{d}{dx_i} - 2c \sum_{j \neq i} \frac{1}{x_i - x_j} (\mathrm{id} - s_{ij})$$

Let us remark that similar degenerations $\mathbb{H}^{\mathrm{trig}}, \mathbb{H}^{\mathrm{rat}}$ exist for a general root system R. This will be the subject of Section 4.

2.4.7. *Braid group action.* The main goal of this section is to produce a braid group action on $\mathbb{H}_n$ by algebra automorphisms.

Lemma 2.4.16. *The following assignment can be extended to an automorphism of $\mathbb{H}_n$:*

$$\rho_1(T_i) = T_i, i = 1, \dots, n-1, \quad \rho_1(X_j) = X_j, j = 1, \dots, n, \quad \rho_1(\pi) = X_1^{-1} \pi \tag{2.4.7}$$

Proof. The only relation that is not immediate to check that it is preserved is $\pi^n T_i = T_i \pi^n$. Using the relations $\pi X_i^{-1} = X_{i+1}^{-1} \pi$ if $i < n$

$$\begin{aligned} (X_1^{-1}\pi) \cdots (X_1^{-1}\pi) = & \quad X_1^{-1} (\pi X_1^{-1}) \cdots (\pi X_1^{-1}) \pi \\ = & \quad X_1^{-1} X_2^{-1} \cdots X_n^{-1} \pi^n \end{aligned}$$

Which is the product of a symmetric polynomial in the $X_i's$ and π^n. Both terms commute with all T_i. From here, the result follows. $\square$

For completeness, let us give a formula for $\rho_1(Y_i)$. Recall that we have $Y_1 = T_1 \cdots T_{n-1} \pi^{-1}$, so $\rho_1(Y_1) = T_1 \cdots T_{n-1} \pi^{-1} X_1 = Y_1 X_1$. Now, using the fact that $T_i^{-1} Y_i T_i^{-1} = Y_{i+1}$ we get:

$$\rho_1(Y_i) = Y_i X_i (T_{i-1}^{-1} \cdots T_1^{-1})(T_1^{-1} \cdots T_{i-1}^{-1}) \tag{2.4.8}$$

The following lemma can be checked similarly to Lemma 2.4.16.

Lemma 2.4.17. *The following assignment can be extended to an automorphism of $\mathbb{H}_n$:*

$$\rho_2(T_i) = T_i, i = 1, \dots, n-1; \quad \rho_2(Y_j) = Y_j, \quad \rho_2(X_j) = X_j Y_j (T_{j-1} \cdots T_1)(T_1 \cdots T_{j-1}). \tag{2.4.9}$$

Remark 2.4.18. *Note that $\rho_2 = \varepsilon \rho_1 \varepsilon$, where $\varepsilon : \mathbb{H} \to \mathbb{H}$ is the $\mathbb{C}$-linear involution defined in Lemma 2.4.9.*

Let us give a formula for ρ_2^{-1}. We have that $\rho_2^{-1}(T_i) = T_i, \rho_2^{-1}(Y_j) = Y_j$, while

$$\rho_2^{-1}(X_j) = X_j(T_{j-1}^{-1} \cdots T_1^{-1})(T_1^{-1} \cdots T_{j-1}^{-1})Y_j^{-1}$$

Lemma 2.4.19. *Consider the braid group on three strands, $B_3 := \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\rangle$. The assignment*

$$\sigma_1 \mapsto \rho_1, \quad \sigma_2 \mapsto \rho_2^{-1}$$

gives an action of B_3 on $\mathbb{H}_n$.

Proof. We need to check that ρ_1, ρ_2^{-1} satisfy the relation:

$$\rho_1\rho_2^{-1}\rho_1 = \rho_2^{-1}\rho_1\rho_2^{-1} \tag{2.4.10}$$

It is obvious that when we evaluate both sides on T_i we just get T_i. Then, thanks to the action relations and the fact that both sides of (2.4.10) are automorphisms, we just need to check that $\rho_1\rho_2^{-1}\rho_1(X_1) = \rho_2^{-1}\rho_1\rho_2^{-1}(X_1)$, and a similar equation for Y_1. We have:

$$\rho_1\rho_2^{-1}\rho_1(X_1) = \rho_1\rho_2^{-1}(X_1) = \rho_1(X_1Y_1^{-1}) = X_1X_1^{-1}Y_1^{-1} = Y_1^{-1}$$
$$\rho_2^{-1}\rho_1\rho_2^{-1}(X_1) = \rho_2^{-1}(Y_1^{-1}) = Y_1^{-1}$$

It is similarly easy to check that $\rho_1\rho_2^{-1}\rho_1(Y_1) = \rho_2^{-1}\rho_1\rho_2^{-1}(Y_1) = Y_1X_1Y_1^{-1}$. The lemma follows. □

Let us remark that the automorphisms ρ_1, ρ_2 descend to the rational degeneration of $\mathbb{H}_n$.

Lemma 2.4.20. *The following define automorphisms of $\mathbb{H}_n^{\mathrm{rat}}$:*

$$\begin{array}{ccc} \rho_1(s_i) = s_i & \rho_1(x_j) = x_j & \rho_1(y_j) = y_j + x_j \\ \rho_2(s_i) = s_i & \rho_2(x_j) = x_j + y_j & \rho_2(y_j) = y_j \end{array}$$

We still have the relations $\rho_1\rho_2^{-1}\rho_1 = \rho_2^{-1}\rho_1\rho_2^{-1}$. Moreover, $(\rho_1\rho_2^{-1}\rho_1)^4 = \mathrm{id}$, so that we have an action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}_n^{\mathrm{rat}}$, cf. Lemma 3.3.5.

It is interesting to note that, according to [C2, 2.12.4], the automorphisms ρ_1, ρ_2 have no trigonometric analogue.

3. The polynomial representation

3.1. **Upper triangularity of Y^λ.** We study the polynomial representation more carefully. Our first goal is to see that the operators Y^λ are upper triangular with respect to a certain partial order on P. First of all, recall that we have the partial order $<$ on P^+, which is defined by $\nu > \mu$ if $\nu - \mu \in P^+$. We extend this order to P.

Definition 3.1.1. *For $\mu \in P$, let $\mu^+ \in P^+$ be the dominant weight lying in the orbit $W\mu$. Define a partial order on the weight lattice P as follows: $\nu \prec \mu$ if $\nu^+ < \mu^+$, or $\nu^+ = \mu^+$ and $\nu > \mu$ (note the change of signs!)*

Let us give some properties of the order $\prec$ that will be useful later.

Lemma 3.1.2. *Let $\mu \in P$ and let $\alpha \in R_+$*

1. *If $\langle \mu, \alpha^\vee \rangle = r > 0$, then $s_\alpha(\mu) \succ \mu$, while $\mu - \alpha, \ldots, \mu - (r-1)\alpha \prec \mu$.*
2. *If $\langle \mu, \alpha^\vee \rangle = -r < 0$, then $\mu + \alpha, \ldots, \mu + (r-1)\alpha, s_\alpha(\mu) \prec \mu$.*

Proof. See e.g. [M, Section 2.6]. □

Recall now from Seth's talk that, if $\lambda \in P_+^\vee$ is such that $t(\lambda) = \pi_r s_{i_\ell} \cdots s_{i_1}$ is a reduced expression, then $Y^\lambda = \pi_r T_{i_l} \cdots T_{i_1}$. We would like to use this to obtain some information about the operator Y^λ. First of all, if $a \in R^a$ is a root, define the operator

$$G(a) := \tau_a + (\tau_a - \tau_a^{-1}) \frac{\mathrm{id} - s_a}{X^{-a} - 1}$$

where $\tau_a = \tau_i$ if $w(a) = a_i$ for some element $w \in W^a$. Note that $T_i = s_i G(a_i)$ and, if $w \in W^a$, $G(w(a)) = wG(a)w^{-1}$. In particular, if s is a reflection we have $G(a)s = sG(s(a))$. Thus, for $\lambda \in P_+^\vee$ we have

$$Y^\lambda = \pi_r s_{i_\ell} G(a_{i_\ell}) s_{i_{\ell-1}} G(a_{i_{\ell-1}}) \cdots s_{i_1} G(a_{i_1}) = t(\lambda) G(a^{(\ell)}) \cdots G(a^{(1)}),$$

where $a^{(j)} = s_{i_1} \cdots s_{i_{j-1}}(a_{i_j})$.

Theorem 3.1.3. *For $\lambda \in P^\vee, \mu \in P$ we have*

$$Y^\lambda(X^\mu) = \sum_{\nu \preceq \mu} c_{\mu,\nu} X^\nu$$

with $c_{\mu,\nu} \in \mathbb{C}_{q,\tau}$.

Proof. Assume first that $\lambda \in P_+^\vee$, so that in particular $Y_\lambda = t(\lambda) G(a^{(\ell)}) \cdots G(a^{(1)})$ with $a^{(i)} = \alpha_i + k_i \delta$, $\alpha_i \in R^+$. So let $a = \alpha + k\delta$ with $\alpha \in R^+$. Then we have:

$$G(a) X^\mu = \tau_a X^\mu + (\tau_a - \tau_a^{-1}) \frac{X^{s_a(\mu)} - X^\mu}{1 - X^{-a}}$$

Now assume that $\langle \alpha^\vee, \mu \rangle = r > 0$, so $s_a(\mu) = \mu - ra$. Thus, we have

$$G(a) X^\mu = \tau_a X^\mu - (\tau_a - \tau_a^{-1})(X^\mu + X^{\mu - a} + \cdots + X^{\mu - (r-1)a}) = \tau_\alpha^{-1} X^\mu + \ldots$$

where the ellipsis stands for lower order terms, see Lemma 3.1.2. The case $\langle \alpha^\vee, \mu \rangle < 0$ is similar, for $\langle \alpha^\vee, \mu \rangle = 0$ we just have $G(a) X^\mu = \tau_a X^\mu$. Since $t(\lambda)$ is diagonal, we have that Y^λ is a composition of upper triangular operators and the result follows.

Now if $\lambda = \lambda' - \lambda''$ with $\lambda', \lambda'' \in P_+^\vee$, then $Y^\lambda = Y^{\lambda'} (Y^{\lambda''})^{-1}$. Since the inverse of an upper triangular operator is again upper triangular, the result follows. $\square$

3.2. **Difference operators.** The goal of this section is to produce some difference operators on the space $\mathbb{C}_{q,\tau}[X]$ using the representation theory of DAHA. Recall from the proof of Theorem 2.2.4 that for every $w \in W^{ae}$, the extended affine Weyl group, the action of T_w on $\mathbb{C}_{q,\tau}[X]$ may be written as

$$T_w = \sum_{\substack{\lambda \in P^\vee \\ w \in W}} g_{\lambda,w} t(\lambda) w, \qquad g_{\lambda,w} \in \mathbb{C}_{q,\tau}(X) \tag{3.2.1}$$

so in particular the same is true for $Y^\lambda \in \mathcal{H}^Y \subseteq \mathbb{H}$. Recall that the center of $\mathcal{H}^Y$ is precisely $\mathbb{C}_{q,\tau}[Y]^W$.

Lemma 3.2.1. *Let $f(Y) \in \mathbb{C}_{q,\tau}[Y]^W$. Then, the action of $f(Y)$ on $\mathbb{C}_{q,\tau}[X]$ preserves the space of W-invariants $\mathbb{C}_{q,\tau}[X]^W$.*

Proof. Note that, from the formula for the action of T_i, $i = 1, \ldots, n$, on $\mathbb{C}_{q,\tau}[X]$ it follows that $p(X) \in \mathbb{C}_{q,\tau}[X]$ is W-invariant if and only if

$$T_i p(X) = \tau_i p(X), i = 1, \ldots, n$$

From here, the result follows easily using the fact that $f(Y)$ commutes with T_i. $\square$

Now let f be an operator on $\mathbb{C}_{q,\tau}[X]$ of the form (3.2.1). We define its restriction by

$$\operatorname{Res}(f) := \sum_{\substack{\lambda\in P^\vee\\ w\in W}} g_{\lambda,w}\tau(\lambda)$$

In particular, $\operatorname{Res}(f)$ is a difference operator on $\mathbb{C}_{q,\tau}[X]$ and $f|_{\mathbb{C}_{q,\tau}[X]^W} = \operatorname{Res}(f)|_{\mathbb{C}_{q,\tau}[X]^W}$. For $f \in \mathbb{C}_{q,\tau}[Y]^W$, we denote $L_f := \operatorname{Res}(f)$. This is a difference operator on $\mathbb{C}_{q,\tau}[X]$ preserving the space of invariants $\mathbb{C}_{q,\tau}[X]^W$.

Corollary 3.2.2. *The operators $L_f, f \in \mathbb{C}_{q,\tau}[Y]^W$, are pairwise commutative and W-invariant.*

Proof. Let $f, g \in \mathbb{C}_{q,\tau}[Y]^W$. Then $L_fL_g = \operatorname{Res}(f)\operatorname{Res}(g)$. Since g is W-invariant, $\operatorname{Res}(f)\operatorname{Res}(g) = \operatorname{Res}(fg) = \operatorname{Res}(gf)$. Since now f is W-invariant, we get $\operatorname{Res}(gf) = \operatorname{Res}(g)\operatorname{Res}(f) = L_gL_f$. $\square$

Remark 3.2.3. *It follows from Theorem 3.1.3 that the operators $L_f : \mathbb{C}_{q,\tau}[X]^W \to \mathbb{C}_{q,\tau}[X]^W$ are upper triangular with respect to the basis formed by $\{x_\lambda := \sum_{\lambda'\in W\lambda} X^{\lambda'}\}_{\lambda\in P^+}$ and the dominance ordering on P^+.*

The operators L_f are intimately related to the theory of Macdonald's polynomials. This will be the subject of a subsequent talk.

3.2.1. *Example: A_1.* Let us consider the example of a root system of type A_1. We keep the notation of Section 2.3.2, with one small caveat. Now we set $X := X^\alpha$, so that $\mathbb{C}_{q,\tau}[X]$ is the algebra of polynomials in $X^{\pm 1/2}$. With this convention, the action of T is given by

$$\tau s + (\tau - \tau^{-1})\frac{s - \mathrm{id}}{X - 1}$$

While the action of $t(\rho)$ is $t(\rho)(X) = q^2X$, so $t(\rho)f(X) = f(q^2X)$ and, in particular, $\pi_\rho(X) = t(\rho)s(X) = q^{-2}X^{-1}$. Note that we have

$$Y^\rho = \quad t(\rho)\left(\tau + (\tau - \tau^{-1})\frac{\mathrm{id} - s}{X^{-1} - 1}\right)$$

Let us now deal with $Y^{-\rho} = T^{-1}\pi_\rho$. Here we will use the reflection s_0 on the affine Weyl group: it is easy to check that we have a relation $\pi_\rho T^{-1}\pi_\rho = T_0^{-1}$, so that $Y^{-\rho} = \pi_\rho T_0^{-1} = \pi_\rho(T_0 + (\tau^{-1} - \tau))$. Thus,

$$\begin{aligned}
Y^{-\rho} = &\quad \pi_\rho\left(\tau_{s_0} + (\tau - \tau^{-1})\frac{s_0 - \mathrm{id}}{q^{-2}X^{-1} - 1} + (\tau^{-1} - \tau)\right)\\
= &\quad t(\rho)s\left(\tau t(\alpha)s + (\tau - \tau^{-1})\frac{t(\alpha)s - q^{-2}X^{-1}}{q^{-2}X^{-1} - 1}\right)\\
= &\quad \tau t(-\rho) + (\tau - \tau^{-1})t(\rho)\frac{t(-\alpha) - q^{-2}Xs}{q^{-2}X - 1}\\
= &\quad \tau t(-\rho) + (\tau - \tau^{-1})\frac{t(-\rho) - Xt(\rho)s}{X - 1}
\end{aligned}$$

Thus, $\operatorname{Res} Y^\rho = \tau t(\rho)$, $\operatorname{Res} Y^{-\rho} = \dfrac{\tau X - \tau^{-1}}{X - 1}t(-\rho) + (\tau - \tau^{-1})\dfrac{1}{X^{-1} - 1}t(\rho)$. So

$$\operatorname{Res}(Y^\rho + Y^{-\rho}) = \frac{\tau X^{-1} - \tau^{-1}}{X^{-1} - 1}t(\rho) + \frac{\tau X - \tau^{-1}}{X - 1}t(-\rho). \tag{3.2.2}$$

This is (a scalar multiple of) Macdonald's difference operator for A_1. The symmetric polynomials here are spanned by binomials of the form $X^{i/2} + X^{-i/2}, i \geq 0$. It is an easy exercise to check that the operator (3.2.2) indeed preserves the space of symmetric polynomials, and that it is upper triangular with respect to the basis $x_i := X^{i/2} + X^{-i/2}$.

3.3. **Spherical DAHA.** We have seen that the operators $L_f, f \in \mathbb{C}_{q,\tau}[X]^W$ define difference operators on the space of W-invariant polynomials on X. We can actually define a smaller algebra than the DAHA $\mathbb{H}$ which includes all the operators L_f and which acts on $\mathbb{C}_{q,\tau}[X]^W$. This is known as the *spherical* DAHA and it is constructed as follows.

Let $\mathbb{C}_\tau$ be the 1-dimensional (over $\mathbb{C}_\tau$) representation of the finite Hecke algebra H where T_i acts by τ_i, $i = 1, \dots, n$. We can realize this representation as $\mathbb{C}_\tau = H\mathbf{e}$, where $\mathbf{e} \in H$ is an idempotent which is constructed as follows. For $w \in W$ let $\tau_w := \tau_{i_1} \cdots \tau_{i_k}$, where $w = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition. Note that τ_w is well-defined, since it is the scalar by which w acts on the representation $\mathbb{C}_\tau$. Now define $\widetilde{\mathbf{e}} := \sum_{w \in W} \tau_w T_w$.

Lemma 3.3.1. *For $i = 1, \dots, n$, we have $T_i \widetilde{\mathbf{e}} = \tau_i \widetilde{\mathbf{e}}$.*

Proof. We will do a direct calculation. We will need the following equation that we have already seen in Seth's talk. In the finite Hecke algebra H:

$$T_i T_w = \begin{cases} T_{s_i w} & \text{if } \ell(s_i w) > \ell(w) \\ T_{s_i w} + (\tau_i - \tau_i^{-1}) T_w & \text{if } \ell(s_i w) < \ell(w) \end{cases}$$

where the length ℓ is the usual one in W, i.e., the length of a reduced expression of w. Thus, we have:

$$T_i \widetilde{\mathbf{e}} = \sum_{\substack{w \in W \\ \ell(s_i w) > \ell(w)}} \tau_w T_{s_i w} + \sum_{\substack{w \in W \\ \ell(s_i w) < \ell(w)}} \tau_w (T_{s_i w} + (\tau_i - \tau_i^{-1}) T_w)$$

Now we find the coefficient of T_w in the previous expression. We have two cases. If $\ell(s_i w) < \ell(w)$, then we have that the coefficient of T_w is $\tau_{s_i w} + \tau_w(\tau_i - \tau_i^{-1}) = \tau_i \tau_w$, since $\tau_w = \tau_i \tau_{s_i w}$. If $\ell(s_i w) > \ell(w)$, then the coefficient of T_w is simply $\tau_{s_i w} = \tau_i \tau_w$. We are done. $\square$

Remark 3.3.2. *Similarly, we can see that $\widetilde{\mathbf{e}} T_i = \tau_i \widetilde{\mathbf{e}}$ for $i = 1, \dots, n$.*

Thanks to the previous lemma, $\widetilde{\mathbf{e}}^2 = \sum_{w \in W} \tau_w T_w \widetilde{\mathbf{e}} = \sum_{w \in W} \tau_w^2 \widetilde{\mathbf{e}}$. Thus

$$\mathbf{e} := \left(\sum_{w \in W} \tau_w^2 \right)^{-1} \widetilde{\mathbf{e}}$$

is an idempotent.

Definition 3.3.3. *Define the* spherical DAHA *as $\mathbb{SH} := \mathbf{e}\mathbb{H}\mathbf{e}$. This is a non-unital subalgebra of $\mathbb{H}$, with unit $\mathbf{e}$.*

Remark 3.3.4. *In the $\mathfrak{gl}_n$ case, note that the automorphisms ρ_1, ρ_2 of $\mathbb{H}_n$ preserve the idempotent $\mathbf{e}$, hence they also preserve the spherical subalgebra. So we have an action of B_3 on $\mathbb{SH}_n$.*

The following result will be important to connect DAHA's to EHA's, which is one of the objectives of the course. First, we recall a well-known result. For a proof, see e.g. [KT, Appendix A].

Lemma 3.3.5. *The group $\mathrm{SL}_2(\mathbb{Z})$ is a quotient of the braid group on three strands $B_3 = \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$. The quotient map $B_3 \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z})$ is given by*

$$\sigma_1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_2 \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

the kernel of this map is generated by $(\sigma_1\sigma_2\sigma_1)^4$.

Theorem 3.3.6. *The braid group action on $\mathbb{SH}_n$ factors through $\mathrm{SL}_2(\mathbb{Z})$, that is, $(\rho_1 \rho_2^{-1} \rho_1)|_{\mathbb{SH}_n}^4 = \mathrm{id}_{\mathbb{SH}_n}$.*

Proof. According to [C2, 3.2.2], $(\rho_1\rho_2^{-1}\rho_1)^4$ is conjugation by $T_{w_0}^{-2}$, where w_0 is the longest element of S_n. Since $T_{w_0}\mathbf{e} = \tau_{w_0}\mathbf{e}$, the result follows easily. $\square$

Note that, if M is a $\mathbb{H}$-module, then $\mathbf{e}M$ becomes a $\mathbb{SH}$-module. For the polynomial representation we have:

$$\mathbf{e}\mathbb{C}_{q,\tau}[X] = \{f \in \mathbb{C}_{q,\tau}[X] : T_i f = \tau_i f, i = 1, \dots, n\} = \mathbb{C}_{q,\tau}[X]^W$$

Now note that, for $f \in \mathbb{C}_{q,\tau}[Y]^W$, we have that $\mathbf{e}L_f\mathbf{e}|_{\mathbb{C}_{q,\tau}[X]^W} = L_f|_{\mathbb{C}_{q,\tau}[X]^W}$. Thus, the action of the spherical DAHA $\mathbb{SH}$ on $\mathbb{C}_{q,\tau}[X]^W$ already includes the operators L_f defined above.

4. Degenerations

In this section, we give definitions that generalize the degenerate (trigonometric and rational) DAHA's from Sections 2.3.4, 2.3.5, 2.4.5, 2.4.6. These algebras can be obtained from $\mathbb{H}$ in a very similar manner to what was done there.

4.1. **Trigonometric degeneration.** Let us first define the trigonometric DAHA. In order to do this, let c_i, $i = 0, \dots, n$ be formal variables such that $c_i = c_j$ whenever s_i and s_j are conjugate. We will also take commuting variables $\hat{y}_1, \dots, \hat{y}_n$ and, for $b \in P^\vee$, we will denote

$$\hat{y}_b := \sum \langle b, \alpha_j \rangle \hat{y}_j.$$

Let us remark that the extended affine Weyl group $W^{ae} = W \ltimes t(P)$ acts on the space $\mathbb{C}[c,t][\hat{y}_1, \dots, \hat{y}_n]$ by algebra automorphisms. Indeed, we need to define the action of $s_1, \dots, s_n$ and $t(\lambda)$, $\lambda \in P$ on elements of the form $\hat{y}_b, b \in P^\vee$. We have that $s_i\hat{y}_b = \hat{y}_{s_i(b)}$ for $i = 1, \dots, n$, while $t(\lambda)\hat{y}_b = \hat{y}_b - \langle \lambda, b \rangle t$.

Definition 4.1.1. *The* trigonometric DAHA, $\mathbb{H}^{\mathrm{trig}}$ *is the $\mathbb{C}[c,t]$-algebra generated by the extended affine Weyl group W^{ae} and pairwise commuting variables $\hat{y}_1, \dots, \hat{y}_n$, subject to the following relations.*

$$s_i\hat{y}_b - \hat{y}_{s_i(b)}s_i = -c_i\langle b, \alpha_i \rangle, \quad s_0\hat{y}_b - s_0(\hat{y}_b)s_0 = c_0\langle b, \theta \rangle, \quad \pi_r\hat{y}_b = \hat{y}_{\pi_r(b)}\pi_r \tag{4.1.1}$$

for $i = 1, \dots, n$, $b \in P^\vee$, and $\pi_r \in \Omega' (\cong P/Q)$.

Let us remark that the variable t appears in disguise in the second relation of (4.1.1).

Since, unlike the nondegenerate and rational cases, the variables X, $\hat{y}$ are not symmetric, the algebra $\mathbb{H}^{\mathrm{trig}}$ admits more than one polynomial representation. First, we have the *differential polynomial representation*, which is given in terms of trigonometric differential Dunkl operators. In order to do this, for $b \in P^\vee$, define the following derivation on the group algebra $\mathbb{C}[c,t][X]$ of the weight lattice P:

$$\partial_b(X^a) = \langle b, a \rangle X^a$$

We have then that $\mathbb{H}^{\mathrm{trig}}$ acts on $\mathbb{C}[c][X]$. The group W acts naturally and y_b acts via the *trigonometric differential Dunkl operator*

$$D_b^{\mathrm{trig}} := t\partial_b + \sum_{\alpha \in R^+} \frac{c_\alpha \langle b, \alpha^\vee \rangle}{1 - X^{-\alpha}}(\mathrm{id} - s_\alpha) - \langle \rho_c, b \rangle$$

where ρ_c is the formal expression $\rho_c := \dfrac{1}{2}\displaystyle\sum_{\alpha \in R^+} c_\alpha \alpha$.

We also have the *difference-rational polynomial representation*, on the algebra $\mathbb{C}[c,t][\hat{y}_1, \dots, \hat{y}_n]$. Recall that the extended affine Weyl group W^{ae} acts on this space by algebra automorphisms. We deform this action by the *Demazure-Lusztig operators*:

$$S_i := s_i + \frac{c_i}{\hat{y}_{\alpha_i}}(s_i - \mathrm{id}), i = 0, \dots, n$$

where $y_{\alpha_0} := -y_\theta + t$. And define, for $w \in W^{ae}$ with $w = \pi_r s_{i_1} \cdots s_{i_\ell}$ a reduced expression, $S_w := \pi_r S_{i_1} \cdots S_{i_\ell}$. According to [C2, 1.6], this still defines an action of W^{ae} on $\mathbb{C}[c,t][\lambda]$. Here we only check that $S_i^2 = \mathrm{id}$. Indeed, we have for $i \neq 0$

$$(4.1.2)\qquad \begin{aligned} S_i\hat{y}_a = & \left(s_i + \frac{c_i}{\hat{y}_{\alpha_i}}(s_i - \mathrm{id})\right)\hat{y}_a \\ = & \ \hat{y}_{s_i(a)} + \frac{c_i}{\hat{y}_{\alpha_i}}\left(\hat{y}_{s_i(a)} - \hat{y}_a\right) \\ = & \ \hat{y}_{s_i(a)} + \frac{c_i}{\hat{y}_{\alpha_i}}\left(\hat{y}_a - \hat{y}_{\langle\alpha_i^\vee, a\rangle\alpha_i} - \hat{y}_a\right) \\ = & \ \hat{y}_{s_i(a)} + \frac{c_i}{\hat{y}_{\alpha_i}}\left(-\langle\alpha_i^\vee, a\rangle\hat{y}_{\alpha_i}\right) \\ = & \ \hat{y}_{s_i(a)} - c_i\langle\alpha_i^\vee, a\rangle.1 \end{aligned}$$

Thanks to (4.1.2), we have that $S_i(\hat{y}_{s_i(a)}) = \hat{y}_a + c_i\langle\alpha_i^\vee, a\rangle.1$. It follows from (4.1.2) again that $S_i^2 = \mathrm{id}$. Let us now treat the case $i = 0$. First of all, note that $s_0\hat{y}_a = \hat{y}_a + \langle a, \theta^\vee\rangle(t - \hat{y}_\theta)$. Then, we have:

$$(4.1.3)\qquad \begin{aligned} S_0\hat{y}_a = & \left(s_0 + \frac{c_0}{t - \hat{y}_\theta}(s_0 - \mathrm{id})\right)\hat{y}_a \\ = & \ \hat{y}_a + \langle a, \theta^\vee\rangle(t - \hat{y}_\theta) + \frac{c_0}{t - \hat{y}_\theta}\langle a, \theta^\vee\rangle(t - \hat{y}_\theta) \\ = & \ \hat{y}_a + \langle a, \theta^\vee\rangle(t - \hat{y}_\theta) + c_0\langle\theta^\vee, a\rangle.1 \end{aligned}$$

It follows from (4.1.3), the fact that S_0 clearly fixes c, t and 1, and that $\langle\theta, \theta^\vee\rangle = 2$, that $S_0^2 = \mathrm{id}$.

Theorem 4.1.2 (See e.g. Proposition 1.6.3 in [C2])**.** *The algebra $\mathbb{H}^{\mathrm{trig}}$ acts on the space $\mathbb{C}[c,t][\hat{y}_1, \dots, \hat{y}_n]$, where elements of the group W act via S_w, and $\hat{y}_b$ acts by multiplication. This representation is faithful and it is known as the* difference-rational polynomial representation*.*

For $b \in P$, the operators $S_{t(b)}$ are known as the *difference-rational Dunkl operators.*

Corollary 4.1.3. *The following are subalgebras of $\mathbb{H}^{\mathrm{trig}}$:*

1. *The group algebra of W, in a natural way.*
2. *The degenerate affine Hecke algebra for W, which is the algebra generated by W and $\hat{y}_1, \dots, \hat{y}_n$.*

4.2. **Rational degeneration.** We also have a rational degeneration. Here, we substitute the group algebras of the lattices P and $P^\vee$ by the vector spaces $V^* \cong V$ where our root systems R, $R^\vee$ are defined.

Definition 4.2.1. *The* rational DAHA*, $\mathbb{H}^{\mathrm{rat}}$, is the $\mathbb{C}[c,t]$-algebra generated by $\mathbb{C}[V], \mathbb{C}[V^*]$ and the group W subject to the relations*

$$wx = w(x)w, \quad wy = w(y)w \quad [y, x] = t\langle y, x\rangle - \textstyle\sum_{\alpha\in R^+} c_\alpha\langle y, \alpha\rangle\langle\alpha^\vee, x\rangle s_\alpha, \quad w \in W, x \in V^*, y \in V$$

The algebra $\mathbb{H}^{\mathrm{rat}}$ admits a polynomial representation on the space $\mathbb{C}[V]$. Here, W acts in a natural way, and $x \in V^*$ acts by multiplication. Now recall that $y \in V$ defines a derivation on $\mathbb{C}[V]$, by setting $\partial_y(x) = \langle y, x\rangle$, $x \in V^*$. Then, we define the *rational Dunkl operator*

$$D_y^{\mathrm{rat}} := t\partial_y - \sum_{\alpha\in R^+} c_\alpha\frac{\langle\alpha, y\rangle}{\alpha}(\mathrm{id} - s_\alpha)$$

Theorem 4.2.2. *The assignment $w \mapsto w$, $x \mapsto x$, $y \mapsto D_y^{\mathrm{rat}}$ defines a representation of $\mathbb{H}^{\mathrm{rat}}$ on $\mathbb{C}[c,t][V]$. This is known as the* polynomial representation*, and it is a faithful representation of $\mathbb{H}^{\mathrm{rat}}$.*

Remark 4.2.3. *The hard part of the previous theorem is to prove that the Dunkl operators commute.*

Corollary 4.2.4. *The algebras $\mathbb{C}[c,t][V], \mathbb{C}[c,t][V^*], \mathbb{C}[c,t]W$ sit naturally as subalgebras of $\mathbb{H}^{\mathrm{rat}}$.*

Let us remark that, unlike $\mathbb{H}$ and $\mathbb{H}^{\mathrm{trig}}$, the definition of the rational DAHA $\mathbb{H}^{\mathrm{rat}}$ can be generalized to the case where W is a group generated by complex reflections acting on a vector space V (so W is not necessarily the Weyl group of a root system). This has been done in [EG].

Remark 4.2.5. *We can also define spherical subalgebras $\mathbb{SH}^{\mathrm{trig}}$, $\mathbb{SH}^{\mathrm{rat}}$ of the degenerate DAHAs. There are defined as $e\mathbb{H}^{\mathrm{trig}}e$, $e\mathbb{H}^{\mathrm{rat}}e$, respectively, where the idempotent e now is the trivial idempotent of the group W, that is, $e = \frac{1}{|W|}\sum_{w\in W} w$.*

Remark 4.2.6. *Let us remark that, just as we did in Section 3.2, we can use the representation theory of DAHA to define a large family of commuting differential (resp. difference) operators on $\mathbb{C}[c,t][X]$ or $\mathbb{C}[c,t][x_1,\dots,x_n]$ (resp. on $\mathbb{C}[c,t][\lambda]$) that restrict to differential (resp. difference) operators on the W-invariant subalgebras. These operators are given by elements in $\mathbb{C}[c,t][\hat{y}]^W$, $\mathbb{C}[c,t][y]^W$ and $\mathbb{C}[c,t][X]^W$, respectively.*

4.3. **Integrable systems.** The degenerate DAHA are connected to the theory of the Olshanetsky-Perelomov integrable systems, aka generalized Calogero-Moser integrable systems. In this section we elaborate on this connection. Here, we treat the differential case (i.e., rational DAHA,) the difference (i.e., trigonometric) case can be done by similar methods, see e.g. [C1]. Recall that we have a root system $R \subseteq V^* \cong V$, where V is a vector space with nondegenerate form $\langle\cdot,\cdot\rangle$. For the rest of these notes, we specialize to $t = 1$.

Definition 4.3.1. *The* quantum Olshanetsky-Perelomov Hamiltonian *of R is the differential operator*

$$H := \Delta_V - \sum_{\alpha\in R^+} \frac{c_\alpha(c_\alpha+1)\langle\alpha,\alpha\rangle}{\alpha^2}$$

where Δ_V is the Laplace operator on V, and $c_\alpha \in \mathbb{C}$ are such that $c_\alpha = c_{w(\alpha)}$ for every $w \in W$.

Example 4.3.2. *Perhaps, the quantum Olshanetsky-Perelomov Hamiltonian has the clearest physical meaning in type A. Here (taking $V = \mathbb{C}^n$ instead of $\mathbb{C}^{n-1}$) we have*

$$H = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{1\leq i<j\leq n} \frac{2c(c+1)}{(x_i-x_j)^2}$$

which is the quantum Hamiltonian for a system of n particles on the line interacting with potential $c(c+1)/(x_i-x_j)^2$.

Our goal is to see that the quantum system defined by the Olshanetsky-Perelomov Hamiltonian is completely integrable. Let us be a bit more explicit about this. Consider the action of the Weyl group W on the symmetric algebra $S(V)$. According to the Chevalley-Shepard-Todd theorem, the algebra of invariants $S(V)^W$ is polynomial, with algebraically homogeneous generators $P_1,\dots,P_n$ of degrees $d_1,\dots,d_n$, respectively. Recall also that we have the symbol map, $D(V) \to S(V^*)\otimes S(V)$, that to each differential operator associates its symbol. Note, however, that we need a slight extension of this: the hamiltonian H does not belong to $D(V)$. We can consider the principal open subset V^{reg} that is the complement of the union of the hyperplanes $\langle\alpha,\cdot\rangle = 0$. Then, we have a symbol map $\sigma : D(V^{reg}) \to \mathbb{C}[V^{reg}]\otimes S(V)$. For example, $\sigma(H) = P$, where $P(p) = \langle p,p\rangle$, and we use the inner product on V^* that is dual to the inner product on V. Note also that $\sigma(H) \in S(V)^W$.

In the sequel, we will assume that V is an irreducible representation of W. So $\sigma(H) = P_1$ where, recall, we denote $P_1, \ldots, P_n$ the algebraically independent homogeneous generators of $S(V)^W$.

Theorem 4.3.3. *The system defined by the quantum Olshanetsky-Perelomov Hamiltonian is completely integrable. More precisely, there exist algebraic differential operators $H_1, \ldots, H_n$ on V^{reg} such that:*

1. $H_1 = H$.
2. $\sigma(H_i) = P_i$.
3. $[H_i, H_j] = 0$.

Remark 4.3.4. *If we do not assume that V is an irreducible representation of W, then Theorem 4.3.3 is still valid with the exception that (1) should be replaced by $H_2 = H$, see e.g. Example 4.3.2, where we have $P_i = \sum_{i=1}^n x^i$.*

The idea to prove this theorem is similar to what we have done in Section 3.2. So, first of all, if $f = \sum_{w\in W} f_w w$ is an operator on V^{reg}, where $f_i \in D(V^{reg})$, define

$$\mathrm{Res}(f) = \sum_{w\in W} f_w$$

So that $\mathrm{Res}(f)$ is a differential operator. Note that if g is W-invariant, then $\mathrm{Res}(fg) = \mathrm{Res}(f)\,\mathrm{Res}(g)$ for any operator f of a similar form. Now let $y_1, \ldots, y_n$ be an orthonormal basis of V. So, considering the algebra $\mathbb{C}[y_1, \ldots, y_n] \subseteq \mathbb{H}^{\mathrm{rat}}$ as an algebra of operators on V^{reg}, which we can do thanks to the Dunkl representation, we have the following result, which is proven similarly to the results in Subsection 3.2.

Lemma 4.3.5. *For every $f \in \mathbb{C}[y_1, \ldots, y_n]^W$, denote $L_f := \mathrm{Res}(f)$. Then, $\{L_f : f \in \mathbb{C}[y_1, \ldots, y_n]^W\}$ form a commuting family of differential operators with coefficients being rational functions on V regular on V^{reg}. Moreover, $\sigma L_{P_i} = P_i$.*

So what remains to do is to relate the operator H to L_{P_1}.

Proposition 4.3.6. *We have*

$$L_{P_1} = \Delta_V - \sum_{\alpha\in R^+} \frac{c_\alpha \langle \alpha, \alpha\rangle}{\alpha} \partial_{\alpha^\vee}$$

Proof. We need to compute $\mathrm{Res}(\sum_{i=1}^n D_{y_i}^2)$, where we denote $D_{y_i} := D_{y_i}^{\mathrm{rat}}$. First of all, note that $\mathrm{Res}(D_{y_i}^2) = \mathrm{Res}(D_{y_i}\partial_{y_i})$. Now, for every $y \in V$ we have

$$\begin{aligned} D_y\partial_y = {} & \partial_y^2 - \textstyle\sum_{\alpha\in R^+} c_\alpha \frac{\langle \alpha, y\rangle}{\alpha}(\mathrm{id} - s_\alpha)\partial_y \\ = {} & \partial_y^2 - \textstyle\sum_{\alpha\in R^+} \frac{\langle \alpha, y\rangle}{\alpha}(\partial_y(\mathrm{id} - s_\alpha) + [\partial_y, s_\alpha]) \\ = {} & \partial_y^2 - \textstyle\sum_{\alpha\in R^+} \frac{\langle \alpha, y\rangle}{\alpha}(\partial_y(\mathrm{id} - s_\alpha) + \langle \alpha, y\rangle \partial_{\alpha^\vee} s) \end{aligned}$$

From where the result follows easily. ∎

Let us denote $\overline{H} := L_{P_1}$. It is not the quantum OP Hamiltonian, but we can get H via an automorphism $\varphi : D(V^{reg}) \to D(V^{reg})$, which is defined by $\varphi(f) = f$, $f \in \mathbb{C}[V^{reg}]$, $\varphi(\partial_y) = \partial_y - \sum_{\alpha\in R^+} c_\alpha \frac{\langle y, \alpha\rangle}{\alpha}$. It is an exercise to check that φ indeed defines an automorphism of $D(V^{reg})$. The next result finishes the proof of Theorem 4.3.3.

Lemma 4.3.7. *We have $\varphi(H) = \overline{H}$.*

Proof. We have

$$\begin{aligned}\varphi(\partial_{y_i}^2) = & \left(\partial_{y_i} - \sum_{\alpha\in R^+} c_\alpha \frac{\langle\alpha,y_i\rangle}{\alpha}\right)^2 \\ = & \partial_{y_i}^2 - \sum_{\alpha\in R^+} c_\alpha\langle\alpha,y\rangle(\partial_{y_i}\alpha^{-1} + \alpha^{-1}\partial_{y_i}) + \sum_{\alpha,\alpha'\in R^+} c_\alpha c_{\alpha'}\frac{\langle\alpha,y_i\rangle\langle\alpha',y_i\rangle}{\alpha\alpha'} \\ = & \partial_{y_i}^2 - 2\sum_{\alpha\in R^+} c_\alpha\langle\alpha,y_i\rangle\alpha^{-1}\partial_{y_i} + \sum_{\alpha\in R^+} c_\alpha\frac{\langle\alpha,y_i\rangle^2}{\alpha^2} + \sum_{\alpha,\alpha'\in R^+} c_\alpha c_{\alpha'}\frac{\langle\alpha,y_i\rangle\langle\alpha',y_i\rangle}{\alpha\alpha'}\end{aligned}$$

So it follows that

$$\begin{aligned}\varphi(\Delta_V) = & \Delta_V - \sum_{\alpha\in R^+}\frac{c_\alpha}{\alpha}\sum_{i=1}^n 2\langle\alpha,y_i\rangle\partial_{y_i} + \sum_{\alpha\in R^+}\frac{c_\alpha}{\alpha^2}\sum_{i=1}^n\langle\alpha,y_i\rangle^2 + \sum_{\alpha,\alpha'\in R^+}\frac{c_\alpha c_{\alpha'}}{\alpha\alpha'}\sum_{i=1}^n\langle\alpha,y_i\rangle\langle\alpha',y_i\rangle \\ = & \Delta_V - \sum_{\alpha\in R^+}\frac{c_\alpha}{\alpha}\langle\alpha,\alpha\rangle\partial_{\alpha^\vee} + \sum_{\alpha\in R^+}\frac{c_s(c_s+1)\langle\alpha,\alpha\rangle}{\alpha^2} + \sum_{\alpha\neq\alpha'\in R^+} c_\alpha c_{\alpha'}\frac{\langle\alpha,\alpha'\rangle}{\alpha\alpha'}\end{aligned}$$

Thus, $\varphi(H) = \overline{H} + \sum_{\alpha\neq\alpha'\in R^+} c_\alpha c_{\alpha'}\frac{\langle\alpha,\alpha'\rangle}{\alpha\alpha'}$, and to prove the lemma (and hence Theorem 4.3.3) we just need to show that this last term, which we denote by P, is 0. First of all, note that the term is clearly W-invariant. Now denote

$$\delta := \prod_{\alpha\in R^+}\alpha$$

which is sign-invariant. So δP is sign-invariant. This is a polynomial of degree $n-2$. But the smallest degree of a nonzero sign-invariant element in $S(V)$ is n. Thus, $\delta P = 0$, and so $P = 0$. ☐

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LECTURE 3 (PART 1): MACDONALD POLYNOMIALS

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ABSTRACT. These are notes for a seminar talk given at the MIT-Northeastern Double Affine Hecke Algebras and Elliptic Hall Algebras (DAHAEHA) seminar (Spring 2017).

CONTENTS

1. GOALS

The purpose of this document is to introduce Macdonald polynomials, and prove the Macdonald conjecture which concerns the value of certain bilinear forms when evaluated on these polynomials. The existence of these polynomials is not trivial; we will see two different approaches. Then we will explain how Affine Hecke Algebras and Double Affine Hecke Algebras can be used to prove the Macdonald conjecture.

2. REVIEW OF NOTATION

2.1. **Root System and Weyl Groups.** We write R for an irreducible finite root system in a vector space V, equipped with inner product $(-,-)$. We write R^a for the associated affine root system. We employ the following notation:

- The set of positive roots of R is denoted R_+, and the set of negative roots is denoted R_-. Similarly we write R^a_+ and R^a_- for the positive and negative roots of R^a, respectively.
- We write $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ for a choice of simple roots in R. The coroot associated to α_i is $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}$.
- We write α_0 for $\delta - \theta$ where θ is the longest root in R. In this way $\{\alpha_0, \alpha_1, \cdots, \alpha_n$ form a set of simple roots for R^a.
- The root lattice is $Q = \mathbb{Z}\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. The coroot lattice is $Q^\vee = \mathbb{Z}\{\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_n^\vee\}$.
- The weight lattice is $P = \mathbb{Z}\{\omega_1, \omega_2, \cdots, \omega_n\}$, where ω_i is the i-th fundamental weight ($P = \{\lambda \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \forall \alpha \in R\}$). The dominant weights are $P_+ = \mathbb{N}\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ ($P_+ = \{\lambda \mid (\lambda, \alpha^\vee) \in \mathbb{N} \forall \alpha \in R\}$). Similarly we have the coweight lattice is $P^\vee = \{\lambda \mid (\lambda, \alpha) \in \mathbb{Z} \forall \alpha \in R\}$, and the dominant coweights are $P^\vee = \{\lambda \mid (\lambda, \alpha) \in \mathbb{N} \forall \alpha \in R\}$.
- The half-sum of positive roots is $\rho = \sum_{\alpha \in R_+} \alpha$, and it is well known that $\rho = \sum_{i=1}^n \omega_i$.

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- We write $W = \langle s_1, s_2, \cdots, s_n\rangle$ for the (finite) Weyl group associated to R, generated by the simple reflections s_i. We also write $W^a = \langle s_0, s_1, \cdots, s_n\rangle$ for the affine Weyl group. We have the isomorphism
$$W^a = W \ltimes t(Q^\vee)$$
Here, as in previous lectures, t indicates translation in $Q^\vee$, so this is a subgroup of the group of invertible affine maps on $Q^\vee$.
- The extended affine Weyl group is $W^{ae} = W \ltimes t(P^\vee)$ (a subgroup of the group of invertible affine linear maps on $P^\vee$).
- We write $\Omega \subset W^{ae}$ for the set of all length zero elements of W^{ae}. It is a subgroup which acts faithfully on the set of simple roots of R^a. Furthermore, Ω is isomorphic to $P^\vee/Q^\vee$ and is in bijection with minuscule weights (to be discussed later).
- We actually have $W^{ae} = \Omega \ltimes W^a$.
- We write λ_+ for the unique dominant weight in the W-orbit of λ. Similarly we write λ^- for the unique anttdominant weight in the orbit of λ
- We will work over the field $\mathbb{C}(q, t)$, which we write $\mathbb{C}_{q,t}$.
- Shortly after the beginning, we will specialise to the case where $t = q^k$ (where $k \in \mathbb{Z}_{\geq 0}$). The notation t appears in parts of the theory directly related to Hecke algebras.

2.2. **Double Affine Hecke Algebras.**

- We write H for the finite Hecke algebra attached to the root system R. Similarly H^a is the affine Hecke algebra, and H^{ae} is the extended affine Hecke algebra.
- $H^{ae} = \Omega \ltimes H^a$
- H^a is generated by $T_0, T_1, \cdots, T_n$ subject to the braid relations and the quadratic relations $(T_i - t_i)(T_i + t_i^{-1}) = 0$.
- Thinking of H^{ae} as a quotient of the affine extended braid group, one has the elements Y^λ coming from the lattice associated to integral coweights.
- $H^{ae} = H \otimes \mathbb{C}(t)[Y]$ as a vector space, where the Y^λ.
- The centre of H^{ae} is precisely $\mathbb{C}(t)[Y]^W$.
- We have Cherednik's basic representation of H^{ae} on $\mathbb{C}_{q,t}[X] = \mathbb{C}_{q,t}[P]$, where T_i acts as $t_i s_i + (t_i - t_i^{-1})\frac{s_i - 1}{X^{-\alpha_i} - 1}$.
- The extended affine Weyl group action on $\mathbb{C}_{q,t}[X]$ satisfies $t_\lambda)(X^\mu) = q^{2(\lambda,\mu)}X^\mu$, where $t_\lambda)$ is translation by λ.

3. Macdonald Polynomials

3.1. **Definition and Proof of Existence.** Recall that the Weyl group acts on the set of weights, P. We may therefore extend the action of W to the group algebra $\mathbb{C}_{q,t}[P]$. We will be concerned with elements of $\mathbb{C}_{q,t}[P]^W$, namely elements of the group algebra which are fixed by the Weyl group action. Note that $\mathbb{C}_{q,t}[P]^W$ is a linear subspace of $\mathbb{C}_{q,t}[P]$. Macdonald polynomials will form a basis of $\mathbb{C}_{q,t}[P]^W$. Note that an obvious basis of $\mathbb{C}_{q,t}[P]^W$ is given by the orbit sums $m_\lambda = \sum_{\mu \in W\lambda} e^\mu$ for $\lambda \in P_+$. By a standard theorem in Lie theory, there is a unique dominant weight in each Weyl group orbit on P, which shows that $\{m_\lambda \mid \lambda \in P_+\}$ is indeed a basis for $\mathbb{C}_{q,t}[P]^W$.

At this point, one might protest that it is unclear how these are polynomials. To answer this, recall that $P = \mathbb{Z}\{\omega_1, \omega_2, \cdots, \omega_n\}$ and therefore $\mathbb{C}_{q,t}[P]$ can be thought of as the algebra of Laurent polynomials in the variables $\omega_1, \omega_2, \cdots, \omega_n$ (with complex coefficients). To conform with standard notation, given $\lambda \in P$, we write e^λ instead of λ for the associated element in $\mathbb{C}_{q,t}[P]$ (this avoids ambiguity between additive and multiplicative notation).

Next, we introduce a bilinear form on $\mathbb{C}_{q,t}[P]$.

Definition 3.1. *If $f \in \mathbb{C}_{q,t}[P]$, write $[f]_0$ for the coefficient of e^0 in f, when expressed in the e^λ basis. Suppose that $f \mapsto \bar{f}$ is the involution of $\mathbb{C}_{q,t}[P]$ defined by $e^\lambda \mapsto e^{-\lambda}$. Let* $\Delta_{q,t} = \prod_{\alpha \in R}\prod_{i=0}^{\infty} \frac{1-q^{2i}e^\alpha}{1-t^2q^{2i}e^\alpha}$

(consider this as a Laurent series in the variables q, t, having coefficients in $\mathbb{C}_{q,t}[P]$). Then, we define the bilinear form $\langle -, -\rangle_{q,t}$ on $\mathbb{C}_{q,t}[P]$ as follows:

$$\langle f, g\rangle_{q,t} = \frac{1}{|W|}[f\Delta_{q,t}\bar{g}]_0$$

We will just be interested in the restriction of the bilinear form to $\mathbb{C}_{q,t}[P]^W$. Motivation for this construction will come later.

Note that if we define $\Delta^+_{q,t} = \prod_{\alpha\in R_+}\prod_{i=0}^{\infty}\frac{1-q^{2i}e^{\alpha}}{1-t^2q^{2i}e^{\alpha}}$, then $\Delta_{q,t} = \Delta^+_{q,t}\overline{\Delta^+_{q,t}}$. This will be convenient in what follows. We are now able to give a definition of Macdonald polynomials, although it will not be immediately clear that they exist.

Theorem 3.2. *For each $\lambda \in P_+$, there exists a unique $P_\lambda \in \mathbb{C}_{q,t}[P]^W$ such that:*
(1) $P_\lambda = m_\lambda + \sum_{\mu<\lambda} a_{\lambda,\mu}m_\mu$
(2) $\langle P_\lambda, P_\mu\rangle_{q,t} = 0$ *whenever* $\lambda \neq \mu$

Here, $\mu < \lambda$ means that $\lambda - \mu \in Q_+$.

We will prove this theorem (at least, in some special cases), but first we discuss it.

Remark 3.3. *Gram-Schmidt orthogonalisation cannot be applied here because $<$ is not a total order on P_+. However, it does imply uniqueness.*

To see why this is could be an interesting construction, we consider some examples.

Example 3.4. *Suppose that $t = 1$, so that $\Delta_{q,t} = 1$. Then $\langle f, g\rangle_{q,t} = [f\bar{g}]_0$, and it is easy to see that $P_\lambda = m_\lambda$ satisfy the statement of the theorem (and this does not depend on q).*

Now suppose that $t = q$. Then $\Delta^+_{q,t} = \prod_{\alpha\in R_+}(1 - e^{\alpha})$ because the product telescopes. Let χ_λ be given by the Weyl Character Formula for $\lambda \in P_+$:

$$\chi_\lambda = \frac{\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R_+}(1 - e^{-\alpha})}$$

We calculate $\langle \chi_{\lambda_1}, \chi_{\lambda_2}\rangle_{q,t}$.

$$\begin{aligned}
\langle \chi_{\lambda_1}, \chi_{\lambda_2}\rangle_{q,t} &= \frac{1}{|W|}[\chi_{\lambda_1}\Delta_{q,t}\overline{\chi_{\lambda_2}}]_0 \\
&= \frac{1}{|W|}[\left(\chi_{\lambda_1}\overline{\Delta^+_{q,t}}\right)(\Delta^+_{q,t}\overline{\chi_{\lambda_2}})]_0 \\
&= \frac{1}{|W|}[\sum_{w_1\in W}\varepsilon(w_1)e^{w_1(\lambda_1+\rho)-\rho}\sum_{w_2\in W}\varepsilon(w_2)e^{-w_2(\lambda_2+\rho)+\rho}]_0 \\
&= \frac{1}{|W|}\sum_{w_1\in W}\sum_{w_2\in W}\varepsilon(w_1w_2)[e^{w_1(\lambda_1+\rho)-w_2(\lambda_2+\rho)}]_0
\end{aligned}$$

Note that the nonzero terms are precisely those for which $w_1(\lambda_1 + \rho) = w_2(\lambda_2 + \rho)$. This is equivalent to $\lambda_1 + \rho = w_1^{-1}w_2(\lambda_2 + \rho)$. Using the fact that each weight has a unique dominant weight in its orbit in the Weyl group, we see that this equation can only hold if $\lambda_1 + \rho = \lambda_2 + \rho$ (the latter is the unique dominant weight in its orbit). So we get zero unless $\lambda_1 = \lambda_2$. Furthermore, the terms which contribute 1 are those for which $w_1^{-1}w_2$ fixes $\lambda_2 + \rho$. Recall the length of a Weyl group element is equal to the number of positive roots that it maps to negative roots, so that

$$w_1^{-1}w_2(\rho) = \frac{1}{2}\sum_{\alpha\in R_+}w_1^{-1}w_2(\alpha) \leq \frac{1}{2}\sum_{\alpha\in R_+}\alpha = \rho$$

with equality if and only if $w_1^{-1}w_2$ is the identity element of W. Since $w_1^{-1}w_2(\lambda) \leq \lambda$, we may add these two inequalities to find that the identity is the only element of W that fixes $\lambda_2 + \rho$. The number of solutions

(w_1, w_2) to $w_1^{-1}w_2 = \mathrm{Id}_W$ is clearly $|W|$, so we obtain

$$\langle \chi_{\lambda_1}, \chi_{\lambda_2} \rangle_{q,t} = \frac{|W|}{|W|} = 1$$

Finally, we write $\chi_\lambda = \sum_\mu a_{\lambda,\mu} m_\mu$, where the $a_{\lambda,\mu}$ correspond to the dimensions of weight spaces in the irreducible representation of the relevant simple Lie algebra of highest weight λ. Recall that the irreducible representation is generated by the action of the lower triangular part of the Lie algebra (usually written $\mathfrak{n}_-$) on a highest weight vector, which itself is unique up to scalar multiplication. This implies that only $\mu \leq \lambda$ appear in the sum, and that $a_{\lambda,\lambda} = 1$. This proves that χ_λ satisfy the conditions of the theorem.

In light of the preceding example, it might be reasonable to view Macdonald polynomials as a deformation of characters of representations of simple Lie algebras.

Example 3.5. *Let $V = \mathbb{R}^n$ with standard basis e_i, and let $R = \{e_i - e_j \mid i \neq j\}$ so that R is a root system of type A_{n-1} and we may take $\alpha_i = e_i - e_{i+1}$. The associated simple Lie algebra is $\mathfrak{sl}_n$, and the Weyl group is $W = S_n$, which acts on V by permutation of coordinates. Since $\alpha = \alpha^\vee$ for all $\alpha \in R$, the weights and the coweights of R are the same. Let $\mathbb{C}_{q,t}[P]$ be presented by letting $x_i = \exp(e_i)$, so that $e^{\alpha_i} = \frac{x_i}{x_{i+1}}$. Then $\mathbb{C}_{q,t}[P]$ is realised as the space of Laurent polynomials in $x_1, x_2, \cdots, x_n$ of total degree zero. Then it is easily seen that the positive roots correspond to $\frac{x_i}{x_j}$ with $i < j$. It is also easy to see that if $\lambda \in P$ corresponds to $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$ (where necessarily the λ_i sum to zero), then the value of the fundamental weight ω_r applied to λ is $\omega_r(\lambda) = \lambda_1 + \lambda_2 + \cdots + \lambda_r$. It is also easy to see that being dominant is equivalent to having the λ_i forming a weakly decreasing sequence, and λ is integral if and only if the λ_i are integers.*

We now prove theorem 3.2 in the case where $t = q^k$, for $k \in \mathbb{Z}_{\geq 0}$, and when there are minuscule weights associated to the root system R (so R cannot be G_2, F_4, E_8). Although these restrictions are not required, they mitigate technical difficulties. The specialisation of the parameter t is the case relevant to the Macdonald conjecture, so not much will be lost to us. The reader who is interested in greater generality is directed to [Mac00].

Proof. Firstly, note that the product in the definition of $\Delta^+_{q,t}$ telescopes:

$$\begin{aligned} \Delta^+_{q,t} &= \prod_{\alpha \in R_+} \prod_{i=0}^{\infty} \frac{1 - q^{2i} e^\alpha}{1 - t^2 q^{2i} e^\alpha} \\ &= \prod_{\alpha \in R_+} \prod_{i=0}^{\infty} \frac{1 - q^{2i} e^\alpha}{1 - q^{2k} q^{2i} e^\alpha} \\ &= \prod_{\alpha \in R_+} \prod_{i=0}^{k-1} (1 - q^{2i} e^\alpha) \end{aligned}$$

In particular, we obtain a finite expression. Next, if $\pi \in P^\vee$ we define $T_\pi(e^\lambda) = q^{2(\pi,\lambda) e^\lambda}$ (and extend linearly), where we may have to include fractional powers of q in our ring. Now let us write

$$D_\pi(f) = \sum_{w \in W} w \left(\frac{T_\pi(\Delta^+_{q,t}(f))}{\Delta^+_{q,t}} \right)$$

In the case where π is a minuscule coweight (i.e. $0 \leq (\lambda, \alpha) \leq 1$ for all positive roots $\alpha \in R_+$), this simplifies as follows.

$$D_\pi(f) = \left(\sum_{w \in W} w \right) \left(\prod_{\alpha \in R_+, (\pi,\alpha)=1} \frac{1 - q^{2k} e^\alpha}{1 - e^\alpha} \right) T_\pi(f) \tag{1}$$

This is clearly the symmetrisation of some rational function, whose denominator is a product of distinct terms of the form $(1 - e^\alpha)$. It is certainly W-invariant. In particular, let $\delta = \prod_{\alpha \in R_+} \frac{1}{e^{\alpha/2} - e^{-\alpha/2}}$ be the Weyl denominator, which is antisymmetric (implicitly we are now working in $e^\rho \mathbb{C}_{q,t}[P]$). To see this, recall that the action of $s_i \in W$ permutes the positive roots except for α_i which it maps to $-\alpha_i$. Thus s_i acts by multiplication by -1 on $\frac{1}{e^{\alpha_i/2}-e^{-\alpha_i/2}}$, and permutes the other factors of δ. Thus $\delta T_\pi(f)$ is antisymmetric with respect to the W-action, and is also a polynomial (we have removed the denominators). In particular, for any s_i, the coefficient of e^λ must be minus the coefficient of $e^{s_i(\lambda)}$. This means that no e^λ fixed by s_i can occur in $\delta T_\pi(f)$, so $\delta T_\pi(f)$ is a linear combination of $e^\lambda - e^{s_i(\lambda)}$. Now observe that

$$e^\lambda - e^{s_i(\lambda)} = e^\lambda - e^{\lambda-(\lambda,\alpha_i^\vee)\alpha_i} = e^\lambda(1-e^{-(\lambda,\alpha_i^\vee)\alpha_i}) = (1-e^{-\alpha_i})(1+e^{-\alpha_i}+\cdots+e^{-((\lambda,\alpha_i^\vee)-1)\alpha_i})$$

This is therefore divisible by $e^{\alpha_i/2}-e^{-\alpha_i/2}$ for each simple root α_i. As a result, the same is true of $\delta D_\pi(f)$. Since each root is in the orbit of a simple root, by applying the action of a suitable element of W we find that $e^{\alpha/2}-e^{-\alpha/2}$ divides $\delta D_\pi(f)$ for all positive roots α. It is not difficult to check that these are coprime in the UFD $\mathbb{C}_{q,t}[P/2]$. Hence, $\delta D_\pi(f)$ is divisible by $\prod_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2}) = \delta$ (in the sense of polynomial divisibility). We conclude that $D_\pi(f)$ is actually a polynomial (rather than a rational function), which preserves $\mathbb{C}_{q,t}[P]^W$.

If we can show that D_π is triangular with respect to the e^λ basis, and is self-adjoint with respect to $\langle -,-\rangle_{q,t}$ with distinct eigenvalues, then the theorem will follow. This is because triangularity allows us to restrict to the finite dimensional subspace spanned by m_μ for $\mu \le \lambda$, whence distinct eigenvalues guarantee diagonalisability. Finally, self adjointness (and distinctness of eigenvalues) implies the eigenvectors are orthogonal. The Macdonald polynomials will be the eigenvectors of this operator.

To see the self-adjoint property, recall that $\Delta_{q,t}$ is W-invariant, as is the m_λ basis:

$$\begin{aligned}
\frac{1}{|W|}[D_\pi(m_\lambda)\Delta_{q,t}\bar{e^\mu}]_0 &= \frac{1}{|W|}[\sum_{w\in W} w\left(\frac{T_\pi(\Delta^+_{q,t}m_\lambda)}{\Delta^+_{q,t}}\right)\Delta_{q,t}\overline{m_\mu}]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W}\left(\frac{T_\pi(\Delta^+_{q,t}m_\lambda)}{\Delta^+_{q,t}}\right)\Delta_{q,t}\overline{m_{w^{-1}\mu}}]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W} T_\pi(\Delta^+_{q,t}m_\lambda)\overline{\Delta^+_{q,t}}\overline{m_\mu}]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W} \Delta^+_{q,t}m_\lambda T_{-\pi}(\overline{\Delta^+_{q,t}}\overline{m_\mu})]_0 \\
&= \frac{1}{|W|}[\sum_{w\in W} m_\lambda\Delta^+_{q,t}\overline{T_\pi((\Delta^+_{q,t}m_\mu))}]_0 \\
&= \frac{1}{|W|}[e^\lambda\Delta_{q,t}\sum_{w\in W}\frac{\overline{T_\pi((\Delta^+_{q,t}m_\mu))}}{\overline{\Delta^+_{q,t}}}]_0 \\
&= \frac{1}{|W|}[e^\lambda\Delta_{q,t}\sum_{w\in W} w^{-1}\overline{\frac{T_\pi(\Delta^+_{q,t}m_\mu)}{\Delta^+_{q,t}}}]_0 \\
&= \frac{1}{|W|}[e^\lambda\Delta_{q,t}\overline{D_\pi(m_\mu)}]_0
\end{aligned}$$

We calculate the leading order term, and in doing so, observe triangularity. For this, we use a deformed version of the Weyl characters χ_λ. Throughout we consider everything as formal series of the form $c_\lambda e^\lambda + \sum_{\mu<\lambda} c_\mu e^\mu$ (the c_λ being constants), where we refer to $c_\lambda e^\lambda$ as the leading term.

$$\chi_\lambda^{(k)} = \frac{\sum_{w\in W}\varepsilon(w)e^{w(\lambda+k\rho)}}{\prod_{\alpha\in R_+}(e^{k\alpha/2}-e^{-k\alpha/2})}$$

This is a formal series rather than a polynomial. The numerator has leading order term $e^{\lambda+k\rho}$, and the denominator has leading order term $e^{k\rho}$, so it is easy to see that the $\chi_\lambda^{(k)}$ (for $\lambda \ge 0$) have leading term e^λ and so are related to the e^λ by a triangular matrix. Thus, it will be enough to consider $D_\pi(\chi_\lambda^{(k)})$ to prove triangularity and calculate the eigenvalues. So, we write

$$D_\pi(\chi_\lambda^{(k)}) = \sum_{w'\in W}\left[w'\left(\prod_{\alpha\in R_+,(\pi,\alpha)=1}\frac{1-q^{2k}e^\alpha}{1-e^\alpha}\right)\right]\left[w'(T_\pi(\chi_\lambda^{(k)}))\right]$$

We calculate the leading order term of each set of square brackets separately. This means we will write it as a constant times e^μ plus terms indexed by weights lower than μ in the dominance order. We first consider the action of w' on each factor of

$$\left(\prod_{\alpha\in R_+,(\pi,\alpha)=1}\frac{1-q^{2k}e^\alpha}{1-e^\alpha}\right) = \left(\prod_{\alpha\in R_+}\frac{1-q^{2k(\pi,\alpha)}e^\alpha}{1-e^\alpha}\right)$$

The action of w' on

$$\frac{1-q^{2k(\pi,\alpha)}e^\alpha}{1-e^\alpha}$$

is

$$\frac{1-q^{2k(\pi,\alpha)}e^{w'(\alpha)}}{1-e^{w'(\alpha)}}$$

If $w'(\alpha)$ is negative root, the leading term is just 1, otherwise we may write this as

$$\frac{q^{2k(\pi,\alpha)}-e^{-w'(\alpha)}}{1-e^{-w'(\alpha)}}$$

whence the leading term is clearly $q^{2k(\pi,\alpha)}$. Thus the total contribution to the leading order term is $q^{(\pi,2k\nu)}$, where ν is the sum of all positive roots α such that $w'(\alpha)$ is also positive.

Upon applying T_π to $\chi_\lambda^{(k)}$, we get

$$T_\pi(\chi_\lambda^{(k)}) = \frac{\sum_{w\in W}\varepsilon(w)q^{2(\pi,w(\lambda+k\rho))}e^{w(\lambda+k\rho)}}{\prod_{\alpha\in R_+}(q^{(\pi,k\alpha)}e^{k\alpha/2}-q^{(\pi,-k\alpha)}e^{-k\alpha/2})}$$

If the Weyl group element w' is applied to this expression, we get

$$w'(T_\pi(\chi_\lambda^{(k)})) = \frac{\sum_{w\in W}\varepsilon(w)q^{2(\pi,w(\lambda+k\rho))}e^{w'(w(\lambda+k\rho))}}{\prod_{\alpha\in R_+}w'(q^{(\pi,k\alpha)}e^{k\alpha/2}-q^{(\pi,-k\alpha)}e^{-k\alpha/2})}$$

We expand the denominator as a series of the form e^μ plus lower order terms. To extract the leading order term from $\prod_{\alpha\in R_+}w'(q^{(\pi,k\alpha)}e^{k\alpha/2}-q^{(\pi,-k\alpha)}e^{-k\alpha/2})$, we first note that we pick up a sign for each α mapped to a negative root, thus obtaining $\varepsilon(w')$. We pick the term corresponding to the positive root in each $w'(\alpha)/2, -w'(\alpha)/2$ pair, obtaining the following:

$$\varepsilon(w')e^{k\rho}q^{(\pi,k\nu')}$$

Here $\nu' = \sum_{\alpha\in R_+}\sigma(w'(\alpha))\alpha$, where $\sigma(\alpha)$ is the sign of a root. Clearly $2\nu-\nu' = 2\rho$. So our leading term so far is $q^{(\pi,2k\rho)}/\varepsilon(w')e^{k\rho}$. The leading term of the numerator arises when $w=(w')^{-1}$, when we get $\varepsilon((w')^{-1})q^{2(\pi,(w')^{-1}(\lambda+k\rho))}e^{\lambda+k\rho}$. Taking the product of these (and noting that $\varepsilon(w')=\varepsilon((w')^{-1})$), we obtain

$$q^{2(\pi,(w')^{-1}(\lambda+k\rho))}q^{(\pi,k(2\rho))}e^\lambda$$

Finally, we sum over $w'\in W$ to get the coefficient

$$q^{2(\pi,k\rho)}\sum_{w\in W}q^{2(\pi,w(\lambda+k\rho))}$$

These are not necessarily distinct for distinct $\lambda\in P_+$. However, one can find a suitable coweight π in types A and B (and E_6 and E_7). In type D it is possible to find a linear combination of the operators D_π with this property. We demonstrate the case of type A below. □

Example 3.6. *Suppose that R is the root system A_{n-1} as before. Then the positive roots are $e_i - e_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ for $1 \leq i < j \leq n$. All coefficients are zero or one, so it is clear that the fundamental weights ω_i are minuscule coweights (note that $\alpha = \alpha^\vee$ for all $\alpha \in R$). Additionally $\rho = (n-1, n-3, \cdots, 1-n)$.*

We are given (a constant independent of λ multiplied by) $\sum_{w\in W} q^{2(\pi, w(\lambda+k\rho))}$, which is equivalent to knowing the multiset of values $(w^{-1}(\pi), \lambda + k\rho)$ as w ranges across W. In general, this does not determine λ, but for certain π, it does. For example, in type A_3, we may choose $\lambda_1 = (4, 0, -2, -2)$ and $\lambda_2 = (2, 2, 0, -4)$, so that $\lambda_1 + k\rho = (3k + 4, k, -k - 2, -3k - 2)$ and $\lambda_2 + k\rho = (3k + 2, k + 2, -k, -3k - 4)$. Then we see that $\pi_2 = (1/2, 1/2, -1/2, -1/2)$ is a fundamental (co)weight. But, one can check that in both cases, the multiset of values of $(w^{-1}(\pi_2), \lambda_i + k\rho)$ is $\{4k + 4, 2k + 2, 2, -2, -2k - 2, -4k - 4\}$. However, if π_1 is used instead, it is easy to restrict λ from the multiset $(w^{-1}(\pi_1), \lambda+k\rho)$. For example, we may add any multiple of $(1, 1, \cdots, 1)$ to π_1 without affecting the inner product, allowing us to assume $\pi_1 = (1, 0, 0, \cdots, 0)$. Thus, the inner product gives the coordinates of the vector $\lambda + k\rho$. Since these are strictly decreasing, they determine $\lambda + k\rho$, and hence λ.

Proposition 3.7. *The D_π operators commute.*

Proof. Let D be the operator with distinct eigenvalues that was used to construct the Macdonald Polynomials. For π a minuscule coweight, consider $D_\pi + cD$, where $c \in \mathbb{C}_{q,t}$. Since D has distinct eigenvalues, this linear combination has district eigenvalues for generic c. This means that this linear combination of operators is diagonalisable, and as before, its eigenvectors are the Macdonald polynomials. Since the Macodonald polynomials are unique, this means that D and $D_\pi + cD$ are diagonalisable with the same eigenbasis. We conclude that D_π is diagonalisable, with Macdonald polynomials as eigenvectors. This means that in the basis of Macdonald polynomials, the D_π are diagonal operators, and hence commute. □

Example 3.8. *Continuing with R being of type A_{n-1} as in example 3.5, we recall that the positive roots correspond to $\frac{x_i}{x_j}$ with $i < j$. In this setting, T_{ω_r} can be taken to send x_i to $q^2 x_i$ if $i \leq r$ and to x_i otherwise. In this way $\frac{x_i}{x_{i+1}}$ is unchanged unless $i = r$, in which case it is multiplied by q^2, which is the correct action. In fact, this makes it easy to write down explicit formulae for D_{ω_r} (acting on $\mathbb{C}_{q,t}[P]^W$) in terms of the "shift operators" T_i. If $I \subset \{1, 2, \cdots, n\}$, we write $T_I = \prod_{i\in I} T_i$ (the order of composition is unimportant since these operators clearly commute). Using equation 1, we have*

$$\begin{aligned} D_{\omega_r} &= \sum_{w\in S_n} w\left(\prod_{1\leq i\leq r<j\leq n} \frac{1 - q^{2k}\frac{x_i}{x_j}}{1 - \frac{x_i}{x_j}} T_{\{1,2,\cdots,r\}}\right) \\ &= \sum_{w\in S_n} w\left(\prod_{1\leq i\leq r<j\leq n} \frac{x_j - q^{2k}x_i}{x_j - x_i} T_{\{1,2,\cdots,r\}}\right) \\ &= r!(n-r)! \sum_{I\subset\{1,2,\cdots,n\}, |I|=r} \left(\prod_{i\in I, j\notin I} \frac{x_j - q^{2k}x_i}{x_j - x_i}\right) T_I \end{aligned}$$

Here we have used the fact that S_n acts transitively on r-element subsets of $\{1, 2, \cdots, n\}$ with stabiliser of size $r!(n - r)!$. It is easy to see that these commute.

Remark 3.9. *The proof in* [Mac00] *begins by introducing the concept of a "quasi-minuscule weight".*

3.2. **Macdonald Polynomials via AHA/DAHA.** We begin by describing a collection of operators that generalise the construction of the first section, without relying on minuscule weights. For the moment, we return to the case of t unspecialised. We recall that $\mathcal{H}_{q,t}$ has a representation $\mathbb{C}_{q,t}[X]$ where π_r acts as π_r and the action of T_i is given by

$$T_i = t_i s_i + (t_i - t_i^{-1})\frac{s_i - 1}{X^{-\alpha_i} - 1}$$

Here, the meaning of the expression is exactly the same as in previous lectures (the action of $s_i - 1$ on any X^λ yields an element divisible by $X^{-\alpha_i} - 1$). Note that $p \in \mathbb{C}_{q,t}[X]$ is W invariant precisely when it is annihilated by each $T_i - t_i$ (for the action of the latter element is to first apply $s_i - 1$, and then do some divisions and multiplications). This action will be of primary importance for our construction of Macdonald polynomials in $\mathbb{C}_{q,t}[X]^W = \mathbb{C}_{q,t}[P]^W$.

We note also that if we have $T_i = s_i G(\alpha_i)$, for

$$G(\alpha) = t + (t - t^{-1})\frac{1 - s_i}{X^\alpha - 1} = \frac{tX^\alpha - t^{-1}}{X^\alpha - 1} - \frac{(t - t^{-1})s_\alpha}{X^\alpha - 1}$$

Here s_α is the reflection associated to the root α. These $G(\alpha)$ satisfy the property that for $wG(\alpha)w^{-1} = G(w(\alpha))$ for $w \in W^{ae}$. We introduce these elements for the following important triangularity fact.

Definition 3.10. *For $\lambda, \mu \in P^\vee$, say that $\lambda \succ \mu$ if $\lambda^+ > \mu^+$ (i.e. $\lambda^+ - \mu^+ \in Q_+$) or $\lambda^+ = \mu^+$ and $\lambda > \mu$. When we need to indicate lower order terms, we write l.o.t..*

Proposition 3.11. *For $\lambda \in P^\vee$ and $\mu \in P_+$, we have:*

$$Y^\lambda X^\mu = \sum_{\mu \succeq \nu} c_{\nu,\mu} X^\nu$$

Moreover,

$$c_{\mu,\mu} = q^{(\lambda, \mu + k\rho)}$$

Proof. Clearly it is enough to prove this for $\lambda \in P_+^\vee$ for otherwise we may write it as the difference of two dominant coweights, and compose a triangular operator corresponding to one, with the inverse of the triangular operator corresponding to the other. Choose a reduced expression for $t_\lambda \in W^{ae}$ of the form $\pi_r s_{i_1} \cdots s_{i_r}$, so that $Y^\lambda = \pi_r T_{i_1} \cdots T_{i_r}$. Then we have:

$$\begin{aligned} Y^\lambda &= \pi_r T_{i_1} \cdots T_{i_{r-1}} T_{i_r} \\ &= \pi_r s_{i_1} G(\alpha_{i_1}) \cdots s_{i_{r-1}} G(\alpha_{i_{r-1}}) s_{i_r} G(\alpha_{i_r}) \\ &= \pi_r s_{i_1} G(\alpha_{i_1}) \cdots s_{i_{r-1}} s_{i_r} G(s_{i_r}(\alpha_{i_{r-1}})) G(\alpha_{i_r}) \\ &= \cdots \\ &= \pi_r s_{i_1} \cdots s_{i_{r-1}} s_{i_r} G(\alpha^{(1)}) \cdots G(\alpha^{(r)}) \\ &= t_\lambda G(\alpha^{(1)}) \cdots G(\alpha^{(r)}) \end{aligned}$$

Here $t_\lambda(X_\mu) = q^{2(\lambda,\mu)}$, and we see that the $\alpha^{(i)}$ that arise are precisely the positive roots that t_λ^{-1} maps to negative roots, i.e. $\{\alpha + k'\delta \mid \alpha \in R_+, 0 \leq k' < (\lambda, \alpha)\}$.

Note that the definition of $G(\alpha + k')$ gives us the following:

$$G(\alpha + k')X^\mu = \begin{cases} tX^\mu + \text{l.o.t.} & (\mu, \alpha^\vee) \geq 0 \\ t^{-1}X^\mu + \text{l.o.t.} & (\mu, \alpha^\vee) < 0 \end{cases}$$

Since $\mu \in P_+$, and $\alpha \in R_+$, we are in the first case. It is clear that the leading order term picks up a factor of $t = q^k$, (λ, α) times for each $\alpha \in R_+$. So the total contribution is $q^{2(\lambda,\rho)}$ which gives the required formula when we include $q^{2(\lambda,\mu)}$ coming from t_λ. □

Proposition 3.12. *If $f(Y) \in \mathbb{C}_{q,t}[Y]^W$, thought of as a central element of the affine Hecke algebra $H^a = H\mathbb{C}_{q,t}[Y]$, then $f(Y)$ preserves the space $\mathbb{C}_{q,t}[X]^W$.*

Proof. It is enough to show that for $p \in \mathbb{C}_{q,t}[X]^W$, $(T_i - t_i)f(Y)p = 0$. But $f(Y)$ and $T_i - t_i$ are elements of $H^a = H\mathbb{C}_{q,t}[Y]$ in which the former is central, which means they commute. So $(T_i - t_i)f(Y)p = f(Y)(T_i - t_i)p = f(Y) \cdot 0 = 0$. □

Since $W^{ae} = P^\vee \rtimes W$, the action of any $w \in W^{ae}$ in $\mathbb{C}_{q,t}[X]$ may be written as

$$\sum_{w \in W, \lambda \in P^\vee} g_{\lambda,w} t_\lambda w$$

here we recall that $t_\lambda(X^\mu) = q^{2(\lambda,\mu)}X^\mu$. The $g_{\lambda,w}$ are rational functions in the X^α, whose denominators are products of terms of the form $X^{\alpha_i} - 1$, for α_i simple roots.

Definition 3.13. *Define the restriction of the action of T_w for $w \in W^{ae}$ via*

$$Res(\sum_{w\in W,\lambda\in P^\vee} g_{\lambda,w}t_\lambda w) = \sum_{w\in W,\lambda\in P^\vee} g_{\lambda,w}t_\lambda$$

(that is, omit the Weyl group action in each term). This is clearly a linear operation. For $f(Y) \in \mathbb{C}_{q,t}[Y]^W$, we define $L_f = Res(f)$.

Proposition 3.14. *The L_f , for $f \in \mathbb{C}_{q,t}[Y]^W$ are W-invariant commuting operators on $\mathbb{C}_{q,t}[X]$.*

Proof. The W-invariance is clear because for $f \in \mathbb{C}_{q,t}[Y]^W$, because the action of $w \in W$ on $Res(f)$ can be obtained by restricting the action of w on f (which is invariant).

Consider the action of f as $\sum_{w\in W,\lambda\in P^\vee} f_{\lambda,w}t_\lambda w$, and that of g in the form $\sum_{w'\in W,\mu\in P^\vee} g_{\mu,w'}t_\mu w'$. Thus, since g is W-invariant, fg acts as

$$\sum_{w\in W,\lambda\in P^\vee} f_{\lambda,w}t_\lambda w \sum_{w'\in W,\mu\in P^\vee} g_{\mu,w'}t_\mu w' = \sum_{w\in W,\lambda\in P^\vee} f_{\lambda,w}t_\lambda \sum_{w'\in W,\mu\in P^\vee} g_{\mu,w'}t_\mu w'w$$

Taking the restriction of this, we obtain $L_f Lg$. Hence $L(f)L(g) = L(fg)$. But $fg = gf$, so we also get $L(fg) = L(gf) = L(g)L(f)$ (where the last step uses the W-invariance of f). □

By proposition 3.11, we see that the L_f ($f \in \mathbb{C}_{q,t}[Y]^W$) are actually triangular operators on $\mathbb{C}_{q,t}[X]^W$, with respect to the m_μ basis. In particular, we have $L_f(m_\mu) = f(q^{2(\mu+k\rho)})m_\mu +$ l.o.t.. This notation means that each Y^λ in f should be replaced with the scalar $q^{2(\lambda,\mu+k\rho)}$. One easily checks that the eigenvalues $\sum_{\nu\in W\lambda} q^{2(\nu,\mu)}$ determine μ (as $\lambda \in P_+^\vee$ varies). Since the L_f commute, we easily see they form a family of simultaneously diagonalisable operators. We thus obtain:

Lemma 3.15. *The operators are L_f are diagonalisable triangular operators on $\mathbb{C}_{q,t}[X]^W$, with respect to the m_μ basis. The eigenvalues of L_f are $f(q^{2(\mu+k\rho)})$. Moreover the eigenvector associated to this may be taken to be of the form $m_\mu +$ l.o.t..*

Remark 3.16. *It can be shown that the operators D_π we constructed for minuscule weights π are the L_f corresponding to $\sum_{w\in W} f^{w(\pi)}$ up to a scalar multiple. We describe the proof. Consider the ordering defined by $\mu \sqsubseteq \lambda$ when $\mu^+ < \lambda^+$, or $\mu^+ = \lambda^+$ and $\mu \geq \lambda$ (note that the last inequality is opposite to that in the definition of $\succeq$). The leading order term of Y^λ turns out to be*

$$\left(\prod_{\alpha\in R_-^a\cap t_\lambda R_+^a} \frac{tX^\alpha - t^{-1}}{X^\alpha - 1}\right) t_\lambda$$

One can check that the leading order term of our operator is of the form $g(X)t_{\pi^-}$, where π^- is the anttdominant coweight in the orbit of π. We see that this can only arise from the term Y^{π^-}, where the coefficient is

$$|W_\pi| \prod_{\alpha\in R,(\alpha,\pi^-)=1} \frac{tX^\alpha - t^{-1}}{X^\alpha - 1} t_{\pi^-}$$

Here, W_π is the stabiliser of π. Since π is minuscule, there are no dominant weights below it, so the remainder of the operator is determined by conjugacy considerations. This gives that

$$Res(\sum_{w\in W} Y^{w(\pi)}) = \sum_{w\in W} w\left(\prod_{\alpha\in R,(\alpha,\pi^-)=1} \frac{tX^\alpha - t^{-1}}{X^\alpha - 1} t_\pi\right)$$

This agrees with D_π up to a scalar.

Independently of the previous remark, we may deduce that the eigenvectors of L_f are actually Macdonald polynomials, for which we will use the same self-adjointness argument that we did before. For this, we introduce a bilinear form.

Definition 3.17. *Define the $\mathbb{C}_{q,t}$-linear involution ι of $\mathbb{C}_{q,t}(q)$ by $\iota(q) = q^{-1}$, and extend it to $\mathbb{C}_{q,t}[X]$ by declaring $\iota(X^\mu) = X^\mu$. Then, definite the bilinear form $\langle -, \rangle'_k$ on $\mathbb{C}_{q,t}[X]$ as follows.*

$$\langle f, g\rangle'_k = [f\mu_k\iota(\bar{g})]_0$$

Similarly to before, $[h]_0$ is the coefficient of $X^0 = 1$ in h, written in the X^μ basis, and μ_k is defined as

$$\mu_k = \prod_{\alpha\in R_+}\prod_{i=1-k}^{k}(q^iX^{\alpha/2} - q^{-i}X^{-\alpha/2})$$

This inner product is neither symmetric, nor W-invariant, but it happens to be exactly the right definition to help prove the Macdonald conjecture.

Definition 3.18. *We define the following quantity, which will be important.*

$$\varphi_k = \prod_{\alpha\in R_+}(q^kX^{\alpha/2} - q^{-k}X^{-\alpha/2})$$

Note that $\varphi_0 = \delta$, the Weyl denominator.

This means that we have

$$\mu_k = (-1)^{k|R_+|}q^{-k(k-1)|R_+|}\Delta_k\frac{\varphi_k}{\delta}$$

Here $\Delta_k = \Delta_{q,t}$ (from the first section, but we identify e^α with X^α), but we emphasise the dependence on k. We make some observations

(1) $\bar{\mu_k} = \iota(\mu_k)$, so $\langle f, g\rangle'_k = \iota(\langle g, f\rangle'_k)$.
(2) At $q = 1$, $\langle -, -\rangle'_k = \pm\langle -, -\rangle_{q,t}$, in particular, the form $\langle -, -\rangle'_k$ is generically non degenerate.

Proposition 3.19. *If $f, g \in \mathbb{C}_{q,t}[X]^W$, then*

$$\langle f, g\rangle'_k = (-1)^{k|R_+|}q^{-k(k-1)|R_+|}\langle f, \iota(g)\rangle_{q,t}d_k$$

where

$$d_k = q^{k|R_+|}\sum_{w\in W}q^{-2k|R_+\cap w^{-1}(R_-)|}$$

Proof. We observe that $[h]_0 = [w(h)]_0$ for any $h \in \mathbb{C}_{q,t}[X]$. It's enough to compute

$$\begin{aligned}[f\overline{\iota(g)}\mu_k]_0 &= [f\overline{\iota(g)}\Delta_k\frac{\varphi_k}{\delta}]_0 \\ &= \frac{1}{|W|}\sum_{w\in W}[w(f\overline{\iota(g)}\Delta_k\frac{\varphi_k}{\delta})]_0 \\ &= \frac{1}{|W|}\sum_{w\in W}[f\overline{\iota(g)}\Delta_k w(\frac{\varphi_k}{\delta})]_0\end{aligned}$$

Here we have used W-invariance of f, g, Δ_k. Note that $\sum_{w\in W} w(\frac{\varphi_k}{\delta}) = \frac{1}{\delta}\sum_{w\in W}\varepsilon(w)w(\varphi_k)$. It is clear that multiplying by δ gives a W-antiinvariant polynomial, hence something divisible by δ. Therefore this quantity is a polynomial. It is

$$\sum_{w}\in W\prod_{\alpha\in R_+}\frac{q^{k\epsilon_\alpha}X^{\alpha/2} - q^{-k\epsilon_\alpha}X^{-\alpha/2}}{X^{\alpha/2} - X^{-\alpha/2}}$$

where ϵ_α is 1 if $w(\alpha) \in R_+$ and -1 otherwise. The leading order term of this is a constant multiple of X^0, so the whole quantity must be scalar. That scalar is

$$\sum_{w\in W}\prod_{\alpha\in R_+}q^{k\epsilon_\alpha} = q^{k|R_+|}\sum_{w\in W}q^{-2k|R_+\cap w^{-1}(R_-)|}$$

Combining this with the scalar factors relating μ_k and Δ_k gives the desired result. ∎

We now prove some statements about Macdonald polynomials before we embark on the proof of their existence.

Proposition 3.20. *We have:*

(1) $\iota(P_\lambda) = P_\lambda$
(2) $\langle P_\mu, P_\nu\rangle'_k = 0$ *if* $\mu \neq \nu$
(3) *The Macdonald polynomials are uniquely defined by* $P_\lambda = m_\lambda + l.o.t.$ *and the above orthogonality property.*

Proof.

$$\Delta_k = \prod_{\alpha\in R_+}\prod_{i=0}^{k-1}(-q^{2i}X^\alpha + 1 + q^{4i} - q^{2i}X^{-\alpha})$$

This immediately implies that $\iota(\Delta_k) = q^{-4k(k-1)|R_+|}\Delta_k$, which gives that $[P_\mu\Delta_k P_\nu]_0 = 0$ implies $[\iota(P_\mu)\Delta_k\iota(P_\nu)]_0 = 0$. So, $\iota(P_\lambda)$ satisfy the definition of the Macdonald polynomials, hence they equal P_λ by uniqueness (so the first statement follows). Since $\langle -,-\rangle'_k$ and $\langle -,-\rangle_{q,t}$ agree up to a scalar, the second statement is true. The third statement follows from the nondegeneracy of $\langle -,-\rangle'_k$. □

Definition 3.21. *For an operator h from $\mathbb{C}_{q,t}$ to itself, define its adjoint h^* (with respect to $\langle -,-\rangle'_k$) in the usual way:*

$$\langle hf, g\rangle'_k = \langle f, h^*g\rangle'_k$$

We also define $h^\dagger$ via $[h(f)\overline{\iota(g)}]_0 = [f\overline{\iota(h^\dagger(g))}]_0$.

Proposition 3.22. *We have the following:*
(1) $h^* = \mu_k^{-1}h^\dagger\mu_k$.
(2) *If $p \in \mathbb{C}_{q,t}[X]$ is identified with the operator of multiplication by p, then $p^\dagger = \overline{\iota(p)}$.*
(3) *For $w \in W^{ae}$, $w^\dagger = w^{-1}$.*
(4) $T_i^\dagger = T_i^{-1}$
(5) $(Y^\lambda)^\dagger = Y^{-\lambda}$

Proof. The first statement is clear from the definition of the bilinear form. The second statement is trivial. The third statement follows from W-invariance of X^0. To prove the fourth statement, we note that $\overline{\iota(t)} = t^{-1}$, since $\iota(q) = q^{-1}$ and $t = q^k$. As $T_i^{-1} = T_i - (t - t^{-1})$, it is enough to show $(T_i - t)^* = T_i - t$. For this, we have:

$$s_i^* = \mu_k^{-1}s_i\mu_k = -\varphi_k s_i\varphi_k = \frac{q^{-k}X^{\alpha_i/2} - q^kX^{-\alpha_i/2}}{q^kX^{\alpha_i/2} - q^{-k}X^{-\alpha_i/2}}s_i$$

where we have used the fact that s_i permutes the set of positive roots different from α_i, so that only the factor corresponding to α_i in the definition of ϕ_k is relevant. One easily checks that

$$T_i - t = \frac{tX^{-\alpha_i/2} - t^{-1}X^{\alpha_i/2}}{X^{-\alpha_i/2} - X^{\alpha_i/2}}(s_i - 1)$$

This is something we know how to take the adjoint of

$$\begin{aligned}
(T_i - t)^* &= (s_i^* - 1)\left(\frac{tX^{-\alpha_i/2} - t^{-1}X^{\alpha_i/2}}{X^{-\alpha_i/2} - X^{\alpha_i/2}}\right)^* \\
&= \frac{tX^{-\alpha_i/2} - t^{-1}X^{\alpha_i/2}}{t^{-1}X^{-\alpha_i/2} - tX^{\alpha_i/2}}\frac{tX^{\alpha_i/2} - t^{-1}X^{-\alpha_i/2}}{X^{\alpha_i/2} - X^{-\alpha_i/2}}s_i - \frac{tX^{-\alpha_i/2} - t^{-1}X^{\alpha_i/2}}{X^{-\alpha_i/2} - X^{\alpha_i/2}} \\
&= \frac{tX^{-\alpha_i/2} - t^{-1}X^{\alpha_i/2}}{X^{-\alpha_i/2} - X^{\alpha_i/2}}(s_i - 1) \\
&= T_i - t
\end{aligned}$$

To prove the fifth part, it is enough to consider $\lambda \in P_+^\vee$, for which the previous part suffices, together with $\pi_r^* = \pi_r^{-1}$ (as Y^λ is a product of π_r and T_i). We already know that $\pi_r^\dagger = \pi_r^{-1}$, so this is equivalent to the statement that π_r preserves μ_k. □

Corollary 3.23. *The mapping from $\mathbb{C}_{q,t}[Y]^W$ to itself defined by $f(q)Y^\lambda \mapsto \iota(f(q))Y^{-\lambda}$ is an involution which agrees with the $*$-adjoint. Therefore $h \mapsto h^*$ is an involution on $\mathbb{C}_{q,t}[Y]^W$.*

Theorem 3.24. *For $f \in \mathbb{C}_{q,t}[Y]^W$, the Macdonald polynomial P_λ is an eigenvector of L_f with eigenvalue $f(q^{2(\lambda+k\rho)})$.*

Proof. We already know eigenvectors with the stated eigenvalues exist, and it suffices to check that they satisfy the definition of Macdonald polynomials. By proposition 3.20, it is enough to check orthogonality (we have already seen triangularity). Observe that for $f = m_\nu \in \mathbb{C}_{q,t}[Y]^W$, $f^* = m_{-\nu^-} = m_{-w_0\nu}$. We check

$$\begin{aligned}
\sum_{\eta \in W\nu} q^{2(\eta,\lambda+k\rho)} \langle P_\lambda, P_\nu \rangle'_k &= \langle L_f P_\lambda, P_\nu \rangle'_k \\
&= \langle P_\lambda, L^*_f P_\nu \rangle'_k \\
&= \iota(\sum_{\eta \in -Ww_0\nu} q^{2(\eta,\mu+k\rho)}) \langle P_\lambda, P_\nu \rangle'_k \\
&= \sum_{\eta \in W\nu} q^{2(\eta,\mu+k\rho)} \langle P_\lambda, P_\nu \rangle'_k
\end{aligned}$$

Since we already know these eigenvalues are distinct (for suitably chosen f), this shows that the P_λ are orthogonal, and the L_f are self-adjoint. □

Corollary 3.25. *The eigenvectors of the L_f operators ($f \in \mathbb{C}_{q,t}[Y]^W$), P_λ, satisfy the definition of Macdonald polynomials. Therefore Macdonald polynomials exist.*

4. The Macdonald Conjecture

This section is dedicated to the proof the following theorem (the Macdonald Conjecture), using DAHA.

Theorem 4.1. *Let P_λ be the Macdonald polynomial associated to $\lambda \in P$, and let $t_\alpha = q^k$. Then*

$$\langle P_\lambda, P_\lambda \rangle_{q,t} = \prod_{\alpha \in R_+} \prod_{i=1}^{k-1} \frac{1 - q^{2(\alpha^\vee, \lambda+k\rho)+2i}}{1 - q^{2(\alpha^\vee, \lambda+k\rho)-2i}}$$

If we write $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$ for the "quantum number n", this may also be written

$$q^{|R|\frac{k(k-1)}{2}} \prod_{\alpha \in R_+} \prod_{i=1}^{k-1} \frac{[(\alpha^\vee, \lambda + k\rho) + i]}{[(\alpha^\vee, \lambda + k\rho) - i]}$$

Remark 4.2. *If $k = 1$, the $P_\lambda = \chi_\lambda$ (the Weyl character); we already know these are orthonormal with respect to $\langle -, - \rangle_{q,t}$ (see example 3.4), which is consistent with the theorem.*

Remark 4.3. *The case where $\lambda = 0$, i.e.*

$$\frac{1}{|W|}[\prod_{\alpha \in R} \prod_{i=0}^{k-1} (1 - q^{2i} e^\alpha)]_0 = q^{|R|\frac{k(k-1)}{2}} \prod_{\alpha \in R_+} \prod_{i=1}^{k-1} \frac{[(\alpha^\vee, k\rho) + i]}{[(\alpha^\vee, k\rho) - i]}$$

is known as the Macdonald constant term conjecture. This special case of the conjecture was not known until the general case was resolved.

Firstly, note that

$$\begin{aligned}
\langle P_\lambda, P_\lambda \rangle_{q,t} &= d_k (-1)^{k|R_+|} q^{-k(k-1)|R_+|} \langle P_\lambda, \iota(P_\lambda) \rangle'_k \\
&= d_k (-1)^{k|R_+|} q^{-k(k-1)|R_+|} \langle P_\lambda, P_\lambda \rangle'_k
\end{aligned}$$

so it is sufficient to find $\langle P_\lambda, P_\lambda \rangle'_k$. Actually we will do this inductively with k, so it will be necessary to distinguish P_λ associated to different k. Therefore we will write $P^{(k)}_\lambda$ for the Macdonald polynomial indexed by λ for $t = q^k$.

Definition 4.4. *We need the following quantities.*

1. $\mathcal{X} = \varphi_{-k} = \prod_{\alpha \in R_+} (q^{-k} X^{\alpha/2} - q^k X^{-\alpha/2})$
2. $\mathcal{Y} = \varphi^\vee_{-k} = \prod_{\alpha \in R_+} (q^{-k} Y^{\alpha^\vee/2} - q^k Y^{-\alpha^\vee/2})$
3. $\hat{\mathcal{Y}} = \varphi^\vee_k = \prod_{\alpha \in R_+} (q^k Y^{\alpha^\vee/2} - q^{-k} Y^{-\alpha^\vee/2})$
4. $G = \mathcal{X}^{-1} \mathcal{Y}$

(5) $\hat{G} = \hat{\mathcal{Y}}\mathcal{X}$
(6) $\mathcal{P} = \frac{1}{|W|}\sum_{w\in W} w$
(7) $\mathcal{P}_- = \frac{1}{|W|}\sum_{w\in W} \varepsilon(w)w$
(8) $\mathcal{P}_-^q = \frac{1}{\sum_{w\in W} t^{-2l(w)}}\sum_{w\in W}(-t)^{-l(w)}T_w$

Lemma 4.5. *We have the following (checking these is trivial):*
(1) $\iota(\mathcal{X}) = (-1)^{|R_+|}\overline{\mathcal{X}} = \varphi_k$
(2) $\mathcal{X}^* = (-1)^{|R_+|}\mathcal{X}$
(3) $\mathcal{Y}^* = (-1)^{|R_+|}\mathcal{Y}$
(4) $\hat{\mathcal{Y}}^* = (-1)^{|R_+|}\hat{\mathcal{Y}}$
(5) $\mathcal{P}_-^+|_{q=t=1} = \mathcal{P}_-$

We first prove the statement that $\mathcal{P}_-(\mathcal{Y}-\hat{\mathcal{Y}})$ vanishes on $\mathbb{C}_{q,t}[X]^W$.

4.1. **Properties of Symmetrisers and Antisymmetrisers.**

Lemma 4.6. *Let V be a finite dimensional representation of W and let $V' = \sum \ker(1-s_i)$. Then V' is W-invariant.*

Proof. It is enough to check that $s_i\ker(1-s_j) \subseteq \ker(1-s_i)+\ker(1-s_j)$. If $v \in s_i\ker(1-s_j)$. then $s_iv \in \ker(1-s_j)$, so in particular, $s_j(s_iv) = s_iv$. Writing $v_\pi = \frac{1}{2}(v \pm s_iv)$, so that $v = v_+ + v_-$, $s_iv = v_+ - v_- \in \ker(1-s_j)$. But $v_+ \in \ker(1-s_i)$, so we are done. ☐

Corollary 4.7. *In the previous lemma, V' is the sum of isotypic components of non-sign representations of W.*

Proof. It is enough to check this on irreducible representations, in which case V' is either zero or all of V. It is zero if and only if each $\ker(1-s_i)$ is empty, i.e. each s_i acts with eigenvalue -1 only (since $s_i^2 = 1$, the eigenvalues of its action must be ± 1). But then s_i acts as minus the identity, so V is a direct sum of sign representations of W. ☐

Corollary 4.8. *We conclude that* $\ker(\mathcal{P}_-) = \sum_i \ker(1-s_i)$.

Proposition 4.9. *We have that $\mathcal{P}_-$ is divisible by $T_i - t$ on both sides. Also, when acting on $\mathbb{C}_{q,t}[X]$, $\ker(\mathcal{P}_-^q) = \ker(\mathcal{P}_-)$, and $Im(\mathcal{P}_-^q) = \mathbb{C}_{q,t}[X]^{-W}$ (antiinvariants). Analogous statements are true for the action on $\mathbb{C}_{q,t}[Y]$.*

Proof. To see the first statement, break W into cosets of $\{1, s_i\}$ (left or right cosets, according to which divisibility condition one wishes to prove). Choosing minimal length coset representatives T_w, we obtain that $\mathcal{P}_-^q$ is a linear combination of terms of the form $(-t)^{-l(w)}T_w(1-t^{-1}T_i)$ (this is the right divisibility case). This is clearly divisible by $T_i - t$. To see the kernel and image of $\mathcal{P}_-^q$, first observe that $\mathbb{C}_{q,t}[X]$ is filtered by finite dimensional spaces (this easily follows from the fact that W-orbits on P are finite). Note that the kernel contains the sum of the kernels of $T_i - t$, and that is the same as the sum of the kernels of $s_i - 1$ (recall that we observed that $\ker(T_i - t) = \ker(s_i - 1)$ when we showed that the intersection of these is the space of invariants). But when $q = 1$, $t = 1$ and $\mathcal{P}_-^q$ becomes $\mathcal{P}_-$ and we have equality of spaces. We must therefore have equality of spaces generically. To see the image statement, observe that because we have left divisibility, and noting that $T_i - t$ is left divisible by $s_i - 1$, multiplying by s_i changes the sign. Therefore the image of $\mathcal{P}_-^q$ is contained in $\mathbb{C}_{q,t}[X]^{-W}$ with equality at $q = 1$ (as in the kernel case). Therefore we have equality generically. The $\mathbb{C}_{q,t}[Y]$ case is essentially the same. ☐

Corollary 4.10. *The operator $\mathcal{P}_-^q$ is a projector. On its image (antiinvariants) T_i acts as $-t^{-1}$. Therefore T_w acts as $(-t)^{-l(w)}$. In particular, $\sum_{w\in W}(-t)^{-l(w)}T_w$ acts as $\sum_{w\in W} t^{-2l(w)}$. Dividing through by that scalar shows that $\mathcal{P}_-^q$ acts as the identity on antiinvariants.*

Theorem 4.11. *For $f \in \mathbb{C}_{q,t}[X]^W$, $\mathcal{P}_-(\mathcal{Y}-\hat{\mathcal{Y}})f = 0$.*

Proof. We have shown that for $f \in \mathbb{C}_{q,t}[X]$, $\mathcal{P}_-^q(f) = 0$ if and only if $\mathcal{P}_-(f) = 0$. So it is enough to prove the statement with $\mathcal{P}_-$ replaced by $\mathcal{P}_-^q$. Let $R = \mathcal{P}_-^q(\mathcal{Y} - \hat{\mathcal{Y}}) \in H^{ae}$. Since $\mathbb{C}_{q,t}[X]$ is a faithful representation of H^{ae}, the statement that R annihilates all invariants is equivalent to R being of the form $\sum_i h_i(T_i - t)$ (for some $h_i \in H^{ae}$). This is because there is a PBW-style theorem for H^{ae} which implies that any element may be written as a linear combination of terms, each of which is Y^λ followed by a product of T_i. Since we can replace T_i by $(T_i - t) + t$, we may rewrite our element at Y^λ followed by a product of $T_i - t$. It is clear that a nonzero sum of Y^λ (with no $T_i - t$ factors following it) does not annihilate invariants. To prove this decomposition in H^{ae}, we may work in any faithful representation of H^{ae} and we choose to do so in $\mathbb{C}_{q,t}[Y]$. We now employ the same reasoning in reverse. The stated decomposition is equivalent to R annihilating $\mathbb{C}_{q,t}[Y]^W$. This is true if and only if $\mathcal{P}_-(\mathcal{Y} - \hat{\mathcal{Y}})$ annihilates invariants. But $\mathcal{P}_-$ is divisible by $(1 + (-1)^{|R_+|}w_0)$ on both sides (recall that w_0 is the longest element of W which sends positive roots to negative roots and vice versa). It is easy to check that the following identities hold:

$$\begin{aligned} w_0(\mathcal{Y}) &= (-1)^{|R_+|}\hat{\mathcal{Y}} \\ w_0(\hat{\mathcal{Y}}) &= (-1)^{|R_+|}\mathcal{Y} \end{aligned}$$

Then it immediately follows that

$$(1 + (-1)^{|R_+|}w_0)(\mathcal{Y} - \hat{\mathcal{Y}}) = 0$$

This implies that $\mathcal{P}_-(\mathcal{Y} - \hat{\mathcal{Y}})$ annihilates invariants, which is what we needed to prove. ☐

4.2. **The Proof.**

Lemma 4.12. *We have the following (as actions on $\mathbb{C}_{q,t}[X]$):*

$$\begin{aligned} (T + t^{-1})\mathcal{X} &= \frac{t^{-1}X^{-\alpha_i/2} - tX^{\alpha_i/2}}{t^{-1}X^{\alpha_i/2} - tX^{-\alpha_i/2}}\mathcal{X}(T_i - t) \\ (T + t^{-1})\mathcal{Y} &= \frac{t^{-1}Y^{-\alpha_i^\vee/2} - tY^{\alpha_i^\vee/2}}{t^{-1}Y^{\alpha_i^\vee/2} - tY^{-\alpha_i^\vee/2}}\mathcal{Y}(T_i - t) \\ (T - t)\hat{\mathcal{Y}} &= \frac{tY^{-\alpha_i^\vee/2} - t^{-1}Y^{\alpha_i^\vee/2}}{tY^{\alpha_i^\vee/2} - t^{-1}Y^{-\alpha_i^\vee/2}}\hat{\mathcal{Y}}(T_i + t^{-1}) \end{aligned}$$

Proof. Recall that T_i acts as $ts_i + (t - t^{-1})\frac{s_i - 1}{X^{-\alpha_i} - 1}$, which is of the form $As_i + B$ (where A, B commute with the X^λ). It is clear that these terms preserve $\prod_{\alpha \in R_+, \alpha \neq \alpha_i}(q^{-k}X^{\alpha_i} - q^k X^{-\alpha_i})$, so it is enough to consider the term $(q^{-k}X^{\alpha_i} - q^k X^{-\alpha_i})$ (the only factor of $\mathcal{X}$ not appearing in the previously mentioned product). One can verify the following equations.

$$\begin{aligned} T_i X^{\alpha_i/2} &= X^{-\alpha_i/2}T_i + (t - t^{-1})X^{\alpha_i/2} \\ X^{\alpha_i/2}T_i &= T_i X^{-\alpha_i/2} + (t - t^{-1})X^{\alpha_i/2} \end{aligned}$$

hence,

$$\begin{aligned} T_i(q^{-k}X^{\alpha_i} - q^k X^{-\alpha_i}) &= (q^{-k}X^{-\alpha_i} - q^k X^{\alpha_i})T_i + (q^{2k} - q^{-2k})X^{\alpha_i/2} \\ t^{-1}(q^{-k}X^{\alpha_i} - q^k X^{-\alpha_i}) &= q^{-2k}X^{\alpha_i} - X^{-\alpha_i} \end{aligned}$$

Summing the last two equations gives

$$(T + t^{-1})(q^{-k}X^{\alpha_i} - q^k X^{-\alpha_i}) = (q^{-k}X^{-\alpha_i} - q^k X^{\alpha_i})(T_i - t)$$

This proves the first equation. The others can be proven using similar computations. ☐

Corollary 4.13. *Since $\mathbb{C}_{q,t}[X]^W$ is the intersection of the kernels of $(T_i - t)$, and $\mathbb{C}_{q,t}[X]^{-W}$ is the the intersection of the kernels of $(T_i + t^{-1})$, the preceding lemma implies the following.*

1. $\mathcal{X}\mathbb{C}_{q,t}[X]^W = \mathbb{C}_{q,t}[X]^{-W}$
2. $\mathcal{Y}\mathbb{C}_{q,t}[X]^W \subseteq \mathbb{C}_{q,t}[X]^{-W}$
3. $\hat{\mathcal{Y}}\mathbb{C}_{q,t}[X]^{-W} \subseteq \mathbb{C}_{q,t}[X]^W$

Proof. The only thing that has not yet been explained is why there is an equality instead of a one-way inclusion in the first statement. The reason for this is that when $q = 1$ we have equality (this reduces to the fact that every antisymmetric polynomial is divisible by the Weyl denominator), hence we have equality generically. $\square$

The point of the previous arguments is to deduce the following.

Corollary 4.14. *The operators $G = \mathcal{X}^{-1}\mathcal{Y}$ and $\hat{G} = \hat{\mathcal{Y}}\mathcal{X}$ preserve $\mathbb{C}_{q,t}[X]^W$.*

Proposition 4.15. *If $f, g \in \mathbb{C}_{q,t}[X]^W$, $\langle Gf, g\rangle'_{k+1} = \frac{d_{k+1}}{d_k}\langle f, \hat{G}g\rangle'_k$.*

Proof. Note that $\mu_{k+1} = \varphi_{k+1}\varphi_{-k}\mu_k = \varphi_{k+1}\mathcal{X}\mu_k$. Since $\mu_k\mathcal{X} = \prod_{\alpha\in R_+}\prod_{i=-k}^{k}(q^i X^{\alpha/2} - q^{-i}X^{-\alpha/2})$, we see that $\mu_k\mathcal{X}$ is antisymmetric (the $i = 0$ term is the Weyl denominator which is antisymmetric, but the product of the $i = \pm j$ gives a symmetric quantity). So,

$$\begin{aligned}
\mathcal{P}(\mu_{k+1}) &= \frac{1}{|W|}\sum_{w\in W} w(\varphi_{k+1}\varepsilon(w)\mathcal{X}\mu_k \\
&= \mathcal{P}_-(\varphi_{k+1})\mathcal{X}\mu_k \\
&= \frac{1}{|W|}d_{k+1}\delta\mathcal{X}\mu_k
\end{aligned}$$

Similar methods show

$$\mathcal{P}(\mathcal{X}^2\mu_k) = \frac{d_k}{|W|}\delta\mathcal{X}\mu_k$$

This gives the (important) equation

$$\mathcal{P}(\mu_{k+1}) = \frac{d_{k+1}}{d_k}\mathcal{P}(\mathcal{X}^2\mu_k)$$

We note that $[f]_0 = [\mathcal{P}f]_0$. Therefore, as $G(f)$ and $\overline{\iota(g)}$ are invariant,

$$\begin{aligned}
\langle Gf, g\rangle'_{k+1} &= [G(f)\overline{\iota(g)}\mu_{k+1}]_0 \\
&= [G(f)\overline{\iota(g)}\mathcal{P}(\mu_{k+1})]_0 \\
&= \frac{d_{k+1}}{d_k}[\mathcal{X}^{-1}\mathcal{Y}(f)\overline{\iota(g)}\mathcal{P}(\mathcal{X}^2\mu_k)]_0 \\
&= \frac{d_{k+1}}{d_k}[\mathcal{P}(\mathcal{X}^{-1}\mathcal{Y}(f)\overline{\iota(g)}\mathcal{X}^2\mu_k)]_0 \\
&= \frac{d_{k+1}}{d_k}[\mathcal{P}_-(\mathcal{Y}(f))\overline{\iota(g)}\mathcal{X}\mu_k]_0
\end{aligned}$$

In this last step we used the fact that $\mathcal{X}\mu_k$ is antisymmetric. We now use the fact that $\mathcal{P}_-(\mathcal{Y} - \hat{\mathcal{Y}})$ vanishes on invariants. We may therefore replace the $\mathcal{Y}$ with $\hat{\mathcal{Y}}$.

$$\begin{aligned}
&= \frac{d_{k+1}}{d_k}[\mathcal{P}_-(\hat{\mathcal{Y}}(f))\overline{\iota(g)}\mathcal{X}\mu_k]_0 \\
&= \frac{d_{k+1}}{d_k}[\mathcal{P}(\hat{\mathcal{Y}}(f)\overline{\iota(g)}\mathcal{X}\mu_k)]_0 \\
&= \frac{d_{k+1}}{d_k}[\hat{\mathcal{Y}}(f)\overline{\iota(g)}\mathcal{X}\mu_k]_0 \\
&= \frac{d_{k+1}}{d_k}\langle\mathcal{X}\hat{\mathcal{Y}}f, g\rangle'_k \\
&= \frac{d_{k+1}}{d_k}\langle f, \hat{\mathcal{Y}}\mathcal{X}g\rangle'_k \\
&= \langle f, \hat{G}g\rangle'_k
\end{aligned}$$

In the second last step we used the adjoint formulae that we had noted previously in lemma 4.5. $\square$

Theorem 4.16. *We have the following equalities.*

$$G(P^{(k)}_{\lambda+\rho}) = q^{k|R_+|}c_k(\lambda)P^{(k+1)}_{\lambda}$$

(this is taken to be zero if $\lambda+\rho \notin P_+$)

$$\hat{G}(P^{(k+1)}_{\lambda}) = q^{-k|R_+|}\hat{c}_k(\lambda)P^{(k+1)}_{\lambda+\rho}$$

where

$$c_k(\lambda) = \prod_{\alpha\in R_+}(q^{-k+(\alpha^\vee,\lambda+(k+1)\rho)} - q^{k-(\alpha^\vee,\lambda+(k+1)\rho)})$$

$$\hat{c}_k(\lambda) = \prod_{\alpha\in R_+}(q^{k+(\alpha^\vee,\lambda+(k+1)\rho)} - q^{-k-(\alpha^\vee,\lambda+(k+1)\rho)})$$

Proof. Recalling that $Y^\lambda X^\mu = q^{2(\lambda,\mu+k\rho)}X^\mu + \text{l.o.t.}$, it is easy to check the triangularity condition $GP^{(k)}_{\lambda+\rho} = q^{k|R_+|}c_k(\lambda)m_\lambda + \text{l.o.t.}$. So, it suffices to check the triangularity condition $\langle GP^{k}_{\lambda+\rho}, m_\mu\rangle'_{k+1} = 0$ for $\mu < \lambda$. This is equivalent to $\langle P^{(k)}_{\lambda+\rho}, \hat{G}m_\mu\rangle'_k = 0$. But, similarly, we see that $\hat{G}m_\mu$ is a constant multiple of $m_{\mu+\rho}$ plus lower order terms, whence the conclusion follows from earlier properties of Macdonald polynomials. The second statement is proved similarly. ☐

Definition 4.17. *Let*

$$M'_k(\lambda) = \langle P^{(k)}_\lambda, P^{(k)}_\lambda\rangle'_k$$

Also let

$$M_k(\lambda) = \langle P^{(k)}_\lambda, P^{(k)}_\lambda\rangle_{q,t} = d_k^{-1}(-1)^{k|R_+|}q^{|R_+|k(k-1)}M'_k(\lambda)$$

Note that we have the following (where we make use of the triangularity properties of G and $\hat{G}$):

$$\begin{aligned}
M'_{k+1}(\lambda) &= \frac{1}{c_k(\lambda)\iota(c_k(\lambda))}\langle GP^{(k)}_{\lambda+\rho}, GP^{(k)}_{\lambda+\rho}\rangle'_k \\
&= \frac{1}{c_k(\lambda)\iota(c_k(\lambda))}\frac{d_{k+1}}{d_k}\langle P^{(k)}_{\lambda+\rho}, \hat{G}GP^{(k)}_{\lambda+\rho}\rangle'_k \\
&= \frac{\iota(\hat{c}_k(\lambda))\iota(c_k(\lambda))}{c_k(\lambda)\iota(c_k(\lambda))}\frac{d_{k+1}}{d_k}\langle P^{(k)}_{\lambda+\rho}, P^{(k)}_{\lambda+\rho}\rangle'_k \\
&= (-1)^{|R_+|}\frac{d_{k+1}}{d_k}\frac{\hat{c}_k(\lambda)}{c_k(\lambda)}M'_k(\lambda+\rho)
\end{aligned}$$

Translating this into $M_k(\lambda)$, we obtain:

$$M_{k+1}(\lambda) = \left(\prod_{\alpha\in R_+}\frac{1-q^{2(\alpha^\vee,\lambda+(k+1)\rho)+2k}}{1-q^{2(\alpha^\vee,\lambda+(k+1)\rho)-2k}}\right)M_k(\lambda+\rho)$$

We iteratively apply this equation $k-1$ times to reduce to the case where $k=1$ where we already know that the Macdonald polynomials are orthonormal (as they are Weyl characters in that case). Note that at each step, the loss of a ρ due to k decreasing is compensated for by λ incrementing by ρ. This gives completes the proof of the Macdonald conjecture. We obtain

$$\langle P^{(k)}_\lambda, P^{(k)}_\lambda\rangle_{q,t} = \prod_{\alpha\in R_+}\prod_{i=1}^{k-1}\frac{1-q^{2(\alpha^\vee,\lambda+(k+1)\rho)+2i}}{1-q^{2(\alpha^\vee,\lambda+(k+1)\rho)-2i}}$$

5. Closing Remarks

Whilst we worked with $t_i = t = q^k$, we could actually have taken $t_i = q^{k_i}$ where $k_i \in \mathbb{Z}_{\geq 0}$. This would have involved introducing several different shift operators (our G, $\hat{G}$) to control the the different parameters separately. In that case, we would replace $k\rho$ with $\rho_k = \sum_{\alpha \in R_+} \frac{k_\alpha}{2}\alpha$, where k_α is equal to k_i for a simple root α_i in the orbit of α.

It is in fact true that (appropriately understood), Macdonald polynomials in type A define symmetric functions (i.e. they have suitable restriction properties for A_n as n decreases). In [Mac95], it is shown that if $\lambda = a_1\omega_1 + \cdots a_{n-1}\omega_{n-1}$ is written as a partition $\lambda = (a_1 + \cdots + a_{n-1}, a_2 + \cdots + a_{n-1}, \cdots, a_{n-1})$, then

$$\langle P_\lambda, P_\lambda \rangle_{q,t} = \prod_{s \in \lambda} \frac{1 - q^{a(s)+1}t^{l(s)}}{1 - q^{a(s)}t^{l(s)+1}}$$

Here $s \in \lambda$ means s is a box in the Young diagram of λ. Also $a(s)$ is the arm length of s, namely the number of boxes to the right of s (not including s), and $l(s)$ is the leg length of s, namely the number of boxes below s (not including s).

References

[Mac95] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, 2nd edition, 1995.

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FROM DAHA TO EHA

ANDREI NEGUȚ

1. Goals

The main purpose of this talk is two connect the two halves of our seminar. Specifically, we will follow the outline below:

- Consider the spherical double affine Hecke algebra (DAHA) of $\mathfrak{gl}_n$
- Define the limit $n \to \infty$
- Identify the generators and relations of the limit with those in the elliptic Hall algebra (EHA)

Note that the first bullet was introduced in José's talks, although we will recall it explicitly with focus on $\mathfrak{gl}_n$. Then we will use things from Chris' talks to work out the second bullet. Finally, the formulas we will work out in the third bullet will be compared with Mitya's talks next week. The reference is Schiffmann–Vasserot [1].

2. The spherical DAHA of $\mathfrak{gl}_n$

Recall (Definition 2.4.6 and Theorem 2.4.8 of José's notes) the DAHA:

$$\mathbb{H}_n = \mathbb{C}(q,v)\Big\langle T_1, ..., T_{n-1}, X_1^{\pm 1}, ..., X_n^{\pm 1}, Y_1^{\pm 1}, ..., Y_n^{\pm 1} \Big\rangle$$

subject to the relations that all X's commute, all Y's commute, and:

$$(T_i - v)(T_i + v^{-1}) = 0 \qquad T_i T_j = T_j T_i \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{1}$$

$$T_i X_j = X_j T_i \qquad T_i Y_j = Y_j T_i \qquad T_i X_i T_i = X_{i+1} \qquad T_i^{-1} Y_i T_i^{-1} = Y_{i+1} \tag{2}$$

$$Y_1 X_1 ... X_n = q X_1 ... X_n Y_1 \qquad Y_2^{-1} X_1 Y_2 X_1^{-1} = T_1^2 \tag{3}$$

where i, j go over all possible indices such that $j \notin \{i-1, i, i+1\}$. Recall the action:

$$SL_2(\mathbb{Z}) \curvearrowright \mathbb{H}_n \tag{4}$$

in which the generators of $SL_2(\mathbb{Z})$ act as:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \begin{cases} T_i \mapsto T_i \\ X_i \mapsto X_i \\ Y_i \mapsto Y_i X_i (T_1 ... T_{i-1})^{-1} (T_{i-1} ... T_1)^{-1} \end{cases} \tag{5}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} : \begin{cases} T_i \mapsto T_i \\ X_i \mapsto X_i Y_i (T_{i-1} ... T_1)(T_1 ... T_{i-1}) \\ Y_i \mapsto Y_i \end{cases} \tag{6}$$

Finally, recall the idempotent:

$$e = \frac{1}{[n]!_v^+} \sum_{\sigma \in S_n} v^{l(\sigma)} T_\sigma$$

where $T_\sigma = T_{i_1} ... T_{i_r}$ corresponds to a reduced decomposition of σ as a product of transpositions. Recall that the v–factorial is defined by setting:

$$[i]_v^\pm = \frac{v^{\pm 2i} - 1}{v^{\pm 2} - 1} \qquad \Longrightarrow \qquad [n]!_v^\pm = [1]_v^\pm \cdot ... \cdot [n]_v^\pm \tag{7}$$

It is easy to show that:

$$e^2 = e \qquad \text{and} \qquad eT_i = T_i e = ve \tag{8}$$

As in Definition 3.3.3 of José's notes, let:

$$\mathbb{SH}_n = e\mathbb{H}_n e$$

denote the **spherical** subalgebra of $\mathbb{H}_n$, which is an algebra in its own right with unit e. As we will see in Proposition 3, the size of this subalgebra is "the same" as the size of $\mathbb{C}(q,v)[X_1^{\pm 1}, ..., X_n^{\pm 1}, Y_1^{\pm 1}, ..., Y_n^{\pm 1}]^{S_n}$. Since the action of $SL_2(\mathbb{Z})$ on $\mathbb{H}_n$ leaves e invariant, we conclude that $SL_2(\mathbb{Z})$ preserves $\mathbb{SH}_n$.

3. The generators

For any $k > 0$, Schiffmann–Vasserot in [1] consider the following elements of $\mathbb{SH}_n$:

$$P_{0,k}^{(n)} = e(Y_1^k + ... + Y_n^k)e \tag{9}$$

They further generalize these elements to arbitrary $(a,b) \in \mathbb{Z}^2 \backslash 0$, by letting $k = \gcd(a,b)$ and defining:

$$P_{a,b}^{(n)} = \begin{pmatrix} * & \frac{a}{g} \\ * & \frac{b}{g} \end{pmatrix} \cdot P_{0,g}^{(n)} \tag{10}$$

where $*$ denote arbitrary integers such that the matrix on the left has determinant 1. We claim that there is no ambiguity here, since the integers denoted $*$ are determined up to multiplying the matrix (10) on the right with powers of the matrix (6). Since the latter preserves both e and the Y's, it preserves the elements (9), and so (10) are uniquely defined for any a and b.

Proposition 1. *For any $a, b \in \mathbb{Z}$, we have:*

$$P_{a,1}^{(n)} = [n]_v^- \cdot eY_1 X_1^a e \tag{11}$$

$$P_{1,b}^{(n)} = q[n]_v^+ \cdot eX_1 Y_1^b e \tag{12}$$

$$P_{-1,b}^{(n)} = [n]_v^- \cdot eY_1^b X_1^{-1} e \tag{13}$$

$$P_{a,-1}^{(n)} = q[n]_v^+ \cdot eX_1^a Y_1^{-1} e \tag{14}$$

Proof. Since $Y_{i+1} = T_i^{-1} Y_i T_i^{-1}$, we can use (8) to infer $eY_{i+1}e = v^{-2} eY_i e$. Iterating this relation gives us:

$$P_{0,1}^{(n)} = (1 + v^{-2} + ... + v^{-2n+2}) eY_1 e = [n]_v^- \cdot eY_1 e$$

Let us now hit this element with various matrices $\in SL_2(\mathbb{Z})$ to obtain (11):

$$P_{a,1}^{(n)} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^a \cdot P_{0,1}^{(n)} = [n]_v^- e \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^a \cdot Y_1 \right] e = [n]_v^- \cdot eY_1 X_1^a e$$

Hitting the case $a = 1$ with the other generator of $SL_2(\mathbb{Z})$ gives us:

$$P_{1,b}^{(n)} = \begin{pmatrix} 1 & 0 \\ b-1 & 1 \end{pmatrix} \cdot P_{1,1}^{(n)} = [n]_v^- e \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{b-1} \cdot Y_1 X_1 \right] e = [n]_v^- \cdot eY_1 X_1 Y_1^{b-1} e$$

Using formula (2.10) of [1] together with (8), we have:

$$Y_1 X_1 = q(T_1 ... T_{n-1})(T_{n-1} ... T_1) X_1 Y_1 \quad \Longrightarrow \tag{15}$$

$$\Longrightarrow \quad eY_1 X_1 Y_1^{b-1} e = qv^{2n-2} eX_1 Y_1^b e$$

and so the above relation implies (12). To obtain (13) and (14), note that:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \text{ sends } X_1 \mapsto X_1 Y_1 X_1^{-1}, Y_1 \mapsto X_1^{-1} \tag{16}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{ sends } X_1 \mapsto Y_1^{-1},\ Y_1 \mapsto Y_1 X_1 Y_1^{-1} \tag{17}$$

Therefore, formulas (11) and (12) imply:

$$P_{-1,b}^{(n)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} P_{b,1}^{(n)} = [n]_v^- \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} eY_1 X_1^b e = [n]_v^- \cdot eY_1^b X_1^{-1} e$$

$$P_{a,-1}^{(n)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} P_{1,a}^{(n)} = q[n]_v^+ \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} eX_1 Y_1^a e = q[n]_v^+ \cdot eY_1^{a-1} X_1^a Y_1^{-1} e$$

thus proving (13) and (14).

□

Proposition 2. *For any $k \in \mathbb{N}$, we have:*

$$P_{0,k}^{(n)} = e \sum_{i=1}^n Y_i^k e \tag{18}$$

$$P_{-k,0}^{(n)} = e \sum_{i=1}^n X_i^{-k} e \tag{19}$$

$$P_{0,-k}^{(n)} = q^k \cdot e \sum_{i=1}^n Y_i^{-k} e \tag{20}$$

$$P_{k,0}^{(n)} = q^k \cdot e \sum_{i=1}^n X_i^k e \tag{21}$$

Proof. The following element of $SL_2(\mathbb{Z})$:

$$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \text{takes} \qquad Y_1 \mapsto Y_1 X_1^{-1} \mapsto X_1^{-1}$$

and $T_i \mapsto T_i$. Then one can iterate (2) to show that this matrix takes any $Y_i \mapsto X_i^{-1}$, and so it takes relation (18) to (19) and (20) to (21). However, note that (18) is simply the definition (9), so it remains to prove (20). To this end, let us consider the following element of $SL_2(\mathbb{Z})$:

$$\Gamma := \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

takes $\Gamma(Y_1) = X_1 Y_1^{-1} X_1^{-1}$. Using (15) and (2), we may rewrite this as:

$$\Gamma(Y_1) = qY_1^{-1} T_1 ... T_{n-1} T_{n-1} ... T_1 = qT_1^{-1} ... T_{n-1}^{-1} Y_n^{-1} T_{n-1} ... T_1$$

Because the product of T's on the left is the inverse of the product on the right, we may raise this relation to the k–th power:

$$\Gamma(Y_1^k) = q^k T_1^{-1} ... T_{n-1}^{-1} Y_n^{-k} T_{n-1} ... T_1$$

Because the idempotent e satisfies $eT_i = T_i e = ve$, see (8), we conclude that:

$$\Gamma(eY_1^k e) = q^k e Y_n^{-k} e \tag{22}$$

As [1] claims, there is a unique polynomial P with coefficients in $\mathbb{C}(q, v)$ such that:

$$P(eY_1 e, ..., eY_1^k e) = e \sum_{i=1}^n Y_i^k e \tag{23}$$

(in fact, this is true if one replaces $\sum_i Y_i^n$ with any other degree k symmetric polynomial in the Y variables), and that the same polynomial satisfies:

$$P(eY_n^{-1} e, ..., eY_n^{-k} e) = e \sum_{i=1}^n Y_i^{-k} e \tag{24}$$

The reason why these two equalities hold for the same polynomial P follows from the automorphism of the single affine Hecke algebra that sends $T_i \mapsto T_{n-i}$ and $Y_i \mapsto Y_{n+1-i}^{-1}$ (this automorphism can be checked either by hand, note that it is closely related to Theorem 3.3.3 of Seth's notes). Note that the degree of P in its first variable, plus twice its degree in the second variable, ... plus k times its degree in the last variable, equals k. Combining (22), (23), (24) yields:

$$\Gamma\left(e \sum_{i=1}^n Y_i^k e\right) = q^k \left(e \sum_{i=1}^n Y_i^{-k} e\right)$$

which is precisely what we needed to prove.

□

Proposition 3. *The elements $P_{a,b}^{(n)}$ generate $\mathbb{SH}_n$ as an algebra.*

Proof. Since all the structure constants in $\mathbb{SH}_n$ are Laurent polynomials in q and v, and $\mathbb{SH}_n$ is free over the ring $\mathbb{C}[q^{\pm 1}, v^{\pm 1}]$ (this was proved by Cherednik) it is enough to prove the proposition in the specialization $q = v = 1$. Note that:

$$\mathbb{SH}_n|_{q=v=1} \cong \mathbb{C}[x_1^{\pm 1}, ..., x_n^{\pm 1}, y_1^{\pm 1}, ..., y_n^{\pm 1}]^{S_n} \tag{25}$$

by sending $eP(X, Y)e$ to the symmetrization (that is, the average over all $n!$ permutations of the variables) of the Laurent polynomial $P(x, y)$. Since the symmetrizations of the polynomials in the right hand sides of (11)–(14) and (18)–(21) generate the right hand side of (25) (this is an exercise), the Proposition follows.

□

4. The relations

In preparation for the stable limit, let us rescale our generators to:

$$u_{a,b} = \frac{v^k q^{-k} - v^{-k}}{k} \cdot P^{(n)}_{a,b} \tag{26}$$

where $k = \gcd(a,b)$. For all coprime a, b, define:

$$1 + \sum_{k=1}^{\infty} \frac{\theta_{ak,bk}}{x^k} = \exp\left(\sum_{k=1}^{\infty} (v^{-k} - v^k) \frac{u_{ak,bk}}{x^k}\right) \tag{27}$$

Proposition 4. *The elements $u_{a,b} \in \mathbb{SH}_n$ satisfy the commutation relations:*

$$[u_{a,b}, u_{a',b'}] = 0 \tag{28}$$

if $ab' = a'b$, and:

$$[u_{a,b}, u_{a',b'}] = \pm\theta_{a+a',b+b'} \cdot \frac{(q^l - 1)(v^l q^{-l} - v^{-l})}{l(v - v^{-1})} \tag{29}$$

if one of the following situations occurs (let $l = \gcd(a,b)$ above):

- *$ab' = a'b \pm k$, $\gcd(a,b) = k$, $\gcd(a',b') = 1$, $\gcd(a+a', b+b') = 1$*
- *$ab' = a'b \pm k$, $\gcd(a,b) = 1$, $\gcd(a',b') = 1$, $\gcd(a+a', b+b') = k$*

Remark 1. *By Pick's theorem, the lattice points (a,b), (a',b') that appear in (29) are those such that the triangle with vertices $(0,0), (a,b), (a+a', b+b')$ has no lattice points inside and on the edges, with the possible exception of the edge $(0,0), (a,b)$ in the case of the first bullet or the edge $(0,0), (a+a', b+b')$ in the case of the second bullet.*

Proof. Since the action of $SL_2(\mathbb{Z})$ on lattice points is transitive, it is enough to check (28) when $a = 0$. In this case, relations (18) and (20) tell us that:

$$u_{0,b} = \text{const} \cdot e \sum_i Y_i^b e \qquad \text{and} \qquad u_{0,b'} = \text{const} \cdot e \sum_i Y_i^{b'} e \tag{30}$$

Because e commutes with symmetric polynomials in the Y_i, (30) commute because the Y_i's commute with each other. By a similar logic, we can use the $SL_2(\mathbb{Z})$ action to make $(a,b), (a',b')$ equal to $(0, \pm k), (1, 0)$ in the case of the first bullet, and $(k, -1), (0, 1)$ in the case of the second bullet. Moreover, using one more rotation, we may assume $k > 0$. Therefore, it remains to prove:

$$[u_{0,\pm k}, u_{1,0}] = \pm u_{1,\pm k} \cdot \frac{(q^k - 1)(v^k q^{-k} - v^{-k})}{k} \tag{31}$$

$$[u_{k,-1}, u_{0,1}] = \theta_{k,0} \cdot \frac{(q-1)(vq^{-1} - v^{-1})}{v - v^{-1}} \tag{32}$$

(we used the fact that $\theta_{a,b} = u_{a,b}(v^{-1} - v)$ if $\gcd(a,b) = 1$). Let us prove the first. In the notation of the previous Subsection, it amounts to:

$$\left[P^{(n)}_{0,\pm k}, P^{(n)}_{1,0}\right] = \pm P^{(n)}_{1,\pm k} \cdot (q^k - 1)$$

When the sign is $\pm = +$, relations (12) and (18) make this relation is equivalent to:

$$\left[e\sum_{i=1}^n Y_i^k e, eX_1e\right] = e\left[\sum_{i=1}^n Y_i^k, X_1\right] e = (q^k-1)eX_1Y_1^k e \tag{33}$$

The first equality holds on general grounds (because $\sum_{i=1} Y_i^k$ is symmetric), while the second equality is proved in a 6–page computation in Appendix A of [1]. When the sign is $\pm = -$, one must apply the following automorphism to (33):

$$T_i \mapsto T_1^{-1}, \qquad X_1 \mapsto Y_1X_1Y_1^{-1}, \qquad Y_i \mapsto Y_i^{-1}, \qquad v \mapsto v^{-1}, \qquad q \mapsto q^{-1}$$

The above is the composition of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$ of (17) and the automorphism:

$$T_i \mapsto T_1^{-1}, \qquad X_i \mapsto Y_i, \qquad Y_i \mapsto X_i, \qquad v \mapsto v^{-1}, \qquad q \mapsto q^{-1}$$

that was introduced in Theorem 2.2.7 of José's notes. As for relation (32), it reads:

$$\left[P_{k,-1}^{(n)}, P_{0,1}^{(n)}\right] = \frac{\theta_{k,0}^{(n)} \cdot (q-1)}{(v-v^{-1})(vq^{-1}-v)}$$

where $\theta_{k,0}^{(n)}$ is defined via the generating series:

$$1 + \sum_{k=1}^\infty \frac{\theta_{k,0}^{(n)}}{x^k} = \exp\left(\sum_{k=1}^\infty (v^{-k}-v^k)(v^kq^{-k}-v^{-k})q^k \cdot \frac{e\sum_{i=1}^n X_i^k e}{kx^k}\right) \in \mathbb{SH}_n[[x^{-1}]]$$

Then we may use (14), (18) and (21) to write the required relation as:

$$\frac{(1-qv^{-2})(v^{2n}-1)}{q-1} \cdot e\left[X_1^kY_1^{-1}, \sum_{i=1}^n Y_i\right] e = \theta_{k,0}^{(n)} \tag{34}$$

which is proved in a 5–page computation in Appendix B of [1].

□

5. The stable limit

Let us define the $\mathbb{C}(q,v)$–algebra:

$$\mathcal{A} = \langle u_{a,b}\rangle_{(a,b)\in\mathbb{Z}} \Big/ \text{relations (28) and (29)}$$

Mitya will discuss this algebra in more depth next week, and then Alexey and Tudor's talks will identify it with the elliptic Hall algebra. Meanwhile, note that Proposition 3 and 4 imply that there exist surjective ring homomorphisms:

$$\mathcal{A} \overset{\phi_n}{\twoheadrightarrow} \mathbb{SH}_n, \qquad\qquad u_{a,b} \mapsto \text{the RHS of (26)} \tag{35}$$

for any $n \in \mathbb{N}$. Our goal is to make the above into an isomorphism by letting $n \to \infty$. Unfortunately, this will only be possible when we restrict to the positive halves of the algebras in question:

$$\mathcal{A} \supset \mathcal{A}^+ = \mathbb{C}(q,v)\,\langle u_{a,b}\rangle_{(a,b)\in\mathbb{Z}^{2,+}}$$

$$\mathbb{SH}_n \supset \mathbb{SH}_n^+ = \mathbb{C}(q,v)\left\langle P_{a,b}^{(n)}\right\rangle_{(a,b)\in\mathbb{Z}^{2,+}}$$

where $\mathbb{Z}^2 \supset \mathbb{Z}^{2,+} = \{(a,b), a>0 \text{ or } a=0, b>0\}$ denotes half of the lattice plane. Then the goal of the remainder of this talk is to prove the following Propositions:

Proposition 5. *There exists a morphism* $\mathbb{SH}_n^+ \to \mathbb{SH}_{n-1}^+$ *given by* $P_{a,b}^{(n)} \mapsto P_{a,b}^{(n-1)}$.

Clearly, the maps ϕ_n of (35) are compatible with the morphisms in Proposition 5.

Proposition 6. *The induced map:*

$$\mathcal{A}^+ \xrightarrow{\theta^+} \lim_{\leftarrow} \mathbb{SH}_n^+$$

given by $u_{a,b} \mapsto (..., P_{a,b}^{(n)}, ...)$, *is an isomorphism.*

Proof. of Proposition 5: Recall Cherednik's **basic representation**:

$$\mathbb{H}_n \hookrightarrow \text{Diff}(\mathbb{A}^{*n}) \rtimes S_n \tag{36}$$

where $\text{Diff}(\mathbb{A}^{*n}) = \mathbb{C}(q,v)[x_1^{\pm 1}, ..., x_n^{\pm 1}, D_1^{\pm 1}, ..., D_n^{\pm n}]$ is the ring of q–difference operators on punctured n–dimensional space, whose generators satisfy the relations:

$$[x_i, x_j] = [D_i, D_j] = 0 \qquad\qquad D_i x_j = q^{\delta_i^j} x_j D_i$$

Specifically, the map (36) is given by:

$$X_i \quad \mapsto \quad \text{multiplication by } x_i \tag{37}$$

$$T_i \quad \mapsto \quad v s_i + \frac{x_{i+1}(v - v^{-1})(s_i - 1)}{x_i - x_{i+1}} \tag{38}$$

$$T_{m-1}^{-1} ... T_i^{-1} Y_i T_{i-1} ... T_1 \quad \mapsto \quad s_{m-1} ... s_1 D_1 \tag{39}$$

where $s_i \in S_n$ denotes the transposition of i and $i+1$. Note that the basic representation was discussed in both Seth's and José's notes (Theorem 2.4.5 of the latter, together with the first formula after Definition 2.4.1). Because of the denominators $x_i - x_{i+1}$, the target of the map (36) is more precisely a certain localization of the ring $\text{Diff}(\mathbb{A}^{*n})$, but there's no need to burden the notation with detail. When we restrict this embedding to the spherical Hall algebra, we obtain the composition:

$$\mathbb{SH}_n \hookrightarrow \text{Diff}(\mathbb{A}^{*n})^{S_n} \rtimes S_n \twoheadrightarrow \text{Diff}(\mathbb{A}^{*n})^{S_n} \tag{40}$$

which is also an embedding. The map on the right is the projection $D \rtimes \sigma \mapsto D$ for all $D \in \text{Diff}(\mathbb{A}^{*n})$ and $\sigma \in S_n$. Let us consider the smaller subalgebras:

$$\mathbb{SH}_n^+ \supset \mathbb{SH}_n^{++} = \mathbb{C}(q,v)\left\langle P_{a,b}^{(n)} \right\rangle_{a,b,\geq 0}$$

$$\text{Diff}(\mathbb{A}^{*n}) \supset \text{Diff}^{++}(\mathbb{A}^{*n}) = \mathbb{C}(q,v)[x_1, ..., x_n, D_1, ..., D_n]$$

We claim that the maps (40) restrict to:

$$\psi_n : \mathbb{SH}_n^{++} \hookrightarrow \text{Diff}^{++}(\mathbb{A}^{*n})^{S_n} \tag{41}$$

Indeed, Propositions 3 and 4 imply that the domain is generated by $P_{0,k}^{(n)}$ and $P_{k,0}^{(n)}$ (this kind of generation statement will be discussed in more detail in Mitya's talk) so it is enough to show that these elements land in $\text{Diff}^{++}(\mathbb{A}^{*n}) \subset \text{Diff}(\mathbb{A}^{*n})$. Comparing formulas (18), (21) with (37), (39), this statement is clear since no negative powers of x_i and D_i come up in the latter formulas.

Lemma 1. *The maps ψ_n of (41) can be completed to a commuting square:*

$$\begin{tikzcd} \mathbb{SH}^{++}_n \arrow[r, "\psi_n"] \arrow[d, dotted] & \text{Diff}^{++}(\mathbb{A}^{*n})^{S_n} \arrow[d, "\gamma"] \\ \mathbb{SH}^{++}_{n-1} \arrow[r, "\psi_{n-1}"] & \text{Diff}^{++}(\mathbb{A}^{*n-1})^{S_{n-1}} \end{tikzcd} \tag{42}$$

where the dotted map on the left takes $P^{(n)}_{a,b} \mapsto P^{(n-1)}_{a,b}$, and the map γ is given by:

$$\gamma(x_i) = x_i \qquad\qquad \gamma(x_n) = 0 \tag{43}$$

$$\gamma(D_i) = \frac{D_i}{v} \qquad\qquad \gamma(D_n) = 0 \tag{44}$$

for all $i \in \{1, ..., n-1\}$. Note that γ is a homomorphism.

Note that the map γ would not have been defined if we had considered anything greater than the $++$ algebras, because the $P^{(n)}_{a,b}$ with $b < 0$ involve inverse powers of Y_i, and we could not have set $D_n \mapsto 0$ in (44). Let us show how this Lemma implies Proposition 5. Consider any relation:

$$0 = \sum \text{const} \prod_i P^{(n)}_{a_i,b_i} \in \mathbb{SH}^+_n \tag{45}$$

for various choices of $a_i > 0$ or $a_i \geq 0, b_i > 0$. Because the sum is finite, we may choose some k large enough so that $b_i + ka_i \geq 0$ for all i that appear in (45). The $SL_2(\mathbb{Z})$ invariance of spherical DAHAs implies that we have a relation:

$$0 = \sum \text{const} \prod_i P^{(n)}_{a_i,b_i+ka_i} \in \mathbb{SH}^{++}_n \tag{46}$$

By Lemma 1, relation (46) also holds with n replaced by $n-1$, and therefore $SL_2(\mathbb{Z})$ invariance implies that so does (45). Having showed that any relation between the generators of $\mathbb{SH}^+_n$ also holds in $\mathbb{SH}^+_{n-1}$, this concludes the proof of Proposition 5.

Proof. of Lemma 1: Relations (28) and (29) imply that for any a, b, there exists a finite polynomial $Q_{a,b}$ in the variables $\alpha_1, \alpha_2, ..., \beta_1, \beta_2, ...$ such that:

$$Q_{a,b}\left(P^{(n)}_{0,1}, P^{(n)}_{0,2}, ..., P^{(n)}_{1,0}, P^{(n)}_{2,0}, ...\right) = P^{(n)}_{a,b}$$

for all n. It's really important that the above relation holds in $\mathbb{SH}_n$ for all n, for a fixed polynomial $Q_{a,b}$ (the combinatorics which establishes this fact will be discussed in more detail by Mitya in the next talk, but it's not hard to believe). Since $\psi_n, \psi_{n-1}, \gamma$ in (42) are homomorphism, to establish the commutativity of the square, it is therefore enough to prove that:

$$\gamma \circ \psi_n\left(P^{(n)}_{0,a}\right) \quad \text{and} \quad \gamma \circ \psi_n\left(P^{(n)}_{a,0}\right) \quad \in \quad \psi_{n-1}\left(\mathbb{SH}^{++}_{n-1}\right)$$

This is obvious for the latter, namely $P^{(n)}_{a,0}$, because its image under ψ_n is $q^k \sum_{i=1}^n x_i^a$. As for the former, it is true that for any symmetric polynomial $f(Y_1, ..., Y_n)$ we have:

$$\gamma \circ \psi_n\left(ef(Y_1, ..., Y_{n-1}, Y_n)e\right) = \gamma \circ \psi_n\left(ef(Y_1, ..., Y_{n-1}, 0)e\right) \tag{47}$$

One way to see this is to chase through the definitions and observe that:

$$\psi_n\left(ef(Y_1, ..., Y_{n-1}, Y_n)e\right) = L_f$$

where the operator $L_f \in \mathrm{Diff}(\mathbb{A}^{*n})^{S_n}$ was introduced in Lemma 4.3.5 of José's talk, or Definition 3.13 of Chris' talk. Then equation (47) is merely the compatibility of L_f's for n and $n-1$ via the homomorphism γ. Alternatively, since both sides of (47) are additive and multiplicative in f, it is enough to check the equality when f is the k–th elementary symmetric function in $Y_1, ..., Y_n$. In this case, Macdonald shows (see Lemma 4.5 of [1]) that:

$$\psi_n\left(e\sum_{1\le i_1<...<i_k\le n} Y_{i_1}...Y_{i_k}e\right) = \sum_{I\subset\{1,...,n\}}^{|I|=k}\prod_{j\notin I}^{i\in I}\frac{x_iv - x_jv^{-1}}{x_i - x_j}\prod_{i\in I} D_i$$

and it is clear that setting $x_n, D_n \mapsto 0$ in the right hand side produces the corresponding expression when n is replaced by $n-1$ (up to a power of v, which is accounted for in (44)). □

□

Proof. of Proposition 6: Since the $P^{(n)}_{a,b}$ generate $\mathbb{SH}^+_n$ by Proposition 4, ι^+ is surjective. To prove it is also injective, it is enough to show that the analogous map:

$$\mathcal{A}^{++} \xrightarrow{\iota^{++}} \lim_{\leftarrow} \mathbb{SH}^{++}_n$$

is injective (we have already seen the reason for this: if there's a relation of the form (45) in the kernel of ι^+, then we could act with an element of $SL_2(\mathbb{Z})$ to turn it into a relation of the form (46) in the kernel of ι^{++}). We claim the following:

- The algebra $\mathcal{A}^{++}$ is graded by $\mathbb{N}_0\times\mathbb{N}_0$, with $u_{a,b}$ in degree (a, b)
- The dimension of $\mathcal{A}^{++}_{a,b}$ is equal to the number of unordered collections:

$$(a_1, b_1), ..., (a_t, b_t) \quad \text{with} \quad \sum a_i = a \text{ and } \sum b_i = b \tag{48}$$

The first bullet is immediate, and the second bullet will be explained by Mitya in more detail. The intuition behind it is the following: elements of $\mathcal{A}^{++}$ are linear combinations of ordered products $u_{a_1,b_1}...u_{a_t,b_t}$ with $a_i, b_i \ge 0$, and the second bullet claims that we can always use relations (28) and (29) to rearrgange the terms in this product such that the lattice points $(a_1, b_1), ..., (a_t, b_t)$ form a convex path. The number of convex paths is equal to the number of unordered collections (48).

Therefore, to prove the injectivity of ι^{++}, it is enough to show that:

$$\dim\left(\lim_{\leftarrow}\mathbb{SH}^{++}_n \text{ in degree } (a, b)\right) = \text{the number in the second bullet} \tag{49}$$

To prove this, we will invoke the argument used in the proof of Proposition 3. Since the integral form of $\mathbb{SH}^{++}_n$ is a free module over the ring $\mathbb{C}[q^{\pm1}, v^{\pm1}]$, its rank can be computed in the specialization $q = v = 1$:

$$\dim\left(\mathbb{SH}^{++}_n \text{ in degree } (a, b)\right) = \dim_{\mathbb{C}}\left(\mathbb{C}[x_1, ..., x_n, y_1, ..., y_n]^{S_n} \text{ in degree } (a, b)\right)$$

where a refers to the degree in the x variables and b refers to the degree in the y variables. The polynomial rings in the right hand side have a well-known inverse limit, the ring of polynomials in infinitely many variables:

$$\dim\left(\lim_{\leftarrow} \mathbb{SH}_n^{++} \text{ in degree } (a,b)\right) = \dim_{\mathbb{C}}\left(\mathbb{C}[x_1,...,y_1,...]^{\text{sym}} \text{ in degree } (a,b)\right)$$

All that remains is to observe that a basis of the space in the RHS is given by:

$$\text{Sym } x_1^{a_1}x_2^{a_2}...y_1^{b_1}y_2^{b_2}...$$

with $\sum a_i = a$, $\sum b_i = b$. The number of such basis vectors is precisely the number in the second bullet, which appears in (49).

$\square$

References

[1] Schiffmann O., Vasserot E., *Cherednik algebras, $W-$algebras and the equivariant cohomology of the moduli space of instantons on* $\mathbb{A}^2$, **Publ. Math. Inst. Hautes Etud. Sci.**, 118 (2013), Issue 1, 213-342

LECTURE 4: ELLIPTIC HALL ALGEBRA BY GENERATORS AND RELATIONS

DMYTRO MATVIEIEVSKYI

Abstract. These are notes for a talk given at the MIT-Northeastern Graduate Student Seminar on Double Affine Hecke Algebras and Elliptic Hall Algebras, Spring 2017.

Contents

1. Goals and structure of the talk

The main goal of the talk is to introduce the elliptic Hall algebra (EHA) and show that it is isomorphic to a quotient of the Ding-Iohara algebra (also known as the quantum toroidal $\mathfrak{gl}_1$). We will start with the notion of quantum affinization for Kac-Moody algebras. After that we will define the elliptic Hall algebra $\mathcal{E}_K$ and its specialization $\mathcal{E}$ that will be one of the main objects of study in our seminar. The first one can be understood as the quantum affinization of $\mathcal{U}_q(\widehat{\mathfrak{gl}_1})$. Next we will move to the Ding-Iohara algebra $\tilde{\mathcal{U}}$ and its quotient $\tilde{\mathcal{E}}$ that is the quantum toroidal $\mathfrak{gl}_1$. We will define it by generators and relations and construct a surjective map $\tilde{\mathcal{E}} \twoheadrightarrow \mathcal{E}$. The main result of this talk is that this map is an isomorphism. We will give a combinatorial proof of this theorem. Interaction with other objects such as the Hall algebra of coherent sheaves on an elliptic curve and shuffle algebra will be discussed in other talks. We finish these notes with a discussion of Hopf algebra structure on the EHA.

2. Quantum affinization

2.1. Quantum Kac-Moody algebra. Let $\mathfrak{g}$ be a Kac-Moody algebra. We set C be the Cartan matrix of $\mathfrak{g}$, $\mathfrak{h}$ – its coroot lattice. Let $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ and $\alpha_1^\vee, \dots, \alpha_n^\vee \in \mathfrak{h}$ be sets of simple roots and simple coroots correspondingly. For this data we define the quantum Kac-Moody algebra $\mathcal{U}_q(\mathfrak{g})$.

Definition 2.1.1. *The quantum Kac-Moody algebra $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra generated by elements k_h for every $h \in \mathfrak{h}$ and $x_1^\pm, x_2^\pm, \dots, x_n^\pm$ with the following set of relations.*

$$\begin{aligned}
&k_{h+h'} = k_h k_{h'},\ k_0 = 1,\\
&k_h x_i^\pm = q^{\pm\alpha_i(h)} x_i^\pm k_h,\\
&[x_i^+, x_j^-] = \delta_{i,j} \frac{k_{\alpha_i^\vee} - k_{-\alpha_i^\vee}}{q - q^{-1}},\\
&\sum_{r=0,1,\dots,1-C_{i,j}} \frac{(-1)^r}{[r!][(1-C_{i,j}-r)!]} \left(x_i^\pm\right)^{1-C_{i,j}-r} x_j^\pm \left(x_i^\pm\right)^r = 0,\ \textit{for } i \neq j.
\end{aligned}$$

In the formulas above we put $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ and $[m]! = [m][m-1]\dots[1]$.

2.1.1. *Quantum Kac-Moody algebra of $\mathfrak{sl}_2$.* Let us show an example and apply the construction above to the Lie algebra $\mathfrak{sl}_2$. We have elements $x_1^+ = e$, $x_1^- = f$ and k_m for $m \in \mathbb{Z}$. Note that $k_m = k_1^m$. Then $\mathcal{U}_q(\mathfrak{sl}_2)$ is generated by $K := k_1$, K^{-1}, e and f with the following set of relations.

$$\begin{aligned}
&KK^{-1} = K^{-1}K = 1,\\
&Ke = q^2 eK,\\
&Kf = q^{-2} fK,\\
&[e, f] = \frac{K - K^{-1}}{q - q^{-1}}.
\end{aligned}$$

2.2. Quantum affinization. Let $\mathfrak{g}$, $\mathfrak{h}$, α_i and $\alpha_i^\vee$ be the same as in the previous section. Let us have variables $x_{i,r}^\pm$ (with $i \in \{1, \dots, n\}$, $r \in \mathbb{Z}$), $h_{i,m}$ (with $i \in \{1, \dots, n\}$, $m \in \mathbb{Z} \setminus \{0\}$). We put $\phi_{i,m}^\pm$ be the elements determined by the formal power series

$$\begin{aligned}
&\sum_{m \geq 0} \phi_{i,\pm m}^\pm z^{\pm m} = k_{\alpha_i^\vee} \exp\left(\pm(q - q^{-1}) \sum_{m' \geq 1} h_{i,\pm m'} z^{\pm m'}\right),\\
&\phi_{i,m}^+ = 0,\ \text{for } m < 0,\\
&\phi_{i,m}^- = 0,\ \text{for } m > 0.
\end{aligned}$$

Let us consider the following series:

$$\begin{aligned}
&x_i^\pm(w) = \sum_{r \in \mathbb{Z}} x_{i,r}^\pm w^r,\\
&\phi_i^\pm(z) = \sum_{m \in \mathbb{Z}_{>0}} h_{i,m}^\pm z^m,\\
&\delta\left(\frac{z}{w}\right) = \sum_{k \in \mathbb{Z}} \left(\frac{z}{w}\right)^k.
\end{aligned}$$

Definition 2.2.1. *The quantum affinization of $\mathcal{U}_q(\mathfrak{g})$ is the $\mathbb{C}$-algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ with the generators $x^{\pm}_{i,r}$, k_h (with $h \in \mathfrak{h}$), $h_{i,m}$ and the relations below.*

$$
\begin{aligned}
&(i) k_{h+h'} = k_h k_{h'},\ k_0 = 1,\ k_h \phi_i^{\pm}(z) = \phi_i^{\pm}(z) k_h,\\
&(ii) \phi_i^{\pm}(w)\phi_j^{\pm}(z) = \phi_j^{\pm}(z)\phi_i^{\pm}(w),\ \phi_i^{+}(w)\phi_j^{-}(z) = \phi_j^{-}(z)\phi_i^{+}(w),\\
&(iii) k_h x_i^{\pm}(z) = q^{\pm\alpha_i(h)} x_i^{\pm}(z) k_h,\\
&(iv) \phi_i^{\pm}(z) x_j^{\pm}(w) = \frac{q^{\pm C_{i,j}} w - z}{w - q^{\pm C_{i,j}} z} x_j^{\pm}(w)\phi_i^{\pm}(z),\\
&(v) [x_i^{+}(z), x_k^{-}(w)] = \frac{\delta_{i,j}}{q-q^{-1}} \delta(\frac{w}{z}) \left(\phi_i^{+}(w) - \phi_i^{-}(w)\right),\\
&(vi) (w - q^{\pm C_{i,j}} z) x_i^{\pm}(z) x_j^{\pm}(w) = (q^{\pm C_{i,j}} w - z) x_j^{\pm}(w) x_i^{\pm}(z),\\
&(vii) \sum_{\sigma\in\Sigma_s}\sum_{k=0,\dots,s} (-1)^k \frac{[s]!}{[k]![s-k]!} x_i^{\pm}(w_{\sigma(1)})\dots x_i^{\pm}(w_{\sigma(k)}) x_i^{\pm}(z) x_i^{\pm}(w_{\sigma(k+1)})\dots x_i^{\pm}(w_{\sigma(s)}) = 0,
\end{aligned}
$$

where $s = 1 - C_{i,j}$. The equation (iv) is expanded for $|w| > |z|$.

Remark 2.2.2. *The correspondence $x_i^{\pm} \to x_{i,0}^{\pm}$ gives a map of algebras $\mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\hat{\mathfrak{g}})$. Therefore $\mathcal{U}_q(\hat{\mathfrak{g}})$ has a structure of $\mathcal{U}_q(\mathfrak{g})$-module.*

Proof. The proof of this statement is straightforward computation of the coefficients in relations above. We left it to the reader. □

2.2.1. *Quantum affine algebra $U(\widehat{\mathfrak{sl}_2})$.* Let us apply this construction to the Lie algebra $\mathfrak{sl}_2$. We have generators $e_r = x^{+}_{1,r}$, $f_r = x^{-}_{1,r}$ for $r \in \mathbb{Z}$, $h_m = h_{1,m}$ for $m \neq 0$, K and K^{-1}. Let us define power series $e(w)$, $f(w)$, $\phi^{\pm}(z)$ as in the general case. We have the following set of relations:

$$
\begin{aligned}
&KK^{-1} = K^{-1}K = 1,\\
&K\phi^{\pm}(z) = \phi^{\pm}(z)K,\\
&Ke = q^2 eK,\\
&Kf = q^{-2} fK,\\
&\phi^{\pm}(z)e(w) = \frac{q^2 w - z}{w - q^2 z} e(w)\phi^{\pm}(z),\\
&\phi^{\pm}(z)f(w) = \frac{q^{-2} w - z}{w - q^{-2} z} f(w)\phi^{\pm}(z),\\
&[e(z), f(w)] = \frac{\delta_{i,j}}{q-q^{-1}} \delta(\frac{w}{z})(\phi^{+}(w) - \phi^{-}(w)).
\end{aligned}
$$

2.3. **The quantum Heisenberg algebra.**

Definition 2.3.1. *The infinite-dimensional Heisenberg algebra $\mathcal{H}$ is the $\mathbb{C}$-algebra generated by elements $a_{\pm n}$ for $n \in \mathbb{Z}_{>0}$ and a central element γ with relations $[a_n, a_m] = \delta_{n,-m} n\gamma$.*

Let us fix complex numbers q_1, q_2 and $q = q_1 q_2$ and set

$$\alpha_k = \frac{(1-q_1^k)(1-q_2^k)(1-q^{-k})}{n}.$$

We consider the algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}_1})$ generated over $\mathbb{C}(q_1, q_2)$ by elements $a_{\pm n}$ for $n \in \mathbb{Z}_{>0}$ and a central element c with relations $[a_n, a_m] = \delta_{n,-m}\frac{c^n - c^{-n}}{\alpha_n}$.
The algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}_1})$ is called the quantum Heisenberg algebra.

2.4. **The elliptic Hall algebra.** The main goal of this section is to construct an algebra $\mathcal{E}_K$ that, in a sense, is the quantum affinization of $\mathcal{U}_q(\widehat{\mathfrak{gl}_1})$ (note that $\widehat{\mathfrak{gl}_1}$ is not a Kac-Moody Lie algebra so the construction of the previous section does not apply literally but serves as motivation).
We set $Z = \mathbb{Z}^2$ and $Z^\times = \mathbb{Z}^2 \setminus (0,0)$. For an element $x = (a,b) \in Z^\times$ let us put $\deg(x) := g.c.d.(a,b) \in \mathbb{Z}_{>0}$. For a pair of elements $(x,y) \in (Z^2)^\times$ we set $\epsilon_{x,y} = sign(\det(x,y)) \in \{\pm 1\}$ and let $\Delta_{x,y}$ be the triangle in Z with vertices $(0,0), x, x+y$.

Definition 2.4.1. *The elliptic Hall algebra $\mathcal{E}_K$ is the $\mathbb{C}$ algebra generated by the set of elements u_x for $x \in Z^\times$ and κ_α for $\alpha \in Z$ subject to the relations given below. We define elements θ_z for $z \in Z^\times$ satisfying the following equations for every $x_0 \in Z^\times$ with $\deg(x_0) = 1$:*

$$\sum_i \theta_{ix_0} s^i = exp(\sum_{r \geq 1} \alpha_r u_{rx_0} s^r).$$

Note that $\theta_{x_0} = \alpha_1 u_{x_0}$, $\theta_{2x_0} = \alpha_2 u_{2x_0} + \frac{\alpha_1}{2} u_{x_0}^2$.
The generating relations of $\mathcal{E}_K$ are as follows.

$$\kappa_\alpha \text{ is central,}$$
$$\kappa_\alpha \kappa_\beta = \kappa_{\alpha+\beta}, \ \kappa_0 = 1,$$
$$[u_x, u_y] = \delta_{x,-y} \frac{\kappa_x - \kappa_x^{-1}}{\alpha_{\deg(x)}} \text{ with } \deg((r,d)) := \gcd(r,d) \text{ if } x, y \text{ belong to the same line,}$$
$$[u_y, u_x] = \epsilon_{x,y} \kappa_{\alpha(x,y)} \frac{\theta_{x+y}}{\alpha_1} \text{ if } \deg(x) = 1 \text{ and } \Delta_{x,y} \text{ has no interior lattice point.}$$

In the expression above

$$\alpha(x,y) = \epsilon_x \frac{(\epsilon_x x + \epsilon_y y - \epsilon_{x+y}(x+y))}{2}, \ \text{if } \epsilon_{x,y} = 1,$$
$$\alpha(x,y) = \epsilon_y \frac{(\epsilon_x x + \epsilon_y y - \epsilon_{x+y}(x+y))}{2}, \ \text{if } \epsilon_{x,y} = -1.$$

Here $\epsilon_x = 1$ for $x = (r,d)$ if $r > 0$ or $r = 0$ and $d > 0$ and $\epsilon_x = -1$ in other case.

For any line L through the origin with a rational slope elements u_x for $x \in L$ satisfy relations of the quantum Heisenberg algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}_1})$. Therefore $\mathcal{E}$ can be understood as the quantum affinization of the quantum Heisenberg algebra.
Note that all κ_z are defined from $\kappa_{0,1}$ and $\kappa_{1,0}$. In this talk we will be interested in specialization $\mathcal{E}$ of this algebra to the case $\kappa_{0,1} = 1$, $\kappa_{1,0} = c$.

Corollary 2.4.2. *The algebra $\mathcal{E}$ (we will also call it the EHA) is the $\mathbb{C}(c^{\pm}1)$ algebra generated by the set of elements u_x for $x \in Z^\times$ subject to the following relations:*

$$(i)[u_x, u_{x'}] = 0 \text{ if } x, x' \text{ belong to the same line and } x \neq -x',$$
$$(i')[u_x, u_{x'}] = \frac{c^r - c^{-r}}{q - q^{-1}} \text{ if } x = (r,d), x' = (-r,-d),$$
$$(ii)[u_y, u_x] = \epsilon_{x,y} \frac{\theta_{x+y}}{\alpha_1} \text{ if } \deg(x) = 1, \epsilon_x = \epsilon_y \text{ and } \Delta_{x,y} \text{ has no interior lattice point,}$$
$$(ii')[u_y, u_x] = \epsilon_{x,y} \frac{c^{\alpha(r,r')}}{\alpha_1} \theta_{x+y} \text{ if } \deg(x) = 1, x = (r,d), y = (r',d'), \epsilon_x \neq \epsilon_y \text{ and } \Delta_{x,y} \text{ has no interior lattice point.}$$

Let us give few examples of commutation relations:

$$[u_{(1,0)}, u_{(0,1)}] = \frac{\theta_{(1,1)}}{\alpha_1} = u_{(1,1)},$$
$$[u_{(1,0)}, u_{(1,2)}] = \frac{\theta_{(2,2)}}{\alpha_1} = \frac{\alpha_2}{\alpha_1}u_{(2,2)} + \frac{\alpha_1}{2}u_{(1,1)},$$
$$[u_{(-1,0)}, u_{(1,2)}] = -\frac{\kappa_{1,0}}{\alpha_1}\theta_{(2,2)} = -\frac{\alpha_2 c}{\alpha_1}u_{(2,2)} - \frac{\alpha_1 c}{2}u_{(1,1)},$$

$[u_{(1,2)}, u_{(1,-1)}]$ is not proportional to $u_{(2,1)}$. This commutator will be computed later.

3. Properties of EHA

3.1. $\mathrm{SL}(2,\mathbb{Z})$**-action.** We have a natural action of $\mathrm{SL}(2,\mathbb{Z})$ on the generators u_x of the algebra $\mathcal{E}$. An element $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ sends u_x to $u_{\gamma(x)}$. If $c \neq 1$ this action does not preserve $\alpha(x,y)$ and therefore does not descend to $\mathcal{E}$. In fact we have an action of the universal cover $\widehat{\mathrm{SL}}(2,\mathbb{Z})$ on $\mathcal{E}$. Nevertheless it sends θ_x to $\theta_{\gamma(x)}$, $\epsilon_{x,y}$ to $\epsilon_{\gamma(x),\gamma(y)}$, preserves degrees and triangles w/o interior lattice points. Suppose that x, y satisfy condition of the commutation relation (iii). Note that assuming $c = 1$ we have the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathcal{E}$. Therefore if x, $\gamma(x)$, y and $\gamma(y)$ lie the right half-plane (so $\alpha(x,y) = 0$), then the commutation relation is preserved by the action of γ. We will use it in Section 3.4.

3.2. **Smaller set of generators.** For any element $x = (a,b) \in Z^\times$ we define its rank as $\mathrm{rank}(x) := a$.

Lemma 3.2.1. *The EHA $\mathcal{E}_K$ (ansd therefore its specialization $\mathcal{E}$) is generated by the elements $u_{\pm1,l}$, $u_{0,\pm k}$ for $l \in \mathbb{Z}$, $k \in \mathbb{Z}_{>0}$.*

Proof. We will prove this lemma by the induction on the rank of an element. Denote by $\mathcal{T}$ the subalgebra generated by $u_{\pm1,l}$, $u_{0,\pm k}$ and assume that $u_{r,s} \in \mathcal{T}$ for any $(r,s) \in Z^\times$ with $n > |r| \geq 1$. It is enough to prove that $u_{n,d} \in \mathcal{T}$, the fact for the $u_{-n,d}$ will follow analogously. Let us denote $z = (n,d)$ and define $x = (r,s)$ be the closest point to the line $0z$ such that $r < n$. By the construction $\Delta_{x,z-x}$ has no interior points (they need to be closer to the line $0z$ then x) and $\epsilon_x = \epsilon_{z-x} = \epsilon_z = 1$. Therefore $\theta_z = \frac{1}{\alpha_1}[t_x, t_{z-x}]$ and ranks of x and $z - x$ are less then n. Therefore $\theta_z \in \mathcal{T}$. Let $z = kz_0$ where $\deg(z_0) = 1$. From the definition $\theta_z = \alpha_k u_z + \sum_{i_1+\ldots+i_m=k} \beta_{i_1,\ldots,i_m} u_{i_1 z_0} \ldots u_{i_m z_0}$ where $\beta_{i_1,\ldots,i_m} = \frac{\prod_{j=1}^m \alpha_{i_j}}{m!}$. Every summand except for $\alpha_k u_z$ belongs to $\mathcal{T}$, so $u_z \in \mathcal{T}$. □

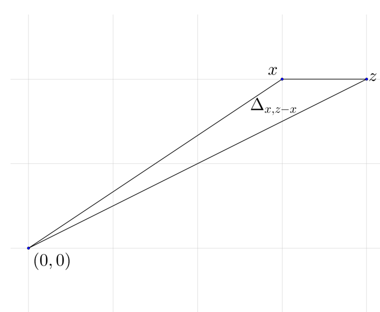

FIGURE 1. The induction step.

3.3. **Basis combinatorial notions.** In the next section we will give a basis of the EHA $\mathcal{E}$ as a vector space. For this purpose we need to introduce more notations.

For an element $z \in Z^\times$ we define its slope $\mu(z)$ to be the angle between the horizontal axis and the ray $0z$. We set $\mu(z) \in]-\frac{\pi}{2}, \frac{3\pi}{2}]$.

For the every sequence $s = (x_1, x_2, \dots, x_n)$ of elements in $Z^\times$ we associate a broken line in Z connecting points $0, x_1, x_1 + x_2, \dots, x_1 + \dots + x_n$. We call two sequences s and $s^{'}$ equivalent if $s^{'}$ can be obtained from s by successive permutations of adjacent vertices $x_i \neq -x_{i+1}$ of the same slope. We will refer to equivalence classes of sequences as paths and denote the set of all paths as Path. To a path $p = (x_1, x_2, \dots, x_n)$ we assign the element $u_p = u_{x_1} u_{x_2} \dots u_{x_n} \in \mathcal{E}$. From the definition of $\mathcal{E}$ we see that the elements u_p generate EHA as a vector space.

We say that the path p represented by a sequence $(x_1, x_2, \dots, x_n)$ is convex if it satisfies

$$-\frac{\pi}{2} < \mu(x_1) \leq \mu(x_2) \leq \dots \leq \mu(x_n) \leq \frac{3\pi}{2}.$$

We denote the set of all convex paths as Conv. For example the path p in Figure 2 is not convex because $\mu(x_i) > \mu(x_2)$ but the path p' is convex. In next section we will prove that u_p for $p \in \mathsf{Conv}$ is a basis of $\mathcal{E}$.

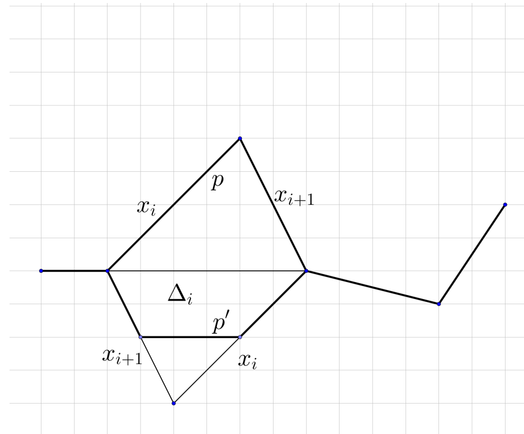

FIGURE 2. The triangle Δ and a local convexification p' of the non-convex path p.

For the pair x_i, x_{i+1} in a path p such that $\mu(x_i) > \mu(x_{i+1})$ we can construct the triangle Δ with vertices $x_1 + \dots + x_{i-1}, x_1 + \dots + x_{i-1} + x_{i+1}, x_1 + \dots + x_{i-1} + x_i + x_{i+1}$. We call the path p' obtained from p by replacing x_i, x_{i+1} by a convex path in the triangle Δ_i *a local convexification* of p.

For the path $p = (x_1, x_2, \dots, x_n)$ we define its *length* $l(p) = n$ and *weight* $|p| = \sum_{i=1}^n x_i$. Note that for every path p there is a unique convex path $p^\sharp$ with the same entries that in p. This path is constructed by permuting entries according to the order given by slope. Note that any two segments of paths p and $p^\sharp$ either do not intersect or coincide. Let us consider the subalgebra $\mathcal{E}^+$ of $\mathcal{E}$ generated by $u_{(a,b)}$ with $a > 0$ or $a = 0$, $b > 0$. It is generated by paths with entries $x_i = (a_i, b_i)$, $a_i > 0$ or $a_i = 0$, $b_i > 0$ for all i. We call them positive paths. Let us denote Conv^+ the set of positive convex paths. We can define the subalgebra $\mathcal{E}^-$ and the subset Conv^- in the same way. Then any positive p and the corresponding positive convex path $p^\sharp$ bound a polygon with all vertices in lattice. Let $a(p)$ to be the area of this polygon, we will abuse the terminology and call $a(p)$ the area of the path p. We have the following lemma.

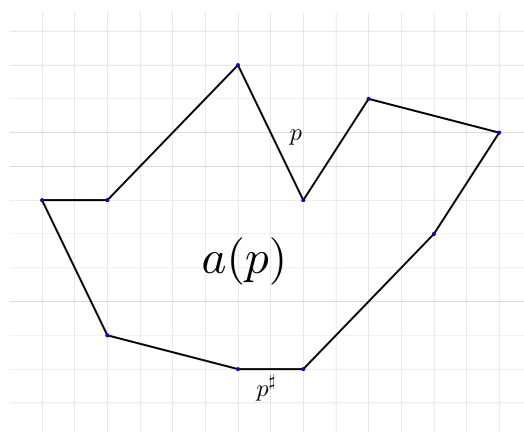

FIGURE 3. The area $a(p)$ of the path p.

Lemma 3.3.1. *Let p be a positive path in $Z^\times$. Then*
i) p is convex if and only if $a(p) = 0$,
ii) for any subpath p' of p we have $a(p') \leq a(p)$,
iii) For any local convexification p' of p we have $a(p') < a(p)$.

i) is obvious.
To prove ii) let us consider the path p^* obtained from p by replacing p' by $p'^\sharp$. Note that $p^{*,\sharp} = p^\sharp$ because p^* is obtained by permuting entries. Therefore $a(p) = a(p') + a(p^*) \Rightarrow a(p') \leq a(p)$.
So it remains to prove the third statement of the lemma.

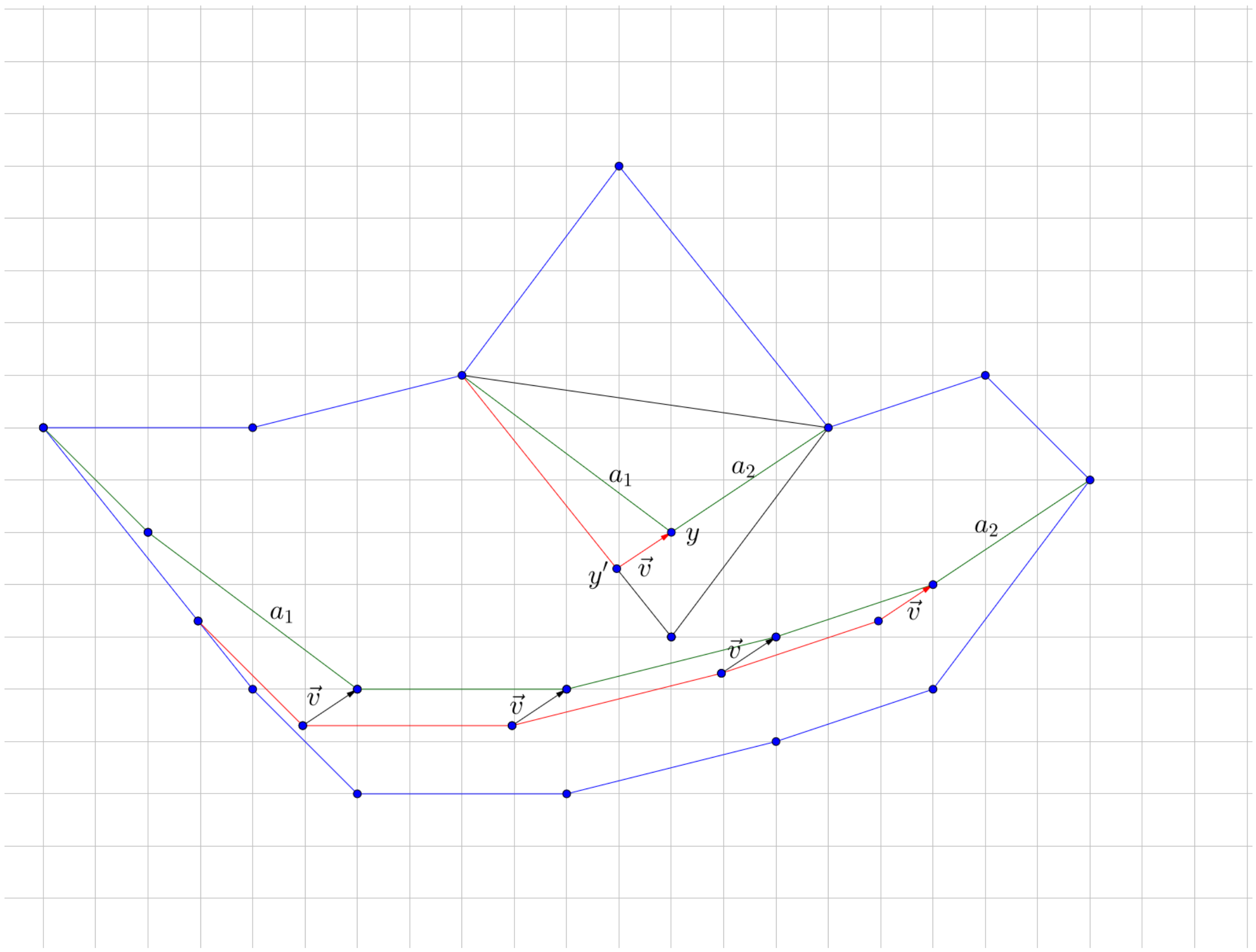

FIGURE 4. The lower green path $p'^{\sharp}$ lies between upper green p' and red $p''^{\sharp}$.

Let p' be a local convexification of p obtained by replacing entries x_i, x_{i+1} by a convex path. We claim that $p'^{\sharp}$ belongs to the polygon bounded by $p^{\sharp}$ and p'. The upper bound is obvious. Suppose that we have points inside the triangle Δ for p'. Let $y \in p'$ be the first such vertex and let p'' be a broken line defined in the following way. We take the edge from y and extend it in the direction of y to the intersection y' with the edge of Δ. We replace y by y' in the path p'. The broken line p'' doesn't have to be a path but we still can define notions of slope, convex broken line and area in the same way. We claim that $p'^{\sharp}$ belongs to the polygon A_y bounded by p' and $p''^{\sharp}$. Indeed let us look Figure 4. Here blue color corresponds to the path p and corresponding convex path $p^{\sharp}$, green color – to p' and $p'^{\sharp}$ and red – to p'' and $p''^{\sharp}$. Note that we are interested only in z with $\mu(x_{i+1}) \leq \mu(z) \leq \mu(x_i)$. Let us introduce some notations. We denote the incoming in y segment of p' by a_1 and the outgoing segment by a_2. Therefore for the first edges in $p'^{\sharp}$ and $p''^{\sharp}$ we have that the slope of the second one is bigger. Therefore up to the segment a_1 we have $p'^{\sharp}$ belongs to A_y. Let us denote by $\vec{v}$ the vector between y' and y. Note that we have this vector (black color on the picture) between corresponding vertices of $p'^{\sharp}$ and $p''^{\sharp}$ after the segment a_1 in $p'^{\sharp}$. After that we have the same segments in both of broken lines up to the segment a_2. And we have $p'^{\sharp}$ and $p''^{\sharp}$ coincide after a_2. Therefore $p'^{\sharp}$ belongs to A_y.
So it is enough to prove the claim for the convexification p' without interior lattice points. The same argument applied to p' gives that it is enough to prove for the case when a point on a side of Δ should be the intersection of it's sides, i.e. the vertex. In this case we get $p'^{\sharp} = p^{\sharp}$.

Let us denote by B the polygon bounded by p and p'. Therefore

$$a(p) \geq a(p') + S(B) \geq a(p') + s(\Delta) \Rightarrow a(p') < a(p).$$

3.4. **Basis of convex paths.** The main goal of this section is to prove an important lemma that u_p for $p \in \mathsf{Conv}^+$ give a basis of $\mathcal{E}^+$ as a vector space. In future talks we will show from another description of the EHA that elements corresponded to convex paths are linearly independent. For now we refer reader for the proof to the paper [BS] of Burban and Shiffmann. So it remains to check that these elements span everything. For these purposes we need to recall a standard fact about area of the polygon with lattice points.

Proposition 3.4.1. ***(Pick's formula.)*** *For the polygon with lattice points P with $i(P)$ interior lattice points and $b(P)$ lattice points on the boundary we have the following formula*

$$S(P) = i(P) + \frac{b(P)}{2} - 1$$

The most important step to prove that convex paths span the whole EHA is to check that every non-convex path of length 2 is generated by convex paths. Let us show that the statement for an arbitrary path p will follow.

Lemma 3.4.2. *Suppose that $[u_x, u_y] \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q$ for any x, y such that $|\det(x,y)| < d$. Then for any positive path p satisfying $a(p) < d$ we have $u_p \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q$*

Proof. For $a(p) = 0$, Lemma 3.3.1 i) states that p is convex, so proposition holds. If $a(p) > 0$ then p is not convex, so we have $\mu(x_1) \leq \mu(x_2) \leq \ldots \leq \mu(x_s) > \mu(x_{s+1})$ for some s. The statement ii) of Lemma 3.3.1 gives us that $det(x_s, x_{s+1}) = a((x_s, x_{s+1})) \leq a(p) < d$, so by the lemma assumption $u_{x_s}u_{x_{s+1}} = \sum_i \beta_i u_{q_i}$ where q_i is a local convexification of p. Therefore it is enough to prove that $u_{q_i} \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q$ for every i. By Lemma 3.3.1 iii) we have $a(q_i) < a(p)$. Note that the area function $a(\bullet)$ takes values of the form $\frac{n}{2}$ for $n \in \mathbb{Z}_{\geq 0}$ and $a(q) < a(p) < d$ for every convexification q. Applying the same procedure to q_i finitely many times we will get a linear combination of convex paths. The lemma follows. $\square$

Ir remains to show that $[u_x, u_y] \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q$ for any two segments $x, y \in \mathbb{Z}^\times$.

Proposition 3.4.3. *For any elements $x, y \in Z^\times$ with $\mu(x) > \mu(y)$ we have $u_x u_y \in \bigoplus_{p \in I_{x,y}} \mathbb{C}u_p$ where $I_{x,y}$ – the set of convex paths inside the triangle $\Delta_{x,y}$.*

Proof. We will prove this lemma by the induction on $\det(y, x)$. If $\det(y, x) = 1$ then by Pick's formula for $S(\Delta_{x,y}) = \frac{1}{2}$ we have $i(\Delta_{x,y}) = b(\Delta_{x,y}) = 0$, so $\deg(x) = \deg(x + y) = 1$ and $\Delta_{x,y}$ has no interior lattice points. It follows that $u_x u_y = u_y u_x + [u_x, u_y] = u_y u_x + \kappa_{\alpha(x,y)} u_{x+y}$.
Let us assume that the statement of the lemma holds for any x', y' with $\det(x', y') < d$ (if $\det(y', x') < 0$ then the path (x', y') is convex) and set $\det(y, x) = d$. Let us first consider the case when $\Delta_{x,y}$ has no interior lattice points.

Lemma 3.4.4. *If $\Delta_{x,y}$ has no interior lattice points then $u_x u_y \in \bigoplus_{p \in I_{x,y}} \mathbb{C}u_p$.*

Proof. We put $y_0 = \frac{y}{\deg(y)}$, $x_0 = \frac{x}{\deg(x)}$ and $(x+y)_0 = \frac{x+y}{\deg(x+y)}$.

1) Assume $\deg(x) \geq 2$, $\deg(y) \geq 2$

Let us consider the point $z = (\deg(x)-1)x_0 + y_0 \in \Delta_{x,y}$. The only case when z is not an interior point is $\deg(x) = \deg(y) = 2$. We can find such $\gamma \in \mathrm{SL}(2,\mathbb{Z})$ that $x = \gamma((0,2))$ and $y = \gamma((2,0))$. As both points x and y are positive action of γ preserves commutation relations. Direct computation shows

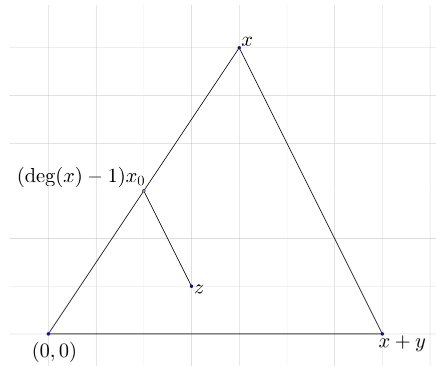

FIGURE 5. The interior lattice point z for case 1).

$$u_{(2,0)} = \frac{\alpha_1}{\alpha_2}[u_{(1,1)}, u_{(1,-1)}] - \frac{\alpha_1^2}{2}u_{(1,0)}^2 \Rightarrow$$

$$[u_{(0,2)}, u_{(2,0)}] = \frac{\alpha_1}{\alpha_2}[u_{(0,2)}, [u_{(1,1)}, u_{(1,-1)}]] - \frac{\alpha_1^2}{2}[u_{(0,2)}, u_{(1,0)}^2]$$

It is enough to check that each summand can be decomposed into a linear combination of monomials corresponding to convex paths. Indeed

$$[u_{(0,2)}, [u_{(1,1)}, u_{(1,-1)}]] =$$
$$[u_{(1,1)}, [u_{(0,2)}, u_{(1,-1)}]] + [u_{(1,-1)}, [u_{(1,1)}, u_{(0,2)}]] =$$
$$-[u_{(1,1)}, u_{(1,1)}] + [u_{(1,-1)}, u_{(1,3)}] = [u_{(1,-1)}, u_{(1,3)}] = \frac{1}{\alpha_1}\theta_{(2,2)}$$

$$[u_{(0,2)}, u_{(1,0)}^2] = [u_{(0,2)}, u_{(1,0)}]u_{(1,0)} + u_{(1,0)}[u_{(0,2)}, u_{(1,0)}] = -u_{(1,2)}u_{(1,0)} - u_{(1,0)}u_{(1,2)} =$$
$$-[u_{(1,2)}, u_{(1,0)}] + 2u_{(1,0)}u_{(1,2} = -\frac{1}{\alpha_1}\theta_{(2,2)} + 2u_{(1,0)}u_{(1,2)} \in \mathbb{C}u_{(2,2)} \oplus \mathbb{C}u_{(1,1)} \oplus \mathbb{C}u_{(1,0)}u_{(1,2)}.$$

Therefore $u_x u_y = u_y u_x + \gamma([u_{(0,2)}, u_{(2,0)}]) \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q$.

2) Suppose $\deg(x) = 1$ or $\deg(y) = 1$. Then

$$u_x u_y = u_y u_x + \frac{\kappa_{\alpha(x,y)}\theta_{x+y}}{\alpha_1} =$$
$$u_y u_x + \kappa_{\alpha(x,y)} \sum_{i_1+\ldots+i_m = \deg(x+y)} \beta_{i_1,\ldots,i_m} u_{i_1(x+y)_0} \ldots u_{i_m(x+y)_0} \in \bigoplus_{q \in \mathsf{Conv}^+} \mathbb{C}u_q.$$

□

Therefore we may assume that there are interior points in $\Delta_{x,y}$. Let $z \in \Delta_{x,y}$ be a point such that the triangle $0xz$ has no interior points and $\deg(z) = \deg(x-z) = 1$. Suppose that $x - z$ is positive. The case when $x - z$ is negative is similar, we will have same expressions but with coefficients depending on κ. We have

$$[u_z, u_{x-z}] = \frac{\theta(x)}{\alpha_1} = \frac{\alpha_{\deg(x)}}{\alpha_1} u_x + f,$$

where f is generated by elements u_{kx_0} for $f < \deg(x)$. We get

$$[u_x, u_y] = \frac{\alpha_1}{\alpha_{\deg(x)}}[[u_z, u_{x-z}], u_y] - [f, u_y] =$$

$$\frac{\alpha_1}{\alpha_{\deg(x)}}[[u_y, u_{x-z}], u_z] - \frac{\alpha_1}{\alpha_{\deg(x)}}[[u_y, u_z], u_{x-z}] - [f, u_y].$$

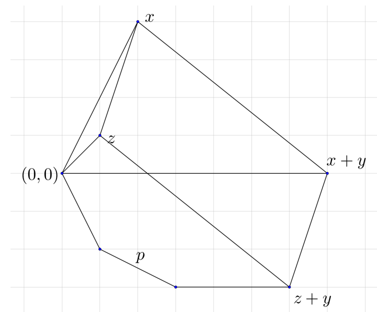

FIGURE 6. The case of interior lattice point $z \in \Delta_{x,y}$.

We see that the triangles $0zy$ and xyz lie inside the triangle $0xy$. Therefore we have $\det(y, z) = 2S(0zy) < 2S(0xy) = \det(y, x)$ and analogously $\det(y, x-z) < \det(y, x)$. By the induction hypothesis we have $[u_y, u_z] \in \bigoplus_{p \in I_{z,y}} \mathbb{C}u_p$ and $[u_y, u_{z-x}] \in \bigoplus_{q \in I_{z-x,y}} \mathbb{C}u_q$.

From the construction $\mu(x-z) < \mu(z) < \mu(y)$, so $(x-z, p)$ is a convex path for all $p \in I_{z,y}$. Therefore the path $(p, x-z)$ is a local convexification of the path $(x, y, x-z)$. By Lemma 3.3.1 iii) we have

$a((p, x-z)) < a((z, y, x-z)) < a(x, y) = \det(y, x) = d$. So $[u_{x-z}, u_p] \in \bigoplus_{t \in I_{x,y}} \mathbb{C}u_t$.

Analogously $[u_q, u_x] \in \bigoplus_{s \in I_{x,y}} \mathbb{C}u_s$. Therefore it is enough to prove that $[f, u_y] \in \bigoplus_{s \in I_{x,y}} \mathbb{C}u_s$. Note that f is a finite sum of paths $u_{i_1x_0}u_{i_2x_0}\dots u_{i_mx_0}$ with some coefficients, where $i_1 + \ldots + i_m = \deg(x)$. We need to show that $[u_{i_1x_0}u_{i_2x_0}\dots u_{i_mx_0}, u_y] \in \bigoplus_{s \in I_{x,y}} \mathbb{C}u_s$. Let us state a stronger fact:

Lemma 3.4.5. *Let $p = (i_1x_0, i_2x_0, \ldots, i_mx_0)$ be a positive path with $i_1 + \ldots + i_m \leq \deg(x)$ and $i_j < \deg(x)$ for all j and q be a convex path between $(i_1 + \ldots + i_m)x_0$ and y. Then $u_pu_q \in \bigoplus_{s \in I_{x,y}} \mathbb{C}u_s$.*

Proof. We prove the proposition by induction on m. If $m = 1$ let us denote $x' = i_1x_0$. Then q is a local convexification of the path $(x', y - x')$, so $a((p, q)) < a((x', y - x')) < a((x, y)) = det(x, y) = d$, so the statement follows from Proposition 3.4.3. Here the first inequality is given by Lemma 3.3.1 iii) and the second one follows from the fact that the triangle $0x'y$ is contained in $0xy$.

Suppose that the proposition holds for all paths p with $k < m$ segments. We have

$$[u_p, u_q] = [u_{i_1x_0}u_{i_2x_0}\dots u_{i_mx_0}, u_q] = [u_{i_1x_0}u_{i_2x_0}\dots u_{i_{m-1}x_0}, u_q]u_{i_mx_0} + u_{i_1x_0}\dots u_{i_{m-1}x_0}[u_{i_mx_0}, u_q]$$

By the induction hypothesis the first summand of the right hand side is a linear combination of the elements $u_tu_{i_mx_0}$ where t is a convex path with all slopes between $\mu(y)$ and $\mu(x)$. Therefore $(t, i_mx_0) \in I_{x,y}$.

The second summand by the proposition for $m = 1$ is a linear combination of the elements $u_{i_1x_0}u_{i_2x_0}\dots u_{i_{m-1}x_0}u_s$ where s is a convex path with all slopes between $\mu(y)$ and $\mu(x)$. By the induction hypothesis it can be rewritten as a linear combination of u_r for $r \in I_{x,y}$. Therefore $[u_p, u_q] \in \bigoplus_{r \in I_{x,y}} \mathbb{C}u_r$, q.e.d. ∎

Applying Lemma 3.4.5 above to $q = (y)$ and all summands in f we get $[f,x] \in \bigoplus_{s\in I_{x,y}} \mathbb{C}u_s$. Therefore the sum $[u_x,u_y] = \frac{\alpha_1}{\alpha_{\deg(x)}}[[u_y,u_{x-z}],u_z] - \frac{\alpha_1}{\alpha_{\deg(x)}}[[u_y,u_z],u_{x-z}] - [f,u_y] \in \bigoplus_{q\in I_{x,y}} \mathbb{C}u_q$, q.e.d. □

Proposition 3.4.6. *The algebra $\mathcal{E}^+$ is isomorphic to $\bigoplus_{p\in \mathit{Conv}^+} \mathbb{C}u_p$ as a vector space.*

Proof. We have already said that elements u_p are linearly independent for convex paths p. Let us consider a non-convex path $q = (x_1, x_2, \dots, x_n)$ and let $\mu(x_1) \le \dots \mu(x_s) > \mu(x_{s+1})$. Then by Proposition 3.4.3 $u_q = \sum_i \beta_i u_{q_i}$ where q_i is a local convexification of q. By Lemma 3.3.1 we have $a(q_i) < a(q)$. We can apply the same procedure to q_i. The area function takes only half-integer non-negative values, so in a finite number of steps we get $u_q \in \mathbb{C}u_p{}_{p\in \mathrm{Conv}^+}$. □

Example 3.4.7. *Let us show the decomposition of $[u_{(1,2)}, u_{(1,-1)}]$ into a linear combination of elements corresponding to convex paths.*

$$u_{(1,-1)} = \frac{1}{\kappa_{0,-1}}[u_{(0,-1)}, u_{(1,0)}] = [u_{(0,-1)}, u_{(1,0)}],$$

$$[u_{(1,2)}, u_{(1,-1)}] = [u_{(1,2)}, [u_{(0,-1)}, u_{(1,0)}]] = [u_{(0,-1)}, [u_{(1,2)}, u_{(1,0)}]] + [u_{(1,0)}, [u_{(0,-1)}, u_{(1,2)}]] =$$

$$[u_{(0,-1)}, \frac{\theta_{(2,2)}}{\alpha_1}] + [u_{(0,1)}, u_{(1,1)}] = u_{(2,1)} - \frac{\alpha_1}{2}[u_{(0,-1)}, u^2_{(1,1)}] - \frac{\alpha_2}{\alpha_1}[u_{(0,-1)}, u_{(2,2)}] =$$

$$\frac{\alpha_1 - \alpha_2}{\alpha_1}u_{2,1} + \frac{\alpha_1}{2}(u_{(1,1)}u_{(1,0)} + u_{(1,0)}u_{(1,1)}) =$$

$$= \frac{\alpha_1 - \alpha_2}{\alpha_1}u_{(2,1)} + \alpha_1 u_{(1,0)}u_{(1,1)} + \frac{\alpha_1}{2}[u_{(1,1)}, u_{(1,0)}] = \frac{\alpha_1 - 2\alpha_2}{2\alpha_1}u_{2,1} + \alpha_1 u_{(1,0)}u_{(1,1)}.$$

3.5. **Triangular decomposition of the EHA.** Let us denote by $\mathcal{E}^>$, $\mathcal{E}^<$ and $\mathcal{E}^0$ subalgebras generated by $u_{1,l}$, $u_{-1,l}$ and $u_{0,\pm k}$ respectively. Note that $\mathcal{E}^> \not\simeq \mathcal{E}^+$ cause we have not elements $u_{0,m}$ in $\mathcal{E}^>$. We have the following important corollary of Proposition 3.4.6.

Proposition 3.5.1. *The EHA $\mathcal{E}$ has a triangular decomposition $\mathcal{E}^> \otimes \mathcal{E}^0 \otimes \mathcal{E}^< \simeq \mathcal{E}$ where the isomorphism is given by the multiplication map.*

Proof. First, we need to check that the multiplication map $m : \mathcal{E}^> \otimes \mathcal{E}^0 \otimes \mathcal{E}^< \to \mathcal{E}$ is surjective. We have the following set of relations

$$(i)[u_{(1,l)}, u_{(0,m)}] = u_{(1,l+m)} \in \mathcal{E}^>,\ [u_{(1,l)}, u_{(0,-m)}] = -\kappa_{0,-m}u_{(1,l+m)} = -u_{(1,l+m)} \in \mathcal{E}^>,$$

$$(ii)[u_{(-1,l)}, u_{(0,m)}] = -\kappa_{0,m}u_{(-1,l+m)} = -u_{(-1,l+m)} \in \mathcal{E}^<,\ [u_{(-1,l)}, u_{(0,-m)}] = u_{(-1,l-m)} \in \mathcal{E}^<,$$

$$(iii)[u_{(1,l)}, u_{(-1,m)}] = \pm\frac{\theta_{(0,l+m)}}{\alpha_1} \in \mathcal{E}^0 \text{ if } l \ne -m,$$

$$(iv)[u_{(1,l)}, u_{(-1,-l)}] = \frac{\kappa_{1,l} - \kappa_{-1,-l}}{\alpha_1} = \frac{c - c^{-1}}{\alpha_1},$$

$$(v)[u_{(0,\pm l)}, u_{(0,\pm m)}] = 0.$$

Therefore in the element u_p for $p = (x_1, \dots, x_n)$ we can move all $x_i \in \mathcal{E}^+$ to the beginning using relations (i), (iii) and (iv) and get a linear combination of elements u_{p_i} for paths $p_i = (q_i, y_1, \dots, y_k)$ where $u_{q_i} \in \mathcal{E}^>$ and all $y_i \in \mathcal{E}^<$ or $y_i \in \mathcal{E}^0$. In the same way using relations (ii) we can move all $x_i \in \mathcal{E}^<$ to the end and get a linear combination of the elements u_{s_i} fo $s_i = (q_i, t_i, r_i)$ where $u_{q_i} \in \mathcal{E}^>$, $u_{t_i} \in \mathcal{E}^0$ and $u_{r_i} \in \mathcal{E}^<$. But all such u_{s_i} belong to the image of m. Therefore m is surjective.

By Proposition 3.4.6 convex paths form a basis in both $\mathcal{E}^>$ and $\mathcal{E}^<$ (the proof is analogous). The basis of $\mathcal{E}^0$ is given by all paths with all segments of form $(0,l)$. We denote the corresponding sets of convex paths $\mathsf{Conv}^>$, Conv^0 and $\mathsf{Conv}^<$. Therefore it is enough to prove that $m(u_{v_i}\otimes u_{t_j}\otimes u_{w_k})$ are linearly independent in $\mathcal{E}$ for all elements $v_i\in\mathsf{Conv}^>$, $t_j\in\mathsf{Conv}^0$ and $w_k\in\mathsf{Conv}^<$. Let us consider two cases.

1) t_j is a path consisting of entries $(0,n)$ for $n\in\mathbb{Z}_{>0}$. Note that for any elements $u_{(r,d)}$ with $r>0$, $u_{(0,n)}$ with $n>0$ and $u_{(r',d')}$ with $r'<0$ we have $\mu((r,d))<\mu((0,n))<\mu((r',d'))$. Therefore $m(u_{v_i}\otimes u_{t_j}\otimes u_{w_k})$ are convex paths that are linearly independent by analogous to Proposition 3.4.6 result for $\mathcal{E}$.

2) Suppose that t_j has an entry $(0,-n)$. Using relations (v) we can set $t_j=((0,a_1),(0,a_2),\dots,(0,a_k))$ with $a_1<a_2<\ldots<a_k$. Let us denote the subalgebra generated by $u_{(0,n)}$ (resp. $u_{(0,-n)}$) as $\mathcal{E}^{0,+}$ (resp, $\mathcal{E}^{0,-}$). We set $u_{t_j^+}=\prod_{a_i>0}u_{(0,a_i)}\in\mathcal{E}^{0,+}$ and $u_{t_j^-}=\prod_{a_i<0}u_{(0,a_i)}\in\mathcal{E}^{0,-}$. Let us show that the multiplication map $m':\mathcal{E}^<\otimes\mathcal{E}^{0,-}\to\mathcal{E}$ is injective.
We call a path $p=(x_1,x_2,\dots,x_r)$ concave if $(x_r,\dots,x_2,x_1)$ is convex. Analogously to Proposition 3.4.6 we can show that elements corresponding to concave paths form a basis in $\mathcal{E}$ and $\mathcal{E}^<$. Then if we choose the concave basis u_p in $\mathcal{E}^<$ and a basis u_q in $\mathcal{E}^{0,-}$ then $m'(u_p\otimes u_q)$ will be elements corresponding to different concave paths and therefore m' is injective.
Let $\sum_q u_q$ be the decomposition of an element $m(u_{v_i}\otimes u_{t_j^-})$ into a linear combination of convex paths. Let p' be a local convexification of a path p that replaces entries x_i,x_{i+1} by $y_1,\dots,y_k$. Proposition 3.4.3 implies that $\mu(x_i)>\mu(y_j)>\mu(x_{i+1})$ for all j. Therefore $m(u_{v_i}\otimes u_{t_j}\otimes u_{w_k})=\sum_q m(u_q\otimes u_{t_j^+}\otimes u_{w_k})$ – linear combination of convex paths. Let Conv_{t_j,w_k} be a set of convex paths p that have form $p=(x_1,x_2,\dots,x_l,t_j,w_k)$ with $\mu(x_i)>\frac{\pi}{2}$, $\forall i$. (We allow l to be 0.) Note that $\mathsf{Conv}=\bigsqcup\limits_{t_j,w_k}\mathsf{Conv}_{t_j,w_k}$. Injectivity of m' implies that all $m'(u_{v_i}\otimes u_{t_j^-})$ ae linearly independent. Therefore $m(m'(u_{v_i}\otimes u_{t_j^-})\otimes u_{t_j^+}\otimes u_{w_k})=m(u_{v_i}\otimes u_{t_j}\otimes u_{w_k})\in\mathsf{Conv}_{t_j,w_k}$ are linearly independent. It follows that m is injective. $\square$

4. Ding-Iohara algebra

4.1. Generators and relations. We give explicit generators and relations for the Ding-Iohara algebra $\mathcal{U}$. Let us fix complex numbers q_1, q_2 and $q=q_1q_2$. Recall that $\alpha_k=(1-q_1^k)(1-q_2^k)(1-q^{-k})$. We set

$$\chi(z,w)=(z-q_1w)(z-q_2w)(z-q^{-1}w),$$
$$\delta(\frac{z}{w})=\sum_{k\in\mathbb{Z}}(\frac{z}{w})^k.$$

We will define an algebra $\tilde{\mathcal{U}}$ by generators $e_k,f_k,h_n^\pm$ where $k\in\mathbb{Z}$ and $n\in\mathbb{Z}_{>0}$ and relations to be specified below.
Let us define generating series

$$e(z):=\sum_{k\in\mathbb{Z}}e_kz^{-k},$$
$$f(z):=\sum_{k\in\mathbb{Z}}f_kz^{-k},$$
$$\psi^\pm(z)=1+\sum_{n\in\mathbb{Z}_{>0}}h_n^\pm z^{\pm n}.$$

Let ϵ, ϵ_1 and ϵ_2 be elements of $\{1,-1\}$. The defining relations of $\tilde{\mathcal{U}}$ are as follows.

$$\begin{aligned}
&\text{(i) } \psi^{\epsilon_1}(z)\psi^{\epsilon_2}(w) = \psi^{\epsilon_2}(w)\psi^{\epsilon_1}(z),\\
&\text{(ii) } \chi(z,w)\psi^{\epsilon_1}(z)e(w) = -\chi(w,z)e(w)\psi^{\epsilon_1}(z),\\
&\text{(iii) } \chi(z,w)e(z)e(w) = -\chi(w,z)e(w)e(z),\\
&\text{(iv) } \chi(w,z)f(z)f(w) = -\chi(z,w)f(w)f(z),\\
&\text{(v) } [f(z),e(w)] = \frac{1}{\alpha_1}\left(\delta(\frac{z}{w})(c\psi^-(z) - c^{-1}\psi^+(z)\right).
\end{aligned}$$

We are interested in its quotient $\tilde{\mathcal{E}}$ by cubic relations. We put $\operatorname*{Res}_{zyw}(a)$ be the coefficient of $(zyw)^{-1}$ in a. Cubic relations are as follows.

$$\operatorname*{Res}_{zyw}\left[(zyw)^m(z+w)(y^2-zw)e(z)e(y)e(w)\right] = 0 \text{ for all } m\in\mathbb{Z},$$
$$\operatorname*{Res}_{zyw}\left[(zyw)^m(z+w)(y^2-zw)f(z)f(y)f(w)\right] = 0 \text{ for all } m\in\mathbb{Z}.$$

The algebra $\tilde{\mathcal{E}}$ is called the quantum toroidal $\mathfrak{gl}_1$.
It will be useful to rewrite last two relations. We denote the coefficient of $z_1^{l_1}z_2^{l_2}z_3^{l_3}$ in a as $\underset{z_1^{l_1}z_2^{l_2}z_3^{l_3}}{C}(a)$.

$$\operatorname*{Res}_{zyw}\left[(zyw)^{-m}(z+w)(y^2-zw)e(z)e(y)e(w)\right] = \underset{(zyw)^{m-1}}{C}\left[(z+w)(y^2-zw)e(z)e(y)e(w)\right] =$$
$$\underset{z^my^{m+1}w^{m-1}}{C}[e(z)e(y)e(w)] - \underset{z^{m+1}y^{m-1}w^m}{C}[e(z)e(y)e(w)] + \underset{z^{m-1}y^{m+1}w^m}{C}[e(z)e(y)e(w)] -$$
$$\underset{z^my^{m-1}w^{m+1}}{C}[e(z)e(y)e(w)] = e_me_{m+1}e_{m-1} - e_{m+1}e_{m-1}e_m + e_{m-1}e_{m+1}e_m - e_me_{m-1}e_{m+1} =$$
$$e_m[e_{m+1},e_{m-1}] - [e_{m+1},e_{m-1}]e_m = [e_m,[e_{m+1},e_{m-1}]]$$

Analogously we have

$$\operatorname*{Res}_{zyw}\left[(zyw)^{-m}(z+w)(y^2-zw)f(z)f(y)f(w)\right] = [f_m,[f_{m+1},f_{m-1}]].$$

So last two relations state that

$$[e_m,[e_{m+1},e_{m-1}]] = 0,$$
$$[f_m,[f_{m+1},f_{m-1}]] = 0.$$

5. The isomorphism between $\mathcal{E}$ and $\tilde{\mathcal{E}}$.

5.1. **The map $\phi : \tilde{\mathcal{E}} \to \mathcal{E}$.** In this subsection we construct a surjective map $\phi : \tilde{\mathcal{E}} \to \mathcal{E}$. We know that the algebra $\tilde{\mathcal{E}}$ is generated by elements e_k, f_k and $h_n^{\pm}$. Let us put $\phi(e_k) = u_{(1,k)}$, $\phi(f_k) = u_{(-1,k)}$ and $\phi(h_n^{\pm}) = \theta_{(0,\pm n)}$. We need to check that ϕ respects sets of relations for $\tilde{\mathcal{E}}$ and $\mathcal{E}$, i.e. elements u_x for $x \in \{(1,k),(-1,k),(0,n)\}$ satisfy relations of $\tilde{\mathcal{E}}$.

Let us consider formal series

$$\begin{aligned}\mathbb{T}_1(z) &= \sum_{l\in\mathbb{Z}} u_{1,l}z^l,\\ \mathbb{T}_{-1}(z) &= \sum_{l\in\mathbb{Z}} u_{-1,l}z^l,\\ \mathbb{T}_0^1(z) &= 1+\sum_{n\in\mathbb{Z}_{>0}} \theta_{0,n}z^n,\\ \mathbb{T}_0^{-1}(z) &= 1+\sum_{n\in\mathbb{Z}_{>0}} \theta_{0,-n}z^{-n},\\ \delta(z) &= \sum_{l\in\mathbb{Z}} z^l.\end{aligned}$$

To show that a map of algebras ϕ is well-defined it is enough to check the following set of relations

(i) $\mathbb{T}_0^{\epsilon_1}(z)\mathbb{T}_0^{\epsilon_2}(w) = \mathbb{T}_0^{\epsilon_2}(w)\mathbb{T}_0^{\epsilon_1}(z)$
(ii) $\chi(z,w)\mathbb{T}_0^{\epsilon_2}(z)\mathbb{T}_{\epsilon_1}(w) = -\chi(w,z)\mathbb{T}_{\epsilon_1}(w)\mathbb{T}_0^{\epsilon_2}(z)$,
(iii) $\chi(z,w)\mathbb{T}_{\epsilon}(z)\mathbb{T}_{\epsilon}(w) = -\chi(w,z)\mathbb{T}_{\epsilon}(w)\mathbb{T}_{\epsilon}(z)$,
(iv) $[\mathbb{T}_{-1}(z),\mathbb{T}_1(w)] = \dfrac{1}{\alpha_1}\delta(\dfrac{z}{w})(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z))$,
(v) $[u_{(1,m)},[u_{(1,m+1)},u_{(1,m-1)}]] = 0$,
(vi) $[u_{(-1,m)},[u_{(-1,m+1)},u_{(-1,m-1)}]] = 0$.

Proposition 5.1.1. *The map $\phi : \tilde{\mathcal{E}} \to \mathcal{E}$ is a well-defined surjective map of algebras.*

Proof. From the relations of the EHA $\mathcal{E}$ we have $u_{(0,l)}u_{(0,m)} = u_{(0,m)}u_{(0,l)}$ and therefore $\mathcal{E}^0$ is a commutative subalgebra. By the definition $\mathbb{T}_0^{\epsilon} \in \mathcal{E}^0$. The relation (i) follows.
We will prove (ii) for $\epsilon_1 = \epsilon_2 = \{+\}$. All other cases are analogous. We need to rewrite this condition. We put $\zeta(x) = \frac{(1-q_1x)(1-q_2x)}{(1-x)(1-qx)}$.

Lemma 5.1.2. $\zeta(x) = \exp\left(\sum_n \frac{x^n(1-q_1^n)(1-q_2^n)}{n}\right)$.

Proof. Let us consider $\log(\zeta(x))$. Note that $\log(1-z) = -\sum_n \frac{z^n}{n}$.

$$\log(\zeta(x)) = \log(1-q_1x)+\log(1-q_2x)-\log(1-x)-\log(1-q_2x) =$$
$$-\sum_n q_1^n\frac{x^n}{n} - \sum_n q_2^n\frac{x^n}{n} + \sum_n \frac{x^n}{n} + \sum_n q^n\frac{x^n}{n} = \sum_n \frac{(1-q_1^n)(1-q_2^n)}{n}x^n.$$

Taking exponent of both sides we get the proposition. ☐

Note that

$$\frac{\chi(z,w)}{\chi(w,z)} = \frac{(z-q_1w)(z-q_2w)(z-q^{-1}w)}{(w-q_1z)(w-q_2z)(w-q^{-1}z)} = \frac{z^2(1-q_1\frac{w}{z})(1-q_2\frac{w}{z})(z-q^{-1}w)}{w^2(1-q_1\frac{z}{w})(1-q_2\frac{z}{w})(w-q^{-1}z)} =$$
$$\frac{z^2}{w^2}\frac{(1-q_1\frac{w}{z})(1-q_2\frac{w}{z})}{q^{-1}z(q\frac{w}{z}-1)}\frac{q^{-1}w(q\frac{z}{w}-1)}{(1-q_1\frac{z}{w})(1-q_2\frac{z}{w})} = \frac{w}{z}\frac{(1-q_1\frac{w}{z})(1-q_2\frac{w}{z})}{(1-q\frac{w}{z})(z-w)}\frac{(1-q\frac{z}{w})(z-w)}{(1-q_1\frac{z}{w})(1-q_2\frac{z}{w})} =$$
$$\frac{(1-q_1\frac{w}{z})(1-q_2\frac{w}{z})}{(1-q\frac{w}{z})(1-\frac{w}{z})}\frac{(1-q\frac{z}{w})(\frac{z}{w}-1)}{(1-q_1\frac{z}{w})(1-q_2\frac{z}{w})} = -\frac{\zeta(w\backslash z)}{\zeta(z\backslash w)}.$$

The relation (ii) can be rewritten as

$$\mathbb{T}_0^+(z)\mathbb{T}_1(w)\zeta(\frac{z}{w}) = \mathbb{T}_1(w)\mathbb{T}_0^+(z)\zeta(\frac{w}{z}).$$

This relation should be understood as an equality of coefficients for these formal series expanded in $|w| > |z|$. Let us look at

$$\zeta(\frac{w}{z}) = \frac{(1-q_1\frac{w}{z})(1-q_2\frac{w}{z})}{(1-\frac{w}{z})(1-q\frac{w}{z})} = \frac{(\frac{z}{w}-q_1)(\frac{z}{w}-q_2)}{(\frac{z}{w}-1)(\frac{z}{w}-q)} = \frac{(q_1^{-1}\frac{z}{w}-1)(q_2^{-1}\frac{z}{w}-1)}{(\frac{z}{w}-1)(q^{-1}\frac{z}{w}-1)} = \frac{(1-q_1^{-1}\frac{z}{w})(1-q_2^{-1}\frac{z}{w})}{(1-\frac{z}{w})(1-q^{-1}\frac{z}{w})} =$$
$$\exp\left(\sum_n \frac{(1-q_1^{-n})(1-q_2^{-n})}{n}\left(\frac{z}{w}\right)^n\right)$$

Then we have the relation (ii) in the following form

$$\mathbb{T}_0^+(z)\mathbb{T}_1(w)\exp\left(\sum_n \frac{z^n(1-q_1^n)(1-q_2^n)}{nw^n}\right) = \mathbb{T}_1(w)\mathbb{T}_0^+(z)\exp\left(\sum_n \frac{z^n(1-q_1^{-n})(1-q_2^{-n})}{nw^n}\right),$$
$$\mathbb{T}_1(w)\mathbb{T}_0^+(z) = \mathbb{T}_0^+(z)\mathbb{T}_1(w)\exp\left(\sum_n\left(\frac{(1-q_1^n)(1-q_2^n)}{n} - \frac{(1-q_1^{-n})(1-q_2^{-n})}{n}\right)\left(\frac{z}{w}\right)^n\right).$$

Note that $\mathbb{T}_0^+(z) = \exp\left(\sum_n \alpha_n u_{(0,n)} z^n\right)$.

$$\mathbb{T}_1(w)\exp\left(\sum_n \alpha_n u_{(0,n)}z^n\right) =$$
$$\exp\left(\sum_n \alpha_n u_{(0,n)}z^n\right)\mathbb{T}_1(w)\exp\left(\sum_n\left(\frac{(1-q_1^n)(1-q_2^n)}{n} - \frac{(1-q_1^{-n})(1-q_2^{-n})}{n}\right)\left(\frac{z}{w}\right)^n\right).$$

Lemma 5.1.3. *Suppose that* $[e,p] = ec \Rightarrow e\exp(p) = \exp(p)e\exp(c)$.

Proof. We need to show that $e\frac{p^k}{k!} = \sum_{a+b=k}\frac{p^a}{a!}e\frac{c^b}{b!}$. We will prove it by the induction by k.
For $k=1$ we have $e = pe + ec$.
Suppose that the statement holds for $k-1$. Then

$$e\frac{p^k}{k!} = \frac{1}{k}ep^{k-1}p = \frac{1}{k}\sum_{a+b=k-1}\frac{p^a}{a!}e\frac{c^b}{b!}p = \frac{1}{k}\sum_{a+b=k-1}\frac{p^a}{a!}ep\frac{c^b}{b!} =$$
$$\frac{1}{k}\sum_{a+b=k-1}\frac{p^{a+1}}{a!}e\frac{c^b}{b!} + \frac{1}{k}\sum_{a+b=k-1}\frac{p^a}{a!}e\frac{c^{b+1}}{b!} = \frac{1}{k}\sum_{a+b=k}\left(\frac{p^a}{(a-1)!}e\frac{c^b}{b!} + \frac{p^a}{a!}e\frac{c^b}{(b-1)!}\right) =$$
$$\frac{1}{k}\sum_{a+b=k}(a+b)\frac{p^a}{a!}e\frac{c^b}{b!} = \sum_{a+b=k}\frac{p^a}{a!}e\frac{c^b}{b!}.$$

∎

Let us compute $[\mathbb{T}_1(w), \sum_n \alpha_n u_{(0,n)}z^n]$.

$$[\mathbb{T}_1(w), \sum_n \alpha_n u_{(0,n)}z^n] = \sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}_{>0}}\alpha_n[u_{(1,k-n)}, u_{(0,n)}]w^{k-n}z^n =$$
$$\sum_{k\in\mathbb{Z}}\sum_{n\in\mathbb{Z}_{>0}}\alpha_n u_{(1,k)}w^k\left(\frac{z}{w}\right)^n = \mathbb{T}_1(w)\sum_{n\in\mathbb{Z}_{>0}}\alpha_n\left(\frac{z}{w}\right)^n = \mathbb{T}_1(w)\sum_{n\in\mathbb{Z}_{>0}}\frac{(1-q_1^n)(1-q_2^n)(1-q^{-n})}{n}\left(\frac{z}{w}\right)^n =$$
$$\mathbb{T}_1(w)\sum_{n\in\mathbb{Z}_{>0}}\frac{(1-q_1^n)(1-q_2^n)-(1-q_1^{-n})(1-q_2^{-n})}{n}\left(\frac{z}{w}\right)^n.$$

Applying Lemma 5.1.3 we have the proof of (ii).
The only known proof of (iii) uses the isomorphism $\mathcal{E}$ with the EHA of elliptic curve and GL_n modular forms. The first fact will be covered in future talks. The second one will lead us too far away from our topic. One can read details in Kapranov's paper [K].
To prove (iv) let us consider the coefficient of $z^k w^l$ in expressions of both sides.

$$\underset{z^k w^l}{C}\left([\mathbb{T}_{-1}(z), \mathbb{T}_1(w)]\right) = [u_{(-1,k)}, u_{(1,l)}],$$
$$\underset{z^k w^l}{C}\left(\frac{1}{\alpha_1}\delta(\frac{z}{w})(c^{-1}\mathbb{T}_0^-(z) - c\mathbb{T}_0^+(z))\right) = \frac{1}{\alpha_1}\underset{z^{k+l}}{C}\left(c^{-1}\mathbb{T}_0^-(z) - c\mathbb{T}_0^+(z)\right).$$

We need to consider three cases.

1) $k+l<0$,

$$[u_{(-1,k)}, u_{(1,l)}] = \kappa_{\alpha((-1,k),(1,l)}\frac{\theta_{(0,l+k)}}{\alpha_1},$$
$$\alpha((-1,k),(1,l) = (\frac{(1,-k)+(1,l)-(0,-k-l)}{2}) = (1,l) \Rightarrow [u_{(-1,k)}, u_{(1,l)}] = c\frac{\theta_{(0,l+k)}}{\alpha_1},$$
$$\frac{1}{\alpha_1}\underset{z^{k+l}}{C}\left(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z)\right) = \frac{1}{\alpha_1}\underset{z^{k+l}}{C}(c\mathbb{T}_0^-(z)) = c\frac{\theta_{(0,l+k)}}{\alpha_1}.$$

2) $k+l=0$,

$$[u_{(-1,k)}, u_{(1,l)}] = [u_{(-1,-l)}, u_{(1,l)}] = \frac{c-c^{-1}}{\alpha_1},$$
$$\frac{1}{\alpha_1}\underset{z^{k+l}}{C}(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z)) = \frac{c-c^{-1}}{\alpha_1}.$$

3) $k+l>0$,

$$[u_{(-1,k)}, u_{(1,l)}] = -\kappa_{\alpha((-1,k),(1,l)}\frac{\theta_{(0,l+k)}}{\alpha_1},$$
$$\alpha((-1,k),(1,l) = -(\frac{(1,-k)+(1,l)-(0,k+l)}{2}) = (-1,k) \Rightarrow [u_{(-1,k)}, u_{(1,l)}] = -c^{-1}\frac{\theta_{(0,l+k)}}{\alpha_1},$$
$$\frac{1}{\alpha_1}\underset{z^{k+l}}{C}\left(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z)\right) = -\frac{1}{\alpha_1}\underset{z^{k+l}}{C}(c^{-1}\mathbb{T}_0^+(z)) = -c^{-1}\frac{\theta_{(0,l+k)}}{\alpha_1}.$$

We see that in all cases

$$\underset{z^k w^l}{C}\left([\mathbb{T}_{-1}(z), \mathbb{T}_1(w)]\right) = \underset{z^k w^l}{C}\left(\frac{1}{\alpha_1}\delta(\frac{z}{w})(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z))\right) \text{ for all } k,l \Rightarrow$$
$$[\mathbb{T}_{-1}(z), \mathbb{T}_1(w)] = \frac{1}{\alpha_1}\delta(\frac{z}{w})(c\mathbb{T}_0^-(z) - c^{-1}\mathbb{T}_0^+(z)).$$

To prove (v) we note that the triangle $(0,(1,l+1),(2,2l))$ has no interior lattice points. Then $[[u_{(1,l+1)}, u_{(1,l-1)}], u_{(1,l)}] = [u_{(2,2l)}, u_{(1,l)}] = 0$.
The proof of (vi) is analogous.
Lemma 3.2.1 implies surjectivity of ϕ. □

We are ready to state the main theorem of this talk.

Theorem 5.1.4. *The map ϕ gives an isomorphism between $\tilde{\mathcal{E}}$ and the EHA $\mathcal{E}$.*

5.2. Properties of $\tilde{\mathcal{E}}$.

The main goal of this subsection is to get the similar result to Lemma 3.5.1 for the algebra $\tilde{\mathcal{E}}$. Let us introduce subalgebras $\tilde{\mathcal{E}}^{>}$ generated by elements e_l, $\tilde{\mathcal{E}}^0$ generated by elements $h_{\pm n}$ and $\tilde{\mathcal{E}}^{<}$ generated by elements f_l. We have the following triangular decomposition.

Proposition 5.2.1. *The multiplication map $\tilde{m} : \tilde{\mathcal{E}}^{>} \otimes \tilde{\mathcal{E}}^0 \otimes \tilde{\mathcal{E}}^{<} \to \tilde{\mathcal{E}}$ is surjective.*

Proof. It is enough to establish that

$$1)[e_k, f_l] \in \tilde{\mathcal{E}}^0,$$
$$2)[e_k, h_{\pm n}] \in \tilde{\mathcal{E}}^{>} + m'(\tilde{\mathcal{E}}^{>} \otimes \tilde{\mathcal{E}}^0), \text{ where } m' : \tilde{\mathcal{E}}^{>} \otimes \tilde{\mathcal{E}}^0 \to \tilde{\mathcal{E}} \text{ is the multipliction map.}$$
$$3)[f_k, h_{\pm n}] \in \tilde{\mathcal{E}}^{<} + m''(\tilde{\mathcal{E}}^0 \otimes \tilde{\mathcal{E}}^{<}), \text{ where } m'' : \tilde{\mathcal{E}}^0 \otimes \tilde{\mathcal{E}}^{<} \to \tilde{\mathcal{E}} \text{ is the multipliction map..}$$

Then the lemma will follow from the same reason as in Lemma 3.5.1. We will prove these facts using the relations of $\tilde{\mathcal{E}}$.
The relation (iv) implies that $[e(z), f(w)] \in \tilde{\mathcal{E}}^0(z, w)$ Note that coefficient of $z^k w^l$ is exactly $[e_k, f_l]$ that proves 1).
Let us prove 2) for h_n. The h_{-n} case is analogous.

Lemma 5.2.2. *We have $[e_l, h_n] = \beta e_{l+n} + \sum_i e_i h_i$ for some e_i, h_i. Analogously $[e_l, h_{-n}] = \beta' e_{l-n} + \sum_i e_i h_i$.*

Proof. First, let us compute the coefficients of $z^k w^l$ in the relation (ii) of $\tilde{\mathcal{E}}$.

$$h_{k-3}e_l - (q_1 + q_2 + q^{-1})h_{k-2}e_{l-1} + (q_1^{-1} + q_2^{-1} + q)h_{k-1}e_{l-2} - h_k e_{l-3} =$$
$$e_l e_{k-3} - (q_1^{-1} + q_2^{-1} + q)e_{l-1}e_{k-2} + (q_1 + q_2 + q^{-1})e_{l-2}e_{k-1} - e_{l-3}e_k.$$

We can rewrite it as

$$[h_{k-3}, e_l] + (q_1^{-1} + q_2^{-2} + q - q_1 - q_2 - q^{-1})e_{l-1}h_{k-2} + (q_1^{-1} + q_2^{-1} + q)[e_{l-1}, h_{k-2}] -$$
$$(q_1^{-1} + q_2^{-2} + q - q_1 - q_2 - q^{-1})e_{l-2}h_{k-1} - (q_1^{-1} + q_2^{-1} + q)[e_{l-2}, h_{k-1}] - [h_k, e_{l-3}] = 0,$$
$$q_1^{-1} + q_2^{-2} + q - q_1 - q_2 - q^{-1} = (1 - q_1)(1 - q_2)(1 - q^{-1}) = \alpha_1, \text{ so}$$
$$[h_{k-3}, e_l] + \alpha_1 e_{l-1}h_{k-2} + (q_1^{-1} + q_2^{-1} + q)[e_{l-1}, h_{k-2}] -$$
$$\alpha_1 e_{l-2}h_{k-1} - (q_1^{-1} + q_2^{-1} + q)[e_{l-2}, h_{k-1}] - [h_k, e_{l-3}] = 0,$$

Note that in the formula above we suppose $h_{-1} = h_{-2} = \ldots = 0$ and $h_0 = 1$ because we compute commutator with $\psi^+(z)$.
Now we are ready to prove this proposition. We use induction by n. For $n = 1$ we have from the relation above

$$[e_{l-3}, h_1] = \alpha_1 e_{l-2}h_0 = \alpha_1 e_{l-2}.$$

Suppose thaw we prove the proposition for all $n < l$. From the computation above $[e_k, h_l]$ is a linear combination of $[e_{k+1}, h_{l-1}]$, $[e_{k+2}, h_{l-2}]$, $[e_{k+3}, h_{l-3}]$, $[e_{k+1}h_{l-1}]$ and $[e_{k+2}h_{l-2}]$. The proposition follows. ☐

3) is analogous to 2). ☐

Now we have the following commutative diagram

$$\begin{tikzcd}
\tilde{\mathcal{E}}^{>} \otimes \tilde{\mathcal{E}}^0 \otimes \tilde{\mathcal{E}}^{<} \arrow[r, "\hat{\phi}"] \arrow[d, "\tilde{m}"] & \mathcal{E}^{>} \otimes \mathcal{E}^0 \otimes \mathcal{E}^{<} \arrow[d, "m"] \\
\tilde{\mathcal{E}} \arrow[r, "\phi"] & \mathcal{E}
\end{tikzcd}$$

From Lemma 3.5.1 the map m is isomorphism. Proposition 5.2.1 states that $\tilde{m}$ is surjective. Proposition 5.1.1 shows that ϕ is surjective. Therefore Theorem 5.1.4 is equivalent to the fact that $\hat{\phi}$ is an isomorphism.

Remark 5.2.3. *Theorem 5.1.4 implies that $\tilde{m}$ is an isomorphism.*

We will denote $\hat{\phi}$ also by ϕ. It is enough to check that every part $\phi^{>}$, ϕ^{0} and $\phi^{<}$ is an isomorphism. For commutative subalgebras $\mathcal{E}^0$ and $\tilde{\mathcal{E}}^0$ the isomorphism is obvious. Cases of $^{>}$ and $^{<}$ are analogous, so we will focus on the first of them. We need to introduce more combinatorical notions.

5.3. **Minimal paths.** We are going to introduce the notion of a minimal path.
For a point $z \in Z^{\times}$ we have a line L going through $(0,0)$ and z. We want to choose a closest parallel to L line $L' \neq L$ that has lattice points. We have two identical options. We set L' be a line lying above L. By a *minimal path* we denote the path $(x, z-x)$ with $x \in L'$. We need two important lemmas about minimal paths.

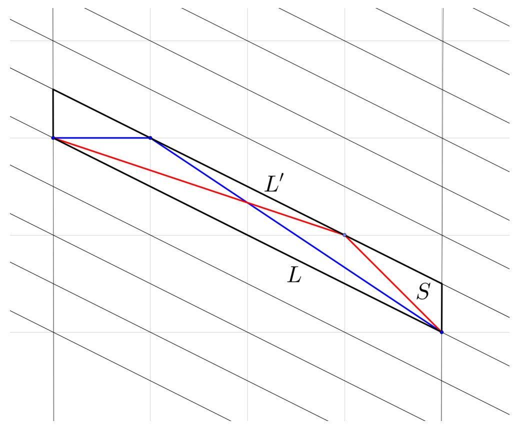

FIGURE 7. Two different minimal paths.

Lemma 5.3.1. *Let $z = (r,d)$ be a positive segment with $r \geq 2$. Then there exists a minimal path $(x, z-x)$ such that* $\mathrm{rank}(x) > 0$ *and* $\mathrm{rank}(z-x) > 0$. *In fact there are* $\gcd(r,d)$ *such paths.*

Proof. Let S be a strip bounded by the vertical lines through origin and z and lines L and L'. Then to prove the lemma it is enough to find a lattice point on L' inside S.
If $\deg(z) > 1$ the statement is obvious. So it is enough to consider $\deg(z) = 1$. The only case when we don't have a point on the line L' inside S is when both intersection points of vertical lines through $(0,0)$ and z with L' are lattice points. We know that $r \geq 2$ and $\deg(z) = 1$, so we have a point $(1, \frac{d}{r}) \in L$ such that $\frac{d}{r} \notin \mathbb{Z}$. Then consider a line L'' parallel to L through $(1, \lceil \frac{d}{r} \rceil)$. It is closer to L then L', so $(x, z-x)$ was not a minimal path and we get a contradiction. □

Lemma 5.3.2. *A positive path $(x, z-x)$ is minimal if and only if* $\deg(x) = \deg(z-x) = 1$ *and the triangle $\Delta_{x,z-x}$ has no interior lattice points.*

Proof. All points inside the triangle $\Delta_{x,z-x}$ or on the sides x and $z-x$ are closer to the line L then L' containing x. Therefore if the path $p = (x, z-x)$ is minimal then there are no such lattice points.
Suppose that we have a not-minimal path $p = (x, z-x)$ satisfying conditions of lemma. Let us choose a minimal path $p = (y, z-y)$. Note that $S(\Delta_{x,z-x}) > S(\Delta_{y,z-y})$ because they have common side z but a point y is closer to L then x. On the other hand by Pick's formula $S(\Delta_{x,z-x}) = \deg(z) = S(\Delta_{y,z-y})$. We get a contradiction. □

Note that for a positive minimal path $(x, z-x)$ we have $[u_x, u_{z-x}] = \frac{\theta_z}{\alpha_1}$.

5.4. **The proof of the main theorem.** In this subsection we will prove Theorem 5.1.4. We have a map $\phi : \tilde{\mathcal{E}}^{>} \to \mathcal{E}^{>}$. Note that for every r we have a vector space $\mathcal{E}_r^{>}$ generated by the paths u_p with $\mathrm{rank}(p) = r$. We are going to construct for each $p \in \mathcal{E}_r^{>}$ a preimage $e_p = \phi^{-1}(u_p)$ satisfying commutation relations of $\mathcal{E}$ and show that such e_p (for all r) generate $\tilde{\mathcal{E}}$ as vector space. We prove it by the induction on r.

For $r = 1$ we have $e_{1,m} = e_m = \phi^{-1}(u_{(1,m)})$ by the definition of ϕ.
Now suppose that we define such preimages for all paths of rank k for all $k < r$. Let us put $e_{k,m} = \phi_{r-1}^{-1}(u_{(k,m)})$. Our goal is to define a preimage of $u_{(r,m)}$. We set $z = (r, m)$ and choose a minimal path $(x, z-x)$. Then we have $[u_x, u_{z-x}] = u_z$. We want to set $e_z = [e_x, e_{z-x}]$. We need the following lemma.

Lemma 5.4.1. *For any two minimal paths $p = (x, z-x)$ and $p' = (y, z-y)$ it holds $[e_x, e_{z-x}] = [e_y, e_{z-y}]$.*

Proof. Set $l = \deg(z)$ and $z_0 = \frac{z}{l} = (r, d)$. As in the definition of minimal path let L be the line through the origin and z and L' – the closest parallel to L line above it with lattice points on it. We can choose the point x to be closest to the vertical line through $(0,0)$ lattice point on L'. If $r > 1$ Then all minimal paths are $(x, lz_0 - x)$, $(x + z_0, (l-1)z_0 - x)$, …, $(x + (l-1)z_0, z_0 - x)$. Note that if $l = 1$ there is a unique minimal path, so the lemma holds. So we may suppose $l > 1$. We have already defined element e_{z-z_0}, so $[e_{x+(i-1)z_0}, e_{(l-i)z_0-x}] = e_{z-z_0} = [e_{x+iz_0}, e_{(l-i-1)z_0-x}]$ holds. We apply the operator $ad(e_{z_0})$ to both sides of the equation.

$$\begin{aligned}
[e_{z_0}, [e_{x+(i-1)z_0}, e_{(l-i)z_0-x}]] &= [[e_{z_0}, e_{x+(i-1)z_0}], e_{(l-i)z_0-x}] - [[e_{z_0}, e_{(l-i)z_0-x}], e_{x+(i-1)z_0}] = \\
&- [e_{x+iz_0}, e_{(l-i)z_0-x}] + [e_{x+(i-1)z_0}, e_{(l-i+1)z_0-x}], \\
[e_{z_0}, [e_{x+iz_0}, e_{(l-i-1)z_0-x}]] &= [[e_{z_0}, e_{x+iz_0}], e_{(l-i-1)z_0-x}] - [[e_{z_0}, e_{(l-i-1)z_0-x}], e_{x+iz_0}] = \\
&- [e_{x+(i+1)z_0}, e_{(l-i-1)z_0-x}] + [e_{x+iz_0}, e_{(l-i)z_0-x}], \\
[e_{x+(i-1)z_0}, e_{(l-i+1)z_0-x}] - [e_{x+iz_0}, e_{(l-i)z_0-x}] &= [e_{x+iz_0}, e_{(l-i)z_0-x}] - [e_{x+(i+1)z_0}, e_{(l-i-1)z_0-x}].
\end{aligned}$$

We need just one additional relation to show that $[e_x, e_{lz_0-x}] = [e_{x+iz_0}, e_{(l-i)z_0-x}]$ for any i. Let us consider three cases.
1) $l = 2$. We have $[e_x, e_{z_0-x}] = e_{z_0}$. Applying $ad(e_{z_0})$ we get

$$0 = [e_{z_0}, [e_x, e_{z_0-x}]] = [[e_{z_0}, e_x], e_{z_0-x}] + [[e_{z_0}, e_{z_0} - x], e_x] = -[e_{x+z_0}, e_{z_0-x}] + [e_x, e_{2z_0} - x].$$

2) $l \geq 3$. By the induction hypothesis $[e_x, e_{(l-2)z_0-x}] = [e_{x+iz_0}, e_{(l-i-2)z_0-x}] = e_{(l-2)z_0}$. We apply the operator $ad(e_2z_0)$. Note that by Lemma 5.3.2 the triangle $0, x+iz_0, (l-i)z_0 - x$ has no interior lattice points and no lattice points on sides $x + iz_0$ and $(l-i)z_0 - x$. Therefore a triangle $\Delta_{x+iz_0, 2z_0}$ has no interior lattice points (it lies inside a parallelogram on sides z and $x + iz_0$) and no lattice points on the side $x + iz_0$.

$$\begin{aligned}
[e_{2z_0}, [e_x, e_{(l-2)z_0-x}]] =& [[e_{2z_0}, e_x], e_{(l-2)z_0-x}] - [[e_{2z_0}, e_{(l-2)z_0-x}], e_x] = \\
&- [e_{x+2z_0}, e_{(l-2)z_0-x}] + [e_x, e_{lz_0-x}], \\
[e_{2z_0}, [e_{x+iz_0}, e_{(l-i-2)z_0-x}]] =& [[e_{2z_0}, e_{x+iz_0}], e_{(l-i-2)z_0-x}] - [[e_{2z_0}, e_{(l-i-2)z_0-x}], e_{x+iz_0}] = \\
&- [e_{x+(i+2)z_0}, e_{(l-i-2)z_0-x}] + [e_{x+iz_0}, e_{(l-i)z_0-x}], \\
[e_x, e_{lz_0-x}] - [e_{x+2z_0}, e_{(l-2)z_0-x}] =& [e_{x+iz_0}, e_{(l-i)z_0-x}] - [e_{x+(i+2)z_0}, e_{(l-i-2)z_0-x}].
\end{aligned}$$

From this relation and all obtained by applying $ad_{e_{z_0}}$ we have $[e_x, e_{lz_0-x}] = [e_{x+iz_0}, e_{(l-i)z_0-x}]$ for all i, q.e.d. ◻

For any sequence $s = (x_1, x_2, \ldots, x_n)$ we set $e_s = e_{x_1} e_{x_2} \ldots e_{x_n}$. We need to check that by this definition we get the same element for two equivalent (i.e. representing the same path) sequences s and s'.

Lemma 5.4.2. *If s and s' are two equivalent sequences then $e_s = e_{s'}$.*

Proof. It is enough to consider the case of $l(s) = 2$. Indeed suppose that we have proved for the case of two segments and let $l(s) > 2$. For every two segments x_i, x_{i+1} in $s = (x_1, \ldots, x_n)$ we have $e_s = e_{s'}$ with $s' = (x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n)$. Let us have $s = (kw, lw)$ and $s' = (lw, kw)$. Let us take two minimal paths $p = (x, (k+l)w - x)$ and $p' = (x + lw, kw - x)$. Note that from Lemma

5.4.1 we have $[e_x, e_{(k+l)w-x}] = [e_{x+lw}, e_{kw-x}]$ and from the induction hypothesis $[e_x, e_{kw-x}] = \frac{\theta_{kw}}{\alpha_1}$ with $\theta(z)$ defined from e_x in the same way as for the EHA $\mathcal{E}$.

$$\begin{aligned}[][\theta_{kw}, e_{lw}] = \alpha_1([[e_x, e_{kw-x}], e_{lw}]) =&\alpha_1([e_x, [e_{kw-x}, e_{lw}]] + [[e_x, e_{lw}], e_{kw-x}]) = \\ &\alpha_1(-[e_x, e_{(k+l)w-x}] + [e_{x+lw}, e_{kw}]) = 0.\end{aligned}$$

But $\theta_{kw} = \alpha_k e_{kw} + t$ where t is a linear combination of e_p with $l(p) \geq 2$. As u_p for convex p are linearly independent we have $[e_{kw}, e_{lw}] = 0$. $\square$

In such way for every convex path p of rank r we construct an element e_p such that $\phi(e_p) = u_p$. We put J be a vector subspace of $\tilde{\mathcal{E}}^{>}$ generated by all e_p. From Proposition 3.4.6 u_p is a basis of $\mathcal{E}^{>}$, so ϕ gives an isomorphism between J and $\mathcal{E}^{>}_r$. It remains to show that $e_p \in J$ for every (not necessarily convex) path p of rank r. We will use the induction by area $a(p)$. If $a(p) = 0$ then p is convex path and the statement holds. Suppose that we have already shown it for all p with $a(p) < n$.
It is enough to consider the case of $l(p) = 2$. If $l(p) > 2$ then for every subpath p' of two segments $a(p') < a(p)$, so we can choose a convexification with smaller area by the induction hypothesis. The same argument on the area function shows that after finite number of convexifications we will get a linear combination of convex paths.

For a path $p = (x, (r, d) - x)$ we consider the region R bounded by the line L going through the origin and (r, d), parallel line through x and vertical lines though $(0, 0)$ and (r, d). The path p divides this region in 3 segments Δ, Δ_x and Δ'. Let us consider two different cases.

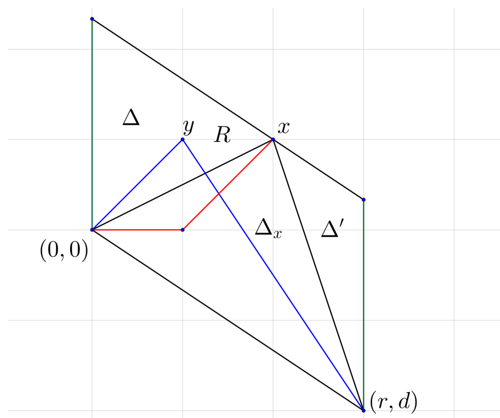

FIGURE 8. Three triangles Δ, Δ_x, Δ', the path p' (blue) and the convexification q (red).

1) There is a lattice point y in one of two triangles Δ and Δ'. We allow this point lie on the left boundary of Δ or right boundary of Δ' (green on Figure 8) but not on the boundary in common with Δ nor on the top boundary (black on Figure 8). Assume that $y \in \Delta$ and consider paths $p' = (y, (r, d) - y)$ and $m = (y, x - y, (r, d) - x)$. By construction we have $a(q) < a(p)$.

Proposition 5.4.3. *$e_m = \beta e_p + t = \beta' e_{p'} + t'$ where $t, t' \in J$ and $\beta, \beta' \in \mathbb{C}(q_1, q_2, c)$.*

Proof. Proposition 3.4.2 we have $u_y u_{x-y} = \beta u_x + \sum_q \beta_q u_q$ where q runs among the set of convexifications of $(y, x - y)$. Note that $\mathrm{rank}(y) < r$ and $\mathrm{rank}(x - y) < r$, so by the induction hypothesis the same holds for $e_y e_{x-y}$. We have

$$e_m = e_y e_{x-y} e_{(r,d)-x} = \beta e_x e_{(r,d)-x} + \sum_q \beta_q e_q e_{(r,d)-x} = \beta e_p + \sum_q \beta_q e_{q'},$$

where q' is the concatenation of q and $(r, d) - x$. Note that $l(q') \geq 3$ and $a(q') \leq a(p)$ because q' is a local convexification of p. Therefore by the induction hypothesis $e_{q'} \in J$. The first equality of the proposition holds.
The equality $e_m = \beta' e_{p'} + t'$ is proved in a similar way.

$$e_m = e_y e_{x-y} e_{(r,d)-x} = \beta' e_y e_{(r,d)-y} + \sum_q \beta_q e_y e_q = \beta e_{p'} + \sum_q \beta_q e_{q'},$$

where $q' \in J$ by the same reasons. □

Therefore we may suppose that p has no interior lattice points in Δ and Δ'.
2) If both Δ and Δ' have no interior lattice points then the same is true for Δ_x that is equal to the sum of these two triangles reflected along common sides. Suppose that Δ_x has lattice points on boundaries,
(i) We have a point on the bottom boundary. Then we have a point z on the upper boundary of the region R. We put $q = (z, (r, d) - z)$ and $q^\sharp = ((r, d) - z, z)$. The computation above shows that $e_p = \beta e_q + t$ where $t \in J$. By the construction triangle $\Delta_{z,(r,d)-z}$ has no interior lattice points and $\deg(z) = \deg((r, d) - z) = 1$. Therefore $u_q = u_{q^\sharp} + \frac{\theta_{(r,d)}}{\alpha_1} \in J$. So we may assume that $\deg((r, d)) = 1$.
(ii) In subsection 3.2.4 we show that if there are no interior lattice points in $\Delta_{x,y}$ then either $\deg(x) = 1$ or $\deg(y) = 1$ or $\deg(x) = \deg(y) = 2$. We apply this statement to the triangle $\Delta_x = \Delta_{x,(r,d)-x}$. In the last case we have $\deg((r, d)) \geq 2$. So we may suppose that $\deg(x) = 1$.
Set $w = \frac{(r,d)-x}{\deg((r,d)-x)}$ and put $y = x - w$. Suppose that y is positive. The triangle $\Delta_{y,w}$ has no interior lattice points because its area is the same as the area of $\Delta_{x,w}$. The area $S(\Delta_{y,(r,d)-x}) =$

$S(\Delta_{x,(r,d)-x})$, so $\Delta_{y,(r,d)-x}$ has no interior lattice points. By the induction hypothesis we get $[e_y, e_w] = e_x,\ [e_y, e_{(r,d)-x}] = e_{(r,d)-w}$. Therefore

$$[e_x, e_{(r,d)-x}] = [[e_y, e_w], e_{(r,d)-x}] = [[e_y, e_{(r,d)-x}], e_w] + [e_y, [e_w, e_{(r,d)-x}]] = [e_{(r,d)-w}, e_w] = \frac{\theta_{(r,d)}}{\alpha_1} \in J,$$

because $((r,d) - w, w)$ is a minimal path. It finishes the proof of the theorem 5.1.4.

Remark 5.4.4. *In fact $y = x - w$ can be not a positive segment. But the cases of $\mathcal{E}^{>}$ and $\mathcal{E}^{<}$ are analogous and* rank$(y) < r$*, so we may apply the induction hypothesis to both subalgebras. Suppose that* rank$(y) = l$*. We get:*

$$[e_y, e_w] = c^{-l} e_x,\ [e_y, e_{(r,d)-x}] = c^{-l} e_{(r,d)-w}. \text{ Therefore}$$
$$[e_x, e_{(r,d)-x}] = c^{-l}[[e_y, e_w], e_{(r,d)-x}] = c^{-l}[[e_y, e_{(r,d)-x}], e_w] + c^{-l}[e_y, [e_w, e_{(r,d)-x}]] =$$
$$c^{-l}[e_{(r,d)-w}, e_w] = \frac{\theta_{(r,d)}}{\alpha_1} \in J.$$

6. Hopf algebra structure.

In terms of Drinfeld generators $\mathbb{T}_1$, $\mathbb{T}_0^{\pm}$ and $\mathbb{T}_{-1}$ it is easy to write down the bialgebra structure on $\mathcal{E}$.

$$\Delta(\mathbb{T}_1(z)) = \mathbb{T}_1(z) \otimes 1 + \mathbb{T}_0^{+}(z) \otimes \mathbb{T}_1(z),$$
$$\Delta(\mathbb{T}_{-1}(z)) = \mathbb{T}_{-1}(z) \otimes 1 + \mathbb{T}_{-1}(z) \otimes \mathbb{T}_0^{-}(z),$$
$$\Delta(\mathbb{T}_0^{\pm}(z)) = \mathbb{T}_0^{\pm}(z) \otimes \mathbb{T}_0^{\pm}(z).$$

It will be shown in next talks that the algebra $\mathcal{E}$ admits a Hopf algebra structure.

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Coherent sheaves on elliptic curves.

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Abstract

We describe the abelian category of coherent sheaves on an elliptic curve, and construct an action of a central extension of $\mathrm{SL}_2(\mathbb{Z})$ on the derived category.

Contents

1 Coherent sheaves on elliptic curve

Definition 1.1. *An elliptic curve* over a field k is a nonsingular projective algebraic curve of genus 1 over k with a fixed k-rational point.

Remark 1.2. If the characteristic of k is neither 2 nor 3, an elliptic curve can be alternatevily defined as the subvariety of $\mathbb{P}^2_k$ defined by an equation $y^2z = x^3 - pxz^2 - qz^3$, where $p, q \in k$, and the polynomial $x^3 - pxz^2 - qz^3$ is square-free. In this case, the fixed point is $(0 : 1 : 0)$.

Remark 1.3. Over the field of complex numbers, there is even a simpler description. An elliptic curve is precisely a quotient $\mathbb{C}/\Lambda$ of $\mathbb{C}$ by a nondegenerate lattice $\Lambda \subset \mathbb{C}$ of rank 2.

Remark 1.4. Any elliptic curve carries a structure of a group, with the fixed point being the identity.

Fix an elliptic curve X over a field k. We do not assume that k is algebraically closed, since the main example is the finite field $\mathbb{F}_q$.

Recall that *a coherent sheaf* $\mathcal{F}$ on X is a sheaf of modules over $\mathcal{O}$ such that for every open affine $U \subset X$ the restriction $\mathcal{F}|_U$ is isomorphic to $\widehat{N}$ for some finitely generated $\mathcal{O}(U)$-module N.

Example 1.5. The structure sheaf $\mathcal{O}$ is indeed a coherent sheaf. Also, one can consider the ideal sheaf $\mathfrak{m}_x = \mathcal{O}(-x)$ corresponding to a closed point $x \in X$. Then the cokernel of the inclusion $\mathcal{O}(-x) \to \mathcal{O}$ is the so called *skyscrapper sheaf* $\mathcal{O}_x$, which is coherent as well.

Theorem 1.6. *Coherent sheaves on X form an abelian category $Coh(X)$.*

Theorem 1.7 (Global version of Serre theorem)**.** *Any coherent sheaf $\mathcal{F}$ on a smooth projective variety of dimension n over a field k admits a resolution $\mathcal{F}_n \to \mathcal{F}_{n-1} \to \ldots \to \mathcal{F}_1 \to \mathcal{F}_0$ where each $\mathcal{F}_i$ is finitely generated and locally free ($\simeq$ vector bundle).*

Theorem 1.8 (Grothendieck's finiteness theorem)**.** *Any coherent sheaf $\mathcal{F}$ on a smooth projective variety of dimension n over a field k has finite dimensional cohomologies over k.*

Corollary 1.9. *For any coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ the space* $\operatorname{Hom}(\mathcal{F}, \mathcal{G})$ *has finite dimension over k, since* $\operatorname{Hom}(\mathcal{F}, \mathcal{G}) = \Gamma(\mathcal{H}om(\mathcal{F}, \mathcal{G}), X) = H^0(\mathcal{H}om(\mathcal{F}, \mathcal{G}), X)$*.*

Theorem 1.10 (Grothendieck's vanishing theorem)**.** *Any coherent sheaf $\mathcal{F}$ on a smooth projective variety of dimension n over a field k has no i-th cohomologies for $i > n$.*

Definition 1.11. An abelian category $\mathcal{C}$ is called *hereditary* if $\operatorname{Ext}^2(-,-) = 0$.

Corollary 1.12. *The category $Coh(X)$ is hereditary.*

2 (Semi)stable sheaves

To a coherent sheaf we can associate two numbers, the Euler characteristic $\chi(\mathcal{F})$ and the rank $\operatorname{rk}(\mathcal{F})$.

Definition 2.1. The Euler characteristic $\chi(\mathcal{F})$ is the alternating sum $\sum_i (-1)^i \dim_k H^i(\mathcal{F}, X)$. In our case, it is equal to $\dim_k H^0(\mathcal{F}, X) - \dim_k H^1(\mathcal{F}, X)$.

Definition 2.2. The rank $\operatorname{rk}(\mathcal{F})$ is the dimension of the stalk $\mathcal{F}_\xi$ of $\mathcal{F}$ at a generic point ξ of X over the residue field. It is independent of ξ.

Example 2.3. We have $\chi(\mathcal{O}) = 0$, $\operatorname{rk}(\mathcal{O}) = 1$, $\chi(\mathcal{O}_x) = 1$, $\operatorname{rk}(\mathcal{O}_x) = 0$.

Proposition 2.4. *Given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we have $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$ and* $\operatorname{rk}(\mathcal{F}) = \operatorname{rk}(\mathcal{F}') + \operatorname{rk}(\mathcal{F}'')$*.*

Definition 2.5. *The slope* $\mu(\mathcal{F})$ of a nontrivial coherent sheaf $\mathcal{F}$ is the quotient $\chi(\mathcal{F})/\operatorname{rk}(\mathcal{F})$. In the case $\operatorname{rk}(\mathcal{F}) = 0$ we set $\mu(\mathcal{F}) = \infty$.

Lemma 2.6. *Given a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, we have three options:*

- $\mu(\mathcal{F}') < \mu(\mathcal{F}) < \mu(\mathcal{F}'')$;
- $\mu(\mathcal{F}') = \mu(\mathcal{F}) = \mu(\mathcal{F}'')$;
- $\mu(\mathcal{F}') > \mu(\mathcal{F}) > \mu(\mathcal{F}'')$.

Proof. We have

$$\mu(\mathcal{F}') = \frac{\chi(\mathcal{F}')}{\mathrm{rk}(\mathcal{F}')},$$
$$\mu(\mathcal{F}'') = \frac{\chi(\mathcal{F}'')}{\mathrm{rk}(\mathcal{F}'')},$$
$$\mu(\mathcal{F}) = \frac{\chi(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} = \frac{\chi(\mathcal{F}') + \chi(\mathcal{F}')}{\mathrm{rk}(\mathcal{F}') + \mathrm{rk}(\mathcal{F}'')}$$

Since both $\mathrm{rk}(\mathcal{F}')$ and $\mathrm{rk}(\mathcal{F}'')$ are nonnegative, we indeed get the lemma. □

Definition 2.7. A coherent sheaf $\mathcal{F}$ is called *stable (resp. semistable)* if for any nontrivial short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ we have $\mu(\mathcal{F}') < \mu(\mathcal{F})$ (resp. $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$).

General theory gives us the following

Theorem 2.8 ([1] Harder-Narasimhan filtration)**.** *For a coherent sheaf $\mathcal{F}$, there is a unique filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \subset \mathcal{F}_{n+1} = \mathcal{F}$$

such that all $\mathcal{A}_i = \mathcal{F}_{i+1}/\mathcal{F}_i$ are semistable and $\mu(\mathcal{A}_i) > \mu(\mathcal{A}_{i+1})$ for each i.

In our case, we can derive much stronger proposition. Before stating it, note two useful statements.

Proposition 2.9. *If $\mathcal{F}$ and $\mathcal{G}$ are semistable sheaves, and $\mu(\mathcal{F}) > \mu(\mathcal{G})$, then $Hom(\mathcal{F}, \mathcal{G}) = 0$.*

Proof. Suppose we have a nontrivial map $f\colon \mathcal{F} \to \mathcal{G}$. Then $\mu(\mathcal{F}) \leq \mu(\mathcal{F}/\ker f) = \mu(\mathrm{im}\, f) \leq \mu(\mathcal{G})$. Contradiction. □

Another property of $Coh(X)$ we will need is

Proposition 2.10 (Calabi-Yau property)**.** *For any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, there is an isomorphism* $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^1(\mathcal{G}, \mathcal{F})^*$.

Proof. From Remark 1.4 we know that the canonical bundle K is trivial, $K \simeq \mathcal{O}$. Also by Serre duality we get

$$\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = \mathrm{Ext}^0(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^1(\mathcal{G}, \mathcal{F} \otimes K)^* = \mathrm{Ext}^1(\mathcal{G}, \mathcal{F})^*.$$

□

We are ready to prove

Theorem 2.11. *Any nontrivial coherent sheaf is a direct sum of indecomposable semistable sheaves.*

Proof. We only need to prove that any indecomposable sheaf is semistable. Suppose some indecomposable sheaf $\mathcal{F}$ is not semistable. Then the Harder-Narasimhan filtration of $\mathcal{F}$ is nontrivial. Consider only the case of length 1 filtration, it captures the main idea. So, we have a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$, where both $\mathcal{F}'$ and $\mathcal{F}''$ are semistable, and $\mu(\mathcal{F}') > \mu(\mathcal{F}'')$. By Proposition 2.9 we get $\operatorname{Hom}(\mathcal{F}', \mathcal{F}'') = 0$. By Proposition 2.10 we obtain $\operatorname{Ext}^1(\mathcal{F}'', \mathcal{F}') = \operatorname{Hom}(\mathcal{F}', \mathcal{F}'')^* = 0$. Therefore the exact sequence splits, contradiction with the assumption that $\mathcal{F}$ is indecomposable. $\square$

Definition 2.12. Denote the full subcategory of semistable coherent sheaves on X of slope μ by C_μ.

Proposition 2.13. *The category C_μ is abelian, artinian, and closed under extensions. The simple objects in C_μ are stable sheaves of slope μ.*

Corollary 2.14. *$Coh(X)$ is the direct sum of all C_μ (on the level of objects).*

3 Euler form

Since rk and χ are well defined on $K_0(Coh(X))$, we can consider

Definition 3.1. *The Euler form* $\langle \mathcal{F}, \mathcal{G} \rangle$ of two elements $\mathcal{F}, \mathcal{G} \in K_0(Coh(X))$ is equal to $\dim \operatorname{Hom}(\mathcal{F}, \mathcal{G}) - \dim \operatorname{Ext}^1(\mathcal{F}, \mathcal{G})$.

Proposition 3.2. *We have $\langle \mathcal{F}, \mathcal{G} \rangle = \operatorname{rk}(\mathcal{F})\chi(\mathcal{G}) - \chi(\mathcal{F}) \operatorname{rk}(\mathcal{G})$.*

Proof. First notice that the RHS only depends on the classes of $\mathcal{F}$ and $\mathcal{G}$ in the Grothendieck group $K_0(Coh(X))$. Therefore it is sufficient to check the equality for some generators of the Grothendieck group, for example, for locally free sheaves. If $\mathcal{F}$ is locally free, the LHS reduces to $\chi(\mathcal{F}^\vee \otimes \mathcal{G})$. Note that in the case of elliptic curve, the Hirzebruch-Riemann-Roch theorem gives us that $\chi(\mathcal{E}) = \deg(\mathcal{E})$ for any coherent sheaf $\mathcal{E}$. Applying it here, we get

$$\begin{aligned} LHS = \chi(\mathcal{F}^\vee \otimes \mathcal{G}) = \deg(\mathcal{F}^\vee \otimes \mathcal{G}) = \operatorname{rk}(\mathcal{F}) \deg(\mathcal{G}) - \deg(\mathcal{F}) \operatorname{rk}(\mathcal{G}) = \\ \operatorname{rk}(\mathcal{F})\chi(\mathcal{G}) - \chi(\mathcal{F}) \operatorname{rk}(\mathcal{G}) = RHS. \end{aligned}$$

$\square$

Definition 3.3. *The charge map* is

$$Z = (\operatorname{rk}, \chi) \colon K_0(Coh(X)) \to \mathbb{Z}^2.$$

It is surjective, since we have both $(1, 0)$ and $(0, 1)$ in the image. We have a canonical nondegenerate volume form on $\mathbb{Z}^2$, $\langle (a, b), (c, d) \rangle = ad - bc$, and it is equal to the push-forward of the Euler form.

Proposition 3.4. *The kernel of the Euler form coincides with the kernel of Z, equivalently, $K_0(Coh(X))/\ker \langle , \rangle \simeq \mathbb{Z}^2$.*

Also we can now write some relations between different C_μ and $C_{\mu'}$.

Proposition 3.5. *Suppose $\mathcal{F}$ and $\mathcal{F}'$ are indecomposable, and $Z(\mathcal{F}) = (r, \chi)$, $Z(\mathcal{F}') = (r', \chi')$.*

- *If $\chi/r > \chi'/r'$, then $\mathrm{Hom}(\mathcal{F}, \mathcal{F}') = 0$, $\dim \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}') = \chi r' - \chi' r$;*
- *If $\chi/r < \chi'/r'$, then $\dim \mathrm{Hom}(\mathcal{F}, \mathcal{F}') = \chi' r - \chi r'$, $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}') = 0$.*

Proof. By Proposition 2.10 and Proposition 2.9 we know that

- if $\chi/r > \chi'/r'$, then $\mathrm{Hom}(\mathcal{F}, \mathcal{F}') = 0$;
- if $\chi/r < \chi'/r'$, then $\mathrm{Ext}^1(\mathcal{F}, \mathcal{F}') = 0$.

Proposition 3.2 concludes the proof. $\square$

4 Derived category of coherent sheaves

Let us show that Corollary 1.12 implies a neat description of the derived category $D^b(Coh(X))$ of bounded complexes of coherent sheaves on X.

Theorem 4.1. *Suppose $\mathcal{C}$ is a hereditary abelian category. Then any object $L \in D^b(\mathcal{C})$ is isomorphic to the sum of its cohomologies, i. e. $L = \bigoplus_i H^i L[-i]$.*

Proof. Let L be a complex $\ldots \xrightarrow{d^{i-1}} L^i \xrightarrow{d^i} L^{i+1} \xrightarrow{d^{i+1}} \ldots$. Fix any i. We have a short exact sequence $0 \to \ker d^{i-1} \to L^{i-1} \to \operatorname{im} d^{i-1} \to 0$. Apply $\mathrm{RHom}(H^i L, -)$. This gives rise to an exact sequence $\mathrm{Ext}^1(H^i L, L^{i-1}) \to \mathrm{Ext}^1(H^i L, \operatorname{im} d^{i-1}) \to \mathrm{Ext}^2(H^i L, \ker d^{i-1})$. Since $Coh(X)$ is hereditary, we obtain a surjection from $\mathrm{Ext}^1(H^i L, L^{i-1})$ to $\mathrm{Ext}^1(H^i L, \operatorname{im} d^{i-1})$. In particular, there exists M^i such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & L^{i-1} \arrow[r] \arrow[d] & M^i \arrow[r] \arrow[d] & H^i L \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{im} d^{i-1} \arrow[r] & \ker d^i \arrow[r] & H^i L \arrow[r] & 0
\end{tikzcd}$$

Then the following morhphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^i L \arrow[r] & 0 \arrow[r] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & L^{i-1} \arrow[r] \arrow[u] & M^i \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

of complexes is a quasi-isomorphism. If we compose its inverse with the morphism

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & L^{i-1} \arrow[r] \arrow[d, equal] & M^i \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & L^{i-2} \arrow[r] & L^{i-1} \arrow[r] & L^i \arrow[r] & L^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

we get a morphism $H^i L[-i] \to L$ in $D^b(Coh(X))$ which is isomorphism in the i-th cohomology, and zero elsewhere. Therefore, if we sum up all this morphisms, we obtain an isomorphism $\bigoplus_i H^i L[-i] \to L$. $\square$

Corollary 4.2. *The derived category $D^b(Coh(X))$ is the direct sum of $\mathbb{Z}$ copies of $Coh(X)$, a sheaf $\mathcal{F}$ in the i-th copy goes to $\mathcal{F}$.*

Since $K_0(D^b(Coh(X))) = K_0(Coh(X))$, Z is defined on $K_0(D^b(Coh(X)))$ as well. Note that $Z(\mathcal{F}[i]) = (-1)^i Z(\mathcal{F})$.

Remark 4.3. The corollary works for any smooth projective curve X. Another example of a hereditary category is the category of representations of a quiver.

5 $\mathrm{SL}_2(\mathbb{Z})$ action

Proposition 3.2 suggests to define $\langle L, M\rangle = \sum_i (-1)^i \dim \mathrm{Hom}(L, M[i])$ for any two objects $L, M \in D^b(Coh(X))$. Therefore the Euler form is preserved by any autoequivalence of $D^b(Coh(X))$. In other words, any autoequivalence $f \in Aut(D^b(Coh(X)))$ gives a corresponding automorphism of $\mathbb{Z}^2$ preserving the volume form, i. e. gives an element $\pi(f) \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 5.1. Say that an object $\mathcal{E} \in D^b(Coh(X))$ is *spherical* if $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = k$ (and consequently $\mathrm{Hom}(\mathcal{E}, \mathcal{E}[1]) = k$).

Example 5.2. The structure sheaf $\mathcal{O}$ and the skyscrapper sheaf at a rational k-point are spherical.

Definition 5.3. *A Forier-Mukai transform* with a kernel $\mathcal{L} \in D^b(Coh(X \times Y))$ is a functor $\Phi_{\mathcal{L}} \colon D^b(Coh(X)) \to D^b(Coh(Y))$ which sends an object $\mathcal{F} \in D^b(Coh(X))$ to $R\pi_{2*}(\pi_1^* \mathcal{F} \otimes^L \mathcal{L})$, where $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to X$ are the natural projections.

Definition 5.4. For a spherical object $\mathcal{E} \in D^b(Coh(X))$, which is a complex of locally free sheaves, we can define *a twist functor* $T_{\mathcal{E}} \colon D^b(Coh(X)) \to D^b(Coh(X))$ to be equal to a Forier-Mukai transform with the kernel $cone(\mathcal{E}^\vee \boxtimes \mathcal{E} \to \mathcal{O}_\Delta) \in D^b(Coh(X \times X))$.

Theorem 5.5 ([2])**.** *For a spherical object $\mathcal{E} \in D^b(Coh(X))$ the twist functor $T_{\mathcal{E}}$ is an exact equivalence which sends an object $\mathcal{F}$ to $cone(\mathrm{RHom}(\mathcal{E}, \mathcal{F}) \otimes^L \mathcal{E} \xrightarrow{ev_{\mathcal{F}}} \mathcal{F})$.*

Remark 5.6. The evaluation works by applying $ev \colon \mathrm{Ext}^i(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}[-i] \to \mathcal{F}$ on each grading.

Let us see how $T_{\mathcal{E}}$ acts on Grothendieck group.

Proposition 5.7. *The action of $T_{\mathcal{E}}$ on $K_0(D^b(Coh(X)))$ is given by $[\mathcal{F}] \mapsto [\mathcal{F}] - \langle \mathcal{E}, \mathcal{F}\rangle [\mathcal{E}]$.*

Proof. Indeed, $[T_{\mathcal{E}}(\mathcal{F})] = [\mathcal{F}] - [\mathrm{RHom}(\mathcal{E}, \mathcal{F}) \otimes^L \mathcal{E}] = [\mathcal{F}] - \langle \mathcal{E}, \mathcal{F}\rangle [\mathcal{F}]$. □

Corollary 5.8. $\pi(T_{\mathcal{O}}) = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$, $\pi(T_{\mathcal{O}_x}) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.

Proof. Since $\mathbb{Z}^2$ are generated by the charges of $\mathcal{O}$ and $\mathcal{O}_x$, we can check this on $\mathcal{O}$ and $\mathcal{O}_x$ only.

$$\begin{aligned} T_{\mathcal{O}}([\mathcal{O}]) &= [\mathcal{O}] - \langle \mathcal{O}, \mathcal{O}\rangle [\mathcal{O}] = [\mathcal{O}], \\ T_{\mathcal{O}}([\mathcal{O}_x]) &= [\mathcal{O}_x] - \langle \mathcal{O}, \mathcal{O}_x\rangle [\mathcal{O}] = [\mathcal{O}_x] - [\mathcal{O}], \\ T_{\mathcal{O}_x}([\mathcal{O}]) &= [\mathcal{O}] - \langle \mathcal{O}_x, \mathcal{O}\rangle [\mathcal{O}_x] = [\mathcal{O}] + [\mathcal{O}_x], \\ T_{\mathcal{O}_x}([\mathcal{O}_x]) &= [\mathcal{O}_x] - \langle \mathcal{O}_x, \mathcal{O}_x\rangle [\mathcal{O}_x] = [\mathcal{O}_x]. \end{aligned}$$

□

Proposition 5.9. *$T_{\mathcal{O}_x}$ is in fact just the tensor product with $\mathcal{O}(x)$.*

Proof. The formula for the adjoint of a Forier-Mukai transform gives that the inverse of $T_{\mathcal{O}_x}$ is the Forier-Mukai transform with the kernel $cocone(\mathcal{O}_\Delta \to \mathcal{O}_{(x,x)})$. The map inside a cocone is nonzero. But any nonzero map $\mathcal{O}_\Delta \to \mathcal{O}_{(x,x)}$ is a nonzero multiple of the natural surjection $\mathcal{O}_\Delta \to \mathcal{O}_{(x,x)}$. Therefore the cocone is equal to the kernel of this map, or just $\mathcal{O}_\Delta \otimes \pi_1^*(\mathcal{O}(-x))$. Now note that the sheaf $\mathcal{O}_\Delta$ in the kernel trivializes all pullbacks and pushforwards we do to the identity maps between sheaves on X and on $\Delta \simeq X$. The proposition follows. $\square$

The matrices $\pi(T_{\mathcal{O}})$ and $\pi(T_{\mathcal{O}_x})$ generate $\mathrm{SL}_2(\mathbb{Z})$, therefore, $\pi\colon Aut(D^b(Coh(X))) \to \mathrm{SL}_2(\mathbb{Z})$ is surjective.

6 Classification of indecomposable sheaves

Note that indecomposable torsion sheaves lie in C_∞, and generate C_∞. Moreover, we have

Theorem 6.1. *Indecomposable torsion sheaves are parametrized by a positive integer $s > 0$ and a closed point $x \in X$. The corresponding torsion sheaf is $\mathcal{O}/\mathcal{O}(-sx)$.*

Proof. Indeed, we reduce to the case of one point, then the local ring is PID, and the claim follows. $\square$

In addition to that, $\mathrm{SL}_2(\mathbb{Z})$ action allows us to prove

Theorem 6.2. *For each $\mu \in \mathbb{Q}$ we have a canonical isomorphism $C_\mu \simeq C_\infty$.*

Proof. Indeed, let μ be equal to a/b for coprime a and b. Choose some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ which sends (a,b) to $(0,1)$, and lift it to an autoequivalence $\tilde{f} \in Aut(D^b(Coh(X)))$ of the derived category. Take any indecomposable sheaf $\mathcal{F} \in C_\mu$. Then $\tilde{f}(\mathcal{F})$ is an indecomposable object in $D^b(Coh(X))$ with the slope ∞. Therefore, it is of form $\mathcal{G}[k]$, where $\mathcal{G}$ is a torsion sheaf, and k is some integer. Denote by $\overline{f}\colon C_\mu \to C_\infty$ a map which sends an indecomposable sheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$ defined in this way. It is easy to see that if we begin with the inverse matrix f^{-1}, then we get a map $\overline{f^{-1}}\colon C_\infty \to C_\mu$ which is inverse to $\overline{f}$. Also $\overline{f}$ does not depend on a lift $\tilde{f}$. So C_μ and C_∞ are canonically isomorphic. $\square$

Summarizing, we have

Theorem 6.3. *Indecomposable sheaves are parametrized by a pair (rk, χ) in the right half of $\mathbb{Z}^2$ and a closed point $x \in X$.*

Let us show how this describes indecomposable sheaves with charges $(1,1)$ and $(1,0)$.

Proposition 6.4.

$$T_{\mathcal{O}}(\mathcal{O}) = \mathcal{O}, \quad T_{\mathcal{O}}(\mathcal{O}(x)) = \mathcal{O}_x,$$
$$T_{\mathcal{O}_x}(\mathcal{O}) = \mathcal{O}(x), \quad T_{\mathcal{O}_x}(\mathcal{O}_x) = \mathcal{O}_x.$$

Proof. The second line is a consequence of Proposition 5.9. The first line is an easy computation based on Theorem 5.5. □

Proposition 6.5. *The indecomposable sheaves of charge* $(1,1)$ *are the sheaves* $\mathcal{O}(x)$*. The indecomposable sheaves of charge* $(1,0)$ *are the sheaves* $\mathcal{O}(x-y)$*.*

Proof. The autoequivalence $T_{\mathcal{O}}^{-1}$ maps the charge $(0,1)$ to $(1,1)$, so we can use it to obtain the indecomposables of charge $(1,1)$. Given an indecomposable $\mathcal{O}_x$ of charge $(0,1)$, its image is $\mathcal{O}(x)$ by Proposition 6.4. The first part follows.

Then we can apply $T_{\mathcal{O}_y}^{-1}$ to the latter indecomposables. We get that the indecomposables of charge $(1,0)$ are $\mathcal{O}(x-y)$. □

7 Braid group relations

For matrices $A = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ we have the following relations

$$ABA = BAB$$
$$(AB)^3 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

We expect similar relations to hold for $T_{\mathcal{O}}$ and $T_{\mathcal{O}_x}$.

Theorem 7.1 ([2])**.**

$$T_{\mathcal{O}}T_{\mathcal{O}_x}T_{\mathcal{O}} \simeq T_{\mathcal{O}_x}T_{\mathcal{O}}T_{\mathcal{O}_x}$$
$$(T_{\mathcal{O}}T_{\mathcal{O}_x})^3 \simeq i^*[1],$$

where $i\colon X \to X$ *is the inverse map of* X*.*

We can prove the braid relation using the following

Proposition 7.2 ([2])**.** *Given two spherical objects* E_1 *and* E_2*, we have*

$$T_{E_1}T_{E_2} = T_{T_{E_1}(E_2)}T_{E_1}$$

Proof. Using the computations in Proposition 6.4, we can write

$$T_{\mathcal{O}}T_{\mathcal{O}_x}T_{\mathcal{O}} = T_{\mathcal{O}}T_{T_{\mathcal{O}_x}(\mathcal{O})}T_{\mathcal{O}_x} = T_{\mathcal{O}}T_{\mathcal{O}(x)}T_{\mathcal{O}_x} = T_{T_{\mathcal{O}}(\mathcal{O}(x))}T_{\mathcal{O}}T_{\mathcal{O}_x} = T_{\mathcal{O}_x}T_{\mathcal{O}}T_{\mathcal{O}_x}.$$

□

This shows that $T_{\mathcal{O}}$ and $T_{\mathcal{O}_x}$ generate the group $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$ in $Aut(D^b(Coh(X)))$, the central extension of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$.

References

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HALL ALGEBRAS

TUDOR PĂDURARIU

Contents

1. Introduction

In this seminar the elliptic Hall algebra (EHA) was introduced as the limit of the spherical double affine Hecke algebras of $\mathfrak{gl}_n$ and we have written an explicit presentation in terms of generators and relations [2]. In this talk, we will define it as an algebra that specializes to a certain subalgebra of the Hall algebra of every elliptic curve over a finite field, definition that will explain its name.

The plan for this talk is the following. We begin by defining the Hall algebra, and explaining when one can construct a coproduct, when this algebra is a bialgebra with respect to these operations, or a Hopf algebra. A Hall algebra $\mathbb{H}_{\mathcal{A}}$ can be defined for any abelian category $\mathcal{A}$ with certain finitary properties, but we will see that this algebra has richer properties for global dimension one categories. A natural supply of such categories are the abelian categories of representations of a quiver over a finite field and of coherent sheaves over a projective curve C over a finite field. We are interested in the curves C of genus one, and inside the Hall algebras of these curves we will find specializations of EHA as defined in the previous talks. In the second part of the talk, we focus on the elliptic curve case, for which we show that the derived equivalences of $D^b\mathrm{Coh}\,(X)$ act by algebra automorphisms on the Drinfeld double of $\mathbb{H}_{\mathcal{A}}$. This action will be used in proving a PBW theorem for certain subalgebras of the Hall algebra, and in identifying this algebra with the EHA defined by generators and relations in [2].

2. Hall algebras

2.1. **Definition of the product and of the coproduct.** We start with a small abelian category $\mathcal{A}$ of finite global dimension. We say that $\mathcal{A}$ is finitary if for all objects M and N in $\mathcal{A}$, we have that

$$|\mathrm{Hom}(M,N)|,\ |\mathrm{Ext}^i(M,N)| < \infty.$$

In most examples, these finitary categories will be linear over a finite field k. Examples of such categories are the categories of $k-$representations of a quiver, or of a finite dimensional algebra over k, and the categories of coherent sheaves on any projective smooth scheme over k.

For two objects M and N of $\mathcal{A}$, define

$$\langle M,N\rangle := (\prod_{i=0}^{\infty} |\mathrm{Ext}^i(M,N)^{(-1^i)}|)^{1/2}.$$

This defines the multiplicative Euler form $\langle , \rangle : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{C}^\times$. When $\mathcal{A}$ is $k-$linear, we have that $\langle M,N\rangle = v^{\chi(M,N)}$, where $v^2 = q$ is a square root of q, the number of elements of the field k.

We are now ready to define the Hall algebra $\mathbb{H}_{\mathcal{A}}$. As a vector space,

$$\mathbb{H}_{\mathcal{A}} := \bigoplus_{M \text{ iso class in } \mathcal{A}} \mathbb{C}M.$$

Given M, N, and R three objects in $\mathcal{A}$, we define $P^R_{M,N}$ to be the number of short exact sequences $0 \to N \to R \to M \to 0$, and $a_p := |\mathrm{Aut}\,(P)|$.

We observe that

$$\frac{P^R_{M,N}}{a_M a_N} = |\{L \subset R | L \cong N, R/L \cong M\}|.$$

Proposition 2.1. *(Ringel) The multiplication*

$$[M]\cdot[N] = \langle M,N\rangle \sum_R \frac{P^R_{M,N}}{a_M a_N}[R]$$

defines an associative algebra structure on $\mathbb{H}_{\mathcal{A}}$*, with unit* $[0]$*, zero object of* $\mathcal{A}$*.*

Proof. Because $\mathrm{Ext}^1(M,N)$ is finite, the definition of $[M]\cdot[N]$ is an element of $\mathbb{H}_{\mathcal{A}}$. It is immediate to check that $[0]$ is the unit. For the associativity, direct computations give that, for three objects M, N, and L of $\mathcal{A}$, we have:

$$([M] \cdot [N]) \cdot [L] = \langle M, N\rangle\langle R, L\rangle\langle M, L\rangle \sum_R c^R_{M,N,L}[R],$$

where $c^R_{M,N,L}$ counts the number of elements of the set $\{0 \subset B \subset A \subset R | B \cong L, A/B \cong N, R/A \cong M\}$. Computing $[M] \cdot ([N] \cdot [L])$ gives the same result. $\square$

Observe that the multiplication of a Hall algebra encodes all the ways in which one can extend one object by another object. One can define a Hall algebra for any exact category with the above finitary conditions.

Further, observe that $\mathbb{H}_\mathcal{A}$ is naturally graded by the classes in the Grothendieck group $K(\mathcal{A})$.

Example. Let $\mathcal{A}$ be a semisimple category with S_i the simple objects. Then for $i \neq j$ we have $[S_i][S_j] = [S_i \oplus S_j] = [S_j][S_i]$, and $[S_i][S_i] = |\mathrm{End}\,(S_i)|^{\frac{1}{2}}(|\mathrm{End}\,(S_i)| + 1)[S_i \oplus S_i]$.

It is natural to ask whether we can define a comultiplication on the vector space $\mathbb{H}_\mathcal{A}$. It should involve all possible ways to break an object into two smaller objects in $\mathcal{A}$. Usually we can break an object in infinitely many ways into two objects, so we need to introduce certain completions of the Hall algebra in order to define the coproduct. We will gloss over some of the details, see [4] for full explanations.

For $a, b \in K(\mathcal{A})$, we define

$$\mathbb{H}_\mathcal{A}[a]\hat{\otimes}\mathbb{H}_\mathcal{A}[b] = \prod_{M \text{of class}\, a, N \text{ of class}\, b} \mathbb{C}[M] \otimes \mathbb{C}[N].$$

Elements of this vector space are simply formal (infinite) linear combinations

$$\sum_{M \text{ of class } a, N \text{ of class } b} c_{M,N}[M] \otimes [N].$$

Further, we define

$$\mathbb{H}_\mathcal{A}\hat{\otimes}\mathbb{H}_\mathcal{A} := \prod_{a,b} \mathbb{H}_\mathcal{A}[a]\hat{\otimes}\mathbb{H}_\mathcal{A}[b].$$

Thus, the elements of this completed tensor product are all formal (infinite) linear combinations $\sum_{M,N} c_{M,N}[M] \otimes [N]$.

Proposition 2.2. *(Green) The coproduct*

$$\Delta[R] = \sum_{M,N} \langle M, N\rangle \frac{P^R_{M,N}}{a_R}[M] \otimes [N]$$

puts on $\mathbb{H}_\mathcal{A}$ the structure of a (topological) coassociative coalgebra with counit $\varepsilon : \mathbb{H}_\mathcal{A} \to \mathbb{C}$ defined by $\varepsilon[M] = \delta_{M,0}$.

Observe that the coproduct takes values in the finite part $\mathbb{H}_{\mathcal{A}} \otimes \mathbb{H}_{\mathcal{A}}$ if and only if for any object R, there exist only finitely many subobjects $N \subset R$. It holds for categories of representations of quivers, but not for categories of coherent sheaves on a projective variety. Indeed, any subrepresentation of the $k-$representation (V_i) of a quiver Q is specified by subspaces of the V_i, of which there are finitely many possibilities, and maps between the corresponding spaces, of which there are finitely many.

It is not clear how to check coassociativity of Δ given this formula, because it is not clear that $(\Delta \otimes 1)\Delta$ makes sense. Fortunately, it makes sense because the only terms in $\mathbb{H}_{\mathcal{A}} \hat{\otimes} \mathbb{H}_{\mathcal{A}}$ that contribute to $[M_1] \otimes [M_2] \otimes [M_3]$ are of the form $[N] \otimes [M_3]$, where N is an extension of M_1 by M_2, of which there are finitely many.

As one last comment about the coproduct, sometimes the Grothendieck group $K(\mathcal{A})$ is not finitely generated, which happens for the category of coherent sheaves on an elliptic curve. It is preferable to work with a smaller $K-$group, like the numerical $K-$group for an arbitrary curve, in these situations. In these cases the definition of the completion $\mathbb{H} \hat{\otimes} \mathbb{H}$ needs to be slightly changed, see [1].

Next, we investigate when these two operations define a bialgebra. One cannot take the product of two elements in the completed product of the Hall algebra, but one can take the product if they are in the image of the comultiplication Δ, see [4] for more details.

In order to state the next theorem, which gives an answer to when these two operations put a bialgebra structure on $\mathbb{H}_{\mathcal{A}}$, we need to twist the multiplication, or, alternatively, we need to add a degree zero piece to $\mathbb{H}_{\mathcal{A}}$. Let $K = \mathbb{C}[K(\mathcal{A})]$ be the group algebra of the Grothendieck group of $\mathcal{A}$, and denote by k_a the class of the element $a \in K(\mathcal{A})$. Define the vector space $\mathbb{H}'_{\mathcal{A}} = \mathbb{H}_{\mathcal{A}} \otimes K$. We want to put an algebra structure on this space extending the algebra structure on the two factors of the tensor product. We only need to explain how k_a and $[M]$ commute, for which we introduce the relation

$$k_a[M]k_a^{-1} = \langle a, M\rangle \langle M, a\rangle [M].$$

We can also extend the comultiplication as follows: $\Delta(k_a) = k_a \otimes k_a$ and

$$\Delta([R]k_a) = \sum_{M,N} \langle M, N\rangle \frac{P^R_{M,N}}{a_R} [M]k_{N+a} \otimes [N]k_a.$$

For the next theorem, we need to assume that the global dimension gldim $(\mathcal{A}) \leqslant 1$. Recall that the global dimension n of an abelian category is the smallest integer n with the property that $\mathrm{Ext}^{n+1}(A, B) = 0$ for all objects $M, N \in \mathcal{A}$. Recall that for $\mathcal{A}$ a category with enough injectives and projectives, this number is the same as the supremum after all projective dimensions of elements in $\mathcal{A}$ and the same as the supremum after all injective resolutions of elements in $\mathcal{A}$.

Theorem 2.3. *The comultiplication map*

$$\Delta : \mathbb{H}'_{\mathcal{A}} \to \mathbb{H}'_{\mathcal{A}} \otimes \mathbb{H}'_{\mathcal{A}}$$

is an algebra morphism.

We can also extend the counit map to $\varepsilon([M]k_a) = \delta_{M,0}$. Then the theorem says that $(\mathbb{H}'_{\mathcal{A}}, i, m, \varepsilon, \Delta)$ is a (topological) bialgebra.

Remember that if $\mathcal{A}$ satisfies the finite subobject condition, there is no need to introduce the completion. Further, if the symmetrized Euler product $\langle M, N\rangle\langle N, M\rangle = 1$, then there is no need to introduce the factor $\mathbb{C}[K(\mathcal{A})]$. For a $k-$linear category $\mathcal{A}$, this happens when $\chi(M, M) = 0$, for all $M \in \mathcal{A}$, for example for $\mathcal{A} = \mathrm{Coh}\,(X)$, where X is an elliptic curve. A final observation is that even if one can define Hall algebras for exact categories, Green's theorem holds for abelian categories only.

It is natural to ask when this bialgebra is actually a Hopf algebra. For this, we would need to construct an antipode map which satisfies the axioms of a Hopf algebra. Xiao managed to construct one such antipode map $S : \mathbb{H}'_{\mathcal{A}} \to \mathbb{H}'_{\mathcal{A}}$ for categories of representations of a quiver.

The Hall algebras for categories of $\mathrm{gldim}\,(\mathcal{A}) \leqslant 1$ come with (at least) one other piece of extra structure: a nondegenerate Hopf pairing:

Proposition 2.4. *The scalar product* $(,) : \mathbb{H}_{\mathcal{A}} \otimes \mathbb{H}_{\mathcal{A}} \to \mathbb{C}$ *defined by*

$$([M]k_a, [N]k_b) = \frac{\delta_{M,N}}{a_M}\langle a, b\rangle\langle b, a\rangle$$

is a non-degenerate Hopf pairing, that is, it satisfies $(xy, z) = (x \otimes y, \Delta(z))$ *for any* $x, y, x \in \mathbb{H}'_{\mathcal{A}}$.

2.2. **Examples.** We have already computed the product of some elements in a semisimple category $\mathcal{A}$. The next easiest example is provided by categories of nilpotent $k-$representations of quiver $\mathrm{Rep}\,(Q)$. Recall that k is a finite field. For us, a quiver will be allowed to have multiple edges or cycles, but no loops. Therefore $\mathrm{gldim}\,\mathrm{Rep}(Q) \leqslant 1$. Such a quiver has simple objects S_i with a one dimensional vector space k in the ith vertex, and zero everywhere else.

Let's compute relations for the A_2 quiver. There are only two simple objects S_1 and S_2, and they have a unique nontrivial extension I_{12} corresponding to $\mathrm{Ext}^1(S_1, S_2) = k$. Observe that $\mathrm{Ext}^1(S_2, S_1) = 0$. Further, the only indecomposable objects are S_1, S_2, and I_{12}. We can prove the following relations in the Hall algebra of this quiver:

(1) $[S_1][S_2] = v^{-1}([S_1 \oplus S_2] + [I_{12}])$ because there is only one subobject isomorphic to S_1 in both $S_1 \oplus S_2$ and I_{12},
(2) $[S_2][S_1] = [S_1 \oplus S_2]$ because there are no extensions of S_1 by S_2,

(3) $[S_2][S_1]^2 = v(v^2+1)[S_2][S_1^2] = v(v^2+1)[S_1^2 \oplus S_2]$ is similar to item (2),

(4) $[S_1]^2[S_2] = v(v^2+1)[S_1^2][S_2] = v^{-1}(v^2+1)([S_1^2 \oplus S_2] + [S_1 \oplus I_{12}])$ is similar to item (1),

(5) $[S_1][S_2][S_1] = [S_1][S_1 \oplus S_2] = (v^2+1)[S_1^2 \oplus S_2] + [S_1 \oplus I_{12}]$ because there are $\mathbb{P}^1(\mathrm{End}\,(S_1, S_1))$ subobjects $S_1 \subset S_1 \oplus S_1$.

Putting the last three relations together, we get that

(6) $[S_1]^2[S_2] - (v+v^{-1})[S_1][S_2][S_1] + [S_2][S_1]^2 = 0.$

Similarly we can also prove that

(7) $[S_2]^2[S_1] - (v+v^{-1})[S_2][S_1][S_2] + [S_1][S_2]^2 = 0.$

In fact, any relation satisfied by $[S_1]$ and $[S_2]$ is a consequence of one of the two above relations. Indeed, relations (5) and (6) provide us with an algebra morphism

$$\Phi : U_v(\mathfrak{b}) \to \mathbb{H}_Q,$$

where in this case Q is the A_2 quiver and $\mathfrak{b} \subset \mathfrak{sl}_2$ is the positive Borel subalgebra$-$ we recall the structure of the quantum group in the next paragraph. The map Φ is automatically a surjection because the Hall algebra is generated by the classes of the simple objects $[S_1]$ and $[S_2]$. To show injectivity, observe that $U_v(\mathfrak{b}) = U_v(\mathfrak{n}) \otimes \mathbb{C}[K_1^{\pm}, K_2^{\pm}]$ and $\mathbb{H}' = \mathbb{H} \otimes \mathbb{C}[k_{S_1}^{\pm}, k_{S_2}^{\pm}]$. The dimensions of the $(n,m)-$graded piece of the quantum group $U_v(\mathfrak{n})$ can be computed by the PBW theorem as the number of ways to write (n,m) as the sum $a_1(1,0) + a_2(0,1) + a_3(1,1)$, where $(1,0), (0,1)$, and $(1,1)$ are the dimension vectors of the positive roots. The dimension of the $(n,m)-$graded piece of $\mathbb{H}_Q$ is the same number, because $(1,0), (0,1)$, and $(1,1)$ are the dimension vectors for the indecomposable representations of Q.

The above connection between a quantum group and the $A-2$ quiver is far from being isolated. Recall that for $\mathfrak{g}$ a Kac-Moody algebra associated to the matrix A, we can define a quantum group $U_v(\mathfrak{g}')$ with positive Borel part $U_v(\mathfrak{b}'_+)$. Here $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$. The quantum group $U_v(\mathfrak{b}'_+)$ is generated by E_i, K_i, K_i^{-1}, for $i \in I$, with the relations

- $K_i K_i^{-1} = K_i^{-1} K_i = 1$,
- $K_i K_j = K_j K_i$,
- $K_i E_j K_i^{-1} = v^{a_{ij}} E_j$, for all $i, j \in I$,
- the quantum Serre relation involving the E_is, see [4].
- the coproduct is defined via $\Delta(K_i) = K_i \otimes K_i$ and $\Delta(E_i) = E_1 \otimes 1 + K_i \otimes E_i$,
- the antipode is defined by $S(K_i) = K_i^{-1}$, $S(E_i) = -K_i^{-1} E_i$.

Theorem 2.5. *(Ringel, Green) The assignment $E_i \to [S_i]$, $k_i \to k_{S_i}$ for $i \in I$ defines an embedding of Hopf algebras*

$$\Phi : U_v(\mathfrak{b}'_+) \to \mathbb{H}'_Q.$$

The map Φ is an isomorphism if and only if Q is of finite type, or, equivalently, if $\mathfrak{g}$ is a simple Lie algebra.

One can prove that the coefficients $P^R_{M,N}$ are polynomials in v with rational coefficients. Using these polynomials as structure constants for multiplication and comultiplication, we can define a universal (or generic) version $\mathbb{H}_Q$ of the Hall algebra over $\mathbb{C}[t^{1/2}, t^{-1/2}]$, which recovers the Hall algebra of a quiver over k when $t = q$ the number of elements of the field k.

Before we start discussing examples coming from geometry, we need to discuss one other quiver example, the Jordan quiver. The reason is the following: the torsion category of sheaves on a smooth projective curve X splits as follows

$$\mathrm{Tor}\,(X) = \prod_{x \in X} \mathrm{Tor}_x.$$

Each category Tor_x is the category of torsion sheaves supported at x, which is the same as the category of finite dimensional modules over the discrete valuation ring $\mathcal{O}_{x,X}$. This category if the same as nilpotent representations of the Jordan quiver over the finite field $k_x = \mathcal{O}_{x,X}/m_{x,X}$.

For k a finite field, denote by N_k the category of nilpotent representations of the Jordan quiver over k. There exists exactly one indecomposable object $I_{(r)}$ of any length $r \in \mathbb{N}$. Further, all objects are of the form $I_\lambda = I_{(\lambda_1)} \oplus \cdots \oplus I_{(\lambda_s)}$ for a partition $\lambda = (\lambda_1, \cdots, \lambda_s)$.

Denote by Λ_t the (Macdonald) ring of symmetric functions over $\mathbb{Q}[t, t^{-1}]$, and by e_λ and p_λ the elementary symmetric functions and the power sum symmetric functions, respectively. Recall the Macdonald ring is defined via the projective limit $\Lambda = \lim \mathbb{C}[t, t^{-1}, x_1, \cdots, x_n]^{\Sigma(n)}$ where the maps between consecutive rings send the biggest index variable to zero, and the multiplication is induced from the multiplication on each of the individual polynomial rings. The coproduct is defined via the inclusion

$$\mathbb{C}[t, t^{-1}x_1, \cdots, x_{2n}]^{\Sigma(2n)} \hookrightarrow \mathbb{C}[t, t^{-1}, x_1, \cdots, x_n]^{\Sigma(n)} \otimes \mathbb{C}[t, t^{-1}, x_{n+1}, \cdots, x_{2n}]^{\Sigma(n)}.$$

Then the following theorem gives us a very explicit description of the Hall algebra associated to N_k.

Theorem 2.6. *(Macdonald)*

The assignment $[I_{(1)^r}] \to u^{r(r-1)} e_r$ extends to a bialgebra isomorphism

$$\Phi_k : \mathbb{H}_{N_k} \to \Lambda_t|_{t=u^2}.$$

Set $F_r := \Phi_l^{-1}(p_r)$. Then:

(i) $F_r = \sum_{|\lambda|=r} n_u(l(\lambda) - 1)[I_\lambda]$, where $n_u(l) := \prod_{i=1}^{l}(1 - u^{-2i})$,

(ii) $\Delta(F_r) = F_r \otimes 1 + 1 \otimes F_r$,

(iii) $(F_r, F_s) = \delta_{rs} \frac{ru^r}{u^{-r} - u^r}$.

2.3. **Hall algebras for projective curves.** Let x be a closed point of the (smooth) projective curve X over a finite field k. denote by $\deg(x)$ the degree of the finite extension k_x/k, where $k_x = \mathcal{O}_{x,X}/m_{x,X}$. Recall that N_{k_x} is equivalent to Tor_x. Let

$$\Phi_{k_k} : \mathbb{H}_{\mathrm{Tor}_x} \to \Lambda_y|_{t=v^{2\deg(x)}}$$

be the isomorphism provided by Macdonald's theorem, where $v^2 = k^{-1}$.

For r a natural number, define

$$\frac{T_{r,x}^{(\infty)}}{[r]} = \frac{\deg(x)}{r} \Phi_{k_x}^{-1}(p_{\frac{r}{\deg(x)}}) \text{ if } \deg(x)|r$$

and by zero otherwise. We put $T^{(\infty)} = \sum_x T_{r,x}^{(\infty)}$ which is a finite sum, as there are only finitely many points on X of a given degree.

For $\mathbb{P}^1$, Kapranov proved a Ringel-Green style theorem, comparing the Hall algebra with the positive part $U_v(L\mathfrak{b}_+) \subset U_v(L\mathfrak{sl}_2)$ of the quantum loop algebra of $\mathfrak{sl}_2$. For more details, see [4][page 64].

Theorem 2.7. *(Kapranov) There exists an embedding of algebras*

$$\Phi : U_v(L\mathfrak{b}_+) \to \mathbb{H}_{\mathbb{P}^1}.$$

2.4. **The Drinfeld double construction.** We have seen in the above examples that the Hall algebra (as defined in the beginning of these notes) recovers the positive nilpotent part of a quantum group. Adding the group algebra of the Grothendieck group $\mathbb{C}[K(\mathcal{A})]$ corresponds to adding the Cartan part to the quantum group. It is natural to try to construct the full quantum group this way. A possible idea is to change the category we are looking at. We know that we should get two copies of the Hall algebra, so we would like to replace the abelian category $\mathcal{A}$ with a variant that contains two copies of $\mathcal{A}$. However, there exists a completely algebraic procedure from which one can double a quantum group, which we will explain in this section.

In our case, we can start with the (topological) bialgebra $\mathbb{H}_X$ and construct another (topological) bialgebra $\mathbb{DH}_X$ which is generated by the Hall algebra $\mathbb{H}_X$ and its dual $\mathbb{H}_X^*$ with opposite coproduct. In our case, we can identify the dual of the Hall algebra with the Hall algebra via the Hopf pairing. Thus $\mathbb{DH}_X$ will be generated by two copies $\mathbb{H}_X^+$ and $\mathbb{H}_X^-$ of the Hall algebra. The set of relations we impose is the following: for any pair (h, g) of elements in $\mathbb{H}_X$, there is one relation $R(h, g)$ given by

$$\sum_{i,j} h_i^{(1)-} g_j^{(2)+} (h_i^{(2)}, g_j^{(1)}) = \sum_{i,j} g_j^{(1)+} h_i^{(2)-} (h_i^{(1)}, g_j^{(2)}),$$

where $\Delta(x) = \sum_i x_i^{(1)} \otimes x_i^{(2)}$.

In our case, even if the coproduct takes values in the completed tensor product, the relations $R(h, g)$ actually contain finitely many terms. It is worth noticing that one can prove a PBW theorem for the Drinfeld double, saying that the multiplication map

$$m : \mathbb{H}_X^- \otimes \mathbb{H}_X^+ \to \mathbb{DH}_X$$

is a vector space isomorphism.

Drinfeld used this technique to construct the full quantum group $U_v(\mathfrak{g})$ as a quotient of Drinfeld double of the quantum group $U_v(\mathfrak{b}_+)$.

3. The Hall algebra of an elliptic curve

3.1. **Coherent sheaves on an elliptic curve.** In this subsection, we recap some of the material from the previous talk [3]. Let X be an elliptic curve over an arbitrary field k. The slope of a sheaf $F \in \mathrm{Coh}\,(X)$ is defined as

$$\mu(F) = \frac{\deg(F)}{\mathrm{Rank}\,(F)} \in \mathbb{Q} \cup \{\infty\}.$$

A sheaf F is called stable/ semistable if for all proper subsheaves $G \subset F$, we have $\mu(G) < (\leqslant) \mu(F)$. Also, any sheaf F has a unique Harder-Narasimhan filtration, that is, a filtration by subsheaves such that the quotients are semistable of strictly increasing slope. Define the category C_μ as the full subcategory of $\mathrm{Coh}\,(X)$ of semistable sheaves of slope μ [1]; one can show C_μ is an abelian subcategory of $\mathrm{Coh}\,(X)$. For two slopes $b < a$, we define $C[b, a]$ to be the full subcategory of $\mathrm{Coh}\,(X)$ whose objects are the elements of C_μ for $b \leqslant \mu \leqslant a$.

Theorem 3.1. *(Atiyah) The following hold:*

(i) the Harder-Narasimhan filtration of any coherent sheaf splits (non canonically). In particular, any indecomposable coherent sheaf is stable.

(ii) the stable sheaves of slope μ are the simple objects in the category C_μ.

(iii) there exist equivalences (canonical using the chosen rational point on the elliptic curve X) of abelian categories

$$\varepsilon_{a,b} : C_a \to C_b$$

for any $a, b \in \mathbb{Q} \cup \{\infty\}$.

To define the extended Hall algebra, we added as a degree zero part the algebra of $K_0(X)$. In our case, the symmetrized Euler form vanishes, so it is not necessary to add this part to the Hall algebra. Also, it is preferable in some situations to work with the numerical $K-$group, which is finitely generated:

$$K_0(X) \to K_0'(X) = \mathbb{Z}^2,$$

which sends a sheaf F to $(\mathrm{rank}\,(F), \deg(F))$.

Recall from last time the discussion about derived equivalences of $D^b\mathrm{Coh}\,(X)$: examples include the shift functor [1], automorphism induced by automorphisms of the curve X itself, tensoring with line bundles, and the Seidel-Thomas autoequivalences. The Seidel-Thomas equivalences are defined as follows: given a spherical object object $E \in \mathrm{Coh}\,(X)$, namely a sheaf E with $\mathrm{Hom}\,(E,E) = \mathrm{Hom}\,(E, E[1]) = k$, define $T_E : D(X) \to D(X)$ by

$$T_E(F) = \mathrm{cone}\,(RHom(E,F) \otimes E \to F).$$

The autoequivalences $T_{\mathcal{O}}$ and $T_{\mathcal{O}_\S}$ satisfy the braid group relation

$$T_{\mathcal{O}_x} T_{\mathcal{O}} T_{\mathcal{O}_x} = T_{\mathcal{O}} T_{\mathcal{O}_x} T_{\mathcal{O}}.$$

The group generated by $T_{\mathcal{O}}, T_{\mathcal{O}_x}$, and [1] is the universal covering $\widetilde{SL(2,\mathbb{Z})}$ of $SL(2,\mathbb{Z})$ [1]. The only derived equivalences we will be interested in will be elements of this group. It is important to keep in mind that given a spherical object E, the autoequivalence T_E descends to an automorphism of $K_0'(X) = \mathbb{Z}^2$, and that this automorphism can be written explicitly, see [1][page 1177] for more details.

The image of the abelian category $\mathrm{Coh}\,(X)$ under a Seidel-Thomas derived equivalence T_E, for E a spherical object, is given by a tilted heart $\mathrm{Coh}_v(X)$; we will first state the definition of $\mathrm{Coh}_v(X)$ and then we will explain how to compute v from E. Recall that $\mathrm{Coh}_{\leqslant v}(X)$ is the full subcategory of $\mathrm{Coh}\,(X)$ consisting of sheaves F whose all direct summands have slope $\leqslant v$; $\mathrm{Coh}_{>v}(X)$ is defined in a similar manner. Now, the tilted heart $\mathrm{Coh}_v(X)$ is the full subcategory of $D^b\mathrm{Coh}\,(X)$ with objects the complexes $F \oplus G[1]$, where $F \in \mathrm{Coh}_{>v}(X)$ and $G \in \mathrm{Coh}_{\leqslant v}(X)$. For a spherical sheaf E of class $(r,d) \in K_0'(X)$ and of slope $\mu = \frac{d}{r}$, the autoequivalence T_E establishes an autoequivalence between $\mathrm{Coh}\,(X)$ and $\mathrm{Coh}_v(X)$ where $v = -\infty$ for $\mu = \infty$ and $v = \mu - \frac{1}{r^2}$ otherwise.

We have mentioned earlier that the most naive approach from a geometrical point of view for doubling the Hall algebra in order to pass from the positive part of a quantum group to the full quantum group is to look for a double version (in some way) of the abelian category $\mathcal{A}$. For this purpose, define the root category $\mathbf{R}_X$ as the orbit category $\mathbf{R}_X = D^b(X)/[2]$; $\mathbf{R}_X$ has a triangulated structure [4]. One can think of $\mathbf{R}_X$ as the category of $2-$periodic complexes of coherent sheaves on X. For any object $A \in \mathrm{Coh}\,(X)$, we denote by A^+ the image of A in $\mathbf{R}_X$ and by A^- the image of $A[-1]$ in $\mathbf{R}_X$. We define the semistable objects of $\mathbf{R}_X$ to be the objects $A^\pm$, where A is semistable in $\mathrm{Coh}\,(X)$. The action of the group $\widetilde{SL(2,\mathbb{Z})}$ breaks to an action of $SL(2,\mathbb{Z}) \times \mathbb{Z}/2\mathbb{Z}$ in $\mathbf{R}_X$ because $[1]^2 = \mathrm{id}$ in $\mathbf{R}_X$. From now on, whenever we refer to the action of $SL(2,\mathbb{Z})$ on $\mathbf{R}_X$, we refer to this particular action. Using tilted hearts, one can prove that the set of semistable objects of the root category $\mathbf{R}_X$ is invariant under the action of $SL(2, \mathbb{Z})$.

3.2. **A PBW theorem for the full Hall algebra.** The algebra we are interested in, the elliptic Hall algebra, will be defined as a subalgebra of the Hall algebra of X. One of the main theorems we will prove about the EHA is a PBW theorem. Before doing it, we need a PBW theorem for $\mathbb{H}_X$ which is significantly easier to prove.

First, we need to fix some notation. Let $\mathbb{H}_X^a \subset \mathbb{H}_X$ be the subspace spanned by classes of sheaves $F \in C_a$. The category C_a is stable under extensions, thus $\mathbb{H}_X^a$ is a subalgebra of the Hall algebra. Further, Atiyah's theorem provides algebra isomorphisms

$$\varepsilon_{a,b} : \mathbb{H}_X^b \to \mathbb{H}_X^a.$$

Proposition 3.2. *(PBW theorem for the Hall algebra) The multiplication map*

$$m : \otimes_a \mathbb{H}_X^a \to \mathbb{H}_X$$

is an isomorphism.

Proof. We have that $\mathrm{Hom}\,(G, F) = \mathrm{Ext}\,(F, G) = 0$ for $F \in C_a$ and $G \in C_b$, where $a < b$. Thus, up to a power of v, we have that

$$[F_1] \cdots [F_r] = [F_1 \oplus \cdots \oplus F_r],$$

where $F_i \in C_{a_i}$ and $a_1 < \cdots < a_r$. Any sheaf can be decomposed in such a direct sum of semistable sheaves by Theorem 3.1.1, so the multiplication map is surjective. To show injectivity, pick a finite sum S with the minimal number of terms possible which sums up to zero $\sum_{\mu(F_1)<\cdots<\mu(F_a)} c[F_1] \cdots [F_a] = 0$, where all the sheaves F_i appearing in the sum are semistable; then $\sum c'[F_1 \sum \cdots \sum F_a] = 0 \in \mathbb{H}_X$. Assume that the not all coefficients are zero. Let F be a sheaf with maximal slope appearing in the sum, and which is not contained in any other sheaf G appearing in the sum. All the terms $[F_1 \cdots \mathcal{F}_a]$ that are equal, up to a constant, to a term that contains the sheaf F must have the last term $F_a = F$. Then $\sum_{F_a=F} c[F_1] \cdots [F_{a-1}] = 0$, which by the minimality of the chosen sum implies that all the coefficients in the sum $\sum_{F_a=F} c[F_1] \cdots [F_{a-1}] = 0$ are zero, and thus that $[F]$ appears with coefficient zero in the sum S, which contradicts our assumption on $[F]$. Thus m is injective and thus an isomorphism. $\square$

3.3. **Drinfeld double of the Hall algebra.** The Drinfeld double of the Hall algebra carries more symmetries than the Hall algebra alone− for example, the derived equivalences of $D(X)$ give algebra automorphisms of the Drinfeld double $\mathbb{DH}_X$. As we have already seen in the previous talks, this symmetry can be used reduce general statements to simpler ones, see the proof of Theorem 3.10.

Before we start proving that derived equivalences give algebra automorphisms of $\mathbb{DH}_X$, we want to find generators and relations for $\mathbb{DH}_X$. We can actually phrase all the relations in terms of semistable sheaves only.

Proposition 3.3. *The Hall algebra $\mathbb{H}_X$ is isomorphic to the $K-$algebra generated by $\{x_F | F \text{ semistable}\}$ subject to the relations $P([F],[G])$:*

$$x_F x_G = v^{-\chi(F,G)} \sum_H c_{F,G}^H \underline{x}_H,$$

where F and G are semistable, and $\underline{x}_H := v^{\sum_{i<j}\langle H_i,H_j\rangle} x_{H_1} \cdots x_{H_r}$, where $H = \oplus H_i$ is the Harder Narasimhan decomposition of H.

We can also rephrase the Drinfeld double relations $R([F],[G])$ in fairly explicit terms. We will not write all these relations, see [1], but here is an example: suppose F is $a-$semistable and G is $b-$semistable and $a < b$. The relation $R([F],[G])$ becomes

$$[F]^-[G]^+ = v^{\langle F,G\rangle} \sum_{B,C} v^{-\langle C,B\rangle} c_{F[1],G}^{B[1]\oplus C} [C]^+[B]^-,$$

where $c_{F[1],G}^{B[1]\oplus C}$ is the number of distinguished triangles

$$G \to B[1] \oplus C \to F[1].$$

In our case, the relation for $a = b$ says that the two halves commute, and this will be used in the proof of Theorem 3.10

Theorem 3.4. *Let Φ be an autoequivalence of $D(X)$ in the group $\widetilde{SL(2,\mathbb{Z})}$. The assignment $[F] \to [\Phi F]$ for F semistable object of $\mathbf{R}_X$ extends to an algebra automorphism of $\mathbb{DH}_X$.*

Proof. The algebra $\mathbb{DH}_X$ is isomorphic to the $K-$algebra generated by $[F]^+$ and $[F]^-$, where F is a semistable sheaf, subject to:

(i) the Hall algebra relations $P([F]^+,[G]^+)$,

(ii) the Drinfeld double relations $R([F],[G])$.

Le F be $a-$semistable and G be $b-$semistable, and Φ a derived equivalence in $SL(2,\mathbb{Z})$. Denote by $\tilde{F}[i]$ and $\tilde{G}[j]$ their images under Φ, and we can assume that i and j are both 0 or -1. The sheaves $\tilde{F}$ and $\tilde{G}$ are semistable, of slopes $\tilde{a}$ and $\tilde{b}$, respectively. We need to check that the relations $P([\tilde{F}],[\tilde{G}])$ and $R([\tilde{F}],[\tilde{G}])$ hold in $\mathbb{DH}_X$. It is clear that after applying the autoequivalence [1] the sheaves will continue to satisfy the two relations.

(1) Assume that $i = j$. Assume that $a > b$; then $\tilde{a} > \tilde{b}$. We thus get an isomorphism of Hall algebras for exact categories

$$\Phi : C[b, a] \to C[\tilde{b}, \tilde{a}]$$

preserving all the Hall algebra constants, and thus the relations $P([F], [G])$ are mapped to $P([\tilde{F}], [\tilde{G}])$.

(2) Assume that i and j are different, and say for simplicity that i is odd and j is even. Also assume that $a > b$. This case can only happen for $E = \mathcal{O}_x$. Once again, an easy computation shows that $\tilde{a} < \tilde{b}$. By the description of the derived equivalences via tilted hearts, there exists an integer $b \leqslant k < a$ such that

$$\Phi(C_\phi) \in \mathrm{Coh}\,(X)[i]$$

for $k < \phi \leqslant a$ and

$$\Phi(C_\phi) \in \mathrm{Coh}\,(X)[i-1]$$

for $b \leqslant \phi \leqslant k$.

Split any extension H of G by F as $H = H_0 \oplus H_1$, where H_0 has all semistable factors with slopes $\leqslant k$, and H_1 has all direct summands with slopes $> k$. Relation $P([F], [G])$ becomes

$$[F][G] = v^{-\langle F,G\rangle} \sum_{H_0,H_1} c_{F,G}^{H_0\oplus H_1} v^{\langle H_0,H_1\rangle}[H_0][H_1].$$

Now, $\Phi(F) = \tilde{F}[i], \Phi(H_1) = \tilde{H}_1[i-1], \Phi(G) = \tilde{G}[i]$, and $\Phi(H_0) = \tilde{H}_0[i]$. We also have $\langle F, G\rangle = -\langle \tilde{F}, \tilde{G}\rangle$ and $\langle H_0, H_1\rangle = -\langle \tilde{H}_0, \tilde{H}_1\rangle$.

The relation $P(\Phi(F), \Phi(G))$ we need to prove is actually

$$[\tilde{F}]^-[\tilde{G}]^+ = v^{\langle \tilde{F},\tilde{G}\rangle} \sum_{\tilde{H}_0,\tilde{H}_1} v^{-\langle \tilde{H}_0,\tilde{H}_1\rangle} c_{\tilde{F}[1],\tilde{G}}^{\tilde{H}_0\oplus\tilde{H}_1[1]}[\tilde{H}_0]^+[\tilde{H}_1]^-.$$

The equality to be proven is simply the Drinfeld double relation for $\tilde{F}$ and $\tilde{G}$.

The relations $R([\tilde{F}], [\tilde{G}])$ can be checked in a similar manner. $\square$

Corollary 3.5. *The universal cover* $\widetilde{SL}(2, \mathbb{Z})$ *acts by algebra automorphisms on* $\mathbb{DH}_X$.

3.4. **The Elliptic Hall Algebra.** Recall the elements $T_r^{(\infty)} = \sum T_{r,x}^{(\infty)}$ defined in section 2.3, $T_r^{(\infty)} \in \mathbb{H}_X^{(\infty)}$. For arbitrary $\mu \in \mathbb{Q}$, define

$$T_r^{(\mu)} = \varepsilon_{\mu,\infty}(T_r^{(\infty)}).$$

Observe that $\varepsilon_{a,b}(T_r^{(b)}) = T_r^{(a)}$, for any slopes a and b.

Definition 3.6. Let $\mathbb{U}_X^+ \subset \mathbb{H}_X^+$ be the subalgebra generated by all elements $T_r^{(\mu)}$, for $r \geqslant 1$ and $\mu \in \mathbb{Q} \cup \{\infty\}$. Define $\mathbb{U}_X^- \subset \mathbb{H}_X^-$ similarly and let $\mathbb{U}_X \subset \mathbb{DH}_X$ be the subalgebra generated by $\mathbb{U}_X^+$ and $\mathbb{U}_X^-$.

We will introduce different notation for the generators of $\mathbb{U}_X$: for $\mu = \frac{l}{n}$ with l and n relatively prime, $n \geqslant 1$, write $T_{(\pm rn, \pm rl)} = (T_r^{(\mu)})^{\pm} \in \mathbb{U}_X^+$, $T_{(0,r)} = (T_r^{(\infty)})^+, T_{(0,0)} = 1$. Also define $(\mathbb{Z}^2)^{\pm} = \{(q,p) \in \mathbb{Z}^2 | \pm q > 0 \text{ or } \pm q = 0, p > 0\}$ and similarly for the minus half. Then $\mathbb{U}_X^{\pm}$ is the subalgebra of $\mathbb{DH}_X$ generated by $T_{(q,p)}$ for $(q,p) \in (\mathbb{Z}^2)^{\pm}$.

The $\widetilde{SL}(2,\mathbb{Z})$ action on $\mathbb{DH}_X$ preserves $\mathbb{U}_X$. This action factors through $SL(2,\mathbb{Z})$, as mentioned earlier, and for $\gamma \in SL(2,\mathbb{Z})$ we have $\gamma T_{(p,q)} = T_{\gamma(p,q)}$.

Finally, in some situations it will be more convenient to use another set of generators for the algebra $\mathbb{U}_X$. For $a \in (\mathbb{Z}^2)^+$ define

$$1_a^{ss} = \sum_{H \text{sstable slope } a} [H] \in \mathbb{H}_X^+[a].$$

For $a = (q,p)$ with q and p relatively prime, a computation using torsion sheaves and Macdonald's theorem shows that

$$1 + \sum_{r \geqslant 1} 1_{ra}^{ss} s^r = \exp\left(\sum_{r \geqslant 1} \frac{T^{ra}}{[r]} s^r\right).$$

This makes transparent that the elements 1_a^{ss} with $a \in (\mathbb{Z}^2)^{\pm}$ generate $\mathbb{U}_X^{\pm}$. In the next part, we list some results about the generators of $\mathbb{U}_X$ that will be used later in the talk. We do not provide (complete) proofs, which can be found in [1]:

(1) If $a = (p,q)$ with p and q coprime, then $T_a = 1_a^{ss}$. This is immediate from the above identity.
(2) Macdonald's theorem implies that $\Delta(T_{(0,n)}) = T_{(0,n)} \otimes 1 + 1 \otimes T_{(0,n)}$.
(3) If we define $1_a = \sum_{F \text{ of class } a} [F] \in \mathbb{H}_X$, we have that $1_{(0,l)} = 1_{(0,l)}^{ss}$ and $1_{(1,l)} = \sum_{F=(1,l)} [F] = 1_{(1,l)}^{ss} + \sum_{d>0} v^d 1_{(1,l-d)}^{ss} 1_{(0,d)}$ as any coherent sheaf on X of rank one splits uniquely as the sum of a line bundle and of a torsion sheaf.
(4) $\Delta_{a,b}(1_{a+b}) = v^{\langle a,b \rangle} 1_a \otimes 1_b$. This is a general result for Hall algebras of abelian categories of gldim $\leqslant 1$. Here, $\Delta_{a,b} : \mathbb{U}_X[a+b] \to \mathbb{U}_X[a] \times \mathbb{U}_X[b]$ is the (a,b) component of the coproduct Δ.

The next two results will be used in the proof of Theorem 3.10 which gives generators and relations for the EHA. We denote by $c_i(X) = \frac{|X(\mathbb{F}_{q^i})| v^i [i]}{i}$.

(5) Let $x = (q, p) \in (\mathbb{Z}^2)^+$, and define $r = \gcd(p, q)$. Then

$$(T_x, T_x) = \frac{c_r(X)}{v^{-1} - v}.$$

One can use the $SL(2, \mathbb{Z})$ action and reduce the computation to the case $x = (0, r)$. By Macdonald's theorem, one can compute explicitly $(T_{r,x}^{(\infty)}, T_{r,x}^{(\infty)}) = \frac{v^r[r]d}{r(v^{-1}-v)}$. Also, recall that Macdonalds theorem also says that $T_{r,x}^{(\infty)}$ are orthogonal to each other.

This computation is used in proving the next result:

(6) For any $n \geqslant 0$ and any $a = (r, d) \in (\mathbb{Z}^2)^+$ we have

$$[T_{(0,n)}, 1_a] = c_n(X) \frac{v^{rn} - v^{-rn}}{v - v^{-1}} 1_{a+(0,n)}.$$

We will only say a few words about the argument, a full proof can be found in [1][Lemma 4.11]. One introduces the elements $1^{vec} = \sum_{F \text{ v.b. class } a} [F] \in \mathbb{U}_X^+$. Because 1_a can be written explicitly in function of 1_a^{vec}, using identities similar to the ones in item (3), one reduces the above statement to the one where 1_a is replaced by 1_a^{vec}. The first part of the proof is showing that $[T_{(0,n)}, 1_a] \in \mathbb{H}_X^{vec}$.

Next, one computes directly the coefficient of a vector bundle $[V]$ in the commutator $[[T], 1_a^{vec}]$, where T is a torsion sheaf. The answer ends up depending on T and the rank of V only. This implies that $[T_{(0,n)}, 1_a^{vec}] = u_r 1_a^{vec}$, where u_r is a constant depending on r only.

The general case can be reduced to the rank one case. In the rank one case, by the $SL(2, \mathbb{Z})$ action we can assume that $a = (1, 0)$. Then u_1 is computed by expressing the scalar product $(T_{(0,n)} T_{(1,0)}, 1_{(1,n)})$ in two different ways, one using the Hopf pairing and item (4), and one using item (3).

3.5. **A PBW theorem for the EHA.** Recall the PBW decomposition we have obtained for the Hall algebra in Section 3.2: the multiplication map induces isomorphisms $\otimes_a \mathbb{H}_X^{a,\pm} \to \mathbb{H}_X^{\pm}$ and $\otimes_a \mathbb{H}_X^{a,+} \otimes \otimes_a \mathbb{H}_X^{a,-} \to \mathbb{H}_X$. In this section, we prove the analogous theorem for the EHA.

Theorem 3.7. *(Burban-Schiffmann) The multiplication map induces isomorphisms of $K-$vector spaces $\otimes_a \mathbb{U}_X^{a,\pm} \to \mathbb{U}_X^{\pm}$ and $\otimes_a \mathbb{U}_X^{a,+} \otimes \otimes_a \mathbb{U}_X^{a,-} \to \mathbb{U}_X$.*

Proof. Let $\mathbb{H}_X^{vec} = m(\otimes_{a<\infty} \mathbb{H}_X^{a,+})$ be the subspace generated by classes of vector bundles.

Claim: $\mathbb{U}_X^+ \subset \mathbb{H}_X^{vec} \otimes \mathbb{U}_X^{\infty}$.

Let's assume the claim for the moment. For any slope a, there exists $c \in SL(2,\mathbb{Z})$ such that $c(a) = \infty$. Recall that $SL(2,\mathbb{Z})$ acts on $\mathbb{DH}_X$, preserves the subalgebra $\mathbb{U}_X$, and permutes the factors $\mathbb{H}_X^{a,+}$. Using the claim, we obtain:

$$c(\mathbb{U}_X^+) \subset \mathbb{U}_X^+ \otimes \mathbb{U}_X^- \subset (\otimes_{a<\infty}\mathbb{H}_X^{a,+} \otimes \mathbb{U}_X^{\infty,+}) \otimes (\otimes_{a<\infty}\mathbb{H}_X^{a,-} \otimes \mathbb{U}_X^{\infty,-}),$$

from which we deduce, after applying c^{-1}, that

$$\mathbb{U}_X^+ \subset \otimes_{\mu<a}\mathbb{H}_X^{\mu,+} \otimes \mathbb{U}_X^{a,+} \otimes \otimes_{\mu>a}\mathbb{U}_X^{a,+}.$$

This is true for all slopes a, and thus

$$\mathbb{U}_X^+ \subset \otimes_a \mathbb{U}_X^{a,+},$$

as desired.

We still need to establish the above claim in order to finish the proof. Consider an element $u \in \mathbb{U}_X^+[a]$, and expand it as $u = \sum_l u_l$, where $u_l = \sum_i u'_{l,i}u''_{l,i}$, where $u'_{l,i} \in \mathbb{H}_X^{vec}$ and $u''_{l,i} \in \mathbb{H}_X^{(\infty)}[(0,l)]$ for all i and l. Denote by $\pi : \mathbb{H}_X \to \mathbb{H}_X^{vec}$. We can compute that

$$(\pi \otimes 1)\Delta_{a-(0,l),(0,l)}(u) = v^{\langle a-(0,l),(0,l)\rangle}u_l.$$

On the other hand, the fact above tells us that

$$\Delta_{a-(0,l),(0,l)}(u) \in \mathbb{U}_X^+[a-(0,l)] \otimes \mathbb{U}_X^+[(0,l)].$$

From these two relations we deduce that $u_l \in \mathbb{H}_X^{vec} \otimes \mathbb{U}_X^+[(0,l)]$, and thus that u has the claimed property.

$\square$

3.6. **EHA via generators and relations.** Once we have proven the PBW theorem, we can start identifying the algebra $\mathbb{U}_X$ defined in these notes with the EHA defined by generators and relations as defined in [2]. To differentiate between the two algebras, we will call the latter $\mathcal{E}$. Let's review the definition of $\mathcal{E}$.

Let o be the origin in $\mathbb{Z}^2$, and let Conv$'$ be the set of all convex paths $p = (x_1, \cdots, x_r)$ satisfying $\angle x_1L_0 \geqslant \cdots \geqslant \angle x_rL_o \geqslant 0$; here, the notation $\angle x_1L_0$ means the angle between the vector ox_1 and the vector L_0 which joins the origin o and the point $(0,-1)$. Two convex paths p and q are equivalent if they are obtained from each other by a permutation of their edges. Let Conv be the set of equivalence classes of convex paths Conv$'$, Conv$^+$ its subset of convex paths with all angles $\geqslant \pi$, and Conv$^-$ its subset with all paths $< \pi$. Concatenation gives an identification

$$\text{Conv} = \text{Conv}^+ \times \text{Conv}^-.$$

The PBW theorem proved in the previous section says that the elements $\{T_p | p \in \text{Conv}^\pm\}$ are a $K-$basis of $\mathbb{U}_X^\pm$, where to a path $p = (x_1, \cdots, x_r)$ we associate the element $T_p := T_{x_1} \cdots T_{x_r} \in \mathbb{U}_X$. For $x \in \mathbb{Z}^2 - o$, define $\deg(x) := \gcd(p, q) \in \mathbb{N}$. Also, for $x, y \in \mathbb{Z}^2 - o$ noncollinear, denote by $\varepsilon(x, y) = \mathrm{sign}(\det(x, y)) \in \{-1, 1\}$.

Definition 3.8. Fix $\sigma, \bar{\sigma} \in \mathbb{C} - \{0, -1, 1\}$, and let $v := (\sigma\bar{\sigma})^{-1/2}$ and $c_i(\sigma, \bar{\sigma}) = (\sigma^{i/2} - \sigma^{-i/2})(\bar{\sigma}^{i/2} - \bar{\sigma}^{-i/2})\frac{[i]_v}{i}$.

Let $\mathcal{E}_{\sigma,\bar{\sigma}}$ be the $\mathbb{C}-$algebra generated by $\{t_x | x \in \mathbb{Z}^2 - o\}$ modulo the following relations:

(1) if x and y belong to the same line in $\mathbb{Z}^2$, then $[t_x, t_y] = 0$,

(2) if x is of degree one and y is another nonzero lattice point such that $\Delta(x, y)$ has no interior lattice points, then

$$[t_y, t_x] = \varepsilon(x, y) c_{\deg(y)}(\sigma, \bar{\sigma}) \frac{\theta_{x+y}}{v^{-1} - v},$$

where the elements θ_z, for $z \in \mathbb{Z}^2 - o$, are defined as follows:

$$\sum_i \theta_{ix_0} s^i = \exp((v^{-1} - v) \sum_{t \geqslant 1} t_{rx_0} s^r),$$

for any degree one element $x_0 \in \mathbb{Z}^2 - o$.

We also denote by $\mathcal{E}^{\pm}_{\sigma,\bar{\sigma}}$ the subalgebras generated by t_x with $x \in (\mathbb{Z}^2)^{\pm}$.

Let X be an elliptic curve over $\mathbb{F}_{q^r}$. Then by Hasse's theorem there exist complex numbers σ and $\bar{\sigma}$ such that $\sigma\bar{\sigma} = q$, and

$$|X(\mathbb{F}_{q^r})| = q^r + 1 - (\sigma^r + \bar{\sigma}^r).$$

Observe that $c_i(\sigma, \bar{\sigma}) = \frac{v^i [i] |X(\mathbb{F}_{q^i})|}{i} = c_i(X)$.

Theorem 3.9. *(Burban-Schiffmann)*

The assignment $\Omega : t_x \to T_x$ for all $x \in \mathbb{Z}^2 - o$ extends to an isomorphism

$$\Omega : \mathcal{E}_{\sigma,\bar{\sigma}} \to \mathbb{U}_X \otimes_K \mathbb{C}.$$

Proof. The proof consists of three parts. First, we need to check that Ω is well defined, that is, we need to check that the generators T_x of the Hall algebra $\mathbb{U}_X$ satisfy relations (1) and (2) from definition 3.9. The second part is a counting dimensions argument which will show that Ω restricts to isomorphisms to both halves. One can use their inverses to construct a map of vector spaces

$$\Omega^{-1} : \mathbb{U}_X \otimes_K \mathbb{C} \to \mathcal{E}_{\sigma,\bar{\sigma}},$$

which restricts to algebra isomorphisms on both halves. The third part is checking that Ω^{-1} is well-defined, that is, that the Drinfeld double relations hold in $\mathcal{E}_{\sigma,\bar{\sigma}}$.

The first step is checking that relations (1) and (2) in definition 3.9 hold for $\mathbb{U}_X$. For (1), by the $SL(2, \mathbb{Z})$ invariance of both algebras, we can assume that $x = (0, r)$

and $y = (0,s)$, Then the relation follows because $\mathbb{H}_X^{(\infty)}$ is commutative [1]. For relation (2), since $\deg(x) = 1$, we cannot have both $\deg(y)$ and $\deg(x+y)$ equal to 2 by arithmetic reasons, or one of them $\geqslant 3$ and the other $\geqslant 2$ by an application of Pick's theorem. Thus we have either $\deg(x+y) = 1$ or $\deg(y) = 1$. We only discuss the first possibility. Using the $SL(2,\mathbb{Z})$ action, we can assume $x = (1,0)$, and if $\det(x,y) = r$, we can further assume $y = (s,r)$ with $0 \leqslant s < r$. Because there are no lattice points in $\Delta(x,y)$, we deduce that $y = (0,r)$. We thus need to check (2) for $T_{(1,0)}$ and $T_{(0,r)}$. The relation we need to prove is

$$[T_{(0,r)}, T_{(1,0)}] = c_r(\sigma, \bar{\sigma}) \frac{\theta_{(1,r)}}{v^{-1} - v}.$$

Observe that $\theta_{(1,r)} = (v^{-1} - v) t_{(1,r)}$, and thus the relation we need to prove becomes

$$[T_{(0,r)}, T_{(1,0)}] = c_r T_{(1,r)},$$

which is exactly relation (6) in section 3.4. This implies that Ω is extends to a surjective $SL(2,\mathbb{Z})$ equivariant algebra morphism.

By the PBW theorem for the algebras $\mathcal{E}^{+-}_{\sigma,\bar{\sigma}}$, we deduce that Ω restricts to isomorphisms on both halves of these two algebras. Denote their inverses by

$$\Omega_{\pm}^{-1} : \mathbb{U}_X^{\pm} \otimes_K \mathbb{C} \to \mathcal{E}^{\pm}_{\sigma,\bar{\sigma}}.$$

Recall that the algebra $\mathbb{U}_X$ is generated by the two halves $\mathbb{U}_X^{\pm}$ modulo the Drinfeld double relations $R(h,g)$, for $h,g \in \mathbb{U}_X^{+}$ both classes of semistable sheaves. We need to check that these relations hold in $\mathcal{E}_{\sigma,\bar{\sigma}}$ in order to conclude that Ω^{-1} is an algebra morphism.

There are two cases to consider. The first one is when the slopes of g and h are equal to μ. With some work, one can see that the relation $R(h,g)$ says that the algebras $\mathbb{U}_X^{+,\mu}$ and $\mathbb{U}_X^{-,\mu}$ commute, which is true in $\mathcal{E}_{\sigma,\bar{\sigma}}$ by relation (1). The second case is when the slopes are different. There exists an element $c \in SL(2,\mathbb{Z})$, such that $c(g) \in \mathbb{U}_X^{+}$ and $c(h) \in \mathbb{U}_X^{-}$. Once again, we want to show that the relation $\Omega^{-1}(R(h,g))$ holds in $\mathcal{E}$. The relation $R(h,g)$ holds in $\mathbb{U}_X$; applying c we obtain that the relation $cR(h,g)$ in $\mathbb{U}_X^{+}$. Because Ω is an algebra isomorphism on the two halves, we obtain the relation $\Omega^{-1}(cR(h,g))$ in $\mathcal{E}^{+}$. By the $SL(2,\mathbb{Z})-$equivariance property of $\mathcal{E}$, we obtain the relation $c\Omega^{-1}(R(h,g))$ in $\mathcal{E}^{+}$, and, consequently, the relation $\Omega^{-1}(R(h,g))$ in $\mathcal{E}$, which is what we wanted to prove.

□

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SHUFFLE ALGEBRA VS EHA

BORIS TSVELIKHOVSKY

Abstract. These are the notes for a talk at the MIT-Northeastern seminar for graduate students on Double Affine Hecke Algebras and Elliptic Hall Algebras, Spring 2017.

Contents

Date:

1. Introduction

The shuffle algebras were introduced by Feigin and Odesskii. These algebras are unital associative subalgebras of $\mathbb{C} \bigoplus_{n\in\mathbb{Z}_{>0}} \mathbb{C}(z_1,\ldots,z_n)^{\mathfrak{S}_n}$ with multiplication defined by

$$f(z_1,\ldots,z_n) * g(z_1,\ldots,z_m) := \mathrm{Sym}\left(f(z_1,\ldots,z_n) g(z_{n+1},\ldots,z_{n+m}) \prod_{\substack{i\in\{1,\ldots n\}\\ j\in\{n+1,\ldots n+m\}}} \mu\left(\frac{z_i}{z_j}\right)\right),$$

for some function μ. In the work of Schiffmann and Vasserot [SV] it was shown that subalgebras in Hall algebras of vector bundles of smooth projective curves generated by $\mathbf{1}_{Pic^d(X)} := \sum_{\mathcal{L}\in Pic^d(X)} \mathcal{L}$, where $Pic^d(X)$ is the set of line bundles over X of degree d are isomorphic to subalgebras of S, generated by elements of degree 1. For a smooth projective curve of genus g, one takes $\mu_g(x) = x^{g-1}\frac{1-qx}{1-x^{-1}}\prod_{i=1}^{g}(1-\alpha_i x^{-1})(1-\overline{\alpha_i}x^{-1})$, where α_i are the roots of the numerator of the zeta function of the curve, i.e. $\zeta_X(t) = \exp(\sum_{d\geq 1} \#X(\mathbb{F}_{q^d})\frac{t^d}{d}) = \frac{\prod_{i=1}^{g}(x-\alpha_i)(x-\overline{\alpha_i})}{(1-t)(1-qt)}$. This isomorphism was made more explicit in case of elliptic curves (elliptic Hall algebras) in [Neg]. The latter paper will be the primary reference for this talk.

. The action of shuffle algebra on the sum of localized equivariant K-groups (with respect to the $T = \mathbb{C}^* \times \mathbb{C}^*$ induced from the action on $\mathbb{C}^2$) of Hilbert schemes of points on $\mathbb{C}^2$ was provided in [FT].

In Section 2 we show that the Hall algebra of locally free sheaves (vector bundles) on projective line is isomorphic to shuffle algebra with $\mu(x) = x^{-1}\frac{1-qx}{1-x^{-1}}$ and formulate the general isomorphism of Schiffmann and Vasserot.

Section 3 recalls the definition and basic properties of the elliptic Hall algebra (EHA) and Sections 4 and 5 are devoted to speculations on the isomorphism of EHA and the corresponding shuffle algebra and their Drinfeld doubles.

2. Shuffle Algebras vs Hall Algebras of Genus g Curves

2.1. **Hall Algebra of the Projective Line.** The goal of this section is to show that the Hall algebra of vector bundles on $\mathbb{P}^1(\mathbb{F}_q)$ is isomorphic to the shuffle algebra S_q with the function $\mu(x) = \frac{1-qx}{x-1}$.

Proposition 2.1. *The shuffle algebra S_q is generated by $\bigoplus_{d\in\mathbb{Z}} S_{d,1}$, the ideal of relations is generated by the following relations of degree 2 (in $\bigoplus_{d\in\mathbb{Z}} S_{d,2}$)*

$$z^{m+1} * z^n - qz^n * z^{m+1} = qz^m * z^{n+1} - z^{n+1} * z^m \tag{2.1}$$

Proof. We notice that the vector space generated by monomials $z^{i_1} * \ldots * z^{i_d}, i_j \in \mathbb{Z}$ forms an ideal inside $\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{\mathfrak{S}_n}$. Indeed, $f(z_1, \ldots, z_n) z^{\alpha_1} * \ldots * z^{\alpha_n} = z^{\alpha_1 + i_1} * \ldots * z^{\alpha_n + i_n}$, where $f(z_1, \ldots, z_n) = Sym(z^{i_1} * \ldots * z^{i_n})$ is in $\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{\mathfrak{S}_n}$.

The next step is to show that the ideal $\underbrace{\bigoplus_{d \in \mathbb{Z}} S_{d,1} * \ldots * \bigoplus_{d \in \mathbb{Z}} S_{d,1}}_{n}$ has no common zeros and therefore must coincide with S_n (we use that $S_n = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]^{\mathfrak{S}_n}$ is finitely generated over $\mathbb{C}$ and, therefore, any maximal ideal must be vanishing at a point). Suppose that all functions from the ideal vanished at a point with coordinates $(\alpha_1, \ldots, \alpha_n)$. It is not hard to show that this implies that there exist $\alpha_{j_1} = q\alpha_{j_2} = \ldots = q\alpha_{j_k} = q\alpha_{j_1}$, i.e. $\alpha_{j_1} = q^k \alpha_{j_1}$. But neither $\alpha_{j_1} = 0$ nor $q^k = 1$. This completes the proof of the first claim.

. **Exercise.** Alternatively, show that $\underbrace{1 * 1 * \ldots * 1}_{n} = c \prod_{i=1}^{n} \frac{q^i - 1}{q - 1} = c[n]_q!$, where $c \in \mathbb{C}$ is a constant. This implies that $1 \in \underbrace{\bigoplus_{d \in \mathbb{Z}} S_{d,1} * \ldots * \bigoplus_{d \in \mathbb{Z}} S_{d,1}}_{n}$, unless q is kth root of unity ($k \leq n$).[1]

The relations (2.1) can be checked directly. They allow to rewrite every monomial $z^{i_1} * \ldots * z^{i_k}$ in such a way that $i_{m+1} \leq i_m + 1 \ \forall m \in \{1, \ldots, k\}$. Indeed, if $i_{m+1} > i_m + 1$, then using (2.1) for $z^{i_m} * z^{i_{m+1}}$, we see that the other three summands have the difference $i'_{m+1} - i'_m - 1$ strictly less than $i_{m+1} - i_m - 1$. In case $i_{m+1} = i_m + 1$, (2.1) becomes

$$q z^{i_m + 1} * z^{i_m} - z^{i_m} * z^{i_m + 1} + q z^{i_m + 1} * z^{i_m} - z^{i_m} * z^{i_m + 1} = 0$$

and allows to swap the two factors. This implies that there no relations, other then those generated by (2.1), as otherwise taking the limit $q \to 1$ we would obtain relations between monomial symmetric Laurent polynomials, which are known to be independent. □

To describe the Hall algebra $\mathcal{H}_{lf}(\mathbb{P}^1)$ of vector bundles on $\mathbb{P}^1(\mathbb{F}_q)$, we first recall the Grothendieck theorem: every vector bundle V of rank k on $\mathbb{P}^1$ splits as a sum of line bundles $V = \bigoplus_{j=1}^{n} \mathcal{O}(i_j)$. The result holds over fields of $char = p$ as well. Therefore, the only indecomposable objects are the line bundles $\mathcal{O}(i)$.

Observation. If $m \leq n + 1$, then $\mathrm{Ext}^1(\mathcal{O}(m), \mathcal{O}(n)) = 0$.

Proof. Using that $\omega_{\mathbb{P}^1} = \mathcal{O}(-2)$ and Serre duality, we conclude $\mathrm{Ext}^1(\mathcal{O}(m), \mathcal{O}(n))^* = \mathrm{Hom}(\mathcal{O}(n), \mathcal{O}(m-2))$, which is zero for $m \leq n + 1$. □

Definition. We introduce $\nu = \sqrt{q}$ and define the Euler form to be $\langle M, N \rangle = \dim(\mathrm{Hom}(M, N)) - \dim(\mathrm{Ext}^1(M, N))$. The product of two elements $[\mathcal{O}(n)]$ and $[\mathcal{O}(m)] \in \mathcal{H}_{lf}(\mathbb{P}^1)$ is defined to

[1]**Hint**: the degree of the polynomial is zero, hence, it is enough to evaluate it at a single point. One convenient choice is the point $(\xi, \xi^2, \ldots, \xi^{n-1}, 1)$, where $\xi = \sqrt[n]{1}$ is the primitive root of unity (use induction on n). Also, compare to formula (i) in Theorem 10 of [BK].

be $[\mathcal{O}(n)] * [\mathcal{O}(m)] := \nu^{\langle M,N\rangle} \sum_R P^R_{\mathcal{O}(n),\mathcal{O}(m)}[R]$, where R is a vector bundle of rank 2 and $P^R_{\mathcal{O}(n),\mathcal{O}(m)} = \frac{L^R_{\mathcal{O}(n),\mathcal{O}(m)}}{|\mathrm{Aut}\mathcal{O}(m)||\mathrm{Aut}\mathcal{O}(n)|}$ with $L^R_{\mathcal{O}(n),\mathcal{O}(m)}$ equal to the number of SES

$$0 \to \mathcal{O}(m) \to R \to \mathcal{O}(n) \to 0.$$

The next lemma shows computations of some structure constants in $\mathcal{H}_{lf}(\mathbb{P}^1)$ (see also **Theorem 10** in [BK]).

Lemma 2.2. *The following relations hold in $\mathcal{H}_{lf}(\mathbb{P}^1)$:*

$$[\mathcal{O}(n)]*[\mathcal{O}(m)] = \nu^{m-n}\left(q^{n-m+1}([\mathcal{O}(m)\oplus\mathcal{O}(n)] + \sum_{s=1}^{\lfloor\frac{n-m}{2}\rfloor}(q^2-1)q^{n-m-1}[\mathcal{O}(m+s)\oplus\mathcal{O}(n-s)])\right),\ n>m \tag{2.2}$$

Proof. We notice that for a nontrivial extension

$$0 \to \mathcal{O}(m) \to \mathcal{O}(p)\oplus\mathcal{O}(q) \to \mathcal{O}(n) \to 0$$

to exist, we must have $\min(p,q) > m$, $\max(p,q) < n$ and $p+q = m+n$, these are precisely the summands of the sum in the r.h.s of (2.2). To compute the coefficient with which $[\mathcal{O}(m+s)\oplus\mathcal{O}(n-s)]$ appears in the product, we notice that it is equal to the number of pairs of coprime polynomials of degrees s and $n-m-s$. Indeed a pair of such polynomials (φ_1,φ_2) defines a map $\psi : 0 \to \mathcal{O}(m) \to \mathcal{O}(m+s)\oplus\mathcal{O}(n-s)$ and $\mathrm{coker}(\psi)$ is locally free (has trivial support) whenever (φ_1,φ_2) are coprime. The number of such pairs of polynomials is computed in the next proposition, which completes the proof of the lemma (one also needs to use that $|\mathrm{Aut}\mathcal{O}(n)| = q-1\ \forall n\in\mathbb{Z}$). □

Proposition 2.3. *The number $\eta(a,b)$ of pairs (J,L), consisting of coprime homogeneous polynomials in $\mathbb{F}_q[x,y]$ of degrees a and b, respectively, is given by*

$$\begin{cases}\eta(a,b) = (q-1)(q^{a+b+1}-1), a=0 \text{ or } b=0\\ \eta(a,b) = (q-1)(q^2-1)q^{a+b-1}, a\geq 1 \text{ and } b\geq 1\end{cases} \tag{2.3}$$

Proof. The first assertion follows from the fact that the space of homogeneous polynomials of degree s in two variables is of dimension $s+1$ (the number of nontrivial linear combinations of vectors from the basis is $q^{s+1}-1$ and the other polynomialal is a nonzero constant).

The second claim is verified by induction on $\min(a,b)$, using that the number of all possible pairs of polynomials in $\mathbb{F}_q[x,y]$ of degrees a and b can be expressed as

$$(q^{a+1}-1)(q^{b+1}-1) = \sum_{d=0}^{\min(a,b)}\frac{q^{d+1}-1}{q-1}\eta(a-d,b-d).$$

□

Corollary 2.4. *The following relations hold in $\mathcal{H}_{lf}(\mathbb{P}^1)$:*

$$\mathcal{O}(m+1)*\mathcal{O}(n) - q\mathcal{O}(n)*\mathcal{O}(m+1) = q\mathcal{O}(m)*\mathcal{O}(n+1) - \mathcal{O}(n+1)*\mathcal{O}(m). \tag{2.4}$$

Proof. We check (2.4) for $m > n$. Then

$$\mathcal{O}(n) * \mathcal{O}(m+1) = \nu^{m-n+2}\mathcal{O}(n) \oplus \mathcal{O}(m+1);$$

$$\mathcal{O}(n+1) * \mathcal{O}(m) = \nu^{m-n}\mathcal{O}(n+1) \oplus \mathcal{O}(m);$$

$$\mathcal{O}(m)*\mathcal{O}(n+1) = \nu^{n-m+2}(q^{m-n}\mathcal{O}(m)\oplus\mathcal{O}(n+1)+ \sum_{s=1}^{\lfloor\frac{m-n-1}{2}\rfloor}(q^2-1)q^{m-n-2}[\mathcal{O}(n+s+1)\oplus\mathcal{O}(m-s)]);$$

$$\mathcal{O}(m+1)*\mathcal{O}(n) = \nu^{n-m}(q^{m-n+2}\mathcal{O}(m+1)\oplus\mathcal{O}(n)+ \sum_{s=1}^{\lfloor\frac{m-n+1}{2}\rfloor}(q^2-1)q^{m-n}[\mathcal{O}(n+s)\oplus\mathcal{O}(m-s+1)])$$

and we get the desired equality.

□

The conclusion of the section is the following result.

Theorem 2.5. *The map $\Upsilon : z^i \to \mathcal{O}(i)$ extends to an isomorphism of algebras S_q and $\mathcal{H}_{lf}(\mathbb{P}^1)$.* □

2.2. **General Result.** Let X be a smooth connected projective curve of genus g over some finite field $\mathbb{F}_q$. Let $\zeta_X(t) \in \mathbb{C}(t)$ be its zeta function:

$$\zeta_X(t) = \exp(\sum_{d\geq 1}\#X(\mathbb{F}_{q^d})\frac{t^d}{d}).$$

Example. For a smooth elliptic curve E we have

$$\zeta_E(t) = \frac{1-a_q(E)t+qt^2}{(1-t)(1-qt)}.$$

. It is known that $\zeta_X(t) = \frac{\prod_{i=1}^{g}(1-\alpha_i t)(1-\overline{\alpha_i}t)}{(1-t)(1-qt)}$ is a rational function of t and the roots α_i of the polynomial in the numerator are such that $|\alpha_i| = q^{\frac{1}{2}}$, so $\alpha_i\overline{\alpha_i} = q$.

We choose $\mu_g(x) = x^{g-1}\frac{1-qx}{1-x^{-1}}\prod_{i=1}^{g}(1-\alpha_i x^{-1})(1-\overline{\alpha_i}x^{-1})$ and denote the corresponding shuffle algebra by S_g. For $d \in \mathbb{Z}$, let $Pic^d(X)$ be the set of line bundles over X of degree d, define

$$\mathbf{1}_{Pic^d(X)} := \sum_{\mathcal{L}\in Pic^d(X)}\mathcal{L}.$$

In [SV] it was shown that the map $\mathbf{1}_{Pic^d(X)} \mapsto z^i$ extends to an isomorphism

$$\Upsilon_X : U_X^{>} \to S_{\mathbf{1}},$$

where $U_X^{>}$ stands for the subalgebra of $\mathcal{H}_{lf}(X)$ generated by $\mathbf{1}_{Pic^d(X)}$ and $S_{\mathbf{1}}$ denotes the subalgebra of S_g, generated by $\mathbb{C}[z^{\pm 1}]$.

3. EHA: a Reminder

We recall the presentation of the elliptic Hall algebra $\mathcal{E}^+$ via generators and relations.

Definition. A triangle with vertices $X = (0, 0)$, $Y = (k_2, d_2)$ and $Z = (k_1 + k_2, d_1 + d_2)$ on the lattice $\mathbb{Z}^2$ is said to be **quasi-empty**, if the following properties hold:

•$k_1, k_2 \in \mathbb{Z}_{>0}$;

•$\frac{d_1}{k_1} > \frac{d_2}{k_2}$;

• There are no lattice points inside the triangle and on at least one of the edges XY, YZ.

If the first two conditions hold and there are no lattice points on both XY and YZ, the triangle is called **empty**.

The positive half $\mathcal{E}^+$ is by definition generated by the elements $u_{k,d}$ (here $k \geq 1$ and $d \in \mathbb{Z}$), with relations:

$$[u_{k_1,d_1}, u_{k_2,d_2}] = 0, \tag{3.1}$$

whenever the points $(k_1, d_1), (k_2, d_2)$ are collinear, and:

$$[u_{k_1,d_1}, u_{k_2,d_2}] = \frac{\theta_{k_1+k_2,d_1+d_2}}{\alpha_1}, \tag{3.2}$$

whenever the triangle with vertices $(0, 0)$, (k_2, d_2) and $(k_1 + k_2, d_1 + d_2)$ is quasi-empty. Here

$$\alpha_n = \frac{(q_1^n - 1)(q_2^n - 1)(q^{-n} - 1)}{n} \tag{3.3}$$

and θ is given by the generating function

$$\sum_{n=0}^{\infty} \theta_{na,nb} t^n = \exp(\sum_{n=0}^{\infty} \alpha_n u_{na,nb} t^n), \tag{3.4}$$

where $\gcd(a, b) = 1$.

The proof of the following result can be found in [SV]

Theorem 3.1. *The map $u_{1,d} \mapsto z^d$ extends to an isomorphism of algebras*

$$\Upsilon : \mathcal{E}^+ \to \tilde{S},$$

where $\tilde{S}$ is the subalgebra of S, generated by $\bigoplus_{d\in\mathbb{Z}} S_{d,1}$.

4. Shuffle Algebra and EHA

We consider the shuffle algebra depending on three parameters q_1, q_2, q, s.t. $q_1q_2 = q$. This is an associative graded unital subalgebra S of the graded space of symmetric rational functions in infinitely many variables endowed with the product

$$F(z_1,\dots,z_n)*G(z_1,\dots,z_m)=\frac{1}{n!m!}\mathrm{Sym}_{S_{n+m}}\left(F(z_1,\dots,z_n)G(z_{n+1},\dots,z_{n+m})\prod_{\substack{i\in\{1,\dots n\}\\ j\in\{n+1,\dots n+m\}}}\mu\left(\frac{z_i}{z_j}\right)\right),$$

where $\mu(x)=\frac{(x-1)(x-q)}{(x-q_1)(x-q_2)}$ and $\mathrm{Sym}_{S_k}\left(H(z_1,\dots,z_k)\right)=\sum_{\sigma\in S_k}sgn(\sigma)H(z_{\sigma(1)},\dots,z_{\sigma(k)})$. Each component S_n consists of rational functions

$$F(z_1,\dots,z_n)=\frac{f(z_1,\dots,z_n)\prod_{1\le i<j\le n}(z_i-z_j)^2}{\prod_{1\le i\neq j\le n}(z_i-q_1z_j)(z_i-q_2z_j)}, \tag{4.1}$$

with $f(z_1,\dots,z_n)$ - a symmetric Laurent polynomial, satisfying the wheel condition.

Definition. A symmetric Laurent polynomial $f(z_1,\dots,z_n)$ satisfies the **wheel condition** if

$$f(z_1,\dots,z_n)=0 \text{ when } \frac{z_i}{z_j}=q_1, \frac{z_j}{z_k}=q_2 \text{ and } \frac{z_k}{z_i}=\frac{1}{q}. \tag{4.2}$$

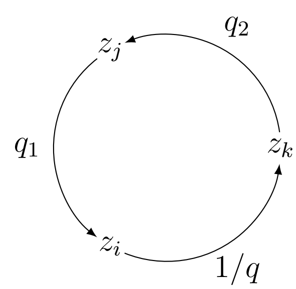

Figure 1. Wheel condition.

Next we verify that S is an algebra.

Proposition 4.1. *S is closed under the product* $(*)$.

Proof. The shuffle product of $F(z_1,\dots,z_n)\in S_n$ and $G(z_1,\dots,z_m)\in S_m$ can be written as

$$F(z_1,\dots,z_n)*G(z_1,\dots,z_m)=\frac{\prod_{1\le i<j\le n+m}(z_i-z_j)^2}{\prod_{1\le i\neq j\le n+m}(z_i-q_1z_j)(z_i-q_2z_j)}\cdot$$

$$\frac{1}{n!m!}\mathrm{Sym}_{S_{n+m}}\left(f(z_1,\dots,z_n)g(z_{n+1},\dots,z_{n+m})\prod_{1\le i\le n<j\le n+m}\frac{(z_i-qz_j)(z_i-q_1z_j)(z_i-q_2z_j)}{(z_i-z_j)}\right).$$

The rational function on the second line of the expression above does not have poles, as the only possible poles are at $z_i=z_j$ and those are simple. However, as the function is symmetric it cannot have poles of odd order and, therefore, is regular. It remains to check that the conditions (4.2) are satisfied. Indeed, if the indices of all three variables are in either $\{1,\dots,n\}$

or $\{n+1,\dots,n+m\}$ simultaneously, this follows from the wheel conditions for f or g. Otherwise the product $\prod\limits_{1\le i\le n<j\le n+m}\frac{(z_i-qz_j)(z_i-q_1z_j)(z_i-q_2z_j)}{(z_i-z_j)}$ vanishes. $\square$

The shuffle algebra $S := \mathbb{C}\bigoplus\limits_{d\in\mathbb{Z},n>0} S_{d,n}$ is bigraded by the degree and the number of variables in f.

The next definition will be of great importance for the proof that S and $\mathcal{E}^+$ are isomorphic.

Definition. An element $F(z_1,\dots,z_n)\in S$ is said to have **slope** $\le\mu$ ($\mu\in\mathbb{R}$) if the limit $\lim\limits_{\xi\to\infty}\frac{F(\xi z_1,\dots,\xi z_i,z_{i+1},\dots,z_n)}{\xi^{\mu i}}$ exists and is finite for all $i\in\{1,\dots,n\}$.

The subspace of elements of $S_{d,n}$ with slope $\le\mu$ will be denoted by S^{μ}_{n}. Notice, that we have inclusions $S^{\mu}_{k,n}\subset S^{\mu'}_{d,n}$ for $\mu\le\mu'$ and $S_{d,n}=\bigcup\limits_{\mu\in\mathbb{R}}S^{\mu}_{d,n}$, thus, an increasing filtration on the infinite-dimensional vector space $S_{d,n}$. It is also true that $S^{\mu}=\mathbb{C}\bigoplus\limits_{k\in\mathbb{Z},n>0}S^{\mu}_{d,n}$ is a subalgebra. One advantage of considering the subspaces $S^{\mu}_{d,n}$ is that they are finite dimensional, the next proposition provides an upper bound on the dimension.

Proposition 4.2. *The dimension of $S^{\mu}_{d,n}$ does not exceed the number of unordered tuples (the order between pairs with $n_i=n_j$ is disregarded) of pairs $(n_1,d_1),\dots,(n_s,d_s)$, such that*

$$\begin{cases} n_1+\ldots+n_s=n\\ d_1+\ldots+d_s=d\\ d_i\le\mu n_i\ \forall\ i\in\{1,\dots,n\},\end{cases}\tag{4.3}$$

where $t,n_i\in\mathbb{N}$ and $d_i\in\mathbb{Z}$.

Proof. Let $\rho=(n_1,\dots,n_s)$ be a partition of n. We consider the map

$$\varphi_\rho: S^{\mu}_{d,n}\to\mathbb{C}[y_1^{\pm1},\dots,y_t^{\pm1}]$$

$$\varphi_\rho(F(z_1,\dots,z_n))=f(qy_1,q^2y_1\dots,q^{k_1}y_1,qy_2,q^2y_2\dots,q^{k_2}y_2,\dots,qy_t,q^2y_t\dots,q^{k_t}y_t)$$

and define

$$S^{\mu,\rho}_{d,n}:=\bigcap_{\rho'\succ\rho}\ker\varphi_{\rho'},$$

where $\succ$ stands for the usual dominance order on partitions, and set $S^{\mu,(n)}_{d,n}=S^{\mu}_{d,n}$. Then the subspaces $S^{\mu,\rho}_{d,n}$ form a filtration of $S^{\mu}_{d,n}$, i.e.

$$\rho\prec\rho'\Rightarrow S^{\mu,\rho}_{d,n}\subset S^{\mu,\rho'}_{d,n}.$$

We take $F\in S^{\mu,\rho}_{d,n}$ and notice that the wheel condition implies $\varphi_\rho(F)$, vanishes, if

$$y_j=q_2q^{a-b}y_i, a\in\{1,\dots,n_i-1\}, b\in\{1,\dots,n_j\}\text{ or}\tag{4.4}$$

$$y_j=q_1q^{a-b}y_i, a\in\{1,\dots,n_i-1\}, b\in\{1,\dots,n_j\},\tag{4.5}$$

for $i<j$ as, for example, (4.4) guarantees that there are $z_i = q^{a+1}y_i$, $z_j = q_2 q^a y_i$ and $z_k = q^a y_i$. As $\varphi_\rho(F)$ also belongs to $S^{\mu,\rho}_{d,n}$, it vanishes, whenever

$$y_j = q^{n_i-b+1}y_i \text{ or } y_j = q^{-b}y_i, \text{ where } b\in\{1,\dots,n_j\}, \tag{4.6}$$

with $i<j$, as well. We conclude that the Laurent polynomial $\varphi_\rho(F)$ is divisible by

$$D_F = \prod_{1\le i<j\le t}\left(\prod_{b=1}^{n_j}(y_j - q^{n_i-b+1}y_i)(y_j-q^{-b}y_i)\prod_{b=1}^{n_j}\prod_{a=1}^{n_i-1}(y_j-q_1q^{a-b}y_i)(y_j-q_2q^{a-b}y_i)\right),$$

a polynomial of degree

$$\deg(D_F) = \sum_{i<j} 2n_j + 2(n_i-1)n_j = 2\sum_{i<j} n_in_j = n^2 - \sum_i n_i^2$$

and of degree in each variable y_i equal to

$$\deg_i(D_F) = \sum_{j\ne i} 2n_in_j = 2n_i\sum_{j\ne i} n_j = 2n_i(n-n_i).$$

. As follows from the definition of elements of S in (4.1), $\deg(f) = \deg(F) + n(n-1) = d + n(n-1)$, and we obtain

$$\deg(\varphi_\rho(F)) = \deg(f) = d + n(n-1).$$

Next we use that the slope of $\varphi_\rho(F)$ is not greater than μ. This and (4.1) provide an upper bound on the degree of $\varphi_\rho(F)$:

$$\deg_i(\varphi_\rho(F)) - (n_i(n_i-1) + 2n_i(n-n_i)) \le \mu n_i$$
$$\deg_i(\varphi_\rho(F)) \le 2nn_i - n_i(n_i+1) + \mu n_i.$$

The above allows to conclude

$$\deg(\varphi_\rho(F)/D_F) = \sum_i n_i(n_i-1) + d$$

$$\deg_i(\varphi_\rho(F)/D_F) \le n_i(n_i-1) + \mu n_i.$$

The basis for such polynomials consists of monomials

$$y_1^{n_1(n_1-1)+d_1},\dots,y_t^{n_t(n_t-1)+d_t},$$

with $d_1+\ldots+d_t = d$ and $d_i \le \mu n_i$. To complete the proof it remains to notice that if $n_i = n_j$, then $\varphi_\rho(F)$ is invariant under the transposition (ij), so the respective order of d_i and d_j can be disregarded. $\square$

Corollary 4.3. *The subspace of $S_{d,n}$, consisting of elements F, s.t.*

$$\lim_{\xi\to\infty}\frac{F(\xi z_1,\dots,\xi z_i,\dots,\xi z_n)}{\xi^{\frac{di}{n}}} = 0 \ \forall i\in\{1,\dots,n\}$$

is at most one-dimensional.

Proof. In this case $\mu = \frac{d}{n}$, so $d_i \leq \frac{d}{n}n_i$ and $d_1 + \ldots + d_t \leq \frac{d}{n}n_1 + \ldots + \frac{d}{n}n_t = d$. Thus, we must have that each $d_i = \frac{d}{n}n_i$. On the other hand, for the limit above to be zero, the inequalities $d_i \leq \frac{d}{n}n_i$ must be strict. Therefore, the only possibility is $n_1 = n$ and $d_1 = d$. □

In [SV] it was shown that the map $u_{1,d} \to z^d$ extends to an isomorphism

$$\Upsilon : \mathcal{E}^+ \to S_{\mathbf{1}},$$

where $S_{\mathbf{1}}$ is the subalgebra of S, generated by $\mathbb{C}[z, z^{-1}]$. This allows us to conclude that the map $\mathcal{E}^+ \to S$ (which we also denote by Υ) is also injective and the next proposition shows that it is surjective as well.

Proposition 4.4. *The map $\Upsilon : \mathcal{E}^+ \to S$ is surjective.*

Proof. We denote by $\mathcal{E}_{n,d} \subset \mathcal{E}^+$ the subspace of elements of bidegree (n, d) and

$$\mathcal{E}^{\mu}_{n,d} = \left\{ \text{sums of products of } u_{n',d'} \text{ with } \frac{d'}{n'} \leq \mu \right\} \subset \mathcal{E}_{n,d}.$$

The dimension of $\mathcal{E}^{\mu}_{n,d}$ equals to the number of convex paths in $\mathbf{Conv}^+$ with slope $\leq \mu$ and such paths are in bijection with the pairs of tuples from proposition 4.2 (see lemma 5.6 of [BS]). Therefore, the dimension of $S^{\mu}_{d,n}$ does not exceed the dimension of $\mathcal{E}^{\mu}_{n,d}$ and (due to invectivety of Υ) it is sufficient to show $\Upsilon(\mathcal{E}^{\mu}_{n,d}) \subset S^{\mu}_{d,n}$. The verification of this can be found in the proof of proposition 3.5 in [Neg]. □

. We introduce $P_{k,d} := \Upsilon(u_{k,d})$.

Corollary 4.5. *S is generated by the first graded component, i.e. $\mathbb{C}[z^{\pm 1}]$.*

5. The Double of Shuffle Algebra

We start with the general construction. Suppose $(\mathcal{A}, *, \triangle)$ is a bialgebra (we assume that $\triangle$ is coassociative and the product and coproduct are compatible in the sense that $\triangle(a * b) = \triangle(a) * \triangle(b)$) with a symmetric non-degenerate form

$$(\cdot, \cdot) : \mathcal{A} \otimes \mathcal{A} \to \mathbb{C},$$

satisfying

$$(a * b, c) = (a \otimes b, \triangle(c)) \ \forall \ a, b, c \in \mathcal{A}. \tag{5.1}$$

Definition. The **Drinfeld double** $D\mathcal{A} = \mathcal{A}^{coop} \otimes \mathcal{A}$ ($\mathcal{A}^{coop}$ has the same product as $\mathcal{A}$, but the coproduct is opposite) is a free product of algebras with both $\mathcal{A}^- = \mathcal{A}^{coop} \otimes 1$ and $\mathcal{A}^+ = 1 \otimes \mathcal{A}$ being subbialgebras, subject to the relations

$$\sum_{i,j} a_i^{(1)-} * b_j^{(2)+}(a_i^{(2)-}, b_j^{(1)+}) = \sum_{i,j} b_j^{(1)+} * a_i^{(2)-}(a_i^{(1)-}, b_j^{(2)+}), \text{ where}$$

$$\triangle(a) = \sum_i a_i^{(1)-} \otimes a_i^{(2)-} \text{ and } \triangle(b) = \sum_j b_j^{(1)+} \otimes b_j^{(2)+} \;\forall\; a \in \mathcal{A}^+,\; b \in \mathcal{A}^-.$$

The bialgebra structure on $D\mathcal{A}$ is determined uniquely.

Our next goal is to endow the Shuffle algebra S with a coproduct. For this we will need to consider a slightly larger algebra $\tilde{S}$. It is generated by S and the set of elements $h_i, i \in \mathbb{Z}_{\geq 0}$ with the following relations

$$[h_i, h_j] = 0$$

$$F(z_1, \ldots, z_n) * h(w) = h(w) * \left(F(z_1, \ldots, z_n) \prod_{i=1}^{n} \Omega\left(\frac{w}{z_i}\right)\right), \tag{5.2}$$

where $h(w) := \sum_{n \geq 0} h_n w^{-n}$ and

$$\Omega(x) = \frac{\mu\left(\frac{1}{x}\right)}{\mu(x)} = \frac{(x - q^{-1})(x - q_1)(x - q_2)}{(x - q)(x - q_1^{-1})(x - q_2^{-1})} = \exp\left(-\sum_{n \geq 1} \alpha_n x^{-n}\right). \tag{5.3}$$

We understand (5.2) by expanding the r.h.s. in negative powers of w and setting the corresponding terms equal. Now we can define the coproduct on $\tilde{S}$ (we skip the proof and refer to the Appendix in [Neg]).

Proposition 5.1. *The following formulas define a coassociative coproduct on $\tilde{S}$:*

$$\triangle(h(w)) := h(w) \otimes h(w)$$

$$\triangle(F(z_1, \ldots, z_n)) = \sum_{i=0}^{n} \frac{\prod_{b>i} h(z_b) F(z_1, \ldots, z_i \otimes z_{i+1}, \ldots, z_n)}{\prod_{a \leq i < b} \mu\left(\frac{z_b}{z_a}\right)} \tag{5.4}$$

. The r.h.s of the second line above should be understood by expanding in nonnegative powers of $\frac{z_a}{z_b}$ for $a \leq i < b$, obtaining an infinite sum of monomials. Then in each summand all h_i's are moved to the left, followed by powers of $z_1, \ldots, z_i$ to the left of $\otimes$ and powers of $z_{i+1}, \ldots, z_n$ - to the right. A typical summand looks like

$$h_{k_i+1} \ldots h_{k_n} z_1^{s_1} \ldots z_i^{s_i} \otimes z_{i+1}^{s_{i+1}} \ldots z_n^{s_n},$$

here $\triangle(F(z_1, \ldots, z_n))$ belongs to the completion $\tilde{S} \hat{\otimes} \tilde{S}$.

Finally, the bialgebra $\tilde{S}$ has a pairing, given by:

$$(h(v), h(w^{-1})) = \Omega\left(\frac{w}{v}\right)$$

$$(F, G) = \frac{1}{\alpha_1^k} : \int : \frac{F(z_1, \ldots, z_n) G(\frac{1}{z_1}, \ldots, \frac{1}{z_n})}{\prod_{1 \leq i \neq j \leq n} \mu\left(\frac{z_i}{z_j}\right)} Dz_1 \ldots Dz_n \tag{5.5}$$

for $F, G \in S_{n,d}$ with $Dz := \frac{1}{2\pi i z}$ and we define the normal-ordered integral $:\int:$ by
(5.6)

$$\left(Sym\left(z_1^{k_1}\dots z_n^{k_n}\prod_{1\le i\neq j\le n}\mu\left(\frac{z_i}{z_j}\right)\right),F\right)=\frac{1}{\alpha_1^k}\int_{|z_1|<<|z_2|<<\dots<<|z_n|}\frac{z_1^{k_1}\dots z_n^{k_n}F(\frac{1}{z_1},\dots,\frac{1}{z_n})}{\prod\limits_{1\le i\neq j\le n}\mu\left(\frac{z_i}{z_j}\right)}Dz_1\dots Dz_n.$$

This sufficient to define the pairing on $\tilde{S}$ since any element can be written as a linear combination of monomials $z^{i_1} * \dots * z^{i_k}$ by the corollary of proposition 4.4.

We refer to [Neg] for the proof of the next proposition.

Proposition 5.2. *The formulas 5.6 above produce a well-defined pairing on bialgebra $\tilde{S}$:*

$$\tilde{S}\otimes\tilde{S}\to\mathbb{C}(q_1,q_2).$$

We denote the Drinfeld double of $\tilde{S}$ with respect to the pairing 5.6 by $D\tilde{S}$.

Next we slightly expand the elliptic Hall algebra $\mathcal{E}^+$ by adding the commuting elements $\{u_{0,i}|i\in\mathbb{Z}\}$ and a central element c with relations

$$[u_{0,d},u_{1,d'}]=u_{1,d+d'}\ \forall d\in\mathbb{Z},d'\in\mathbb{Z}_{>0}$$

and denote the is algebra by $\tilde{\mathcal{E}}^+$.

The coproduct is given by

$$\triangle(u_{0,d})=u_{0,d}\otimes 1+1\otimes u_{0,d},\ \ \triangle(u_{1,d})=u_{1,d}\otimes 1+c\sum_{n\ge 0}\theta_{0,n}\otimes u_{1,d-n},$$

where $\theta_{0,n}$ are computed according to 3.4. It remains to define a pairing on $\tilde{\mathcal{E}^+}$, which is done by setting

$$(u_{0,d},u_{0,d})=\frac{1}{\alpha_d},(u_{1,d},u_{1,d})=\frac{1}{\alpha_1}.$$

Theorem 5.3. *The morphism of proposition 4.4 can be extended to $\Upsilon:\tilde{\mathcal{E}}^+\to\tilde{S}$ by*

$$\Upsilon(c)=h_0,\ \textit{and}\ \Upsilon(u_{0,d})=p_d,$$

where $p_1,p_2,\dots$ are obtained from the series

$$h(w)=h_0\,exp\left(\sum_{n=1}^{\infty}\alpha_n p_n w^{-n}\right).$$

Thus extended Υ preserves the coproduct and bialgebra pairing and, therefore, induces the isomorphism of Drinfeld doubles:

$$\tilde{\Upsilon}:D\tilde{\mathcal{E}}^+\to D\tilde{S}.$$

Proof. First one needs to show that the formulas above indeed extend the isomorphism defined in proposition 4.4, i.e. respect the relations between elements added to the algebras. Next, we need to check that Υ preserves the pairing. It is enough to show this for generators, provided it satisfies conditions 5.1 (this is shown on page 24 of [Neg]). For example,

$$(z_1^d, z_1^d) = \frac{1}{\alpha_1}\frac{1}{2\pi i}\int\limits_{|z_1|<1} \frac{z_1^d z_1^{-d}}{z_1} dz_1 = \frac{1}{\alpha_1} = (u_{1,d}, u_{1,d})$$

□

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