1. Examples of fibre products

It turns out that the fibre product is extremely useful.

Definition 1.1. *Let $f\colon X \longrightarrow S$ be a morphism of schemes, and let $s \in S$ be a point of S. The* ***fibre over*** *s is the fibre product over the morphism f and the inclusion of s in S, where the point s is given a scheme structure by taking the residue field $\kappa(s)$.*

It is interesting to see what happens in some specific examples. First consider a family of conics in the plane,

$$X = \operatorname{Spec} \frac{k[x, y, t]}{\langle ty - x^2 \rangle}.$$

The inclusion

$$k[t] \longrightarrow \frac{k[x, y, t]}{\langle ty - x^2 \rangle},$$

realises X as a family over the affine line over k,

$$f\colon X \longrightarrow \mathbb{A}^1_k.$$

Pick a point $p \in \mathbb{A}^1$. If the point is closed, this is the same as picking a scalar, and of course the residue field is nothing more than k. If we pick a non-zero scalar a, then we just get the conic defined by $ay - x^2$ in $k[x, y]$ (since tensoring by k won't change anything),

$$X_p = \operatorname{Spec} \frac{k[x, y]}{\langle ay - x^2 \rangle}.$$

But now suppose that $a = 0$. In this case the above reduces to

$$X_0 = \operatorname{Spec} \frac{k[x, y]}{\langle x^2 \rangle},$$

a double line. It is also interesting to consider the fibre over the generic point ξ, corresponding to the maximal ideal $\langle 0 \rangle$. In this case the residue field is $k(t)$, and the **generic fibre** is

$$X_\xi = \operatorname{Spec} \frac{k(t)[x, y]}{\langle ty - x^2 \rangle},$$

which is the conic $V(ty - x^2) \subset \mathbb{A}^2_{k(t)}$ over the field $k(t)$.

Similarly, if we pick the family

$$X = \operatorname{Spec} \frac{k[x, y, t]}{\langle xy - t \rangle}.$$

then, for $a \neq 0$, the fibre is a smooth conic, but for $t = 0$ the fibre is a pair of lines.

Once again, the point is that there are some more exotic examples, which can be treated in a similar fashion. Consider for example $\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}[x]$. Once again this is a scheme over $\operatorname{Spec} \mathbb{Z}$, and once again it is interesting to compute the fibres. Suppose first that we take the generic point. Then this has residue field $\mathbb{Q}$. If we tensor $\mathbb{Z}[x]$ by $\mathbb{Q}$, then we get $\mathbb{Q}[x]$. If we take Spec of this, we get the affine line over $\mathbb{Q}$. Now suppose that we take a maximal ideal $\langle p \rangle$. In this case the residue field is $\mathbb{F}_p$ the finite field with p elements. Tensoring by this field we get $\mathbb{F}_p[x]$ and taking Spec we get the affine line over the finite field with p elements.

It is also possible to figure out all the prime ideals in $\mathbb{Z}[x]$. They are

(1) $\langle 0 \rangle$
(2) $\langle p \rangle$, p a prime number.
(3) $\langle f(x) \rangle$, $f(x)$ irreducible over $\mathbb{Q}$, with content one,
(4) maximal ideals of the form $\langle p, f(x) \rangle$, where $f(x)$ is a polynomial, with content one, whose reduction modulo p is irreducible.

Note that the zero ideal is the generic point, and the closure of the ideal $\langle p \rangle$ is the fibre over the same ideal downstairs. The closure of an ideal of type (3) is perhaps the most interesting. It will consist of all maximal points $\langle p, g \rangle$, where the reduction of $g(x)$ is a factor of the reduction of $f(x)$ inside $\mathbb{F}_p[x]$.

It is now possible to consider closed subschemes of $\mathbb{A}^1_{\mathbb{Z}}$. For example consider

$$X = \operatorname{Spec} \frac{\mathbb{Z}[x]}{\langle 3x - 16 \rangle}.$$

Fibre by fibre, we get a collection of subschemes of $\mathbb{A}^1_{\mathbb{F}_p}$. If we reduce modulo 5, that is, tensor by $\mathbb{F}_5$ then we get

$$X = \operatorname{Spec} \frac{\mathbb{F}_5[x]}{\langle 3x - 1 \rangle},$$

a single point. However something strange happens over the prime 3, since we get an equation which cannot be satisfied. If we think of this as the graph of the rational map $16/3$, then we have a pole at 3, which cannot be removed. Of course over 2, this rational function is zero.

Now suppose that we consider $x^2 - 3$. Then we get a conic. In fact, this is the same as considering

$$\frac{\mathbb{Z}[x]}{\langle x^2 - 3 \rangle} = \mathbb{Z}[\sqrt{3}].$$

So the seemingly strange picture we had before becomes a little more clear.

Consider the residue fields. Recall that there are three cases.

(1) If p divides the discriminant of $K/\mathbb{Q}$ (which in this case is 12), that is $p = 2$ or 3, then the ideal $\langle p \rangle$ is a square in A.

$$\langle 2 \rangle A = (\langle 1 + \sqrt{3} \rangle)^2,$$

and

$$\langle 3 \rangle A = (\langle \sqrt{3} \rangle)^2.$$

(2) If 3 is a square modulo p, the prime $\langle p \rangle$ factors into a product of distinct primes,

$$\langle 11 \rangle A = \langle 4 + 3\sqrt{3} \rangle \langle 4 - 3\sqrt{3} \rangle,$$

or

$$\langle 13 \rangle A = \langle 4 + \sqrt{3} \rangle \langle 4 - \sqrt{3} \rangle,$$

(3) If $p > 3$ and 3 is not a square mod p (e.g $p = 5$ and 7), the ideal $\langle p \rangle$ is prime in A.

Let us consider the coordinate rings in all three cases. In the first case we get

$$A/\mathfrak{p}^2,$$

and the residue field is $\mathbb{F}_p$. In the second case there are two points with coordinate rings $\mathbb{F}_p$. Finally in the third case there is a single point with coordinate ring

$$\mathbb{F}_{p^2},$$

the unique finite field with p^2 elements. Note that in all three cases, the coordinate ring of the inverse image has length two over the coordinate ring of the base (in our case $\mathbb{F}_p$). In fact this is the general picture. Finite maps have a degree, and the length of the coordinate ring over the base is equal to this degree.

Now suppose that we consider a plane conic in $\mathbb{A}^2_{\mathbb{Z}}$,

$$X = \operatorname{Spec} \frac{\mathbb{Z}[x, y]}{\langle x^2 - y^2 - 5 \rangle}.$$

Over the typical prime, we get a smooth conic in the corresponding affine plane over a finite field. But now consider what happens over $\langle 2 \rangle$ and $\langle 5 \rangle$. Modulo two, we have

$$x^2 - y^2 - 5 = (x + y + 1)^2,$$

and modulo 5 we have

$$x^2 - y^2 - 5 = (x - y)(x + y).$$

Thus we get a double line over $\langle 2 \rangle$ and a pair of lines over $\langle 5 \rangle$.

Another useful way to think of the fibre product, is as a base change. In arithmetic, one always wants to compare what happens over different fields, or even different rings.

Definition 1.2. *Let S be a scheme. As $\operatorname{Spec}\mathbb{Z}$ is a terminal object in the category of schemes, there is a unique morphism $S \longrightarrow \operatorname{Spec}\mathbb{Z}$* ***Affine*** n***-space*** *over S is the scheme obtained by base change from $\mathbb{A}^n_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x_1, x_2, \dots, x_n]$, so that*

$$\mathbb{A}^n_S = \mathbb{A}^n_{\mathbb{Z}} \underset{\operatorname{Spec}\mathbb{Z}}{\times} S.$$

Now consider an interesting example over a non-algebraically close field. Consider the inclusion $\mathbb{R} \longrightarrow \mathbb{C}$. This gives a morphism of schemes,

$$f\colon X = \operatorname{Spec}\mathbb{C} \longrightarrow Y = \operatorname{Spec}\mathbb{R},$$

where X and Y are schemes with only one point, but the first has sheaf of rings given by $\mathbb{C}$ and the second $\mathbb{R}$. Now consider what happens when we make the base change f over f. Then we get a scheme

$$X \underset{Y}{\times} X.$$

Note that this has degree two over X. Since $\mathbb{C}$ is algebraically closed, in fact this must consist of two points, even though f only has one point in the fibre. Algebraically,

$$\mathbb{C} \underset{\mathbb{R}}{\otimes} \mathbb{C} \simeq \mathbb{C}^2,$$

and the spectrum has two points.

In particular, the property of being irreducible is not preserved by base change. Consider also the example of $\operatorname{Spec} k[x,t]/\langle x^2 - t\rangle \subset \mathbb{A}^2_k$ over the affine line, with coordinate t, say over an algebraically closed field k. Then the fibre over every closed point, except zero, is reducible. But the fibre over the generic point is irreducible, since $x^2 - t$ won't factor, even if you invert every polynomial in t. However suppose that we make a base change of the affine line by the affine line given by

$$\mathbb{A}^1_k \longrightarrow \mathbb{A}^1_k \qquad \text{given by} \qquad t \longrightarrow t^2.$$

After base change, the new scheme is given by $x^2 - t^2$. But this factors, even over the generic point

$$x^2 - t^2 = (x-t)(x+t).$$

Definition 1.3. *Let X be a scheme over a field k. We say that X is* ***geometrically irreducible*** *if $X \underset{\operatorname{Spec} k}{\times} \operatorname{Spec}\bar{k}$ is irreducible.*

Note that the property of being geometrically irreducible is preserved under base change.

2. Rational maps

It is often the case that we are given a variety X and a morphism defined on an open subset U of X. As open sets in the Zariski topology are very large, it is natural to view this as a map on the whole of X, which is not everywhere defined.

Definition 2.1. *A* ***rational map*** $\phi\colon X \dashrightarrow Y$ *between quasi-projective varieties is a pair (f,U) where U is a dense open subset of X and $f\colon U \longrightarrow Y$ is a morphism of varieties. Two rational maps (f_1,U_1) and (f_2,U_2) are considered equal if there is a dense open subset $V \subset U_1 \cap U_2$ such that the two functions $f_1|_V$ and $f_2|_V$ are equal.*

It is customary to avoid using the pair notation and to leave U unspecified. We often say in this case that ϕ is defined on U. Note that if U and V are two dense open sets, and (f,U), (g,V) represent the same rational map, then $(h, U \cup V)$ also represents the same map, where h is defined in the obvious way. By Noetherian induction, it follows that there is a largest open set on which ϕ is defined, which is called the **domain of** ϕ. The complement of the domain is called the **locus of indeterminancy**.

One way to get a picture of a rational map, is to consider the graph.

Definition 2.2. *Let $\phi\colon X \dashrightarrow Y$ be a rational map.*

The ***graph of*** ϕ *is the closure of the graph of f, where the pair (f,U) represents ϕ.*

The ***image of*** ϕ *is the image of the graph of ϕ under the second projection.*

Note that the domain of ϕ is precisely the locus where the first projection map is an isomorphism.

Definition 2.3. *Let $\phi\colon X \dashrightarrow Y$ and $\psi\colon Y \dashrightarrow Z$ be two rational maps. Suppose that $\phi = (f,U)$ and $\psi = (g,V)$ and that $f^{-1}(V)$ is dense (if X is irreducible this is equivalent to the requirement that $f(U) \cap V$ is non-empty). Then we may define the composition of ϕ and ψ by taking the pair $(g \circ f, f^{-1}(V))$.*

Note that in general, we cannot compose rational maps. The problem might be that the image of the first map might lie in the locus where the second map is not defined. However there will never be a problem if X is irreducible and ϕ is dominant:

Definition 2.4. *We say that ϕ is* ***dominant*** *if the the image of ϕ is dense in Y.*

Note that this gives us a category, the category of irreducible varieties and dominant rational maps.

Definition 2.5. *We say that a dominant rational map $\phi\colon X \dashrightarrow Y$ of irreducible quasi-projective varieties is birational if it has an inverse. In this case we say that X and Y are **birational**. We say that X is **rational** if it is birational to $\mathbb{P}^n$.*

It is interesting to see an example. Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the map

$$[X : Y : Z] \longrightarrow [YZ : XZ : XY].$$

This map is clearly a rational map. It is called a **Cremona transformation**. Note that it is a priori not defined at those points where two coordinates vanish. To get a better understanding of this map, it is convenient to rewrite it as

$$[X : Y : Z] \longrightarrow [1/X : 1/Y : 1/Z].$$

Written as such it is clear that this map is an involution, so that it is in particular a birational map.

It is interesting to check whether or not this map really is well defined at the points $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$. To do this, we need to look at the graph.

Consider the following map,

$$\mathbb{A}^2 \dashrightarrow \mathbb{A}^1,$$

which assigns to a point $p \in \mathbb{A}^2$ the slope of the line connecting the point p to the origin,

$$(x, y) \longrightarrow y/x.$$

Now this map is not defined along the locus where $x = 0$. Replacing $\mathbb{A}^1$ with $\mathbb{P}^1$ we get a map

$$(x, y) \longrightarrow [x : y].$$

Now the only point where this map is not defined is the origin. We consider the graph,

$$\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Consider how the graph sits over $\mathbb{A}^2$. Outside the origin, the first projection is an isomorphism. Over the origin, the graph is contained in a copy of the image, that is, $\mathbb{P}^1$. Consider any line $y = tx$, through the origin. Then this line, minus the origin, is sent to the point with slope t. It follows that the closure of this line is sent to the point with slope t. Varying t, it follows that any point of the fibre over $\mathbb{P}^1$ is a point of the graph.

Thus the morphism $p\colon \Gamma \longrightarrow \mathbb{A}^2$ is an isomorphism outside the origin and contracts a whole copy of $\mathbb{P}^1$ to a point. For this reason, we call p a blow up.

Definition 2.6. *Let $\phi\colon X \dashrightarrow \mathbb{P}^k$ be a rational map, which is given locally by $f_1, f_2, \dots, f_k$. Let I be the ideal spanned by $f_1, f_2, \dots, f_k$. The induced morphism $p\colon \Gamma \longrightarrow X$ is called the* ***blow up of the ideal*** *I.*

Clearly p is always birational, as it is an isomorphism outside $V(I)$.

In our case $I = \langle x, y\rangle$, the maximal ideal of p, so that we call p the blow up of a point. Suppose we have coordinates $[S : T]$ on $\mathbb{P}^1$. Then outside of the origin, the graph satisfies the equation $xT = yS$. Thus the closure must satisfy the same equation. Since this equation determines the graph outside the origin, in fact the graph is defined by this equation (as the whole fibre over the origin lives in the graph, we don't need anymore equations).

The inverse image of the origin is called the **exceptional divisor**.

Definition 2.7. *Let $\pi\colon X \longrightarrow Y$ be a birational morphism. The locus where π is not an isomorphism is called the* ***exceptional locus****. If $V \subset Y$, the inverse image of V is called the* ***total transform****. Let Z be the image of the exceptional locus. Suppose that V is not contained in Z. The* ***strict transform of*** *V is the closure of the inverse image of $V - Z$.*

It is interesting to compute the strict transform of some planar curves. We have already seen that lines through the origin lift to curves that sweep out the exceptional divisor. In fact the blow up separates the lines through the origin. These are then the fibres of the second morphism.

Let us now take a nodal cubic,

$$y^2 = x^2 + x^3.$$

We want to figure out its strict transform, so that we need the inverse image in the blow up. Outside the origin, there are two equations to be satisfied,

$$y^2 = x^2 + x^3 \qquad \text{and} \qquad xT = yS.$$

Passing to the coordinate patch $y = xt$, where $t = T/S$, and substituting for y in the first equation we get

$$x^2t^2 - x^2 - x^3 = x^2(t^2 - x - 1).$$

Now if $x = 0$, then $y = 0$, so that in fact locally $x = 0$ is the equation of the exceptional divisor. So the first factor just corresponds to the exceptional divisor. The second factor will tell us what the closure of our curve looks like, that is, the strict transform. Now over the origin, $x = 0$, so that $t^2 = 1$ and $t = \pm 1$. Thus our curve lifts to a curve which intersects the exceptional divisor in two points. (If we compute in the coordinate patch $x = sy$, we will see that the curve does not meet the point at infinity). These two points correspond to the fact that the nodal cubic has two tangent lines at the origin, one of slope 1 and one of slope -1.

Now consider what happens for the cuspidal cubic, $y^2 = x^3$. In this case we get

$$(xt)^2 - x^3 = x^2(t^2 - x).$$

Once again the factor of x^2 corresponds to the fact that the inverse image surely contains the exceptional divisor. But now we get the equation $t^2 = 0$, so that there is only one point over the origin, as one might expect from the geometry.

Let us go back to the Cremona transformation. To compute what gets blown up and blown down, it suffices to figure out what gets blown down, by symmetry. Consider the line $X = 0$. If $bc \neq 0$, the point $[0 : b : c]$ gets mapped to $[0 : 0 : 1]$. Thus the strict transform of the line $X = 0$ in the graph gets blown down to a point. By symmetry the strict transforms of the other two lines are also blown down to points. Outside of the union of these three lines, the map is clearly an isomorphism.

Thus the involution blows up the three points $[0 : 0 : 1]$, $[0 : 1 : 0]$, and $[1 : 0 : 0]$ and then blows down the three disjoint lines. Note that the three exceptional divisors become the three new coordinate lines.

One of the most impressive results of the nineteenth century is the following characterisation of the birational automorphism group of $\mathbb{P}^2$.

Theorem 2.8 (Noether)**.** *The birational automorphism group is generated by a Cremona transformation and* PGL(3)*.*

This result is very deceptive, since it is known that the birational automorphism group is, by any standards, very large.

3. Rational Varieties

Definition 3.1. *A **rational function** is a rational map to $\mathbb{A}^1$.*

*The set of all rational functions, denoted $K(X)$, is called the **function field**.*

Lemma 3.2. *Let X be an irreducible variety.*

Then the function field is a field. If $U \subset X$ is any open affine subset, then the function field is precisely the field of fractions of the coordinate ring of U.

Proof. Clear, since on an irreducible variety, any rational function is determined by its restriction to any open subset, and locally any morphism is given by a rational function. $\square$

Proposition 3.3. *Let K be an algebraically closed field.*

Then there is an equivalence of categories between the category of irreducible varieties over K with morphisms the dominant rational maps, and the category of finitely generated field extensions of K.

Proof. Define a functor F from the category of varieties to the category of fields as follows. Given a variety X, let $K(X)$ be the function field of X. Given a rational map $\phi\colon X \dashrightarrow Y$, define $F(\phi)\colon K(Y) \longrightarrow K(X)$ by composition. If f is a rational function on Y, then $\phi \circ f$ is a rational function on X.

We have to check that F is essentially surjective and fully faithful. Suppose that L is a finitely generated field extension of K. Then $L = K(\alpha_1, \alpha_2, \dots, \alpha_n)$. Let $A = K[\alpha_1, \alpha_2, \dots, \alpha_n]$. Let X be any affine variety with coordinate ring A. Then X is irreducible as A is an integral domain and the function field of X is precisely L as this is the field of fractions of A.

The fact that F is fully faithful is proved in the same way as before. $\square$

Proposition 3.4. *Let X and Y be two irreducible varieties.*

Then the following are equivalent

1. *X and Y are birational.*
2. *X and Y contain isomorphic open subsets.*
3. *The function fields of X and Y are isomorphic.*

Proof. We have already seen that (1) and (3) are equivalent and clearly (2) implies (1) (or indeed (3)). It remains to prove that if X and Y are birational then they contain isomorphic open subsets.

Let $\phi\colon X \dashrightarrow Y$ be a birational map with inverse $\psi\colon Y \dashrightarrow X$. Suppose that ϕ is defined on U and ψ is defined on V. Then $\psi \circ \phi$ is defined on $\phi^{-1}(V)$ and it is equal to the identity there, since it is the identity on some dense open subset. Similarly $\phi \circ \psi$ is the identity on $\phi^{-1}(U)$. Then the open subset $\phi^{-1}(\psi^{-1}(U))$ of X and $\psi^{-1}(\phi^{-1}(V))$ of Y are isomorphic open subsets. $\square$

Corollary 3.5. *Let X be an irreducible variety.*
Then the following are equivalent
(1) X is rational.
(2) X contains an open subset of $\mathbb{P}^n$.
(3) The function field of X is a purely transcendental extension of K.

Proof. Immediate from (3.4). $\square$

Let us consider some examples. I claim that the curve
$$C = V(y^2 - x^2 - x^3)$$
is rational. We have already seen that there is a morphism
$$\mathbb{A}^1 \longrightarrow C \qquad \text{given by} \qquad t \longrightarrow (t^2 - 1, t(t^2 - 1)).$$
We want to show that it is a birational map. One way to proceed is to construct the inverse. In fact the inverse map is
$$C \dashrightarrow \mathbb{A}^1 \qquad \text{given by} \qquad (x, y) \longrightarrow y/x.$$
Another way to proceed is to prove that the function field is purely transcendental. Now the coordinate ring is
$$K[x, y]/\langle y^2 - x^2 - x^3\rangle.$$
So the fraction field is $K(x, y)$, where $y^2 = x^2 + x^3$. Consider $t = y/x$. I claim that $K(t) = K(x, y)$. Clearly there is an inclusion one way. Now
$$t^2 = y^2/x^2 = 1 + x \qquad \text{and so} \qquad x = t^2 - 1 \in K(t).$$
But $y = tx$, so that we do indeed have equality $K(t) = K(x, y)$. Thus C is rational.

Perhaps a more interesting example is to consider the Segre variety $V \subset \mathbb{P}^3$. Consider projection π from a point p of the Segre variety,
$$\pi \colon V \dashrightarrow \mathbb{P}^2.$$
Clearly the only possible point of indeterminancy is the point p. Since a line, not contained in V, meets the Segre variety in at most two points, it follows that this map is one to one outside p, unless that line is contained in V. On the other hand, if $q \in \mathbb{P}^2$, the line $\langle p, q\rangle$ will meet the Segre variety in at least two points, one of which is p.

Now through the point p, there passes two lines l and m (one line of each ruling). These get mapped to two separate points, say q and r. It follows that p is indeed a point of indeterminancy. To proceed further, it is useful to introduce coordinates. Suppose that $p = [0 : 0 : 0 : 1]$, where $V = V(XW - YZ)$.

Now projection from $p \in \mathbb{P}^3$ defines a rational map

$$\phi\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2,$$

whose exceptional locus is a copy of $\mathbb{P}^2$. Indeed the graph of ϕ lies in $\mathbb{P}^3 \times \mathbb{P}^2$ and as before over the point p, we get a copy of the whole of the image $\mathbb{P}^2$, as can be seen by looking at lines through p. Working on the affine chart $W \neq 0$, V is locally defined as $x = yz$. If $[R : S : T]$ are coordinates on $\mathbb{P}^2$, the equations for the blow up of $\mathbb{P}^3$ are given as

$$xS = yR \qquad xT = zR \qquad yT = zS.$$

The blow up of V at p is given as the strict transform of V in the blow up of $\mathbb{P}^3$. We work in the patch $T \neq 0$. Then $x = rz$ and $y = sz$ so that the we get the equation

$$rz - sz^2 = z(r - sz) = 0.$$

Now $z = 0$ corresponds to the whole exceptional locus so that $r = sz$ defines the strict transform. In this case $z = 0$, means $r = 0$, so that we get a line in the exceptional $\mathbb{P}^2$.

In other words the graph of π is the blow up of p, with an exceptional divisor isomorphic to $\mathbb{P}^1$. The graph of π then blows down the strict transform of the two lines. Note that the image of the exceptional divisor, is precisely the line connecting the two points q and r.

To see that π is birational, we write down the inverse,

$$\psi\colon \mathbb{P}^2 \dashrightarrow V.$$

Given $[R : S : T]$, we send this to $[R : S : T : ST/R]$. Clearly this lies on the quadric $XW - YZ$ and is indeed the inverse map. Note that the inverse map blows up q and r then blows down the line connecting them to p.

In fact it turns out that the picture above for rational maps on surfaces is the complete picture.

Theorem 3.6 (Elimination of Indeterminancy)**.** *Let $\phi\colon S \dashrightarrow Z$ be a rational map from a smooth surface.*

Then there is an iterated sequence of blow ups of points $p\colon T \longrightarrow S$ such that the induced rational map $\psi\colon T \longrightarrow Z$ is a morphism.

Theorem 3.7. *Let $\phi\colon S \dashrightarrow T$ be a birational map of smooth surfaces.*

Then there is an iterated sequence of blow ups of points $p\colon W \longrightarrow S$ such that the induced map $q\colon W \longrightarrow T$ is also an iterated sequence of blow up of points, composed with an isomorphism.

In fact it turns out that both of these results generalise to all dimensions. In the first result, one must allow blowing up the ideal of any smooth subvariety. In the second result, one must allow mixing up the sequence of blowing up and down, although it is conjectured that the one can perform first a sequence of blow ups and then a sequence of blow downs.

Another way to proceed, is to compute the field of fractions. The coordinate ring on the affine piece $W \neq 0$ is

$$K[x, y, z]/\langle x - yz\rangle = k[y, z].$$

The field of fractions is visibly then $K(y, z)$. However perhaps the easiest way to proceed is to observe that $\mathbb{P}^1 \times \mathbb{P}^1$ contains $\mathbb{A}^1 \times \mathbb{A}^1 \simeq \mathbb{A}^2$, so that the Segre Variety is clearly rational.

In fact it turns out in general to be a vary hard problem to determine which varieties are rational. As an example of this consider Lüroth's problem.

Definition 3.8. *We say that a variety X is **unirational** if there is a dominant rational map $\phi\colon \mathbb{P}^n \dashrightarrow X$.*

Question 3.9 (Lüroth)**.** *Is every unirational variety rational?*

Note that one way to restate Lüroth's problem is to ask if every subfield of a purely transcendental field extension is purely transcendental. It turns out that the answer is yes in dimension one, in all characteristics. This is typically a homework problem in a course on Galois Theory.

In dimension two the problem is already considerably harder, and it is false if one allows inseparable field extensions. The first step is in fact to establish (3.6) and (3.7).

In dimension three it was shown to be false even in characteristic zero, in 1972, using three different methods.

One proof is due to Artin and Mumford. It had been observed by Serre that the cohomology ring of a smooth unirational threefold is indistinguishable from that of a rational variety (for $\mathbb{P}^3$ one gets $\frac{\mathbb{Z}[x]}{\langle x^4\rangle}$, and the cohomology ring varies in a very predictable under blowing up and down) except possibly that there might be torsion in $H^3(X, \mathbb{Z})$. They then give a reasonably elementary construction of a threefold with non-zero torsion in H^3.

Another proof is due to Clemens and Griffiths. It is not hard to prove that every smooth cubic hypersurface in $\mathbb{P}^4$ is unirational. On the other hand they prove that some smooth cubics are not rational. To prove this consider the family of lines on the cubic. It turns out that this is a two dimensional family, and that a lot of the geometry of the cubic is controlled by the geometry of this surface.

The third proof is due to Iskovskikh and Manin. They prove that every smooth quartic in $\mathbb{P}^4$ is not rational. On the other hand, some quartics are unirational. In fact they show, in an amazing tour de force, that the birational automorphism group of a smooth quartic is finite. Clearly this means that a smooth quartic is never rational.

4. Images of Varieties

Given a morphism $f\colon X \longrightarrow Y$ of quasi-projective varieties, a basic question might be to ask what is the image of a closed subset $Z \subset X$. Replacing X by Z we might as well assume that $Z = X$.

At first this question seems quite hopeless; indeed our first hope is that the image of X is always a quasi-projective subvariety. Unfortunately this is definitely not true. For example, take $X = Y = \mathbb{A}^2$. Let $\pi\colon X \longrightarrow Y$ be the morphism $(a, b) \longrightarrow (a, ab)$. Let us determine the image. Pick $(x, y) \in \mathbb{A}^2$. If $x \neq 0$, then take $a = x$ and $b = y/x$. Then $(a, ab) = (x, y)$. Thus the image contains the complement of the x-axis. Now if $y \neq 0$ and $x = 0$, then (x, y) is surely not in the image. However $(0, 0)$ is in the image; indeed it is the image of $(0, 0)$. Thus the image is equal to the complement of the x-axis union the origin.

In fact, it turns out that this is as bad as it gets. The first case to deal with, in fact the crucial case, which is of interest in its own right, is the case when X is projective.

Definition 4.1. *Let $f\colon X \longrightarrow Y$ be a function between two topological spaces. We say that f is **proper** if f takes closed sets to closed sets.*

Theorem 4.2. *Every morphism $\pi\colon X \longrightarrow Y$ of varieties, where X is projective, is proper.*

Definition 4.3. *Let $i\colon X \longrightarrow Y$ be a morphism. We say that i is **closed** if the image of X is closed. We say that i is a **closed embedding** if i is closed and i is an isomorphism onto its image.*

Definition 4.4. *Let $\pi\colon X \longrightarrow Y$ be a morphism.*

*We say that π is a **projective morphism** if it can be factored into a closed embedding $i\colon X \longrightarrow Y \times \mathbb{P}^n$ and the projection morphism $Y \times \mathbb{P}^n \longrightarrow Y$.*

Obvious examples of projective morphisms are blow ups. Also

Lemma 4.5. *Every morphism from a projective variety is projective.*

Proof. Just take the graph. □

Clearly closed embeddings are proper and the composition of proper maps is proper. Thus to prove (4.2) it suffices to prove:

Theorem 4.6. *Every projective morphism is proper.*

Moreover we may assume that $X \subset Y \times \mathbb{P}^n$ and that we are projecting onto the first factor. The trick is to reduce to the case $n = 1$. The idea is that projective space $\mathbb{P}^n$, via projection, is very close to the product $\mathbb{P}^1 \times \mathbb{P}^{n-1}$.

Definition 4.7. *Let $\pi\colon X \longrightarrow Y$ be a morphism of varieties. We say that π is a fibre bundle, with fibre F, if we can find a cover of the base Y, such that over each open subset U of the cover, $\pi^{-1}(U) \simeq U \times F$.*

Note that if π is a fibre bundle then every fibre of π is surely a copy of F. It is convenient to denote $\pi^{-1}(U)$ by $X|_U$.

Lemma 4.8. *The graph of the projection map from a point p defines a morphism $\Gamma \longrightarrow \mathbb{P}^{n-1}$, which is a fibre bundle, with fibre $\mathbb{P}^1$.*

Proof. The rational map given by projection from a point p

$$\pi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$

is clearly defined everywhere, except at the point of projection. Moreover this map is clearly constant on any line through p. Thus the morphism $\Gamma \longrightarrow \mathbb{P}^{n-1}$ has fibres equal to the lines trough p.

Pick two hyperplanes H_1 and H_2, neither of which contain p. Under projection, we may indentify H_1 with the base $\mathbb{P}^{n-1}$. Let $V = H_1 \cap H_2$. Then the image of V is a hyperplane in $\mathbb{P}^{n-1}$. Let U be the complement. Projection from V defines a rational map down to $\mathbb{P}^1$. This rational map is an isomorphism on every line l through p which does not intersect V.

Define a morphism $\psi\colon \Gamma|_U \longrightarrow \mathbb{P}^1 \times U \subset \mathbb{P}^1 \times \mathbb{P}^{n-1}$ via these two projection maps. It is not hard to see that ψ is an isomorphism. Fixing H_1 and varying H_2 it is clear that we get a cover of $\mathbb{P}^{n-1}$ in this way. $\square$

One nice property of both (4.2) and (4.6) is that they may be checked locally on the base.

Lemma 4.9. *Let U_α be an open cover of Y.*

Then $\pi(X)$ is closed if and only if $\pi(X|_{U_\alpha})$ is closed, for every α.

Proof. Clear, since a subset $A \subset Y$ is closed if and only if its intersection with every element of the open cover is closed. $\square$

Lemma 4.10. *To prove* (4.6) *we may assume that $n = 1$.*

Proof. Let $X \subset Y \times \mathbb{P}^n$. If $X = Y \times \mathbb{P}^n$ then there is nothing to prove, since the image is the whole of Y. Otherwise pick a point p such that X is not contained in $Y \times \{p\}$. Let $q\colon Y \times \Gamma \longrightarrow Y \times \mathbb{P}^n$ be the blow up of $Y \times \{p\}$ (equivalently blow up $\mathbb{P}^n$ at p and then cross with Y). Let X' be the strict transform of X. Then the image of X' and X in Y coincide.

Now by (4.8), the morphism $Y \times \Gamma \longrightarrow Y$ factors through $Y \times \mathbb{P}^{n-1}$. By induction on n, it suffices to prove that the image of $X \subset Y \times \Gamma$ inside in $Y \times \mathbb{P}^{n-1}$ is closed.

By (4.9) we are free to replace $Y \times \mathbb{P}^{n-1}$ by any open subset. Then by (4.8) we may assume $\Gamma = \mathbb{P}^{n-1} \times \mathbb{P}^1$. Replacing Y by $Y \times \mathbb{P}^{n-1}$ we are done. $\square$

The idea now is to work locally on Y and think of $Y \times \mathbb{P}^1$ as being $\mathbb{P}^1$ over a funny field.

Lemma 4.11. *Let $X \subset Y \times \mathbb{P}^1$ be a closed subset.*

Then locally about every point of Y, X is defined by polynomials $F(S,T)$, where $[S : T]$ are homogeneous coordinates on $\mathbb{P}^1$ and the coefficients of F belong to the coordinate ring of Y.

Proof. This is easy. If Y is affine, then we can cover $Y \times \mathbb{P}^1$ by two open affine sets $Y \times U_0$ and $Y \times U_1$. In this case X is locally defined, on each piece, by polynomials $f(s)$ and $g(t)$, where $s = S/T$ and $t = T/S$ and the coefficients of f and g belong to $A(Y)$. Since $f(s) = g(t)$ on $Y \times (U_0 \cap U_1)$ it follows that there is a global polynomial $F(S,T)$ with coefficients in $A(Y)$ which on each piece affine piece reduces to $f(s)$ and $g(t)$. $\square$

In other words, we only need to consider polynomials $F(S,T) \in A(Y)[S,T]$. Given $y \in Y$, let $F_y = F_y(S,T) \in K[S,T]$ be the polynomial we obtain by subsituting in $y \in Y$ to the coefficients.

Lemma 4.12. *Let $X \subset Y \times \mathbb{P}^1$.*

Then $y \in \pi(X)$ if and only if for every pair of functions $F(S,T)$ and $G(S,T) \in A(Y)[S,T]$ vanishing on X, both $F_y(S,T)$ and $G_y(S,T)$ have a common zero on $\{y\} \times \mathbb{P}^1$.

Proof. One inclusion is clear. So suppose that $y \notin \pi(X)$. Pick $F(S,T)$ that does not vanish on $\{y\} \times \mathbb{P}^1$. Then $F_y(S,T)$ has only finitely many zeroes. For each such zero p_i, we may find $G^i(S,T)$ such that $G^i_y(S,T)$ does not vanish at p_i. Taking an appropriate linear combination of the G^i gives us a polynomial G such that F_y and G_y do not have a common zero. $\square$

Lemma 4.13. *To prove* (4.6) *we may assume that X is defined by two polynomials F and G.*

To finish off, the idea is to use elimination theory.

Definition-Lemma 4.14. *Let A be a ring, and let F and G be two polynomials in $A[S,T]$, of degrees d and e.*

Let $R(F,G) \in A$ be the determinant of the following matrix

$$\begin{vmatrix} f_0 & f_1 & f_2 & \dots & f_{d-1} & f_d & \dots & \dots \\ 0 & f_0 & f_1 & f_2 & \dots & f_{d-1} & f_d & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & f_0 & f_1 & f_2 & \dots & f_d \\ g_0 & g_1 & g_2 & \dots & g_{e-1} & g_e & \dots & \dots \\ 0 & g_0 & g_1 & g_2 & \dots & g_{e-1} & g_e & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & g_0 & g_1 & g_2 & \dots & g_e \end{vmatrix},$$

where $f_0, f_1, \dots, f_d$ and $g_0, g_1, \dots, g_e$ are the coefficients of F and G.

Then for every maximal ideal $\mathfrak{m}$ of A, $\bar{R}(F,G) = 0$ in the quotient ring $S/\mathfrak{m}$ if and only if the two polynomials $\bar{f}$ and $\bar{g}$ have a common zero.

Proof. Since expanding a determinant commutes with passing to the quotient $A/\mathfrak{m}$, we might as well assume that $S = K$ is a field.

Now note that the rows of this matrix correspond to the polynomials $S^iT^{e-1-i}F$ and $S^jT^{d-1-j}G$, where $0 \leq i \leq e-1$ and $0 \leq j \leq d-1$, expanded in the standard basis of the vector space P_{d+e-1} of polynomials of degree $d+e-1$. Thus the determinant is zero if and only if the polynomials $B = \{S^iT^{e-1-i}F, S^jT^{d-1-j}G\}$ are dependent, inside P_{d+e-1}.

To finish off then it suffices to prove that this happens only when the two polynomials share a common zero. Now note that P_{d+e-1} has dimension $d+e$. Thus the $d+e$ polynomials B are independent if and only if they are a basis. Suppose that they share a common zero. Then the space spanned by B is contained in the vector subspace of all polynomials vanishing at the given point, and so B does not span. Now suppose that they are dependent. Collecting terms, there are then two polynomials P and Q of degrees $e-1$ and $d-1$ such that

$$PF + QG = 0.$$

Suppose that $d \leq e$. Every zero of G must be a zero of PF. As G has e zeroes and P has at most $e-1$ zeroes, it follows that one zero of G must be a zero of F. $\square$

Proof of (4.6). By (4.13) it suffices to prove the result when $n = 1$ and X is defined by two polynomials F and G. In this case $\pi(X)$ is precisely given by the resultant of F and G, which is an element of $A(Y)$. $\square$

(4.2) has the following very striking consequence.

Corollary 4.15. *Every regular function on a connected projective variety is constant.*

Proof. By definition a regular function is a morphism $f\colon X \longrightarrow \mathbb{A}^1$. Now by (4.2) the image of X is closed in $\mathbb{A}^1$. The only closed subsets of $\mathbb{A}^1$ are finite sets of points or the whole of $\mathbb{A}^1$. On the other hand f extends in an obvious way to a morphism $g\colon X \longrightarrow \mathbb{P}^1$. We haven't changed the image, but the image is now also a closed subset of $\mathbb{P}^1$. Thus the image cannot be $\mathbb{A}^1$.

Thus the image is a finite set of points. As X is connected, the image is connected and so the image is a point. □

Corollary 4.16. *Let X be a closed and connected subset of an affine variety.*

If X is also projective then X is a point.

Proof. By assumption $X \subset \mathbb{A}^n$. Suppose that X contains at least two points. Then at least one coordinate must be different. Let f be the function on $\mathbb{A}^n$ corresponding to this coordinate. Then f restricts to a non-constant regular function on X, which contradicts (4.15). □

Corollary 4.17. *Let $X \subset \mathbb{P}^n$ be a closed subset and let H be a hypersurface.*

If X is not a finite set of points, then $H \cap X$ is non-empty.

Proof. Suppose not. Let G be the defining equation of H. Pick F of degree equal to the degree of G. Then F/G is a regular function on X, since G is nowhere zero on X. But this contradicts (4.15). □

We can now answer our original question.

Definition 4.18. *Let X be a topological space. A subset $Z \subset X$ is said to be* ***constructible*** *if it is the finite union of locally closed subsets.*

Note that constructible sets are closed under complements and finite intersections and unions.

Lemma 4.19. *Let X be a Noetherian topological space and let Z be a subset.*

Then Z is constructible if and only if it is of the form

$$Z = Z_0 - (Z_1 - (Z_2 - \cdots - Z_k)),$$

where Z_i are closed and decreasing subsets.

Proof. Suppose that Z is constructible. Let Z_0 be the closure of Z. Then Z is dense in Z_0 and Z_0 is closed. As Z is constructible, it contains a dense open subset of Z_0. Clearly the difference $Z_0 - Z$ is constructible. Let Z_1 be the closure. Then Z_1 is a proper closed subset of Z_0. Continuing in this way, we construct a decreasing sequence of closed subsets,

$$Z_0 \supset Z_1 \supset \dots \supset Z_k \supset \dots$$

As X is Noetherian this sequence must terminate.

Now suppose that Z is an alternating difference of closed subsets,

$$Z = Z_0 - (Z_1 - (Z_2 - \dots - Z_{2k+1})).$$

Then

$$Z = (Z_0 - Z_1) \cup (Z_2 - Z_3) \cup \dots \cup (Z_{2k} - Z_{2k+1}). \qquad \square$$

Theorem 4.20 (Chevalley's Theorem). *Let $\pi\colon X \longrightarrow Y$ be a morphism of quasi-projective varieties.*

Then the image of a constructible set is constructible.

Proof. As the image of a union is the union of the images, it suffices to prove that the image of a locally closed subset is constructible. Suppose that Z is a locally closed subset. Replacing X by the closure of Z and Y by the closure of the image, we may assume that $\pi|_Z$ is dominant. Suppose that $\pi(Z)$ contains an open subset. Replacing X by the complement of the inverse image, we are then done by Noetherian induction.

Thus we are reduced to proving that $\pi(Z)$ contains an open subset. Replacing X by an open subset, we may assume that X is affine. Replacing X by its graph and applying induction on n, we may assume that $Z \subset \mathbb{A}^n$ and that the map is the restriction of the projection map

$$\mathbb{A}^n \longrightarrow \mathbb{A}^{n-1}$$

where

$$(x_1, x_2, \dots, x_n) \longrightarrow (x_1, x_2, \dots, x_{n-1}).$$

Thus we may assume that $Z \subset Y \times \mathbb{A}^1$ and that we are projecting onto Y. Clearly we may replace $\mathbb{A}^1$ by $\mathbb{P}^1$. Working locally, we may assume that every closed subset of $Y \times \mathbb{P}^1$ is defined by polynomials of the form $F(S,T)$.

Let X be the closure of Z and let V be the complement, so that $X = Z - V$ and both Z and V are closed. Suppose that $X = Y \times \mathbb{P}^1$. In this case it suffices to prove that

$$V_Y = \{\, y \in Y \mid \{y\} \times \mathbb{P}^1 \subset V \,\},$$

is contained in a proper closed subset. But V is a proper closed subset so that there is a polynomial G vanishing on V. In this case, V contains the whole fibre if and only if every coefficient of G_y vanishes. Thus the locus V_Y is contained in the vanishing locus of all the coefficients of G.

So we may assume that X is a proper closed subset of $Y \times \mathbb{P}^1$. Thus there is a polynomial F vanishing on X. Since X is closed, its image is closed, whence the whole of Y. It suffices to prove that $\pi(V)$ is a proper closed subset. It is certainly closed, as V is closed. But $R(F, G)$ is a non-zero polynomial that vanishes on the image. $\square$

5. Some naive enumerative geometry

Question 5.1. *How many lines meet four fixed lines in $\mathbb{P}^3$?*

Let us first check that this question makes sense, that is, let us first check that the answer is finite.

Definition 5.2. $\mathbb{G}(k, n)$ *denotes the space of* r*-dimensional linear subspaces of* $\mathbb{P}^n$.

We will assume that we have constructed the **Grassmannian** as a variety. The first natural question then is to determine the dimension of $\mathbb{G}(1, 3)$. We do so in an ad hoc manner. A line l in $\mathbb{P}^3$ is specified by picking two points p and q. Now the set of choices for two points p and q is equal to $\mathbb{P}^3 \times \mathbb{P}^3 - \Delta$, where Δ is the diagonal. Thus the set of choices of pairs of distint points is six dimensional.

Fix a line l. Then if we pick any two points p and q of this line, they give us the same line l. Thus the the Grassmannian of lines in $\mathbb{P}^3$ is $6 = 2 = 4$-dimensional.

It might help to look at this differently. Let

$$\Sigma = \{\, (P, l) \mid P \in l \,\} \subset \mathbb{G}(1, 3) \times \mathbb{P}^3.$$

Then Σ is a closed subset of the product $\mathbb{G}(1, 3) \times \mathbb{P}^3$. There are two natural projection maps.

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^3 \\ \mathbb{G}(1,3). & \end{tikzcd}$$

In fact Σ (together with this diagram) is called an **incidence correspondence**. It is interesting to consider the two morphisms p and q. First p. Pick a line $l \in \mathbb{G}(1, 3)$. Then the fibre of p over l consists of all points P that are contained in l, so that the fibres of p are all isomorphic to $\mathbb{P}^1$. Now consider the morphism q. Fix a point P. Then the fibre of q over P consists of all lines that contain P. Again the fibres of q are isomorphic.

To compute the dimension of $\mathbb{G}(1, 3)$, we compute the dimension of Σ in two ways, borrowing the following result from later in the class.

Theorem 5.3. *Let* $\pi\colon X \longrightarrow Y$ *be a dominant morphism of irreducible varieties.*

Then there is an open subset U *of* Y*, such that for every* $y \in U$*, the dimension of the fibre of* π *over is equal to* k*, a constant. Moreover the dimension of* X *is equal to the dimension of* Y *plus* k*.*

We first apply (5.3) to q. The dimension of the base is 3. As every fibre is isomorphic, to compute k, we can consider any fibre. Pick any point P. Pick an auxiliary plane, not passing through P. Then the set of lines containing P is in bijection with the points of this auxiliary plane, so that the dimension of a fibre is two. Thus the dimension of $\Sigma = 3 + 2 = 5$.

Now we apply (5.3) to p. The dimension of any fibre is one. Thus the dimension of the Grassmannian is $5 - 1 = 4$, as before.

The next question is to determine how many conditions it is to meet a fixed line l_1. I claim it is one condition. The easiest way to see this, is to just to imagine swinging a sword around in space. This will cut any line into two. Thus any one dimensional family of lines meets a given line finitely many times.

More formally, carry out the computation above, replacing Σ with Σ_1, the space of lines which meet l_1. The fibre of q over a point P is now a copy of $\mathbb{P}^1$ (parametrised by l_1). Thus Σ_1 has dimension 4 and the space of lines which meets l_1 has dimension $4 - 1 = 3$.

Thus we have a threefold in the fourfold $\mathbb{G}(1, 3)$. Clearly we expect that four of them will intersect in a finite set of points.

There are two ways to proceed. Here is one which uses the Segre variety:

Lemma 5.4. *Let l_1, l_2 and l_3 and m_1, m_2 and m_3 be two sequences of skew lines in $\mathbb{P}^3$.*

Then there is an element of $\mathrm{PGL}_4(K)$ carrying the first sequence to the second.

Proof. We may as well assume that the first set is given as

$$X = Y = 0 \qquad Z = W = 0 \qquad \text{and} \qquad X - Z = Y - W = 0.$$

Clearly we may find a transformation carrying m_1 to l_1 and m_2 to l_2. For example, pick four points on both sets of lines, and use the fact that any four sets of points in linear general position are projectively equivalent.

Consider the two planes $X = 0$ and $W = 0$. m_3 cannot lie in either of these planes, else either the lines m_1 and m_3 or the lines m_2 and m_3 would not be skew. Consider the two points $[0 : a : b : c]$ and $[d : e : f : 0]$ where m_3 intersects the planes $X = 0$ and $W = 0$.

Clearly m_3 is determined by these points, and in the case of l_3, we have $a = c = d = f = 1$, $b = e = 0$. Pick an element $\phi \in \mathrm{PGL}_4(K)$ and represent it as a 4×4 matrix. If we decompose this 4×4 matrix in block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where each block is a 2×2 matrix, the condition that ϕ fix $l_1 = m_1$ and $l_2 = m_2$ is equivalent to the condition that $B = C = 0$. By choosing A and D appropriately, we reduce to the case that $b = e = 0$. In this case the matrix

$$\begin{pmatrix} 1/d & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \\ 0 & 0 & 1/f & 0 \\ 0 & 0 & 0 & 1/c \end{pmatrix}$$

carries the two points to the standard two points, so that it carries m_3 to l_3. □

This result has the following surprising consequence.

Lemma 5.5. *Let l_1, l_2 and l_3 be three skew lines in $\mathbb{P}^3$.*

Then the family of lines that meets all three lines sweeps out a quadric surface in $\mathbb{P}^3$.

Proof. By (5.4) we may assume that the three lines are any set of three skew lines in $\mathbb{P}^3$. Now the Segre variety V in $\mathbb{P}^1 \times \mathbb{P}^1$ contains three skew lines (just choose any three lines of one of the rulings). Moreover any line of the other ruling certainly meets all three lines. So the set of lines meeting all three lines, certainly sweeps out at least a quadric surface.

To finish, suppose we are given a line l that meets l_1, l_2 and l_3. Then l meets V in three points. As V is defined by a quadratic polynomial, it follows that l is contained in V. Thus any line that meets all three lines, is contained in V. □

Theorem 5.6. *There are two lines that meet four general skew lines in $\mathbb{P}^3$.*

Proof. Fix the first three lines l_1, l_2 and l_3. We have already seen that the set l of lines that meets all three of these lines is precisely the set of lines of one ruling of the Segre variety (up to choice of coordinates).

Pick a line l_4 that meets V transversally in two points. Now for a line l of one ruling to meet the fourth line l_4, it must meet l_4 at a point $P = l \cap l_4$ of V. Moreover this point determines the line l. □

Here is an entirely different way to answer (5.1). Consider using the principle of continuity. Take two of the four lines and deform them so they become concurrent (or what comes to exactly the same thing, coplanar). Similarly take the other pair of lines and degenerate them until they also become concurrent.

Now consider how a line l can meet the four given lines.

Lemma 5.7. *Let l be a line that meets two concurrent lines l_1 and l_2 in $\mathbb{P}^3$.*

Then either l contains $l_1 \cap l_2$ or l is contained in the plane $\langle l_1, l_2 \rangle$.

Proof. Suppose that l does not contain the point $l_1 \cap l_2$. Then l meets l_i, $i = 1$, 2 at two points p_i contained in the plane $\langle l_1, l_2 \rangle$. □

Thus if l meets all four lines, there are three possibilties.

(1) l contains both points of intersection.
(2) l is contained in both planes.
(3) l contains one point and is contained in the other plane.

Clearly there is only one line that satisfies (1). It is not so hard to see that there is only one line that satisfies (2), it is the intersection of the two planes. Finally it is not so hard to see that (3) is impossible. Just choose the point outside of the plane.

Thus the answer is two. It is convenient to introduce some notation to compute these numbers, which is known as Schubert calculus. Let l denote the condition that we meet a fixed line. We want to compute l^4. We proceed formally. We have already seen that

$$l^2 = l_p + l_\pi$$

where l_p denotes the condition that a line contains a point, and l_π is the condition that l is contained in π.

Thus

$$\begin{aligned} l^4 &= (l^2)^2 \\ &= (l_p + l_\pi)^2 \\ &= l_p^2 + 2l_p l_\pi + l_\pi^2 \\ &= 1 + 2 \cdot 0 + 1 = 2, \end{aligned}$$

where the last line is computed as before.

6. Grasmannians

We first treat Grassmanians classically. Fix an algebraically closed field K. We want to parametrise the space of k-planes W in a vector space V. The obvious way to parametrise k-planes is to pick a basis $v_1, v_2, \dots, v_k$ for W. Unfortunately this does not specify W uniquely, as the same vector space has many different bases. However, the line spanned by the vector

$$\omega = v_1 \wedge v_2 \wedge \dots \wedge v_k \in \bigwedge^k V,$$

is invariant under re-choosing a basis.

Definition 6.1. *The* ***Grassmannian*** $G(k,V)$ ***of*** k***-planes in*** V *is the set of rank one vectors in* $\mathbb{P}(\bigwedge^k V)$.

We set $G(k,n) = G(k,K^n)$ and $\mathbb{G}(k,n) = G(k+1,n+1)$. The latter may be thought of as the set of k-planes in $\mathbb{P}^n$.

The embedding of the Grassmannian inside $\mathbb{P}(\bigwedge^k V)$ is known as the Plücker embedding. If we choose a basis $e_1, e_2, \dots, e_n$ for V, then a general element of $\bigwedge^k V$ is given by

$$\sum_I p_I e_I,$$

where I ranges over all collections of increasing sequences of integers between 1 and n,

$$i_1 < i_2 < \dots < i_k,$$

and e_I is shorthand for the wedge of the corresponding vectors,

$$e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}.$$

The coefficients p_I are naturally coordinates on $\mathbb{P}(\bigwedge^k V)$, which are known as the Plücker coordinates.

There is another way to look at the construction of the Grassmannian which is very instructive. If we pick a basis $e_1, e_2, \dots, e_n$ for V, then let A be the $k \times n$ matrix whose rows are $v_1, v_2, \dots, v_k$, in this basis. As before, this matrix does not uniquely specify $W \subset V$, since we could pick a new basis for W. However the operation of picking a new basis corresponds to taking linear combinations of the rows of our matrix, which in turn is the same as multiplying our matrix by a $k \times k$ invertible matrix on the left. In other words the Grassmannian is the set of equivalence classes of $k \times n$ matrices under the action of $\mathrm{GL}_k(K)$ by multiplication on the left.

It is not hard to connect the two constructions. Given the matrix A, then form all possible $k \times k$ determinants. Any such determinant is determined by specifying the columns to pick, which we indicate by a multindex I. In terms of $\bigwedge^k V$, this is the same as picking a basis and expanding our vector as a sum

$$\sum_I p_I e_I,$$

where, as before, e_I is the wedge of the corresponding vectors. For example consider the case $k = 2$, $n = 4$ (lines in $\mathbb{P}^3$). We have a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

The corresponding plane is given as the span of the rows. We can form six two by two determinants. Clearly these are invariant, up to scalars, under the action of $\mathrm{GL}_2(K)$.

The Grassmannian has a natural cover by open affine subsets, isomorphic to affine space, in much the same way that projective space has a cover by open affines, isomorphic to affine space. Pick a linear space U of dimension $n - k$, and consider the set of linear spaces W of dimension k which are complementary to U, that is, which meet U only at the origin. Identify V with the sum $V/U + U$. Then a linear space W complementary to U can be identified with the graph of a linear map

$$V/U \longrightarrow U.$$

It follows that the subset of all linear spaces W complementary to U is equal to

$$\mathrm{Hom}(V/U, U) \simeq K^{k(n-k)} \simeq \mathbb{A}_K^{k(n-k)}.$$

Another way to see this is as follows. Consider the first $k \times k$ minor. Suppose that the corresponding determinant is non-zero, that is, the corresponding vectors are independent. In this case the $k \times k$ minor is equivalent to the identity matrix, and the only element of $\mathrm{GL}_k(K)$ which fixes the identity, is the identity itself. Thus we have a canonical representative of the matrix A for the linear space W. We are free to choose the other $k \times (n - k)$ block of the matrix, which gives us an affine space of dimension $k(n - k)$. The condition that the first $k \times k$ minor has non-zero determinant is an open condition, and this gives us an open affine cover by affine spaces of dimension $k(n - k)$. Note that the condition that the first $k \times k$ minor is invertible is equivalent to the condition that we do not meet the space given by the vanishing of the first k coordinates, which is indeed a linear space of dimension $n - k$.

It is interesting to write down the equations cutting out the image of the Grassmannian under the Plücker embedding, although this turns out to involve some non-trivial multilinear algebra. The problem is to characterise the set of rank one vectors ω in $\bigwedge^k V$.

Definition 6.2. *Let $\omega \in \bigwedge^k V$. We say that ω is* ***divisible*** *by $v \in V$ if there is an element $\phi \in \bigwedge^k V$ such that $\omega = \phi \wedge v$.*

Lemma 6.3. *Let $\omega \in \bigwedge^k V$.*

Then ω is divisible by v if and only if $\omega \wedge v = 0$.

Proof. This is easy. If $\omega = \phi \wedge v$, then

$$\begin{aligned} \omega \wedge v &= \phi \wedge v \wedge v \\ &= 0. \end{aligned}$$

To see the other direction, extend v to a basis $v = e_1, e_2, \dots, e_n$ of V. Then we may expand ω in this basis.

$$\omega = \sum p_I e_I.$$

On the other hand

$$v \wedge e_I = \begin{cases} e_J & \text{if } 1 \notin I, \text{ where } J = \{1\} \cup I \\ 0 & \text{if } 1 \in I. \end{cases}$$

Thus $\omega \wedge v = 0$ if and only if $p_I \neq 0$ implies $1 \in I$ if and only if v divides ω. □

Lemma 6.4. *Let $\omega \in \bigwedge^k V$.*

Then ω has rank one if and only if the linear map

$$\phi(\omega)\colon V \longrightarrow \bigwedge^{k+1} V \qquad\qquad v \longrightarrow \omega \wedge v,$$

has rank at most $n - k$.

Proof. Indeed $\phi(\omega)$ has rank at most $n - k$ if and only if the linear subspace of vectors dividing ω has dimension at least k if and only if ω has rank one. □

Now the map

$$\phi\colon \bigwedge^{k} V \longrightarrow \operatorname{Hom}(V, \bigwedge^{k+1} V),$$

is clearly linear. Thus the map ϕ can be interpreted as a matrix whose entries are linear coordinates of $\bigwedge^k V$ and the locus we want is given by the vanishing of the $(n-k+1) \times (n-k+1)$ minors.

It follows that the Grassmannian is a closed subset of $\mathbb{P}(\bigwedge^k V)$. Unfortunately the equations we get in this way won't be best possible. In particular they won't generate the ideal of the Grassmannian (they only cut out the Grassmannian set theoretically). To find equations that generate the ideal, we have to work quite a bit harder.

Lemma 6.5. *There is a natural pairing between* $\bigwedge^k V$ *and* $\bigwedge^{n-k} V^*$. *This pairing is well-defined up to scalars and preserves the rank.*

Proof. There is a natural pairing

$$\bigwedge^k V \times \bigwedge^{n-k} V \longrightarrow \bigwedge^n V,$$

which sends

$$(\omega, \eta) \longrightarrow \omega \wedge \eta.$$

On the other hand, $\bigwedge^n V$ is one dimensional so that it is non-canonically isomorphic to K and $(\bigwedge^{n-k} V)^*$ is isomorphic to $\bigwedge^{n-k} V^*$. □

Given ω, let ω^* be the corresponding element of $\bigwedge^{n-k} V^*$. Now there is a natural map

$$\psi(\omega^*) \colon V^* \longrightarrow \bigwedge^{n-k+1} V^*$$

which sends

$$v^* \longrightarrow \omega^* \wedge v^*.$$

Further ω has rank one if and only if ω^* has rank one, which occurs if and only if $\psi(\omega^*)$ has rank at most k.

Moreover the kernel of $\phi(\omega)$, namely W, is precisely the annihilator of the kernel of $\psi(\omega^*)$. Dualising, we get maps

$$\phi^*(\omega) \colon \bigwedge^{k+1} V^* \longrightarrow V^* \qquad \text{and} \qquad \psi^*(\omega) \colon \bigwedge^{n-k+1} V \longrightarrow V,$$

whose images annihilate each other if and only if ω has rank one.

Thus ω has rank one if and only if for every $\alpha \in \bigwedge^{k+1} V^*$ and $\beta \in \bigwedge^{n-k+1} V^*$,

$$\Xi_{\alpha,\beta}(\omega) = \langle \phi^*(\omega)(\alpha), \psi^*(\omega)(\beta) \rangle = 0.$$

Now $\Xi_{\alpha,\beta}$ are quadratic polynomials, which are known as the Plücker relations. It turns out that they do indeed generate the ideal of the Grassmannian.

It is interesting to see what happens when $k = 2$:

Lemma 6.6. *Let* $\omega \in \bigwedge^2 V$.

Then ω *has rank one if and only if* $\omega \wedge \omega = 0$.

Proof. One direction is clear, in fact for every k, if ω has rank one then $\omega \wedge \omega = 0$.

To see the other direction, we need to prove that if ω has rank at least two, then $\omega \wedge \omega \neq 0$. First observe that if ω has rank at least two, then we may find a projection down to a vector space of dimension four, such that the image has rank two. Thus we may assume that V has dimension four and ω has rank two. In this case, up to change of coordinates,

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4,$$

and by direct computation, $\omega \wedge \omega$ is not zero. $\square$

Now

$$\omega = \sum_{i,j} p_{i,j} e_i \wedge e_j.$$

Suppose that $n = 4$. If we expand

$$\omega \wedge \omega,$$

then everything is a multiple of $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We need to pick a term from each bracket, so that the union is $\{1, 2, 3, 4\}$. In other words, the coefficient of the expansion is a sum over all partitions of $\{1, 2, 3, 4\}$ into two equal parts. By direct computation, we get

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

In particular, $\mathbb{G}(1, 3)$ is a quadric in $\mathbb{P}^5$ of maximal rank.

7. The universal family

As with the space of conics in $\mathbb{P}^2$, the main point of the Grassmannian, is that it comes with a universal family. We first investigate what this means in the baby case of quasi-projective varieties before we move on to the more interesting case of schemes.

Definition 7.1. *A* ***family of*** k***-planes in*** $\mathbb{P}^n$ ***over*** B *is a closed subset* $\Sigma \subset B \times \mathbb{P}^n$ *such that the fibres, under projection to the first factor, are identified with* k*-planes in* $\mathbb{P}^n$*.*

Definition 7.2. *Let* F *be the functor from the category of varieties to the category of sets, which assigns to every variety, the set of all (flat) families of* k*-planes in* $\mathbb{P}^n$*, up to isomorphism.*

Theorem 7.3. *The Grassmannian* $\mathbb{G}(k, n)$ *represents the functor* F*.*

It might help to unravel some of the definitions. Suppose that we are given a variety B. Essentially we have to show that there is a natural bijection of sets,

$$F(B) = \operatorname{Hom}(B, \mathbb{G}(k, n)).$$

The set on the left is nothing more than the set of all families of k-planes in $\mathbb{P}^n$, with base B. In particular given a morphism $f\colon B \longrightarrow \mathbb{G}(k, n)$, we are supposed to produce a family of k-planes over B. Here is how we do this. Suppose that we have constructed the natural family of k-planes over $\mathbb{G}(k, n)$,

$$\begin{tikzcd} \Sigma \arrow[r, hook] & \mathbb{G}(k,n) \times \mathbb{P}^n \arrow[d] \\ & \mathbb{G}(k,n), \end{tikzcd}$$

so that the fibre over $[\Lambda] \in \mathbb{G}(k, n)$ is exactly the set,

$$\{[\Lambda]\} \times \Lambda \subset \{[\Lambda]\} \times \mathbb{P}^n$$

that is, the k-plane Λ sitting inside $\mathbb{P}^n$. Then we obtain a family of k-planes over B, simply by taking the fibre square,

$$\begin{tikzcd} \Sigma' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Sigma \arrow[d] \\ B \arrow[r, "f"] & \mathbb{G}(k,n). \end{tikzcd}$$

For this reason, we call the family $\Sigma \longrightarrow \mathbb{G}(k, n)$ the universal family. Note that we can reverse this process. Suppose that $\mathbb{G}(k, n)$ represents the functor F. By considering the identity morphism $\mathbb{G}(k, n) \longrightarrow \mathbb{G}(k, n)$, we get a family $\Sigma \longrightarrow \mathbb{G}(k, n)$, which is universal, in the sense that to obtain any other family, over any other base, we simply pullback Σ along the morphism $f\colon B \longrightarrow \mathbb{G}(k,n)$, whose existence is guaranteed by the universal property of $\mathbb{G}(k,n)$ (that is, that it represents the functor). To summarise the previous discussion: to prove (7.3) it suffices to construct the natural family over $\mathbb{G}(k,n)$ and prove that it is the universal family.

8. Prim and proper

Okay prim is not a property of schemes, but proper and separated are. In this section we want to extend the intuitive notions of being Hausdorff and compact to the category of schemes.

First we come up with a formal definition of both properties and then we investigate how to check the formal definitions in practice. We start with the definition of separated, which should be thought as corresponding to Hausdorff.

Definition 8.1. *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. The* ***diagonal morphism*** *is the unique morphism $\Delta\colon X \longrightarrow X \underset{Y}{\times} X$, given by applying the universal property of the fibre product to the identity map $X \longrightarrow X$, twice.*

We say that the morphism f is ***separated*** *if the diagonal morphism is a closed immersion.*

Example 8.2. *Consider the line X, with a double origin, obtained by gluing together two copies of $\mathbb{A}^1_k$, without identifying the origins. Consider the fibre square over k, $X \underset{k}{\times} X$. This is a doubled affine plane, which has two x-axes, two y-axes and four origins. The diagonal morphism, only hits two of those four origins, whilst the closure contains all four origins.*

Proposition 8.3. *Every morphism of affine schemes is separated.*

Proof. Suppose we are given a morphism of affine schemes $f\colon X \longrightarrow Y$, where $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Then the diagonal morphism is given by,

$$A \underset{B}{\otimes} A \longrightarrow A \qquad \text{where} \qquad a \otimes a' \longrightarrow aa'.$$

As this is a surjective ring homomorphism, it follows that Δ is a closed immersion. $\square$

Corollary 8.4. *$f\colon X \longrightarrow Y$ is separated if and only if the image of the diagonal morphism is a closed subset.*

Proof. One direction is clear. So suppose that the image of the diagonal morphism is closed. We need to prove that $\Delta\colon X \longrightarrow \Delta(X)$ is a homeomorphism and that $\mathcal{O}_{X \underset{Y}{\times} X} \longrightarrow \Delta_* \mathcal{O}_X$ is surjective. Consider the first projection $p_1\colon X \underset{Y}{\times} X \longrightarrow X$. As the composition $p_1 \circ \Delta$ is the identity, it is immediate that Δ is a homeomorphism onto its image.

To check surjectivity of sheaves, we may work locally. Pick $p \in X$ and an open affine neighbourhood $V \subset Y$ of the image $q \in Y$. Let U

be an open affine neighbourhood of p contained in the inverse image of V. Then $U \underset{V}{\times} U$ is an open affine neighbourhood of $\Delta(p)$, and by (8.3), $\Delta\colon U \longrightarrow U \underset{V}{\times} U$ is a closed immersion. But then the map of sheaves is surjective on stalks at p. $\square$

The idea of how to characterise both properties (separated and proper), is based on the idea of probing with curves. After all, the clasic example of the doubled origin, admits an open immersion with two different extensions. Firstly, we need to work more locally than this, so that we want to work with local rings. However, our schemes are so general, that we also need to work with something more general than a curve. We recall some basic facts about valuations and valuation rings.

Definition 8.5. *Let K be a field and let G be a totally ordered abelian group. A* ***valuation*** *of K with values in G, is a map*

$$\nu\colon K - \{0\} \longrightarrow G,$$

such that for all x and $y \in K - \{0\}$ we have:
(1) $\nu(xy) = \nu(x) + \nu(y)$.
(2) $\nu(x + y) \geq \min(\nu(x), \nu(y))$.

Definition-Lemma 8.6. *If ν is a valuation, then the set*

$$R = \{\, x \in K \mid \nu(x) \geq 0 \,\} \cup \{0\},$$

is a subring of K, which is called the ***valuation ring*** *of ν. The set*

$$\mathfrak{m} = \{\, x \in K \mid \nu(x) > 0 \,\} \cup \{0\},$$

is an ideal in R and the pair $(R, \mathfrak{m})$ is a local ring.

Proof. Easy check. $\square$

Definition 8.7. *A valuation is called a* ***discrete valuation*** *if $G = \mathbb{Z}$. The corresponding valuation ring is called a* ***discrete valuation ring****.*

Let X be a variety. There is essentially one way to get a discrete valuation of the function field of X.

Example 8.8. *Let X be a smooth variety, and let x be a point of X. Then every element of $K = K(X)$ is of the form $f/g \in \mathcal{O}_{X,x}$. We define the* ***order of vanishing*** *of f/g along x to be the difference*

$$a - b,$$

where $f \in m^a - m^{a+1}$ and $g \in m^b - m^{b+1}$. Then the order of vanishing defines a valuation ν of K.

Definition 8.9. *Let A and B be two local rings, with the same field of fractions. We say that B* ***dominates*** *A if $A \subset B$ and $m_A = m_B \cap A$. Now let X be a variety and let ν be a valuation on X. We say that x is the* ***centre*** *of ν on X if $\mathcal{O}_{X,x}$ is dominated by the valuation ring of ν.*

Lemma 8.10. *Let R be a local ring which is an integral domain with field of fractions K. Then R is a valuation ring if and only if it is maximal with respect to dominance. Every local ring in K is contained a valuation ring.*

The centre, if it exists, is unique. In the example above, the unique centre is x. It is easy to see, however, that the centre does not determine the valuation.

Example 8.11. *Let S be a smooth surface. Let $T \longrightarrow S$ be any sequence of blow ups with centre x. Let E be any exceptional divisor with generic point $t \in T$. Then t determines a valuation ν on T, whence on S. The centre of ν is x.*

Note however that if the centre of a discrete valuation is a divisor, then there is essentially only one way to define the valuation, as the order of vanishing. Given this, the next natural question to ask, is if it is true that every discrete valuation is associated to a divisor.

Definition 8.12. *A discrete valuation ν of X is called* ***algebraic*** *if there is a birational model Y of X such that the centre of ν on Y is a divisor.*

Example 8.13. *Consider the affine plane over $\mathbb{C}$. Considering only the closed points, the curve $y = e^x$, the graph of the exponential function, defines a discrete valuation of the local ring $\mathcal{O}_{S,p}$, where $S = \mathbb{A}^2_{\mathbb{C}}$ and p is the origin. Given $f \in \mathcal{O}_{S,p}$ we just consider to what order f approximates the curve above.*

Put differently the smooth curve $y = e^x$ determines an infinite sequence of blow ups with centre p. At each stage we blow up the unique point where the strict transform of $y = e^x$ meets the new exceptional divisor. The valuation ν then counts how many points the blow up of $f = 0$ shares.

It is clear that ν is not an algebraic valuation.

Definition 8.14. *Let X be a topological space. We say that x_0 is a* ***specialisation*** *of x_1 if $x_0 \in \overline{\{x_1\}}$.*

Lemma 8.15. *Let R be a valuation ring with quotient field K. Let $T = \operatorname{Spec} R$ and let $U = \operatorname{Spec} K$. Let X be any scheme.*

(1) To give a morphism $U \longrightarrow X$ is equivalent to giving a point $x_1 \in X$ and an inclusion of fields $k(x_1) \subset K$.
(2) To give a morphism $T \longrightarrow X$ is equivalent to giving two points x_0, $x_1 \in X$, with x_0 a specialisation of x_1 and an inclusion of fields $k(x_1) \subset K$, such that R dominates the local ring $\mathcal{O}_{Z,x_0}$, in the subscheme $Z = \overline{\{x_1\}}$ of X, with its reduced induced structure.

Proof. We have already seen (1). Let t_0 be the closed point of T and let t_1 be the generic point. If we are given a morphism $T \longrightarrow X$, then let x_i be the image of t_i. As T is reduced, we have a factorisation $T \longrightarrow Z$. Moreover $k(x_1)$ is the function field of Z, so that there is a morphism of local rings $\mathcal{O}_{Z,x_0} \longrightarrow R$ compatible with the inclusion $k(x_1) \subset K$. Thus R dominates $\mathcal{O}_{Z,x_0}$.

Conversely suppose given x_0 and x_1. The inclusion $\mathcal{O}_{Z,x_0} \longrightarrow R$ gives a morphism $T \longrightarrow \operatorname{Spec} \mathcal{O}_{Z,x_0}$, and composing this with the natural map $\operatorname{Spec} \mathcal{O}_{Z,x_0} \longrightarrow X$ gives a morphism $T \longrightarrow X$. □

Lemma 8.16. *Let $f\colon X \longrightarrow Y$ be a compact morphism of schemes (that is, for every open affine subsheme $U \subset Y$, $f^{-1}(U)$ is compact).*

Then $f(X)$ is closed if and only if it is stable under specialisation.

Proof. Let us in addition suppose that f is of finite type and that X and Y are noetherian. Then $f(X)$ is constructible by Chevalley's Theorem, whence closed. For the general case, see Hartshorne, II.4.5. □

Now we are ready to state:

Theorem 8.17 (Valuative Criterion of Separatedness)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of schemes, and assume that X is Noetherian. Then f is separated if and only if the following condition holds:*

For any field K and for any valuation ring R with quotient field K, let $T = \operatorname{Spec} R$, let $U = \operatorname{Spec} K$ and let $i\colon U \longrightarrow T$ be the morphism induced by the inclusion $R \subset K$. Given morphisms $T \longrightarrow Y$ and $U \longrightarrow X$ which makes a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

there is at most one morphism $T \longrightarrow X$ which makes the diagram commute.

Proof. Suppose that f is separated, and that we are given two morphisms $h\colon T \longrightarrow X$ and $h'\colon T \longrightarrow X$, which make the diagram commute.

Then we get a morphism $h'' \colon T \longrightarrow X \underset{Y}{\times} X$. As the restrictions of h and h' to U agree, it follows that h'' sends the generic point t_1 of T to a point of the diagonal $\Delta(X)$. Since the diagonal is closed, it follows that t_0 is sent to a point of the diagonal. But then the images of t_0 and t_1, under h and h', are the same points x_0 and $x_1 \in X$. Since the inclusion $k(x_1) \subset K$ comes out the same, it follows that $h = h'$.

Now let us prove the other direction. It suffices to prove that $\Delta(X)$ is closed in $X \underset{Y}{\times} X$, which in turn is equivalent to proving that it is stable under specialisation. Suppose that $\xi_1 \in \Delta(X)$ and suppose that ξ_0 is in the closure of $\{\xi_1\}$. Let $K = k(\xi_1)$ and let A be the local ring of ξ_0 in the closure of ξ_1. Then $A \subset K$ and so there is a valuation ring R which dominates A. Then by (8.15) there is a morphism $T \longrightarrow X \underset{Y}{\times} X$, where $T = \operatorname{Spec} R$, sending t_i to ξ_i. Composing with either projection down to X, we get two morphisms $T \longrightarrow X$, which give the same morphism to Y and whose restrictions to U are the same, as $\xi_1 \in \Delta(X)$. By assumption then, these two morphisms agree, and so the morphism $T \longrightarrow X \underset{Y}{\times} X$ must factor through Δ. But then $\xi_0 \in \Delta(X)$, whence $\Delta(X)$ is closed. $\square$

Corollary 8.18. *Assume that all schemes are Noetherian*

(1) Open and closed immersions are separated.
(2) A composition of separated morphisms is separated.
(3) Separated morphisms are stable under base change.
(4) If $f \colon X \longrightarrow Y$ and $f' \colon X' \longrightarrow Y'$ are two separated morphisms over a scheme S, then the product morphism $f \times f' \colon X \underset{S}{\times} X' \longrightarrow Y \underset{S}{\times} Y'$ is also separated.
(5) If $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$ are two morphisms, such that $g \circ f$ is separated, then f is separated.
(6) A morphism $f \colon X \longrightarrow Y$ is separated if and only if Y can be covered by open subsets V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is separated for each i.

Proof. These all follow from (8.17). For example, consider the proof of (2). We are given $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$, two separated morphisms. By assumption we are given two morphisms $h \colon T \longrightarrow X$ and $h' \colon T \longrightarrow X$, as in (8.17). By composition with f, these give two morphisms $k \colon T \longrightarrow Y$ and $k' \colon T \longrightarrow Y$. As g is separated, these morphisms agree. But then as f is separated, $h = h'$. $\square$

Now we turn to the notion of properness.

Definition 8.19. *A morphism $f\colon X \longrightarrow Y$ is **proper** if it is separated, of finite type, and universally closed.*

Example 8.20. *The affine line $\mathbb{A}^1_k$ is certainly separated and of finite type over k. However it is not proper, since it is not universally closed. Indeed consider $\mathbb{A}^2_k = \mathbb{A}^1_k \underset{k}{\times} \mathbb{A}^1_k$. The image of the hyperbola under projection down to $\mathbb{A}^1_k$ is not closed.*

Theorem 8.21 (Valuative Criterion of Properness)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of finite type, with X Noetherian. Then f is proper if and only if for every valuation ring R and for every pair of morphisms $U \longrightarrow Y$ and $T \longrightarrow Y$ which form a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

there is a unique morphism $h\colon T \longrightarrow X$ making the diagram commute.

Proof. Suppose that f is proper. Then f is certainly separated, so h, if it exists, is surely unique. Consider the base change given by $T \longrightarrow Y$, and set $X_T = X \underset{Y}{\times} T$. We get a morphism $U \longrightarrow X_T$, applying the universal property to the morphisms $U \longrightarrow X$ and $U \longrightarrow T$.

$$\begin{tikzcd} U \arrow[r] & X_T \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ & T \arrow[r] & Y. \end{tikzcd}$$

Let $\xi_1 \in X_T$ be the image of the point $t_1 \in U$. Let $Z = \overline{\{\xi_1\}}$. As f is proper, f' is closed and so $f'(Z) \subset T$ is closed. Thus $f'(Z) = T$, as $f'(Z)$ contains the generic point of T. Pick $\xi_0 \in Z$ such that $f(\xi_0) = t_0$. Then we get a morphism of local rings $R \longrightarrow \mathcal{O}_{Z,\xi_0}$. The function field of Z is $k(\xi_1)$ which by construction is a subfield of K. On the other hand, R is maximal with respect to dominance in K. Thus $R \simeq \mathcal{O}_{Z,\xi_0}$. Hence by (8.15) there is a morphism $T \longrightarrow X_T$ sending t_i to ξ_i. Now compose with the natural map $X_T \longrightarrow X$.

Now suppose that f satisfies the given condition. Let $Y' \longrightarrow Y$ be an arbitrary base change, and let $X' \longrightarrow X$ be the induced morphism. Pick a closed subset $Z \subset X'$, imbued with the reduced induced structure:

$$\begin{tikzcd} Z \arrow[r, phantom, "\subset"] & X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ & Y' \arrow[r] & Y. \end{tikzcd}$$

We want to prove that $f(Z)$ is closed. f is of finite type by assumption, so that f' is of finite type. It suffices to show that $f(Z)$ is closed under specialisation, by (8.16).

Pick a point $z_1 \in Z$ and let $y_1 = f(z_1)$. Suppose that y_0 is a specialisation of y_1. Let S be the local ring of the closure of y_1 at y_0. Then the quotient field of S is $k(y_1)$ which is a subfield of $K = k(z_1)$. Pick a valuation ring R contained in K which dominates S. Then by (8.15), we get a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & Z \arrow[d] \\ T \arrow[r] & Y'. \end{tikzcd}$$

Composing with the morphisms $Z \longrightarrow X' \longrightarrow X$ and $Y' \longrightarrow Y$ we get morphisms $U \longrightarrow X$ and $T \longrightarrow Y$. By hypothesis, there is a morphism $T \longrightarrow X$ which makes the diagram commute. By the universal property of a fibre product, this lifts to a morphism $T \longrightarrow X$. As Z is closed, this factors into $T \longrightarrow Z$. Let z_0 be the image of t_0. Then z_0 maps to y_0, as required. $\square$

Corollary 8.22. *Assume that all schemes are Noetherian*

(1) *A closed immersion is proper.*
(2) *Composition of proper morphisms is proper.*
(3) *Proper morphisms are stable under base change.*
(4) *If $f\colon X \longrightarrow Y$ and $f'\colon X' \longrightarrow Y'$ are two proper morphisms over a scheme S, then the product morphism $f \times f'\colon X \underset{S}{\times} X' \longrightarrow Y \underset{S}{\times} Y'$ is also proper.*
(5) *If $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ are two morphisms, such that $g \circ f$ is proper and g is separated, then f is proper.*
(6) *A morphism $f\colon X \longrightarrow Y$ is proper iff Y can be covered by open subsets V_i such that $f^{-1}(V_i) \longrightarrow V_i$ is proper for each i.*

Proof. These all follow from (8.21). $\square$

9. Varieties as schemes

Now we turn to the definition of projective schemes.

The definition mirrors that for affine schemes. First we start with a graded ring S,

$$S = \bigoplus_{d \in \mathbb{N}} S_d.$$

We set

$$S_+ = \bigoplus_{d > 0} S_d,$$

and we let $\operatorname{Proj} S$ denote the set of all homogeneous prime ideals of S, which do not contain S_+. We put a topology on $\operatorname{Proj} S$ analogously to the way we put a topology on $\operatorname{Spec} S$; if $\mathfrak{a}$ is a homogeneous ideal of S, then we set

$$V(\mathfrak{a}) = \{\, \mathfrak{p} \in \operatorname{Proj} S \mid \mathfrak{a} \subset \mathfrak{p} \,\}.$$

The Zariski topology is the topology where these are the closed sets. If $\mathfrak{p}$ is a homogeneous prime ideal, then $S_{(\mathfrak{p})}$ denotes the elements of degree zero in the localisation of S at the set of homogenous elements which do not belong to $\mathfrak{p}$. We define a sheaf of rings $\mathcal{O}_X$ on $X = \operatorname{Proj} S$ by considering, for an open set $U \subset X$, all functions

$$s\colon U \longrightarrow \coprod_{\mathfrak{p} \in U} S_{(\mathfrak{p})},$$

such that $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$, which are locally represented by quotients. That is given any point $\mathfrak{q} \in U$, there is an open neighbourhood V of $\mathfrak{p}$ in U and homogeneous elements a and f in S of the same degree, such that for every $\mathfrak{p} \in V$, $f \notin \mathfrak{p}$ and $s(\mathfrak{p})$ is represented by the class of $a/f \in S_{(\mathfrak{p})}$.

Proposition 9.1. *Let S be a graded ring and set $X = \operatorname{Proj} S$.*

(1) For every $\mathfrak{p} \in X$, the stalk $\mathcal{O}_{X,\mathfrak{p}}$ is isomorphic to $S_{(\mathfrak{p})}$.

(2) For any homogeneous element $f \in S_+$, set

$$U_f = \{\, \mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p} \,\}.$$

Then U_f is open in $\operatorname{Proj} S$, these sets cover X and we have an isomorphism of locally ringed spaces

$$(U_f, \mathcal{O}_X|_{U_f}) \simeq \operatorname{Spec} S_{(f)}.$$

where $S_{(f)}$ consists of all elements of degree zero in the localisation S_f.

In particular $\operatorname{Proj} S$ is a scheme.

Proof. The proof of (1) follows similar lines to the affine case and is left as an exercise for the reader. $U_f = X - V(\langle f\rangle)$ and so U_f is certainly open and these sets certainly cover X. We are going to define an isomorphism

$$(g, g^{\#})\colon (U_f, \mathcal{O}_X|_{U_f}) \longrightarrow \operatorname{Spec} S_{(f)}.$$

If $\mathfrak{a}$ is any homogeneous ideal of S, consider the ideal $\mathfrak{a}S_f \cap S_{(f)}$. In particular if $\mathfrak{p}$ is a prime ideal of S, then $\phi(\mathfrak{p}) = \mathfrak{p}S_f \cap S_{(f)}$ is a prime ideal of $S_{(f)}$. It is easy to see that ϕ is a bijection. Now $\mathfrak{a} \subset \mathfrak{p}$ iff

$$\mathfrak{a}S_f \cap S_{(f)} \subset \mathfrak{p}S_f \cap S_{(f)} = \phi(\mathfrak{p}),$$

so that ϕ is a homeomorphism. If $\mathfrak{p} \in U_f$ then $S_{(\mathfrak{p})}$ and $(S_{(f)})_{\phi(\mathfrak{p})}$ are naturally isomorphic. This induces a morphism $g^{\#}$ of sheaves which is easily seen to be an isomorphism. □

Definition 9.2. *Let R be a ring.* ***Projective n-space over R****, denoted $\mathbb{P}^n_R$, is the proj of the polynomial ring $R[x_1, x_2, \dots, x_n]$.*

Note that $\mathbb{P}^n_R$ is a scheme over $\operatorname{Spec} R$.

Definition-Lemma 9.3. *If X is a topological space, then let $t(X)$ be the set of irreducible closed subsets of X. Then $t(X)$ is naturally a topological space and if we define a map $\alpha\colon X \longrightarrow t(X)$ by sending a point to its closure then α induces a bijection between the closed sets of X and $t(X)$.*

Proof. Observe that

- If $Y \subset X$ is a closed subset, then $t(Y) \subset t(X)$,
- if Y_1 and Y_2 are two closed subsets, then $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$, and
- if Y_α is any collection of closed subsets, then $t(\cap Y_\alpha) = \cap t(Y_\alpha)$.

The defines a topology on $t(X)$ and the rest is clear. □

Theorem 9.4. *Let k be an algebraically closed field. Then there is a fully faithful functor t from the category of varieties over k to the category of schemes. For any variety V, the set of points of V may be recovered from the closed points of $t(V)$ and the sheaf of regular functions is the restriction of the structure sheaf to the set of closed points.*

Proof. We will show that $(t(V), \alpha_* \mathcal{O}_V)$ is a scheme, where $\mathcal{O}_V$ is the sheaf of regular functions on V. As any variety has an open affine cover, it suffices to prove this for an affine variety, with coordinate ring

A. Let X be the spectrum of A. We are going to a define a morphism of locally ringed spaces,

$$\beta = (f, f^{\#})\colon (V, \mathcal{O}_V) \longrightarrow (X, \mathcal{O}_X).$$

If $p \in V$, then let $f(p) = m_p \in X$ be the maximal ideal of elements of A vanishing at p. By the Nullstellensatz, f induces a bijection between the closed points of X and the points of V. It is easy to see that f is a homeomorphism onto its image. Now let $U \subset X$ be an open set. We need to define a ring homomorphism

$$f^{\#}(U)\colon \mathcal{O}_X(U) \longrightarrow f_*\mathcal{O}_V(f^{-1}(U)).$$

Let $s \in \mathcal{O}_X(U)$. We want to define $r = f^{\#}(U)(s)$. Pick $p \in U$. Then we define $r(p)$ to be the image of $s(m_p) \in A_{m_p}$ inside the quotient

$$A_{m_p}/m_p \simeq k.$$

It is easy to see that r is a regular function and that $f^{\#}(U)$ is a ring isomorphism. As the irreducible subsets of V are in bijection with the prime ideals of A, it follows that $(X, \mathcal{O}_X)$ is isomorphic to $(t(V), \alpha_*\mathcal{O}_V)$, and so the latter is an affine scheme.

Note that there is a natural inclusion

$$k \subset A,$$

which associates to a scalar the constant function on V. But then X is a scheme over $\operatorname{Spec} k$. It is easy to check that t is fully faithful. □

Before we check that projective morphisms are proper, let's do a warm up case:

Theorem 9.5. *Let $f\colon \Delta^* \longrightarrow \mathbb{P}^n$ be a meromorphic map of the punctured disk into projective space over $\mathbb{C}$.*

Then f extends to a holomorphic map $g\colon \Delta \longrightarrow \mathbb{P}^n$.

Proof.

$$f(z) = [f_0(z) : f_1(z) : \cdots : f_n(z)].$$

Each meromorphic function $f_i(z) = z^{m_i}h_i(z)$, where m_i is an integer and $h_i(z)$ is holomorphic, $h_i(0)$ is non-zero. Let m be the minimum of $m_0, m_1, \dots, m_n$. Then

$$g_i(z) = z^{-m}f_i(z) = z^{m_i - m}h_i(z)$$

is holomorphic and at least one of $g_i(0)$ is non-zero. On the other hand

$$\begin{aligned} g(z) &= [g_0(z) : g_1(z) : \cdots : g_n(z)] \\ &= [z^{-m}f_0(z) : z^{-m}f_1(z) : \cdots : z^{-m}f_n(z)] \\ &= f(z), \end{aligned}$$

whenever z is non-zero, so that $g\colon \Delta \longrightarrow \mathbb{P}^n$ is a holomorphic map extending f. $\square$

Definition 9.6. *Let $\pi\colon X \longrightarrow S$ be a morphism.*

*We say that π is a **projective morphism** if it can be factored into a closed embedding $i\colon X \longrightarrow \mathbb{P}^n_S$ and the projection morphism $\mathbb{P}^n_S \longrightarrow S$.*

Theorem 9.7. *A projective morphism is proper.*

Proof. Since a closed immersion is of finite type, and using the results of §8, it suffices to prove that $X = \mathbb{P}^n_{\operatorname{Spec}\mathbb{Z}}$ is proper over $\operatorname{Spec}\mathbb{Z}$. Now X is covered by open affines of the form $U_i = \operatorname{Spec}\mathbb{Z}[x_1, x_2, \dots, x_n]$. Thus X is certainly of finite type over $\operatorname{Spec}\mathbb{Z}$. We check the valuative criteria. Suppose we have a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dashed] & \operatorname{Spec}\mathbb{Z}. \end{tikzcd}$$

Let $\xi_1 \in X$ be the image of the unique point of U. By induction on n, we may assume that ξ_1 is not contained in any of the $n+1$ standard hyperplanes, so that $\xi_1 \in U = \bigcap U_i$. Thus the functions x_i/x_j are all invertible on U.

There is an inclusion $k(\xi_1) \subset K$. Let f_{ij} be the image of x_i/x_j. Then

$$f_{ik} = f_{ij} f_{jk}.$$

Let $\nu\colon K \longrightarrow G$ be the valuation associated to R. Let $g_i = \nu(f_{i0})$ and pick k such that g_k is minimal. Then

$$\nu(f_{ik}) = g_i - g_k \geq 0.$$

Hence $f_{ik} \in R$. Define a ring homomorphism

$$\mathbb{Z}[x_0/x_k, x_1/x_k, \dots, x_n/x_k] \longrightarrow R \qquad \text{by sending} \qquad x_i/x_k \longrightarrow f_{ik}.$$

This gives a morphism $T \longrightarrow U_k$ and by composition a morphism $T \longrightarrow X$. $\square$

Using this, we can finally characterise the image of the functor t.

Proposition 9.8. *Fix an algebraically closed field k. Then the image of the functor t is precisely the set of integral quasi-projective schemes, and the image of a projective variety is an integral projective scheme.*

In particular for every variety V, $t(V)$ is an integral separated scheme of finite type over k.

Proof. It only suffices to prove that every integral projective scheme Y is the image of a variety. Let Y be a closed subscheme of $\mathbb{P}^n_k$. Then the set of closed points V is a closed subset of the variety $\mathbb{P}^n$. If Y is irreducible, as V is dense in Y, it follows that V is irreducible. If Y is reduced, it is easy to see that $t(V) = Y$, since they have the same support and they are both reduced. $\square$

Definition 9.9. *A* ***variety*** *is an integral separated scheme of finite type over an algebraically closed field. If in addition it is proper, then we say that is a* ***complete variety****.*

10. Affine toric varieties

First some stuff about algebraic groups:

Definition 10.1. *Let G be a group. We say that G is an* ***algebraic group*** *if G is a quasi-projective variety and the two maps $m\colon G\times G\longrightarrow G$ and $i\colon G\longrightarrow G$, where m is multiplication and i is the inverse map, are both morphisms.*

It is easy to give examples of algebraic groups. Consider the group $G=\mathrm{GL}_n(K)$. In this case G is an open subset of $\mathbb{A}^{n^2}$, the complement of the zero locus of the determinant, which expands to a polynomial. Matrix multiplication is obviously a morphism, and the inverse map is a morphism by Cramer's rule. Note that there are then many obvious algebraic subgroups; the orthogonal groups, special linear group and so on. Clearly $\mathrm{PGL}_n(K)$ is also an algebraic group; indeed the quotient of an algebraic group by a closed normal subgroup is an algebraic group. All of these are affine algebraic groups.

Definition 10.2. *Let G be an algebraic group. If G is affine then we say that G is a* ***linear algebraic group****. If G is projective and connected then we say that G is an* ***abelian variety****.*

Note that any finite group is an algebraic group (both affine and projective). It turns out that any affine group is always a subgroup of a matrix group, so that the notation makes sense.

Definition 10.3. *The group $\mathbb{G}_m$ is $\mathrm{GL}_1(K)$. The group $\mathbb{G}_a$ is the subgroup of $\mathrm{GL}_2(K)$ of upper triangular matrices with ones on the diagonal.*

Note that as a group $\mathbb{G}_m$ is the set of units in K under multiplication and $\mathbb{G}_a$ is equal to K under addition, and that both groups are affine of dimension 1; in fact they are the only linear algebraic groups of dimension one, up to isomorphism.

Note that if we are given a linear algebraic group G, we get a Hopf algebra A. Indeed if A is the coordinate ring of G, then A is a K-algebra and there are maps

$$A\longrightarrow A\otimes A \qquad \text{and} \qquad A\longrightarrow A,$$

induced by the multiplication and inverse map for G (if you don't know what a Hopf algebra is, you can unwind the definitions and take this as the definition of a Hopf algebra).

It is not hard to see that the product of two algebraic groups is an algebraic group.

Definition 10.4. *The algebraic group* $\mathbb{G}_m^k$ *is called a* ***torus***.

So a torus in algebraic geometry is just a product of copies of $\mathbb{G}_m$.
In fact one can define what it means to be a group scheme:

Definition 10.5. *Let* $\pi\colon X \longrightarrow S$ *be a morphism of schemes. We say that* X *is a* ***group scheme*** *over* S*, if there are three morphisms,* $e\colon S \longrightarrow X$*,* $\mu\colon X \underset{S}{\times} X \longrightarrow X$ *and* $i\colon X \longrightarrow X$ *over* S *which satisfy the obvious axioms.*

We can define a group scheme $\mathbb{G}_{m,\operatorname{Spec}\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$, by taking

$$\operatorname{Spec}\mathbb{Z}[x, x^{-1}].$$

Given any scheme S, this gives us a group scheme $\mathbb{G}_{m,S}$ over S, by taking the fibre square. In the case when $S = \operatorname{Spec} k$, k an algebraically closed field, then $\mathbb{G}_{m,\operatorname{Spec} k}$ is $t(\mathbb{G}_m)$, the scheme associated to the quasi-projective variety $\mathbb{G}_m$. We will be somewhat sloppy and not be too careful to distinguish the two cases.

Similarly we can take

$$\mathbb{G}_{a,\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x].$$

Definition 10.6. *Let* G *be an algebraic group and let* X *be a variety acted on by* G*,* $\pi\colon G \times X \longrightarrow X$*. We say that the action is* ***algebraic*** *if* π *is a morphism.*

For example the natural action of $\mathrm{PGL}_n(K)$ on $\mathbb{P}^n$ is algebraic, and all the natural actions of an algebraic group on itself are algebraic.

Definition 10.7. *We say that a quasi-projective variety* X *is a* ***toric variety*** *if* X *is irreducible and normal and there is a dense open subset* U *isomorphic to a torus, such that the natural action of* U *on itself extends to an action on the whole of* X*.*

For example, any torus is a toric variety. $\mathbb{A}^n_k$ is a toric variety. The natural torus is the complement of the coordinate hyperplanes and the natural action is as follows

$$((t_1, t_2, \dots, t_n), (a_1, a_2, \dots, a_n)) \longrightarrow (t_1 a_1, t_2 a_2, \dots, t_n a_n).$$

More generally, $\mathbb{P}^n$ is a toric variety. The action is just the natural action induced from the action above. A product of toric varieties is toric.

One thing to keep track of are the closures of the orbits. For the torus there is one orbit. For affine space and projective space the closure of the orbits are the coordinate subspaces.

Definition 10.8. *Let M be a lattice and let $N = \operatorname{Hom}(M, \mathbb{Z})$ be the dual lattice.*

*A **strongly convex rational polyhedral cone** $\sigma \subset N_{\mathbb{R}} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is*

- *a **cone**, that is, if $v \in \sigma$ and $\lambda \in \mathbb{R}$, $\lambda \geq 0$ then $\lambda v \in \sigma$;*
- ***polyhedral**, that is, σ is the intersection of finitely many half spaces;*
- ***rational**, that is, the half spaces are defined by equations with rational coefficients;*
- ***strongly convex**, that is, σ contains no linear spaces other than the origin.*

One can reformulate some of the parts of the definition of a strongly rational polyhedral cone. For example, σ is a polyhedral cone if and only if σ is the intersection of finitely many half spaces which are defined by homogeneous linear polynomials. σ is a strongly convex polyhedral cone if and only if σ is a cone over finitely many vectors which lie in a common half space (in other words a strongly convex polyhedral cone is the same as a cone over a polytope). And so on.

We first give the recipe of how to go from a fan to an affine toric variety. Suppose we start with σ. Form the dual cone

$$\check{\sigma} = \{\, u \in M_{\mathbb{R}} \mid \langle u, v\rangle \geq 0, v \in \sigma \,\}.$$

Now take the integral points,

$$S_\sigma = \check{\sigma} \cap M.$$

Then form the (semi)group algebra,

$$A_\sigma = K[S_\sigma].$$

Finally form the affine variety,

$$U_\sigma = \operatorname{Spec} A_\sigma.$$

Given a semigroup S, to form the semigroup algebra $K[S]$, start with a K-vector space with basis χ^u, as u ranges over the elements of S. Given u and $v \in S$ define the product

$$\chi^u \cdot \chi^v = \chi^{u+v},$$

and extend this linearly to the whole of $K[S]$.

Note that $K[S]$ is an integral domain so that U_σ is irreducible.

Example 10.9. *For example, suppose we start with $M = \mathbb{Z}^2$, σ the cone spanned by $(1, 0)$ and $(0, 1)$, inside $N_{\mathbb{R}} = \mathbb{R}^2$. Then $\check{\sigma}$ is spanned by the same vectors in $M_{\mathbb{R}}$. Therefore $S_\sigma = \mathbb{N}^2$, the group algebra is*

$\mathbb{C}[x, y]$ *and so we get* $\mathbb{A}^2$*. Similarly if we start with the cone spanned by* $e_1, e_2, \ldots, e_n$ *inside* $N_{\mathbb{R}} = \mathbb{R}^n$ *then we get* $\mathbb{A}^n$*.*

Now suppose we start with $\sigma = \{0\}$ *in* $\mathbb{R}$*. Then* $\check{\sigma}$ *is the whole of* $M_{\mathbb{R}}$*,* S_σ *is the whole of* $M = \mathbb{Z}$ *and so* $\mathbb{C}[M] = \mathbb{C}[x, x^{-1}]$*. Taking* Spec *we get the torus* $\mathbb{G}_m$*.*

More generally we always get a torus of dimension n if we take the origin in $\mathbb{R}^n$. Note that if $\tau \subset \sigma$ is a face then $\check{\sigma} \subset \check{\tau}$ is also a face so that $U_\tau \subset U_\sigma$ is an open subset. In fact

Lemma 10.10. *Let* $\tau \subset \sigma \subset N_{\mathbb{R}}$ *be a face of the cone* σ*.*

Then we may find $u \in S_\sigma$ *such that*

(1) $\tau = \sigma \cap u^\perp$,

(2)
$$S_\tau = S_\sigma + \mathbb{Z}^+(-u),$$

(3) A_τ *is a localisation of* A_σ*, and*

(4) U_τ *is a principal open subset of* U_σ*.*

Proof. The fact that every face of a cone is cut out by a hyperplane is a standard fact in convex geometry and this is (1). For (2) note that the RHS is contained in the LHS by definition of a cone. If $w \in S_\tau$ then $w + p \cdot u$ is in $\check{\sigma}$ for any p sufficiently large. If we take p to be also an integer this shows that w belongs to the RHS.

Let χ^u be the monomial corresponding to u. (2) implies that A_τ is the localisation of A_σ along χ^u. This is (3) and (4) is immediate from (3). □

Since the cone $\{0\}$ is a face of every cone, the affine scheme associated to a cone always contains a torus, which is then dense.

Definition 10.11. *Let* $S \subset T$ *be a subsemigroup of the semigroup* T*. We say that* S *is* ***saturated*** *in* T *if whenever* $u \in T$ *and* $p \cdot u \in S$ *for some positive integer* p*, then* $u \in S$*.*

Given a subsemigroup $S \subset M$ saturation is always with respect to M.

Lemma 10.12. *Suppose that* $S \subset M$*.*

Then $K[S]$ *is integrally closed if and only if* S *is saturated.*

Proof. Suppose that $K[S]$ is integrally closed.

Pick $u \in M$ such that $v = p \cdot u \in S$ for some positive integer p. Let $b = \chi^u$ and $a = \chi^v \in K[S]$. Then

$$b^p = \chi^{pu} = \chi^v = a,$$

so that b is a root of the monic polynomial $x^p - a \in K[S][x]$. As we are assuming that $K[S]$ is integrally closed this implies that $b \in K[S]$ which implies that $u \in S$. Thus S is saturated.

Now suppose that S is saturated. As $K[S] \subset K[M]$ and the latter is integrally closed, the integral closure L of $K[S]$ sits in the middle, $K[S] \subset L \subset K[M]$. The torus acts naturally on $K[M]$ and this action fixes $K[S]$, so that it also fixes L. L is therefore a direct sum of eigenspaces, which are all one dimensional (a set of commuting diagonalisable matrices are simultaneously diagonalisable), that is, L has a basis of the form χ^u, as u ranges over some subset of M. It suffices to prove that $u \in S$.

Since $b = \chi^u$ is integral over $K[S]$, we may find $k_1, k_2, \dots, k_p \in K[S]$ such that

$$b^p + k_1 b^{p-1} + \cdots + k_p = 0.$$

We may assume that every term is in the same eigenspace as b^p. We may also assume that $k_p \neq 0$. As b^p and $k_p \neq 0$ belong to the same eigenspace, which is one dimensional, we get $b^p \in K[S]$. Thus $pu \in S$ and so $u \in S$ as S is saturated. Thus $b \in K[S]$ and $K[S]$ is integrally closed. $\square$

Note that S_σ is automatically saturated, as $\check{\sigma}$ is a rational polyhedral cone. In particular U_σ is normal.

Example 10.13. *Suppose that we start with the semigroup S generated by 2 and 3 inside $M = \mathbb{Z}$. Then*

$$K[S] = K[t^2, t^3] = K[x, y]/\langle y^2 - x^3 \rangle.$$

Note that this does come with an action of $\mathbb{G}_m$;

$$(t, x, y) \longrightarrow (t^2 x, t^3 y).$$

However the curve $V(y^2 - x^3) \subset \mathbb{A}^2$ is not normal.

In fact some authorities drop the requirement that a toric variety is normal.

An action of the torus corresponds to a map of algebras

$$A_\sigma \longrightarrow A_\sigma \underset{K}{\otimes} A_0,$$

which is naturally the restriction of

$$A_0 \longrightarrow A_0 \underset{K}{\otimes} A_0.$$

It is straightforward to check that the restricted map does land in $A_\sigma \underset{K}{\otimes} A_0$.

Lemma 10.14 (Gordan's Lemma)**.** *Let $\sigma \subset \mathbb{N}_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.*

Then S_σ is a finitely generated semigroup.

Proof. Pick vectors $v_1, v_2, \dots, v_n \in S_\sigma$ which generate the cone $\check{\sigma}$. Consider the set

$$K = \{\, v \in M \mid v = \sum t_i v_i, t_i \in [0,1] \,\}.$$

Then K is compact. As M is discrete $K \cap M$ is finite. I claim that the elements of $K \cap M$ generate the semigroup S_σ. Pick $u \in S_\sigma$. Since $u \in \check{\sigma}$ and $v_1, v_2, \dots, v_n$ generate the cone, we may write

$$u = \sum \lambda_i v_i,$$

where $\lambda_i \in \mathbb{Q}$. Let $n_i = \llcorner \lambda_i \lrcorner$. Then

$$u - \sum n_i v_i = \sum (\lambda_i - \llcorner \lambda_i \lrcorner) v_i \in K \cap M.$$

As $v_1, v_2, \dots, v_n \in K \cap M$ the result follows. $\square$

Gordan's lemma (10.14) implies that U_σ is of finite type over K. So U_σ is an affine toric variety.

Example 10.15. *Suppose we start with the cone spanned by e_2 and $2e_1 - e_2$. The dual cone is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the semigroup are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. The corresponding group algebra is $A_\sigma = K[x, xy, xy^2]$. Suppose we call $u = x$, $v = xy$ and $w = xy^2$. Then $v^2 = x^2y^2 = x(xy^2) = uw$. Therefore the corresponding affine toric variety is given as the zero locus of $v^2 - uw$ in $\mathbb{A}^3$.*

11. Toric varieties

Definition 11.1. *A **fan** in $N_{\mathbb{R}}$ is a set F of finitely many strongly convex rational polyhedra, such that*

- *every face of a cone in F is a cone in F, and*
- *the intersection of any two cones in F is a face of each cone.*

It turns out that the set of toric varieties, up to isomorphism, are in bijection with fans, up to the action of $\mathrm{GL}(n, \mathbb{Z})$.

Given a fan F, we get a collection of affine toric varieties, one for every cone of F. It remains to check how to glue these together to get a toric variety. Suppose we are given two cones σ and τ belonging to F. The intersection is a cone ρ which is also a cone belonging to F. Since ρ is a face of both σ and τ there are natural inclusions

$$U_\rho \subset U_\sigma \qquad \text{and} \qquad U_\rho \subset U_\tau.$$

We glue U_σ to U_τ using the natural identification of the common open subset U_ρ. Compatibility of gluing follows automatically from the fact that the identification is natural and from the combinatorics of the fan (see (2.12) of Hartshorne). It is clear that the resulting scheme is of finite type over the groundfield. Separatedness follows from:

Lemma 11.2. *Let σ and τ be two cones whose intersection is the cone ρ.*

If ρ is a face of each then the diagonal map

$$U_\rho \longrightarrow U_\sigma \times U_\tau,$$

is a closed embedding.

Proof. This is equivalent to the statement that the natural map

$$A_\sigma \otimes A_\tau \longrightarrow A_\rho,$$

is surjective. For this, one just needs to check that

$$S_\rho = S_\sigma + S_\tau.$$

One inclusion is easy; the RHS is contained in the LHS. For the other inclusion, one needs a standard fact from convex geometry, which is called the separation lemma: there is a vector $u \in S_\sigma \cap S_{-\tau}$ such that simultaneously

$$\rho = \sigma \cap u^\perp \qquad \text{and} \qquad \rho = \tau \cap u^\perp.$$

By the first equality and the fact that $u \in S_\sigma$, we have $S_\rho = S_\sigma + \mathbb{Z}(-u)$. As $u \in S_{-\tau}$ we have $-u \in S_\tau$ and so the LHS is contained in the RHS. $\square$

So we have shown that given a fan F we can construct a normal variety $X = X(F)$. It is not hard to see that the natural action of the torus corresponding to the zero cone extends to an action on the whole of X. Therefore $X(F)$ is indeed a toric variety.

Let us look at some examples.

Example 11.3. *Suppose that we start with $M = \mathbb{Z}$ and we let F be the fan given by the three cones $\{0\}$, the cone spanned by e_1 and the cone spanned by $-e_1$ inside $N_{\mathbb{R}} = \mathbb{R}$. The two big cones correspond to $\mathbb{A}^1$. We identify the two $\mathbb{A}^1$'s along the common open subset isomorphic to K^*. Now the first $\mathbb{A}^1 = \operatorname{Spec} K[x]$ and the second is $\mathbb{A}^1 = \operatorname{Spec} K[x^{-1}]$. So the corresponding toric variety is $\mathbb{P}^1$ (if we have homogeneous coordinates $[X : Y]$ on $\mathbb{P}^1$ then coordinates on U_0 are $x = X/Y$ and coordinates on U_1 are $y = Y/X = 1/x$).*

Example 11.4. *Now suppose that we start with three cones in $N_{\mathbb{R}} = \mathbb{R}^2$, σ_1, σ_2 and σ_3. We let σ_1 be the cone spanned by e_1 and e_2, σ_2 be the cone spanned by e_2 and $-e_1 - e_2$ and σ be the cone spanned by $-e_1 - e_2$ and e_1. Let F be the fan given as the faces of these three cones. Note that the three affine varieties corresponding to these three cones are all copies of $\mathbb{A}^2$. Indeed, any two of the vectors, e_1, e_2 and $-e_1 - e_2$ are a basis not only of the underlying vector space but they also generate the standard lattice. We check how to glue two such copies of $\mathbb{A}^2$.*

The dual cone of σ_1 is the cone spanned by f_1 and f_2 in $M_{\mathbb{R}} = \mathbb{R}^2$. The dual cone of σ_2 is the cone spanned by $-f_1$ and $-f_1 + f_2$. So we have $U_1 = \operatorname{Spec} K[x, y]$ and $U_2 = \operatorname{Spec} K[x^{-1}, x^{-1}y]$. On the other hand, if we start with $\mathbb{P}^2$ with homogeneous coordinates $[X : Y : Z]$ and the two basic open subsets $U_0 = \operatorname{Spec} K[Y/X, Z/X]$ and $U_1 = \operatorname{Spec} K[X/Y, Z/Y]$, then we get the same picture, if we set $x = Y/X$, $y = Z/X$ (since then $X/Y = x^{-1}$ and $Z/Y = Z/X \cdot X/Y = yx^{-1}$). With a little more work one can check that we have $\mathbb{P}^2$.

Example 11.5. *More generally suppose we start with $n + 1$ vectors $v_1, v_2, \dots, v_{n+1}$ in $N_{\mathbb{R}} = \mathbb{R}^n$ which sum to zero such that the first n vectors $v_1, v_2, \dots, v_n$ span the standard lattice. Let F be the fan obtained by taking all the cones spanned by all subsets of at most n vectors. One can check that the resulting toric variety is $\mathbb{P}^n$.*

Example 11.6. *Now suppose that we take the four vectors e_1, e_2, $-e_1$ and $-e_2$ in $N_{\mathbb{R}} = \mathbb{R}^2$ and let F be the fan consisting of all cones spanned by at most two vectors (but not pairs of inverse vectors, that is, neither e_1 and $-e_1$ nor e_2 and $-e_2$). Then we get four copies of $\mathbb{A}^2$. It is easy to check that the resulting toric variety is $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed the top two fans glue together to get $\mathbb{P}^1 \times \mathbb{A}^1$ and so on.*

We have already seen that cones correspond to open subsets. In fact cones also correspond (in some sort of dual sense) to closed subsets, the closure of the orbits. First observe that given a fan F, we can associate a closed point x_σ to any cone σ. To see this, observe that one can spot the closed points of U_σ using semigroups:

Lemma 11.7. *Let $S \subset M$ be a semigroup. Then there is a natural bijection,*

$$\operatorname{Hom}(K[S], K) \simeq \operatorname{Hom}(S, K).$$

Here the RHS is the set of semigroup homomorphisms, where $K = \{0\} \cup K^*$ is the multiplicative subsemigroup of K (and not the additive).

Proof. Suppose we are given a ring homomorphism

$$f \colon K[S] \longrightarrow K.$$

Define

$$g \colon S \longrightarrow K,$$

by sending u to $f(\chi^u)$. Conversely, given g, define $f(\chi^u) = g(u)$ and extend linearly. □

Consider the semigroup homomorphism:

$$S_\sigma \longrightarrow \{0, 1\},$$

where $\{0, 1\} \subset \{0\} \cup K^*$ inherits the obvious semigroup structure. We send $u \in S_\sigma$ to 1 if $u \in \sigma^\perp$ and send it 0 otherwise. Note that as $\sigma^\perp$ is a face of $\check{\sigma}$ we do indeed get a homomorphism of semigroups. By (11.7) we get a surjective ring homomorphism

$$K[S_\sigma] \longrightarrow K.$$

The kernel is a maximal ideal of $K[S_\sigma]$, that is, a closed point x_σ of U_σ, with residue field K.

To get the orbits, take the orbits of these points. It follows that the orbits are in correspondence with the cones in F. Let $O_\sigma \subset U_\sigma$ be the orbit of x_σ and let $V(\sigma)$ be the closure of O_σ.

Example 11.8. *For the fan corresponding to $\mathbb{P}^1$, the point corresponding to $\{0\}$ is the identity, and the points corresponding to e_1 and $-e_1$ are 0 and ∞. For the fan corresponding to $\mathbb{P}^2$ the three maximal cones give the three coordinate points, the three one dimensional cones give the three coordinate lines (in fact the lines spanned by the points corresponding to the two maximal cones which contain them). As before the zero cone corresponds to the identity point. The orbit is the whole torus and the closure is the whole of $\mathbb{P}^2$.*

Suppose that we start with the cone σ spanned by e_1 and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. We have already seen that this gives the affine toric variety $\mathbb{A}^2$. Now suppose we insert the vector $e_1 + e_2$. We now get two cones σ_1 and σ_2, the first spanned by e_1 and $e_1 + e_2$ and the second spanned by $e_1 + e_2$ and e_2. Individually each is a copy of $\mathbb{A}^2$. The dual cones are spanned by f_2, $f_1 - f_2$ and f_1 and $f_2 - f_1$. So we get $\operatorname{Spec} K[y, x/y]$ and $\operatorname{Spec} K[x, x/y]$.

Suppose that we blow up $\mathbb{A}^2$ at the origin. The blow up sits inside $\mathbb{A}^2 \times \mathbb{P}^1$ with coordinates (x, y) and $[S : T]$ subject to the equations $xT = yS$. On the open subset $T \neq 0$ we have coordinates s and y and $x = sy$ so that $s = x/y$. On the open subset $S \neq 0$ we have coordinates x and t and $y = xt$ so that $t = y/x$. So the toric variety above is nothing more than the blow up of $\mathbb{A}^2$ at the origin. The central ray corresponds to the exceptional divisor E, a copy of $\mathbb{P}^1$.

A couple of definitions:

Definition 11.9. *Let G and H be algebraic groups which act on varieties X and Y. Suppose we are given an algebraic group homomorphism, $\rho\colon G \longrightarrow H$. We say that a morphism $f\colon X \longrightarrow Y$ is ρ-**equivariant** if f commutes with the action of G and H. If X and Y are toric varieties and G and H are the tori contained in X and Y then we say that f is a **toric morphism**.*

It is easy to see that the morphism defined above is toric. We can extend this picture to other toric surfaces. First a more intrinsic description of the blow up. Suppose we are given a toric surface and a two dimensional cone σ such that the primitive generators v and w of the two one dimensional faces of σ generate the lattice (so that up the action of $\mathrm{GL}(2, \mathbb{Z})$, σ is the cone spanned by e_1 and e_2). Then the blow up of the point corresponding to σ is a toric surface, which is obtained by inserting the sum $v + w$ of the two primitive generators and subdividing σ in the obvious way (somewhat like the barycentric subdivision in simplicial topology).

Example 11.10. *Suppose we start with $\mathbb{P}^2$ and the standard fan. If we insert the two vectors $-e_1$ and $-e_2$ this corresponds to blowing up two invariant points, say $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Note that now $-e_1 - e_2$ is the sum of $-e_1$ and $-e_2$. So if we remove this vector this is like blowing down a copy of $\mathbb{P}^1$. The resulting fan is the fan for $\mathbb{P}^1 \times \mathbb{P}^1$.*

Note that this is an easy way to see the birational map between the quadric $Q \subset \mathbb{P}^3$ and $\mathbb{P}^2$ given by projection from a point.

12. Coherent Sheaves

Definition 12.1. *If $(X, \mathcal{O}_X)$ is a locally ringed space, then we say that an $\mathcal{O}_X$-module $\mathcal{F}$ is* ***locally free*** *if there is an open affine cover $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is isomorphic to a direct sum of copies of $\mathcal{O}_{U_i}$. If the number of copies r is finite and constant, then $\mathcal{F}$ is called* ***locally free of rank*** *r.*

A ***sheaf of ideals*** *$\mathcal{I}$ is any subsheaf of $\mathcal{O}_X$.*

Definition 12.2. *Let $X = \operatorname{Spec} A$ be an affine scheme and let M be an A-module. $\tilde{M}$ is the $\mathcal{O}_X$-module which assigns to every open subset U_f the module M_f.*

Remark 12.3. *To realise (12.2) as a sheaf, note that $\tilde{M}(U)$ is the set of functions*

$$s\colon U \longrightarrow \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}},$$

which can be locally represented at $\mathfrak{p}$ as a/g, $a \in M$, $g \in R$, $\mathfrak{p} \notin U_g \subset U$.

Definition 12.4. *An $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is called* ***quasi-coherent*** *if there is an open cover $\{U_i = \operatorname{Spec} A_i\}$ by affines and isomorphisms $\mathcal{F}|_{U_i} \simeq \tilde{M}_i$. If in addition M_i is a finitely generated A_i-module then we say that $\mathcal{F}$ is* ***coherent****.*

Proposition 12.5. *Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for every open affine $U = \operatorname{Spec} A \subset X$, $\mathcal{F}|_U = \tilde{M}$. If in addition X is noetherian then $\mathcal{F}$ is coherent if and only if M is a finitely generated A-module.*

Theorem 12.6. *Let $X = \operatorname{Spec} A$ be an affine scheme.*

The assignment $M \longrightarrow \tilde{M}$ defines an equivalence of categories between the category of A-modules to the category of quasi-coherent sheaves on X, which respects exact sequences, direct sum and tensor product, and which is functorial with respect to morphisms of affine schemes, $f\colon X = \operatorname{Spec} A \longrightarrow Y = \operatorname{Spec} B$. If in addition A is noetherian, this functor restricts to an equivalence of categories between the category of finitely generated A-modules to the category of coherent sheaves on X.

Theorem 12.7. *Let X be a scheme. Suppose that we are given a short exact sequence of $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

1. *$\mathcal{G}$ is quasi-coherent if and only if $\mathcal{F}$ and $\mathcal{H}$ are quasi-coherent.*
2. *If X is noetherian then $\mathcal{G}$ is coherent if and only if $\mathcal{F}$ and $\mathcal{H}$ are coherent.*

Proof. Since this result is local, we may assume that $X = \operatorname{Spec} A$ is affine. The only non-trivial thing is to show that if $\mathcal{F}$ and $\mathcal{H}$ are quasi-coherent then so is $\mathcal{G}$. By (II.5.6) of Hartshorne, there is an exact sequence on global sections,

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

It follows that there is a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \tilde{F} \arrow[r] \arrow[d] & \tilde{G} \arrow[r] \arrow[d] & \tilde{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0,
\end{tikzcd}$$

whose rows are exact. By assumption, the first and third vertical arrow are isomorphisms, and the 5-lemma implies that the middle arrow is an isomorphism. □

Lemma 12.8. *Let $f\colon X \longrightarrow Y$ be a scheme.*

(1) If $\mathcal{G}$ is a quasi-coherent sheaf (respectively X and Y are noetherian and $\mathcal{G}$ is coherent) on X then $f^*\mathcal{G}$ is quasi-coherent (respectively coherent).
(2) If $\mathcal{F}$ is a quasi-coherent sheaf on Y and either f is compact and separated or X is noetherian (respectively X and Y are noetherian, $\mathcal{F}$ is coherent and f is proper) then $f_*\mathcal{F}$ is quasi-coherent (respectively coherent).

Definition-Lemma 12.9. *Let X be a scheme and let $\mathcal{I} \subset \mathcal{O}_X$ be a sheaf of ideals. Then $\mathcal{I}$ is a quasi-coherent sheaf, which is coherent if X is noetherian. Moreover $\mathcal{I}$ defines a closed subscheme Y of X and there is a short exact sequence*

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Conversely, if $Y \subset X$ is a closed subscheme, then the kernel of the morphism of sheaves

$$\mathcal{O}_X \longrightarrow \mathcal{O}_Y,$$

defines an ideal sheaf $\mathcal{I}_Y$, called the ***ideal sheaf of** Y **in** X.*

Remark 12.10. *Let $i\colon Y \longrightarrow X$ be a closed subscheme. If $\mathcal{F}$ is a sheaf on Y, then $\mathcal{G} = i_*\mathcal{F}$ is a sheaf on X, whose support is contained in Y. Conversely, given any sheaf $\mathcal{G}$ on X, whose support is contained in Y, then there is a unique sheaf $\mathcal{F}$ on Y such that $i_*\mathcal{F} = \mathcal{G}$.*

For this reason, it is customary, as in (12.9)*, to abuse notation, and to not distinguish between sheaves on Y and sheaves on X, whose support is contained in Y.*

Most of what we have done with algebras and modules, makes sense for graded algebras and graded modules, in which case we get sheaves on Proj of the graded ring.

Definition 12.11. *Let S be a graded ring and let M be a graded S-module. If $\mathfrak{p} \triangleleft S$ is a homogeneous ideal, then $M_{(\mathfrak{p})}$ denotes those elements of the localisation $M_{\mathfrak{p}}$ of degree zero.*

$\tilde{M}$ is the sheaf on $\operatorname{Proj} S$, which given an open subset $U \subset \operatorname{Proj} S$, then $\tilde{M}(U)$ denotes those functions

$$s\colon U \longrightarrow \coprod_{\mathfrak{p}\in U} M_{(\mathfrak{p})},$$

which are locally fractions of degree zero.

Proposition 12.12. *Let S be a graded ring, let M be a graded S-module and let $X = \operatorname{Proj} S$.*

(1) For any $\mathfrak{p} \in X$, $(\tilde{M})_{\mathfrak{p}} \simeq M_{(\mathfrak{p})}$.
(2) If $f \in S$ is homogeneous,

$$(\tilde{M})_{U_f} \simeq \tilde{M}_{(f)}.$$

(3) $\tilde{M}$ is a quasi-coherent sheaf. If S is noetherian and M is finitely generated then $\tilde{M}$ is a coherent sheaf.

Definition 12.13. *Let $X = \operatorname{Proj} S$, where S is a graded ring. If n is any integer, then set*

$$\mathcal{O}_X(n) = S(n)\tilde{}.$$

If $\mathcal{F}$ is any sheaf of $\mathcal{O}_X$-modules,

$$\mathcal{F}(n) = \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(n).$$

Let

$$\Gamma_*(X, \mathcal{F}) = \bigoplus_{m\in\mathbb{N}} \Gamma(X, \mathcal{F}(n)).$$

Lemma 12.14. *Let S be a graded ring, $X = \operatorname{Proj} S$ and let M be a graded S-module.*

(1) $\mathcal{O}_X(n)$ is an invertible sheaf.
(2) $\tilde{M}(n) \simeq M(n)\tilde{}$. In particular $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \simeq \mathcal{O}_X(m+n)$.
(3) Formation of the twisting sheaf is functorial with respect to morphisms of graded rings.

Proposition 12.15. *Let A be a ring, let $S = A[x_0, x_1, \dots, x_r]$ and let $X = \mathbb{P}^r_A = \operatorname{Proj} A[x_0, x_1, \dots, x_r]$.*

Then

$$\Gamma_*(X, \mathcal{O}_X) \simeq S.$$

Lemma 12.16. *Let S be a graded ring, generated as an S_0-algebra by S_1.*

If $X = \operatorname{Proj} S$ and $\mathcal{F}$ is a quasi-coherent sheaf on X, then

$$\Gamma_*(X, \mathcal{F})\tilde{} = \mathcal{F}.$$

Theorem 12.17. *Let A be a ring.*

(1) *If $Y \subset \mathbb{P}^n_A$ is a closed subscheme then $Y = \operatorname{Proj} S/I$, for some homogeneous ideal $I \triangleleft S = A[x_1, x_2, \dots, x_n]$.*
(2) *Y is projective over $\operatorname{Spec} A$ if and only if it is isomorphic to $\operatorname{Proj} T$ for some graded ring, for which there are finitely many elements of T_1 which generate T as a $T_0 = A$-algebra.*

Proof. Let $\mathcal{I}_Y$ the ideal sheaf of Y in X. Then there is an exact sequence,

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Twisting by $\mathcal{O}_X(n)$ is exact (in fact $\mathcal{O}_X(n)$ is an invertible sheaf), so we get an exact sequence

$$0 \longrightarrow \mathcal{I}_Y(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Taking global sections is left exact, so we get an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{I}_Y(n)) \longrightarrow H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)).$$

Taking the direct sum, there is therefore an injective map

$$I = \Gamma_*(X, \mathcal{I}_Y) = \Gamma_*(X, \mathcal{O}_X) \simeq S.$$

It follows that $I \triangleleft S$ is a homogeneous ideal. Let $\tilde{I}$ be the associated sheaf. Since $\mathcal{I}_Y$ is quasi-coherent, in fact $\tilde{I} = \mathcal{I}_Y$ (see Hartshorne (II.5.15)). But then the subscheme determined by I is equal to Y. Hence (1).

If Y is projective over $\operatorname{Spec} A$ then we may assume that $Y \subset \mathbb{P}^n_A$. By (1) $Y \simeq \operatorname{Proj} S/I$, and if $T = S/I$, then $T_0 \simeq A$ and the image of $x_0, x_1, \dots, x_n \in T_1$ generate T. Conversely, any such algebra is the quotient of S. The kernel I is a homogeneous ideal and $Y \simeq \operatorname{Proj} S/I$. $\square$

Definition 12.18. *Let Y be a scheme. $\mathcal{O}_Y(1) = g^*\mathcal{O}_{\mathbb{P}^r}(1)$ is the sheaf on $\mathbb{P}^r_Y$, where $g \colon \mathbb{P}^r_Y \longrightarrow \mathbb{P}^r_{\operatorname{Spec} \mathbb{Z}}$ is the natural morphism.*

*We say that a morphism $i \colon X \longrightarrow Z$ is an **immersion** if i induces an isomorphism of X with a locally closed subset of Y.*

*We say that an invertible sheaf $\mathcal{L}$ on a scheme X over Y is **very ample** if there is an immersion $i \colon X \longrightarrow \mathbb{P}^r_Y$ over Y, such that $\mathcal{L} \simeq i^*\mathcal{O}_Y(1)$.*

Lemma 12.19. *Let X be a scheme over Y.*

Then X is projective over Y if and only if X is proper over Y and there is a very ample sheaf on X.

Proof. One direction is clear; if X is projective over Y, then it is proper and we just pullback $\mathcal{O}_Y(1)$.

If X is proper over Y then the image of X in $\mathbb{P}^r_Y$ is closed, and so X is projective over Y. □

Definition 12.20. *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is* ***globally generated*** *if there are elements $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that for every point $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$, generate the stalk as an $\mathcal{O}_{X,x}$-module.*

Lemma 12.21. *Let X be a scheme and*

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1. *$\mathcal{F}$ is globally generated.*
2. *The natural map*
$$H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \longrightarrow \mathcal{F},$$
is surjective.
3. *$\mathcal{F}$ is a quotient of a free sheaf.*

Proof. Clear. □

Lemma 12.22 (Push-pull)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be a locally free $\mathcal{O}_Y$-module.*
$$f_*(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} f^*\mathcal{G}) = f_*\mathcal{F} \underset{\mathcal{O}_Y}{\otimes} \mathcal{G}.$$

Theorem 12.23 (Serre)**.** *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

Then there is a positive integer $n_0 \geq 0$ such that $\mathcal{F}(n)$ is globally generated for all $n \geq n_0$.

Proof. By assumption there is a closed immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. Let $\mathcal{G} = i_*\mathcal{F}$. Then (12.22) implies that
$$\mathcal{G}(n) = i_*\mathcal{F}(n).$$
Then $\mathcal{F}(n)$ is globally generated if and only if $\mathcal{G}(n)$ is globally generated. As i is a closed immersion it is a proper morphism; as $\mathcal{F}$ is coherent and i is proper, $\mathcal{G}$ is coherent. Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$ we may assume that $X = \mathbb{P}^r_A$.

Consider the standard open affine cover U_i, $0 \leq i \leq r$ of $\mathbb{P}^r_A$. Since $\mathcal{F}$ is coherent, $\mathcal{F}_i = \mathcal{F}|_{U_i} = \tilde{F}_i$, for some finitely generated

$A[x_0/x_i, x_1/x_i, \dots, x_r/x_i]$-module F_i. Pick generators s_{ij} of F_i. For each j, we may lift $x_i^{n_{ij}} s_{ij}$ to t_{ij}, for some n_{ij} (see (II.5.14)). By finiteness, we may assume that $n = n_{ij}$ does not depend on i and j. Now the natural map

$$x_i^n \colon \mathcal{F} \longrightarrow \mathcal{F}(n),$$

is an isomorphism over U_i. Thus t_{ij} generate the stalks of $\mathcal{F}$. □

Corollary 12.24. *Let X be a scheme projective over a noetherian ring A and let $\mathcal{F}$ be a coherent sheaf.*

Then $\mathcal{F}$ is a quotient of a direct sum of line bundles of the form $\mathcal{O}_X(n_i)$.

Proof. Pick $n > 0$ such that $\mathcal{F}(n)$ is globally generated. Then

$$\bigoplus_{i=1}^{k} \mathcal{O}_X \longrightarrow \mathcal{F}(n),$$

is surjective. Now just tensor by $\mathcal{O}_X(-n)$. □

13. Dimension of schemes

Our aim in this section is give a formal definition of the dimension of a variety, to compute the dimension in specific examples and to prove some of the interesting properties of the dimension.

Definition 13.1. *Let X be a topological space.*

The ***dimension of*** *X is equal to the supremum of the length n of strictly increasing sequences of irreducible closed subsets of X,*

$$\varnothing \neq Z_0 \subset Z_1 \subset \cdots \subset Z_n.$$

We will call a chain ***maximal*** *if it cannot be extended to a longer chain.*

Note that if X is Noetherian then the dimension of X is, by definition, equal to the maximal dimension of an irreducible component. Note that also that the dimension of X is equal to the dimension of any dense open subset, and that the dimension of any subset is at most the dimension of X.

In general this notion of dimension is a little unwieldy, even for Noetherian topological spaces (in fact, it is pretty clear that this definition is useless for any topological space that is not Noetherian or at least close to Noetherian).

For quasi-projective varieties it is much better behaved. For example,

Theorem 13.2. *Let X be a quasi-projective variety.*

Then the dimension of X is equal to the length of any maximal chain of irreducible subvarieties.

Definition 13.3. *Let $f\colon X \longrightarrow I$ be a map from a topological space to an ordered set I. We say that f is* ***upper semi-continuous****, if for every $a \in I$, the set*

$$\{\, x \in X \mid f(x) \geq a \,\},$$

is closed in X.

The key result is:

Theorem 13.4. *Let $\pi\colon X \longrightarrow Y$ be a dominant morphism of quasi-projective varieties. Then the function*

$$\mu\colon X \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\mu(p)$ is the local dimension of the fibre $X_p = \pi^{-1}(\pi(p))$ at p. Moreover if X_0 is any irreducible component of X and Y_0 is the closure of the image, we have

$$\dim(X_0) = \dim(Y_0) + \mu_0,$$

where μ_0 is the minimum value of μ on X_0.

Note that semi-continuity of μ is equivalent to saying that the dimension can jump up on closed subsets, but not down. For example, consider what happens for the blow up of a point. In this case, μ is equal to zero outside of the exceptional divisor and it jumps up to one on the exceptional divisor.

We will prove these two results in tandem. Let $d = \dim X$. We will need an intermediary result, which is of independent interest:

Lemma 13.5. *Assume* $(13.2)_d$.

If $X \subset \mathbb{P}^n$ is a closed subset of dimension d and $H \subset \mathbb{P}^n$ is a hypersurface then

$$\dim(X \cap H) \geq \dim(X) - 1,$$

with equality if and only if $H \cap X$ does not contain a component of X of maximal dimension.

Proof. We might as well assume that X is irreducible and that H does not contain X. Pick a maximal chain of irreducible subvarieties of X which contains a component Y of $X \cap H$,

$$\varnothing \neq Z_0 \subset Z_1 \subset \cdots \subset Z_e.$$

Then $X = Z_e$ and $Y = Z_i$, some i. As we are assuming $(13.2)_d$, $d = e$ and $\dim Y = i$.

Suppose $Z \neq X$ is irreducible and

$$Y \subset Z \subset X.$$

I claim that $Z = Y$. To see this, if we pass to an open affine subset then Z and Y are defined by ideals $J \subset I \subset A$, where A is the coordinate ring, $I = \langle f \rangle$ is principal and J is a prime ideal. Pick $g \in J$, $g \neq 0$. Write $g = g_1 g_2 \dots g_k$ as a product of irreducibles. As J is a prime ideal, $g_i \in J$ for some i. As $g_i \in I$, $g_i = uf$, and u must be a unit as g_i is irreducible. But then $I = J$ and $Z = Y$.

It follows that $i = d - 1$ and so $\dim Y = d - 1$. □

Lemma 13.6. $(13.2)_{d-1}$ *implies* $(13.4)_d$.

Proof. The result is local on X, so we might as well assume that X and Y are irreducible and affine. We first show that

$$\mu(p) \geq \dim(X) - \dim(Y),$$

for every point of $p \in X$. If $e = \dim(Y) = \dim(X) = d$ there is nothing to prove. So we may assume that $e = \dim(Y) < d = \dim(X)$. Let $q = \pi(p)$. By (13.5) we may embed $Y \subset \mathbb{A}^n$ and pick a hyperplane $q \in H \subset Y$ such that $\dim(H \cap Y) = \dim(Y) - 1$. By an obvious induction, we may pick $\dim(Y)$ hyperplanes $H_1, H_2, \dots, H_e$, whose intersection is a finite set containing q. Working locally about q, we may assume that q is the only point in the intersection. Let $f_1, f_2, \dots, f_e$ be the corresponding polynomials. Then the fibre X_p is defined by the polynomials $g_1, g_2, \dots, g_e$, where $g_i = \pi^* f_i$. So

$$\dim(X_p) \geq \dim(X) - \dim(Y),$$

as required.

To finish the proof, by Noetherian induction applied to X, it suffices to prove that there is an open subset U of X such that

$$\mu(p) \leq \dim(X) - \dim(Y),$$

for every $p \in U$. As usual, we may assume that $X \subset Y \times \mathbb{A}^n$ and that π is projection onto the second factor. Factoring π into the product of n projections, we may assume that $n = 1$, by induction on n. We may assume that $X \subset Y \times \mathbb{A}^1$ is closed. If $X = Y \times \mathbb{A}^1$ then $\mu_0 = 1$ and it is clear that $\dim X \geq \dim Y + 1$. As we have already proved the reverse inequality, $\dim X = \dim Y + 1$.

Otherwise there is a fibre of dimension zero. As X is a proper subset of Y, $\dim X = \dim Y$ and $\mu_0 = 0$. Working locally, we may assume that X is defined by polynomials of the form $F \in A(Y)[S, T]$. Further there is a polynomial $F \in A(Y)[S, T]$ vanishing on X, such that F_y is not the zero polynomial, for at least one $y \in Y$. In this case, the set of points where F_y is not the zero polynomial, is an open subset of Y, and for any point in this open subset, the fibre has dimension zero. □

Lemma 13.7. $(13.4)_d$ *implies* $(13.2)_d$.

Proof. We may assume that X is affine. Pick a finite projection down to $\mathbb{A}^n$. As we are assuming $(13.4)_d$, $n = d$. It clearly suffices to prove the result for $X = \mathbb{A}^d$. Consider projection down to $\mathbb{A}^{d-1}$. Given a maximal chain of irreducible subsets

$$\varnothing \neq Z_0 \subset Z_1 \subset \cdots \subset Z_n = \mathbb{A}^d,$$

let

$$\varnothing \neq Y_0 \subset Y_1 \subset \cdots \subset Y_n = \mathbb{A}^{d-1},$$

be the image in $\mathbb{A}^{d-1}$. Then there is an index i such that Z_i contains the general fibre and Z_{i-1} does not contain the general fibre. Other than that, Y_j determines Z_j and the result follows by induction on d. □

Proof of (13.2) *and* (13.4). Immediate from (13.6) and (13.7). □

Corollary 13.8. *Let $\pi\colon X \longrightarrow Y$ be a surjective and projective morphism of quasi-projective varieties. Then the function*

$$\lambda\colon Y \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\lambda(q)$ is the dimension of the fibre $X_q = \pi^{-1}(q)$ at q. Moreover if X_0 is any irreducible component of X, with image Y_0, then we have

$$\dim(X_0) = \dim(Y_0) + \lambda_0,$$

where λ_0 is the minimum value of λ on Y_0.

Proof. π is proper as it is projective. Therefore the set

$$\{\, y \in Y \mid \lambda(y) \geq k \,\},$$

is closed as it is the image of the set

$$\{\, x \in X \mid \mu(x) \geq k \,\},$$

which is closed by (13.4). □

Note that we cannot discard the hypothesis that π is projective in (13.8). For example, let X be the disjoint union of $\mathbb{A}^2$ minus the y-axis and a single point p. Define a morphism $\pi\colon X \longrightarrow Y = \mathbb{A}^1$ by sending the extra point to the origin and otherwise taking the projection onto the x-axis. Then the fibre dimension is one at every point of Y, other than at the origin, where it is zero. In particular λ is not upper semi-continuous in this example. On the other hand, μ is upper semi-continuous, by virtue of the fact that the extra point is isolated in X.

One rather beautiful consequence of (13.4) is the following:

Corollary 13.9. *Let $\pi\colon X \longrightarrow Y$ be a morphism of projective varieties.*

If Y is irreducible and every fibre of π is irreducible and of the same dimension, then X is irreducible.

Proof. Let $X = X_1 \cup X_2 \cup \cdots \cup X_k$ be the decomposition of X into its irreducible components. Let $\pi_i = \pi|_{X_i}\colon X_i \longrightarrow Y_i$, where Y_i is the image of X_i and let $\lambda_i\colon X_i \longrightarrow \mathbb{N}$ be the function associated to π_i, as in (13.8). Let

$$Z_i = \{\, y \in Y_i \mid \lambda_i(y) \geq \lambda_0 \,\}.$$

(13.8) implies that the closed sets $Z_1, Z_2, \dots, Z_k$ cover Y. As Y is irreducible it follows that there is an index i, say $i = 1$, such that $Z_1 = Y_1 = Y$. But then the fibres of π_1 and π are equal, as they are of the same dimension and the fibres of π are irreducible. This is only possible if $X = X_1$. □

14. Transcendence degree

Example 14.1. *$\mathbb{P}^n$ has dimension n. More generally a toric variety containing a torus $\mathbb{G}_m^n$ has dimension n. In particular the toric variety corresponding to a fan F in N is equal to the rank of the free abelian group N.*

Consider $\mathbb{G}(k,n)$. Then this contains an open subset U isomorphic to $\mathbb{A}^{(k+1)(n-k)}$. So $\mathbb{G}(k,n)$ has dimension $(k+1)(n-k)$. For example, $\mathbb{G}(1,3)$ has dimension $2 \cdot 2 = 4$.

Suppose that X and Y are quasi-projective varieties. Then the dimension of $X \times Y$ is the sum of the dimensions.

We can use (**??**) to calculate the dimension using different methods. One way is to project onto a linear subspace. If we repeatedly project from a point (which is the same as projecting once from a linear space of positive dimension) then the induced morphism $X \longrightarrow \mathbb{P}^k$ will eventually become dominant. At this point the morphism is finite over an open subset and the dimension of X is then k. Note that if we go back one step, then the closure of the image of X will be a hypersurface in $\mathbb{P}^{k+1}$.

Equivalently, if $X \subset \mathbb{P}^n$ and X has dimension d then a general linear space of dimension $n-d-1$ is disjoint from X and a general linear space of dimension $n-d$ meets X in a finite set of points. Note that general means that the linear space belongs to an open set of the corresponding Grasmannian. If X is closed, we can do slightly better, since if X is closed of dimension d, then every linear space of dimension $n-d$ must intersect X.

To calculate the dimension of an algebraic variety one can also use:

Definition 14.2. *Let L/K be a field extension. The **transcendence degree** of L/K is equal to the supremum of the length $x_1, x_2, \dots, x_k$ of algebraically independent elements of L/K.*

It is easy to prove:

Theorem 14.3. *Let X be an irreducible quasi-projective variety.*

Then the dimension of X is equal to the transcendence degree of $K(X)/K$.

One trick to calculate dimensions is to use the generic point of a variety. If we have a morphism $\pi\colon X \longrightarrow Y$ of irreducible varieties then μ_0 is actually the dimension of the generic fibre X_η, over the residue field of the generic point η of Y. Indeed the generic point ξ of X maps to the generic point of Y and so ξ is also the generic point of the generic fibre. The dimension of the generic fibre is the transcendence degree of the residue field of ξ over the residue field of η. The dimension of X is the transcendence degree of the residue field of ξ over K. But the transcendence degree is additive on extensions.

Perhaps an easy example will make all of this clear. Consider $\mathbb{A}^2_K$. Suppose the generic point is ξ, with residue field $K(x,y)$. This has transcendence degree two over K. If we take a projection down to $\mathbb{A}^1_K$, with generic point η and residue field $K(y)$ then the transcendence degree of $K(x,y)/K(y)$ is one, the dimension of the generic fibre. $K(y)/K$ also has transcendence degree one and $\mathbb{A}^1_K$ has dimension one, as expected.

Now let's turn to calculating the dimension of some more examples, using these new techniques. Let us first calculate the dimension of the universal family over the Grassmannian.

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^n \\ \mathbb{G}(k,n). & \end{tikzcd}$$

Note that there are two ways to proceed; we can either use the morphism p or q.

First we use the morphism p. If we fix an element $[\Lambda] \in \mathbb{G}(k,n)$ then the fibre of p will be a copy of the k-plane Λ. Thus every fibre of p is isomorphic to $\mathbb{P}^k$. It follows that Σ has dimension $k + (k+1)(n-k)$.

Now let us use the morphism q. If we fix point $x \in \mathbb{P}^n$, then the fibre of q is equal to the set of k-planes in $\mathbb{P}^n$, containing x. This is isomorphic to a Grassmannian $\mathbb{G}(k-1, n-1)$. Thus the dimension of Σ is equal to $n + k(n-k)$, which is easily seen to be equal to the previous expression.

Note that also we can prove that Σ is irreducible. Either way, it fibres over an irreducible base, with irreducible fibres of the same dimension.

Similarly the universal family of conics has dimension six (=five+one=two+four) and this space is irreducible. It is perhaps more interesting to figure out the dimension of the secant variety and the space of incident l-planes to an irreducible projective variety $X \subset \mathbb{P}^n$.

First the space $\mathcal{C}_l(X)$ of l-planes which meets a closed subset X of $\mathbb{P}^n$. In this case the universal family over $\mathcal{C}_l(X)$ has dimension equal to

$$\dim X + l(n-l),$$

where the second factor is equal to the dimension of the space of l-planes which contains a point. Since we have already seen that this is a variety isomorphic to $\mathbb{G}(l-1,n)$, it follows that the universal family is irreducible, provided X is irreducible.

In particular suppose that X has dimension k, and suppose that $l \leq n-k-1$. Then a typical l-plane which meets X, will only meet X in one point. Thus the map from the universal family to $\mathbb{G}(l,n)$ is in fact birational, and the dimension of $\mathcal{C}_l(X)$ is

$$k + l(n-1).$$

In other words the codimension of $\mathcal{C}_l(X)$ is

$$n - l - k.$$

Thus if $l = n-k-1$, $\mathcal{C}_l(X)$ is a hypersurface in $\mathbb{G}(l,n)$.

Question 14.4. *Fix d. What is the smallest positive integer k such that any polynomial $f(x)$ of degree d over the field $\mathbb{C}$ is a sum of k dth powers of linear forms?*

One way to answer this problem is to use the secant variety to the rational normal curve of degree d. Let V be a two dimensional complex vector space. Then $\mathbb{P}^1 = \mathbb{P}(V)$ and the rational normal curve is the set of pure dth powers in the vector space $\mathbb{P}^d = \mathbb{P}(\operatorname{Sym}^d V)$. A polynomial $f(x)$ of degree d corresponds to a point of $\mathbb{P}^d$ and it is a sum of k dth powers if and only if belongs to the locus of $k-1$-planes which intersect C in k points. We want to know when this locus is the whole of $\mathbb{P}^n$. In this case its dimension is n.

It turns out that even when look at the locus of secant lines that this problem is very hard for a general variety X. In general, we have a rational morphism

$$X \times X \dashrightarrow \mathbb{G}(1,n)$$

Now note that if $l \subset \mathbb{G}(1,n)$ is a point of the image, then this map is not finite over l if and only if l is contained in X. Since the only subvariety with the property that the line through every two points is contained in the subvariety, is a linear space, we may assume that this map has finite fibres over an open set of the image. Then the image has dimension $2k$, where k is the dimension of X. Then the universal family over the image, has dimension $2k+1$ and the dimension of the image in $\mathbb{P}^n$ then has dimension $2k+1$ as well, provided that through a general point of the secant variety (the closure of the set of lines that meet X in at least two points), there passes only finitely secant lines.

Thus the expected dimension of the secant variety is $2k+1$, provided this dimension is at most n. For example, the secant variety to a space curve is expected to be the whole of $\mathbb{P}^3$ and the secant variety to a surface in $\mathbb{P}^5$ is expected to be the whole of $\mathbb{P}^5$.

Definition 14.5. *Let X be a closed irreducible non-degenerate (that is, X is not contained in a proper linear subspace) subvariety of $\mathbb{P}^n$.*

*The **deficiency of** X, denoted $\delta(X)$, is equal to the dimension of the family of secant lines passing through a general point of the secant variety.*

We have already seen then that the dimension of the secant variety is equal to $2k + 1 - \delta(X)$.

Let us calculate the secant variety to the d-uple embedding, at least in characteristic zero. Recall that if $X = \mathbb{P}(V) = \mathbb{P}^k$ then X is embedded in $\mathbb{P}(\mathrm{Sym}^d(V))$, as the space of rank one symmetric tensors (the pure powers). The secant variety then consists of all rank at most two symmetric tensors, that is, anything which is a sum of two rank one symmetric tensors.

In the case of the Veronese, we get the space of rank two quadratic forms. As there are quadratic forms of rank three, it follows than the secant variety to the Veronese is a proper subset of $\mathbb{P}^5$. In fact the space of rank two symmetric tensors is a hypersurface in $\mathbb{P}^5$, given as the vanishing of a determinant. Expanding it follows that the secant variety is defined by a cubic polynomial. Note that the deficiency is equal to 1 in this case.

It is interesting to look at the dimension of some more exotic schemes. $\operatorname{Spec}\mathbb{Z}$ has dimension one. Consider $\mathbb{A}^1_{\mathbb{Z}}$. This has dimension one over $\operatorname{Spec}\mathbb{Z}$ and absolute dimension two. Consider $\mathbb{A}^2_{\mathbb{Z}}$. This has dimension two over $\operatorname{Spec}\mathbb{Z}$ and so it has absolute dimension three.

15. Cubics I

In this section we give a geometric application of some of the ideas of the previous sections. Recall the definition of a rational variety.

Definition 15.1. *A variety X over* $\operatorname{Spec} k$ *is* ***rational*** *if it birational to $\mathbb{P}^n_k$, for some n.*

Theorem 15.2. *Every smooth cubic $C \subset \mathbb{P}^2$ is irrational.*

We will prove (15.2) later.

Theorem 15.3. *Every smooth cubic surface $S \subset \mathbb{P}^3$ is rational.*

The key to the proof of (15.3) is the following celebrated:

Theorem 15.4. *Every smooth cubic surface $S \subset \mathbb{P}^3$ contains twenty seven lines.*

Example 15.5. *Let $S \subset \mathbb{P}^3$ be the cone over a cubic curve $C \subset \mathbb{P}^2$. Then S contains infinitely many lines.*

Lemma 15.6. *Every cubic surface $S \subset \mathbb{P}^3$ contains a line.*

Proof. A cubic is specified by choosing the coefficients of a homogeneous cubic in four variables of which there are $\binom{6}{3} = 20$; the space of all cubics is therefore naturally parametrised by $\mathbb{P}^{19}$. Consider the incidence correspondence

$$\Sigma = \{\, (l, F) \in \mathbb{G}(1,3) \times \mathbb{P}^{19} \,|\, l \subset V(F) \,\} \subset \mathbb{G}(1,3) \times \mathbb{P}^{19}.$$

This is a closed subset of $\mathbb{G}(1,3) \times \mathbb{P}^{19}$ and the two natural projections $f\colon \Sigma \longrightarrow \mathbb{G}(1,3)$ and $g\colon \Sigma \longrightarrow \mathbb{P}^{19}$ are proper, since they are projective.

Let $l \in \mathbb{G}(1,3)$ and consider $f^{-1}(l)$. This is the space of cubics containing the line l. There are two ways to figure out what the fibre looks like.

One can change coordinates so that $l = V(X_2, X_3)$, so that the points of l are $[a : b : 0 : 0]$. In this case the coefficients of X^3, X^2Y, XY^2 and Y^3 must all vanish. The fibre is a copy of a linear subspace of dimension 15 in $\mathbb{P}^{19}$.

Aliter: Pick four distinct points p_1, p_2, p_3 and p_4 of l. Suppose $F(p_i) = 0$, for $1 \leq i \leq 4$. Then $F|_l$ is a cubic polynomial in two variables, vanishing at four points. Thus $F|_l$ is the zero polynomial. It follows that $l \subset V(F)$ if and only if F vanishes at p_i, for $1 \leq i \leq 4$.

The condition that $F(p_i) = 0$ imposes one linear constraint. One can check that these four points impose independent conditions, so that that the space of cubics containing all four points is a linear subspace of dimension 15.

Either way, Σ fibres over an irreducible base with irreducible fibres of the same dimension. It follows that Σ is irreducible of dimension $4 + 15 = 19$. It suffices then to exhibit a single cubic with finitely many lines, since then the morphism g must be dominant, whence surjective. It is a fun exercise to compute the twenty seven lines on $X^3 + Y^3 + Z^3 + T^3 = 0$. $\square$

Lemma 15.7. *If $S \subset \mathbb{P}^3$ is a smooth cubic surface and $l \subset S$ is a line then there are ten lines meeting l.*

In particular S contains two skew lines.

Proof. Consider the planes $H \subset \mathbb{P}^3$ containing l. Then $H \cap S = l \cup C$, where $C \subset H \simeq \mathbb{P}^2$ is a plane curve of degree two.

First observe that C is never a double line n. Indeed, if F and G are the linear polynomials which define l and F and H are the linear polynomials defining n, so that $F = 0$ is the plane spanned by l and n, then the equation of S has the form

$$FQ + GH^2,$$

for some quadratic polynomial Q. But then S is singular at the two points where $F = H = Q = 0$ (just compute partials).

Suppose that m is a line that intersects l. Then $C = m \cup n$, where n is another line, which also meets l. Thus lines that intersect l come in concurrent pairs and we just have to show that there are five such pairs.

We may suppose that l is given by $Z = T = 0$. Then S is defined by an equation of the form

$$AX^2 + 2BXY + CY^2 + 2DX + 2EY + F,$$

where A, B, C, D, E and F are homogeneous polynomials in Z and T.

The pencil of planes containing l is given by $Z = \lambda T$. Note that $C = C_\lambda$ is a pair of lines if and only if C is singular. C_λ is singular if and only if the determinant

$$\begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}$$

is zero. The determinant is a homogeneous polynomial of degree 5 in Z and T and so it suffices to show it has no repeated roots.

Supppose that $Z = 0$ is a root. There are two cases. If the singular point s of C_0 is not a point of l then we may assume that C_0 is given by $XY = 0$. Then every entry of the matrix above is divisible by Z, except B. On the other hand, as s is not a singular point of S it follows that Z^2 does not divide F. Thus Z^2 does not divide the determinant.

If s is a point of l then we may assume that C_0 is given by $X^2 - T^2 = 0$ and one can check that Z^2 does not divide the determinant. $\square$

Proof of (15.4). We just prove that S contains a pair of skew lines. (15.6) implies that S contains at least one line l. (15.7) implies that there are ten other lines meeting l. Pick one of them l'. Of the ten lines meeting l', at most one of them intersects l. Thus we may find a line m meeting l' not intersecting l. $\square$

Proof of (15.3). By assumption S contains two skew lines l and m. Define a rational map

$$\phi \colon l \times m \dashrightarrow S,$$

by sending the point (p, q) to the intersection of the line $n = \langle p, q \rangle$ with $S - (l \cup m)$. Since a cubic intersects a typical line in three points, and the line n intersects S at $p \in l$ and $q \in m$, there is an open subset of $l \times m$ such that the line n intersects S at one further point $r = \phi(p, q)$.

Define a rational map

$$\psi \colon S \dashrightarrow l \times m,$$

by sending $r \in S - (l \cup m)$ to (p, q), where p is the intersection point of the plane $\langle p, m \rangle$ with l and q is the intersection point of the plane $\langle p, l \rangle$ with m.

It is easy to check that ϕ and ψ are inverse. It follows that ϕ is birational. As $\mathbb{P}^1 \times \mathbb{P}^1 \simeq l \times m$ is rational, S is rational. $\square$

16. Cubics II

It turns out that the question of which varieties are rational is one of the subtlest geometric problems one can ask. Since the problem of determining whether a variety is rational or not is so delicate, various intermediary notions have been introduced.

Definition 16.1. *We say that a variety X is* ***unirational*** *if there is a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$.*

Theorem 16.2. *Every smooth cubic threefold $V \subset \mathbb{P}^4$ is unirational.*

Note some basic properties of unirational varieties.

Lemma 16.3. *Let X be a variety over k. The following are equivalent:*

(1) X is unirational.
(2) There is a dominant generically finite morphism $\phi\colon Y \longrightarrow X$, where Y is rational.
(3) The function field of X is contained in a purely transcendental field extension of k.
(4) There is a finite extension of the function field of X which is a purely transcendental field extension of k.

Proof. The fact that (1) and (3) are equivalent, follows from the equivalence of categories between dominant rational maps and inclusions of function fields, and (4) follows from (2) in a similar fashion.

So suppose that $\phi\colon \mathbb{P}^n_k \dashrightarrow X$ is a dominant rational map. Replacing $\mathbb{P}^n$ by the normalisation of the graph of ϕ, we may assume that there a quasi-projective variety Y and a dominant morphism $Y \longrightarrow X$. If the dimension of the generic fibre is greater than zero, then pick a hyperplane $H \subset \mathbb{P}^n$, whose inverse image in Y dominates X. Continuing in this way, we reduce to the case where is generically finite. □

Thus to prove (16.2) we are looking for a dominant rational map $\mathbb{P}^3 \dashrightarrow V$. The trick is to consider low degree rational curves on V.

Lemma 16.4. *Every smooth cubic threefold V be in $\mathbb{P}^4$ contains a two dimensional family F of lines.*

Proof. Consider the incidence correspondence

$$\Sigma = \{\, (l, H) \mid l \subset S = H \cap V \,\} \subset \mathbb{G}(1, 3) \times \mathbb{P}^4.$$

This has two morphisms, $p\colon \Sigma \longrightarrow \mathbb{G}(1, 3)$ and $q\colon \Sigma \longrightarrow \mathbb{P}^3$. Let H be a general hyperplane in $\mathbb{P}^4$. Then $S = H \cap V$ is a smooth cubic surface in $\mathbb{P}^3$. But then we have already seen that S contains a finite number of lines. Thus the minimum dimension of the fibres of p is zero. It follows that Σ has dimension four.

If we fix l then there is a two dimensional family of hyperplanes containing l (in fact a copy of $\mathbb{P}^2$). Since the fibres of q are two dimensional, it follows that $F_1 = q(\Sigma)$ has dimension two. $\square$

It is interesting to observe that there is a four dimensional family of conics. If you fix a line l and look at the family of planes containing the line then this will cut the cubic in a plane cubic curve. Part of this curve is the line l, and the residual curve is a conic. The family of planes containing the line is a copy of $\mathbb{P}^2$ and so a four dimensional family of conics.

The idea to prove (16.2) is to exploit the family of conics residual to a line.

Definition 16.5. *A **conic bundle** is a projective morphism, $\pi\colon X \longrightarrow S$, between quasi-projective varieties, where the fibres are conics in $\mathbb{P}^2$. A **rational conic bundle**, is any morphism, which is a conic bundle over an open subset of the base.*

Of course the fibres of any conic bundle have three types

- a smooth conic,
- a pair of lines,
- a double line.

Note that a morphism is a rational conic bundle if and only if the generic fibre is a smooth conic in $\mathbb{P}^2_K$, where K is the function field of the base. We will change our conventions a little; for now on in this section a variety is a separated scheme of finite type over a field, not necessarily algebraically closed. If the groundfield is not algebraically closed, then this question can become very tricky, even in low dimensions.

Lemma 16.6. *Let $\pi\colon X \longrightarrow S$ be a morphism of quasi-projective varieties.*

If the generic fibre is rational and S is rational then X is rational.

Proof. By assumption the function field of S is a purely transcendental extension of the groundfield k, $K = K(S) \simeq k(x_1, x_2, \dots, x_m)$. Equivalently the residue field of the generic point η of S is purely transcendental over k. Let X_η be the generic fibre. The function field of X_η is a purely transcendental extension of K. This is the residue field of the generic point ξ of X_η, which is also the residue field of X.

Thus the function field of X is a purely transcendental extension of a purely transcendental extension, so that it is a purely transcendental extension of k. Thus X is rational. $\square$

Lemma 16.7. *Let $V \subset \mathbb{P}^4$ be a smooth cubic.*

Then the blow up of V along a line is a rational conic bundle over $\mathbb{P}^2$.

Definition 16.8. *A k-**rational point** of a scheme X over S is any point which is the image of a morphism $\operatorname{Spec} k \longrightarrow X$ over S. The set of all k-rational points is denoted $X(k)$.*

In other words a k-rational point is simply a point whose residue field is a subfield of k.

Example 16.9. *Let $X = \mathbb{A}^1_{\mathbb{R}}$. Then $p = \langle x^2 + a \rangle \in X$, where $a \in \mathbb{R}$ and $a > 0$, corresponds to two $\mathbb{C}$-valued points. Indeed, there are two scheme maps*

$$\operatorname{Spec} \mathbb{C} \longrightarrow X,$$

whose image is p, corresponding to the fact that there are two automorphisms of $\operatorname{Spec} \mathbb{C}$ over $\operatorname{Spec} \mathbb{R}$, given by the identity and complex conjugation.

Lemma 16.10. *Let $C \subset \mathbb{P}^2_k$ be a smooth conic, over a field k.*

Then $C \simeq \mathbb{P}^1_k$ if and only if C contains a k-rational point

Proof. One direction is clear as $\mathbb{P}^1_k$ certainly contains k-rational points.

Now suppose that C contains a k-rational point. After applying an element of $\mathrm{PGL}(3, k)$, we may assume that this point is $[0 : 0 : 1]$. Consider projection from this point. This defines a morphism $C - [0 : 0 : 1] \longrightarrow \mathbb{P}^1$, which is surely defined over k (indeed it is the restriction of $[x : y : z] \longrightarrow [x : y]$). It is then straightforward to check that this morphism extends to an isomorphism. □

Example 16.11. *The conic $C = V(x^2 + y^2 - z^2) \subset \mathbb{P}^2_{\mathbb{R}}$ is not rational over $\operatorname{Spec} \mathbb{R}$.*

Definition 16.12. *Let $\pi \colon X \longrightarrow S$ be a morphism of schemes. A **section** of π is a morphism $\sigma \colon S \longrightarrow X$ such that $\sigma \circ \pi$ is the identity. A **rational section** is a section defined on some open subset U of S.*

Lemma 16.13. *Let $\pi \colon X \longrightarrow S$ be a morphism of integral schemes, of finite type. Then picking a rational section of π is equivalent to picking a rational point of the generic fibre.*

Proof. Let K be the function field of S. We may as well assume that both $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are affine, so that K is the field of fractions of A. The generic fibre has coordinate ring $B \underset{A}{\otimes} K$. Suppose that we have a rational section. Then we may as well assume that we have a section. But this is equivalent to a ring homomorphism $B \longrightarrow A$. In turn this induces a ring homomorphism $B \underset{A}{\otimes} K \longrightarrow K$

which is equivalent to a morphism $\operatorname{Spec} K \longrightarrow X_\xi$, where ξ is the generic point of S. But this is exactly the same as a rational point of the general fibre.

Now suppose that we have a rational point of the generic fibre. This is equivalent to a ring homorphism $B \underset{A}{\otimes} K \longrightarrow K$. Since we have a morphism of finite type, B is a finitely generated A-algebra. Pick generators $b_1, b_2, \dots, b_k$. Denote the image of b_i in K by c_i/d_i, where c_i and d_i are elements of A. Passing to the open affine subset U_d of S, where d is the product $d_1 \cdot d_2 \cdot \dots \cdot d_k$, we may assume that $d_i = 1$, so that we get a morphism $B \longrightarrow A$. But this is equivalent to a section of π. □

Proposition 16.14. *Let $\pi \colon X \longrightarrow S$ be a rational conic bundle, between two varieties, over an algebraically closed field k. Let $T \subset X$ be a subvarety of X which dominates S.*

1. *If T is unirational, then so is X.*
2. *If $T \longrightarrow S$ is birational and T is rational, then so is X.*

Proof. Consider the base change $T \longrightarrow S$ of X. Let Y be a component of the base change of maximal dimension, which dominates X. Then $Y \longrightarrow T$ is a conic bundle. Moreover, there is a natural morphism $T \longrightarrow Y$ which is a section. Possibly base changing further, we may assume that the base is rational, and that there is a rational section. Thus it suffices to prove (2).

Consider the generic fibre. By assumption it is a smooth conic in $\mathbb{P}^2_K$, where K is not algebraically closed. The rational section implies that this conic has a rational point. But then this conic is isomorphic to $\mathbb{P}^1_K$. The function field of this conic is then $K(t)$. The generic point of X is also the ceneric point of the generic fibre. It follows that the function field of X is isomorphic to $K(t)$. Since K is purely transcendental over k the groundfield, this implies that the field of fractions of X is purely transendental over k. But then X is rational. □

Proof of (16.2). Let V be a smooth cubic threefold and let l be a line in V and let X be the blow up of V along l, $f \colon X \longrightarrow V$. Then there is a conic bundle $\pi \colon X \longrightarrow \mathbb{P}^2$. Let E be the excpetional divisor of the blow up. Then E is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Thus E is rational. But E dominates $\mathbb{P}^2$ and we are done by (16.14). □

In fact $E \longrightarrow \mathbb{P}^2$ is a two to one map.

Question 16.15 (Lüroth's problem). *Is every unirational variety rational?*

Note that one way to restate Lüroth's problem is to ask if every subfield of a purely transcendental field extension is purely transcendental. It turns out that the answer is yes in dimension one, in all characteristics. This is typically a homework problem in a course on Galois Theory. There is also a simple geometric proof of this fact (essentially the Riemann-Hurwitz formula). In dimension two the problem is already considerably harder, and it is false if one allows inseparable field extensions.

In dimension three it was shown to be false even in characteristic zero, in 1972, using three different methods.

One proof is due to Artin and Mumford. It had been observed by Serre that the cohomology ring of a smooth unirational threefold is indistinguishable from that of a rational variety (for $\mathbb{P}^3$ one gets $\mathbb{Z}[x]/\langle x^3 \rangle$ and the cohomology ring varies in a very predictable under blowing up and down) except possibly that there might be torsion in $H_3(X, \mathbb{Z})$. They then give a reasonably elementary construction of a threefold with non-zero torsion in H_3.

Another proof is due to Clemens and Griffiths. (16.2) implies that every smooth cubic hypersurface in $\mathbb{P}^4$ is unirational. On the other hand they prove that some smooth cubics are not rational. A lot of the geometry of the cubic is controlled by the geometry of the Fano surface of lines.

The third proof is due to Iskovskikh and Manin. They prove that every smooth quartic in $\mathbb{P}^4$ is not rational. On the other hand, some quartics are unirational. In fact they show, in an amazing tour de force, that the birational automorphism group of a smooth quartic is finite. Clearly this means that a smooth quartic is never rational.

Since it is so hard to distinguish between rational and unirational, yet another closely related notion has been introduced.

Definition 16.16. *Let X be a variety, over an algebraically closed field of characteristic zero. We say that X is* ***rationally connected*** *if for two general points x and y of X, we may find a rational curve connecting x and y.*

One convenient way to restate this condition, is that for two general points x and y, we may find a morphism

$$f \colon \mathbb{P}^1_k \longrightarrow X,$$

such that $f(0) = x$ and $f(\infty) = y$. Indeed the nonconstant image of $\mathbb{P}^1_k$ is always birational to $\mathbb{P}^1_k$.

It is interesting to consider what happens in higher dimension. The space of cubic fourfolds $X \subset \mathbb{P}^5$ is a copy of projective space of dimension

$$\binom{3+5}{3} - 1 = 55.$$

Every cubic fourfold is unirational. Some cubic fourfolds are rational. For example, it is possible to write down smooth cubic fourfolds X which contain a pair of skew planes, L and M. This defines a rational map

$$\phi\colon L \times M \dashrightarrow X,$$

which assigns to a point (p, q) the point of intersection of the line $\langle p, q\rangle$ with $X - (L \cup M)$. The rational map

$$\psi\colon X \dashrightarrow L \times M,$$

which assigns to every point r the point (p, q), where p is the intersection of the 3-plane $\langle r, M\rangle$ with L and q is the intersection of the 3-plane $\langle r, L\rangle$ with M, is the inverse of ϕ. Thus X is birational to

$$L \times M = \mathbb{P}^2 \times \mathbb{P}^2 \simeq \mathbb{P}^4$$

The locus of cubics which contain a pair of skew planes has codimension two in $\mathbb{P}^{55}$. One can also write down other configurations of subvarieties of X which guarantee that X is rational.

Conjecture 16.17. *The locus of smooth rational cubic fourfolds inside the open subset of $\mathbb{P}^{55}$ consisting of all smooth cubics is a countable union of closed subsets of codimension two.*

In particular there are smooth irrational cubic fourfolds.

It is interesting to note the following

Theorem 16.18. *Fix a positive integer d.*

Then there is a positive integer n_0 such that if $n \geq n_0$ then every smooth hypersurface of degree d in $\mathbb{P}^n$ is unirational.

By contrast, there is no example of smooth hypersurface of degree $d \geq 4$ which is rational.