Applications of Deligne's Weil II

We start by recalling the main theorem of this seminar (Weil II for curves) and illustrating some of its arithmetic consequences. Then we introduce the theory of Lefschetz pencils and deduce the last part (Riemann hypothesis) of the Weil conjectures. Finally, we prove the geometric semisimplicity of lisse pure sheaves, and use it to deduce the hard Lefschetz theorem.

This is an expanded note prepared for a STAGE talk, Spring 2014. Our main references are [1], [2] and [3].

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Weil II for curves

Recall that we have proved the following target theorem.

Theorem 1 Let k be a finite field and $\ell\ne\mathrm{char}(k)$ be a prime number. Fix an embedding $\tau:\overline{\mathbb{Q}_\ell}\hookrightarrow\mathbb{C}$. Let U/k be smooth geometrically connected curve. Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}_\ell}$-sheaf on U, τ-pure of weight w. Then

a. $H_c^0(U_{\bar k},\mathcal{F})$ is τ-pure of weight w.
b. $H_c^1(U_{\bar k},\mathcal{F})$ is τ-mixed of weight $\le w+1$.
c. $H_c^2(U_{\bar k},\mathcal{F})$ is τ-pure of weight $w+2$.

Remark 1 Recall the strategy of the proof:

a. Via a series of elementary reductions, reduce to the case $U=\mathbb{A}^1$ (at the cost of making the sheaf $\mathcal{F}$ more complicated) and $\mathcal{F}$ is lisse geometrically irreducible pure of weight 0.
b. Fix $N\gg 0$, put $\mathcal{F}$ in a 2-variable family $\mathcal{F}\otimes\mathcal{L}_{\psi(sT^N)}$ over $\mathbb{A}^2_{(s,T)}$ by Artin-Schreier twists. It suffcies to show that that $H_c^1(\mathbb{A}^1,\mathcal{F}\otimes\mathcal{L}_{\psi(sT^N)})$ is pure of weight 1 (the *purity theorem*) for $s\ne 0$ and take the limit $s\to 0$ (*weight dropping*).
c. Now fix s and N. To show the purity theorem, put $\mathcal{F}\otimes\mathcal{L}_{\psi(sT^N)}$ in a 2-variable family $\mathcal{G}$ over $\mathbb{A}^2_{(a,b)}$ encoding $H_c^1(\mathbb{A}^1,\mathcal{F}\otimes\mathcal{L}_{\psi(sT^N+aT+bT^2)})$ at each fiber. $\mathcal{G}$ is lisse and has geometric monodromy either a finite irreducible subgroup of GL_N or a finite index subgroup containing SL_N (the *monodromy theorem*). The monodromy theorem reduces to the computation of the 4th moment of $\mathcal{G}$, which in turn results from the computation $\dim H_c^4(\mathbb{A}^2,\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G}^\vee\otimes\mathcal{G}^\vee)=2$.
d. Choosing a space filling curve C in $\mathbb{A}^2_{(a,b)}$, the big monodromy forces $H_c^2(C,\mathcal{G}\otimes\mathcal{G}^\vee)$ to be pure of weight 2 (Frobenius acts on the geometric coinvariants via *roots of unity*) and so the L-series $L(C,\mathcal{G}\otimes\mathcal{G}^\vee)(T)$ is analytic in $|T|<1/q$. Now Rankin's trick implies that $L(C,\mathcal{G}\otimes\mathcal{G}^\vee)(T)$ dominates each term $\det(1-T\,\mathrm{Frob}_{a,b}\,|\mathcal{G}\otimes\mathcal{G}^\vee)^{-1}$, hence the latter is also holomorphic in $|T|<1/q$. The tensor power trick then shows that $\mathcal{G}$ is pure of weight 1, i.e., the purity theorem is true.

Corollary 1 Let $j:U\hookrightarrow X$ be a smooth compactification over k. Then $H^i(X_{\bar k},j_*\mathcal{F})$ is τ-*pure* of weight $w+i$ for $i=0,1,2$.

Proof

a. $i=0$: By Leray or Gysin, we have $H^0(X, j_*\mathcal{F}) = H^0(U, \mathcal{F})$, which consists of the $\pi_1^g(U)$-invariants, is τ-pure of weight w. (Note: cohomology groups are always understood as taking the cohomology of the base change $X_{\bar{k}}$).

b. $i=1$: By Leray or Gysin, we have $H^1(X, j_*\mathcal{F}) = \mathrm{im}(H^1_c(U, \mathcal{F}) \to H^1(U, \mathcal{F}))$. Since $H^1_c(U, \mathcal{F})$ is τ-mixed of weight $\le w+1$, and by Poincare duality, $H^1(U, \mathcal{F})$ is τ-mixed of weight $\ge w+1$, it follows $H^1(X, j_*\mathcal{F})$ is τ-pure of weight $w+1$.

c. $i=2$: by birational invariance of top H_c (or long exact sequence associated to $0 \to j_!\mathcal{F} \to j_*\mathcal{F} \to i_*i^*j_*\mathcal{F} \to 0$), we have $H^2(X, j_*\mathcal{F}) = H^2_c(U, \mathcal{F})$, which is τ-pure of weight $w+2$. □

Remark 2 Deligne's Weil II proved a stronger version of Theorem 1: replace $U \to \operatorname{Spec} k$ by any morphism $f: X \to Y$ between any separated $\mathbb{Z}[1/\ell]$-schemes of finite type and $\mathcal{F}$ by any τ-mixed constructible $\overline{\mathbb{Q}_\ell}$-sheaf of weight $\le w$. Then the constructible $\overline{\mathbb{Q}_\ell}$-sheaf $R^i f_! \mathcal{F}$ on Y is τ-mixed of weight $\le w+i$ for any i. The weaker Theorem 1 is nevertheless strong enough to deduce the last part (the Riemann hypothesis) of the Weil conjectures.

Several remarkable arithmetic consequences follows.

Theorem 2 (Riemann Hypothesis over finite fields) Let X/k be a smooth projective geometrically connected variety. Then for any $\tau: \overline{\mathbb{Q}_\ell} \hookrightarrow \mathbb{C}$ and any i, $H^i(X_{\bar{k}}, \overline{\mathbb{Q}_\ell})$ is τ-pure of weight i.

Example 1 We will deduce this theorem in the next section. The Riemann hypothesis illustrates the following surprising slogan: the arithmetic of a smooth projective variety X over a finite field $\mathbb{F}_q$ is controlled by the topology of the corresponding complex manifold $X(\mathbb{C})$. If we factorize the zeta function of X as

$$Z(X,T) = \frac{\prod_i (1-\alpha_i T)}{\prod_j (1-\beta_j T)},$$

here α_i (resp. β_i) are the Frob_q eigenvalues on the odd (resp. even) degree cohomology groups $H^k(X_{\overline{\mathbb{F}_q}}, \overline{\mathbb{Q}_\ell})$, then by the Lefschetz trace formula,

$$\#X(\mathbb{F}_{q^m}) = \sum \beta_j^m - \sum \alpha_i^m.$$

The Riemann hypothesis provides the key to understand these α_i and β_j: it allows us to obtain very good estimates of $\#X(\mathbb{F}_{q^m})$ as long as we know enough about the cohomology of X. For example, when $X = E$ is an elliptic curve over $\mathbb{F}_q$, we obtain

$$\#E(\mathbb{F}_q) = 1 + q - (\alpha_1 + \alpha_2).$$

By Corollary 1, we know that α_1, α_2, as Frob_q eigenvalues on $H^1(E, \overline{\mathbb{Q}_\ell})$, has pure weight 1. In this way we recover the classical Hasse-Weil bound,

$$|\#E(\mathbb{F}_q) - (1+q)| \le 2\sqrt{q}.$$

Example 2 (Ramanujan conjecture) The Ramanujan τ-function is defined to be the coefficients of the q-expansion of the weight 12 cusp eigenform

$$\Delta(z) = q\prod_n (1-q^n)^{24} = \sum_{n \ge 1} \tau(n) q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + \cdots.$$

Ramanujan famously observed (!) without proof that $|\tau(p)| \le 2p^{11/2}$. This turns out to be a general phenomenon for coefficients of a cusp eigenform f of weight k and level N, as a consequence of Weil II together with the algebro-geometric incarnation of cusp eigenforms. Let $j: U = Y_0(N) \hookrightarrow X_0(N)$ be the modular curves over $\mathbb{Z}[1/N]$ with the universal family of elliptic curves $\pi: \mathcal{E} \to U$. By the Eichler-Shimura relation, the Hecke eigenvalue $a(p)$ ($p \nmid N$) of f matches up with the trace of Frob_p on the (2-dimensional) f-isotypic component of

$$H^1(X_0(N), j_* \operatorname{Sym}^{k-2} R^1\pi_* \overline{\mathbb{Q}_\ell}).$$

Since $R^1\pi_*\overline{\mathbb{Q}_\ell}$ is lisse on U of pure weight 1, by Corollary 1, the above H^1 is pure of weight $(k-2)+1 = k-1$. Therefore both Frob_p eigenvalues have absolute values $p^{(k-1)/2}$ and so $|a(p)| \le 2p^{(k-1)/2}$.

Notice in this example, it is convenient to work with non-constant coefficient systems. Here is another typical example.

Example 3 (Kloosterman sum) Let $a \in \mathbb{F}_p^\times$. The classical Kloosterman sum is defined to be

$$\mathrm{Kl}_m(a) = \sum_{\substack{x_1 \cdots x_m = a \\ x_i \in \mathbb{F}_p^\times}} e^{2\pi i \cdot \frac{x_1 + \cdots x_m}{p}}.$$

This can be interpreted geometrically as follows. Let $\mathcal{L}_\psi$ be the Artin-Schrier sheaf on $\mathbb{A}^1$ associated to the additive character

$$\psi : \mathbb{F}_p \to \overline{\mathbb{Q}_\ell}^\times \cong \mathbb{C}^\times, \quad x \mapsto e^{2\pi i x/p},$$

Let X be the smooth affine variety of dimension $m-1$ defined by the equation $x_1 \cdots x_m = a$ and define

$$f : X \to \mathbb{A}^1, \quad (x_1, \ldots, x_m) \mapsto x_1 + \cdots + x_m.$$

Then

$$\mathrm{Kl}_m(a) = \sum_{x \in X(\mathbb{F}_p)} \mathrm{Tr}(\mathrm{Frob}_x | (f^* \mathcal{L}_\psi)_x).$$

Deligne computed that

$$\dim H_c^i(X, f^*\mathcal{L}_\psi) = \begin{cases} 0, & i \neq m-1, \\ m, & i = m-1. \end{cases}$$

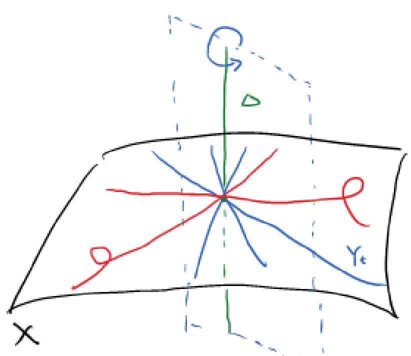
Hence by the strong version of Weil II, we obtain the estimate $|\mathrm{Kl}_m(a)| \leq m p^{(m-1)/2}$, which is certainly not easy to obtain using elementary methods.

Lefschetz pencils and Riemann Hypothesis

When X is a curve, the Riemann hypothesis follows from Corollary 1, since $\overline{\mathbb{Q}_\ell}$ is τ-pure of weight 0 for any τ. For the general case, we induct on $n = \dim X$ via the theory of Lefschetz pencils.

Definition 1 Let $X/k \subseteq \mathbb{P}^m$ be a smooth projective variety of dimension n. A *Lefschetz pencil of hyperplanes* on X, is a family of hyperplanes $(H_t)_{t \in \mathbb{P}^1}$, where $H_{[a:b]} = \{a \cdot f + b \cdot g = 0\}$, such that

- The hyperplane section $Y_t = X \cap H_t$ is smooth for all t in an open dense subset $U \subseteq \mathbb{P}^1$.
- For $t \notin U$, Y_t has only one singular point and the singularity is an ordinary double point, i.e., its complete local ring of is the form $k[[T_1, \ldots, T_{n+1}]]/(q(T_1, \ldots, T_{n+1}))$, where q is a non-degenerate quadratic form.
- The axis of the pencil $\Delta = \{f = g = 0\}$ (of codimension 2 in $\mathbb{P}^m$) intersects with X transversely (so $\Delta \cap X$ has dimension $n-2$).



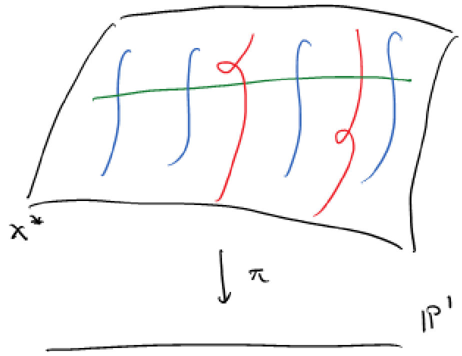
Using incidence correspondences and the Bertini theorem, one can show the existence of Lefschetz pencils.

Theorem 3 There exists a Lefschetz pencil of hyperplanes defined over k on X, after possibly a finite extension of the base field and possibly replacing the projective embedding from $\mathcal{O}_{\mathbb{P}^m}(1)$ by $\mathcal{O}_{\mathbb{P}^m}(d)$ for some $d \geq 2$.

Let X^* be the blow up of X along $\Delta \cap X$, then we obtain a projective morphism $\pi : X^* \to \mathbb{P}^1$ with smooth fibers over $\mathbb{P}^1 - S$, where S is a finite set of points. After possibly a finite base extension, we may assume S consists of k-rational points. The Leray spectral sequence for the blow up $X^* \to X$ implies that

$$H^i(X^*, \overline{\mathbb{Q}_\ell}) \cong H^i(X, \overline{\mathbb{Q}_\ell}) \oplus H^{i-2}(\Delta \cap X, \overline{\mathbb{Q}_\ell})(-1).$$

So it suffices to prove the purity statement for X^*.



Due to the simple nature of singularities, it is possible to describe both the local and global monodromy actions on the cohomology. In the complex setting, this is classically known as the Picard-Lefschetz theory. In the ℓ-adic setting, this is done in SGA 7 and is briefly summarized as follows.

Theorem 4 Let I_s be the local monodromy group at $s \in S$ (i.e., the tame quotient of the etale fundamental group of s). Let Y_η be the geometric generic fiber and Y_s be the geometric fiber at s. Then

a. For $i \neq n$, $H^i(Y_s) \cong H^i(Y_\eta)$ and I_s acts on $H^i(Y_\eta), \overline{\mathbb{Q}_\ell})$ trivially. Namely, away from the middle degree, the singularity at $s \in S$ is not seen.

b. The action of I_s on $H^n(Y_\eta)$ is described by the Picard-Lefschetz formula in terms of the intersection pairing with the vanishing cycle at s. When n is odd, $H^n(Y_s) \cong H^n(Y_\eta)^{I_s}$; when n is even, either $H^n(Y_s) \cong H^n(Y_\eta)^{I_s}$, or we have an exact sequence
$$0 \to \overline{\mathbb{Q}_\ell}(-n/2) \to H^n(Y_s) \to H^n(Y_\eta)^{I_s} \to 0.$$

c. $H^n(X) \cong H^n(Y_\eta)^{\pi_1^g(\mathbb{P}^1-S)}$ (the invariants under the global geometric monodromy).

Now we can finish the induction step. For simplicity let us assume n is odd (the even case is similar). Let $\mathcal{F}_i = R^i\pi_*\overline{\mathbb{Q}_\ell}$ and $j: \mathbb{P}^1 - S \hookrightarrow \mathbb{P}^1$. Then by the property of the Lefschetz pencil (Theorem 3 a), b)), we have $\mathcal{F}_i \cong j_*j^*\mathcal{F}_i$. The Leray spectral sequence implies that
$$H^a(\mathbb{P}^1, \mathcal{F}_b) \Longrightarrow H^{a+b}(X^*, \overline{\mathbb{Q}_\ell}).$$
By induction and proper base change, $j^*\mathcal{F}_b$ is lisse on $\mathbb{P}^1 - S$, pure of weight b. So Corollary 1 implies that each term $H^a(\mathbb{P}^1, j_*j^*\mathcal{F}_b)$ is pure of weight $a+b$. Hence $H^{a+b}(X^*, \overline{\mathbb{Q}_\ell})$ is pure of weight $a+b$ as well. This completes the proof of Theorem 2.

The hard Lefschetz theorem

Theorem 5 (Geometric semisimplicity) Let k be a finite field and $\ell \neq \mathrm{char}(k)$ be a prime number. Let X/k be a smooth geometrically connected variety. Let $\mathcal{F}$ be a lisse τ-pure $\overline{\mathbb{Q}_\ell}$-sheaf on X. Then the representation ρ of $\pi_1^g(X)$ associated to $\mathcal{F}$ is semisimple.

Proof As we have already seen in Koji's talk, we can replace X by a space filling curve without changing the geometric monodromy group, the smallest algebraic group containing the image of $\rho(\pi_1^g)$. So it suffices to treat the curve case. We are going to induct on the length of ρ as a π_1-representation. When ρ is irreducible as a $\pi_1(X)$-representation, as in Kestutis's talk, ρ is semisimple as a $\pi_1^g(X)$-representation (since $\pi_1^g(X) \subseteq \pi_1(X)$ is normal). Now suppose $\mathcal{F}$ is an extension of lisse τ-pure sheaves on X,
$$0 \to \mathcal{A} \to \mathcal{F} \to \mathcal{B} \to 0,$$
we would like to show that there is a section (as $\pi_1^g(X)$-representations). In other words, we would like to show that the element $f \in H^0(X_{\bar{k}}, \mathcal{B} \otimes \mathcal{B}^\vee)$ corresponding to the identity morphism $\mathcal{B} \to \mathcal{B}$ lies in the image of the first map in the following sequence,
$$H^0(X_{\bar{k}}, \mathcal{F} \otimes \mathcal{B}^\vee) \to H^0(X_{\bar{k}}, \mathcal{B} \otimes \mathcal{B}^\vee) \to H^1(X_{\bar{k}}, \mathcal{A} \otimes \mathcal{B}^\vee).$$
The crucial thing is the *mismatch of weights* in the second map: by Theorem 1, the source has weight 0 but the target has weight 1. Since f is fixed by Frob_k, it must die in $H^1(X_{\bar{k}}, \mathcal{A} \otimes \mathcal{B}^\vee)$ and hence comes from some element of $H^0(X_{\bar{k}}, \mathcal{F} \otimes \mathcal{B}^\vee)$. □

Next we will see how the geometric semisimplicity grew out of the *arithmetic* consideration of weights can help us to understand the fundamental *geometric* structure of smooth projective varieties.

Theorem 6 (Hard Lefschetz) Let X be a projective smooth connected variety over an algebraically closed field (of any characteristic). Let $\mathcal{L}$ be an ample line bundle on X and $L = c_1(\mathcal{L}) \in H^2(X, \overline{\mathbb{Q}_\ell})(-1)$. Then for any $1 \le i \le n$, the i-th iterated cup product

$$L^i : H^{n-i}(X, \overline{\mathbb{Q}_\ell}) \to H^{n+i}(X, \overline{\mathbb{Q}_\ell}(-i))$$

is an isomorphism.

Proof Since X and $\mathcal{L}$ are defined over a finitely generated subfield k_0 of k, using the defining equations, we obtain a morphism of scheme $\mathcal{X} \to S$ and an ample line bundle on $\mathcal{X}$ with the generic fiber X/k_0 and $\mathcal{L}$, where S is scheme of a finite type over $\mathbb{Z}$. After possibly shrinking S, we may assume $\mathcal{X} \to S$ has projective smooth connected fibers. To show L^i is an isomorphism on over generic point, it suffices to check it is an isomorphism on each closed point. This puts us in the situation where k is the algebraic closure of a *finite field* (even if we work with $\mathbb{C}$ at the beginning).

The case $n = 0$ is trivial. We are going to induct on the dimension of X. Take a Lefschetz pencil $(Y_t)_{t \in \mathbb{P}^1}$ on X . Let $f : Y = Y_{t_0} \hookrightarrow X$ be a smooth hyperplane section. By the Lefschetz hyperplane theorem,

$$f^* : H^k(X) \to H^k(Y)$$

is an isomorphism when $k \le n-2$ and is an injection when $k = n-1$ (I am going to omit all the Tate twists due to my laziness). Taking Poincare dual,

$$f_* : H^k(Y) \to H^{k+2}(X),$$

is an isomorphism when $k \ge n$ and is surjective when $k = n-1$. Now by the projection formula, L^i can be decomposed as

$$\begin{tikzcd}
H^{n-i}(X) \arrow[r, "L^{i-1}"] \arrow[d, "f^*"] & H^{n+i-2}(X) \arrow[r, "L"] \arrow[d, "f^*"] & H^{n+i}(X) . \\
H^{n-i}(Y) \arrow[r, "L^{i-1}"] & H^{n+i-2}(Y) \arrow[ur, "f_*"] &
\end{tikzcd}$$

When $i \ge 2$, f^* and f_* are isomorphisms. By induction hypothesis, L^{i-1} is an isomorphism and it follows that L^i is also an isomorphism. It remains to treat the key case $i = 1$. In this case, f^* is an injection and f_* is a surjection. By Poincare duality, $x \mapsto Lx$ is an isomorphism is equivalent to that the pairing

$$H^{n-1}(X) \times H^{n-1}(X) \to \overline{\mathbb{Q}_\ell}, \quad (a, b) \mapsto a \cup Lb$$

is non-degenerate. Using the injection f^*, it is equivalent to that the pairing

$$H^{n-1}(Y) \times H^{n-1}(Y) \to \overline{\mathbb{Q}_\ell}, \quad (a, b) \mapsto a \cup b$$

is non-degenerate on the image of $H^{n-1}(X)$. By the property of the Lefschetz pencil (Theorem 3 c)), we have

$$H^{n-1}(X) \cong H^{n-1}(Y)^{\pi_1^g(\mathbb{P}^1 - S)}$$

Now we use Theorem 5: the π_1^g-action on $H^{n-1}(Y)$ is *semisimple* (this is the only place we use Weil II). So we obtain a π_1^g-equivariant decomposition

$$H^{n-1}(Y) \cong H^{n-1}(X) \oplus W,$$

for some W without trivial π_1^g-constituents. Hence the non-degenerate cup product pairing on $H^{n-1}(Y)$ decomposes accordingly and in particular restricts to a non-degenerate pairing on $H^{n-1}(X)$, as desired. □

We mention one immediate geometric consequence of the hard Lefschetz theorem to end this talk.

Corollary 2 The i-th Betti number $b_i = \dim H^i(X, \overline{\mathbb{Q}_\ell})$ is even for odd i.

Proof The hard Lefschetz together with the Poincare duality provides a non-degenerate pairing on $H^i(X)$, which is *alternating* when i is odd. □

References

[1] N. Katz, *L-functions and monodromy: four lectures on Weil II,* 2000, http://web.math.princeton.edu/~nmk/arizona34.pdf.

[2] Milne, James S., *Lectures on Etale Cohomology (v2.10),* Available at www.jmilne.org/math/.

[3] Deligne, Pierre, *La conjecture de Weil. II,* Inst. Hautes &EACUTE;tudes Sci. Publ. Math. (1980), no.52, 137--252.

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