1. Affine toric varieties

First some stuff about algebraic groups:

Definition 1.1. *Let G be a group. We say that G is an* ***algebraic group*** *if G is a quasi-projective variety and the two maps $m\colon G\times G \longrightarrow G$ and $i\colon G \longrightarrow G$, where m is multiplication and i is the inverse map, are both morphisms.*

It is easy to give examples of algebraic groups. Consider the group $G = \mathrm{GL}_n(K)$. In this case G is an open subset of $\mathbb{A}^{n^2}$, the complement of the zero locus of the determinant, which expands to a polynomial. Matrix multiplication is obviously a morphism, and the inverse map is a morphism by Cramer's rule. Note that there are then many obvious algebraic subgroups; the orthogonal groups, special linear group and so on. Clearly $\mathrm{PGL}_n(K)$ is also an algebraic group; indeed the quotient of an algebraic group by a closed normal subgroup is an algebraic group. All of these are affine algebraic groups.

Definition 1.2. *Let G be an algebraic group. If G is affine then we say that G is a* ***linear algebraic group****. If G is projective and connected then we say that G is an* ***abelian variety****.*

Note that any finite group is an algebraic group (both affine and projective). It turns out that any affine group is always a subgroup of a matrix group, so that the notation makes sense.

Definition 1.3. *The group $\mathbb{G}_m$ is $\mathrm{GL}_1(K)$. The group $\mathbb{G}_a$ is the subgroup of $\mathrm{GL}_2(K)$ of upper triangular matrices with ones on the diagonal.*

Note that as a group $\mathbb{G}_m$ is the set of units in K under multiplication and $\mathbb{G}_a$ is equal to K under addition, and that both groups are affine of dimension 1; in fact they are the only linear algebraic groups of dimension one, up to isomorphism.

Note that if we are given a linear algebraic group G, we get a Hopf algebra A. Indeed if A is the coordinate ring of G, then A is a K-algebra and there are maps

$$A \longrightarrow A \otimes A \qquad \text{and} \qquad A \longrightarrow A,$$

induced by the multiplication and inverse map for G (if you don't know what a Hopf algebra is, you can unwind the definitions and take this as the definition of a Hopf algebra).

It is not hard to see that the product of two algebraic groups is an algebraic group.

Definition 1.4. *The algebraic group $\mathbb{G}_m^k$ is called a* ***torus****.*

So a torus in algebraic geometry is just a product of copies of $\mathbb{G}_m$.
In fact one can define what it means to be a group scheme:

Definition 1.5. *Let $\pi\colon X \longrightarrow S$ be a morphism of schemes. We say that X is a* ***group scheme*** *over S, if there are three morphisms, $e\colon S \longrightarrow X$, $\mu\colon X \underset{S}{\times} X \longrightarrow X$ and $i\colon X \longrightarrow X$ over S which satisfy the obvious axioms.*

We can define a group scheme $\mathbb{G}_{m,\operatorname{Spec}\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$, by taking

$$\operatorname{Spec}\mathbb{Z}[x, x^{-1}].$$

Given any scheme S, this gives us a group scheme $\mathbb{G}_{m,S}$ over S, by taking the fibre square. In the case when $S = \operatorname{Spec} k$, k an algebraically closed field, then $\mathbb{G}_{m,\operatorname{Spec} k}$ is $t(\mathbb{G}_m)$, the scheme associated to the quasi-projective variety $\mathbb{G}_m$. We will be somewhat sloppy and not be too careful to distinguish the two cases.

Similarly we can take

$$\mathbb{G}_{a,\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x].$$

Definition 1.6. *Let G be an algebraic group and let X be a variety acted on by G, $\pi\colon G \times X \longrightarrow X$. We say that the action is* ***algebraic*** *if π is a morphism.*

For example the natural action of $\mathrm{PGL}_n(K)$ on $\mathbb{P}^n$ is algebraic, and all the natural actions of an algebraic group on itself are algebraic.

Definition 1.7. *We say that a quasi-projective variety X is a* ***toric variety*** *if X is irreducible and normal and there is a dense open subset U isomorphic to a torus, such that the natural action of U on itself extends to an action on the whole of X.*

For example, any torus is a toric variety. $\mathbb{A}_k^n$ is a toric variety. The natural torus is the complement of the coordinate hyperplanes and the natural action is as follows

$$((t_1, t_2, \dots, t_n), (a_1, a_2, \dots, a_n)) \longrightarrow (t_1a_1, t_2a_2, \dots, t_na_n).$$

More generally, $\mathbb{P}^n$ is a toric variety. The action is just the natural action induced from the action above. A product of toric varieties is toric.

One thing to keep track of are the closures of the orbits. For the torus there is one orbit. For affine space and projective space the closure of the orbits are the coordinate subspaces.

Definition 1.8. *Let M be a lattice and let $N = \operatorname{Hom}(M, \mathbb{Z})$ be the dual lattice.*

*A **strongly convex rational polyhedral cone** $\sigma \subset N_{\mathbb{R}} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is*

- *a **cone**, that is, if $v \in \sigma$ and $\lambda \in \mathbb{R}$, $\lambda \geq 0$ then $\lambda v \in \sigma$;*
- ***polyhedral**, that is, σ is the intersection of finitely many half spaces;*
- ***rational**, that is, the half spaces are defined by equations with rational coefficients;*
- ***strongly convex**, that is, σ contains no linear spaces other than the origin.*

One can reformulate some of the parts of the definition of a strongly rational polyhedral cone. For example, σ is a polyhedral cone if and only if σ is the intersection of finitely many half spaces which are defined by homogeneous linear polynomials. σ is a strongly convex polyhedral cone if and only if σ is a cone over finitely many vectors which lie in a common half space (in other words a strongly convex polyhedral cone is the same as a cone over a polytope). And so on.

We first give the recipe of how to go from a fan to an affine toric variety. Suppose we start with σ. Form the dual cone

$$\check{\sigma} = \{\, u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0, v \in \sigma \,\}.$$

Now take the integral points,

$$S_\sigma = \check{\sigma} \cap M.$$

Then form the (semi)group algebra,

$$A_\sigma = K[S_\sigma].$$

Finally form the affine variety,

$$U_\sigma = \operatorname{Spec} A_\sigma.$$

Given a semigroup S, to form the semigroup algebra $K[S]$, start with a K-vector space with basis χ^u, as u ranges over the elements of S. Given u and $v \in S$ define the product

$$\chi^u \cdot \chi^v = \chi^{u+v},$$

and extend this linearly to the whole of $K[S]$.

Note that $K[S]$ is an integral domain so that U_σ is irreducible.

Example 1.9. *For example, suppose we start with $M = \mathbb{Z}^2$, σ the cone spanned by $(1, 0)$ and $(0, 1)$, inside $N_{\mathbb{R}} = \mathbb{R}^2$. Then $\check{\sigma}$ is spanned by the same vectors in $M_{\mathbb{R}}$. Therefore $S_\sigma = \mathbb{N}^2$, the group algebra is*

$\mathbb{C}[x,y]$ and so we get $\mathbb{A}^2$. Similarly if we start with the cone spanned by $e_1, e_2, \dots, e_n$ inside $N_{\mathbb{R}} = \mathbb{R}^n$ then we get $\mathbb{A}^n$.

Now suppose we start with $\sigma = \{0\}$ in $\mathbb{R}$. Then $\check{\sigma}$ is the whole of $M_{\mathbb{R}}$, S_σ is the whole of $M = \mathbb{Z}$ and so $\mathbb{C}[M] = \mathbb{C}[x, x^{-1}]$. Taking Spec *we get the torus $\mathbb{G}_m$.*

More generally we always get a torus of dimension n if we take the origin in $\mathbb{R}^n$. Note that if $\tau \subset \sigma$ is a face then $\check{\sigma} \subset \check{\tau}$ is also a face so that $U_\tau \subset U_\sigma$ is an open subset. In fact

Lemma 1.10. *Let $\tau \subset \sigma \subset N_{\mathbb{R}}$ be a face of the cone σ.*

Then we may find $u \in S_\sigma$ such that

(1) $\tau = \sigma \cap u^\perp$,
(2)
$$S_\tau = S_\sigma + \mathbb{Z}^+(-u),$$
(3) A_τ is a localisation of A_σ, and
(4) U_τ is a principal open subset of U_σ.

Proof. The fact that every face of a cone is cut out by a hyperplane is a standard fact in convex geometry and this is (1). For (2) note that the RHS is contained in the LHS by definition of a cone. If $w \in S_\tau$ then $w + p \cdot u$ is in $\check{\sigma}$ for any p sufficiently large. If we take p to be also an integer this shows that w belongs to the RHS.

Let χ^u be the monomial corresponding to u. (2) implies that A_τ is the localisation of A_σ along χ^u. This is (3) and (4) is immediate from (3). □

Since the cone $\{0\}$ is a face of every cone, the affine scheme associated to a cone always contains a torus, which is then dense.

Definition 1.11. *Let $S \subset T$ be a subsemigroup of the semigroup T. We say that S is* ***saturated*** *in T if whenever $u \in T$ and $p \cdot u \in S$ for some positive integer p, then $u \in S$.*

Given a subsemigroup $S \subset M$ saturation is always with respect to M.

Lemma 1.12. *Suppose that $S \subset M$.*

Then $K[S]$ is integrally closed if and only if S is saturated.

Proof. Suppose that $K[S]$ is integrally closed.

Pick $u \in M$ such that $v = p \cdot u \in S$ for some positive integer p. Let $b = \chi^u$ and $a = \chi^v \in K[S]$. Then
$$b^p = \chi^{pu} = \chi^v = a,$$

so that b is a root of the monic polynomial $x^p - a \in K[S][x]$. As we are assuming that $K[S]$ is integrally closed this implies that $b \in K[S]$ which implies that $u \in S$. Thus S is saturated.

Now suppose that S is saturated. As $K[S] \subset K[M]$ and the latter is integrally closed, the integral closure L of $K[S]$ sits in the middle, $K[S] \subset L \subset K[M]$. The torus acts naturally on $K[M]$ and this action fixes $K[S]$, so that it also fixes L. L is therefore a direct sum of eigenspaces, which are all one dimensional (a set of commuting diagonalisable matrices are simultaneously diagonalisable) that is L has a basis of the form χ^u, as u ranges over some subset of M. It suffices to prove that $u \in S$.

Since $b = \chi^u$ is integral over $K[S]$, we may find $k_1, k_2, \dots, k_p \in K[S]$ such that

$$b^p + k_1 b^{p-1} + \cdots + k_p = 0.$$

We may assume that every term is in the same eigenspace as b^p. We may also assume that $k_p \neq 0$. As b^p and $k_p \neq 0$ belong to the same eigenspace, which is one dimensional, we get $b^p \in K[S]$. Thus $pu \in S$ and so $u \in S$ as S is saturated. Thus $b \in K[S]$ and $K[S]$ is integrally closed. $\square$

Note that S_σ is automatically saturated, as $\check{\sigma}$ is a rational polyhedral cone. In particular U_σ is normal.

Example 1.13. *Suppose that we start with the semigroup S generated by 2 and 3 inside $M = \mathbb{Z}$. Then*

$$K[S] = K[t^2, t^3] = K[x, y]/\langle y^2 - x^3 \rangle.$$

Note that this does come with an action of $\mathbb{G}_m$;

$$(t, x, y) \longrightarrow (t^2 x, t^3 y).$$

However the curve $V(y^2 - x^3) \subset \mathbb{A}^2$ is not normal.

In fact some authorities drop the requirement that a toric variety is normal.

An action of the torus corresponds to a map of algebras

$$A_\sigma \longrightarrow A_\sigma \underset{K}{\otimes} A_0,$$

which is naturally the restriction of

$$A_0 \longrightarrow A_0 \underset{K}{\otimes} A_0.$$

It is straightforward to check that the restricted map does land in $A_\sigma \underset{K}{\otimes} A_0$.

Lemma 1.14 (Gordan's Lemma)**.** *Let $\sigma \subset \mathbb{N}_{\mathbb{R}}$ be a strongly convex rational polyhedral cone.*

Then S_σ is a finitely generated semigroup.

Proof. Pick vectors $v_1, v_2, \dots, v_n \in S_\sigma$ which generate the cone $\check{\sigma}$. Consider the set

$$K = \{ v \in M \mid v = \sum t_i v_i, t_i \in [0,1] \}.$$

Then K is compact. As M is discrete $K \cap M$ is finite. I claim that the elements of $K \cap M$ generate the semigroup S_σ. Pick $u \in S_\sigma$. Since $u \in \check{\sigma}$ and $v_1, v_2, \dots, v_n$ generate the cone, we may write

$$u = \sum \lambda_i v_i,$$

where $\lambda_i \in \mathbb{Q}$. Let $n_i = \llcorner \lambda_i \lrcorner$. Then

$$u - \sum n_i v_i = \sum (\lambda_i - \llcorner \lambda_i \lrcorner) v_i \in K \cap M.$$

As $v_1, v_2, \dots, v_n \in K \cap M$ the result follows. $\square$

Gordan's lemma (1.14) implies that U_σ is of finite type over K. So U_σ is an affine toric variety.

Example 1.15. *Suppose we start with the cone spanned by e_2 and $2e_1 - e_2$. The dual cone is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the semigroup are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. The corresponding group algebra is $A_\sigma = K[x, xy, xy^2]$. Suppose we call $u = x$, $v = xy$ and $w = xy^2$. Then $v^2 = x^2y^2 = x(xy^2) = uw$. Therefore the corresponding affine toric variety is given as the zero locus of $v^2 - uw$ in $\mathbb{A}^3$.*

2. Toric varieties

Definition 2.1. *A* ***fan*** *in $N_{\mathbb{R}}$ is a set F of finitely many strongly convex rational polyhedra, such that*

- *every face of a cone in F is a cone in F, and*
- *the intersection of any two cones in F is a face of each cone.*

We will see that the set of toric varieties, up to isomorphism, are in bijection with fans, up to the action of $\mathrm{SL}(n, \mathbb{Z})$.

Given a fan F, we get a collection of affine toric varieties, one for every cone of F. It remains to check how to glue these together to get a toric variety. Suppose we are given two cones σ and τ belonging to F. The intersection is a cone ρ which is also a cone belonging to F. Since ρ is a face of both σ and τ there are natural inclusions

$$U_\rho \subset U_\sigma \qquad \text{and} \qquad U_\rho \subset U_\tau.$$

We glue U_σ to U_τ using the natural identification of the common open subset U_ρ. Compatibility of gluing follows automatically from the fact that the identification is natural and from the combinatorics of the fan (see (2.12) of Hartshorne). It is clear that the resulting scheme is of finite type over the groundfield. Separatedness follows from:

Lemma 2.2. *Let σ and τ be two cones whose intersection is the cone ρ.*

If ρ is a face of each then the diagonal map

$$U_\rho \longrightarrow U_\sigma \times U_\tau,$$

is a closed embedding.

Proof. This is equivalent to the statement that the natural map

$$A_\sigma \otimes A_\tau \longrightarrow A_\rho,$$

is surjective. For this, one just needs to check that

$$S_\rho = S_\sigma + S_\tau.$$

One inclusion is easy; the RHS is contained in the LHS. For the other inclusion, one needs a standard fact from convex geometry, which is called the separation lemma: there is a vector $u \in S_\sigma \cap S_{-\tau}$ such that simultaneously

$$\rho = \sigma \cap u^\perp \qquad \text{and} \qquad \rho = \tau \cap u^\perp.$$

By the first equality and the fact that $u \in S_\sigma$, we have $S_\rho = S_\sigma + \mathbb{Z}(-u)$. As $u \in S_{-\tau}$ we have $-u \in S_\tau$ and so the LHS is contained in the RHS. $\square$

So we have shown that given a fan F we can construct a normal variety $X = X(F)$. It is not hard to see that the natural action of the torus corresponding to the zero cone extends to an action on the whole of X. Therefore $X(F)$ is indeed a toric variety.

Let us look at some examples.

Example 2.3. *Suppose that we start with $M = \mathbb{Z}$ and we let F be the fan given by the three cones $\{0\}$, the cone spanned by e_1 and the cone spanned by $-e_1$ inside $N_{\mathbb{R}} = \mathbb{R}$. The two big cones correspond to $\mathbb{A}^1$. We identify the two $\mathbb{A}^1$'s along the common open subset isomorphic to K^*. Now the first $\mathbb{A}^1 = \operatorname{Spec} K[x]$ and the second is $\mathbb{A}^1 = \operatorname{Spec} K[x^{-1}]$. So the corresponding toric variety is $\mathbb{P}^1$ (if we have homogeneous coordinates $[X : Y]$ on $\mathbb{P}^1$ coordinates on U_0 are $x = X/Y$ and $y = Y/X = 1/x$).*

Now suppose that we start with three cones in $N_{\mathbb{R}} = \mathbb{R}^2$, σ_1, σ_2 and σ_3. We let σ_1 be the cone spanned by e_1 and e_2, σ_2 be the cone spanned by e_2 and $-e_1 - e_2$ and σ be the cone spanned by $-e_1 - e_2$ and e_1. Let F be the fan given as the faces of these three cones. Note that the three affine varieties corresponding to these three cones are all copies of $\mathbb{A}^2$. Indeed, any two of the vectors, e_1, e_2 and $-e_1 - e_2$ are a basis not only of the underlying vector space but they also generate the standard lattice. We check how to glue two such copies of $\mathbb{A}^2$.

The dual cone of σ_1 is the cone spanned by f_1 and f_2 in $M_{\mathbb{R}} = \mathbb{R}^2$. The dual cone of σ_2 is the cone spanned by $-f_1$ and $-f_1 + f_2$. So we have $U_1 = \operatorname{Spec} K[x, y]$ and $U_2 = \operatorname{Spec} K[x^{-1}, x^{-1}y]$. On the other hand, if we start with $\mathbb{P}^2$ with homogeneous coordinates $[X : Y : Z]$ and the two basic open subsets $U_0 = \operatorname{Spec} K[Y/X, Z/X]$ and $U_1 = \operatorname{Spec} K[X/Y, Z/Y]$, then we get the same picture, if we set $x = Y/X$, $y = Z/X$ (since then $X/Y = x^{-1}$ and $Z/Y = Z/X \cdot X/Y = yx^{-1}$). With a little more work one can check that we have $\mathbb{P}^2$.

More generally, suppose we start with $n+1$ vectors $v_1, v_2, \dots, v_{n+1}$ in $N_{\mathbb{R}} = \mathbb{R}^n$ which sum to zero such that the first n vectors $v_1, v_2, \dots, v_n$ span the standard lattice. Let F be the fan obtained by taking all the cones spanned by all subsets of at most n vectors. One can check that the resulting toric variety is $\mathbb{P}^n$.

Now suppose that we take the four vectors e_1, e_2, $-e_1$ and $-e_2$ in $N_{\mathbb{R}} = \mathbb{R}^2$ and let F be the fan consisting of all cones spanned by at most two vectors (but not pairs of inverse vectors, that is, neither e_1 and $-e_1$ nor e_2 and $-e_2$). Then we get four copies of $\mathbb{A}^2$. It is easy to check that the resulting toric variety is $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed the top two fans glue together to get $\mathbb{P}^1 \times \mathbb{A}^1$ and so on.

We have already seen that cones correspond to open subsets. In fact cones also correspond (in some sort of dual sense) to closed subsets, the closure of the orbits. First observe that given a fan F, we can associate a closed point x_σ to any cone σ. To see this, observe that one can spot the closed points of U_σ using semigroups:

Lemma 2.4. *Let $S \subset M$ be a semigroup. Then there is a natural bijection,*

$$\operatorname{Hom}(K[S], K) \simeq \operatorname{Hom}(S, K).$$

Here the RHS is the set of semigroup homomorphisms, where $K = \{0\} \cup K^*$ is the multiplicative subsemigroup of K (and not the additive).

Proof. Suppose we are given a ring homomorphism

$$f \colon K[S] \longrightarrow K.$$

Define

$$g \colon S \longrightarrow K,$$

by sending u to $f(\chi^u)$. Conversely, given g, define $f(\chi^u) = g(u)$ and extend linearly. $\square$

Consider the semigroup homomorphism:

$$S_\sigma \longrightarrow \{0, 1\},$$

where $\{0, 1\} \subset \{0\} \cup K^*$ inherits the obvious semigroup structure. We send $u \in S_\sigma$ to 1 if $u \in \sigma^\perp$ and send it 0 otherwise. Note that as $\sigma^\perp$ is a face of $\check{\sigma}$ we do indeed get a homomorphism of semigroups. By (2.4) we get a surjective ring homomorphism

$$K[S_\sigma] \longrightarrow K.$$

The kernel is a maximal ideal of $K[S_\sigma]$, that is a closed point x_σ of U_σ, with residue field K.

To get the orbits, take the orbits of these points. It follows that the orbits are in correspondence with the cones in F. Let $O_\sigma \subset U_\sigma$ be the orbit of x_σ and let $V(\sigma)$ be the closure of O_σ.

Example 2.5. *For the fan corresponding to $\mathbb{P}^1$, the point corresponding to $\{0\}$ is the identity, and the points corresponding to e_1 and $-e_1$ are 0 and ∞. For the fan corresponding to $\mathbb{P}^2$ the three maximal cones give the three coordinate points, the three one dimensional cones give the three coordinate lines (in fact the lines spanned by the points corresponding to the two maximal cones which contain them). As before the zero cone corresponds to the identity point. The orbit is the whole torus and the closure is the whole of $\mathbb{P}^2$.*

3. Divisors on toric varieties

We start with computing the class group of a toric variety. Recall that the class group is the group of Weil divisors modulo linear equivalence. We denote the class group either by $\mathrm{Cl}(X)$ or $A_{n-1}(X)$.

When talking about Weil divisors, we will always assume we have a scheme which is:

$(*)$ noetherian, integral, separated, and regular in codimension one.

This is never a problem for toric varities. If X is a toric variety, by assumption there is a dense open subset $U \simeq \mathbb{G}_m^n$. The complement Z is a closed invariant subset.

Lemma 3.1. *Suppose that X satisfies $(*)$, let Z be a closed subset and let $U = X \setminus Z$.*

Then there is an exact sequence

$$\mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0,$$

where s is the number of components of Z which are prime divisors.

Proof. If Y is a prime divisor on X then $Y' = Y \cap U$ is either a prime divisor on U or empty. This defines a group homomorphism

$$\rho\colon \mathrm{Div}(X) \longrightarrow \mathrm{Div}(U).$$

If $Y' \subset U$ is a prime divisor, then let Y be the closure of Y' in X. Then Y is a prime divisor and $Y' = Y \cap U$. Thus ρ is surjective. If f is a rational function on X and $Y = (f)$, then the image of Y in $\mathrm{Div}(U)$ is equal to $(f|_U)$, so ρ descends to a map of class groups.

If $Z = Z' \cup \bigcup_{i=1}^s Z_i$ where Z' has codimension at least two and $Z_1, Z_2, \dots, Z_s$ is a prime divisor, then the map which sends $(m_1, m_2, \dots, m_s)$ to $\sum m_i Z_i$ generates the kernel. $\square$

Example 3.2. *Let $X = \mathbb{P}^2_K$ and C be an irreducible curve of degree d. Then $\mathrm{Cl}(\mathbb{P}^2 - C)$ is equal to $\mathbb{Z}_d$. Similarly $\mathrm{Cl}(\mathbb{A}^n_K) = 0$.*

Back to assuming that X is a toric variety. It follows by (3.1) that there is an exact sequence

$$\mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0.$$

Applying this to $X = \mathbb{A}^n_K$ it follows that $\mathrm{Cl}(U) = 0$. So we get an exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

We want to identify the kernel K. This is equal to the set of principal divisors which are supported on the invariant divisors. If f is a rational function such that (f) is supported on the invariant divisors then f has no zeroes or poles on the torus; it follows that $f = \lambda\chi^u$, where $\lambda \in K^*$ and $u \in M$.

Hence there is an exact sequence

$$M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Recall that the invariant divisors are in bijection with the one dimensional cones τ of the fan F. Now, given a one dimensional cone τ, there is a unique vector $v \in \tau \cap M$ such that if w also belongs to $\tau \cap M$ and we write $w = p \cdot v$ then $p \geq 1$. We call v a **primitive generator** of τ.

Lemma 3.3. *Let $u \in M$. Suppose that X is the affine toric variety associated to a cone $\sigma \subset N_{\mathbb{R}}$. Let v be a primitive generator of a one dimensional ray τ of σ and let D be the corresponding invariant divisor.*

Then $\mathrm{ord}_D(\chi^u) = \langle u, v\rangle$. In particular

$$(\chi^u) = \sum_i \langle u, v_i\rangle D_i,$$

where the sum ranges over the invariant divisors.

Proof. We can calculate the order on the open set $U_\tau = \mathbb{A}^1_k \times \mathbb{G}_m^{n-1}$, where D corresponds to $\{0\} \times \mathbb{G}_m^{n-1}$. In this case we can ignore the factor $\mathbb{G}_m^{n-1}$ and we are reduced to the one dimensional case. So $N = \mathbb{Z}$, $v = 1$ and $u \in M = \mathbb{Z}$. In this case χ^u is the monomial x^u and the order of vanishing at the origin is exactly u. $\square$

It follows that if $X = X(F)$ is the toric variety associated to a fan F which spans $N_{\mathbb{R}}$ then we have a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Example 3.4. *Let σ be the cone spanned by $2e_1 - e_2$ and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. There are two invariant divisors D_1 and D_2. The principal divisor associated to $u = f_1 = (1, 0)$ is $2D_1$ and the principal divisor associated to $u = f_2 = (0, 1)$ is $D_2 - D_1$. So the class group is $\mathbb{Z}_2$.*

Note that the dual cone $\check{\sigma}$ is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the monoid $S_\sigma = \check{\sigma} \cap M$ are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. So the group algebra

$$A_\sigma = k[x, xy, xy^2] = \frac{k[u, v, w]}{\langle v^2 - uw\rangle},$$

and $X = U_\sigma$ is the quadric cone.

Now suppose we take the standard fan associated to $\mathbb{P}^2$. The invariant divisors are the three coordinate lines, D_1, D_2 and D_3. If $f_1 = (1, 0)$

and $f_2 = (0,1)$ *then*

$$(\chi^{f_1}) = D_1 - D_3 \qquad \textit{and} \qquad (\chi^{f_2}) = D_2 - D_3.$$

So the class group is $\mathbb{Z}$*.*

We now turn to calculating the Picard group of a toric variety X.

Definition 3.5. *Let* X *be a scheme.*

The set of invertible sheaves forms an abelian group $\mathrm{Pic}\,(X)$*, where multiplication corresponds to tensor product and the inverse to the dual.*

Recall that if X is a normal variety, every Cartier divisor D on X determines a Weil divisor

$$\mathrm{ord}_V(D)V,$$

where sum runs over all prime divisors of X. Thus the set of Cartier divisors embeds in the set of Weil divisors. We say that X is **factorial** if every Weil divisor is Cartier.

Let's consider which Weil divisors on a toric variety are Cartier. We classify all Cartier divisors whose underlying Weil divisor is invariant; we dub these Cartier divisors T-Cartier. We start with the case of the affine toric variety associated to a cone $\sigma \subset N_{\mathbb{R}}$. It suffices to classify all invertible subsheaves $\mathcal{O}_X(D) \subset \mathcal{K}$, where $\mathcal{K}$ is the sheaf of total quotient rings of $\mathcal{O}_X$. Taking global sections, since we are on an affine variety, it suffices to classify all fractional ideals,

$$I = H^0(X, \mathcal{O}_X(D)) \subset H^0(X, \mathcal{K}).$$

Invariance of D implies that I is graded by M, that is, I is a direct sum of eigenspaces. As D is Cartier, I is principal at the distinguished point x_σ of U_σ, so that $I/\mathfrak{m}I$ is one dimensional, where

$$\mathfrak{m} = \sum k \cdot \chi^u.$$

Pick $U \in M$ such that the image of χ^u generates this one dimensional vector space. Nakayama's Lemma implies that $I = A_\sigma \chi^u$, that is I is the ideal generated by χ^u, so that $D = (\chi^u)$ is principal. As every Weil divisor is linearly equivalent to a Weil divisor supported on the invariant divisors, every Cartier divisor is linearly equivalent to a T-Cartier divisor. Hence, the only Cartier divisors are the principal divisors and X is factorial if and only if the Class group is trivial.

Example 3.6. *The quadric cone* Q*, given by* $xy - z^2 = 0$ *in* $\mathbb{A}^3_k$ *is not factorial. We have already seen* (3.4) *that the class group is* $\mathbb{Z}_2$*.*

If $\sigma \subset N_{\mathbb{R}}$ is not maximal dimensional then every Cartier divisor on U_σ whose associated Weil divisor is invariant is of the form (χ^u) but

$$(\chi^u) = (\chi^{u'}) \qquad \text{if and only if} \qquad u - u' \in \sigma^\perp \cap M = M(\sigma).$$

So the T-Cartier divisors are in correspondence with $M/M(\sigma)$.

Now suppose that $X = X(F)$ is a general toric variety. Then a T-Cartier divisor is given by specifying an element $u(\sigma) \in M/M(\sigma)$, for every cone σ in F. This defines a divisor $(\chi^{-u(\sigma)})$; equivalently a fractional ideal

$$I = H^0(X, \mathcal{O}_X(D)) = A_\sigma \cdot \chi^{u(\sigma)}.$$

These maps must agree on overlaps; if τ is a face of σ then $u(\sigma) \in M/M(\sigma)$ must map to $u(\tau) \in M/M(\tau)$.

Note that it is somewhat hard to keep track of the T-Cartier divisors. We look for a way to repackage the same combinatorial data into a more convenient form. As usual, this means we should look at the dual picture.

The data

$$\{\, u(\sigma) \in M/M(\sigma) \,|\, \sigma \in F \,\},$$

for a T-Cartier divisor D determines a continuous piecewise linear function ϕ_D on the support $|F|$ of F. If $v \in \sigma$ then let

$$\phi_D(v) = \langle u(\sigma), v \rangle.$$

Compatibility of the data implies that ϕ_D is well-defined and continuous. Conversely, given any continuous function ϕ, which is linear and integral (that is, given by an element of M) on each cone, we can associate a unique T-Cartier divisor D. If $D = \sum a_i D_i$ the function is given by $\phi_D(v_i) = -a_i$, where v_i is the primitive generator of the ray corresponding to D_i.

Note that

$$\phi_D + \phi_E = \phi_{D+E} \qquad \text{and} \qquad \phi_{mD} = m\phi_D.$$

Note also that $\phi_{(\chi^u)}$ is the linear function given by $-u$. So D and E are linearly equivalent if and only if ϕ_D and ϕ_E differ by a linear function.

If X is any variety which satisfies $(*)$ then the natural map

$$\operatorname{Pic}(X) \longrightarrow \operatorname{Cl}(X),$$

is an embedding. It is an interesting to compare $\operatorname{Pic}(X)$ and $\operatorname{Cl}(X)$ on a toric variety. Denote by $\operatorname{Div}_T(X)$ the group of T-Cartier divisors.

Proposition 3.7. *Let $X = X(F)$ be the toric variety associated to a fan F which spans $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \mathrm{Div}_T(X) \arrow[r] \arrow[d, hook] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbb{Z}^s \arrow[r] & \mathrm{Cl}(X) \arrow[r] & 0.
\end{tikzcd}$$

In particular

$$\rho(X) = \mathrm{rank}(\mathrm{Pic}(X)) \leq \mathrm{rank}(\mathrm{Cl}(X)) = s - n.$$

Further $\mathrm{Pic}(X)$ *is a free abelian group.*

Proof. We have already seen that the bottom row is exact. If L is an invertible sheaf then $L|_U$ is trivial. Suppose that $L = \mathcal{O}_X(E)$. Pick a rational function such that $(f)|_U = E|_U$. Let $D = E - (f)$. Then D is T-Cartier, since it is supported away from the torus and exactness of the top row is easy.

Finally, $\mathrm{Pic}(X)$ is represented by equivalence classes of continuous, piecewise integral linear functions modulo linear functions. Clearly if $m\phi$ is linear then so is ϕ, so that $\mathrm{Pic}(X)$ is torsion free. □

4. Ample line bundles on toric varieties

It is interesting to find the ample line bundles on a toric variety. Suppose that $X = X(F)$ is the toric variety associated to the fan $F \subset N_{\mathbb{R}}$. Recall that we can associate to a T-Cartier divisor $D = \sum a_i D_i$, a continuous piecewise linear function

$$\phi_D \colon |F| \longrightarrow \mathbb{R},$$

where $|F| \subset N_{\mathbb{R}}$ is the support of the fan, which is specified by the rule that $\phi_D(v_i) = -a_i$.

We can also associate to D a rational polyhedron

$$\begin{aligned} P_D &= \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \quad \forall i \,\} \\ &= \{\, u \in M_{\mathbb{R}} \,|\, u \geq \phi_D \,\}. \end{aligned}$$

Lemma 4.1. *If X is a toric variety and D is T-Cartier then*

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{u \in P_D \cap M} k \cdot \chi^u.$$

Proof. Suppose that $\sigma \in F$ is a cone. We can identify $H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D))$ as the set of rational functions f on X which have a pole no worse than D:

$$(f) + D \geq 0.$$

This gives us a vector space of rational functions which, as usual, decomposes into eigenspaces. Now f has no poles along the torus, so we may assume that f belongs to the Laurent polynomial ring

$$K[x_1, x_1^{-1}, x_2, x_2^{-1}, x_3, x_3^{-1}, \dots, x_n, x_n^{-1}].$$

Therefore the eigenspaces are given by χ^u, $u \in M$ and we want

$$(\chi^u) + D \geq 0.$$

Writing this out in components, we have

$$\langle u, v_i \rangle + a_i \geq 0 \qquad \text{for all} \qquad v_i \in \sigma.$$

Thus we have

$$H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D)) = \bigoplus_{u \in P_D(\sigma) \cap M} k \cdot \chi^u,$$

where

$$P_D(\sigma) = \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \quad \forall v_i \in \sigma \,\}.$$

These identifications are compatible on overlaps. Since

$$H^0(X, \mathcal{O}_X(D)) = \bigcap_{\sigma \in F} H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D))$$

and

$$P_D = \bigcap_{\sigma \in F} P_D(\sigma),$$

the result is clear. $\square$

It is interesting to compute some examples. Let's start with $\mathbb{P}^1$. A T-Cartier divisor is a sum $ap + bq$ (p and q are the fixed points, zero and infinity). We want those rational functions which have a pole no worse than $-a$ at p and a pole no worse than $-b$ at q. Consider the general monomial $f = x^i$. If $i \geq 0$ then f is regular at p and has a pole of order i at q. So $i \leq a$. If $i \leq 0$ then f has a pole of order $-i$ at p and f is regular at q. So $-i \leq b$, that is $i \geq -b$.

The polytope corresponding to $ap+bq$ is $[-b, a]$ and a general rational function with poles no worse than $ap + bq$ has the form

$$c_{-b}x^{-b} + c_{-b+1}x^{-b+1} + \cdots + c_{-1}x^{-1} + c_0 + c_1 x + \cdots + c_a x^a.$$

The corresponding piecewise linear function is

$$\phi(x) = \begin{cases} -ax & x > 0 \\ bx & x < 0. \end{cases}$$

Now consider $\mathbb{P}^2$ and dD_3. We are looking at rational functions $x^i y^j$ which are regular on the standard open affine $U_0 = \mathbb{A}^2_K$. So $i \geq 0$ and $j \geq 0$. Since we have a pole no worse than d along D_3, we must have $i + j \leq d$. Therefore P_D is the convex hull of $(0,0)$, $(d,0)$ and $(0,d)$. The number of integral points is

$$\frac{(d+1)^2}{2} + \frac{d+1}{2} = \frac{(d+2)(d+1)}{2},$$

which is the usual formula for the number of homogeneous polynomials of degree d in three variables.

Let D be a Cartier divisor on a toric variety $X = X(F)$ given by a fan F. It is interesting to consider when the complete linear system $|D|$ is base point free. Since any Cartier divisor is linearly equivalent to a T-Cartier divisor, we might as well suppose that $D = \sum a_i D_i$ is T-Cartier. Note that the base locus of the complete linear system of any Cartier divisor is invariant under the action of the torus. Since there are only finitely many orbits, it suffices to show that for each cone $\sigma \in F$ the point $x_\sigma \in U_\sigma$ is not in the base locus. It is also clear that if x_σ is not in the base locus of $|D|$ then in fact one can find a T-Cartier divisor $D' \in |D|$ which does not contain x_σ.

Now the invariant Weil divisor D_i contains x_σ precisely when $v_i \in \sigma$. So we want an invariant Weil divisor $D' = \sum b_i D_i$ such that $b_i \geq 0$

with strict equality if $v_i \notin \sigma$. As $D' = D + (\chi^u)$, if x_σ is not in the base locus of $|D|$ then we can find $u \in M$ such that

$$\langle u, v_i \rangle \geq -a_i,$$

with strict equality if $v_i \in \sigma$. The interesting thing is that we can reinterpret this condition using ϕ_D.

Definition 4.2. *The function $\phi\colon V \longrightarrow \mathbb{R}$ is (upper)* ***convex*** *if*

$$\phi(\lambda v + (1-\lambda)w) \geq \lambda\phi(v) + (1-\lambda)w \qquad \forall v, w \in V.$$

When we have a fan F and ϕ is linear on each cone σ, then ϕ is called ***strictly convex*** *if the linear functions $u(\sigma)$ and $u(\sigma')$ are different, for different maximal cones σ and σ'.*

Theorem 4.3. *Let $X = X(F)$ be the toric variety associated to a T-Cartier divisor D.*

Then

(1) $|D|$ is base point free if and only if ψ_D is convex.

(2) D is very ample if and only if ψ_D is strictly convex and the semigroup S_σ is generated by

$$\{\, u - u(\sigma) \,|\, u \in P_D \cap M \,\}.$$

Proof. (1) follows from the remarks above. (2) is proved in Fulton's book. □

For example if $X = \mathbb{P}^1$ and

$$\phi(x) = \begin{cases} -ax & x > 0 \\ bx & x < 0. \end{cases}$$

so that $D = ap + bq$ then ϕ is convex if and only if $a + b \geq 0$ in which case D is base point free. D is very ample if and only if $a + b > 0$. When ϕ is continuous and linear on each cone σ, we may restate the convex condition as saying that the graph of ϕ lies under the graph of $u(\sigma)$. It is strictly convex if it lies strictly under the graph of $u(\sigma)$ outside of σ, for all n-dimensional cones σ.

Using this, it is altogether too easy to give an example of a smooth proper toric variety which is not projective. Let $F \subset N_{\mathbb{R}} = \mathbb{R}^3$ given by the edges $v_1 = -e_1$, $v_2 = -e_2$, $v_3 = -e_3$, $v_4 = e_1 + e_2 + e_3$, $v_5 = v_3 + v_4$, $v_6 = v_1 + v_4$ and $v_7 = v_2 + v_4$. Now connect v_1 to v_5, v_3 to v_7 and v_2 to v_6 and v_5 to v_6, v_6 to v_7 and v_7 to v_5.

It is not hard to check that X is smooth and proper (proper translates to the statement that the support $|F|$ of the fan is the whole of $N_{\mathbb{R}}$).

Suppose that ψ is strictly convex. Let

$$w = \frac{v_1 + v_3 + v_4}{3},$$

the barycentric centre of the triangle with vertices v_1, v_3 and v_4. Then

$$w = \frac{v_1 + v_5}{3} = \frac{v_3 + v_6}{3}.$$

Since v_1 and v_5 belong to the same maximal cone, ψ is linear on the line connecting them. In particular

$$\psi(w) = \psi(\frac{v_1 + v_5}{3}) = \frac{1}{3}\psi(v_1) + \frac{1}{3}\psi(v_5).$$

Since v_1, v_5 and v_3 belong to the same cone and v_6 does not, by strict convexity,

$$\psi(w) = \psi(\frac{v_3 + v_6}{3}) > \frac{1}{3}\psi(v_3) + \frac{1}{3}\psi(v_6).$$

Putting all of this together, we get

$$\psi(v_1) + \psi(v_5) > \psi(v_2) + \psi(v_6).$$

By symmetry

$$\begin{aligned}
\psi(v_1) + \psi(v_5) &> \psi(v_3) + \psi(v_6) \\
\psi(v_2) + \psi(v_6) &> \psi(v_1) + \psi(v_7) \\
\psi(v_3) + \psi(v_7) &> \psi(v_2) + \psi(v_5).
\end{aligned}$$

But adding up these three inequalities gives a contradiction.

5. Relative proj and projective bundles

We want to define a relative version of Proj, in pretty much the same way we defined a relative version of Spec. We start with a scheme X and a quasi-coherent sheaf $\mathcal{S}$ sheaf of graded $\mathcal{O}_X$-algebras,

$$\mathcal{S} = \bigoplus_{d \in \mathbb{N}} \mathcal{S}_d,$$

where $\mathcal{S}_0 = \mathcal{O}_X$. It is convenient to make some simplifying assumptions:

(†) X is Noetherian, $\mathcal{S}_1$ is coherent, $\mathcal{S}$ is locally generated by $\mathcal{S}_1$.

To construct relative Proj, we cover X by open affines $U = \operatorname{Spec} A$. With a view towards what comes next, we denote global sections of $\mathcal{S}$ over U by $H^0(U, \mathcal{S})$. Then $\mathcal{S}(U) = H^0(U, \mathcal{S})$ is a graded A-algebra, and we get $\pi_U \colon \operatorname{Proj} \mathcal{S}(U) \longrightarrow U$ a projective morphism. If $f \in A$ then we get a commutative diagram

$$\begin{tikzcd} \operatorname{Proj} \mathcal{S}(U_f) \arrow[r] \arrow[d, "\pi_{U_f}"'] & \operatorname{Proj} \mathcal{S}(U) \arrow[d, "\pi_U"'] \\ U_f \arrow[r] & U. \end{tikzcd}$$

It is not hard to glue π_U together to get $\pi \colon \mathbf{Proj}\, \mathcal{S} \longrightarrow X$. We can also glue the invertible sheaves together to get an invertible sheaf $\mathcal{O}(1)$.

The relative consruction has some similarities to the old construction.

Example 5.1. *If X is Noetherian and*

$$\mathcal{S} = \mathcal{O}_X[T_0, T_1, \dots, T_n],$$

then satisfies (†) *and* $\mathbf{Proj}\, \mathcal{S} = \mathbb{P}^n_X$.

Given a sheaf $\mathcal{S}$ satisfying (†), and an invertible sheaf $\mathcal{L}$, it is easy to construct a quasi-coherent sheaf $\mathcal{S}' = \mathcal{S} \star \mathcal{L}$, which satisfies (†). The graded pieces of $\mathcal{S}'$ are $\mathcal{S}_d \otimes \mathcal{L}^d$ and the multiplication maps are the obvious ones. There is a natural isomorphism

$$\phi \colon P' = \mathbf{Proj}\, \mathcal{S}' \longrightarrow P = \mathbf{Proj}\, \mathcal{S},$$

which makes the diagram commute

$$\begin{tikzcd} P' \arrow[rr, "\phi"] \arrow[dr, "\pi'"] & & P \arrow[dl, "\pi"'] \\ & X, & \end{tikzcd}$$

and

$$\phi^* \mathcal{O}_P(1) \simeq \mathcal{O}_{P'}(1) \otimes \pi'^* \mathcal{L}.$$

Note that π is always proper; in fact π is projective over any open affine and properness is local on the base. Even better π is projective if X has an ample line bundle; see (II.7.10).

There are two very interesting family of examples of the construction of relative Proj. Suppose that we start with a locally free sheaf $\mathcal{E}$ of rank $r \geq 2$. Note that

$$\mathcal{S} = \bigoplus \operatorname{Sym}^d \mathcal{E},$$

satisfies (†). $\mathbb{P}(\mathcal{E}) = \mathbf{Proj}\, \mathcal{S}$ is the **projective bundle** over X associated to $\mathcal{E}$. The fibres of $\pi \colon \mathbb{P}(\mathcal{E}) \longrightarrow X$ are copies of $\mathbb{P}^n$, where $n = r - 1$. We have

$$\bigoplus_{l=0}^{\infty} \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) = \mathcal{S},$$

so that in particular

$$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}.$$

Also there is a natural surjection

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Indeed, it suffices to check both statements locally, so that we may assume that X is affine. The first statement is standard and proved in 18.725, and the second statement reduces to the statement that the sections $x_0, x_1, \dots, x_n$ generate $\mathcal{O}_P(1)$. The most interesting result is:

Proposition 5.2. *Let $g \colon Y \longrightarrow X$ be a morphism.*

Then a morphism $f \colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X is the same as giving an invertible sheaf $\mathcal{L}$ on Y and a surjection $g^* \mathcal{E} \longrightarrow \mathcal{L}$.

Proof. One direction is clear; if $f \colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ is a morphism over X, then the surjective morphism of sheaves

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1),$$

pullsback to a surjective morphism

$$g^* \mathcal{E} = f^*(\pi^* \mathcal{E}) \longrightarrow \mathcal{L} = f^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Conversely suppose we are given an invertible sheaf $\mathcal{L}$ and a surjective morphism of sheaves

$$g^* \mathcal{E} \longrightarrow \mathcal{L}.$$

I claim that there is then a unique morphism $f \colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X, which induces the given surjection. By uniqueness, it suffices to prove this result locally. So we may assume that $X = \operatorname{Spec} A$ is affine and

$$\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}_X,$$

is free. In this case surjectivity reduces to the statement that the images $s_0, s_1, \ldots, s_n$ of the standard sections generate $\mathcal{L}$, and the result reduces to one we have already proved. $\square$

6. Blowing up

Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the map

$$[X : Y : Z] \longrightarrow [YZ : XZ : XY].$$

This map is clearly a rational map. It is called a **Cremona transformation**. Note that it is a priori not defined at those points where two coordinates vanish. To get a better understanding of this map, it is convenient to rewrite it as

$$[X : Y : Z] \longrightarrow [1/X : 1/Y : 1/Z].$$

Written as such it is clear that this map is an involution, so that it is in particular a birational map.

It is interesting to check whether or not this map really is well defined at the points $[0 : 0 : 1]$, $[0 : 1 : 0]$ and $[1 : 0 : 0]$. To do this, we need to look at the closure of the graph.

To get a better picture of what is going on, consider the following map,

$$\mathbb{A}^2 \dashrightarrow \mathbb{A}^1,$$

which assigns to a point $p \in \mathbb{A}^2$ the slope of the line connecting the point p to the origin,

$$(x, y) \longrightarrow x/y.$$

Now this map is not defined along the locus where $y = 0$. Replacing $\mathbb{A}^1$ with $\mathbb{P}^1$ we get a map

$$(x, y) \longrightarrow [x : y].$$

Now the only point where this map is not defined is the origin. We consider the closure of the graph,

$$\Gamma \subset \mathbb{A}^2 \times \mathbb{P}^1.$$

Consider how Γ sits over $\mathbb{A}^2$. Outside the origin the first projection is an isomorphism. Over the origin the graph is contained in a copy of the image, that is, $\mathbb{P}^1$. Consider any line $y = tx$ through the origin. Then this line, minus the origin, is sent to the point with slope t. It follows that the closure of this line is sent to the point with slope t. Varying t, it follows that any point of the fibre over $\mathbb{P}^1$ is a point of the graph.

Thus the morphism $\pi\colon \Gamma \longrightarrow \mathbb{A}^2$ is an isomorphism outside the origin and contracts a whole copy of $\mathbb{P}^1$ to a point. For this reason, we call π a blow up. The inverse image of the origin is called the **exceptional divisor**.

Definition 6.1. *Let $\pi\colon X \longrightarrow Y$ be a birational morphism. The locus where π is not an isomorphism is called the* ***exceptional locus****. If $V \subset Y$, the inverse image of V is called the* ***total transform****. Let Z be the image of the exceptional locus. Suppose that V is not contained in Z. The* ***strict transform of*** *V is the closure of the inverse image of $V - Z$.*

It is interesting to compute the strict transform of some planar curves. We have already seen that lines through the origin lift to curves that sweep out the exceptional divisor. In fact the blow up separates the lines through the origin. These are then the fibres of the second morphism.

Let us now take a nodal cubic,

$$y^2 = x^2 + x^3.$$

We want to figure out its strict transform, so that we need the inverse image in the blow up. Outside the origin, there are two equations to be satisfied,

$$y^2 = x^2 + x^3 \qquad \text{and} \qquad xT = yS.$$

Passing to the coordinate patch $y = xt$, where $t = T/S$, and substituting for y in the first equation we get

$$x^2t^2 - x^2 - x^3 = x^2(t^2 - x - 1).$$

Now if $x = 0$, then $y = 0$, so that in fact locally $x = 0$ is the equation of the exceptional divisor. So the first factor just corresponds to the exceptional divisor. The second factor will tell us what the closure of our curve looks like, that is, the strict transform. Now over the origin, $x = 0$, so that $t^2 = 1$ and $t = \pm 1$. Thus our curve lifts to a curve which intersects the exceptional divisor in two points. (If we compute in the coordinate patch $x = sy$, we will see that the curve does not meet the point at infinity). These two points correspond to the fact that the nodal cubic has two tangent lines at the origin, one of slope 1 and one of slope -1.

Now consider what happens for the cuspidal cubic, $y^2 = x^3$. In this case we get

$$(xt)^2 - x^3 = x^2(t^2 - x).$$

Once again the factor of x^2 corresponds to the fact that the inverse image surely contains the exceptional divisor. But now we get the equation $t^2 = 0$, so that there is only one point over the origin, as one might expect from the geometry.

Let us go back to the Cremona transformation. To compute what gets blown up and blown down, it suffices to figure out what gets blown down, by symmetry. Consider the line $X = 0$. If $bc \neq 0$, the point $[0 : b : c]$ gets mapped to $[0 : 0 : 1]$. Thus the strict transform of the line $X = 0$ in the graph gets blown down to a point. By symmetry the strict transforms of the other two lines are also blown down to points. Outside of the union of these three lines, the map is clearly an isomorphism.

Thus the involution blows up the three points $[0 : 0 : 1]$, $[0 : 1 : 0]$, and $[1 : 0 : 0]$ and then blows down the three disjoint lines. Note that the three exceptional divisors become the three new coordinate lines.

One of the most impressive results of the nineteenth century is the following characterisation of the birational automorphism group of $\mathbb{P}^2$.

Theorem 6.2 (Noether)**.** *The birational automorphism group is generated by a Cremona transformation and* PGL(3)*.*

This result is very deceptive, since it is known that the birational automorphism group is, by any standards, very large.

One can blow up points on other varieties. For example, suppose we take $\mathbb{A}^3$, and consider the lines through the origin. This gives us a rational map to $\mathbb{P}^2$. The closure of the graph is the blow up of the origin; the exceptional divisor is a copy of $\mathbb{P}^2$. In coordinates we have (x, y, z) and $[S : T : U]$ and the equations for the graph are $xT = yS$, $xU = zS$ and $yU = zT$.

The blow up features in many interesting geometric constructions. Let $(XY - ZT = 0) \subset \mathbb{P}^3$ be the smooth quadric Q. Let $P_0 = [0 : 0 : 0 : 1]$, a point of the quadric. Consider projection from this point. We get a rational map to $\mathbb{P}^2$:

$$f : Q \dashrightarrow \mathbb{P}^2.$$

This map is in fact birational. A line meets the quadric in two points, one of which is P_0. f sends $[X : Y : Z : T]$ to $[X : Y : Z]$. The map

$$g : \mathbb{P}^2 \dashrightarrow Q,$$

which sends $[X : Y : Z]$ to $[X : Y : Z : XY/Z] = [XZ : YZ : Z^2 : XY]$ is the inverse. Let $\tilde{Q}$ be the strict transform of Q under the blow up. Locally about P_0, affine coordinates are (x, y, z), and the quadric is $z = xy$.

Let $[A : B : C]$ be coordinates on the exceptional divisor. Suppose we work on the affine patch $A \neq 0$. Then $y = bx$ and $z = cx$, where $b = B/A$ and $c = C/A$. Equations for the total transform of Q are

$$z - xy = cx - bx^2 = x(c - bx).$$

Equations for $\tilde{Q}$ are $c = bx$, smooth. When $b = 0$ we get $c = 0$, the equation of a line. $\tilde{Q} \longrightarrow Q$ contracts a copy E of $\mathbb{P}^1$ to a point.

Note that the induced morphism $h\colon Q \longrightarrow \mathbb{P}^2$ blows down the strict transform of the two lines passing through P_0 to two different points Q and R of $\mathbb{P}^2$. The image of the exceptional divisor E is the line connecting Q and R; with some patience one can check all of this in local coordinates.

We now consider how to define the blow up for an arbitrary scheme. Recall

(†) X is Noetherian, $\mathcal{S}_1$ is coherent, $\mathcal{S}$ is locally generated by $\mathcal{S}_1$.

Definition 6.3. *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Let*

$$\mathcal{S} = \bigoplus_{d=0}^{\infty} \mathcal{I}^d,$$

where $\mathcal{I}^0 = \mathcal{O}_X$ and $\mathcal{I}^d$ is the dth power of $\mathcal{I}$. Then $\mathcal{S}$ satisfies (†).

*$\pi\colon \mathbf{Proj}\,\mathcal{S} \longrightarrow X$ is called the **blow up** of $\mathcal{I}$ (or Y, if Y is the subscheme of X associated to $\mathcal{I}$).*

Example 6.4. *Let $X = \mathbb{A}^n_k$ and let P be the origin. We check that we get the usual blow up. Let*

$$A = k[x_1, x_2, \dots, x_n].$$

As $X = \operatorname{Spec} A$ is affine and the ideal sheaf $\mathcal{I}$ of P is the sheaf associated to $\langle x_1, x_2, \dots, x_n\rangle$,

$$Y = \mathbf{Proj}\,\mathcal{S} = \operatorname{Proj} S,$$

where

$$S = \bigoplus_{d=0}^{\infty} I^d.$$

There is a surjective map

$$A[y_1, y_2, \dots, y_n] \longrightarrow S,$$

of graded rings, where y_i is sent to x_i. $Y \subset \mathbb{P}^n_A$ is the closed subscheme corresponding to this morphism. The kernel of this morphism is

$$\langle y_i x_j - y_j x_i\rangle,$$

which are the usual equations of the blow up.

Definition 6.5. *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. We are going to define the **inverse image ideal sheaf** $\mathcal{I}' \subset \mathcal{O}_Y$. First we take the inverse image of the sheaf $f^{-1}\mathcal{I}$, where we just think of f as being a continuous map. Then $f^{-1}\mathcal{I} \subset f^{-1}\mathcal{O}_Y$. Let $\mathcal{I}' = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ be the ideal generated by the image of $f^{-1}\mathcal{I}$ under the natural morphism $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$.*

Theorem 6.6 (Universal Property of the blow up)**.** *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent ideal sheaf.*

If $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ then $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ is an invertible sheaf. Moreover π is universal amongst all such morphisms. If $f\colon Z \longrightarrow X$ is any morphism such that $f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is invertible then there is a unique induced morphism $g\colon Z \longrightarrow Y$ which makes the diagram commute

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "f"'] & Y \arrow[d, "\pi"] \\ & X. \end{tikzcd}$$

Proof. By uniqueness, we can check this locally. So we may assume that $X = \operatorname{Spec} A$ is affine. As $\mathcal{I}$ is coherent, it corresponds to an ideal $I \subset A$ and

$$X = \operatorname{Proj} \bigoplus_{d=0}^{\infty} I^d.$$

Now $\mathcal{O}_Y(1)$ is an invertible sheaf on Y. It is not hard to check that $\pi^{-1}\mathcal{I} \cdot \mathcal{O}_Y = \mathcal{O}_Y(1)$.

Now show that we are given $f\colon Z \longrightarrow X$. We first construct g, then show that if g is any factorisation of f, the pullback ideal sheaf is invertible and then finally show that g is unique.

Pick generators $a_0, a_1, \dots, a_n$ for I. This gives rise to a surjective map of graded A-algebras

$$\phi\colon A[x_0, x_1, \dots, x_n] \longrightarrow \bigoplus_{d=0}^{\infty} I^d,$$

whence to a closed immersion $Y \subset \mathbb{P}^n_A$. The kernel of ϕ is generated by all homogeneous polynomials $F(x_0, x_1, \dots, x_n)$ such that $F(a_0, a_1, \dots, a_n) = 0$.

Now the elements $a_0, a_1, \dots, a_n$ pullback to global sections $s_0, s_1, \dots, s_n$ of the invertible sheaf $\mathcal{L} = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ and $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. So we get a morphism

$$g\colon Z \longrightarrow \mathbb{P}^n_A,$$

over X, such that $g^*\mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{L}$ and $g^{-1}x_i = s_i$. Suppose that $F(x_0, x_1, \dots, x_n)$ is a homogeneous polynomial in the kernel of ϕ. Then $F(a_0, a_1, \dots, a_n) = 0$ so that $F(s_0, s_1, \dots, s_n) = 0$ in $H^0(Z, \mathcal{L}^d)$. It follows that g factors through Y.

Now suppose that $f\colon Z \longrightarrow X$ factors through $g\colon Z \longrightarrow Y$. Then

$$f^{-1}\mathcal{I} \cdot \mathcal{O}_Z = g^{-1}(\mathcal{I} \cdot \mathcal{O}_Y) \cdot \mathcal{O}_Z = g^{-1}\mathcal{O}_Y(1) \cdot \mathcal{O}_Z.$$

Thus $\mathcal{L} = f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is an invertible sheaf.

Therefore there is a surjective map

$$g^*\mathcal{O}_Y(1) \longrightarrow \mathcal{L}.$$

But then this map must be an isomorphism and so $g^*\mathcal{O}_Y(1) = \mathcal{L}$. $s_i = g^*x_i$ and uniqueness follows. $\square$

Note that by the universal property, the morphism π is an isomorphism outside of the subscheme V defined by $\mathcal{I}$. We may put the universal property differently. The only subscheme with an invertible ideal sheaf is a Cartier divisor (local generators of the ideal, give local equations for the Cartier divisor). So the blow up is the smallest morphism which turns a subscheme into a Cartier divisor. Perhaps surprisingly, therefore, blowing up a Weil divisor might give a non-trivial birational map.

If X is a variety it is not hard to see that π is a projective, birational morphism. In particular if X is quasi-projective or projective then so is Y. We note that there is a converse to this:

Theorem 6.7. *Let X be a quasi-projective variety and let $f\colon Z \longrightarrow X$ be a birational projective morphism.*

Then there is a coherent ideal sheaf $\mathcal{I}$ and a commutative diagram

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "\pi"] \\ & X, & \end{tikzcd}$$

where $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ and the top row is an isomorphism.

7. Blowing up and toric varieties

Suppose that we start with the cone σ spanned by e_1 and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. We have already seen that this gives the affine toric variety $\mathbb{A}^2$. Now suppose we insert the vector $e_1 + e_2$. We now get two cones σ_1 and σ_2, the first spanned by e_1 and $e_1 + e_2$ and the second spanned by $e_1 + e_2$ and e_2. Individually each is a copy of $\mathbb{A}^2$. The dual cones are spanned by f_2, $f_1 - f_2$ and f_1 and $f_2 - f_1$. So we get $\operatorname{Spec} K[y, x/y]$ and $\operatorname{Spec} K[x, x/y]$.

Suppose that we blow up $\mathbb{A}^2$ at the origin. The blow up sits inside $\mathbb{A}^2 \times \mathbb{P}^1$ with coordinates (x, y) and $[S : T]$ subject to the equations $xT = yS$. On the open subset $T \neq 0$ we have coordinates s and y and $x = sy$ so that $s = x/y$. On the open subset $S \neq 0$ we have coordinates x and t and $y = xt$ so that $t = y/x$. So the toric variety above is nothing more than the blow up of $\mathbb{A}^2$ at the origin. The central ray corresponds to the exceptional divisor E, a copy of $\mathbb{P}^1$.

A couple of definitions:

Definition 7.1. *Let G and H be algebraic groups which act on varieties X and Y. Suppose we are given an algebraic group homomorphism, $\rho\colon G \longrightarrow H$. We say that a morphism $f\colon X \longrightarrow Y$ is ρ-**equivariant** if f commutes with the action of G and H. If X and Y are toric varieties and G and H are the tori contained in X and Y then we say that f is a **toric morphism**.*

It is easy to see that the morphism defined above is toric. We can extend this picture to other toric surfaces. First a more intrinsic description of the blow up. Suppose we are given a toric surface and a two dimensional cone σ such that the primitive generators v and w of the two one dimensional faces of σ generate the lattice (so that up the action of $\mathrm{GL}(2, \mathbb{Z})$, σ is the cone spanned by e_1 and e_2). Then the blow up of the point corresponding to σ is a toric surface, which is obtained by inserting the sum $v + w$ of the two primitive generators and subdividing σ in the obvious way (somewhat like the barycentric subdivision in simplicial topology).

Example 7.2. *Suppose we start with $\mathbb{P}^2$ and the standard fan. If we insert the two vectors $-e_1$ and $-e_2$ this corresponds to blowing up two invariant points, say $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Note that now $-e_1 - e_2$ is the sum of $-e_1$ and $-e_2$. So if we remove this vector this is like blowing down a copy of $\mathbb{P}^1$. The resulting fan is the fan for $\mathbb{P}^1 \times \mathbb{P}^1$.*

Note that this is an easy way to see the birational map between the quadric $Q \subset \mathbb{P}^3$ and $\mathbb{P}^2$ given by projection from a point.

Example 7.3. *Suppose we start with* $\mathbb{P}^2$ *and the standard fan,* $v_1 = e_1$*,* $v_2 = e_2$ *and* $v_3 = -e_1 - e_2$*. Suppose we insert* $w_0 = e_1 + e_2$*,* $w_1 = -e_1$ *and* $w_2 = -e_2$*, that is, suppose we blow up the three coordinate points.*

In the resulting fan, with six one dimensional cones, note that $v_1 = (-e_2) + (e_1 + e_2) = w_0 + w_1$*,* $v_2 = (e_1 + e_2) + (-e_1) = w_0 + w_2$ *and* $v_3 = -e_1 - e_2 = w_1 + w_2$*. It follows that we may blow down the strict transform of the three lines to get another copy of* $\mathbb{P}^2$*, with the upside down fan* w_0*,* w_1 *and* w_2*.*

This represents the standard Cremona transformation.

We can generalise this to higher dimensions. For example suppose we start with the standard cone for $\mathbb{A}^3$ spanned by e_1 and e_2 and e_3. If we insert the vector $e_1 + e_2 + e_3$ (thereby creating three maximal cones) this corresponds to blowing up the origin. (In fact there is a simple recipe for calculating the exceptional divisor; mod out by the central $e_1 + e_2 + e_3$; the quotient vector space is two dimensional and the three cones map to the three cones in the quotient two dimensional vector space which correspond to the fan for $\mathbb{P}^2$). Suppose we insert the vector $e_1 + e_2$. Then the exceptional locus is $\mathbb{P}^1 \times \mathbb{A}^1$. In fact this corresponds to blowing up one of the axes (the axis is a copy of $\mathbb{A}^1$ and over every point of the axis there is a copy of $\mathbb{P}^1$).

It is interesting to figure out the geometry behind the example of a toric variety which is not projective. To warm up, suppose that we start with $\mathbb{A}^3_k$. This is the toric variety associated to the fan spanned by e_1, e_2, e_3. Imagine blowing up two of the axes. This corresponds to inserting two vectors, $e_1 + e_2$ and $e_1 + e_3$. However the order in which we blow up is significant. Let's introduce some notation. If we blow up the x-axis $\pi\colon Y \longrightarrow X$ and then the y-axis, $\psi\colon Z \longrightarrow Y$, let's call the exceptional divisors E_1 and E_2, and let E_1' denote the strict transform of E_1 on Z. E_1 is a $\mathbb{P}^1$-bundle over the x-axis. The strict transform of the y-axis in Y intersects E_1 in a point p. When we blow up this curve, $E_1' \longrightarrow E_1$ blows up the point p. The fibre of E_1' over the origin therefore consists of two copies Σ_1 and Σ_2 of $\mathbb{P}^1$. Σ_1 is the strict transform of the fibre of E_1 over the origin and Σ_2 is the exceptional divisor. The fibre Σ of E_2 over the origin is a copy of $\mathbb{P}^1$. Σ and Σ_2 are the same curve in Z.

The example of a toric variety which is not projective is obtained from $\mathbb{P}^3$ by blowing up three coordinate axes, which form a triangle. The twist is that we do something different at each of the three coordinate points. Suppose that $\pi\colon X \longrightarrow \mathbb{P}^3$ is the birational morphism down to $\mathbb{P}^3$, and let E_1, E_2 and E_3 be the three exceptional divisors. Over one point we extract E_1 first then E_2, over the second point we extract first E_2 then E_3 and over the last point we extract first E_3 then E_1.

To see what has gone wrong, we need to work in the homology and cohomology groups of X. Any curve C in X determines an element of $[C] \in H_2(X, \mathbb{Z})$. Any Cartier divisor D in X determines a class $[D] \in H^2(X, \mathbb{Z})$. We can pair these two classes to get an intersection number $D \cdot C \in \mathbb{Z}$. One way to compute this number is to consider the line bundle $\mathcal{L} = \mathcal{O}_X(D)$ associated to D. Then

$$D \cdot C = \deg \mathcal{L}|_C.$$

If D is ample then this intersection number is always positive. This implies that the class of every curve is non-trivial in homology.

Suppose the reducible fibres of E_1, E_2 and E_3 over their images are $A_1 + A_2$, $B_1 + B_2$ and $C_1 + C_3$. Suppose that the general fibres are A, B and C. We suppose that A_1 is attached to B, B_1 is attached to C and C_1 is attached to A. We have

$$\begin{aligned}
[A] &= [A_1] + [A_2] \\
&= [B] + [A_2] \\
&= [B_1] + [B_2] + [A_2] \\
&= [C] + [B_2] + [A_2] \\
&= [C_1] + [C_2] + [B_2] + [A_2] \\
&= [A] + [C_2] + [B_2] + [A_2],
\end{aligned}$$

in $H_2(X, \mathbb{Z})$, so that

$$[A_2] + [B_2] + [C_2] = 0 \in H_2(X, \mathbb{Z}).$$

Suppose that D were an ample divisor on X. Then

$$0 = D \cdot ([A_2] + [B_2] + [C_2]) > D \cdot [A_2] + D \cdot [B_2] + D \cdot [C_2] > 0,$$

a contradiction.

There are a number of things to say about this way of looking at things, which lead in different directions. The first is that there is no particular reason to start with a triangle of curves. We could start with two conics intersecting transversally (so that they lie in different planes). We could even start with a nodal cubic, and just do something different over the two branches of the curve passing through the node. Neither of these examples are toric, of course. It is clear that in the first two examples, the morphism

$$\pi \colon X \longrightarrow \mathbb{P}^3,$$

is locally projective. It cannot be a projective morphism, since $\mathbb{P}^3$ is projective and the composition of projective is projective. It also follows that π is not the blow up of a coherent sheaf of ideals on $\mathbb{P}^3$. The third example is not even a variety. It is a complex manifold (and in fact it is something called an algebraic space). In particular the notion of the blow up in algebraic geometry is more delicate than it might first appear.

8. Kähler differentials

Let A be a ring, B an A-algebra and M a B-module.

Definition 8.1. *An A-**derivation** of B into M is a map $\mathrm{d}\colon B \longrightarrow M$ such that*

(1) $\mathrm{d}(b_1 + b_2) = \mathrm{d}b_1 + \mathrm{d}b_2$.
(2) $\mathrm{d}(bb') = b'\mathrm{d}b + b\mathrm{d}b'$.
(3) $\mathrm{d}a = 0$.

Definition 8.2. *The module of **relative differentials**, denoted $\Omega_{B/A}$, is a B-module together with an A-derivation, $\mathrm{d}\colon B \longrightarrow \Omega_{B/A}$, which is universal with this property:*

If M is a B-module and $\mathrm{d}'\colon B \longrightarrow M$ is an A-derivation then there exists a unique B-module homomorphism $f\colon \Omega_{B/A} \longrightarrow M$ which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_{B/A} \arrow[d, "f"] \\ & M. \end{tikzcd}$$

One can construct the module of relative differentials in the usual way; take the free B-module, with generators

$$\{\, \mathrm{d}b \mid b \in B \,\},$$

and quotient out by the three obvious sets of relations

(1) $\mathrm{d}(b_1 + b_2) - \mathrm{d}b_1 - \mathrm{d}b_2$,
(2) $\mathrm{d}(bb') - b'\mathrm{d}b - b\mathrm{d}b'$, and
(3) $\mathrm{d}a$.

The map $\mathrm{d}\colon B \longrightarrow M$ is the obvious one.

Example 8.3. *Let $B = A[x_1, x_2, \dots, x_n]$. Then $\Omega_{B/A}$ is the free B-module generated by $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_n$.*

Proposition 8.4. *Let A' and B be A-algebras and $B' = B \underset{A}{\otimes} A'$. Then*

$$\Omega_{B'/A'} = \Omega_{B'/A} \underset{B}{\otimes} B'$$

Furthermore, if S is a multiplicative system in B, then

$$\Omega_{S^{-1}B/A} = S^{-1}\Omega_{B/A}.$$

Suppose that $X = \operatorname{Spec} B \longrightarrow Y = \operatorname{Spec} A$ is a morphism of affine schemes. The **sheaf of relative differentials** $\Omega_{X/Y}$ is the quasi-coherent sheaf associated to the module of relative differentials $\Omega_{B/A}$.

Example 8.5. *Let $X = \operatorname{Spec} \mathbb{R}$ and $Y = \operatorname{Spec} \mathbb{Q}$. Then $\mathrm{d}\pi \in \Omega_{X/Y} = \Omega_{\mathbb{R}/\mathbb{Q}}$ is a non-zero differential.*

One could use the affine case to construct the sheaf of relative differentials globally. A better way to proceed is to use a little bit more algebra (and geometric intuition):

Proposition 8.6. *Let B be an A-algebra. Let*

$$B \underset{A}{\otimes} B \longrightarrow B,$$

be the diagonal morphism $b \otimes b' \longrightarrow bb'$ and let I be the kernel. Consider $B \underset{A}{\otimes} B$ as a B-module by multiplication on the left. Then I/I^2 inherits the structure of a B-module. Define a map

$$\mathrm{d} \colon B \longrightarrow \frac{I}{I^2},$$

by the rule

$$\mathrm{d}b = 1 \otimes b - b \otimes 1.$$

Then I/I^2 is the module of differentials.

Now suppose we are given a morphism of schemes $f \colon X \longrightarrow Y$. This induces the diagonal morphism

$$\Delta \colon X \longrightarrow X \underset{Y}{\times} X.$$

Then Δ defines an isomorphism of X with its image $\Delta(X)$ and this is locally closed in $X \underset{Y}{\times} X$, that is, there is an open subset $W \subset X \underset{Y}{\times} X$ and $\Delta(X)$ is a closed subset of W (it is closed in $X \underset{Y}{\times} X$ if and only if X is separated).

Definition 8.7. *Let $\mathcal{I}$ be the sheaf of ideals of $\Delta(X)$ inside W. The* ***sheaf of relative differentials***

$$\Omega_{X/Y} = \Delta^* \left(\frac{\mathcal{I}}{\mathcal{I}^2} \right).$$

Theorem 8.8 (Euler sequence)**.** *Let A be a ring, let $Y = \operatorname{Spec} A$ and $X = \mathbb{P}^n_A$.*

Then there is a short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Proof. Let $S = A[x_0, x_1, \dots, x_n]$ be the homogeneous coordinate ring of X. Let E be the graded S-module $S(-1)^{n+1}$, with basis $e_0, e_1, \dots, e_n$ in degree one. Define a (degree 0) homomorphism of graded S-modules

$E \longrightarrow S$ by sending $e_i \longrightarrow x_i$ and let M be the kernel. We have a left exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow S.$$

This gives rise to a short exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \tilde{M} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Note that even though $E \longrightarrow S$ is not surjective, it is surjective in all non-negative degrees, so that the map of sheaves is surjective.

It remains to show that $\tilde{M} \simeq \Omega_{X/Y}$. First note that if we localise at x_i, then $E_{x_i} \longrightarrow S_{x_i}$ is a surjective homomorphism of free S_{x_i}-modules, so that M_{x_i} is a free S_{x_i}-module of rank n, generated by

$$\{\, e_j - \frac{x_j}{x_i} e_i \,|\, j \neq i \,\}.$$

It follows that if U_i is the standard open affine subset of X defined by x_i then $\tilde{M}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module of rank n generated by the sections

$$\{\, \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i \,|\, j \neq i \,\}.$$

(We need the extra factor of x_i to get elements of degree zero.)

We define a map

$$\phi_i \colon \Omega_{X/Y}|_{U_i} \longrightarrow \tilde{M}|_{U_i},$$

as follows. As $U_i = \operatorname{Spec} k[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}]$, it follows that $\Omega_{X/Y}$ is the free $\mathcal{O}_{U_i}$-module generated by

$$\mathrm{d}\left(\frac{x_0}{x_i}\right), \mathrm{d}\left(\frac{x_1}{x_i}\right), \dots, \mathrm{d}\left(\frac{x_n}{x_i}\right).$$

So we define ϕ_i by the rule

$$\mathrm{d}\left(\frac{x_j}{x_i}\right) \longrightarrow \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i.$$

ϕ_i is clearly an isomorphism. We check that we can glue these maps to a global isomorphism,

$$\phi \colon \Omega_{X/Y} \longrightarrow \tilde{M}.$$

On $U_i \cap U_j$, we have

$$\left(\frac{x_k}{x_i}\right) = \left(\frac{x_k}{x_j}\right)\left(\frac{x_j}{x_i}\right).$$

Hence in $(\Omega_{X/Y})|_{U_i \cap U_j}$ we have

$$\mathrm{d}\left(\frac{x_k}{x_i}\right) - \frac{x_k}{x_j}\mathrm{d}\left(\frac{x_j}{x_i}\right) = \frac{x_j}{x_i}\mathrm{d}\left(\frac{x_k}{x_j}\right).$$

If we apply ϕ_i to the LHS and ϕ_j to the RHS, we get the same thing, namely

$$\frac{1}{x_i x_j}\left(x_j e_k - x_k e_j\right).$$

Thus the isomorphisms ϕ_i glue together. $\square$

Definition 8.9. *A variety is **smooth** (aka non-singular) if all of its local rings are regular local rings.*

Theorem 8.10. *The localisation of any regular local ring at a prime ideal is a regular local ring.*

Thus to check if a variety is smooth it is enough to consider only the closed points.

Theorem 8.11. *Let X be an irreducible separate scheme of finite type over a an algebraically closed field k.*

Then $\Omega_{X/k}$ is locally free of rank $n = \dim X$ if and only if X is a smooth variety over k.

If $X \longrightarrow Z$ is a morphism of schemes and $Y \subset X$ is a closed subscheme, with ideal sheaf $\mathcal{I}$. Then there is an exact sequence of sheaves on Z,

$$\frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/Z} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/Z} \longrightarrow 0.$$

Theorem 8.12. *Let X be a smooth variety of dimension n. Let $Y \subset X$ be an irreducible closed subscheme with sheaf of ideals $\mathcal{I}$.*

Then Y is smooth if and only if

(1) $\Omega_{Y/k}$ is locally free, and
(2) the sequence above is also left exact:

$$0 \longrightarrow \frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0.$$

Furthermore, in this case, $\mathcal{I}$ is locally generated by $r = \operatorname{codim}(Y, X)$ elements and $\frac{\mathcal{I}}{\mathcal{I}^2}$ is locally free of rank r on Y.

Proof. Suppose (1) and (2) hold. Then $\Omega_{Y/k}$ is locally free and so we only have to check that its rank q is equal to the dimension of Y. Then $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $n-q$. Nakayama's lemma implies that $\mathcal{I}$ is locally generated by $n-q$ elements and so $\dim Y \geq n-(n-q) = q$. On the other hand, if $y \in Y$ is any closed point $q = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ and so $q \geq \dim Y$. Thus $q = \dim Y$. This also establishes the last statement.

Now suppose that Y is smooth. Then $\Omega_{Y/k}$ is locally free of rank $q = \dim Y$ and so (1) is immediate. On the other hand, there is an exact sequence

$$\frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0.$$

Pick a closed point $y \in Y$. As $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $r = n - q$, we may pick sections $x_1, x_2, \dots, x_r$ of $\mathcal{I}$ such that $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_r$ generate the kernel of the second map.

Let $Y' \subset X$ be the corresponding closed subscheme. Then, by construction, $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_r$ generate a free subsheaf of rank r of $\Omega_{X/k} \otimes \mathcal{O}_{Y'}$ in a neighbourhood of y. It follows that for the exact sequence for Y'

$$\frac{\mathcal{I}'}{\mathcal{I}'^2} \longrightarrow \Omega_{X/Z} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y'/Z} \longrightarrow 0,$$

the first map is injective and $\Omega_{Y'/k}$ is locally free of rank $n - r$. But then Y' is smooth and $\dim Y' = \dim Y$. As $Y \subset Y'$ and Y' is integral, we must have $Y = Y'$ and this gives (2). $\square$

Theorem 8.13 (Bertini's Theorem)**.** *Let $X \subset \mathbb{P}^n_k$ be a closed smooth projective variety. Then there is a hyperplane $H \subset \mathbb{P}^n_k$, not containing X, such that $H \cap X$ is regular at every point.*

Furthermore the set of such hyperplanes forms an open dense subset of the linear system $|H| \simeq \mathbb{P}^n_k$.

Proof. Let $x \in X$ be a closed point. Call a hyperplane H **bad** if either H contains X or H does not contain X but it does contain x and $X \cap H$ is not regular at x. Let B_x be the set of all bad hyperplanes at x. Fix a hyperplane H_0 not containing x, defined by $f_0 \in V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Define a map

$$\phi_x \colon V \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}^2,$$

as follows. Given by $f \in V$, f/f_0 is a regular function on $X - X \cap X_0$. Send f to the image of f/f_0 to its class in the quotient $\mathcal{O}_{X,x}/\mathfrak{m}^2$.. Now $x \in X \cap H$ if and only if $\phi_x(f) \in \mathfrak{m}$. Now $x \in X \cap H$ is a regular point if and only if $\phi_x(f) \neq 0$.

Thus B_x is precisely the kernel of ϕ_x. Now as k is algebraically closed and x is a closed point, ϕ_x is surjective. If $\dim X = r$ then $\mathcal{O}_{X,x}/\mathfrak{m}^2$ has dimension $r + 1$ and so B_x is a linear subspace of $|H|$ of dimension $n - r - 1$.

Let $B \subset X \times |H|$ be the set of pairs (x, H) where $H \in B_x$. Then B is a closed subset. Let $p \colon B \longrightarrow X$ and $q \colon B \longrightarrow |H|$ denote projection onto either factor. p is surjective, with irreducible fibres of dimension $n - r - 1$. It follows that B is irreducible of dimension $r + (n - r - 1) = n - 1$. The image of q has dimension at most $n - 1$. Hence $q(B)$ is a proper closed subset of $|H|$. $\square$

Remark 8.14. *We will see later that $H \cap X$ is in fact connected, whence irreducible, so that in fact $Y = H \cap X$ is a smooth subvariety.*

Definition 8.15. *Let X be a smooth variety. The* ***tangent sheaf***

$$T_X = \mathbf{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X).$$

Note that the tangent sheaf is a locally free sheaf of rank equal to the dimension of X.

9. Algebraic versus analytic geometry

An analytic variety is defined in a very similar way to a scheme. First of all, given an open subset of $U \subset \mathbb{C}^n$, we say $X \subset U$ is an analytic closed subset if locally X is defined by the vanishing of holomorphic (equivalently analytic functions). A regular analytic function on X is then something which is the restriction of a holomorphic (equivalently analytic) functions from $U \subset \mathbb{C}^n$, so that the ring of regular functions on X is

$$\frac{\mathcal{O}_U^{\mathrm{an}}(U)}{I},$$

where $\mathcal{O}_U^{\mathrm{an}}(U)$ is the ring of holomorphic functions on U.

Globally, we have a locally ringed space $(X, \mathcal{O}_X^{\mathrm{an}})$, where X is locally isomorphic to an analytic closed subset of some open subset $U \subset \mathbb{C}^n$ together with its sheaf of analytic functions.

Theorem 9.1 (Chow's Theorem). *Let $X \subset \mathbb{P}^n$ be a closed analytic subset of projective space.*

Then X is a projective subscheme.

More generally, given a (n algebraic) scheme $(X, \mathcal{O}_X)$ of finite type over $\mathbb{C}$, we can construct an analytic variety $(X^{\mathrm{an}}, \mathcal{O}_X^{\mathrm{an}})$ in a fairly obvious way. To get X^{an} we just have to ditch the points which are not closed and enrich the topology.

The resulting functor, from the category of schemes of finite type over $\mathbb{C}$ to the category of analytic spaces, induces an equivalence of categories between projective schemes and compact analytic subschemes of projective space. The key point is that a morphism of schemes or analytic spaces is represented by the graph; the graph sits inside the product so that if the domain and range are projective then so is the graph and then one just applies (9.1).

Note that if we drop the condition that $X \subset \mathbb{P}^n$ is an analytic closed subset then there is no longer an equivalence of categories. For example $\mathbb{C}$ has lots of holomorphic functions which are nowhere near algebraic.

If X is an analytic space whose local rings are all regular then X is locally modeled on open subsets of $\mathbb{C}^n$, so that X is a complex manifold.

A basic result in the theory of C^∞-maps is Sard's Theorem, which states that the set of points where a map is singular is a subset of measure zero (of the base). Since any holomorphic map between complex manifolds is automatically C^∞, and the derivative of a polynomial is the same as the derivative as a holomorphic function, it follows that any morphism between varieties, over $\mathbb{C}$, is smooth over an open subset. In fact by the Lefschetz principle, this result extends to any variety over $\mathbb{C}$.

Theorem 9.2. *Let $f\colon X \longrightarrow Y$ be a morphism of varieties over a field of characteristic zero.*

Then there is a dense open subset U of Y such that if $q \in U$ and $p \in f^{-1}(q) \cap X_{sm}$ then the differential $df_p\colon T_pX \longrightarrow T_qY$ is surjective. Further, if X is smooth, then the fibres $f^{-1}(q)$ are smooth if $q \in U$.

Let us recall the Lefschetz principle. First recall the notion of a first order theory of logic. Basically this means that one describes a theory of mathematics using a theory based on predicate calculus. For example, the following is a true statement from the first order theory of number theory,

$$\forall n \forall x \forall y \forall z \quad n \geq 3 \implies x^n + y^n \neq z^n.$$

One basic and desirable property of a first order theory of logic is that it is complete. In other words every possible statement (meaning anything that is well-formed) can be either proved or disproved. It is a very well-known result that the first order theory of number theory is not complete (Gödel's Incompleteness Theorem). What is perhaps more surprising is that there are interesting theories which are complete.

Theorem 9.3. *The first order logic of algebraically closed fields of characteristic zero is complete.*

Notice that a typical statement of the first order logic of fields is that a system of polynomial equations does or does not have solution. Since most statements in algebraic geometry turn on whether or not a system of polynomial equations have a solution, the following result is very useful.

Principle 9.4 (Lefschetz Principle)**.** *Every statement in the first order logic of algebraically closed fields of characteristic zero, which is true over $\mathbb{C}$, is in fact true over any algebraically closed field of characteristic.*

In fact this principle is immediate from (9.3). Suppose that p is a statement in the first order logic of algebraically closed fields of characteristic zero. By completeness, we can either prove p or not p. Since p holds over the complex numbers, there is no way we can prove not p. Therefore there must be a proof of p. But this proof is valid over any field of characteristic zero, so p holds over any algebraically closed field of characteristic zero.

A typical application of the Lefschetz principle is (9.2). By Sard's Theorem, we know that (9.2) holds over $\mathbb{C}$. On the other hand, (9.2), can be reformulated in the first order logic of algebraically closed fields of characteristic zero. Therefore by the Lefschetz principle, (9.2) is true over algebraically closed field of characteristic zero.

Perhaps even more interesting, is that (9.2) fails in characteristic p. Let $f\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the morphism $t \longrightarrow t^p$. If we fix s, then the equation

$$x^p - s$$

is purely inseparable, that is, has only one root. Thus f is a bijection. However, df is the zero map, since $dz^p = pz^{p-1}dz = 0$. Thus df_p is nowhere surjective. Note that the fibres of this map, as schemes, are isomorphic to zero dimensional schemes of length p.

10. The canonical bundle and divisor

Definition 10.1. *Let X be a smooth variety of dimension n over a field k. The* ***canonical sheaf****, denoted ω_X, is the highest wedge of the sheaf of relative differentials,*

$$\omega_X = \bigwedge^n \Omega_{X/k}.$$

Note that ω_X is an invertible sheaf on X. We may write $\omega_X = \mathcal{O}_X(K_X)$, for some Cartier divisor K_X. The interesting thing is that we may generalise this:

Definition 10.2. *Let X be a normal variety over a field k. Let $U \subset X$ be the smooth locus, an open subset of X, whose complement has codimension at least two.*

The ***canonical divisor****, denoted K_X, is the Weil divisor obtained by picking a Weil divisor representing the invertible sheaf ω_U and then taking the closure.*

Note that the canonical divisor is only defined up to linear equivalence.

Definition 10.3. *Let X be a smooth projective variety over a field k. The* ***geometric genus*** *of X, denoted $p_g(X)$, is the dimension of the k-vector space $H^0(X, \omega_X)$. The m-th* ***plurigenus****, denoted $P_m(X)$, is the dimension of the k-vector space $H^0(X, \mathcal{O}_X(mK_X))$. The* ***irregularity*** *of X, denoted $q(X)$, is the dimension of the k-vector space $H^0(X, \Omega_{X/k})$.*

Note that $p_g(X) = P_1(X)$. If X is a curve, then $p_g(X) = P_1(X) = q(X)$.

Theorem 10.4. *Let X and X' be two smooth projective varieties over a field k.*

If X and X' are birational then $p_g(X) = p_g(X')$, $P_n(X) = P_n(X')$ and $q(X) = q(X')$.

Proof. We will just prove that the geometric genus is a birational invariant. By symmetry, it suffices to show that $p_g(X') \leq p_g(X)$. By assumption there is a birational map $\phi X \dashrightarrow X'$. Let $V \subset X$ be the largest open subset of X for which this map restricts to a morphism, $f \colon V \longrightarrow X'$. This induces a map of sheaves,

$$f^*\Omega_{X'/k} \longrightarrow \Omega_{V/k}.$$

Since these are both locally free of the same rank $n = \dim V$, taking the highest wedge, we get

$$f^*\omega'_X \longrightarrow \omega_V.$$

Since f is birational there is an open subset $U \subset V$ such that $f(U)$ is open in X' and f induces an isomorphism $U \longrightarrow f(U)$. Since a non-zero section of an invertible sheaf cannot vanish on any non-empty open subset, we have an injection on global sections

$$H^0(X', \omega_{X'}) \longrightarrow H^0(V, \omega_V).$$

So it suffices to show that the natural restriction map

$$H^0(X, \omega_X) \longrightarrow H^0(V, \omega_V),$$

is an isomorphism.

First off, we note that the codimension of the complement $X - V$ is at least two. Indeed, let P be a codimension one point. Then $\mathcal{O}_{X,P}$ is a DVR, as X is smooth. We already have a map of the generic point of X to X'. As X' is projective it is proper, so that there is a unique morphism $\operatorname{Spec} \mathcal{O}_{X,P} \longrightarrow X'$ compatible with ϕ. This morphism extends to a neighbourhood of P, so that f is defined in a neighbourhood of P, that is $P \in V$.

To show that the restriction map is bijective, it suffices to show that if $U \subset X$ is an open subset for which $\omega_X|_U \simeq \mathcal{O}_U$, we have

$$H^0(U, \mathcal{O}_U) \longrightarrow H^0(U \cap V, \mathcal{O}_{U \cap V}).$$

But this follows as $U - V$ has codimension at least two and X is normal; any function on X which is regular in codimension two is regular. $\square$

Definition 10.5. *Let Y be a smooth subvariety of a smooth variety X over a field k, with ideal sheaf $\mathcal{I}$. The locally free sheaf $\mathcal{I}/\mathcal{I}^2$ is called the* ***conormal sheaf****. Its dual*

$$\mathcal{N}_{Y/X} = \mathbf{Hom}_{\mathcal{O}_Y}(\frac{\mathcal{I}}{\mathcal{I}^2}, \mathcal{O}_Y),$$

is called the ***normal sheaf*** *of Y in X.*

Note that by taking duals of the usual exact sequence on Y we get

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0.$$

Theorem 10.6 (Adjunction formula)**.** *Let Y be a smooth subvariety of codimension r of a smooth variety X over a field k. Then*

$$\omega_Y \simeq \omega_X \otimes \bigwedge^r \mathcal{N}_{Y/X}.$$

If $r = 1$ then if we consider Y as a divisor on X and put $\mathcal{L} = \mathcal{O}_X(Y)$, we get

$$\omega_Y \simeq \omega_X \otimes \mathcal{L} \otimes \mathcal{O}_Y.$$

In terms of divisors,

$$K_Y = (K_X + Y)|_Y.$$

Proof. Follows from the exact sequence above, after taking highest wedge and then the dual. $\square$

It is interesting to calculate the canonical divisor in the case of a smooth toric variety. To calculate the canonical divisor, we need to write down a rational (or meromorphic in the case of $\mathbb{C}$) differential form. Note that if $z_1, z_2, \dots, z_n$ are coordinates on the torus then

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

is invariant under the action of the torus, so that the associated divisor is supported on the invariant divisor.

To calculate the zeroes and poles of this meromorphic differential, we may work locally about any invariant divisor. So we may assume that $X = U_\sigma$ is affine, isomorphic to $\mathbb{A}^1 \times \mathbb{G}_m^{n-1}$. As usual, we reduce to the case when $n = 1$, in which case we have

$$\frac{\mathrm{d}z}{z},$$

which has a simple pole at 0.

Thus this rational form has a simple pole along every invaraint divisor, that is

$$K_X + D \sim 0,$$

where D is a sum of the invariant divisors. For example,

$$-K_{\mathbb{P}^n} = H_0 + H_1 + \cdots + H_n \sim (n+1)H.$$

One can check this with the formula one gets using the Euler sequence.

11. Homological algebra and derived functors

Let X be any analytic space. Then there is an exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X^{\mathrm{an}} \longrightarrow \mathcal{O}_X^* \longrightarrow 0,$$

where $\mathcal{O}_X^*$ is the sheaf of nowhere zero holomorphic functions under multiplication. The map

$$\mathcal{O}_X^{\mathrm{an}} \longrightarrow \mathcal{O}_X^*,$$

sends a holomorphic function f to $e^{2\pi i f}$, so that the kernel is clearly $\mathbb{Z}$, the sheaf of locally constant, integer valued functions. Given a nowhere zero holomorphic function g, locally we can always find a function f mapping to g, since locally we can always take logs. Thus the map of sheaves is surjective as it is surjective on stalks.

Now suppose we take global sections. We get an exact sequence

$$0 \longrightarrow H^0(X, \mathbb{Z}) \longrightarrow H^0(X, \mathcal{O}_X^{\mathrm{an}}) \longrightarrow H^0(X, \mathcal{O}_X^*),$$

but in general the last map is not surjective.

For example, let $X = \mathbb{C} - \{0\}$. Then z is a nowhere zero function which is not the exponential of any holomorphic function; the logarithm is not a globally well-defined function on the whole punctured plane.

Sheaf cohomology is introduced exactly to fix lack of exactness on the left.

Definition 11.1. *An* ***abelian category*** $\mathcal{U}$ *is a category such that for every pair of objects A and B,* $\mathrm{Hom}(A, B)$ *has the structure of an abelian group and the composition law is linear; finite direct sums exist; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and every morphism can be factored into a epimorphism followed by a monomorphism.*

Example 11.2. *Here are some examples of abelian categories:*

1. *The category of abelian groups.*
2. *The category of modules over a ring A.*
3. *The category of sheaves of abelian groups on a topological space X.*
4. *The category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.*
5. *The category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on a scheme X.*
6. *The category of coherent sheaves of $\mathcal{O}_X$-modules on a noetherian scheme X.*

A **complex** $A^\bullet$ of objects in an abelian category is a sequence of objects, indexed by $\mathbb{Z}$, together with coboundary maps

$$d^i \colon A^i \longrightarrow A^{i+1},$$

such that the composition of any two is zero.

The ith **cohomology** of the complex, is obtained in the usual way:

$$h^i(A^\bullet) = \operatorname{Ker} d^i / \operatorname{Im} d^{i+1}.$$

A morphism of complexes $f \colon A^\bullet \longrightarrow B^\bullet$ is simply a collection of morphisms $f^i \colon A^i \longrightarrow B^i$ compatible with the coboundary maps. They give rise to maps

$$h^i(f) \colon h^i(A^\bullet) \longrightarrow h^i(B^\bullet).$$

Two morphisms f and g are **homotopic** if there are maps $k^i \colon A^i \longrightarrow B^{i-1}$ such that

$$f - g = dk + kd.$$

If f and g are homotopic then $h^i(f) = h^i(g)$.

A functor F from one abelian category $\mathcal{U}$ to another $\mathcal{B}$ is called **additive** if for any two objects A and B in $\mathcal{U}$, the induced map

$$\operatorname{Hom}(A, B) \longrightarrow \operatorname{Hom}(FA, FB),$$

is a group homomorphism. F is **left exact** if in addition, given an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

we get an exact sequence,

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC.$$

Example 11.3. *Fix an object T. The functor*

$$A \longrightarrow \operatorname{Hom}(A, T),$$

is (contravariant) left exact, so that we have an exact sequence

$$0 \longrightarrow \operatorname{Hom}(C, T) \longrightarrow \operatorname{Hom}(B, T) \longrightarrow \operatorname{Hom}(A, T).$$

An **injective** object I for $\mathcal{U}$ if the functor

$$A \longrightarrow \operatorname{Hom}(A, I),$$

is exact. An **injective resolution** of an object A is a complex $I^\bullet$, such that I is zero in negative degrees, I^i is an injective object and there is a morphism $A \longrightarrow I^0$ such that the obvious complex is exact.

We say that $\mathcal{U}$ has **enough injectives** if every object embeds into an injective object. In this case, every object has an injective resolution and any two such are homotopic.

Given a category with enough injectives, define the **right derived functors** of an left exact functor F by fixing an injective resolution and then let

$$R^i F(A) = h^i(F(I^\bullet)).$$

Now it is a well-known result that the category of modules over a ring has enough injectives.

Proposition 11.4. *Let $(X, \mathcal{O}_X)$ be a ringed space.*

Then the category of $\mathcal{O}_X$-modules has enough injectives.

Proof. Let $\mathcal{F}$ be a sheaf. For every $x \in X$ embed the stalk into an injective $\mathcal{O}_{X,x}$-module, $\mathcal{F}_x \longrightarrow I_x$. Let j denote the inclusion of $\{ x \}$ into X and let

$$\mathcal{I} = \prod_{x \in X} j_*(I_x).$$

Now suppose that we have a sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules. Then

$$\begin{aligned} \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) &= \prod_{x \in X} \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_*(I_x)) \\ &= \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x) \end{aligned}$$

In particular there is a natural map $\mathcal{F} \longrightarrow \mathcal{I}$, which is injective, as it is injective on stalks. Secondly the functor

$$\mathcal{G} \longrightarrow \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}),$$

is the direct product over all $x \in X$ of the functor which sends $\mathcal{G}$ to its stalk, which is exact, followed by the functor,

$$\mathcal{G}_x \longrightarrow \mathbf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x),$$

which is exact, as I_x is an injective module. Thus

$$\mathcal{G} \longrightarrow \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}),$$

is exact, which means that $\mathcal{I}$ is injective. □

Corollary 11.5. *Let X be a topological space.*

Then the category of sheaves on X has enough injectives.

Proof. The category of sheaves on X is equivalent to the category of $\mathcal{O}_X$-modules on the ringed space $(X, \mathbb{Z})$. □

If X is a topological space then we define $H^i(X, \mathcal{F})$ to be the right derived functor of

$$\mathcal{G} \longrightarrow \Gamma(X, \mathcal{G}).$$

Given an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$

we get a long exact sequence of cohomology, so that we indeed fix the lack of exactness.

It is interesting to examine why we don't work with projective sheaves instead of injective sheaves. After all, projective modules are much easier to understand than injective modules. A module is projective if and only if it is direct summand of a free module, so any free module is projective.

However if X is a topological space there are almost never enough projectives in the category of sheaves on X. Suppose we have a topological space with the following property. There is a closed point $x \in X$ such that for any neighbourhood V of x in X there is a smaller connected open neighbourhood U of x, that is,

$$x \in U \subset V \subset X,$$

where $U \neq V$ is connected. Let $\mathcal{F} = \mathbb{Z}_{\{x\}}$ be the extension by zero of the constant sheaf $\mathbb{Z}$ on x, so that

$$\mathcal{F}(W) = \begin{cases} \mathbb{Z} & \text{if } x \in W \\ 0 & \text{otherwise.} \end{cases}$$

I claim that $\mathcal{F}$ is not the quotient of a projective sheaf. Suppose that

$$\mathcal{P} \longrightarrow \mathcal{F},$$

is a morphism of sheaves. Let V be any connected open neighbourhood of x. Pick

$$x \in U \subset V \subset X,$$

where $U \neq V$ is open. Let $\mathcal{G} = \mathbb{Z}_U$ be the extension by zero of the locally constant sheaf $\mathbb{Z}$ on U, so that

$$\mathcal{G}(W) = \begin{cases} \mathbb{Z} & \text{if } W \subset U \\ 0 & \text{otherwise.} \end{cases}$$

As $\mathcal{P}$ is projective we have a commutative diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, dashed] \arrow[dr] & \mathcal{G} \arrow[d] \\ & \mathcal{F}. \end{tikzcd}$$

But $\mathcal{G}(V) = \mathbb{Z}_U(V) = 0$, so that the map $\mathcal{P}(V) \longrightarrow \mathcal{F}(V)$ is the zero map. But then the map on stalks is zero, so that the map $\mathcal{P} \longrightarrow \mathcal{F}$ is not surjective and $\mathcal{F}$ is not the quotient of a projective sheaf.

Note that if $(X, \mathcal{O}_X)$ is a ringed space then there are potentially two different ways to take the right derived functors of $\Gamma(X, \mathcal{F})$, if $\mathcal{F}$ is an $\mathcal{O}_X$-module. We could forget that X is a ringed space or we could work in the smaller category of $\mathcal{O}_X$-modules. We check that it does not matter in which category we work.

Definition 11.6. *Let $\mathcal{F}$ be a sheaf. We say that $\mathcal{F}$ is **flasque** if for every pair of open subsets $V \subset U \subset X$ the natural map*

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V),$$

is surjective.

For any open subset $U \subset X$ let $\mathcal{O}_U$ be the extension by zero of the structure sheaf on U, $\mathcal{O}_X|_U$.

Lemma 11.7. *If $(X, \mathcal{O}_X)$ is a ringed space then every injective $\mathcal{O}_X$-module $\mathcal{I}$ is flasque.*

Proof. Let $V \subset U$ be open subsets. Then we have an inclusion $\mathcal{O}_V \longrightarrow \mathcal{O}_U$ of sheaves of $\mathcal{O}_X$-modules. As $\mathcal{I}$ is injective we get a surjection

$$\operatorname{Hom}(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U) \longrightarrow \operatorname{Hom}(\mathcal{O}_V, \mathcal{I}) = \mathcal{I}(V). \qquad \square$$

Lemma 11.8. *If $\mathcal{F}$ is a flasque sheaf on a topological space X then*

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > 0$.

Proof. Embed $\mathcal{F}$ into an injective sheaf and take the quotient to get a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0.$$

As $\mathcal{I}$ is injective it is flasque and so $\mathcal{G}$ is flasque. As $\mathcal{F}$ is flasque there is an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow 0,$$

which taking the long exact sequence of cohomology, shows that

$$H^1(X, \mathcal{F}) = H^1(X, \mathcal{I}),$$

and

$$H^i(X, \mathcal{F}) = H^{i-1}(X, \mathcal{G}),$$

which is zero by induction on i. $\square$

Proposition 11.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the derived functors of*

$$\mathcal{F} \longrightarrow \operatorname{Hom}(X, \mathcal{F}),$$

for either the category of sheaves of $\mathcal{O}_X$-modules or simply category of sheaves on X to the category of abelian groups coincide.

Proof. Take an injective resolution of $\mathcal{F}$ in the category of $\mathcal{O}_X$-modules. Injective is flasque and flasque is acyclic, so this gives us a resolution by acylics in the category of sheaves on X and this is enough to calculate the right derived functors. □

Suppose that X is scheme over an affine scheme

$$X \longrightarrow S = \operatorname{Spec} A.$$

Now

$$H^0(S, \mathcal{O}_S) = A,$$

and all higher cohomology of any $\mathcal{O}_X$-module $\mathcal{F}$ is naturally an A-module.

12. Higher vanishing

Theorem 12.1. *Let X be a noetherian topological space of dimension n.*

Then

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > n$ and any sheaf of abelian groups.

The basic idea is to reduce to the case of the quotient of $\mathbb{Z}_U$. The first thing is to reduce to the finitely generated case. Recall that any ring is the direct limit of its finitely generated subrings, so really all we need is a couple of standard results about direct limits.

Suppose A is a directed set and $(\mathcal{F}_\alpha)$ is a direct system of sheaves indexed by A. Then we may take the direct limit $\varinjlim \mathcal{F}_\alpha$.

Lemma 12.2. *On a noetherian topological space, the direct limit of flasque is flasque.*

Proof. Suppose $(\mathcal{F}_\alpha)$ is a direct system of flasque sheaves. Suppose that $V \subset U$ are open subsets. For each i we have a surjection

$$\mathcal{F}_\alpha(U) \longrightarrow \mathcal{F}_\alpha(V).$$

Now $\varinjlim$ is an exact functor, so

$$\varinjlim \mathcal{F}_\alpha(U) \longrightarrow \varinjlim \mathcal{F}_\alpha(V),$$

is surjective. But on a noetherian topological space we have

$$(\varinjlim \mathcal{F}_\alpha)(U) = \varinjlim \mathcal{F}_\alpha(U),$$

and so

$$(\varinjlim \mathcal{F}_\alpha)(U) \longrightarrow (\varinjlim \mathcal{F}_\alpha)(V),$$

is surjective, so that $\varinjlim \mathcal{F}_\alpha$ is flasque. $\square$

Proposition 12.3. *Let X be a noetherian topological space and let $(\mathcal{F}_\alpha)$ be a direct system of abelian sheaves indexed by A.*

Then there are natural isomorphisms

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha)$$

Proof. By definition of the limit, for each α, there are maps $\mathcal{F}_\alpha \longrightarrow \varinjlim \mathcal{F}_\alpha$. This gives a map on cohomology

$$H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha).$$

Taking the limit of these maps gives a morphism

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha).$$

It is easy to check that this is an isomorphism for $i = 0$.

The general case follows using the notion of a δ-functor; see (III.2.9). $\square$

Lemma 12.4. *Let $Y \subset X$ be a closed subset of a topological space, let $\mathcal{F}$ be a sheaf of abelian groups on Y and let $j\colon Y \hookrightarrow X$ be the natural inclusion.*

Then

$$H^i(Y, \mathcal{F}) \simeq H^i(X, j_*\mathcal{F}).$$

Proof. Suppose that $\mathcal{I}^\bullet$ is a flasque resolution of $\mathcal{F}$. Then $j_*\mathcal{I}^\bullet$ is a flasque resolution of $j_*\mathcal{F}$. $\square$

It is customary to abuse notation and consider $\mathcal{F}$ as a sheaf on X, without bothering to write $j_*\mathcal{F}$.

Proof of (12.1). We introduce some convenient notation. Let $\mathcal{F}$ be a sheaf on X. If $Y \subset X$ is a closed subset then $\mathcal{F}_Y$ denotes the extension by zero of the sheaf $\mathcal{F}|_Y$; similarly if $U \subset X$ is an open subset, then $\mathcal{F}_U$ denotes the extension by zero of the sheaf $\mathcal{F}|_U$. Note that if $U = X - Y$ then there is an exact sequence

$$0 \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0.$$

The proof proceeds by Noetherian induction and induction on $n = \dim X$.

Step 1: We reduce to the case X is irreducible. We proceed by induction on the number m of irreducible components of X. If $m = 1$ there is nothing to prove. Otherwise let Y be an irreducible subset and let $U = X - Y$. Let Z be the closure of U. Then $\mathcal{F}_U$ can be considered as a sheaf on Z, which has $m-1$ irreducible components. By induction on m,

$$H^i(Y, \mathcal{F}|_Y) = H^i(Z, \mathcal{F}|_U) = 0,$$

for all $i > n$, and so

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > n$, by considering the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0.$$

So we may assume that X is irreducible.

Step 2: Suppose that $n = 0$. Then X and the empty set are the only open subsets of X. In this case, to give a sheaf on X is the same as to give an abelian group, and it clear that taking global sections is an exact functor. But then

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > 0$.

Thus we may assume that $n > 0$.

Step 3: Let

$$B = \bigcup_{U} \mathcal{F}(U),$$

and let A be the set of all finite subsets of B. Given $\alpha \in A$, let $\mathcal{F}_\alpha$ be the subsheaf of $\mathcal{F}$ generated by the elements of α. Then A is a directed set and $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$. By (12.3) we may therefore assume that $\mathcal{F}$ is finitely generated.

Step 4: Suppose that $\beta \subset \alpha$ and let r be the cardinality of the difference. Then there is an exact sequence

$$0 \longrightarrow \mathcal{F}_\beta \longrightarrow \mathcal{F}_\alpha \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is generated by r elements. So by induction on r and the long exact sequence of cohomology associated to the short exact sequence above, we are reduced to the case when $\mathcal{F}$ is generated by a single element, so that $\mathcal{F}$ is a quotient of $\mathbb{Z}_U$ for some open subset U,

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathbb{Z}_U \longrightarrow \mathcal{F} \longrightarrow 0.$$

Step 5: We reduce to the case when $\mathcal{F} = \mathbb{Z}_U$.

For each $x \in U$, $\mathcal{R}_x \subset \mathbb{Z}_x = \mathbb{Z}$. If $\mathcal{R} = 0$ there is nothing to prove; otherwise let d be the smallest positive integer which appears in $\mathcal{R}_x$. Then

$$\mathcal{R}|_V = d \cdot \mathbb{Z}|_V,$$

for some non-empty open subset $V \subset U$. In this case, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_V \longrightarrow \mathcal{R} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Now $\mathcal{Q}$ is supported on a smaller set and so

$$H^i(X, \mathcal{Q}) = 0,$$

for all $i \geq n$. Taking the long exact sequence of cohomology, we see that

$$H^i(X, \mathcal{R}) = H^i(X, \mathbb{Z}_V),$$

for all $i > n$.

Thus we may assume that $\mathcal{F} = \mathbb{Z}_U$.

Step 6: Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}_U \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_Y \longrightarrow 0,$$

where $Y = X - U$. By induction on the dimension,

$$H^i(X, \mathbb{Z}_Y) = 0,$$

for all $i \geq n$. Thus
$$H^i(X, \mathbb{Z}_U) = H^i(X, \mathbb{Z}),$$
for all $i > n$. But $\mathbb{Z}$ is a locally constant sheaf on an irreducible space, so that $\mathbb{Z}$ is flasque, and flasque is acyclic. $\square$

13. Cohomology of affine scheme

Proposition 13.1. *Let I be an injective module over a noetherian ring A.*

Then the sheaf $\tilde{I}$ on $X = \operatorname{Spec} A$ is flasque.

Corollary 13.2. *Let X be a noetherian scheme.*

Then ever quasi-coherent sheaf $\mathcal{F}$ on X can be embedded in a flasque, quasi-coherent sheaf $\mathcal{G}$.

Proof. Let $U_i = \operatorname{Spec} A_i$ be a finite open affine cover of X and let $\mathcal{F}|_{U_i} = \tilde{M}_i$ for each i. Pick an embedding of M_i into an injective A_i-module I_i. Let $f_i \colon U_i \longrightarrow X$ be the inclusion and let

$$\mathcal{G} = \bigoplus_i f_{i*}\tilde{I}_i.$$

Now for each i there is an injective map $\mathcal{F}|_{U_i} \longrightarrow \tilde{I}_i$, which induces a map $\mathcal{F} \longrightarrow f_{i*}\tilde{I}_i$. This induces a map $\mathcal{F} \longrightarrow \mathcal{G}$, which is clearly injective.

But $\tilde{I}_i$ is flasque and quasi-coherent on U_i, so that $f_{i*}\tilde{I}_i$ is flasque and quasi-coherent on X. But then $\mathcal{G}$ is flasque and quasi-coherent. □

Theorem 13.3 (Serre)**.** *Let X be a Noetherian scheme.*

TFAE

(1) X is affine,
(2) $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves,
(3) $H^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.

Proof. Suppose X is affine. Let $M = H^0(X, \mathcal{F})$ and take an injective resolution $I^\bullet$ of M in the category of A-modules. Then $\tilde{I}^\bullet$ is a flasque resolution of $\mathcal{F}$ on X. If we take global sections we get back the original injective resolution of A, so that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. Thus (i) implies (ii).

(ii) implies (iii) is easy. Suppose that $H^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.

Fix a closed point p of X together with an open neighbourhood U of p and let $Y = X - U$. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{I}_{Y \cup \{ P \}} \longrightarrow \mathcal{I}_Y \longrightarrow k(P) \longrightarrow 0.$$

This gives us an exact sequence

$$H^0(X, \mathcal{I}_Y) \longrightarrow H^0(X, k(P)) \longrightarrow H^1(X, \mathcal{I}_{Y \cup \{ P \}}) \longrightarrow 0.$$

But then there is regular function $f \in A = H^0(U, \mathcal{O}_U)$ which is not zero at p, so that $p \in X_f \subset U$ is an open neighbourhood of p. As $X_f = U_f$ it follows that X_f is affine.

As X is noetherian, it is compact, so that we can cover X by finitely many open affines, X_{f_i}, where $f_1, f_2, \dots, f_r \in A$.

Finally we check that $f_1, f_2, \dots, f_r$ generate the unit ideal. There is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^r \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

The last map α sends $(a_1, a_2, \dots, a_r)$ to $\sum a_i f_i$. It is surjective as it is surjective on stalks. $\mathcal{F}$ is then the kernel of α.

There is a filtration of $\mathcal{F}$ as follows:

$$\mathcal{F} \cap \mathcal{O}_X \subset \mathcal{F} \cap \mathcal{O}_X^2 \subset \mathcal{F} \cap \mathcal{O}_X^3 \subset \mathcal{F} \cap \mathcal{O}_X^r = \mathcal{F}.$$

The quotients are naturally $\mathcal{O}_X$-submodules of $\mathcal{O}_X$, that is, the quotients are coherent sheaves of ideals. Taking the long exact sequence of cohomology (r times), we get that $H^1(X, \mathcal{F}) = 0$. Taking the long exact sequence of cohomology of the sequence above, we get that α is surjective. But then

$$1 = \alpha(a_1, a_2, \dots, a_r) = \sum a_i f_i,$$

in the ideal generated by $f_1, f_2, \dots, f_r$. (II.2.17) shows that X is affine. $\square$

14. Čech Cohomology

We would like to have a way to compute sheaf cohomology. Let X be a topological space and let $\mathcal{U} = \{U_i\}$ be an open cover, which is locally finite. The group of k-cochains is

$$C^k(\mathcal{U}, \mathcal{F}) = \bigoplus_I \Gamma(U_I, \mathcal{F}),$$

where I runs over all $(k+1)$-tuples of indices and

$$U_I = \bigcap_{i \in I} U_i,$$

denotes intersection. k-cochains are skew-commutative, so that if we switch two indices we get a sign change.

Define a coboundary map

$$\delta^k \colon C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^{k+1}(\mathcal{U}, \mathcal{F}).$$

Given $\sigma = (\sigma_I)$, we have to construct $\tau = \delta(\sigma) \in C^{k+1}(\mathcal{U}, \mathcal{F})$. We just need to determine the components τ_J of τ. Now $J = \{j_0, j_1, \dots, j_k\}$. If we drop an index, then we get a k-tuple. We define

$$\tau_J = \left(\sum_{i=0}^{k} (-1)^i \sigma_{J - \{i_i\}} \right) \Bigg|_{U_J}.$$

The key point is that $\delta^2 = 0$. So we can take cohomology

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = Z^i(\mathcal{U}, \mathcal{F}) / B^i(\mathcal{U}, \mathcal{F}).$$

Here Z^i denotes the group of i-cocycles, those elements killed by δ^i and B^i denotes the group of coboundaries, those cochains which are in the image of δ^{i-1}. Note that $\delta^i(B^i) = \delta^i \delta^{i-1}(C^{i-1}) = 0$, so that $B^i \subset Z^i$.

The problem is that this is not enough. Perhaps our open cover is not fine enough to capture all the interesting cohomology. A **refinement** of the open cover $\mathcal{U}$ is an open cover $\mathcal{V}$, together with a map h between the indexing sets, such that if V_j is an open subset of the refinement, then for the index $i = h(j)$, we have $V_j \subset U_i$. It is straightforward to check that there are maps,

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^i(\mathcal{V}, \mathcal{F}),$$

on cohomology. Taking the (direct) limit, we get the Čech cohomology groups,

$$\check{H}^i(X, \mathcal{F}).$$

For example, consider the case $i = 0$. Given a cover, a cochain is just a collection of sections, (σ_i), $\sigma_i \in \Gamma(U_i, \mathcal{F})$. This cochain is a cocycle if $(\sigma_i - \sigma_j)|_{U_{ij}} = 0$ for every i and j. By the sheaf axiom, this means that there is a global section $\sigma \in \Gamma(X, \mathcal{F})$, so that in fact

$$H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}).$$

It is also sometimes possible to untwist the definition of $\check{H}^1$. A 1-cocycle is precisely the data of a collection

$$(\sigma_{ij}) \in \Gamma(\mathcal{U}, \mathcal{F}),$$

such that

$$\sigma_{ij} - \sigma_{ik} + \sigma_{jk} = 0.$$

In general of course, one does not want to compute these things using limits. The question is how fine does the cover have to be to compute the cohomology? As a first guess one might require that

$$\check{H}^i(U_j, \mathcal{F}) = 0,$$

for all j, and $i > 0$. In other words there is no cohomology on each open subset. But this is not enough. One needs instead the slightly stronger condition that

$$\check{H}^i(U_I, \mathcal{F}) = 0.$$

Theorem 14.1 (Leray). *If X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups and $\mathcal{U}$ is an open cover such that*

$$\check{H}^i(U_I, \mathcal{F}) = 0,$$

for all $i > 0$ and indices I, then in fact the natural map

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \simeq \check{H}^i(X, \mathcal{F}),$$

is an isomorphism.

Finally, we need to construct the coboundary maps. Suppose that we are given a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

We want to define

$$\check{H}^i(X, \mathcal{H}) \longrightarrow \check{H}^{i+1}(X, \mathcal{F}).$$

Cheating a little, we may assume that we have a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & C^i(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & C^i(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d] & C^i(\mathcal{U}, \mathcal{H}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{F}) \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{G}) \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{H}) \arrow[r] & 0.
\end{tikzcd}$$

Suppose we start with an element $t \in \check{H}^i(X, \mathcal{H})$. Then t is the image of $t' \in \check{H}^i(\mathcal{U}, \mathcal{H})$, for some open cover $\mathcal{U}$. In turn t' is represented by $\tau \in Z^i(\mathcal{U}, \mathcal{H})$. Now we may suppose our cover is sufficiently fine, so that $\tau_I \in \Gamma(U_I, \mathcal{H})$ is the image of $\sigma_I \in \Gamma(U_I, \mathcal{G})$ (and this fixes the cheat). Applying the boundary map, we get $\delta(\sigma) \in C^{i+1}(\mathcal{U}, \mathcal{G})$. Now the image of $\delta(\sigma)$ in $C^{i+1}(\mathcal{U}, \mathcal{H})$ is the same as $\delta(\tau)$, which is zero, as τ is a cocycle. But then by exactness of the bottom rows, we get $\rho \in C^{i+1}(\mathcal{U}, \mathcal{F})$. It is straightforward to check that ρ is a cocycle, so that we get an element $r' \in \check{H}^{i+1}(\mathcal{U}, \mathcal{F})$, whence an element r of $\check{H}^{i+1}(X, \mathcal{F})$, and that r does not depend on the choice of σ.

One can check that Čech Cohomology coincides with sheaf cohomology. In the case of a scheme, we already know that it suffices to work with any cover $\mathcal{U}$ such that U_I is affine. From now on, we won't bother to distinguish between sheaf cohomology and Čech Cohomology.

15. Cohomology of projective space

Let us calculate the cohomology of projective space.

Theorem 15.1. *Let A be a Noetherian ring. Let $X = \mathbb{P}^r_A$.*

(1) The natural map $S \longrightarrow \Gamma_*(X, \mathcal{O}_X)$ is an isomorphism.

(2)

$$H^i(X, \mathcal{O}_X(n)) = 0 \qquad \text{for all} \qquad 0 < i < r \text{ and } n.$$

(3)

$$H^r(X, \mathcal{O}_X(-r-1)) \simeq A.$$

(4) The natural map

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathcal{O}_X(-r-1)) \simeq A,$$

is a perfect pairing of finitely generated free A-modules.

Proof. Let

$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n).$$

Then $\mathcal{F}$ is a quasi-coherent sheaf. Let $\mathcal{U}$ be the standard open affine cover. As every intersection is affine, it follows that we may compute sheaf cohomology using this cover. Now

$$\Gamma(U_I, \mathcal{F}) = S_{x_I},$$

where

$$x_I = \prod_{i \in I} x_i.$$

Thus Čech cohomology is the cohomology of the complex

$$\prod_{i=0}^{r} S_{x_i} \longrightarrow \prod_{i<j}^{r} S_{x_i x_j} \longrightarrow \dots \longrightarrow S_{x_0 x_1, \dots x_r}.$$

The kernel of the first map is just $H^0(X, \mathcal{F})$, which we already know is S. Now let us turn to $H^r(X, \mathcal{F})$. It is the cokernel of the map

$$\prod_i S_{x_0 x_1 \dots \hat{x}_i \dots x_r} \longrightarrow S_{x_0 x_1 \dots x_r}.$$

The last term is the free A-module with generators all monomials in the Laurent ring (that is, we allow both positive and negative powers).

The image is the set of monomials where x_i has non-negative exponent for at least one i. Thus the cokernel is naturally identified with the free A-module generated by arbitrary products of reciprocals x_i^{-1},

$$\{\, x_0^{l_0} x_1^{l_1} \dots x_r^{l_r} \mid l_i < 0 \,\}.$$

The grading is then given by

$$l = \sum_{i=0}^{r} l_i.$$

In particular

$$H^r(X, \mathcal{O}_X(-r-1)),$$

is the free A-module with generator $x_0^{-1}x_1^{-1}\dots x_r^{-1}$. Hence (3).

To define a pairing, we declare

$$x_0^{l_0}x_1^{l_1}\dots x_r^{l_r},$$

to be the dual of

$$x_0^{m_0}x_1^{m_1}\dots x_r^{m_r} = x_0^{-1-l_0}x_1^{-1-l_1}\dots x_r^{-1-l_r}.$$

As $m_i \geq 0$ if and only if $l_i < 0$ it is straightforward to check that this gives a perfect pairing. Hence (4).

It remains to prove (2). If we localise the complex above with respect to x_r, we get a complex which computes $\mathcal{F}|_{U_r}$, which is zero in positive degree, as U_r is affine. Thus

$$H^i(X, \mathcal{F})_{x_r} = 0,$$

for $i > 0$ so that every element of $H^i(X, \mathcal{F})$ is annihilated by some power of x_r.

To finish the proof, we will show that multiplication by x_r induces an inclusion of cohomology. We proceed by induction on the dimension. Suppose that $r > 1$ and let $Y \simeq \mathbb{P}^{r-1}_A$ be the hyperplane $x_r = 0$. Then

$$\mathcal{I}_Y = \mathcal{O}_X(-Y) = \mathcal{O}_X(-1).$$

Thus there are short exact sequences

$$0 \longrightarrow \mathcal{O}_X(n-1) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Now $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < r-1$, by induction, and the natural restriction map

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective (every polynomial of degree n on Y is the restriction of a polynomial of degree n on X). Thus

$$H^i(X, \mathcal{O}_X(n-1)) \simeq H^i(X, \mathcal{O}_X(n)),$$

for $0 < i < r-1$, and even if $i = r-1$, then we get an injective map. But this map is the one induced by multiplication by x_r. □

Theorem 15.2 (Serre vanishing). *Let X be a projective variety over a Noetherian ring and let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf.*

(1) $H^i(X, \mathcal{F})$ *are finitely generated* A*-modules.*
(2) *There is an integer* n_0 *such that* $H^i(X, \mathcal{F}(n)) = 0$ *for all* $n \geq n_0$ *and* $i > 0$.

Proof. By assumption there is an immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. As X is projective, it is proper and so i is a closed immersion. If $\mathcal{G} = i_*\mathcal{F}$ then

$$H^i(\mathbb{P}^r_A, \mathcal{G}) \simeq H^i(X, \mathcal{F}).$$

Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$ we may assume that $X = \mathbb{P}^r_A$.

If $\mathcal{F} = \mathcal{O}_X(q)$ then the result is given by (15.1). Thus the result also holds is $\mathcal{F}$ is a direct sum of invertible sheaves. The general case proceeds by descending induction on i. Now

$$H^i(X, \mathcal{F}) = 0,$$

if $i > r$, by Grothendieck's vanishing theorem. On the other hand, $\mathcal{F}$ is a quotient of a direct sum $\mathcal{E}$ of invertible sheaves. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{R}$ is coherent. Twisting by $\mathcal{O}_X(n)$ we get

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{F}(n) \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we get isomorphisms

$$H^i(X, \mathcal{F}(n)) \simeq H^{i+1}(X, \mathcal{R}(n)),$$

for n large enough, and we are done by descending induction on i. $\square$

Theorem 15.3. *Let* A *be a Noetherian ring and let* X *be a proper scheme over* A*. Let* $\mathcal{L}$ *be an invertible sheaf on* X*. TFAE*

(1) $\mathcal{L}$ *is ample.*
(2) *For every coherent sheaf* $\mathcal{F}$ *on* X *there is an integer* n_0 *such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for $n > n_0$.

Proof. Suppose that (1) holds. Pick a positive integer m such that $\mathcal{M} = L^{\otimes m}$ is very ample. Let $\mathcal{F}_r = \mathcal{F} \otimes \mathcal{L}^r$, for $0 \leq r \leq m-1$. By (15.2) we may find n_r depending on r such that $H^i(X, \mathcal{F}_r \otimes \mathcal{M}^n) = 0$ for all $n > n_r$ and $i > 0$. Let p be the maximum of the n_r. Given $n > n_0 = pm$, we may write $n = qm + r$, for some $0 \leq r \leq m-1$ and $q > p$. Then

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = H^i(X, \mathcal{F}_i \otimes \mathcal{M}^q) = 0,$$

for any $i > 0$. Hence (1) implies (2).

Now suppose that (2) holds. Let $\mathcal{F}$ be a coherent sheaf. Let $p \in X$ be a closed point. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \longrightarrow 0,$$

where $\mathcal{I}_p$ is the ideal sheaf of p. If we tensor this exact sequence with $\mathcal{L}^n$ we get an exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p \longrightarrow 0.$$

By hypotheses we can find n_0 such that

$$H^1(X, \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for all $n \geq n_0$. It follows that the natural map

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p),$$

is surjective, for all $n \geq n_0$. It follows by Nakayama's lemma applied to the local ring $\mathcal{O}_{X,p}$ that that the stalk of $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. As $\mathcal{F}$ is a coherent sheaf, for each integer $n \neq n_0$ there is an open subset U, depending on n, such that sections of $H^0(X, \mathcal{F} \otimes \mathcal{L}^n)$ generate the sheaf at every point of U.

If we take $\mathcal{L} = \mathcal{O}_X$ it follows that there is an integer n_1 such that $\mathcal{L}^{n_1}$ is generated by global sections over an open neighbourhood V of p. For each $0 \leq r \leq n_1 - 1$ we may find U_r such that $\mathcal{F} \otimes \mathcal{L}^{n_0+r}$ is generated by global sections over U_r. Now let

$$U_p = V \cap U_0 \cap U_1 \cap \cdots \cap U_{n_1-1}.$$

Then

$$\mathcal{F} \otimes \mathcal{L}^n = (\mathcal{F} \otimes \mathcal{L}^{n_0+r}) \otimes (\mathcal{L}^{n_1})^m,$$

is generated by global sections over the whole of U_p for all $n \neq n_0$.

Now use compactness of X to conclude that we can cover X by finitely many U_p. $\square$

Theorem 15.4 (Serre duality)**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field. Then there is an invertible sheaf ω_X such that*

(1) $h^n(X, \omega_X) = 1$.
(2) Given any other invertible sheaf $\mathcal{L}$ there is a perfect pairing

$$H^i(X, \mathcal{L}) \times H^{n-i}(X, \omega_X \otimes \mathcal{L}^*) \longrightarrow H^n(X, \omega_X).$$

Example 15.5. *Let $X = \mathbb{P}^r_k$. Then $\omega_X = \mathcal{O}_X(-r-1)$ is a dualising sheaf.*

In fact, on any smooth projective variety, the dualising sheaf is precisely the canonical sheaf. This expresses a remarkable coincidence between the dualising sheaf, which is something defined in terms of sheaf cohomology and the determinant of the sheaf of Kähler differentials, which is something which comes from calculus on the variety.

Theorem 15.6. *Let $X = X(F)$ be a toric variety over $\mathbb{C}$ and let D be a T-Cartier divisor. Given $u \in M$ let*

$$Z(u) = \{\, v \in |F| \mid \langle u, v\rangle \geq \psi_D(v) \,\}.$$

Then

$$H^p(X, \mathcal{O}_X(D)) = \bigoplus_{u \in M} H^p(X, \mathcal{O}_X(D))_u \qquad \text{where} \qquad H^p(X, \mathcal{O}_X(D))_u = H^p_{Z(u)}(|F|).$$

Some explanation is in order. Note that the cohomology groups of X are naturally graded by M. (15.6) identifies the graded pieces.

$$H^p_{Z(u)}(|F|) = H^p(|F|, |F| - Z(u), \mathbb{C}).$$

denotes local cohomology. This comes with a long exact sequence for the pair. If X is an affine toric variety then both $|F|$ and $Z(u)$ are convex and the local cohomology vanishes. More generally, if D is ample, then then both $|F|$ and $Z(u)$ are convex and the local cohomology vanishes. This gives a slightly stronger result than Serre vanishing in the case of an arbitrary variety.

16. The normalisation

We start to consider the problem of resolution of singularities. There are two extremes of what one could hope to prove. Here is the weakest statement one could hope for:

Conjecture 16.1 (Resolution of singularities, weak form)**.** *Given a finitely generated field extension K/k there is always a smooth projective variety X over k with function field K.*

Here is essentially the strongest form:

Conjecture 16.2 (Resolution of singularities, strong form)**.** *Let X be a variety over a field k. Then we may find a projective birational morphism $\pi\colon Y \longrightarrow X$ where*

(1) Y is smooth,
(2) π is an isomorphism outside the singular locus of X,
(3) the locus where π is not an isomorphism is a divisor E in Y,
(4) there is a divisor A on Y supported on E which is ample over X,
(5) every component of E is smooth and the tangent spaces to each component intersect in the expected dimension, and
(6) π is invariant under automorphisms of X.

Over $\mathbb{C}$ (5) is equivalent to requiring that E look like the coordinate axes locally analytically (that is, locally in the Euclidean topology not just the Zariski topology). For (6) we actually require invariance under local analytic isomorphism and even invariance under the Galois group of a field extension. More about this later.

In practice, with our current understanding of the problem, if one can prove (16.1) then the same methods can be pushed to prove some form of (16.2).

There are quite a few interesting geometric and algebraic approaches to resolution of singularities and in this section we review some of them. Even though these methods don't always work, they introduce ideas and techniques which are of considerable independent interest.

Definition 16.3. *Let X be an integral scheme. We say that X is* ***normal*** *if all of the local rings $\mathcal{O}_{X,p}$ are integrally closed.*

The ***normalisation of*** *X is a morphism $Y \longrightarrow X$ from a normal scheme, which is universal amongst all such morphisms. If $Z \longrightarrow X$ is a morphism from a normal scheme Z, then there is a unique morphism*

$Z \longrightarrow Y$ which make the diagram commute:

$$\begin{tikzcd} Z \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X. \end{tikzcd}$$

One can always construct the normalisation of a scheme as follows. By the universal property, it suffices to construct the normalisation locally. If $X = \operatorname{Spec} A$, then $Y = \operatorname{Spec} B$, where B is the integral closure of A inside the field of fractions. Note that if X is quasi-projective variety then the normalisation $Y \longrightarrow X$ is a finite and birational morphism.

Definition 16.4. *Let X be a scheme. We say that X satisfies* ***condition*** *S_2 if every regular function defined on an open subset U whose complement has codimension at least two, extends to the whole of X.*

Lemma 16.5 (Serre's criterion)**.** *Let X be an integral scheme.*

Then X is normal if and only if it is regular in codimension one (condition R_1) and satisfies condition S_2.

Note that this gives a simple method to resolve singularities of curves. If C is a curve, the normalisation $C' \longrightarrow C$ is smooth in codimension one, which is to say that C' is smooth.

Note that lots of surface singularities are normal. For example, every hypersurface singularity is S_2, so that a hypersurface singularity is normal if and only if it is smooth in codimension one. Similarly, every quotient singularity is normal.

Before we pass on to other methods, it is interesting to write down some example of varieties which are R_1 but not normal, that is, which are not S_2.

Example 16.6. *Let S be the union of two smooth surfaces S_1 and S_2 joined at a single point p. For example, two general planes in $\mathbb{A}^4$ which both contain the same point p. Let $U = S - \{p\}$. Then U is the disjoint union of $U_1 = S_1 - \{p\}$ and $S_2 - \{p\}$, so U is smooth and the codimension of the complement is two. Let $f\colon U \longrightarrow k$ be the function which takes the value 1 on U_1 and the value 0 on U_2. Then f is regular, but it does not even extend to a continuous function, let alone a regular function, on S.*

Let C be a projection of a rational normal quartic down to $\mathbb{P}^3$, for example the image of

$$[S : T] \longrightarrow [S^4 : S^3T : ST^3 : T^4] = [A : B : C : D].$$

Let S be the cone over C. Then S is regular in codimension one, but it is not S_2. Indeed,

$$\frac{B^2}{A} = S^2T^2 = \frac{C^2}{D}.$$

is a regular function whose only pole is along $A = 0$ and $D = 0$, that is, only at $(0,0,0,0)$ of S.

Note that the coordinate ring

$$k[S^4, S^3T, ST^3, T^4] = \frac{k[A, B, C, D]}{\langle AD - BC, B^3 - A^2C, C^3 - BD^2\rangle},$$

is indeed not integrally closed in its field of fractions. Indeed,

$$\alpha = \frac{B^2}{A},$$

is a root of the monic polynomial $u^2 - BC$.

17. The Albanese method

Beyond the dimension of the Zariski tangent space, perhaps the most basic invariant of any singular point is:

Definition 17.1. *Let $X \subset M$ be a subvariety of a smooth variety. The* ***multiplicity of*** *X* ***at*** *$p \in M$ is the largest μ such that $\mathcal{I}_p \subset \mathfrak{m}^\mu$ where $\mathfrak{m}$ is the maximal ideal of M at p in $\mathcal{O}_{M,p}$ and $\mathcal{I}$ is the ideal sheaf of X in M.*

Example 17.2. *Let $X \subset \mathbb{A}^{n+1}$ be defined by a single equation*

$$f(x_1, x_2, \dots, x_n) = 0.$$

The multiplicity of X at the origin is the degree of as a power series, that is, the smallest degree of a monomial which appears in f.

The multiplicity has two basic properties. X is smooth at p if and only if the multiplicity is one and the multiplicity is upper semi-continuous in families.

We next describe the method of Albanese. Start with $X \subset \mathbb{P}^n$. Now re-embed X by the very ample line bundle $\mathcal{O}_X(m)$, where m is very large, so that $X = X_0 \subset \mathbb{P}^r$, where r is large. Pick a point $p = p_0 \in X_0$, where the multiplicity is largest, to get $X_1 \subset \mathbb{P}^{r-1}$. Now pick a point $p_1 \in X_1$ of largest multiplicity and project down to get $X_2 \subset \mathbb{P}^{r-2}$. Continuing in this way, always projecting from a point of maximal mulitplicity, we construct $X_i \subset \mathbb{P}^{r-i}$.

Theorem 17.3. *If*

$$\deg X_0 < (n! + 1)(r + 1 - n),$$

then the Albanese algorithm stops with a variety X_k and a generically finite map $f_k \colon X_0 \dashrightarrow X_k$, such that either

(1) $\deg f_k \operatorname{mult}_p(X_k) \leq n!$, *or*
(2) *X_k is a cone and* $\deg f_k \leq n!$.

Corollary 17.4. *Assume that every variety of dimension at most $n-1$ is birational to a smooth projective variety.*

Then every projective variety is birational to a projective variety with singularities of multiplicity at most $n!$.

Note that this resolves singularities for curves, since $1! = 1$ and a point of multiplicity one is a smooth point of X. Even for surfaces we get down to points of multiplicity two, which are not so bad. Starting with threefolds, the situation is not nearly so rosy, especially when one realises that if f is a hypersurface singularity of arbitrary multiplicity, then the suspension of f, x^2+f, is a hypersurface singularity of multiplicity two. It is pretty clear that resolving x^2+f entails resolving f.

Unfortunately it seems impossible to improve the bound given in (17.3).

We will need:

Theorem 17.5. *Let $X \subset \mathbb{P}^r$ be an irreducible projective variety of degree d and dimension n.*

If X is not contained in a hyperplane, then

$$d \geq r+1-n.$$

Proof of (17.3). We will prove by induction on k that if after the first k steps we don't have a cone and (1) never holds, that

$$\deg f_k \cdot \operatorname{mult}_p(X_k) \leq (n!+1)(r-k+1-n).$$

Suppose that p is a point of maximal multiplicity μ. If X_k is a cone with vertex p, then there is nothing to prove. Otherwise let X_{k+1} be the closure of the image of p under projection, and let $\pi\colon X_k \dashrightarrow X_{k+1}$ be the resulting rational map. As X_k is not a cone over p, π is generically finite.

Let d_k be the degree of X_k. The degree of π is the number of times a general line through p and another point of X_k meets X_k outside p. The degree d_{k+1} of X_{k+1} is the number of points a general space Λ of dimension $n+k+1-r$ will meet $X_{k+1} \subset \mathbb{P}^{r-k-1}$. Let $\Lambda' = \langle \Lambda, p\rangle$ be the span of Λ and p. This will meet X_{k+1} in $d_k-\mu$ points, other than p. So, we have

$$\deg \pi \cdot d_{k+1} = d_k - \mu.$$

If

$$\deg f_k \cdot \mu > n!,$$

then

$$\begin{aligned} \deg f_{k+1} \cdot d_{k+1} &= \deg f_k \deg \pi \cdot d_{k+1} \\ &= \deg f_k \cdot d_k - \deg f_k \mu \\ &\leq \deg f_k \cdot d_k - (n!+1) \\ &\leq (n!+1)(r-k+1-n) - (n!+1) \\ &\leq (n!+1)(r-(k+1)+1-n). \end{aligned}$$

This completes the induction.

It follows that eventually either

$$\deg f_k \cdot \operatorname{mult}_p X_k \leq n!,$$

which is case (1) or we get a cone (in the extreme case when $k = r - n$, so that $X_k = \mathbb{P}^n$ then we have a cone, since $\mathbb{P}^n$ is cone).

As $X_k \subset \mathbb{P}^{r-k}$ is not contained in a hyperplane, we have

$$d_k \geq (r - k + 1 - n).$$

It follows that if X_k is a cone, then

$$\deg f_k \leq n!. \qquad \square$$

Notice how truly bizarre this argument is; presumably projecting from a point will introduce all sorts of bad singularities (corresponding to secant lines and so on), but just by projecting from the point of maximal multiplicity works.

Example 17.6. *Let*

$$m_1 \leq m_2 \leq \cdots \leq m_r,$$

be a sequence of positive integers. Let C be the image of

$$t \longrightarrow (t^{m_1}, t^{m_2}, t^{m_3}, \dots, t^{m_r}),$$

inside $\mathbb{A}^r$. If we project from $(1, 0, 0, \dots, 0)$, then we get the image of

$$t \longrightarrow (t^{m_2 - m_1}, t_2^{m_3 - m_1}, t_3^{m_4 - m_1}, \dots, t_r^{m_r - m_1}),$$

inside $\mathbb{A}^{r-1}$. It is intuitively clear that the projection of C is less singular than C, but it is hard to say exactly why; for example the multiplicity might go up.

18. Riemann-Roch

Let us turn to the proof of (17.4).

Definition 18.1. *Let $P(z) \in \mathbb{Q}[z]$ be a polynomial. We say that $P(z)$ is* ***numerical*** *if $P(n) \in \mathbb{Z}$ for any sufficiently large integer n.*

Lemma 18.2.

(1) *If $P(z)$ is a numerical polynomial then we may find integers $c_0, c_1, \dots, c_r$ such that*
$$P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r.$$
In particular $P(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$.

(2) *If $f \colon \mathbb{Z} \longrightarrow \mathbb{Z}$ is any function and there is a numerical polynomial $Q(z)$ such that $\Delta(f) = f(n+1) - f(n) = Q(n)$ for n sufficiently large then there is a numerical polynomial $P(z)$ such that $f(n) = P(n)$ for n sufficiently large.*

Proof. We prove (1) by induction on the degree r of P. Since
$$\binom{z}{r} = \frac{z(z-1)\cdots(z-r+1)}{r!} = \frac{z^r}{r!} + \dots,$$
is a polynomial of degree n, they form a basis for all polynomials and we may certainly find rationals $c_0, c_1, \dots, c_r$ such that
$$P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r.$$
Note that
$$Q(z) = \Delta P(z) = P(z+1) - P(z) = c_0\binom{z}{r-1} + c_1\binom{z}{r-2} + \dots + c_{r-1},$$
is a numerical polynomial. By induction on the degree, $c_0, c_1, \dots, c_{r-1}$ are integers. It follows that c_r is an integer, as $P(n)$ is an integer for n large. This is (1).

For (2), suppose that
$$Q(Z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r,$$
for integers $c_0, c_1, \dots, c_r$. Let
$$P(z) = c_0\binom{z}{r+1} + c_1\binom{z}{r} + \dots + c_r\binom{z}{1}.$$
Then $\Delta P(z) = Q(z)$ so that $(f - P)(n)$ is a constant c_{r+1} for any n sufficiently large, so that $f(n) = P(n) + c_{r+1}$ for any n sufficiently large. $\square$

Theorem 18.3 (Asymptotic Riemann-Roch)**.** *Let X be a normal projective variety of dimension n and let $\mathcal{O}_X(1)$ be a very ample line bundle. Suppose that $X \subset \mathbb{P}^k$ has degree d.*

Then

$$h^0(X, \mathcal{O}_X(m)) = \frac{dm^n}{n!} + ...,$$

is a polynomial of degree n, for m large enough, with the given leading term.

Proof. First suppose that X is smooth. Let Y be a general hyperplane section. Then Y is smooth by Bertini. The trick is to compute $\chi(X, \mathcal{O}_X(m))$ by looking at the exact sequence

$$0 \longrightarrow \mathcal{O}_X(m-1) \longrightarrow \mathcal{O}_X(m) \longrightarrow \mathcal{O}_Y(m) \longrightarrow 0.$$

The Euler characteristic is additive so that

$$\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m-1)) = \chi(Y, \mathcal{O}_Y(m)).$$

(18.2) implies that $\chi(X, \mathcal{O}_X(m))$ is a polynomial of degree n, with the given leading term. Now apply Serre vanishing.

For the general case we need that if X is normal and Y is a general hyperplane section, then Y is a normal projective variety of degree d. Y is regular in codimension one by a Bertini type argument and one can check that Y is S_2. □

We will only need (18.3) for the method of Albanese, but it is fun to use similar arguments to prove special cases of Riemann-Roch.

Theorem 18.4 (Riemann-Roch for curves)**.** *Let C be a smooth projective curve of genus g and let D be a divisor of degree d.*

$$h^0(X, \mathcal{O}_C(D)) = d - g + 1 + h^0(C, \mathcal{O}_C(K_C - D)).$$

Proof. We first check that

$$\chi(C, \mathcal{O}_C(D)) = d - g + 1.$$

We may write

$$D = \sum a_i p_i.$$

We proceed by induction on $\sum |a_i|$. Let $p = p_1$. If $a_1 > 0$ then consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(D-p) \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

The Euler characteristic is additive, so that

$$\chi(C, \mathcal{O}_C(D)) = \chi(C, \mathcal{O}_C(D-p)) + 1.$$

The LHS is equal to $(d-1)-g+1+1=d-g+1$ by induction. If $a_1<0$ then consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_C(D+p) \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

The Euler characteristic is additive, so that

$$\chi(C, \mathcal{O}_C(D-p)) = \chi(C, \mathcal{O}_C(D+p)) - 1.$$

The RHS is equal to $d-g+1-1=(d-1)-g+1$ by induction.

So we are reduced to the case when $d=0$. Note that

$$h^1(C, \mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C(K_C - D)),$$

by Serre duality. In particular

$$\chi(C, \mathcal{O}_C) = 1-g,$$

which completes the induction. □

To state Riemann-Roch for surfaces, we will need intersection numbers. Suppose we work over $\mathbb{C}$. Consider the long exact sequence associated to the exponential sequence:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X^{\mathrm{an}} \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

The relevant part we are interested in is the group homomorphism:

$$c_1 \colon \operatorname{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}).$$

Here we identified

$$H^1(X, \mathcal{O}_X^*),$$

with the group of line bundles. One can use this to define intersection numbers, using cup product of cohomology. If $\mathcal{L} = \mathcal{O}_X(L)$, so that L is the divisor of zeroes and poles of a rational section of the invertible sheaf $\mathcal{L}$, we will use the notation

$$c_1(\mathcal{L})^n = L^n,$$

to denote the top self-intersection.

Theorem 18.5 (Riemann-Roch for surfaces)**.** *Let S be a smooth projective surface of irregularity q and geometric genus p_g over an algebraically closed field of characteristic zero. Let D be a divisor on S.*

$$\chi(S, \mathcal{O}_S(D)) = \frac{D^2}{2} - \frac{K_S \cdot D}{2} + 1 - q + p_g.$$

Proof. Pick a very ample divisor H such that $H+D$ is very ample. Let C and Σ be general elements of $|H|$ and $|H+D|$. Then C and Σ are smooth. There are two exact sequences

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D+H) \longrightarrow \mathcal{O}_C(D+H) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(D+H) \longrightarrow \mathcal{O}_\Sigma(D+H) \longrightarrow 0.$$

As the Euler characteristic is additive we have

$$\begin{aligned}\chi(S, \mathcal{O}_S(D+H)) &= \chi(S, \mathcal{O}_S(D)) + \chi(C, \mathcal{O}_C(D+H)) \\ \chi(S, \mathcal{O}_S(D+H)) &= \chi(S, \mathcal{O}_S) + \chi(\Sigma, \mathcal{O}_\Sigma(D+H)).\end{aligned}$$

Subtracting we get

$$\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) = \chi(\Sigma, \mathcal{O}_\Sigma(D+H)) - \chi(C, \mathcal{O}_C(D+H)).$$

Now

$$\begin{aligned}\chi(\Sigma, \mathcal{O}_\Sigma(D+H)) &= (D+H)\cdot\Sigma - \deg K_\Sigma/2 \\ \chi(C, \mathcal{O}_C(D+H)) &= (D+H)\cdot C - \deg K_C/2,\end{aligned}$$

applying Riemann-Roch for curves to both C and Σ. We have

$$(D+H)\cdot\Sigma = (D+H)\cdot H + (D+H)\cdot D,$$

and by adjunction

$$K_\Sigma = (K_S + \Sigma)\cdot\Sigma \qquad \text{and} \qquad K_C = (K_S + C)\cdot C.$$

So putting all of this together we get

$$\begin{aligned}\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) &= (D+H)\cdot D + \frac{1}{2}((K_S+C)\cdot C - (K_S+\Sigma)\cdot\Sigma) \\ &= (D+H)\cdot D + \frac{1}{2}K_S\cdot(C-\Sigma) + \frac{1}{2}(H\cdot H - (H+D)\cdot(H+D)) \\ &= \frac{D\cdot D}{2} - \frac{1}{2}K_S\cdot D.\end{aligned}$$

We have

$$c = \chi(S, \mathcal{O}_S) = h^0(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^2(S, \mathcal{O}_S) = 1 - q + p_g.$$

Here we used the highly non-trivial fact that

$$h^1(S, \mathcal{O}_S) = h^0(S, \Omega^1_S) = q,$$

from Hodge theory and Serre duality

$$h^2(S, \mathcal{O}_S) = h^0(S, \omega_S) = p_g. \qquad \square$$

Definition 18.6. *Let X be a quasi-projective variety and let K be the function field of X. Let L/K be a finite field extension.*

*The **normalisation of** X **in** L is a finite morphism $Y \longrightarrow X$, where Y is a normal quasi-projective variety and the function field of Y is L.*

One can consruct Y in much the same way that one constructs the normalisation. It suffices to construct Y locally, in which case we may assume that $X = \operatorname{Spec} A$ is affine. In this case one simply takes $Y = \operatorname{Spec} B$, where B is the integral closure of A inside L.

Lemma 18.7. *Let $\pi\colon Y \longrightarrow X$ be a finite morphism.*
If $\pi(q) = p$, then

$$\operatorname{mult}_q Y = \deg \pi \cdot \operatorname{mult}_p X.$$

Proof of (17.4). By (18.3) we may pick m sufficiently large such that if

$$\deg X_0 \subset \mathbb{P}^r$$

is the embedding given by $\mathcal{O}_X(m)$, then

$$d_0 \leq (n! + 1)(r + 1 - n).$$

By (17.3) we may find a generically finite morphism $f\colon X \dashrightarrow W$ such that either

$$\deg f \operatorname{mult}_w W \leq n!,$$

or W is a cone and

$$\deg f \leq n!.$$

If W is a cone, then W is birational to a product $\mathbb{P}^1 \times W'$. By our induction hypothesis, W' is birational to a smooth projective variety W''. Then W is birational to $W'' \times \mathbb{P}^1$. Replacing W by $W'' \times \mathbb{P}^1$, we may assume that W is smooth.

Let $\pi\colon Y \longrightarrow W$ be the normalisation of W in the field $L = K(X)/K(W)$. Then Y is birational to X and $\deg f = \deg \pi$. By (18.7),

$$\operatorname{mult}_y Y \leq n!. \qquad \square$$

19. Projective geometry

Definition 19.1. *Let $S \subset \mathbb{P}^n$ be a set of points.*

We say that S is in ***linear general position*** *if any subset of $k \leq n$ points spana a $(k-1)$-plane Λ.*

Remark 19.2. *Note that if S has at least $n+1$ points then S is in linear general position if every subset of $n+1$ points spans $\mathbb{P}^n$.*

Lemma 19.3. *Any two sequences $p_0, p_1, \dots, p_{n+1}$ and $q_0, q_1, \dots, q_{n+2}$ of $n+2$ points in linear general position in $\mathbb{P}^n$ are projectively equivalent, that is, there is an element $\phi \in \operatorname{Aut}(\mathbb{P}^n) = \operatorname{PGL}(n+1)$ such that $\phi(p_i) = q_i$. Furthermore, ϕ is unique.*

Proof. Since we are just saying there is one orbit on the set of $n+2$ points in linear general position, we might as well take

$$p_0 = [1 : 0 : \dots : 0], p_1 = [0 : 1 : \dots : 0], \dots, p_n = [0 : 0 : \dots : 1], p_{n+1} = [1 : 1 : \dots : 1].$$

In terms of existence, note that ϕ corresponds to an $(n+1) \times (n+1)$ matrix with entries in K. The first $n+1$ points correspond to $n+1$ linearly independent vectors in K^{n+1}. There is a unique matrix A sending one set of vectors to the others. At this point $p_i = q_i$, $0 \leq i \leq n$ and $q_i = [a_0 : a_1 : \dots : a_n]$. Since $q_0, q_1, \dots, q_{n+2}$ are in linear general position, it follows that $a_i \neq 0$ for all i. But then the diagonal matrix with $1/a_i$ in the ith spot fixes p_j, $0 \leq j \leq n$ and takes q_{n+1} to p_{n+1}.

As for uniqueness, it is enough to show that the only ϕ which fixes the sequence $p_0, p_1, \dots, p_{n+1}$ is the identity. The fact that the corresponding matrix fixes the first $n+1$ vectors, implies that the matrix is diagonal. The fact it fixes p_{n+1} means the matrix is a scalar multiple of the identity; but then ϕ is the identity. ∎

In the case $n = 1$, the three standard points p_0, p_1 and p_2 correspond to 0, ∞ and 1.

A little bit of notation. We will say that a curve $C \subset \mathbb{P}^n$ is a **rational normal curve** if it is projectively equivalent to the curve

$$[S : T] \longrightarrow [S^n : S^{n-1}T : \dots : T^n].$$

Lemma 19.4. *Let $p_1, p_2, \dots, p_{n+3}$ be $n+3$ points in linear general position in $\mathbb{P}^n$.*

Then there is a unique rational normal curve $C \subset \mathbb{P}^n$ containing $p_1, p_2, \dots, p_{n+3}$.

Proof. We only prove existence. This will follow from the following way to construct rational normal curves.

Let $G(S,T)$ be a homogeneous polynomial of degree $n+1$. Then $G(S,T)$ factors,

$$G(S,T) = \prod_{i=0}^{n} (\mu_i S_i - \lambda_i T_i).$$

Assume that $G(S,T)$ has distinct roots, meaning that $[\lambda_i : \mu_i] \in \mathbb{P}^1$ are $n+1$ different points of $\mathbb{P}^1$. Consider the $n+1$ polynomials

$$G_i(S,T) = \frac{G(S,T)}{\mu_i S_i - \lambda_i T_i},$$

of degree n. Note that $G_0, G_1, \dots, G_n$ are independent in the space of polynomials of degree n; indeed if

$$\sum a_i G_i(S,T) = 0,$$

then we see that $a_i = 0$ after plugging in $[\lambda_i : \mu_i] \in \mathbb{P}^1$.

It follows that the curve C given parametrically by

$$[S : T] \longrightarrow [G_0 : G_1 : \dots : G_n],$$

is a rational normal curve.

We may rewrite this parametrisation as

$$[S : T] \longrightarrow [\frac{1}{\mu_0 S_0 - \lambda_0 T_0} : \frac{1}{\mu_1 S_1 - \lambda_1 T_1} : \dots : \frac{1}{\mu_n S_n - \lambda_n T_n}].$$

Written this way, we see that C passes through the $n+1$ coordinate points. Parametrically we send the zeroes of G to these points.

Now given any set of $n+3$ points in linear general position we have already seen that we can choose the first $n+1$ points to be the coordinate points. This leaves two more points, p_{n+2} and p_{n+3}. The image of $[1 : 0]$ is

$$[\frac{1}{\mu_0} : \frac{1}{\mu_1} : \dots : \frac{1}{\mu_n}]$$

and the image of $[0 : 1]$ is

$$[\frac{1}{\lambda_0} : \frac{1}{\lambda_1} : \dots : \frac{1}{\lambda_n}].$$

Now we can always choose p_{n+2} to be the point $[1 : 1 : \dots : 1]$, in which case we choose $\mu_i = 1$ for all i. Finally the fact that the points are in linear general position implies that the coordinates of p_{n+3} are distinct and non-zero and we can choose $\lambda_0, \lambda_1, \dots, \lambda_n$ accordingly. $\square$

Example 19.5. *There is a simple proof of* (19.4) *in the case when $n = 2$. All rational normal curves are projectively equivalent, so in this case a rational normal curve is the same as a conic.*

In this case we want to prove that there is a unique smooth conic through any five points of $\mathbb{P}^2$, no three of which are collinear.

A conic is given by

$$aX^2 + bY^2 + cZ^2 + dYZ + eXZ + fXY,$$

and the space of all conics is naturally a copy of $\mathbb{P}^5$. The set of conics passing through a fixed point p corresponds to a hyperplane $H_p \subset \mathbb{P}^5$.

The set of conics through five points is then the intersection of five hyperplanes, which is always non-empty, so that there is always at least one conic through any five points.

If the conic is not smooth then it is either a pair of lines or a double line. A pair of lines can contain at most four points, if no three are collinear and a double line can only contain two points. So there must be a smooth conic containing the five points.

Suppose that there is more than one conic. Suppose that F and G are the defining polynomials. Then the pencil of conics given by

$$\lambda F + \mu G = 0,$$

where $[\lambda : \mu] \in \mathbb{P}^1$ also contains all five points. But any pencil of conics must contain a singular conic, and we have just seen that this is impossible.

The next natural thing to look at are quadrics $X \subset \mathbb{P}^n$, the zero sets of quadratic polynomials F. The **rank** of X is the rank of F, that is, the rank of the associated symmetric form.

Proposition 19.6. *(1) Two quadrics are projectively equivalent if and only if they have the same rank.*
(2) If X has maximal rank $n+1$ and $n > 1$ then X is rational.
(3) If X has rank less than $n+1$ and $n > 1$ then X is the cone over a quadric in $\mathbb{P}^{n-1}$.

Proof. (1) follows from the classification of symmetric bilinear forms over an algebraically closed field.

By (1) every quadric is projectively equivalent to a quadric

$$X_0^2 + X_1^2 + \cdots + X_k^2,$$

where $r = k + 1$ is the rank. If $k = n > 1$ then X is irreducible. If we pick a point p of X and project from that point then the resulting rational map

$$\pi \colon X \longrightarrow \mathbb{P}^{n-1},$$

is birational. Geometrically, we pick an auxiliary hyperplane $H \simeq \mathbb{P}^{n-1} \subset \mathbb{P}^n$ and we send a point $q \in X$ to the point $r \in \mathbb{P}^{n-1}$ where the line $\langle p, q\rangle$ meets H. Any line through p only meets X in one further point q and so π is generically one to one, and so has to be birational.

Algebraically, if $H = (X_r = 0)$ and $p = [0 : 0 : \cdots : 0 : 1]$ and we change coordinates to that F is

$$X_0^2 + X_1^2 + \cdots + X_{n-2}^2 + X_{n-1}X_n,$$

then π is the map

$$[X_0 : X_1 : \cdots : X_n] \longrightarrow [X_0 : X_1 : \cdots : X_{n-1}].$$

The inverse rational map is the map

$$[Y_0 : Y_1 : \cdots : Y_{n-1}] \longrightarrow [Y_0 : Y_1 : \cdots : Y_{n-1} : 1/Y_{n-1}(Y_0^2 + Y_1^2 + \cdots + Y_{n-2}^2)].$$

If $k < n$ then X is visibly a cone over quadric in the first $n - 1$ variables. $\square$

The next thing to consider is cubics, that is, varieties defined by a single cubic polynomial.

We start with cubic curves C in $\mathbb{P}^2$. We already know that if C is smooth then C is not rational, since the genus is 1. If C is irreducible but not smooth then projection from the singular point show that C is rational, that is, birational to $\mathbb{P}^1$. In fact with a little bit more work, one can show that C is projectively equivalent either to

$$Y^2Z = X^2 + X^3 \qquad \text{or} \qquad Y^2Z = X^3,$$

a nodal cubic or a cuspidal cubic.

It is interesting to consider what happens if one projects from a point p of a smooth cubic. A general line passes through two more points q and r of the cubic and we get a double cover of $\mathbb{P}^1$. If we only get one point $q = r$ then the line through q is tangent to the curve.

It is a fact that if we choose $p \in C$ general then there are only finitely many lines through p which are tangent to C. For the cubic there are four such lines, and this gives four points in $\mathbb{P}^1$. We can always choose the first three points to to be 0, 1 and ∞ but the last point λ gives moduli.

In fact the space of all cubics is nine dimensional,

$$\binom{3+2}{2} - 1 = 9.$$

PGL(3) has dimension $3 \times 3 - 1 = 8$. So we expect a one dimensional family of non-projectively equivalent cubics.

The next thing to consider are cubic surfaces in $\mathbb{P}^3$.

20. Cubic surfaces

The next thing to consider are cubic surfaces in $\mathbb{P}^3$.

For this we will need to work with the Grassmannian, which is the variety parametrising lines in $\mathbb{P}^3$, equivalently planes in K^4.

Definition 20.1. *The* ***Grassmannian*** $G(k, V)$ ***of*** k***-planes in*** V *denotes the set of all* k *planes in* V.

This set is naturally a variety. We set $G(k, n) = G(k, K^n)$ and $\mathbb{G}(k, n) = G(k+1, n+1)$. The latter may be thought of as the set of k-planes in $\mathbb{P}^n$. The Grassmannian comes with a universal family, an incidence correspondence

$$\Sigma = \{\, (\Lambda, x) \,|\, x \in \Lambda \,\} \subset \mathbb{G}(k, n) \times \mathbb{P}^n,$$

which is a Zariski closed subset. There are two natural projections:

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^n \\ \mathbb{G}(k, n). & \end{tikzcd}$$

Using this we can deduce many of its properties. I claim that $\mathbb{G}(k, n)$ is irreducible and has dimension $(k + 1)(n - k)$. We prove this by induction. Fix a point $x \in \mathbb{P}^n$. The fibre of q is the set of k-planes containing a point. This is isomorphic to the set of $k - 1$-planes in $\mathbb{P}^{n-1}$, which is irreducible of dimension $k(n - k)$. It follows that Σ is irreducible of dimension $n + k(n - k)$. But then $\mathbb{G}(k, n)$ is certainly irreducible. If we fix Λ then the fibre of p is the set of points in Λ. This is Λ, a copy of $\mathbb{P}^k$, so that the fibres of p are irreducible of dimension k. Thus

$$\dim \mathbb{G}(k, n) = n + k(n - k) - k = (k + 1)(n - k).$$

One can use the universal family to make some interesting constructions. For example, suppose we are given a closed subset $X \subset \mathbb{P}^n$. Then $p(q^{-1}(X))$ is a closed subvariety of $\mathbb{G}(k, n)$, consisting of all k-planes in $\mathbb{P}^n$ which intersect X. The first interesting case is that of a curve C in $\mathbb{P}^3$. In this case the general line does not meet the curve C. In fact we get a codimension one subvariety of $\mathbb{G}(1, 3)$. Conversely suppose we are given a closed subvariety Φ of $\mathbb{G}(k, n)$. Then $q(p^{-1}(\Phi))$ is a closed subvariety of $\mathbb{P}^n$, equal to

$$X = \bigcup_{\Lambda \in \Phi} \Lambda.$$

Note that X has the interesting property that through every point of X there passes a k-plane. Classically such varieties are called **scrolls**. Perhaps the first interesting example of a scroll is the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$.

Let us give some more constructions of scrolls. Suppose that we are given two subvarieties X and Y of $\mathbb{P}^n$. Define a rational map

$$\phi \colon X \times Y \dashrightarrow \mathbb{G}(1, n),$$

by sending

$$([v], [w]) \longrightarrow [v \wedge w].$$

The subvariety in $\mathbb{P}^n$, corresponding to the image, is called the **join**. It is the closure of the union of all lines obtained by taking the span of a point of X and a point of Y. Note that ϕ is a morphism if X and Y are disjoint and in this case we don't need to take the closure. If we take $X = Y$, then we get the **secant variety of** X, which is the closure of all the lines which join two points of X.

Suppose that we are given a morphism $f \colon X \longrightarrow Y$, with the property that there is a point $x \in X$ such that $f(x) \neq x$. Consider the morphism

$$\psi \colon X \longrightarrow \mathbb{G}(1, n),$$

which is the composition of

$$X \longrightarrow X \times Y \qquad \text{given by} \qquad x \longrightarrow (x, f(x)),$$

and the morphism ϕ above. As before this gives us a scroll in $\mathbb{P}^n$, by taking the image. Note that all of this generalises to products of k varieties.

Definition 20.2. *Pick complimentary linear spaces $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ of dimensions $n_1, n_2, \dots, n_k$ in $\mathbb{P}^n$, where*

$$n + 1 = \sum_i (n_i + 1).$$

Pick rational normal curves $C_i \subset \Lambda_i$ in and pick identifications

$$\phi_i \colon \mathbb{P}^1 \longrightarrow C_i.$$

Let

$$X = \bigcup_{p \in \mathbb{P}^1} \langle \phi_1(p), \phi_2(p), \dots, \phi_k(p) \rangle.$$

*Then X is called a **rational normal scroll**.*

It is interesting to give some examples. Suppose that we pick two skew lines l and m in $\mathbb{P}^3$. Then we get a surface in $\mathbb{P}^3$, swept out by lines, meeting l and m. Suppose we pick coordinates such that

$l = V(X, Y)$ and $m = V(Z, W)$. Identify $(0, 0, a, b)$ with $(a, b, 0, 0)$. Then it is not hard to see that we get the surface $V(XW - YZ)$.

The next case is when we take a line and a complimentary plane in $\mathbb{P}^4$. The resulting surface in $\mathbb{P}^4$ is called the cubic scroll.

Another intriguing method was proposed by Nash:

Definition 20.3. *Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension n. The* ***Gauss map*** *is the rational map*

$$X \dashrightarrow \mathbb{G}(n, N) \qquad \text{given by} \qquad x \longrightarrow T_x X,$$

which sends a point to its (projective) tangent space.

The ***Nash blow up*** *is given by taking the graph of this rational map.*

Conjecture 20.4. *We can always resolve any variety by successively taking the Nash blow up and normalising.*

Despite the very appealing nature of this conjecture (consider for example the case of curves, when we don't even need to normalise) we only know (20.4) in very special cases. The one very nice feature of the Nash blow up is that it does not involve any choices. Unfortunately it is known that one needs to normalise, and this messes up any sort of induction.

Back to cubic surfaces. The space of cubic surfaces has dimension

$$\binom{3+3}{3} - 1 = 19.$$

So the space of cubic surfaces is a copy of $\mathbb{P}^{19}$. Consider the incidence correspondence

$$\Sigma = \{\, (S, l) \,|\, l \subset S \,\} \subset \mathbb{P}^{19} \times \mathbb{G}(1, 3)$$

There are two natural projections:

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{G}(1,3) \\ \mathbb{P}^{19} & \end{tikzcd}$$

The fibres of q are the space of cubics containing a fixed line. I claim that this is a copy of $\mathbb{P}^{15}$. There are two ways to see this. Fix four points of the line. To contain any one of those points impose one linear condition. To contain all four imposes at most four conditions. But we can find a cubic containing any three points but not the fourth, so in fact they impose exactly four conditions.

Alternatively, we can always choose coordinates so that the line is $[a : b : 0 : 0]$, that is $Z = T = 0$ (using homogeneous coordinates

$[X : Y : Z : T]$). The cubic $F = 0$ contains this line if and only if the coefficients of X^3, X^2Y, XY^2 and Y^3 vanish.

Thus Σ is irreducible and has dimension 19. We would know that every cubic contains a line provided we know that one fibre of p is zero dimensional; in fact we need even less we just need one cubic and a line on the cubic which doesn't deform on the cubic. One can check that the Fermat cubic

$$X^3 + Y^3 + Z^3 + T^3 = 0,$$

has isolated lines. With a little bit more work one can show that the Fermat cubic contains exactly 27 lines. This shows that a general cubic contains at least 27 lines and every cubic contains at least one line.

Let S be a smooth cubic and let $l \subset S$ be a line. It is easy to see that l cannot deform, that is, S is not a scroll. Consider the family of planes Π containing l. The family of planes Π containing l is parametrised by $\mathbb{P}^1$. There are many ways to see this. E.g choose an auxliary skew line m. Then Π intersects m in a single point, which determines Π.

A plane will intersect the cubic S in the line l and a residual conic C. The conic intersects the line l in two points. One can check that this conic becomes singular five times, in which case the conic is the union of two lines, neither of which are l. Thus there are ten lines meeting l. Continuing in this way, one can show that S contains two skew lines l and m. Define a rational map

$$\phi \colon l \times m \dashrightarrow S,$$

as follows. Given $(p, q) \in l \times m$ the line $\langle p, q\rangle$ will intersect S in one further point r, at least for some open subset of $l \times m = \mathbb{P}^1 \times \mathbb{P}^1$. Send (p, q) to r. Then ϕ is birational and S is rational.

It is interesting to consider what happens in higher dimensions.

Theorem 20.5 (Clemens-Griffiths)**.** *If $V \subset \mathbb{P}^4$ is a smooth cubic then V is irrational.*

Starting with fourfolds the situation is quite murky. There are smooth rational cubic fourfolds. One can write down smooth cubic fourfolds which contain two skew planes. The same construction as above yields a birational map to $\mathbb{P}^2 \times \mathbb{P}^2$, which is rational. But the locus of cubic fourfolds which contain a plane in the space of all cubic fourfolds is a proper Zariski closed subset. In fact there are other special cubic fourfolds which are rational (containing ever more exotic configurations of closed subvarieties) which form a countable union of closed subvarieties. Conjecturally, however there are smooth cubic fourfolds which are irrational.