Vector bundles on the Fargues-Fontaine curve

There are expanded notes prepared for a talk in a learning seminar on Fargues' UChicago notes *Geometrization of the local Langlands correspondence*, January 2016 at Columbia. Our main goal is to state the classification theorem of vector bundles on the Fargues-Fontaine curve and give a sketch of the proof. To put things in context, we first review the moduli space of vector bundles on curves and discuss the analogy between the Fargues-Fontaine curve and $\mathbb{P}^1$.

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Review: vector bundles on curves

Let $X/\mathbb{C}$ be a smooth projective curve. There is a nice moduli space parameterizing isomorphism classes of line bundles on X, its the Picard variety. Unlike the case of line bundles, isomorphism classes vector bundles of higher rank in general do not form nice moduli space, e.g., the jump phenomenon shows that it is not even separated. To resolve this issue, one can either remove the word "isomorphism classes" and work directly with the moduli stack of vector bundles Bun_X. Or more concretely, restrict one's attention to those vector bundles which are *semi-stable* and construct a nice moduli space of semi-stable vector bundles using Mumford's GIT. The latter coincides with the coarse moduli space of the open substack of Bun_X consisting of semi-stable vector bundles.

We briefly recall the notion of (semi-)stability and the Harder-Narasimhan filtration.

Definition 1 The *slope* of a vector bundle E on X is the ratio $\mu(E)=\deg E/\operatorname{rk} E$. If we draw the vector $(\operatorname{rk} E, \deg E)$ in the plane then $\mu(E)$ is literally its slope. Since both the degree and the rank are additive in a short exact sequence, we know the three vectors in an extension satisfy the parallel rule. In particular, if $F\subseteq E$ is a subbundle, then $\mu(E)$ is squeezed between $\mu(F)$ and $\mu(E/F)$.

Definition 2 We say a vector bundle E is *stable* (resp. *semi-stable*) if its slope is strictly bigger (resp. bigger) than that of any of its proper subbundle. Equivalently, by the squeeze property, E is stable (resp. semi-stable) if its slope is strictly smaller (resp. smaller) than that of any of its proper quotient bundle. By definition any line bundle is stable.

The following facts are not so difficult to prove:

Theorem 1

- The category $\mathcal{C}(\lambda)$ of semi-stable bundles of a fixed slope λ is an abelian category: in particular, kernels and cokernels are still semi-stable bundles with the same slope.
- For any semi-stable bundle E, there is a *Jordan-Holder filtration* $E=E_0\supseteq E_1\supseteq\cdots\supseteq E_k=0$, such that each successive quotient is stable (necessarily of the same slope by the first part). In particular, the simple objects of $\mathcal{C}(\lambda)$ are the stable bundles.
- For any vector bundle E, there is a unique filtration, known as the *Harder-Narasimhan filtration* or the *slope filtration* $E=E_0\supseteq E_1\supseteq\cdots E_k=0$ such that the successive quotients are all semi-stable with slopes strictly increasing: $\mu(E_0/E_1)<\mu(E_1/E_2)<\cdots$. (The construction starts off: pick the maximal subbundle $E_1\subseteq E$ among all subbunldes of maximal slope)

Let us look at the case $X=\mathbb{P}^1$ to illustrate these notions.

Example 1 By Grothendieck's theorem, each vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles. It follows that

- The only stable bundles are line bundles. Every semi-stable bundle has integer slope d and is a direct sum of $\mathcal{O}(d)$.
- The Harder-Narasimhan filtration of any vector bundle is split.
- The map $(d_1, \ldots, d_n) \mapsto \bigoplus_{i=1}^n \mathcal{O}(d_i)$ gives a bijection between integral sequences $d_1 \geq \cdots \geq d_n$ and $\mathrm{Bun}_{\mathbb{P}^1}/\mathrm{iso}$.

Remark 1 The Narasimhan-Seshadri theorem asserts that there is an equivalence between the category of semi-stable holomorphic vector bundles of slope zero on a Riemann surface X and the category of unitary representations of $\pi_1(X)$. As the simple objects, the stable vector bundles correspond exactly to the irreducible representations. For $X = \mathbb{P}^1$, the category of representations of $\pi_1(X) = \{1\}$ indexed by integers $n \geq 1$, which correspond to the trivial vector bundles $\mathcal{O}^n$. Donaldson gives a conceptual proof of Narasimhan-Seshadri theorem by constructing a flat unitary connection on such vector bundles and the corresponding unitary representation is its monodromy representation.

We will see soon that the classification of vector bundles on the Fargues-Fontaine curve remarkably resembles that of $\mathbb{P}^1$ (and one may even think the Fargues-Fontaine curve as a "twisted $\mathbb{P}^1$"!).

The Fargues-Fontaine curve is like $\mathbb{P}^1$, but not quite

Let E be a discretely valued non-archimedean field with uniformizer π and residue field $\mathbb{F}_q$. Let $F/\mathbb{F}_q$ be a perfectoid field with uniformizer ϖ_F. We have constructed the Fargues-Fontaine curve (a.k.a. the fundamental curve of p-adic Hodge theory) $X = X_{F,E}$. Recall:

- $\mathbb{A}$ is the unique π-adically complete π-torsion free lift of $\mathcal{O}_F$ as an $\mathbb{F}_q$-algebra. Concretely, $\mathbb{A} = W_{\mathcal{O}_E}(\mathcal{O}_F)$ if $\mathrm{char}\, E = 0$ and $\mathbb{A} = \mathcal{O}_F[[\pi]]$ if $\mathrm{char}\, E = p$.
- $Y = \mathrm{Spa}(\mathbb{A}) - V(\pi[\varpi_F])$, where $[\varpi_F] \in \mathbb{A}$ is the Teichmuller lift. This is an adic space: the structure presheaf is actually a sheaf, thanks to Scholze.
- $\mathcal{O}(Y) = B$ is a Frechet algebra given by the completion of $\mathbb{A}[1/\pi, 1/[\varpi_F]]$ with respect to a family of norms indexed by compact intervals in $(0,1)$. The ring B can be thought of (as least in the equal characteristic case) as holomorphic functions on the punctured open unit disk (with variable π and coefficient in F).
- Let ϕ acts Y by the (lift of) Frobenius on F. The action is properly discontinuous and so the quotient $X^{\mathrm{ad}} = Y/\phi^{\mathbb{Z}}$ makes sense and becomes an adic space over E.

Remark 2 The role of the perfectoid field $F/\mathbb{F}_q$ can be thought of as a test scheme over an absolute base "$\mathrm{Spec}\, \mathbb{F}_1$", for the curve "$\mathrm{Spec}\, E \to \mathrm{Spec}\, \mathbb{F}_1$". So $X_{F,E}$ can be thought of as "$\mathrm{Spec}\, E \times_{\mathrm{Spec}\, \mathbb{F}_1} \mathrm{Spec}\, F$".

Remark 3 It turns out that the coverings $X_{F',E'} \to X_{F,E}$ when F'/F, E'/E vary over finite extensions form a universal covering and hence its arithmetic etale fundamental $\pi_1(X) \cong G_F \times G_E$ and its geometric etale fundamental group $\pi_1(X) \cong G_F$. Thus the following can be thought of as an analogue of Narasimhan-Seshadri theorem: there is an equivalence between the category of semi-stable vector bundles of slope 0 on X and the category of G_F-representations over E.

For any integer d, we constructed the line bundle $\mathcal{O}(d)$ on X. Geometrically it is given by $Y \times_{\phi^{\mathbb{Z}}} \mathbb{A}^1_E \to Y/\phi^{\mathbb{Z}}$, where ϕ acts on $\mathbb{A}^1_E$ by π^{-d}. Its global section is then given by $H^0(X^{\mathrm{ad}}, \mathcal{O}(d)) = B^{\phi = \pi^d}$. We defined the schematic curve

$$X = \mathrm{Proj}\left(\bigoplus_{d \geq 0} H^0(X, \mathcal{O}(d))\right).$$

It is a scheme over E, noetherian, regular, dimensional one but not of finite type.

From now on assume F is algebraically closed. Let us see the first resemblance of X to $\mathbb{P}^1$ by computing the Picard group of X. We claim that the degree map gives an isomorphism

$$\mathrm{Pic}(X) \cong \mathbb{Z}, \quad \mathcal{O}(d) \mapsto d.$$

In fact, let $t \in H^0(X, \mathcal{O}(1)) - \{0\}$ be a section whose divisor is a closed point $\infty_t \in |X|$. Then

$$X - \{\infty_t\} = \mathrm{Spec}\, B[1/t]^{\phi = \mathrm{Id}}.$$

It turns out (requires some work) that $B_e := B[1/t]^{\phi=\mathrm{Id}}$ is a PID. It then follows that

$$\mathrm{Pic}^0(X-\{\infty_t\})=0,\quad \mathrm{Pic}(X)\cong\mathbb{Z}.$$

Remark 4 This is analogous to the fact that the function $1/x$ has divisor $\infty\in\mathbb{P}^1$ and $\mathbb{P}^1-\{\infty\}=\mathrm{Spec}\, k[x]$ is a PID and hence $\mathrm{Pic}(\mathbb{P}^1)\cong\mathbb{Z}$.

Remark 5 In fact B_e is *almost Euclidean* for the degree function $\deg=-\mathrm{ord}_{\infty_t}$. Namely for any two nontrivial elements $x,y\in B$, there exists $a,b\in B$ such that

$$x=ay+b,\quad \deg b\le\deg y.$$

Notice Euclidean means that the *strict* inequality $\deg b<\deg y$ holds.

Let us see another resemblance to $\mathbb{P}^1$ by showing the "genus" of X is zero, i.e., $H^1(X,\mathcal{O})=0$. We have an affine covering $U_1=X-\{\infty\}$ and U_2 an infinitesimal neighborhood of ∞. The cohomology of coherent sheaf $\mathcal{E}$ on X can be computed by the Cech complex

$$\mathcal{E}(U_1)\oplus\mathcal{E}(U_2)\to\mathcal{E}(U_1\cap U_2),\quad (f,g)\mapsto f-g.$$

Namely

$$H^0(X,\mathcal{E})=\mathcal{E}(U_1)\cap\mathcal{E}(U_2),\quad H^1(X,\mathcal{E})=\mathcal{E}(U_1\cap U_2)/\mathcal{E}(U_1)+\mathcal{E}(U_2).$$

For $\mathcal{E}=\mathcal{O}_X$, since $\mathcal{O}_X(U_2)=\widehat{\mathcal{O}_{X,\infty}}=B_{\mathrm{dR}}^+$ and $\mathcal{O}_X(U_1\cap U_2)=B_{\mathrm{dR}}:=B_{\mathrm{dR}}[1/t]$, it reads

$$H^0(X,\mathcal{O})=B_e\cap B_{\mathrm{dR}}^+=E,\quad H^1(X,\mathcal{O})=B_{\mathrm{dR}}/(B_{\mathrm{dR}}^++B_e)=0.$$

The latter has to do with the fact that B_e is almost Euclidean.

Remark 6 Warning: however, X fails to satisfy Riemann-Roch:

$$H^1(X,\mathcal{O}(-1))=B_{\mathrm{dR}}/(tB_{\mathrm{dR}}^++B_e)\cong B_{\mathrm{dR}}^+/(tB_{\mathrm{dR}}^++E)\cong C_E\neq0,$$

which has to do with the fact that B_e is not Euclidean. This is the main difference causing the classification of vector bundles on X to be more complicated than the case of $\mathbb{P}^1$.

Vector bundles on the Fargues-Fontaine curve

Let E_n/E be the degree F unramified extension. Notice if we replace E by E_n then Y stays the same but the Frobenius changes since the residue field of E changes. Thus we have a natural degree F unramified cover

$$\pi_n: X_n:=X_{F,E_n}=Y/\phi^{n\mathbb{Z}}\to X=X_{F,E}=Y/\phi^{\mathbb{Z}}.$$

Definition 3 We can push forward the line bundle $\mathcal{O}_{X_n}(d)$ along π_n to get a vector bundle $\pi_{n,*}\mathcal{O}_{X_n}(d)$ of rank F and degree d. Its slope is $\lambda=d/n\in\mathbb{Q}$. We denote it by $\mathcal{O}(\lambda)$ when $(d,n)=1$.

We have following easy properties analogous to the $\mathbb{P}^1$ case:

Proposition 1

a. $\mathcal{O}(\lambda)\otimes\mathcal{O}(\mu)=\bigoplus\mathcal{O}(\lambda+\mu)$.
b. $H^0(X,\mathcal{O}(\lambda))=0$ if and only $\lambda<0$. In particular, $\mathrm{Hom}(\mathcal{O}(\lambda),\mathcal{O}(\mu))=0$ if and only $\lambda>\mu$.
c. $H^1(X,\mathcal{O}(\lambda))=0$ if $\lambda\ge0$. In particular, by (a), there is no nontrivial extension of $\mathcal{O}(\lambda)$ by $\mathcal{O}(\mu)$ if $\lambda\le\mu$.

Now we can state the main classification theorem.

Theorem 2 Suppose F is algebraically closed.

a. The semi-stable stable vector bundles of slope λ are direct sums of $\mathcal{O}(\lambda)$.
b. The Harder-Narasimhan filtration of any vector bundle on X is split.
c. The map $(\lambda_1,\ldots,\lambda_k)\mapsto\bigoplus_{i=1}^k\mathcal{O}(\lambda_i)$ gives a bijection between sequences $\lambda_1\ge\cdots\ge\lambda_k$ ($\lambda_i\in\mathbb{Q}$) and $\mathrm{Bun}_X/\mathrm{iso}$.

Notice (a) implies (b) by the third property in Proposition 1; (a,b) together implies (c).

Remark 7 In the equal characteristic, this theorem is due to Hartl-Pink's classification of ϕ-equivariant vector bundles over $\mathbb{D}_F^*$. In the mixed characteristic, this theorem is equivalent to Kedlaya's classification of ϕ-modules over the Robba ring $\mathcal{R}=\mathcal{O}(\mathcal{Y}_{(0,0+)})$ (by the expanding property of ϕ). We are going to discuss a more geometric proof due to Fargues.

Remark 8 This classification may remind you the Dieudonne-Manin classification of isocrystals. In fact, let $L = \breve{E}$ and let D be an isocrystal over L. Then the vector bundle corresponding to $-(\lambda_1, \ldots, \lambda_k)$ (the minus sign comes from normalization) can be realized geometrically as $Y \times_{\phi^{\mathbb{Z}}} D \to Y/\phi^{\mathbb{Z}}$!

Reduction to degree one modifications of vector bundles

The main goal today is to reduce to the classification theorem 2 to the following two statements about modification of vector bundles on the Fargues-Fontaine curve.

Theorem 3

a. If $\mathcal{O}(1/n)$ is an increasing modification of $\mathcal{E}$ of degree one, i.e., there is an exact sequence
$$0 \to \mathcal{E} \to \mathcal{O}(1/n) \to \mathcal{F} \to 0,$$
with $\mathcal{F} = i_{x,*}k(x)$ for a closed point x, then $\mathcal{E} \cong \mathcal{O}^n$.

b. If $\mathcal{E}$ is an increasing modification of $\mathcal{O}^n$ of degree one,
$$0 \to \mathcal{O}^n \to \mathcal{E} \to \mathcal{F} \to 0,$$
then $\mathcal{E} \cong \mathcal{O}^{n-r} \oplus \mathcal{O}(1/r)$ for some $r \in \{1, \ldots, n\}$.

Remark 9 The modifications of a vector bundle $\mathcal{E}$ at ∞ are given by specifying a lattice in $D \otimes_L B_{\mathrm{dR}}^+$, where D is the isocrystal with the same slopes as $\mathcal{E}$. Moreover, the degree one modification are given by the lattices satisfying the minuscule condition. Then Theorem 3 are proved by showing that all lattices corresponding to the desired modification can be realized as the period lattices of p-divisible groups, which can be explicitly described. It thus boils down to the study of period maps on certain Rapoport-Zink deformation spaces of p-divisible groups. Part (a) reduces to the surjectivity of the de Rham period map from the Lubin-Tate space to $\mathbb{P}^{n-1}$ (due to Gross-Hopkins). Part (b) reduces to that any point in Drinfeld's half space $\Omega \subseteq \mathbb{P}^{n-1}$ comes from the Hodge-Tate period of the dual of a Lubin-Tate formal group, which in turn reduces to that the image of the de Rham period of the Rapoport-Zink space of special formal $\mathcal{O}_B$-modules of height n^2 is exactly Ω (due to Drinfeld).

Our remaining goal is show that Theorem 2 is equivalent to Theorem 3, in a spirit similar to Grothendieck's proof for $\mathbb{P}^1$. One direction is easy to verify.

Proof (Theorem 2 $\Rightarrow$ 3)

a. Since $\mathcal{F}$ has degree 1 and rank 0, we know that $\mathcal{E}$ has degree 0 and rank F. Suppose $\mathcal{E} = \oplus \mathcal{O}(\lambda_i)$, then $\lambda_i \le 1/n$ by the second property. Since $\operatorname{rank} \mathcal{O}(\lambda_i) \le n$, we know that $\lambda_i = 0$ or $1/n$. But $\deg \mathcal{E} = 0$, so $\lambda_i = 0$.

b. Similarly, we have $\deg \mathcal{E} = 1$ and $\operatorname{rk} \mathcal{E} = n$. Suppose $\mathcal{E} = \oplus \mathcal{O}(\lambda_i)$, then $\lambda_i \ge 0$ by the second property. Therefore one $\lambda_i = 1/r$ and the others are all 0. □

The other direction is harder. We reduce to the following lemma.

Lemma 1 Theorem 2 is equivalent to the following statement: for any $n \ge 1$, if
$$0 \to \mathcal{O}(-1/n) \to \mathcal{E} \to \mathcal{O}(1) \to 0$$
is exact, then $H^0(X, \mathcal{E}) \ne 0$.

Proof One direction is clear since if $\deg \mathcal{E} = 0$ and $\mathcal{E} = \oplus \mathcal{O}(\lambda_i)$, then some $\lambda_i \ge 0$ and hence $H^0(X, \mathcal{E}) \ne 0$ by second property.

For the other direction, we need to show that every semi-stable vector bundle $\mathcal{E}$ is a direct sum of $\mathcal{O}(\lambda)$. It turns out that $\mathcal{E}/X$ is semi-stable stable if and only if $\pi_n^* \mathcal{E}/X_n$ is semi-stable and $\mathcal{E}$ is such a direct sum of $\mathcal{O}(\lambda)$ if and only if $\pi_n^* \mathcal{E}$ is a direct sum $\mathcal{O}(n\lambda)$. So we may assume that $\lambda \in \mathbb{Z}$ by pulling back along π_n. Twisting by the line bundle $\mathcal{O}(\lambda)$ we may assume $\deg \mathcal{E} = 0$. We need to show that $\mathcal{E} = \mathcal{O}^{n+1}$.

Let us only consider the case $\operatorname{rk} \mathcal{E} = 2$ (i.e., $n = 1$). Let $\mathcal{L} \subseteq \mathcal{E}$ be the sub line bundle of maximal degree. It has degree ≤ 0 since $\mathcal{E}$ is semi-stable of degree 0. Write $\mathcal{L} = \mathcal{O}(-d)$. We know that
$$0 \to \mathcal{O}(-d) \to \mathcal{E} \to \mathcal{O}(d) \to 0.$$
If $d = 0$, then $\mathcal{E} = \mathcal{O}^2$ by the third property. If $d = -1$, then by assumption that $H^0(X, \mathcal{E}) \ne 0$, hence there is an injection $\mathcal{O}_X \to \mathcal{E}$, which contradicts the maximality of $\deg \mathcal{L}$. More generally, if $d \ge 1$, then

$-d+2 \leq d$. There is an injection $\mathcal{O}(-d+2) \to \mathcal{O}(d)$. Pullback the exact sequence we obtain a new exact sequence

$$0 \to \mathcal{O}(-d) \to \mathcal{E}' \to \mathcal{O}(-d+2) \to 0.$$

Hence by assumption $H^0(X, \mathcal{E}'(d-1)) \neq 0$, which gives an injection $\mathcal{O} \to \mathcal{E}'(d-1) \to \mathcal{E}(d-1)$, i.e. $\mathcal{O}(1-d) \to \mathcal{E}$, which contradicts the maximality of $\deg \mathcal{L}$. □

Remark 10 The general case is done similarly by induction on the rank, which starts off by finding the maximal degree sub line bundle of $\pi_n^* \mathcal{E}$ in order to use the hypothesis when $d = -1$.

Now it remains to prove that Theorem 3 implies the statement in Lemma 1.

Proof We still focus on the rank 2 case. Suppose

$$0 \to \mathcal{O}(-1) \to \mathcal{E} \to \mathcal{O}(1) \to 0.$$

We need to show that $H^0(X, \mathcal{E}) \neq 0$. Choose an injection $\mathcal{O}(-1) \to \mathcal{O}(1)$. Then pushing out gives

$$0 \to O(1) \to \mathcal{E}' \to \mathcal{O}(1) \to 0.$$

Hence $\mathcal{E}' \cong \mathcal{O}(1)^2$ by the third property and we have

$$0 \to \mathcal{E} \to \mathcal{O}(1)^2 \to \mathcal{F} \to 0.$$

Here $\mathcal{F}$ is degree 2 torsion sheaf (cokernel of $\mathcal{O}(-1) \to \mathcal{O}(1)$). Choose a degree 1 subsheaf $\mathcal{F}' \subseteq \mathcal{F}$ and pullback we obtain a degree one modification

$$0 \to \mathcal{E} \to \mathcal{E}'' \to \mathcal{F}' \to 0.$$

Hence

$$0 \to \mathcal{E}'' \to \mathcal{O}(1)^2 \to \mathcal{F}/\mathcal{F}' \to 0.$$

Taking dual $\mathcal{H}om(-, \mathcal{O})$ and twist by $\mathcal{O}(1)$ we obtain another degree one modification

$$0 \to \mathcal{O}^2 \to \mathcal{E}''^{\vee}(1) \to (\mathcal{F}/\mathcal{F}')^{\vee} \to 0.$$

By Theorem 3 (b) we know that either $\mathcal{E}'' = \mathcal{O} \oplus \mathcal{O}(1)$ or $\mathcal{O}(1/2)$. In either case:

a. We have

$$0 \to \mathcal{E} \to \mathcal{O} \oplus \mathcal{O}(1) \to \mathcal{F}' \to 0.$$

Notice $\ker(\mathcal{O}(1) \to \mathcal{F}') \subseteq \mathcal{E}$ is either $\mathcal{O}(1)$ or $\mathcal{O}$, so $H^0(X, \mathcal{E}) \neq 0$.

b. We have

$$0 \to \mathcal{E} \to \mathcal{O}(1/2) \to \mathcal{F}' \to 0.$$

By Theorem 3 (1) we know that $\mathcal{E} \cong \mathcal{O}^2$, hence $H^0(X, \mathcal{E}) \neq 0$. □

Remark 11 The general case is done similarly, which starts off by choosing an injection $\mathcal{O}(-1/n) \to \mathcal{O}(1)^n$ to get a degree $n+1$ modification

$$0 \to \mathcal{E} \to \mathcal{O}(1)^{n+1} \to \mathcal{F} \to 0.$$

Then one write this as a sequence of degree one modifications and use Theorem 3 (a) or (b) at each step.