1. Rank one

We are going to switch freely between the holomorphic and algebraic perspective. This is justified by the following wonderful:

Theorem 1.1 (GAGA)**.** *Fix a projective variety X over* $\operatorname{Spec}\mathbb{C}$.

Then there is an equivalence of categories between the category of coherent sheaves on X and the category of coherent analytic sheaves on the underlying complex analytic space X^{an}.

Note that if the sheaves are the same then so is the cohomology. In particular global sections are the same. Even in the case of $\mathbb{P}^1$ and the structure sheaf this is quite striking. There are many more holomorphic functions on $\mathbb{C}$ than polynomial functions but every meromorphic function on $\mathbb{P}^1$ is given by a rational function.

Definition 1.2. *A* ***holomorphic vector bundle*** *E on a complex projective variety X is a complex manifold together with a holomorphic map $\pi\colon E \longrightarrow X$ and an open cover $\{ U_\alpha \}$ of X such that*

$$E|_{U_\alpha} = \pi^{-1}(U_\alpha)$$

is isomorphic to the product $U_\alpha \times \mathbb{C}^r$ over U_α,

$$\begin{tikzcd} E|_{U_\alpha} \arrow[rr] \arrow[dr] & & U_\alpha \times \mathbb{C}^r \arrow[dl] \\ & U_\alpha, & \end{tikzcd}$$

such that on the overlap

$$U_{\alpha\beta} = U_\alpha \cap U_\beta,$$

the transition functions are linear functions on $\mathbb{C}^r$.

The ***rank*** *of E is r.*

In fact, vector bundles make sense in almost any geometric context. One can perform any operation on vector bundles, that makes sense for vector spaces. In particular, we can take the direct sum of two vector bundles, tensor product, Hom, dual, etc.

Given a holomorphic vector bundle E, we get a sheaf of sections,

Definition 1.3. *If E is a holomorphic vector bundle on a projective variety X then the associated* ***sheaf of sections*** *is the sheaf $\mathcal{O}_X(E)$ which assigns to the open subset $U \subset X$ the set of all holomorphic sections,*

$$\sigma\colon U \longrightarrow E$$

Note that the sheaf of sections is a locally free sheaf of rank r, the rank of E. Indeed, E is trivial over U then the sheaf of sections is a direct sum of r copies of $\mathcal{O}_U$.

There is an equivalence of categories between the category of holomorphic vector bundles on X and the category of locally free holomorphic sheaves on X. This equivalence respects the basic operations (direct sum, etc). The key point is that a vector bundle and a sheaf are both determined by a cover and the (same) transition functions.

By GAGA the holomorphic sheaf $\mathcal{O}_X(U)$ corresponds to a locally free algebraic sheaf (which, by abuse of notation, we will use the same symbol). Putting all this together, classifying holomorphic vector bundles on X is the same as classifying locally free coherent sheaves on X.

We recall the classification of line bundles on X, that is, rank one vector bundles. Suppose that L is a line bundle on X. By assumption we may find a cover $\{ U_\alpha \}$ of X such that $L_\alpha \simeq U_\alpha \times \mathbb{C}$. On overlaps we get a linear transformation of one dimensional vector spaces, that is, a nowhere zero holomorphic function, a one by one invertible matrix,

$$f_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow \mathbb{C}^*.$$

On triple overlaps we have the following compatibility,

$$f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1.$$

By convention $f_{\alpha\alpha} = 1$ and $f_{\beta\alpha} = f_{\alpha\beta}^{-1}$.

In this way we get a 1-cocycle, with values in the sheaf of nowhere zero $\mathcal{O}_X^*$ holomorphic functions. Vice-versa, given a 1-cocycle $\sigma \in H^1(X, \mathcal{O}_X^*)$, by definition we are given an open cover $\{ U_\alpha \}$ of X and nohwere zero holomorphic functions

$$f_{\alpha\beta} \colon U_{\alpha\beta} \longrightarrow \mathbb{C}^*.$$

subject to the rule

$$f_{\alpha\beta} f_{\beta\gamma} f_{\gamma\alpha} = 1.$$

Using this data, one can construct a holomorphic vector bundle with the given transition functions. (One can also use GAGA and construct the associated rank one locally free sheaf on the variety X).

On the other hand, one can take two line bundles and take the tensor product to get another line bundle. At the level of transition functions, one is just multiplying the transition functions. The trivial line bundle $X \times \mathbb{C}$, corresponding to the trivial sheaf $\mathcal{O}_X$, acts as the identity. The dual line bundle, $\operatorname{Hom}(L, X \times \mathbb{C})$ acts as the inverse; it is the line bundle with transition functions the reciprocal of the transition functions of L.

Definition 1.4. *Let X be a projective variety. The group of line bundles on X is called the* ***Picard group*** *and is denoted* $\mathrm{Pic}(X)$.

Theorem 1.5. *If X is a projective variety then*

$$\mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*).$$

One advantage of working over $\mathbb{C}$ is that there are more exact sequences. The exponential sequence is the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^*. \longrightarrow 0.$$

The map from the sheaf of holomorphic functions to the sheaf of nowhere holomorphic functions is the exponential,

$$f \longrightarrow \exp(2\pi i f).$$

The kernel is the locally constant sheaf $\mathbb{Z}$, the sheaf of integer valued holomorphic functions. As usual, a sequence of sheaves is exact if it is exact on stalks. Thus the exponential is surjective, as locally we can take logs.

If we take the long exact sequence of cohomology we get

$$H^1(X, \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}).$$

Note that the sheaf cohomology groups

$$H^i(X, \mathbb{Z})$$

compute the usual topological cohomology. In particular these groups are finitely generated abelian groups. We have already observed that

$$H^1(X, \mathcal{O}_X^*) \simeq \mathrm{Pic}(X).$$

On the other hand,

$$H^1(X, \mathcal{O}_X)$$

is a finite dimensional vector space. The map

$$c_1 \colon \mathrm{Pic}(X) \longrightarrow H^2(X, \mathbb{Z}),$$

is called the first chern class. It is a group homomorphism which assigns to every line bundle a cohomology class. If the line bundle L has a global section,

$$\sigma \in H^0(X, \mathcal{O}_X(L)),$$

we can assign the divisor D of zeroes of σ. Locally, just trivialise L and take the divisor of zeroes of the corresponding holomorphic function. On overlaps, the transition functions are nowhere zero holomorphic functions, so that even if we get different holomorphic functions, we get the same divisor of zeroes. In this case, the first chern class is the cocycle $[D]$ associated to the divisor D. In general, if H is an ample divisor then $L(kH)$ has global sections for k large enough. The first chern class of L is then the difference of $c_1(L(kH)$ and $[kH] = k[H]$, by linearity. Equivalently, every line bundle on a projective variety has a rational section, and the first chern class is the topological class of the divisor of zeroes minus poles of this rational section.

The kernel of the first chern class is the image of the vector space $H^1(X, \mathcal{O}_X)$. By Hodge theory, the free part of the abelian group $H^1(X, \mathbb{Z})$ is embedded as a lattice in $H^1(X, \mathcal{O}_X)$, and the quotient is an abelian variety, a projective algebraic group.

In fact, Grothendieck gave $\mathrm{Pic}(X)$ the structure of a topological group. The quotient

$$\frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})}$$

is isomorphic to $\mathrm{Pic}^0(X)$, the connected component of the identity. The quotient

$$\frac{\mathrm{Pic}(X)}{\mathrm{Pic}^0(X)}$$

is the group of connected components of $\mathrm{Pic}(X)$ and it is embedded in $H^2(X, \mathbb{Z})$ by the first chern class.

To every Cartier divisor D, we can associate a line bundle $\mathcal{O}_X(D)$. In fact the data of a Cartier divisor gives rise a to 1-cocycle, which in turn gives rise to a line bundle. If $D \geq 0$ then $\mathcal{O}_X(D)$ comes with a section whose zero locus is precisely D. Two divisors D_1 and D_2 have isomorphic line bundles if and only if $D_1 \sim D_2$ are linearly equivalent. Thus the group of line bundles is isomorphic to the group of Cartier divisors modulo linear equivalence.

Theorem 1.6. $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$.

Proof. The sheaf cohomology group

$$H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0.$$

On the other hand,

$$H^2(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z},$$

generated by the class of a hyperplane. The sheaf locally free sheaf of rank one, $\mathcal{O}_{\mathbb{P}^n}(1)$ has a section with zero locus a hyperplane. Thus the first chern class map is an isomorphism. $\square$

2. The Tangent bundle and projective bundle

Let us give the first non-trivial example of a vector bundle on $\mathbb{P}^n$. Recall that given any smooth projective variety one can construct the tangent bundle. Geometrically a tangent vector at $x \in X$ is an equivalence class of paths,

$$\gamma\colon (-\epsilon, \epsilon) \longrightarrow X$$

such that $\gamma(0) = x$. Two paths are considered equivalent if they have the same first derivative (one can make sense of this in a way which is not circular). The set of all tangent vectors based at x is a vector space of dimension n, $T_x X$. The tangent TX bundle is the set of all tangent vectors. There is an obvious projection down to X, $\pi\colon TX \longrightarrow X$. The fibre over a point is the tangent bundle. Since X is locally isomorphic to an open subset of $\mathbb{R}^n$ and the tangent bundle of $\mathbb{R}^n$ is a product, it is clear that the tangent bundle is locally a product. The transition functions are given by the Jacobian of the coordinate change. Thus the tangent bundle is a bundle.

The algebraic approach to the construction of the tangent bundle proceeds from a different direction. If X is a variety and $x \in X$ is a point, with local ring $\mathcal{O}_{X,x}$, then the Zariski tangent space is the dual of

$$T_x X = \frac{\mathfrak{m}}{\mathfrak{m}^2}^*,$$

where $\mathfrak{m}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}$. In terms of schemes, one can look at the set of maps

$$\operatorname{Hom}(\operatorname{Spec} \frac{\mathbb{C}[\epsilon]}{\langle \epsilon^2 \rangle}, X),$$

where the unique point of

$$\operatorname{Spec} \frac{\mathbb{C}[\epsilon]}{\langle \epsilon^2 \rangle}$$

is sent to x. This is again the dual of the Zariski tangent space. If X is affine, one can construct the sheaf of differentials. One can globalise this construction in a somewhat bizarre way. Let

$$\Delta\colon X \longrightarrow X \times X,$$

be the diagonal morphism. Let $\mathcal{I}$ be the ideal sheaf of the diagonal sitting inside the product. Then

$$\frac{\mathcal{I}}{\mathcal{I}^2},$$

is naturally supported on the diagonal, a copy of X. In fact if X is smooth this sheaf is locally free and the pullback to X is the cotangent sheaf Ω^1_X, the sheaf of sections of the cotangent bundle.

There is a standard way to construct the tangent and cotangent bundles on projective space. Recall that as a manifold, $\mathbb{P}^n$ is the set of lines in an $(n+1)$ dimensional vector space $V \simeq \mathbb{C}^{n+1}$. Almost by definition there is a universal sublinebundle

$$S \subset V \times \mathbb{P}^n$$

There is a quotient vector bundle Q, so that we get an exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow \mathbb{P}^n \times V \longrightarrow Q \longrightarrow 0.$$

A morphism to projective space is given by a line bundle and a choice of $n+1$ sections which don't vanish simultaneously (the universal property of projective space). In this case the line bundle is the pullback of S^*. So a morphism of

$$\operatorname{Spec} \frac{\mathbb{C}[\epsilon]}{\langle \epsilon^2 \rangle}$$

is given by a choice of deformation of the line bundle S. But if you deform in the direction of S nothing happens. So the Zariski tangent space to $\mathbb{P}^n$ is

$$\operatorname{Hom}(S, Q) \simeq Q \otimes S^*.$$

The exact sequence

$$0 \longrightarrow S \longrightarrow \mathbb{P}^n \times V \longrightarrow Q \longrightarrow 0.$$

is the **Euler sequence**. At the level of sheaves we have

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow T_X(-1) \longrightarrow 0.$$

If we tensor by $\mathcal{O}_{\mathbb{P}^n}$ we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \longrightarrow T_X \longrightarrow 0.$$

This second map sends $(l_0, l_1, \dots, l_n)$ to

$$\sum l_i \frac{\partial}{\partial x_i}.$$

The kernel is $(x_0, x_1, \dots, x_n)$ as

$$\sum x_i \frac{\partial}{\partial x_i}$$

is radial.

If E is a vector bundle of rank r we can associate a projective bundle over X.

Definition 2.1. *A **projective bundle** Y on a complex projective variety X is a projective variety together with a holomorphic map $\pi\colon Y \longrightarrow X$ and an open cover $\{ U_\alpha \}$ of X such that*

$$Y|_{U_\alpha} = \pi^{-1}(U_\alpha)$$

is isomorphic to the product $U_\alpha \times \mathbb{P}^r$ over U_α,

$$\begin{tikzcd} E|_{U_\alpha} \arrow[rr] \arrow[dr] & & U_\alpha \times \mathbb{P}^r \arrow[dl] \\ & U_\alpha, & \end{tikzcd}$$

such that on the overlap

$$U_{\alpha\beta} = U_\alpha \cap U_\beta,$$

the transition functions are linear functions on $\mathbb{P}^r$.

If E is a vector bundle then one can construct the associated projective bundle, $\mathbb{P}(E)$. By definition of E, we can find an open cover $\{ U_\alpha \}$ of X such that $E_\alpha \simeq X \times \mathbb{C}^r$. For the associated projective bundle, $Y = \mathbb{P}(E)$, let $Y_\alpha \simeq X \times \mathbb{P}^{r-1}$. As the transition functions of E are given by linear functions then so are the transition functions for Y. Thus Y is a projective bundle.

One can also make this construction algebraically. Y comes with a locally free sheaf $\mathcal{O}_Y(1)$ of rank one. Fibre by fibre it restricts to the sheaf $\mathcal{O}_{\mathbb{P}^{r-1}}(1)$. Note that two vector bundles E_1 and E_2 give rise to isomorphic projective bundles Y_1 and Y_2 if and only if there is a line bundle L such that $E_1 = L_2 \underset{\mathcal{O}_X}{\otimes} E_2$. In fact one direction is clear, since tensoring by a line bundle won't change the fibres of the projective bundle, the transition functions of Y_1 and Y_2 are the same. Thus Y_1 and Y_2 are isomorphic. Note however that the tautological rank one sheaves differ,

$$\mathcal{O}_{Y_2}(1) = \mathcal{O}_{Y_1}(1) \underset{\mathcal{O}_{Y_1}}{\otimes} \pi^* L.$$

In general, a projective bundle Y over X won't come from a vector bundle. It will come from a vector bundle if the open cover trivialising Y over X are Zariski open subsets and X is smooth. In this case, there is a divisor D on Y, which restricts to the general fibre of π as a hyperplane. Just take the closure of the inverse image of $U_\alpha \times H$, where H is a hyperplane in $\mathbb{P}^{r-1}$. Consider the associated rank one locally free sheaf $\mathcal{O}_Y(D)$. Standard results imply that

$$\mathcal{E} = \pi_*(\mathcal{O}_Y(D)),$$

is a locally free sheaf of rank r.

However there are examples of projective bundles which are trivial in the Euclidean topology which don't come from vector bundles. Consider the exact sequence of algebraic groups,

$$0 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{GL}(r) \longrightarrow \mathrm{PGL}(r) \longrightarrow 0.$$

One can sheafify this sequence to get

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathrm{GL}(r) \longrightarrow \mathrm{PGL}(r) \longrightarrow 0.$$

Taking the long exact sequence of cohomology we get

$$H^1(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \mathrm{GL}(r)) \longrightarrow H^1(X, \mathrm{PGL}(r)) \longrightarrow H^2(X, \mathcal{O}_X^*).$$

Note that is does make sense to take cohomology of a sheaf of non-abelian groups. Note however that higher cohomology is no longer a group, just a pointed set. The cohomology set

$$H^1(X, \mathrm{GL}(r))$$

classifies vector bundles of rank r. The cohomology set

$$H^1(X, \mathrm{PGL}(r))$$

classifies projective bundles of rank $r-1$. The map between them is the natural map which assigns to a vector bundle the associated projective bundle. The kernel of this map is

$$H^1(X, \mathcal{O}_X^*)$$

which as we have already seen classifies line bundles on X. However the last map

$$H^1(X, \mathrm{PGL}(r)).H^2(X, \mathcal{O}_X^*).$$

is not always zero. The image is the Brauer group; it classifies projective bundles over X which are not Zariski trivial.

There is a fun example of a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. Let

$$Y = \{\, (x, L) \mid x \in L \,\} \subset \mathbb{P}^2 \times \mathbb{P}^2$$

be the incidence correspondence between points and lines on $\mathbb{P}^2$. In coordinates $[x : y : z]$ on the first $\mathbb{P}^2$ and $[a : b : c]$ on the second $\mathbb{P}^2$, Y is given by the bihomogeneous equation

$$ax + by + cz = 0.$$

Consider projection π of Y down to the first $\mathbb{P}^2$. The fibre over a point $[x : y : z]$ is the set of all lines this point. Fix the point $p = [0 : 0 : 1]$. The set of lines through p is given by $c = 0$, so that we get the line $[a : b : 0] \subset \mathbb{P}^2$. Thus the fibres of π are copies of $\mathbb{P}^1$. Now suppose we look at the affine open subset $z \neq 0$ of $\mathbb{P}^2$.

We can use point-slope to see that Y is trivial over $U = \mathbb{A}^2 = (z \neq 0)$. More geometrically, a line through the point $[x : y : z]$ will meet the line L_2, given by $z = 0$, at a unique point. Since a line is specified by two points, it is easy to see that Y is isomorphic to $U \times L_2 \simeq U \times \mathbb{P}^1$.

3. Chern classes

We have already seen that the first chern class gives a powerful way to connect line bundles, sections of line bundles and divisors. We want to generalise this to higher rank.

Given any vector bundle we can define higher chern classes. There are many ways to view chern classes, all of which are useful. We present two ways to look at them.

The first is topological. One can view chern classes as (partial) obstructions to the vector bundle being trivial. The first case is a line bundle. If a line bundle is trivial, that is, isomorphic to a product, then we can find a global non-vanishing section of the line bundle. One direction is clear, the trivial line bundle has the section $(x, 1)$, which is nowhere vanishing. On the other hand, if σ is a non-vanishing section of L then define a map

$$X \times \mathbb{C} \longrightarrow L$$

by sending (x, λ) to $\lambda\sigma(x)$.

Note that there are two different ways in which a vector bundle might be trivial. It might be topological trivial, that is, the isomorphism is only a continuous map. Or the isomorphism might be holomorphic. This reflects the two different types of first chern class, the topological first chern class and the more refined first chern class, which takes values in the space of Cartier divisors modulo linear equivalence. Either way, the first chern is defined by taking the equivalence class of the zero locus of a section. The fact that this equivalence class is non-zero means we cannot alter the section and make it nowhere vanishing, so that we get an obstruction to triviality.

Now suppose we have a vector bundle E of higher rank r. There is more than way to find obstructions to trivialising the bundle. Consider the problem of finding a nowhere zero section. We expect a section of a vector bundle of rank r to vanish in codimension r. Indeed, locally the vector bundle is trivial and a section of a vector bundle of rank r is a tuple of r holomorphic functions, which we expect to have a common zero in codimension r.

If L is an ample line bundle then results of Serre imply that $E \otimes L^k$ is globally generated for k sufficiently large and we can always find a section which vanishes in codimension r. We can then use linearity (more about this later) to define the rth chern class $c_r(E)$ of E. Topologically it takes values in $H^{2r}(X, \mathbb{Z})$ and there is a more refined version which takes values in the space of codimension r cycles, modulo rational equivalence.

At the other extreme, suppose the vector bundle were trivial. Then we could find r sections which fibre by fibre are a basis for each fibre. These r sections would then define a non-vanishing section of the highest wedge of E,

$$L = \bigwedge^r E.$$

Note that L is a line bundle, known as the determinant line bundle and we are simply asking if we can find a non-vanishing section of L, that is, we are asking if L is the trivial vector bundle. We define the first chern of E as the first chern class of L,

$$c_1(E) = c_1(\bigwedge^r E).$$

More generally still, the kth chern of a vector bundle E is a measure of how hard it is to find $k - r$ independent sections. The kth chern class of a vector bundle of rank r is a cycle that lives in either $H^{2k}(X, \mathbb{Z})$, of the space of codimension r cycles modulo rational equivalence.

To proceed further, it is convenient to introduce the second way to look at chern classes. This takes a more algebraic approach. We first bundle all of the chern classes together to get the total chern class

$$c(E) = c_0(E) + c_1(E) + c_2(E) + \cdots + c_r(E).$$

Grothendieck observed that the total chern class is unique, given the following axioms:

(1) $c_0(E) = 1$, the class of X.

(2) $c_1(\mathcal{O}_X(D)) = [D]$.

(3) If $f\colon Y \longrightarrow X$ is a morphism then

$$f^*(c(E)) = c(f^*E).$$

(4) If

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

is a short exact sequence of locally free sheaves then

$$c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G}).$$

As a baby case of (4), note that if E_1 and E_2 are vector bundles then

$$c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2).$$

In fact, here is how to define the chern classes, using these properties. Given the vector bundle E, let $Y = \mathbb{P}(E)$ be the associated projective bundle. Fibre by fibre, $\pi\colon Y \longrightarrow X$ is a family of projective spaces $\mathbb{P}^{r-1}$. The cohomology of $\mathbb{P}^{r-1}$ is

$$\frac{\mathbb{Z}[x]}{\langle x^r \rangle},$$

where x is in degree 2, the class of a hyperplane. The universal line bundle $\mathcal{O}_Y(1)$ restricts to a line bundle whose first chern class is x. So the first chern class ξ of $\mathcal{O}_Y(1)$ restricts to the generator x on each fibre. Consider the first $r+1$ powers of ξ. Some linear combination of these sums to zero in the cohomology of Y,

$$\xi^r - c_1\xi^{r-1} + c_2\xi^{r-2} - \cdots + (-1)^r c_r(E) = 0.$$

Let's compute the chern classes of the tangent bundle. We have the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1}(1) \longrightarrow T_X \longrightarrow 0.$$

It follows that

$$c(\mathcal{O}_{\mathbb{P}^n})c(T_X) = c(\mathcal{O}_{\mathbb{P}^n}^{n+1}(1)).$$

Now the total chern class of a trivial line bundle is trivial

$$c(\mathcal{O}_{\mathbb{P}^n}) = 1 \qquad \text{and} \qquad c(\mathcal{O}_{\mathbb{P}^n}(1)) = 1 + H,$$

where H is the class of a hyperplane. Thus

$$\begin{aligned} c(T_X) &= c(\mathcal{O}_{\mathbb{P}^n}^{n+1}(1)) \\ &= \prod_{i=0}^{n}(1+H) \\ &= (1+H)^{n+1} \\ &= 1 + (n+1)H + \frac{(n+1)n}{2}H^2 + \dots. \end{aligned}$$

Consider what happens on $\mathbb{P}^2$. The tangent bundle has rank two. Its total chern class is

$$1 + 3H + 3H^2.$$

(Note that our computation of the second chern class is consistent with Gauss-Bonnett, since the topological Euler characteristic is indeed $3 = 1 + 1 + 1$).

If it were isomorphic to a direct sum then its chern classes would be

$$\begin{aligned} c(\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)) &= c(\mathcal{O}_{\mathbb{P}^2}(a))c(\mathcal{O}_{\mathbb{P}^2}(b)) \\ &= (1+aH)(1+bH) \\ &= 1 + (a+b)H + abH^2. \end{aligned}$$

Thus $a + b = 3$ and $ab = 3$. But this is not possible for integers. Thus the tangent bundle does not split.

The chern classes of a vector bundle provide a useful way to chart out the territory of all vector bundles.

Splitting principle One can use (4) to compute chern classes in many situations. If we want to compute some chern classes, in most cases we can pullback to a situation where the vector bundle splits and the pullback map is injective. Thus in many cases we can compute as though the vector bundles splits.

One can use the splitting principle to compute the chern classes of tensor products.

Question 3.1. *What are the chern classes of the tensor product of a vector bundle and a line bundle?*

Suppose the vector bundle is E and the line bundle is L. We want compute to

$$c(E \otimes L).$$

We use the splitting principle. Assume that

$$E = L_1 \oplus L_2 \oplus L_3 \oplus \cdots \oplus L_r.$$

Then

$$\begin{aligned} c(E) &= c(L_1 \oplus L_2 \oplus L_3 \oplus \cdots \oplus L_r) \\ &= \prod_{i=1}^{r} c(L_i) \\ &= \prod_{i=1}^{r} (1 + \alpha_i), \end{aligned}$$

where $\alpha_i = c_1(L_i)$. Suppose that

$$c_1(L) = \beta.$$

Then

$$\begin{aligned} c(E \otimes L) &= c((L_1 \oplus L_2 \oplus L_3 \oplus \cdots \oplus L_r) \otimes L) \\ &= c((L_1 \otimes L) \oplus (L_2 \otimes L) \oplus (L_3 \otimes L) \oplus \cdots \oplus (L_r \otimes L)) \\ &= \prod_{i=1}^{r} c(L_i \otimes L) \\ &= \prod_{i=1}^{r} (1 + \alpha_i + \beta) \\ &= 1 + (\sum \alpha_i + r\beta) + (\sum_{i \neq j} \alpha_i \alpha_j + (r-1)\beta(\sum_i \alpha_i) + \binom{r}{2}\beta^2) + \dots \\ &= 1 + c_1(E) + rc_1(L) + c_2(E) + (r-1)c_1(E)c_1(L) + \binom{r}{2} c_1^2(L) + \dots \end{aligned}$$

Since the formula for the tensor product is actually quite involved, it is natural to exponentiate to get a simpler formula. Formally, if

$$c(E) = \prod_{i=1}^{r}(1 + \alpha_i),$$

so that the chern classes of E are the symmetric functions in $\alpha_1, \alpha_2, \dots, \alpha_r$ then the chern character is

$$\operatorname{ch}(E) = \sum_{i=1}^{r} e^{\alpha_i},$$

where we use the usual formula for the exponential. Note that then chern character is additive on exact sequences and multiplicative on tensor products.

$$\begin{aligned} \operatorname{ch}(E_1 \oplus E_2) &= \operatorname{ch}(E_1) + \operatorname{ch}(E_2) \\ \operatorname{ch}(E_1 \otimes E_2) &= \operatorname{ch}(E_1)\operatorname{ch}(E_2). \end{aligned}$$

The first few terms of the chern character are

$$\operatorname{ch}(E) = r + c_1(E) + \frac{1}{2}(c_1^2 - c_2) + \dots$$

4. Splitting type

We start with a result due to Grothendieck:

Theorem 4.1 (Grothendieck)**.** *Every vector bundle E on $\mathbb{P}^1$ splits as a direct sum of line bundles,*

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}(E) \simeq \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i).$$

If we impose the condition $a_i \geq a_{i+1}$ then the integers $a_1, a_2, \dots, a_r$ are unique.

Proof. We proceed by induction on the rank r of the vector bundle $\mathcal{E}$ on $\mathbb{P}^1$.

We may suppose that $r > 1$ and that the result is true for all smaller values of r. Let

$$d = \inf\{\, k \in \mathbb{Z} \mid H^0(\mathbb{P}^1, \mathcal{E}(k)) \neq 0 \,\}.$$

Note that if k is sufficiently large then $\mathcal{E}(k)$ is globally generated. In particular $d < \infty$. Note also that if $H^0(\mathbb{P}^1, \mathcal{E}(k)) \neq 0$ then there is an injective map of sheaves

$$\mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E}(k)$$

so that

$$\mathcal{O}_{\mathbb{P}^1}(-k) \longrightarrow \mathcal{E}.$$

In particular

$$h^0(\mathbb{P}^1, \mathcal{E}) \geq h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-k)).$$

If $k < 0$ then the LHS is fixed and the RHS goes to ∞ as k gets smaller. Thus $d > -\infty$ and the infimum is a minimum. Note that $\mathcal{E}$ splits if and only if $\mathcal{E}(d)$ splits, so that, replacing $\mathcal{E}$ by $\mathcal{E}(d)$ there is no harm in assuming that $d = 0$. In this case we have

$$H^0(\mathbb{P}^1, \mathcal{E}(-1)) = 0.$$

Pick $\sigma \in H^0(\mathbb{P}^1, \mathcal{E})$. Locally $\mathcal{E}$ is trivial and σ is an r-tuple of holomorphic functions on an open neighbourhood of $0 \in \mathbb{C}$. If σ vanishes at the origin then

$$\sigma(z) = (zf_1(z), zf_2(z), \dots, zf_r(z)).$$

In this case σ defines a section of $\mathcal{I}_p \underset{\mathcal{O}_{\mathbb{P}^1}}{\otimes} \mathcal{E}$. But

$$\mathcal{I}_p \underset{\mathcal{O}_{\mathbb{P}^1}}{\otimes} \mathcal{E} \simeq \mathcal{E}(-1),$$

and so we get a non-zero function

$$\tau \in H^0(\mathbb{P}^1, \mathcal{E}(-1)),$$

which contradicts our choice of d. Thus locally

$$\sigma(z) = (f_1(z), f_2(z), \dots, f_r(z)),$$

where at least one $f_i(z)$ does not vanish at 0. Thus σ defines an injection

$$\mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E}$$

and the quotient is a locally free sheaf $\mathcal{F}$ of rank $r-1$. Thus we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

Twist this exact sequence by $\mathcal{O}_{\mathbb{P}^1}(-1)$ to get the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{E}(-1) \longrightarrow \mathcal{F}(-1) \longrightarrow 0.$$

Consider the long exact sequence of cohomology. The relevant piece is

$$H^0(\mathbb{P}^1, \mathcal{E}(-1)) \longrightarrow H^0(\mathbb{P}^1, \mathcal{F}(-1)) \longrightarrow H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)).$$

Note that

$$\begin{aligned} h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) &= h^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(1)) \\ &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1-2)) \\ &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \\ &= 0, \end{aligned}$$

by Serre duality. On the other hand

$$h^0(\mathbb{P}^1, \mathcal{E}(-1)) = 0$$

so that

$$h^0(\mathbb{P}^1, \mathcal{F}(-1)) = 0.$$

Now $\mathcal{F}$ is locally free of rank $r-1$. By induction $\mathcal{F}$ is isomorphic to a direct sum of locally free sheaves of rank one,

$$\mathcal{F} \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(a_i),$$

for some integers a_i. Thus

$$\mathcal{F}(-1) \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(a_i - 1).$$

It follows that $a_i - 1 < 0$ for all i, so that $a_i \leq 0$.

Now take the dual of the first exact sequence. We get a short exact sequence of the dual locally free sheaves,

$$0 \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0.$$

I claim that this sequence splits. We have to map the last sheaf back into $\mathcal{E}^*$. It suffices to show that the global section 1 is in the image of the last map; in this case we just send 1 to anything mapping to 1.

Consider the long exact sequence of cohomology. The relevant piece is

$$H^0(\mathbb{P}^1, \mathcal{E}^*) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \longrightarrow H^1(\mathbb{P}^1, \mathcal{F}^*).$$

We have

$$\mathcal{F}^* \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}_{\mathbb{P}^1}(-a_i).$$

By Serre duality,

$$\begin{aligned} h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-a_i)) &= h^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}(a_i)) = 0 \\ &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a_i - 2)) \\ &= 0, \end{aligned}$$

for all i, as $a_i \leq 0$. Thus

$$H^1(\mathbb{P}^1, \mathcal{F}^*) = 0,$$

and so

$$H^0(\mathbb{P}^1, \mathcal{E}^*) \longrightarrow H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}),$$

is surjective. It follows that the short exact sequence

$$0 \longrightarrow \mathcal{F}^* \longrightarrow \mathcal{E}^* \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0,$$

splits. As $\mathcal{F}^*$ splits it follows that $\mathcal{E}^*$ splits and so $\mathcal{E}$ splits. This completes the induction and the proof of the existence of a splitting.

Now suppose that

$$\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i) \simeq \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(b_i),$$

where $a_i \geq a_{i+1}$ and $b_i \geq b_{i+1}$. Suppose that the two sequences are different. Let j be the first index such that $a_j \neq b_j$, so that $a_1 = b_1$, $a_2 = b_2$, ..., $a_{j-1} = b_{j-1}$.

Possibly switching the LHS and the RHS, we may assume that $a_j > b_j$. If we tensor both sides with $-a_j$ and take global sections then the LHS has more sections than the RHS, a contradiction. $\square$

Note that not every sequence of vector bundles on $\mathbb{P}^1$ splits. For example, consider the rank two vector bundle $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. The global section

$$\sigma = (X, Y) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$$

has no zeroes. Therefore we get a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(2) \longrightarrow 0.$$

Note that the last sheaf is locally free, since σ has no zeroes. It obviously has rank one and it must have degree two, simply by considering the first chern class. If this sequence split then we would have an isomorphism

$$\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2),$$

which contradicts uniqueness of the splitting, (4.1).

It is a theorem in topology that the only topological invariant of a rank r vector bundle on $\mathbb{P}^1$ is its first chern class (the second chern class is zero, as we are on a curve). Note that the first chern class is simply the sum

$$\sum a_i.$$

Thus the topological classification is much coarser than the holomorphic.

Finally, there is a more direct way to argue that

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0.$$

splits. One can use global Ext. The obstruction to splitting this sequence lives in

$$\operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{F}, \mathcal{O}_{\mathbb{P}^1}) \simeq H^1(\mathbb{P}^1, \mathcal{F}^* \underset{\mathcal{O}_{\mathbb{P}^1}}{\otimes} \mathcal{O}_{\mathbb{P}^1}).$$

To compute the last sheaf, we apply Serre duality as before. For the exact sequence above that does not split, note that splits. One can use global Ext. The obstruction to splitting this sequence lives in

$$\operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1}) \simeq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0,$$

as expected.

Note that given a vector bundle on $\mathbb{P}^n$, we can always restrict this vector bundle to a line and consider the splitting type. We carry this out for the tangent bundle on $\mathbb{P}^n$. First observe that the tangent bundle is:

Definition 4.2. *We say that a vector bundle E is* ***homogeneous*** *if $\phi^* E \simeq E$ for every element $\phi \in \operatorname{Aut}(\mathbb{P}^n)$.*

Note that a homogeneous vector bundle is **uniform**, meaning that the splitting type is independent of the line, since $\operatorname{Aut}(\mathbb{P}^n)$ is acts transitively on lines.

Note that from the Euler sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}^{n+1}_{\mathbb{P}^n}(1) \longrightarrow T_X \longrightarrow 0,$$

we know that the restriction of the first chern class of the tangent bundle is $n+1$.

We claim that the splitting type is $(2, 1, 1, \dots, 1)$, the most uniform way to distribute the numbers $a_1, a_2, \dots, a_n$ and get the sum $n+1$.

We check the claim. Pick a hyperplane H. The normal bundle of H inside $\mathbb{P}^n$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(1)$ so that there is an exact sequence

$$0 \longrightarrow T_H \longrightarrow T_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0.$$

If we restrict to a line contained in H, the first sheaf becomes

$$\mathcal{O}_{\mathbb{P}^1}(2) \bigoplus_{i=2}^{n-1} \mathcal{O}_{\mathbb{P}^1}(1).$$

and the last sheaf becomes

$$\mathcal{O}_{\mathbb{P}^1}(1).$$

We check this sequence splits. When we compute Ext^1, we have to compute the first cohomology of

$$\mathcal{O}_{\mathbb{P}^1}(1-2) \bigoplus_{i=2}^{n-1} \mathcal{O}_{\mathbb{P}^1}(1-1).$$

But this vanishes, by Serre duality. So the sequence splits and we are done by induction.

5. Rank two vector bundles on $\mathbb{P}^2$

We are going to construct some interesting vector bundles on $\mathbb{P}^2$. First observe the following easy result:

Lemma 5.1. *If E is a rank two vector bundle on $\mathbb{P}^r$ then E splits if and only if E contains a subline bundle.*

Proof. One direction is clear, if E splits there is a surely a subline bundle.

For the other direction, we use the language of sheaves. We may assume that $r > 1$. Let

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^r}(E).$$

If E has a subline bundle then there is an exact sequence of locally free sheaves

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0,$$

where $\mathcal{L}$ and $\mathcal{M}$ both have rank one. It follows that there are integers a and b such that

$$\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^r}(a) \qquad \text{and} \qquad \mathcal{M} \simeq \mathcal{O}_{\mathbb{P}^r}(b).$$

The obstruction to splitting this exact sequence lies in

$$\operatorname{Ext}^1_{\mathbb{P}^k}(\mathcal{M}, \mathcal{L}) \simeq H^1(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(a-b)) = 0.$$

Thus the exact sequence splits and so E splits. $\square$

Suppose that V is a rank two vector bundle that does not split. If k is large enough, then $V(k)$ is globally generated and we may find a global section σ that has m simple zeroes, $x_1, x_2, \dots, x_m$. Let $\pi\colon X \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at these m points, with exceptional divisors $E_1, E_2, \dots, E_m$. Then $\pi^*\sigma$ is a global section of π^*V which vanishes along each exceptional divisor $E_1, E_2, \dots, E_m$ with multiplicity one,

$$\pi^*\sigma \in H^0(X, \pi^*V \otimes \mathcal{I}_E),$$

where $E = \sum E_i$. Thus we get a subline bundle

$$\mathcal{I}_E^* = \mathcal{O}_X(E) \longrightarrow \sigma^*V.$$

This gives us a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \sigma^*V \longrightarrow Q \longrightarrow 0,$$

where Q has rank one.

Now $E_1, E_2, \dots, E_m$ are -1-curves, so that $E_i \simeq \mathbb{P}^1$ and $E_i^2 = -1$. Thus when we restrict to E_i we get the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}^2_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

Here we identified $Q|_{E_i}$ by using the first chern class.

The idea is simply to reverse all of this. Look for exact sequences

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow V' \longrightarrow \mathcal{O}_X(-E) \longrightarrow 0$$

on X which restrict on each exceptional to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}^2_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

Hopefully V' is then the pullback of some rank two vector bundle V on $\mathbb{P}^2$.

Theorem 5.2. *Let $x_1, x_2, \dots, x_m$ be points of $\mathbb{P}^2$.*

There is a rank two vector bundle V on $\mathbb{P}^2$ such that if $L \subset \mathbb{P}^2$ is a line then

$$V|_L \simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$$

if and only if L contains a points of the m points $x_1, x_2, \dots, x_m$.

Proof. Let $\pi\colon X \longrightarrow \mathbb{P}^2$ blow up the points $x_1, x_2, \dots, x_m$, let E_i be the exceptional divisor over x_i and let $E = \sum E_i$, so that

$$\mathcal{O}_X(E) = \bigotimes_{i=1}^{m} \mathcal{O}_X(E_i).$$

There is a short exact sequence

$$0 \longrightarrow \mathcal{I}_E \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

Now extensions of $\mathcal{O}_X(-E)$ by $\mathcal{O}_X(E)$ are given by

$$\operatorname{Ext}^1_X(\mathcal{O}_X(-E), \mathcal{O}_X(E)) = H^1(X, \mathcal{O}_X(2E))$$

and extensions of $\mathcal{O}_E(-E)$ by $\mathcal{O}_E(E)$ are given by

$$\operatorname{Ext}^1_E(\mathcal{O}_E(-E), \mathcal{O}_E(E)) = H^1(E, \mathcal{O}_E(2E))$$

If we tensor the previous exact sequence by $\mathcal{O}_X(2E)$ then we get the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_X(2E) \longrightarrow \mathcal{O}_E(2E) \longrightarrow 0,$$

Taking the long exact sequence of cohomology we get

$$H^1(X, \mathcal{O}_X(E)) \longrightarrow H^1(X, \mathcal{O}_X(2E)) \longrightarrow H^1(E, \mathcal{O}_E(2E)) \longrightarrow H^2(X, \mathcal{O}_E(E)).$$

Note that

$$\begin{aligned} H^2(X, \mathcal{O}_X(E)) &= H^0(X, \omega_X(-E)) \\ &= H^0(X, \mathcal{O}_X(-K_X - E)). \end{aligned}$$

But $\pi_*(-K_X) = -K_{\mathbb{P}^2} = -3L$, so that the last group is zero (one can also use (5.3) at this point).

It follows that we can lift any extension class on E to an extension on X, so that we can choose a vector bundle on X to be any chosen extension on each E_i. But

$$\operatorname{Ext}^1(\mathcal{O}_E(-E), \mathcal{O}_E(E)) = \bigoplus_{i=1}^{m} \operatorname{Ext}^1(\mathcal{O}_{E_i}(-E_i), \mathcal{O}_{E_i}(E_i)).$$

Let ξ be the extension class, which component by component of E gives

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}^2_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

It follows that there is a vector bundle V' on X which is an extension of

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow V' \longrightarrow \mathcal{O}_X(-E) \longrightarrow 0$$

on X which restrict on each exceptional divisor to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}^2_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

By (5.4), there is a vector bundle V on $\mathbb{P}^2$ such that $V' = \pi^* V$.

It remains to check the splitting type of V on a line L. Suppose that L is a line that contains the first a points $x_1, x_2, \dots, x_a$ of $x_1, x_2, \dots, x_m$. Let M be the strict transform of L. We have

$$\begin{aligned} L \cdot E &= L \cdot (E_1 + E_2 + \cdots + E_m) \\ &= L \cdot (E_1 + E_2 + \cdots + E_a) \\ &= a. \end{aligned}$$

If we restrict

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow V' \longrightarrow \mathcal{O}_X(-E) \longrightarrow 0$$

to M we get

$$0 \longrightarrow \mathcal{O}_M(a) \longrightarrow V' \longrightarrow \mathcal{O}_M(-a) \longrightarrow 0.$$

This sequence splits as $a \geq 0$. It follows that

$$\begin{aligned} V|_L &\simeq \sigma^* V|_M \\ &= V'|_M \\ &\simeq \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(-a). \end{aligned}$$ □

Note that the stratification, into type, of the set of lines in $\mathbb{P}^2$, which is a copy of $\mathbb{P}^2$, decomposes $\mathbb{P}^2$ into a union of locally closed subsets. There is the open subset of lines which avoid $x_1, x_2, \dots, x_m$ and for these the splitting type is $(0, 0)$. There are the lines which meet one point and for these the splitting type is $(1, -1)$. Inside the dual $\mathbb{P}^2$ the lines through a point is a line. Otherwise there are finitely many lines for which the splitting type is of type $(a, -a)$ for $a > 1$.

This part of a more general phenomena, where the splitting type can only jump up, in the lexigraphic order.

Lemma 5.3. *Let $\pi\colon X \longrightarrow Y$ be a birational morphism of smooth varieties.*

Then $R^*\pi_*\mathcal{O}_X = \mathcal{O}_Y$. In particular $H^*(X, \mathcal{O}_X) \simeq H^*(Y, \mathcal{O}_Y)$.

Proof. By weak factorisation, we may assume that π is factored into a sequence of smooth blow ups and smooth blow downs. By induction we may assume that there is one blow up.

There are two ways to proceed.

For the first, consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-(j+1)E) \longrightarrow \mathcal{O}_X(-jE) \longrightarrow \mathcal{O}_E(-(j+1)E) \longrightarrow 0.$$

If we take the long exact sequence of cohomology we get

$$R^i\pi_*\mathcal{O}_X(-(j+1)E) \longrightarrow R^i\pi_*\mathcal{O}_X(-jE) \longrightarrow R^i\pi_*\mathcal{O}_E(-jE).$$

Suppose that $j \geq 0$ and $i > 0$. $E \longrightarrow \pi(E)$ is a projective bundle and $\mathcal{O}_E(-jE)$ is relatively nef. Thus the cohomology of $\mathcal{O}_E(-jE)$ restricted to any fibre is zero. Thus $R^i\pi_*\mathcal{O}_E(-jE) = 0$. If j is small enough then the first group vanishes by relative Serre vanishing. Thus the middle group vanishes for all $j \geq 0$ by induction.

Alternatively, we can compute locally. So we may assume that Y is the unit ball in $\mathbb{C}^n$ and π blows up the intersection with a coordinate subspace. The result is then an easy direct computation.

The last statement follows, as the Leray-Serre spectral sequence degenerates at the E_2-level. □

Lemma 5.4. *Let $\pi\colon X \longrightarrow \mathbb{C}^2$ be the blow up of $(0,0) \in \mathbb{C}^2$ with exceptional divisor E, so that*

$$X = \{\,(x, y, [u : v]) \mid [x : y] = [u : v]\,\} \subset \mathbb{C}^2 \times \mathbb{P}^1.$$

If V' is a vector bundle on X which is an extension of

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow V' \longrightarrow \mathcal{O}_X(-E) \longrightarrow 0$$

on X which restrict on the exceptional divisor E to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}^2_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0,$$

then V' is the trivial vector bundle.

In particular there is a vector bundle V on $\mathbb{C}^2$ such that $V' = \pi^*V$.

Proof. Note that the trivial bundle $\mathcal{O}^2_X$ is also an extension

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}^2_X \longrightarrow \mathcal{O}_X(-E) \longrightarrow 0,$$

which restricts to the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^2 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(1) \longrightarrow 0.$$

So it suffices to show that

$$\mathrm{Ext}^1(\mathcal{O}_X(-E), \mathcal{O}_X(E)) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_E(-E), \mathcal{O}_E(E)),$$

is injective, that is, we have to show

$$H^1(X, \mathcal{O}_X(2E)) \longrightarrow H^1(E, \mathcal{O}_E(2E)),$$

is injective. Considering the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_X(2E) \longrightarrow \mathcal{O}_E(2E) \longrightarrow 0,$$

it suffices to show that

$$H^1(X, \mathcal{O}_X(E)) = 0.$$

Consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(E) \longrightarrow \mathcal{O}_E(E) \longrightarrow 0.$$

If we take the long exact sequence of cohomology we get

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X(E)) \longrightarrow H^1(E, \mathcal{O}_E(E)).$$

The last group is equal to

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0,$$

which is zero by Serre duality. The first group is trivial, since $H^1(\mathbb{C}, \mathcal{O}_{\mathbb{C}^2}) = 0$ and the blow up won't change H^1 by (5.3). $\square$

6. A splitting criteria

Theorem 6.1. *A vector bundle E on $\mathbb{P}^n$ splits if and only if $H^i(\mathbb{P}^n, E(k)) = 0$ for all $0 < i < n$ and all integers k.*

Proof. One direction is clear, if E splits then we get vanishing, as

$$H^i(\mathbb{P}^n \mathcal{O}_{\mathbb{P}^n}(k)) = 0$$

for all $0 < i < n$.

Now suppose that we have vanishing. We proceed by induction on n. The case $n = 1$ is Grothendieck's result. Otherwise, pick a hyperplane and consider the short exact sequence of sheaves,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

If we tensor this with $\mathcal{E}(k)$ we get

$$0 \longrightarrow \mathcal{E}(k-1) \longrightarrow \mathcal{E}(k) \longrightarrow \mathcal{E}(k)|_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

If we take the long exact sequence of cohomology we get

$$H^i(\mathbb{P}^n, \mathcal{E}(k)) \longrightarrow H^i(\mathbb{P}^{n-1}, \mathcal{E}|_{\mathbb{P}^{n-1}}(k)) \longrightarrow H^{i+1}(\mathbb{P}^n, \mathcal{E}(k-1)).$$

It follows that

$$H^i(\mathbb{P}^{n-1}, \mathcal{E}|_{\mathbb{P}^{n-1}}(k)) = 0,$$

for all $0 < i < n-1$ and every integer k.

It follows by induction that $\mathcal{E}|_{\mathbb{P}^{n-1}}$ splits as a direct sum, so that

$$\mathcal{E}|_{\mathbb{P}^{n-1}} \simeq \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{r-1}}(a_i),$$

for some integers $a_1, a_2, \dots, a_r$. Let

$$\mathcal{F} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^r}(a_i).$$

We want to show that $\mathcal{E}$ is isomorphic to $\mathcal{F}$.

Pick an isomorphism

$$\phi \colon \mathcal{F}|_{\mathbb{P}^{n-1}} \longrightarrow \mathcal{E}|_{\mathbb{P}^{n-1}}.$$

Claim 6.2. *We can extend ϕ to a homomorphism*

$$\Phi \colon \mathcal{F} \longrightarrow \mathcal{E}.$$

Proof of (6.2). If we tensor

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

with

$$\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E})$$

then we get the following short exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E})(-1) \longrightarrow \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E}) \longrightarrow \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E})|_{\mathbb{P}^{n-1}} \longrightarrow 0..$$

Note that the last sheaf cohomology group is

$$\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^{n-1}}}(\mathcal{F}|_{\mathbb{P}^{n-1}}, \mathcal{E}|_{\mathbb{P}^{n-1}}).$$

If we take the long exact sequence of cohomology the obstruction to lifting lies in

$$H^1(\mathbb{P}^n, \operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E})(-1)).$$

But

$$\operatorname{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{F}, \mathcal{E})(-1) \simeq \mathcal{F}^* \otimes \mathcal{E}(-1),$$

which splits a direct sum of copies twists of $\mathbb{E}$. Thus H^1 vanishes and we can lift ϕ. □

By functoriality, Φ induces a homomorphism

$$\det \Phi \colon \det \mathcal{F} \longrightarrow \det \mathcal{E}.$$

Both line bundles have the same first chern class (since the first chern class is really just a number and this number is determined by its restriction to a hyperplane). Thus we can interpret

$$\begin{aligned} \det \Phi &\in H^0(\mathbb{P}^n, \det \mathcal{F}^* \otimes \det \mathcal{E}) \\ &\in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}). \end{aligned}$$

Thus $\det \Phi$ is a scalar. It restriction to a hyperplane is non-zero and so $\det \Phi$ is a non-zero constant. As $\det \Phi$ is nowhere zero, it follows that Φ is an isomorphism. □

Corollary 6.3. *A vector bundle E on $\mathbb{P}^n$ splits if and only if it restriction to any plane splits.*

Proof. One direction is clear; if E splits its restriction to a plane splits.

For the other direction, pick a hyperplane and consider the short exact sequence of sheaves,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

If we tensor this with $\mathcal{E}(k)$ we get

$$0 \longrightarrow \mathcal{E}(k-1) \longrightarrow \mathcal{E}(k) \longrightarrow \mathcal{E}(k)|_{\mathbb{P}^{n-1}} \longrightarrow 0.$$

By induction we know that $\mathcal{E}(k)|_{\mathbb{P}^{n-1}}$ splits and so by (6.1) we know that

$$H^i(\mathbb{P}^{n-1}, \mathcal{E}(k)|_{\mathbb{P}^{n-1}}) = 0,$$

for every integer k and for all $0 < i < n-1$. If we take the long exact sequence of cohomology we see that

$$H^i(\mathbb{P}^n, \mathcal{E}(k-1)) \longrightarrow H^i(\mathbb{P}^n, \mathcal{E}(k))$$

is surjective for $i < n-1$ and injective for $i > 1$. By Serre vanishing

$$H^i(\mathbb{P}^n, \mathcal{E}(k)) = H^{n-i}(\mathbb{P}^n, \omega_{\mathbb{P}^n} \otimes \mathcal{E}^*(-k))^*$$

for all $0 < i < n$ and for all $|k|$ sufficiently large . Thus

$$H^i(\mathbb{P}^n, \mathcal{E}(k)) = 0,$$

for all k and all $0 < i < n$, by ascending induction if $i < n-1$ and by descending induction if $i > 1$. But then (6.1) implies that E splits. $\square$

7. Uniform bundles

We want to study vector bundles whose splitting type does not depend on the line. Recall that the Grassmannian $\mathbb{G}(1,n)$ parametrises the lines in $\mathbb{P}^n$. The incidence correspondence is the closed subset

$$I = \{\, (p,L) \in \mathbb{P}^n \times \mathbb{G}(1,n) \,|\, p \in L \,\} \subset \mathbb{P}^n \times \mathbb{G}(1,n).$$

There are two natural projections

$$\begin{tikzcd} I \arrow[r, "g"] \arrow[d, "f"'] & \mathbb{G}(1,n). \\ \mathbb{P}^n & \end{tikzcd}$$

The fibre of g over a point $[L]$ is the whole line L. The fibre of f over a point p is the set of lines through p. Pick an auxiliary hyperplane H, not passing through p. Then a line L through intersects H in a unique point q. Vice-versa, given a point $q \in H$ there is a unique line $L = \langle p, q\rangle$ through p and q, and this line meets H in the point q. Thus the fibre over f is a copy of H.

If we look at the subset $G(p) \subset \mathbb{G}(1,n)$ of all lines containing p then the inverse image $B(p)$ of $G(p)$ inside I is almost by definition the blow up of $\mathbb{P}^n$ at the point p.

g realises I as the universal line over the Grassmannian. There is a natural associated rank two vector bundle S to I,

$$S \subset \mathbb{G}(1,n) \times V,$$

where $\mathbb{P}^n = \mathbb{P}(V)$. S is the universal rank two sub bundle of the trivial vector bundle of rank $n+1$ on $\mathbb{G}(1,n)$. I is the projectivisation of S.

Theorem 7.1. *Let E be a vector bundle of rank r on $\mathbb{P}^n$. Fix a point p.*

If $E|_L$ is the trivial vector bundle of rank r for every line L containing p then E is the trivial vector bundle.

Proof. Let $\phi\colon B(p) \longrightarrow \mathbb{P}^n$ be the restriction of f to $B(p)$ so that ϕ is the blow up of p. Let $\gamma\colon B(p) \longrightarrow G(p)$ be the restriction of g to $B(p)$. The fibres of γ are mapped isomorphically by ϕ to lines through p. It follows that ϕ^*E is trivial on the fibres of γ.

Claim 7.2. *There is a vector bundle F on $\mathbb{G}(1,n)$ such that*

$$\phi^*E = \gamma^*F.$$

Assume (7.2). Let D be the exceptional divisor of ϕ. Then the restriction of ϕ^*E to D is the trivial vector bundle of rank r. Let

$i\colon D \longrightarrow B(p)$ be the inclusion of D inside $B(p)$. Then $\gamma \circ i\colon D \longrightarrow G(p)$ is an isomorphism. $(\gamma \circ i)^* F$ is isomorphic to the restriction of F to D, which is the trivial vector bundle of rank r. Therefore $\phi^* E$ is the trivial vector bundle of rank r.

Proof of (7.2). Consider $\gamma_* \phi^* E$. γ is a $\mathbb{P}^1$-bundle. Therefore γ is smooth and so it is flat. Hence $\phi^* E$ is flat over $G(p)$. Let $M = \gamma^{-1} L$. As

$$\begin{aligned} h^0(M, \phi^* E) &= h^0(L, E|_L) \\ &= h^0(L, \mathcal{O}_L^r), \end{aligned}$$

is constant, it follows by the base change theorem that $F = \gamma_* \phi^* E$ is locally free of rank r. Consider the natural map

$$\gamma^* \gamma_* \phi^* E \longrightarrow \phi^* E$$

The first sheaf restricted to M is

$$(\gamma^* \gamma_* \phi^* E)|_M \simeq \mathcal{O}_M \otimes H^0(M, \phi^* E|_M)$$

and this map becomes the evluation map

$$\mathcal{O}_M \otimes H^0(M, \phi^* E|_M) \longrightarrow (\phi^* E)|_M,$$

which is an isomorphism. □

□

Corollary 7.3. *If E is a globally generated bundle on $\mathbb{P}^n$ then E is trivial if and only if $c_1(E) = 0$.*

Proof. One direction is clear; if E is trivial then $c_1(E) = 0$. As E is globally generated, it follows that there is an exact sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}_{\mathbb{P}^n}^N \longrightarrow E \longrightarrow 0.$$

If we restrict to a line L we get

$$0 \longrightarrow K|_L \longrightarrow \mathcal{O}_L^N \longrightarrow E|_L \longrightarrow 0.$$

If $(a_1, a_2, \dots, a_r)$ is the splitting type, then we see that $a_i \geq 0$, for all i. On the other hand, as the first chern class is zero, we have $\sum a_i = 0$, so that $a_1 = a_2 = \cdots = a_r = 0$. It follows that E is trivial on every line and so we can apply (7.1). □

Lemma 7.4. *Let E be a vector bundle of rank r.*

Then the splitting type is upper semi continuous on $\mathbb{G}(1, n)$.

Proof. It is not hard to see that the splitting type decomposes $\mathbb{G}(1,n)$ into constructible subsets. We just have to show that the splitting type never goes down under specialisation. Suppose that L is a line with splitting type $(a_1, a_2, \dots, a_r)$. By an obvious induction it suffices to prove that the initial part $(a_1, a_2, \dots, a_k)$ is upper semi continuous. We may assume that we are given a curve $C \subset \mathbb{G}(1,n)$ and we may assume that $(a_1, a_2, \dots, a_{k-1})$ is constant. Let a be the generic value of a_k over the curve C. Since global sections of $E(-a)$ can only jump up, it follows that a_k can only jump up. $\square$

Theorem 7.5. *Let E be a uniform vector bundle of rank r on $\mathbb{P}^n$.*
If $r < n$ then E splits.

Proof. We proceed by induction on the rank r. If $r = 1$ there is nothing to prove.

Let $(a_1, a_2, \dots, a_r)$ be the splitting type. Suppose that $a_1 = a_2 = \cdots = a_k$ and $a_{k+1} < a_k$. Replacing E by $E(-a_1)$ we may assume that $a_k = 0$. If $k = r$ we may apply (7.1) to conclude that E splits.

Our goal is to find a uniform rank vector bundle F of smaller rank than E and an exact sequence

$$0 \longrightarrow F \longrightarrow E \longrightarrow Q \longrightarrow 0,$$

where Q is a uniform vector bundle.

Suppose we can find such a short exact sequence. By induction F and Q split. It follows that $F \otimes Q^*$ splits, so that the extension splits as

$$\operatorname{Ext}^1_{\mathbb{P}^n}(Q, F) \simeq H^1(\mathbb{P}^n, F \otimes Q^*) = 0.$$

To construct F, proceed as before. Consider f^*E. For the fibres M of f, we have

$$f^*E|_M \simeq \mathcal{O}_M^k \oplus E_1,$$

where E_1 is a direct sum of line bundles with negative twists, so that it has no global sections. Thus $\gamma_* f^*E$ is a vector bundle of rank k. As before, consider the natural map

$$g^* g_* f^* E \longrightarrow f^* E$$

The first sheaf restricted to M is

$$(g^* g_* f^* E)|_M \simeq \mathcal{O}_M \otimes H^0(M, f^*E|_M)$$

and this map becomes the evluation map

$$\mathcal{O}_M \otimes H^0(M, f^*E|_M) \longrightarrow (f^*E)|_M,$$

which gives a rank k sub vector bundle. This gives us a short exact sequence of vector bundles on the blow up,

$$0 \longrightarrow F_1 \longrightarrow f^*E \longrightarrow Q_1 \longrightarrow 0,$$

We check that all three bundles are pulled back from $\mathbb{P}^n$. As before it suffices to prove that F_1 and Q_1 are trivial on the fibres of p. Suppose we restrict to $I_p = f^{-1}(p)$. We get an exact sequence

$$0 \longrightarrow F_1|_{I_p} \longrightarrow \mathcal{O}^r_{I_p} \longrightarrow Q_1|_{I_p} \longrightarrow 0,$$

If we take total chern classes we get

$$c(F_1|_{I_p})c(Q_1|_{I_p}) = c(\mathcal{O}^r_{I_p}) = 1.$$

As $r < n$ this forces

$$c(F_1|_{I_p}) = c(Q_1|_{I_p}) = 1.$$

In particular

$$c_1(F_1|_{I_p}) = c_1(Q_1|_{I_p}) = 0.$$

As $Q_1|_{I_p}$ is globally generated, we have $Q_1|_{I_p})$ is a trivial bundle. Taking duals, we get the same conclusion for F_1. Thus they are both pulled back from $\mathbb{P}^n$. $\square$

There is an interesting way to represent uniform vector bundles on $\mathbb{P}^n$. Suppose that E is a uniform bundle on $\mathbb{P}^n$ and suppose that we think of the splitting type as a partition, so that we have pairs (a_i, r_i), where $a_1 > a_2 > a_3 > \cdots > a_k$ and E has r_i direct summands of the form $\mathcal{O}_{\mathbb{P}^n}(a_i)$. Then there is a filtration

$$0 = F^0 \subset F^1 \subset F^2 \subset \dots F^k = f^*E$$

of f^*E by sub bundles F^i such that

$$\frac{F^i}{F^{i-1}} = g^*G_i \otimes \mathcal{O}_{\mathbb{P}^n}(a_i)$$

This filtration is called the **Harder-Narasimhan filtration** of f^*E.

It is constructed as follows. Let

$$F^1 = g^*(g_*f^*E(-a_1)) \otimes f^*\mathcal{O}_{\mathbb{P}^n}(a_i).$$

Note that $g_*f^*E(-a_1)$ is a vector bundle of rank k_1 on $\mathbb{G}(1,n)$. As before, F^1 is a sub bundle of f^*E. Let

$$Q_1 = \frac{f^*E}{F^1}$$

be the quotient. The idea is simply to keep going. Let

$$\pi\colon f^*E \longrightarrow Q_1,$$

be the quotient map. Then

$$F^2 = \pi^{-1}(g^* g_*(Q_1(-a_2)) \otimes f^*\mathcal{O}_{\mathbb{P}^n}(a_2))$$

is a sub bundle of f^*E that contains F^1. Form the quotient

$$Q_2 = \frac{f^*E}{F^2}$$

and keep going.

The other direction is much easier. If we have such a filtration, its restriction to a line L defines a sequence of short exact sequences, which must all split.

8. Uniform heterogeneous examples

We are going to give an example of a bundle which is uniform, meaning that the splitting type is constant, but not homogeneous, so that the bundle is not fixed under the action of the automorphism group of $\mathbb{P}^n$.

Definition 8.1. *We say that a vector bundle E is k-**homogeneous** if $\phi_1^* E \simeq \phi_2^* E$ for all linear maps $\phi_1 \colon \mathbb{P}^k \longrightarrow \mathbb{P}^n$ and $\phi_2 \colon \mathbb{P}^k \longrightarrow \mathbb{P}^n$.*

Note that E is homogeneous if and only if it is n-homogeneous and it is uniform if and only if it is 1-homogeneous. Since every linear map can be extended from $\mathbb{P}^k$ to $\mathbb{P}^{k+1}$, for $k < n$ it follows that $(k+1)$-homogeneous implies k-homogeneous.

Definition 8.2. *The maximum k such that E is k-homogeneous, denoted $h(E)$, is called the **extent** of E.*

If the rank r of E is less than n, $r < n$, then either E splits, in which case E is homogeneous or E is not uniform. Thus

$$h(E) = 0 \qquad \text{or} \qquad h(E) = n,$$

when $r < n$.

Theorem 8.3. *Let $n \neq 2$. For every $0 \leq e < n-1$ there is a holomorphic vector bundle E on $\mathbb{P}^n$ with extent e.*

Proof. We start with the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

The first bundle is the universal sub line bundle. So the bundle

$$T_{\mathbb{P}^n}(-1)$$

is the quotient of the trivial bundle $\mathbb{P}^n \times V$ by the universal sub line bundle S, where $\mathbb{P}^n = \mathbb{P}(V)$.

Pick a basis $w_0, w_1, \dots, w_n$ of V. These determine sections

$$s_i \in H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1))$$

which at the point p takes the value w_i/S_p.

As the $w_0, w_1, \dots, w_m$ are linearly independent, it follows that $s_0, s_1, \dots, s_m$ have no common zeroes. Thus we get an inclusion

$$\mathcal{O}_{\mathbb{P}^n} \longrightarrow T_{\mathbb{P}^n}(-1)^{\oplus m+1}.$$

Let E be the quotient vector bundle, so that there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow T_{\mathbb{P}^n}(-1)^{\oplus m+1} \longrightarrow E \longrightarrow 0.$$

Note that E has rank

$$(m+1)n-1.$$

Consider what happens if we restrict E to a linear subspace $\mathbb{P}(W) \subset \mathbb{P}(V)$.

Claim 8.4. *Let W_0 be the span of the vectors $w_0, w_1, \dots, w_m$, let $\Lambda_0 = \mathbb{P}(W_0)$ and let $\Lambda \subset \mathbb{P}^n$ be a k-dimensional linear subspace.*

(1) If Λ_0 is not contained in Λ, then

$$E|_\Lambda \simeq T_\Lambda(-1)^{\oplus m+1} \oplus \mathcal{O}_\Lambda^{\oplus(n-k)(m+1)-1}.$$

(2) If Λ_0 is contained in Λ, then

$$E|_\Lambda \simeq E' \oplus \mathcal{O}_\Lambda^{\oplus(n-k)(m+1)},$$

where E' is a bundle on Λ such that $h^0(\Lambda, E^*) = 0$.

If we assume the claim then note that

$$h^0(\Lambda, E^*|_\Lambda) = \begin{cases} (n-k)(m+1)-1 & \text{otherwise} \\ (n-k)(m+1) & \text{if } \Lambda_0 \subset \Lambda. \end{cases}$$

In particular if $k = m-1$ then we are always in the first case, so that E is $(m-1)$-homogeneous. If $k = m$ then there are two possibilities for $h^0(\Lambda, E|_\Lambda)$ so that the E is not m-homogeneous. Thus the extent of E is $m-1$.

Proof of (8.4). First suppose that Λ_0 is not contained in Λ. Then we may assume that w_0 is not contained in W. It follows that the restriction of

$$s_0 \in H^0(\mathbb{P}^n, T_{\mathbb{P}^n}(-1))$$

to Λ is nowhere zero. s_0 defines a short exact sequence

$$0 \longrightarrow \mathcal{O}_\Lambda \longrightarrow T_{\mathbb{P}^n}(-1))|_\Lambda \longrightarrow Q \longrightarrow 0,$$

where Q is a vector bundle of rank $n-1$ on Λ. There is also an exact sequence

$$0 \longrightarrow T_\Lambda(-1) \longrightarrow T_{\mathbb{P}^n}(-1)|_\Lambda \longrightarrow \mathcal{O}_\Lambda{}^{\oplus(n-k)} \longrightarrow 0.$$

This gives a commutative diagram with exact rows and colummns

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_\Lambda(-1) & = & T_\Lambda(-1) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_\Lambda & \to & T_{\mathbb{P}^n}(-1)|_\Lambda & \to & Q & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_\Lambda & \to & \mathcal{O}_\Lambda^{\oplus n-k} & \to & Q' & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The bottom row yields the isomorphism

$$Q' \simeq \mathcal{O}_\Lambda^{\oplus n-k-1}$$

and so the right column gives the isomorphism

$$Q \simeq T_\Lambda(-1) \oplus \mathcal{O}_\Lambda^{\oplus n-k-1}$$

since

$$H^1(\Lambda, T_\Lambda(-1)) = 0.$$

We now consider another similar diagram.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T_{\mathbb{P}^n}(-1)^{\oplus m}|_\Lambda & = & T_{\mathbb{P}^n}(-1)^{\oplus m}|_\Lambda & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_\Lambda & \to & T_{\mathbb{P}^n}(-1)^{\oplus m+1}|_\Lambda & \to & E|_\Lambda & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_\Lambda & \to & T_{\mathbb{P}^n}(-1)|_\Lambda & \to & Q & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Now the last column splits as

$$H^1(\Lambda, Q^* \otimes T_\Lambda(-1)^{\oplus m}|_\Lambda) = 0.$$

It follows then that

$$\begin{aligned} E|_\Lambda &= T_{\mathbb{P}^n}(-1)^{\oplus m}|_\Lambda \oplus T_\Lambda(-1) \oplus \mathcal{O}_\Lambda^{\oplus n-k-1} \\ &= T_\Lambda(-1)^{\oplus m+1}|_\Lambda \oplus \mathcal{O}_\Lambda^{\oplus (n-k)(m+1)-1}. \end{aligned}$$

Now suppose that $\Lambda_0 \subset \Lambda$.

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_\Lambda & \to & T_\Lambda(-1)^{\oplus m+1} & \to & E' & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}_\Lambda & \to & T_{\mathbb{P}^n}(-1)^{\oplus m+1}|_\Lambda & \to & E|_\Lambda & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{O}_\Lambda^{\oplus(n-k)(m+1)} & = & \mathcal{O}_\Lambda^{\oplus(n-k)(m+1)} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

From the top row we get

$$\begin{aligned} h^0(\Lambda, E'^*) &= 0 \\ h^1(\Lambda, E') &= 0. \end{aligned}$$

From the last column we then deduce

$$E|_\Lambda \simeq E' \oplus \mathcal{O}_\Lambda^{\oplus(n-k)(m+1)}. \qquad \square$$

$\square$

If we take $m = 2$ then the rank of E is $3n - 1$ and E is not homogeneous but it is uniform.

9. Simple bundles

Definition 9.1. *A vector bundle is called* ***simple*** *if*

$$h^0(\mathbb{P}^n, E^* \otimes E) = 1.$$

As

$$E^* \otimes E = \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(E, E),$$

a bundle is simple if and only if its only endomorphisms are homotheties.

Definition 9.2. *A vector bundle E on $\mathbb{P}^n$ is* ***decomposable*** *if it is isomorphic to a direct sum $F \oplus G$, where F and G have smaller rank than E.*

If E is decomposable then it has non-trivial endomorphisms, given by different homotheties on both factors. Thus simple bundles are always indecomposable.

Lemma 9.3. *The tangent bundle on $\mathbb{P}^n$ is simple.*

Proof. We start with the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

If we tensor this with $\Omega^1_{\mathbb{P}^n}(1)$ then we get the exact sequence

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \Omega^1_{\mathbb{P}^n}(1)^{\oplus n+1} \longrightarrow \Omega^1_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n} \longrightarrow 0.$$

Taking the long exact sequence of cohomology we get

$$H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(1)^{\oplus n+1}) \longrightarrow H^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) \longrightarrow H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n})$$

Now

$$h^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}(1)^{\oplus n+1}) = 0 \qquad \text{and} \qquad h^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) = 1.$$

But then

$$1 \leq h^0(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) \leq h^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) = 1.$$ □

It follows that the bundles $T_{\mathbb{P}^n}(k)$ and $\Omega^1_{\mathbb{P}^n}(k)$ are all simple, for any integer k.

Consider the rank two bundles we constructed on $\mathbb{P}^2$. We started with points in $p_1, p_2, \dots, p_k$ in $\mathbb{P}^2$, looked at the surface $\pi \colon X \longrightarrow \mathbb{P}^2$ you get by blowing up these points, and modified the extension

$$0 \longrightarrow \mathcal{O}_X(C) \longrightarrow E' \longrightarrow \mathcal{O}_X(-C) \longrightarrow 0,$$

where C is the sum of the exceptionals, so that E' is the pullback of a bundle E on $\mathbb{P}^2$, $E' = \pi^* E$.

Recall that if L is a line that contains a points then the splitting type is $(a, -a)$. In particular E is indecomposable. We want to show that E is not simple. We will need:

Lemma 9.4. *If E is a vector bundle of rank $r > 1$ and both $h^0(\mathbb{P}^n, E) > 0$ and $h^0(\mathbb{P}^n, E^*) > 0$ then E is not simple.*

Proof. Pick non-zero sections

$$\sigma \in H^0(\mathbb{P}^n, E) \qquad \text{and} \qquad \tau \in H^0(\mathbb{P}^n, E^*).$$

Then

$$\sigma \otimes \tau \in H^0(\mathbb{P}^n, E^* \otimes E)$$

is not a homothety, as it has rank one when restricted to any fibre. □

Note that

$$\begin{aligned} E^* &\simeq E \otimes \det E^* \\ &\simeq E, \end{aligned}$$

since $c_1(E) = 0$. On the other hand,

$$h^0(\mathbb{P}^2, E) > 0,$$

by construction, since E' contains the sub line bundle $\mathcal{O}_X(C)$.

We want to construct a simple bundle N of rank $n-1$ on $\mathbb{P}^n$, for any odd integer n, called the **null correlation bundle**. We will construct N as the kernel of a surjective map

$$T_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1).$$

We get a short exact sequence

$$0 \longrightarrow N \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0.$$

We first check that N is simple. We tensor the above sequence by N^* to get

$$h^0(\mathbb{P}^n, N^* \otimes N) \leq h^0(\mathbb{P}^n, N^* \otimes T_{\mathbb{P}^n}(-1)).$$

Now tensor the dual of the short exact sequence above with $T_{\mathbb{P}^n}(-1)$ to get

$$0 \longrightarrow T_{\mathbb{P}^n}(-2) \longrightarrow T_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n} \longrightarrow N^* \otimes T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

If we take the associated long exact sequence of cohomology we get

$$H^0(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n}) \longrightarrow H^0(\mathbb{P}^n, N^* \otimes T_{\mathbb{P}^n}(-1)) \longrightarrow H^1(\mathbb{P}^n, T_{\mathbb{P}^n}(-2)).$$

Now

$$H^1(\mathbb{P}^n, T_{\mathbb{P}^n}(-2)) = 0,$$

and so

$$h^0(\mathbb{P}^n, N^* \otimes T_{\mathbb{P}^n}(-1)) \leq h^0(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n}) = 1.$$

Thus

$$1 \leq h^0(\mathbb{P}^n, N^* \otimes N) \leq h^0(\mathbb{P}^n, N^* \otimes T_{\mathbb{P}^n}(-1)) \leq 1.$$

Now consider the goal of finding a surjective map

$$T_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1).$$

Dualising, it suffices to find a nowhwere zero section of $\Omega^1_{\mathbb{P}^n}(2)$.

Consider the projective bundle

$$\mathbb{P}(\Omega^1_{\mathbb{P}^n})$$

over $\mathbb{P}^n$. Point in this bundle correspond to hyperplanes in the tangent bundle, so that

$$\mathbb{P}(\Omega^1_{\mathbb{P}^n}) \simeq \{\, (x, H) \,|\, x \in H \,\} \subset \mathbb{P}^n \times \mathbb{P}^n.$$

Let

$$p\colon \mathbb{P}(\Omega^1_{\mathbb{P}^n}) \longrightarrow \mathbb{P}^n$$

be the natural projection. As n is odd, $n+1 = 2m$ for some integer m. Let A be the block diagonal $2m \times 2m$ matrix with the 2×2 matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then A is invertible and for all $x \in \mathbb{C}^{n+1}$ we have

$$\langle Ax, x \rangle = 0,$$

where

$$\langle x, y \rangle = \sum x_i y_i,$$

is the standard inner product. Pick homogeneous coordinates $[x_0 : x_1 : \dots : x_n]$ on $\mathbb{P}^n$ and $[\xi_0 : \xi_1 : \dots : \xi_n]$ on the dual $\mathbb{P}^n$. In these coordinates, A defines an isomorphism

$$\Phi\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$$

such that $x \in H = \Phi(x)$. The graph of Φ defines a section

$$g\colon \mathbb{P}^n \longrightarrow \mathbb{P}(\Omega^1_{\mathbb{P}^n}),$$

which sends x to $(x, \Phi(x))$. This gives a sub line bundle of $\Omega^1_{\mathbb{P}^n}$,

$$\mathcal{O}_{\mathbb{P}^n}(a) \longrightarrow \Omega^1_{\mathbb{P}^n}.$$

Lemma 9.5. $a = -2$.

Proof. Note that

$$\Omega^1_{\mathbb{P}^n}(-a)$$

has a non-vanishing section and so

$$c_n(\Omega^1_{\mathbb{P}^n}(-a)) = 0.$$

We compute

$$\begin{aligned}
0 &= c_n(\Omega^1_{\mathbb{P}^n}(-a)) \\
&= -c_n(T_{\mathbb{P}^n}(a)) \\
&= -\sum_{i=0}^{n} c_i(T_{\mathbb{P}^n})a^{n-i} \\
&= -\sum_{i=0}^{n} \binom{n+1}{i} a^{n-i}.
\end{aligned}$$

If we multiply by a we get

$$\begin{aligned}
0 &= \sum_{i=0}^{n} \binom{n+1}{i} a^{n+1-i} \\
&= (1+a)^{n+1} - 1.
\end{aligned}$$

It follows that $1 + a = \pm 1$. As $c_n(T_{\mathbb{P}^n}) = n + 1 \neq 0$, we cannot be in the case $a = 0$. Thus $a = -2$. $\square$

Putting all of this together, we can construct the null correlation bundle N.

Note that

$$\begin{aligned}
c(N) &= \frac{c(T_{\mathbb{P}^n}(-1))}{1+h} \\
&= \frac{1}{(1+h)(1-h)} \\
&= 1 + h^2 + h^4 + h^6 + \cdots + h^{n-1}.
\end{aligned}$$

Thus we have proved:

Theorem 9.6. *For every odd integer n there is a simple bundle N on $\mathbb{P}^n$ with total chern class:*

$$c(N) = 1 + h^2 + h^4 + h^6 + \cdots + h^{n-1}.$$

10. The Tango bundle

We want construct a simple bundle of rank $n-1$ over $\mathbb{P}^n$ for any n.

Lemma 10.1 (Serre)**.** *If E is a globally generated bundle of rank r on $\mathbb{P}^n$ and $r > n$ then there is an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r-n} \longrightarrow E \longrightarrow F \longrightarrow 0,$$

where F is a bundle of rank n.

Proof. As E is globally generated, there is an exact sequence

$$0 \longrightarrow K \longrightarrow H^0(\mathbb{P}^n, E) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow 0.$$

The kernel K is the bundle

$$K = \{\, (x, \sigma) \in \mathbb{P}^n \times H^0(\mathbb{P}^n, E) \,|\, \sigma(x) = 0 \,\}.$$

If we projectivise we get a morphism

$$f \colon \mathbb{P}(K) \longrightarrow \mathcal{O}_{\mathbb{P}^n}\mathbb{P}(\times H^0(\mathbb{P}^n, E))\mathbb{P}(H^0(\mathbb{P}^n, E)).,$$

where the second map is just projection onto the second factor. The fibres of f are just the zero sets of the corresponding function,

$$f^{-1}(\sigma) = \{\, x \in \mathbb{P}^n \,|\, \sigma(x) = 0 \,\}.$$

Suppose that $h^0(\mathbb{P}^n, E) = N + 1$. If $m = n + N - r$ then m is the dimension of $\mathbb{P}(K)$. Thus the image of f has codimension at least $r - n$.

It follows that we can find a $(r - n - 1)$ dimensional linear space $\mathbb{P}(V)$ which avoids the image. The composition

$$V \otimes \mathbb{P}^n \longrightarrow E$$

defines a trivial subbundle of rank $r - n$. $\square$

Corollary 10.2. *If E is a vector bundle of rank r on $\mathbb{P}^n$ and $r > n$ then there is an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(a)^{\oplus r-n} \longrightarrow E \longrightarrow F \longrightarrow 0,$$

where F is a bundle of rank n.

Lemma 10.3. *Let E be a globally generated bundle of rank r.*

Then E contains a trivial bundle of rank one if and only if $c_r(E) = 0$.

Proof. One direction is clear. If there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow F \longrightarrow 0$$

where F is a vector bundle then $c(E) = c(F)$ so that $c_r(E) = 0$.

For the other direction we may assume that $r \leq n$. We consider the same exact sequence as before,

$$0 \longrightarrow K \longrightarrow H^0(\mathbb{P}^n, E) \otimes \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow 0.$$

As before we get we get a morphism

$$f \colon \mathbb{P}(K) \longrightarrow \mathbb{P}(H^0(\mathbb{P}^n, E)),$$

Suppose that f is surjective. Suppose that $h^0(\mathbb{P}^n, E) = N + 1$. If $m = n + N - r$ then m is the dimension on $\mathbb{P}(K)$ and so the general fibre $Z = f^{-1}(\sigma)$ has dimension $n - r$. But then a general section vanishes in codimension r, a contradiction. It follows that f is not surjective. If σ is not in the image then σ is nowhwere vanishing, so that σ defines a trivial sub line bundle. □

We now turn to Tango's construction. We start with the Euler sequence,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

Suppose we take the $\wedge^{n-1}$th power of this sequence.

$$0 \longrightarrow \bigwedge^{n-2} T_{\mathbb{P}^n}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus \binom{n+1}{n-1}} \longrightarrow \bigwedge^{n-1} T_{\mathbb{P}^n}(-1) \longrightarrow 0.$$

Now for the last term we have

$$\begin{aligned} \bigwedge^{n-1} T_{\mathbb{P}^n}(-1) &\simeq \Omega^1_{\mathbb{P}^n}(1) \otimes \det T_{\mathbb{P}^n}(-1) \\ &\simeq \Omega^2_{\mathbb{P}^n}(2). \end{aligned}$$

Let

$$E = (\bigwedge^{n-2} T_{\mathbb{P}^n}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1))^*.$$

The dual sequence to the sequence above is

$$0 \longrightarrow T_{\mathbb{P}^n}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus \binom{n+1}{2}} \longrightarrow E \longrightarrow 0.$$

Thus E is globally generated of rank

$$r = \binom{n+1}{2} - n = \binom{n}{2}.$$

If $n \geq 3$ then $r \geq n$ and so there is an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r-n} \longrightarrow E \longrightarrow E' \longrightarrow 0,$$

for some vector bundle E' of rank n. E' is globally generated as it is a quotient of a globally generated vector bundle. The top chern of E' is

$$\begin{aligned} c_n(E') &= c_n(E) \\ &= 0. \end{aligned}$$

Thus E' contains a trivial subbundle. Let F be the quotient,

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow E' \longrightarrow F \longrightarrow 0.$$

Thus F is a vector bundle of rank $n-1$ with total chern class,

$$\begin{aligned} c(F) &= c(E') \\ &= c(E) \\ &= \frac{1}{c(T_{\mathbb{P}^n}(-2)} \\ &= \frac{1-2h}{(1-h)^{n+1}}. \end{aligned}$$

Finally we show that F is simple.

We will need the following three exact sequences:

$$0 \longrightarrow T_{\mathbb{P}^n}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus\binom{n+1}{2}} \longrightarrow E \longrightarrow 0.$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus r-n} \longrightarrow E \longrightarrow E' \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow E' \longrightarrow F \longrightarrow 0.$$

If we take the dual of the last exact sequence and tensor by F then we get

$$h^0(\mathbb{P}^n, F^* \otimes F) \leq h^0(\mathbb{P}^n, E'^* \otimes F).$$

Now take the dual of the second sequence and tensor by F to get

$$h^0(\mathbb{P}^n, E'^* \otimes F) \leq h^0(\mathbb{P}^n, E^* \otimes F).$$

So we are down to showing:

$$h^0(\mathbb{P}^n, E^* \otimes F) \leq 1.$$

Now tensor the second and third exact sequences with E^* to get

$$0 \longrightarrow E^* \longrightarrow E^* \otimes E' \longrightarrow E^* \otimes F \longrightarrow 0,$$

and

$$0 \longrightarrow E^{*\oplus r-n} \longrightarrow E^* \otimes E \longrightarrow E^* \otimes E' \longrightarrow 0,$$

Thus it remains to show that E is simple and $h^1(\mathbb{P}^n, E^*) = 0$.

Now

$$\begin{aligned} E^* &= \bigwedge^{n-2} T_{\mathbb{P}^n}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \\ &\simeq \bigwedge^{2} \Omega^1_{\mathbb{P}^n}(1) \otimes \det T_{\mathbb{P}^n}(-1) \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \\ &\simeq \Omega^2_{\mathbb{P}^n}(2). \end{aligned}$$

Thus

$$h^0(\mathbb{P}^n, E^*) = h^1(\mathbb{P}^n, E^*) = 0.$$

If we tensor the first exact sequence with E^* then we get

$$0 \longrightarrow E^* \otimes T_{\mathbb{P}^n}(-2) \longrightarrow E^{*\oplus\binom{n+1}{2}} \longrightarrow E^* \otimes E \longrightarrow 0.$$

Taking the long exact sequence of cohomology gives

$$h^0(\mathbb{P}^n, E^* \otimes E) = h^1(\mathbb{P}^n, E^* \otimes T_{\mathbb{P}^n}(-2)).$$

Finally we tensor the Euler sequence with $E^*(-1)$ to get

$$0 \longrightarrow E^*(-2) \longrightarrow E^{*\oplus n+1} \longrightarrow E * \otimes T_{\mathbb{P}^n}(-2) \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we are done, provided,

$$h^1(\mathbb{P}^n, E^*(-1)) = 0 \qquad \text{and} \qquad h^2(\mathbb{P}^n, E^*(-2)) = 0.$$

This follows from the Bott formula, since

$$E^* = \Omega^1_{\mathbb{P}^n}.$$

Putting all of this together we have

Theorem 10.4 (Tango)**.** *For every n there is a simple vector bundle of rank $(n-1)$ F on $\mathbb{P}^n$ with*

$$c(F) = \frac{1-2h}{(1-h)^{n+1}}.$$

11. The Serre construction

Suppose we are given a globally generated rank two vector bundle E on $\mathbb{P}^n$. Then the general global section σ of E vanishes in codimension two on a smooth subvariety Y. If E is decomposable then $\sigma = (F, G)$ where F and G are homogeneous polynomials, so that Y is the zero locus of F and G, a complete intersection. In fact we just need to know that Y has codimension two, in which case it has local complete intersection singularities, for this to work.

We want to reverse this process. Given a subvariety Y, with local complete intersection singularities, we want to construct a vector bundle E on Y and a global section which vanishes on Y. The idea is to extend the normal bundle (as Y is a lci, the normal sheaf is locally free) to a rank two vector bundle on $\mathbb{P}^n$.

Suppose we are given a rank two vector bundle E and a section $\sigma \in H^0(\mathbb{P}^n, E)$. We suppose that the zero locus Y of σ has codimension two. Locally E is trivial. If U is an open subset over which E is trivial, then σ corresponds to pair of regular functions f and g. It follows that Y is a local complete intersection and the ideal sheaf $\mathcal{I}_Y$ of Y is locally generated by f and g. In this case the conormal sheaf

$$N_Y^* = \frac{I_Y}{I_Y^2},$$

is locally free, with local generators f and g. Note that Y need not even be reduced.

Now we can write down a free resolution of the ideal sheaf on U.

$$0 \longrightarrow \mathcal{O}_U \longrightarrow \mathcal{O}_U \oplus \mathcal{O}_U \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

If the first map is α and the second β then we have

$$\alpha(r) = (-fr, gr) \qquad \text{and} \qquad \beta(s,t) = fs + gt.$$

It is easy to check this sequence is exact, since we can check it is exact on stalks and use the fact that f and g is a regular sequence.

We can globalise to the following short exact sequence

$$0 \longrightarrow \det E^* \longrightarrow E^* \longrightarrow \mathcal{I}_Y \longrightarrow 0,$$

where

$$\alpha(\phi_1 \wedge \phi_2) = \phi_1(\sigma)\phi_2 - \phi_2(\sigma)\phi_1 \qquad \text{and} \qquad \beta(\phi) = \phi(\sigma).$$

This sequence is called the **Koszul complex** for σ. If Y has codimension two then it gives a global resolution of $\mathcal{I}_Y$ by locally free sheaves.

If we restrict this exact sequence to Y we get

$$(\det E^*)|_Y \longrightarrow E^*|_Y \longrightarrow \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2} \longrightarrow 0.$$

Note that the first map is in fact the zero map, as can be checked locally. It follows that we get an isomorphism

$$E^*|_Y \simeq \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2}.$$

Theorem 11.1 (Serre). *Let Y be a local complete intersection of codimension two in $\mathbb{P}^n$. Suppose that the determinant of the normal bundle is the restriction of a line bundle on $\mathbb{P}^n$,*

$$\det N_{Y/\mathbb{P}^n} \simeq \mathcal{O}_Y(k) \qquad \text{for some} \qquad k \in \mathbb{Z}.$$

Then there is a rank two vector bundle E on $\mathbb{P}^n$ a global section σ with zero locus Y and there is an exact sequence induced by σ

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow \mathcal{I}_Y(k) \longrightarrow 0.$$

The chern classes of Y are given by

$$\begin{aligned} c_1(E) &= k \\ c_2(E) &= \deg Y. \end{aligned}$$

Proof. If there is a bundle with these properties then

$$\det N^*_{Y/\mathbb{P}^n} = \det E^*|_Y.$$

In this case

$$\det E^* = \mathcal{O}_{\mathbb{P}^n}(-k)$$

so that

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-k) \longrightarrow E^* \longrightarrow \mathcal{J}_Y \longrightarrow 0.$$

Now extensions of $\mathcal{J}_Y$ by $\mathcal{O}_{\mathbb{P}^n}(-k)$ are controlled by

$$\operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{J}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)).$$

There is a spectral sequence who E_2-term is

$$E_2^{p,q} = H^p(\mathbb{P}^n, \mathbf{Ext}^q_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)))$$

and whose E_∞-term is

$$E_\infty^{p+q} = \operatorname{Ext}^{p+q}_{\mathbb{P}^n}(\mathcal{J}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)).$$

Chasing the spectral sequence we get an exact sequence

$$\begin{aligned} 0 &\longrightarrow H^1(\mathbb{P}^n, \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \longrightarrow \operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{J}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \\ &H^0(\mathbb{P}^n, \mathbf{Ext}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \longrightarrow H^2(\mathbb{P}^n, \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))). \end{aligned}$$

On the other hand, the short exact sequence

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

gives to long exact sequence of ext,

$$\begin{aligned} 0 \longrightarrow \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \\ \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))). \end{aligned}$$

Since Y is a local complete intersection, it follows that it is Cohen-Macaulay. Therefore

$$\mathbf{Ext}^i_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) = 0,$$

for $i = 0$ and 1. Thus

$$\begin{aligned} \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) &= \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(-k)) \\ &= \mathcal{O}_{\mathbb{P}^n}(-k). \end{aligned}$$

Thus the long exact sequence we got from the spectral sequence becomes

$$\begin{aligned} 0 \longrightarrow H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \longrightarrow \\ H^0(\mathbb{P}^n, \mathbf{Ext}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \longrightarrow H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-k)). \end{aligned}$$

In particular, if $n > 3$ or $n = 2$ and $k < 3$ then

$$\mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq H^0(\mathbb{P}^n, \mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))).$$

Otherwise, we just get an exact sequence

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^2}(-k)) \longrightarrow H^0(\mathbb{P}^n, \mathbf{Ext}_{\mathbb{P}^2}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^2}(-k))) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-k)).$$

We turn to calculating

$$\mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)).$$

From the long exact sequence associated to

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

we get

$$\mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq \mathbf{Ext}^2_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)).$$

As Y is a local complete intersection of codimension two, we have the local fundamental isomorphism

$$\mathbf{Ext}^2_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq \mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\det \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2}, \mathcal{O}_Y(-k)).$$

By assumption

$$\frac{\mathcal{I}_Y}{\mathcal{I}_Y^2} \simeq \mathcal{O}_Y(-k),$$

so that
$$\mathbf{Hom}_{\mathcal{O}_{\mathbb{P}^n}}(\det \frac{\mathcal{I}_Y}{\mathcal{I}_Y^2}, \mathcal{O}_Y(-k)) \simeq \mathcal{O}_Y.$$

Putting all of this together we have
$$\mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq \mathcal{O}_Y.$$

It follows that
$$\mathrm{Ext}^1_{\mathbb{P}^n}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k)) \simeq H^0(\mathbb{P}^n, \mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))) \simeq H^0(\mathcal{O}_Y, \mathcal{O}_Y).$$

Let F be the extension corresponding to $1 \in H^0(Y, \mathcal{O}_Y)$,
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-k) \longrightarrow F \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$
Then F is a coherent sheaf.

Claim 11.2. *F is a locally free sheaf.*

Proof of (11.2). Pick $x \in \mathbb{P}^n$. Then the image 1_x of 1 in $\mathcal{O}_{\mathbb{P}^n,x}$ lives in
$$\mathbf{Ext}^1_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-k))_x = \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^n,x}}(\mathcal{I}_{Y,x}, \mathcal{O}_{\mathbb{P}^n,x}(-k)).$$
This defines the extension
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n,x}(-k) \longrightarrow F_x \longrightarrow \mathcal{I}_{Y,x} \longrightarrow 0.$$
Since the 1_x generates the $\mathcal{O}_{\mathbb{P}^n,x}$-module
$$\mathrm{Ext}^1_{\mathcal{O}_{\mathbb{P}^n,x}}(\mathcal{I}_{Y,x}, \mathcal{O}_{\mathbb{P}^n,x}(-k)) \simeq \mathcal{O}_{Y,x},$$
it follows that F_x is a free $\mathcal{O}_{\mathbb{P}^n,x}$-module by (11.3). □

□

Lemma 11.3 (Serre)**.** *Let A be a Noetherian local ring and let $I \triangleleft A$ be an ideal with free resolution of length* 1*:*
$$0 \longrightarrow A^p \longrightarrow A^q \longrightarrow I \longrightarrow 0.$$
If
$$0 \longrightarrow A \longrightarrow M \longrightarrow I \longrightarrow 0$$
represents $e \in \mathrm{Ext}^1_A(I, A)$ then M is locally free if and only if e generates the A-module $\mathrm{Ext}^1_A(I, A)$.

Proof. If we start with the short exact sequence
$$0 \longrightarrow A \longrightarrow M \longrightarrow I \longrightarrow 0$$
then we get a long exact sequence
$$\mathrm{Hom}_A(A, A) \longrightarrow \mathrm{Ext}^1_A(I, A) \longrightarrow \mathrm{Ext}^1_A(M, A) \longrightarrow \mathrm{Ext}^1_A(A, A) = 0.$$
Thus $\mathrm{Ext}^1_A(M, A) = 0$ if and only if the first map δ is surjective. Since $\delta(1) = e$, δ is surjective if and only if e generates the A-module $\mathrm{Ext}^1_A(I, A)$.

It remains to prove that if $\mathrm{Ext}^1_A(M, A) = 0$ then M is free. We have a pair of exact sequences

$$0 \longrightarrow A^p \longrightarrow A^q \longrightarrow I \longrightarrow 0.$$

and

$$0 \longrightarrow A \longrightarrow M \longrightarrow I \longrightarrow 0.$$

We lift the map $\phi\colon A^q \longrightarrow I$ to a map $\Phi\colon A^q \longrightarrow M$. Now define

$$\psi\colon (x, v) = \alpha(x) + \Phi(y),$$

where $\alpha\colon A \longrightarrow M$ is the first map. This gives us a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \oplus A^q \arrow[r] \arrow[d, "\psi"'] & A^q \arrow[r] \arrow[d, "\psi"'] \arrow[dl, "\Phi"'] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"'] & M \arrow[r, "\beta"'] & I \arrow[r] & 0
\end{tikzcd}$$

It follows that $\mathrm{Ker}\,\psi \simeq \mathrm{Ker}\,\phi \simeq A^p$ and $\mathrm{Coker}\,\psi = 0$. Thus we get an exact sequence

$$0 \longrightarrow A^p \longrightarrow A^r \longrightarrow M \longrightarrow 0$$

where $r = q + 1$. As $\mathrm{Ext}^1_A(M, A) = 0$, this sequence splits. Thus M is a direct summand of A^r, so that M is projective. As A is local it follows that M is free. $\square$

12. Examples

We start with a useful:

Lemma 12.1. *Let E be a rank two vector bundle on $\mathbb{P}^n$ and let σ be a global section of E such that the local complete intersection $Z(\sigma) = Y$ has codimension two.*

Then E is decomposable if and only if Y is a complete intersection.

Proof. Suppose that E is decomposable. Then $E \sim \mathcal{O}_{\mathbb{P}^n}(a) \oplus \mathcal{O}_{\mathbb{P}^n}(b)$ for some a and b. Then $\sigma = (f, g)$ where f and g are homogeneous polynomials of degrees a and b. In this case $Y = Z(f) \cap Z(g)$.

Now suppose that Y is the intersection of the hypersurfaces W and V. Then there are homogeneous polynomials f and g of degrees a and b such that $W = Z(f)$ and $V = Z(g)$. Thus

$$f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a)) \qquad \text{and} \qquad g \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(b)).$$

The Koszul complex of the section

$$\sigma = (f, g) \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a)) \oplus H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(b))$$

gives the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(a+b)) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(a)) \oplus \mathcal{O}_{\mathbb{P}^n}(-(b)) \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

Thus we get a non-zero element of

$$\operatorname{Ext}^1_{\mathbb{P}^n}(\mathcal{I}_Y, \mathcal{O}_{\mathbb{P}^n}(-(a+b))) \simeq H^0(Y, \mathcal{O}_Y).$$

It is enough to show that the last group is one dimensional. In this case there is only one non-trivial extension of $\mathcal{I}_Y$ by $\mathcal{O}_{\mathbb{P}^n}(-(a+b))$, so that E is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-(a)) \oplus \mathcal{O}_{\mathbb{P}^n}(-(b))$.

Claim 12.2. $h^0(Y, \mathcal{O}_Y) = 1$ *for every complete intersection Y of codimension two in $\mathbb{P}^n$, $n \geq 3$.*

Proof of (12.2). From the long exact sequence of cohomology associated to

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(a+b)) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-(a)) \oplus \mathcal{O}_{\mathbb{P}^n}(-(b)) \longrightarrow \mathcal{I}_Y \longrightarrow 0,$$

we see that

$$h^0(Y, \mathcal{I}_Y) = h^1(Y, \mathcal{I}_Y) = 0.$$

From the long exact sequence of cohomology associated to

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathbb{P}^n \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

we see that

$$h^0(Y, \mathcal{O}_Y) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 1$$

□

□

Lemma 12.3. *Let Y be a local complete intersection in $\mathbb{P}^n$.*

Then
$$\omega_Y \simeq \omega_{\mathbb{P}^n} \otimes \det N_{Y/\mathbb{P}^n}.$$
In particular $\det N_{Y/\mathbb{P}^n}$ *is the restriction of a line bundle if and only if* ω_Y *is the restriction of a line bundle.*

Proof. The first result is adjunction. The second result follows, as
$$\omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1).$$
□

It is not hard to write down local complete intersection curves in $\mathbb{P}^3$, who canonical divisor is the restriction of a line bundle on $\mathbb{P}^3$ and yet the curve is not a complete intersection.

We start with an easier case.

Example 12.4. *Let Y be m reduced points $p_1, p_2, \dots, p_m$ in $\mathbb{P}^2$.*

As Y is zero dimensional it follows that every vector bundle on Y is trivial. Recall that we can apply Serre's result in $\mathbb{P}^2$ provided that $k < 3$. Thus we get vector bundles E of rank two on $\mathbb{P}^2$, which are extensions:
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow E \longrightarrow \mathcal{I}_Y(k) \longrightarrow 0.$$
Note that E has Chern classes
$$\begin{aligned} c_1(E) &= k \\ c_2(E) &= m. \end{aligned}$$

Suppose that $k = 1$. If L is a line that does not meet Y then the exact sequence above reduces to
$$0 \longrightarrow \mathcal{O}_L \longrightarrow E|_L \longrightarrow \mathcal{O}_L(1) \longrightarrow 0.$$
This sequence splits, as
$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0.$$
Thus the splitting type is $(1,0)$ and this is the generic splitting type.

If $k = 2$ then this argument breaks down as
$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \neq 0.$$
In fact the generic splitting type is $(1,1)$.

If L is a line which meets one point x_i of S then $E|_L$ has a section $\sigma|_L$ which vanishes at x_i and nowhere else. In this case, the restriction of the short exact sequence above becomes
$$0 \longrightarrow \mathcal{O}_L(1) \longrightarrow E|_L \longrightarrow \mathcal{O}_L(1) \longrightarrow 0.$$
This splits, as
$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0,$$

so that the splitting type of L is $(1,1)$. Thus the generic splitting type must be $(1,1)$.

In the case $k=2$ the set of jump lines is precisely the set of lines which meet two or more points of Y. The case $k=1$ is more subtle and in fact the set of jump lines is a curve of degree $m-1$ in the dual $\mathbb{P}^2$.

Example 12.5. *Let Y be a union of $d>1$ distinct lines $L_1, L_2, \ldots, L_d$ in $\mathbb{P}^3$.*

As $L_i = H_i \cap G_i$ is the intersection of two hyperplanes it follows that

$$\begin{aligned} N_{Y/\mathbb{P}^3}|_{L_i} &= N_{L_i/\mathbb{P}^3} \\ &= N_{H_i/\mathbb{P}^3}|_{L_i} \oplus N_{G_i/\mathbb{P}^3}|_{L_i} \\ &= \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1). \end{aligned}$$

It follows that $N_{Y/\mathbb{P}^3}|_{L_i} = \mathcal{O}_1 2.$ and so $N_{Y/\mathbb{P}^3} = \mathcal{O}_Y(2)$. By Serre's construction there is a rank two vector indecomposable bundle E, which fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow E \longrightarrow \mathcal{I}_Y(2) \longrightarrow 0,$$

with Chern classes

$$\begin{aligned} c_1(E) &= 2 \\ c_2(E) &= d. \end{aligned}$$

If we restrict E to a line that meets one of the L_i transversally, arguing as above, we see that the generic splitting type is $(1,1)$. If L is a line that meets two of the lines of Y transversally then the L is a jumping line, so that the locus of jumping lines contains a codimension two subvariety of $\mathbb{G}(1,3)$.

Example 12.6. *Let Y be the union of r elliptic curves C_i of degree d_i in $\mathbb{P}^3$.*

For any such curve, there is an exact sequence

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^3} \longrightarrow N_{C/\mathbb{P}^3} \longrightarrow 0.$$

As the tangent bundle of an elliptic curve is trivial it follows that

$$\begin{aligned} \det N_{C/\mathbb{P}^3} &\simeq \det T_{\mathbb{P}^3}|_C \\ &= \mathcal{O}_C(4). \end{aligned}$$

Therefore

$$N_{Y/\mathbb{P}^3} = \mathcal{O}_Y(4).$$

There is then an associated rank two vector bundle E with

$$\begin{aligned} c_1(E) &= 4 \\ c_2(E) &= \sum d_i. \end{aligned}$$

As a special case, if we take two plane elliptic curves C_1 and C_2 sitting in different planes H_1 and H_2 then $d_1 = d_2 = 3$ and $F = E(-2)$ is a rank two vector bundle with

$$\begin{aligned} c_1(F) &= 0 \\ c_2(F) &= 2. \end{aligned}$$

Example 12.7. *Let Y be the union of r disjoint conics $D_1, D_2, \dots, D_r$ in $\mathbb{P}^3$.*

If $D \subset H$ is a conic sitting in a plane H then there is an exact sequence

$$0 \longrightarrow N_{D/H} \longrightarrow N_{D/\mathbb{P}^3} \longrightarrow N_{H/\mathbb{P}^3}|_D \longrightarrow 0.$$

It follows that

$$\begin{aligned} \det N_{D/\mathbb{P}^3} &\simeq N_{D/H} \otimes N_{H/\mathbb{P}^3}|_D \\ &\simeq \mathcal{O}_D(2) \otimes \mathcal{O}_D(1) \\ &= \mathcal{O}_D(3). \end{aligned}$$

and so

$$N_{Y/\mathbb{P}^3} = \mathcal{O}_Y(3).$$

The rank two vector bundle E associated to Y has Chern classes

$$\begin{aligned} c_1(E) &= 3 \\ c_2(E) &= 2r. \end{aligned}$$

The generic splitting type is $(2, 1)$.

Example 12.8. *Now suppose we pick a union Y of complete intersection curves Y_i in $\mathbb{P}^3$.*

Pick r pairs of natural numbers (a_i, b_i), with $a_i + b_i = p$ constant. Pick polynomials

$$f_i \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(a_i)) \qquad \text{and} \qquad g_i \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(b_i)).$$

Let $Y_i = Z(f_i) \cap Z(g_i)$. If we pick $f_1, f_2, \dots, f_r$ and $g_1, g_2, \dots, g_r$ appropriately, $Y_1, Y_2, \dots, Y_r$ are smooth pairwise disjoint curves. Let Y be their union.

The Koszul complex for Y_i is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-(a_i + b_i)) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-a_i) \oplus \mathcal{O}_{\mathbb{P}^3}(-b_i) \longrightarrow \mathcal{I}_{Y_i} \longrightarrow 0.$$

It follows that

$$\begin{aligned}\det N_{Y_i/\mathbb{P}^3} &\simeq \mathcal{O}_{\mathbb{P}^3}(-(a_i+b_i))|_{Y_i}\\ &\simeq \mathcal{O}_{\mathbb{P}^3}(-p)|_{Y_i}.\end{aligned}$$

Thus

$$\det N_{Y/\mathbb{P}^3} = \mathcal{O}_Y(p).$$

The associated rank 2 vector bundle E associated to Y has

$$\begin{aligned}c_1(E) &= p\\ c_2(E) &= \sum a_i b_i.\end{aligned}$$

13. Topological versus holomorphic classification I

It is interesting to compare the topological versus the holomorphic classification of vector bundles on $\mathbb{P}^n$. Of course every holomorphic vector bundle gives rise to a topological vector bundle. A priori, topological vector bundle means that the transition functions are continuous but it is not hard to show that we can smooth the transition functions and that two smooth vector bundles that are topologically equivalent are isomorphic as smooth vector bundles.

We first review the topological classification. Most of the time we will not provide any proofs. Recall that topological line bundles on $\mathbb{P}^n$ are classified by their first chern class c_1, which is an integer. Thus the topological and the holomorphic classification coincide for line bundles.

We already showed that topological vector bundles on $\mathbb{P}^1$ are classified by their first chern class. Holomorphic vector bundles or rank r are classified by a decreasing sequence of integers

$$a_1 \geq a_2 \geq \cdots \geq a_r.$$

Two such are topologically equivalent if

$$\sum a_i = \sum b_i.$$

Schwarzenberger showed that the chern classes of holomorphic vector bundles must satisfy various congruences, as a consequence of the (Hirzebruch)-Riemann-Roch theorem. On the other hand, if one applies the Atiyah-Singer index theorem the Schwarzenberger conditions also hold for any topological vector bundle.

Suppose that we encode the chern classes as a polynomial with integer coefficients

$$c_t(E) = 1 + c_1(E)t + \cdots + c_r(E)t^r \in \mathbb{Z}[t].$$

$c_t(E)$ is called the **chern polynomial** of E. Suppose that we factor this polynomial over the complex numbers

$$c_t(E) = \prod_{i=1}^{r}(1 + x_i t).$$

The numbers $c_1, c_2, \dots, c_r$ must then satisfy the **Schwarzenberger condition**,

$$(S^r_n) \qquad \sum_{i=1}^{r} \binom{n + x_i + s}{s} \in \mathbb{Z} \qquad \text{for every} \qquad s \in \mathbb{Z}.$$

It is straightforward to calculate what these conditions reduce to for low values of r and n. S^1_n and S^2_2 impose no conditions at all. S^2_3 is equivalent to

$$c_1 c_2 \equiv 0 \mod 2.$$

S_4^2 is equivalent to

$$c_2(c_2 + 1 - 3c_1 - 2c_1^2) \equiv 0 \mod 12.$$

Finally, S_3^3 is equivalent to

$$c_1 c_2 \equiv c_3 \mod 2.$$

Note that if E is a rank r bundle over $\mathbb{P}^n$ and $r \geq n$ then E has $n-r$ linearly independent sections. These linearly independent sections define a sub vector bundle of rank $n-r$ and this bundle splits off as a direct summand (using partitions of unity). Thus

$$E \simeq E' \oplus (\mathbb{P}^n \times \mathbb{C}^{n-4})$$

where E' is a vector bundle of rank n.

Topological vector bundles are completely classified. There is one bundle for each collection of chern classes which satisfy the Schwarzenberger conditions.

So for $n = 2$ the topological classification of rank two vector bundles corresponds to pairs of integers (c_1, c_2). For $n = 3$ the classification of rank three vector bundles corresponds to triples of integers (c_1, c_2, c_3) subject to $c_1 c_2 \equiv c_3 \mod 2$.

Now consider rank two vector bundles over $\mathbb{P}^3$. The Schwarzenberger condition is

$$c_1 c_2 \equiv 0 \mod 2.$$

It is known that there always a rank two vector bundle with these chern classes. If c_1 is odd there is one and if c_1 is even the are two topologically inequivalent bundles. These two bundles are distinguished by the α-invariant. If $c_1(E) = 2k$ then $c_1(E(-k)) = 0$. In this case the structure group can be reduced to $\mathrm{Sp}(1) \subset U(2)$.

Thus we just have to classify the symplectic line bundles on $\mathbb{P}^3$. Symplectic line bundles are topologically stable, classified by the group $\mathbb{Z} \oplus \mathbb{Z}_2$. Let π be the projection onto the second factor. The α invariant of a vector bundle E such that $c_1(E) = 2k$ is then given by

$$\alpha(E) = \pi(E(-k)) \in \mathbb{Z}_2.$$

If E is a holomorphic rank two vector bundle on $\mathbb{P}^3$ with $c_1(E) = 2k$ then

$$\alpha(E) \equiv h^0(\mathbb{P}^3, E(-k-2)) + h^1(\mathbb{P}^3, E(-k-2)) \mod 2.$$

Let us turn to the problem of which topological vector bundles have a holomorphic structure. We first treat the case of rank two on $\mathbb{P}^2$.

Recall that there is one topological vector bundle for every pair of integers (c_1, c_2).

Theorem 13.1 (Schwarzenberger)**.** *For every pair of integers* $(c_1, c_2) \in \mathbb{Z}^2$ *there is a holomorphic vector bundle* E *or rank two on* $\mathbb{P}^2$*.*

Proof. Let $\pi \colon X \longrightarrow \mathbb{P}^2$ blow up four points x_1, x_2, x_3 and x_4 of $\mathbb{P}^2$. Let E_i be the exceptional divisor over x_i and let

$$L = \mathcal{O}_X(D)$$

be the line bundle associated to the divisor $D \sum k_i E_i \geq 0$.

We consider extensions

$$0 \longrightarrow L \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(b) \longrightarrow V' \longrightarrow L^* \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(a) \longrightarrow 0.$$

If $V' = \pi^* V$ for some rank two vector bundle V then V has chern classes

$$\begin{aligned} c_1(V) &= a + b \\ c_2(V) &= c_2(\pi^* V) \\ &= (D + b\pi^* H)(-D + a\pi^* H) \\ &= \sum k_i^2 + ab, \end{aligned}$$

where we used the fact that

$$E_i \cdot E_j = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

and $E_i \cdot \pi^* H = 0$.

As every positive integer is the sum of four squares, note that we can always choose a, b, $k_1 \geq 0$, $k_2 \geq 0$, $k_3 \geq 0$ and $k_4 \geq 0$ such that

$$\begin{aligned} c_1 &= a + b \\ c_2 &= \sum k_i^2 + a + b, \end{aligned}$$

where $a - b < 0$.

Proceeding in a similar manner to before, one can check that a vector bundle V' given as an extension

$$0 \longrightarrow L \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(b) \longrightarrow V' \longrightarrow L^* \otimes \pi^* \mathcal{O}_{\mathbb{P}^2}(a) \longrightarrow 0.$$

is the pullback of a vector bundle V from $\mathbb{P}^2$ if and only if its restriction to any of the exceptional curves E_i is of the form

$$0 \longrightarrow \mathcal{O}_{E_i}(-k_i) \longrightarrow \mathcal{O}_{E_i}^{\oplus 2} \longrightarrow \mathcal{O}_{E_i}(-k_i) \longrightarrow 0.$$

(Recall that one shows that there is only one such extension over a neighbourhood of each exceptional.)

We are thus reduced to

Claim 13.2. *For a and $b \in \mathbb{Z}$, $a - b < 0$ and $k_i \geq 0$, $i = 1, 2, 3$ and 4 there is an extension*

$$0 \longrightarrow L \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(b) \longrightarrow V' \longrightarrow L^* \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a) \longrightarrow 0.$$

with

$$V'|_{E_i} \simeq \mathcal{O}_{E_i}^{\oplus 2}.$$

Proof of (13.2). We may assume that $b = 0$. In this case, extensions of the form

$$0 \longrightarrow L \longrightarrow V' \longrightarrow L^* \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a) \longrightarrow 0.$$

are classified by

$$\operatorname{Ext}^1_X(L^* \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a), L) = H^1(X, L^{\otimes 2} \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(-a)).$$

Let $E = E_1 + E_2 + E_3 + E_4$. Since there are extensions

$$0 \longrightarrow \mathcal{O}_E(D) \longrightarrow \mathcal{O}_E^{\oplus 2} \longrightarrow \mathcal{O}_E(-D) \longrightarrow 0,$$

it suffices to show that

$$H^1(X, L^{\otimes 2} \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(-a)) \longrightarrow H^1(E, L^{\otimes 2})$$

is surjective. It suffices to show that

$$H^2(X, \mathcal{O}_X(2D - E) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(-a)) = 0$$

As

$$K_X = \pi^*K_{\mathbb{P}^2} + E,$$

it follows that

$$\begin{aligned} h^2(X, \mathcal{O}_X(2D - E) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(-a)) &= h^0(X, \mathcal{O}_X(2E - 2D) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a - 3)) \\ &\leq h^0(X, \mathcal{O}_X(2E) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a - 3)) \end{aligned}$$

If we take the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{O}_X((k-1)E) \longrightarrow \mathcal{O}_X(kE) \longrightarrow \mathcal{O}_E(kE) \longrightarrow 0,$$

we see that

$$h^0(X, \mathcal{O}_X(kE) \otimes \pi^*\mathcal{O}_{\mathbb{P}^2}(a - 3)) = h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^2}(a - 3)),$$

for all $k \geq 0$. But

$$h^0(X, \pi^*\mathcal{O}_{\mathbb{P}^2}(a - 3)) \leq h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a - 3)) = 0,$$

as $a < 0$. □

□

14. Topological versus holomorphic classification II

We now turn to the problem of putting a holomorphic structure on every topological rank two vector bundle on $\mathbb{P}^3$. This means that given integers c_1 and c_2 such that $c_1c_2 \equiv 0 \mod 2$ then we have to find a holomorphic rank two vector bundle E with chern classes c_1 and c_2 and if c_1 is even, in addition we have to pick E so that

$$\alpha(E) \equiv h^0(\mathbb{P}^3, E(-k-2)) + h^1(\mathbb{P}^3, E(-k-2)) \mod 2$$

is both odd and even.

If we put

$$d(m) = \begin{cases} 1 & \text{if } m \geq 0 \text{ and } m \equiv 0 \mod 4 \\ 0 & \text{otherwise} \end{cases}$$

then note that

$$d(m) \equiv h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(m)) \mod 2.$$

We are going to use the Serre construction.

Lemma 14.1. *Let $Y \subset \mathbb{P}^3$ be the codimension two zero locus of a section $\sigma \in H^0(\mathbb{P}^3, E)$ of a rank two vector bundle E such that $c_1(E) = 2k$.*

Then

$$\alpha(E) \equiv h^0(Y, \mathcal{O}_Y(k-2)) + d(k-2) \mod 2.$$

Proof. Consider the Koszul complex of the section σ

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2k) \longrightarrow E^* \longrightarrow \mathcal{I}_Y \longrightarrow 0.$$

By twisting with $\mathcal{O}_{\mathbb{P}^3}(k-2)$ we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-k-2) \longrightarrow E^*(k-2) \longrightarrow \mathcal{I}_Y(k-2) \longrightarrow 0.$$

Now

$$\begin{aligned} E^*(k-2) &\simeq E \otimes \det E^* \otimes \mathcal{O}_{\mathbb{P}^3}(-k-2) \\ &\simeq E(-k-2). \end{aligned}$$

Therefore the long exact sequence of cohomology associated to the short exact sequence gives

$$0 \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-k-2)) \longrightarrow H^0(\mathbb{P}^3, E(-k-2)) \longrightarrow H^0(\mathbb{P}^3, \mathcal{I}_Y(k-2)) \longrightarrow 0$$

and

$$0 \longrightarrow H^1(\mathbb{P}^3, E(-k-2)) \longrightarrow H^1(\mathbb{P}^3, \mathcal{I}_Y(k-2)) \longrightarrow 0.$$

Finally from the exact sequence

$$0 \longrightarrow \mathcal{I}_Y(k-2) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(k-2) \longrightarrow \mathcal{O}_Y(k-2) \longrightarrow 0,$$

one obtains

$$0 \to H^0(\mathbb{P}^3, \mathcal{I}_Y(k-2)) \longrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k-2)) \longrightarrow H^0(Y, \mathcal{O}_Y(k-2)) \longrightarrow H^1(\mathbb{P}^3, \mathcal{I}_Y(k-2)) \to 0.$$

Putting all of this together we get

$$\begin{aligned}
\alpha(E) &= h^0(\mathbb{P}^3, E(-k-2)) + h^1(\mathbb{P}^3, E(-k-2)) \\
&= h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-k-2)) + h^0(\mathbb{P}^3, \mathcal{I}_Y(k-2)) + h^1(\mathbb{P}^3, \mathcal{I}_Y(k-2)) \\
&\equiv h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(k-2)) + h^0(Y, \mathcal{O}_Y(k-2)) \\
&\equiv h^0(Y, \mathcal{O}_Y(k-2)) + d(k-2) \mod 2.
\end{aligned}$$

□

Example 14.2. *Let Y be the intersection of two hypersurfaces V_a and V_b of degrees a and b, where $a \leq b$ and $a + b = 2k$ is even.*

The associated rank two vector bundle E splits as a direct sum $\mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b)$. We get

$$\begin{aligned}
&h^0(\mathbb{P}^3, E(-k-2)) = h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(\frac{b-a}{2} - 2)) \\
&h^1(\mathbb{P}^3, E(-k-2)) = 0
\end{aligned}$$

and so

$$\alpha(E) = d(\frac{b-a}{2} - 2).$$

Lemma 14.3. *Let σ be a section of a rank two vector bundle E, where $c_1(E) = 2k$. If the zero scheme Y of σ is a disjoint union of r complete intersections $Y_i = V_{a_i} \cap V_{b_i}$, $i = 1, 2, 3, \ldots, r$ with $a_i \leq b_i$ and $a_i + b_i = 2k$ then*

$$\alpha(E) \equiv (r-1)d(k-2) + \sum_{i=1}^{r} d(k - a_i - 2) \mod 2.$$

Proof. We already know that

$$\begin{aligned}
&\alpha(E) + d(k-2) \equiv h^0(Y, \mathcal{O}_Y(k-2)) \mod 2 \qquad \text{so that} \\
&\alpha(E) + d(k-2) \equiv \sum_{i=1}^{r} h^0(Y_i, \mathcal{O}_{Y_i}(k-2)) \mod 2.
\end{aligned}$$

Let $E_i = \mathcal{O}_{\mathbb{P}^3}(a_i) \oplus \mathcal{O}_{\mathbb{P}^3}(b_i)$ be the rank two vector bundle associated to Y_i. Then

$$\alpha(E_i) + d(k-2) \equiv h^0(Y_i, \mathcal{O}_{Y_i}(k-2)) \mod 2,$$

so that

$$\alpha(E) + d(k-2) \equiv \sum_{i=1}^{r} (\alpha(E_i) + d(k-2)) \mod 2$$

that is,

$$\alpha(E) \equiv (r-1)d(k-2) + \sum_{i=1}^{r} \alpha(E_i) \mod 2.$$

However, we proved in the example that

$$\alpha(E_i) \equiv d(k-a_i-2) \mod 2. \qquad \square$$

Define a function

$$\Delta\colon \mathbb{Z}^2 \longrightarrow \mathbb{Z},$$

by the formula

$$\Delta(c_1, c_2) = c_1^2 - 4c_2.$$

Then define

$$\Delta(E) = \Delta(c_1(E), c_2(E)).$$

Note that

$$c_1(E(k)) = c_1(E) + 2k \qquad \text{and} \qquad c_2(E(k)) = c_2(E) + kc_1(E) + k^2,$$

and so

$$\begin{aligned}\Delta(E(k)) &= (c_1(E) + 2k)^2 - 4(c_2(E) + kc_1(E) + k^2)) \\ &= c_1(E)^2 - 4c_2(E) \\ &= \Delta(E).\end{aligned}$$

Further

$$\Delta(c_1, c_2) \equiv \begin{cases} 0 \mod 4 & \text{when } c_1 \text{ is even} \\ 1 \mod 4 & \text{when } c_1 \text{ is odd.} \end{cases}$$

Theorem 14.4 (Atiyah, Horrocks, Rees)**.** *Every topological rank two vector bundle on $\mathbb{P}^3$ has a holomorphic structure.*

Proof. Suppose that we have a pair of integers (c_1, c_2) such that $c_1c_2 \equiv 0 \mod 2$.

We construct a holomorphic rank two vector bundle E with

$$\begin{aligned}c_1(E) &= c_1 \\ c_2(E) &= c_2.\end{aligned}$$

Let Y be the union of disjoint complete intersection curves of type (a_i, b_i) where $a_i + b_i = p$ is constant. If E is the associated rank two vector bundle then E has chern classes

$$\begin{aligned}c_1(E) &= p \\ c_2(E) &= \sum a_i b_i.\end{aligned}$$

We assume that $a_i \leq b_i$ so that $a_i \leq p/2$.

If we fix c_1 and c_2 then it is not hard to argue that we may find p, r and $a_1, a_2, \dots, a_r$ such

$$p^2 - 4\sum_{i=1}^{r} a_i b_i = \Delta(c_1, c_2).$$

Further we may arrange for p to have the same parity as c_1. Let

$$E' = E\left(\frac{c_1 - p}{2}\right).$$

Then

$$\begin{aligned} c_1(E') &= c_1(E) + c_1 - p = c_1 \\ c_2(E') &= c_2(E) + \frac{c_1 - p}{2} c_1(E) + \frac{(c_1 - p)^2}{4} = c_2. \end{aligned}$$

Now suppose that c_1 is even. We have to realise both values of the α-invariant. The α invariant of E' is

$$\begin{aligned} \alpha(E') &\equiv \alpha(E) \\ &\equiv (r-1)d(p/2 - 2) + \sum_{i=1}^{r} d(p/2 - a_i - 2) \mod 2. \end{aligned}$$

It is possible to show, by elementary but involved calculations in number theory, that we can both parities, as we vary p, r and $a_1, a_2, \dots, a_r$. □

We now consider the case of rank three.

Theorem 14.5 (Vogelaar)**.** *Every topological rank three bundle on $\mathbb{P}^3$ has a holomorphic structure.*

In particular

Corollary 14.6. *Every topological bundle on $\mathbb{P}^3$ has a holomorphic structure.*

We will need an extension of the Serre construction from codimension two to codimension three. The key point in the Serre construction is that since the line bundle

$$\det N_{Y/\mathbb{P}^n}(-k)$$

is trivial then there is extension

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow \mathcal{I}_Y(k) \longrightarrow 0$$

such that E is locally free.

For codimension three, the idea is to consider extensions

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n} \longrightarrow E \longrightarrow \mathcal{I}_Y(k) \longrightarrow 0$$

and to try to prove that E is locally free.

Theorem 14.7. *Let Y be a codimension three local complete intersection.*

If the bundle

$$\det N_{Y/\mathbb{P}^n}(-k)$$

is generated by two global sections t_1 and t_2 then there is a vector bundle E of rank three with two sections s_1 and s_2 that are linearly dependent over Y.

15. An argument of Ran

We recall a famous conjecture of Hartshorne:

Conjecture 15.1 (Hartshorne)**.** *Let $Y \subset \mathbb{P}^n$ be a smooth subvariety. If $2 \operatorname{codim} Y < \dim Y$ then Y is a complete intersection.*

The first thing to say is that (15.1) is sharp. For example, consider the Grassmannian $\mathbb{G}(1, 4)$ of lines in $\mathbb{P}^4 = \mathbb{P}(V)$. Under the Plücker embedding this gets mapped into

$$Y \subset \mathbb{P}^9 = \mathbb{P}(\bigwedge^2 V).$$

Y has dimension 6 and codimension 3, but the ideal of Y is generated by quadrics.

It is interesting to specialise this conjecture to the case of codimension two. In this case the conjecture becomes interesting if Y has dimension at least five, that is, $n \geq 7$. By the Serre correspondence this translates to:

Conjecture 15.2. *Every vector bundle of rank two on $\mathbb{P}^7$ splits.*

There is very little evidence for (15.2). We present an argument of Ziv Ran which gives the best results.

Theorem 15.3. *Let $Y \subset \mathbb{P}^{m+2}$ be a locally complete intersection subvariety of codimension two. Let $N = N_{Y/\mathbb{P}^{m+2}}$ be the normal bundle. Let $d = c_2(N)$. Suppose that $\det N \simeq \mathcal{O}_Y(k)$.*

If

(1)

$$k \geq \frac{d}{m} + m,$$

or

(2) $d \leq m$

then Y is a complete intersection.

There is an argument due to Barth that if Y is smooth, the characteristic is zero and $m \geq 4$ that the condition $\det N \simeq \mathcal{O}_Y(d)$ is vacuous.

If $\det N \simeq \mathcal{O}_Y(k)$ then by adjunction we have

$$\omega_X = \mathcal{O}_Y(k - m - 3).$$

Therefore (i) means that ω_X is large relative to d and m.

The idea of the proof of (15.3) goes back to a simple observation of Severi. If Y is contained in a hyersurface X of degree u then every $(u+1)$-secant line to Y must be contained in X. Therefore there is no $(u+1)$-secant line to Y through a general point of projective space.

The idea is to try to reverse this argument. Severi showed that if Y is a surface in $\mathbb{P}^4$ and the 2-secants to Y don't span the whole of $\mathbb{P}^4$ then Y is contained in a quadric. We are going to generalise this argument. If the $(m+1)$-secants to Y don't pass through a general point of $\mathbb{P}^{m+2}$ then Y is a contained in a hyersurface of degree at most m.

Let E be the rank two vector bundle associated to Y. We have the Koszul complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{m+2}} \longrightarrow E \longrightarrow \mathcal{I}_Y(k) \longrightarrow 0$$

We have

$$\begin{aligned} c_1(E) &= k \\ c_2(E) &= d. \end{aligned}$$

Define a function

$$e\colon \mathbb{Z} \longrightarrow \mathbb{Z}$$

by the rule

$$\begin{aligned} e(t) &= c_2(E(-t)) \\ &= c_2(E) - c_1(E)t + t^2 \\ &= d - t(k-t). \end{aligned}$$

Consider the incidence correspondence for the Grassmannian $\mathbb{G}(1, m+2))$ of lines in $\mathbb{P}^{m+2}$,

$$I = \{\, (p, L) \in \mathbb{P}^{m+2} \times \mathbb{G}(1, m+2) \,|\, p \in L \,\} \subset \mathbb{P}^{m+2} \times \mathbb{G}(1, m+2).$$

There are two natural projections

$$\begin{tikzcd} I \arrow[r, "g"] \arrow[d, "f"'] & \mathbb{G}(1, m+2). \\ \mathbb{P}^{m+2} & \end{tikzcd}$$

Let p be a general point of $\mathbb{P}^{m+2}$ and let

$$\Sigma_u = \Sigma_{u,p} = \{\, L \in \mathbb{G}(1, m+2) \,|\, \text{the length of } L \cap Y \text{ is at least } k \,\}.$$

We think of Σ_u as the set of u-secant lines to Y. If the intersection $L \cap Y$ is reduced then L is u-secant. If $u = 2$ and $L \cap Y$ is not reduced then L is tangent to Y at the point $L \cap Y$ and of course tangent lines are limits of secant lines.

There are strong analogies between the splitting type of a line and the number of times it is secant. The fact that Σ_u is non-empty is akin to the existence of jumping lines.

Proposition 15.4. *If none of the integers from* 0 *to* u *are roots of the polynomial* $e(t)$ *and* $u \leq m$ *then* Σ_{u+1} *is non-empty.*

Proof. We will prove the stronger statement that $\dim \Sigma_i = 2(m+1)-i$, for $i \leq u+1$.

Note that Σ_0 has dimension $2(m+1)$. So by induction it suffices to prove that $\Sigma_{u+1} \subset \Sigma_u$ is a divisor. For this it suffices to show that if that $C \subset \Sigma_u$ is an irreducible curve and C does not intersect Σ_{u+1} then $e(u) = 0$.

Our hypotheses imply that the length of $L \cap Y$ is equal to u for all $L \in C$. Let $\tilde{C} \longrightarrow C$ be the normalisation of C and let $\gamma \colon S \longrightarrow \tilde{C}$ be the pullback of the $\mathbb{P}^1$ bundle $g \colon I \longrightarrow \mathbb{G}(1, m+2)$. Let $\phi \colon S \longrightarrow \mathbb{P}^{m+2}$ be the natural map. Let $D = \pi^{-1}(Y)$, where we pullback Y as a scheme.

Consider the map $D \longrightarrow \tilde{C}$. By assumption the length of a fibre is equal to u, a constant. Therefore D is flat over $\tilde{C}$, hence Cohen-Macaulay. But then D is a Cartier divisor in S.

Let F be fibre of π and let Z be the section of π contracted down by ϕ to p. Let H be the pullback of a hyperplane. Then there is a numerical equivalence

$$D \equiv uH + lF,$$

for some l. But as $D \cdot Z = 0$, it follows that $l = 0$, so that

$$D \equiv uH.$$

As the map of sheaves

$$\mathcal{O}_S \longrightarrow \phi^* E.$$

vanishes on D, there is a short sequence

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \phi^* E \longrightarrow Q \longrightarrow 0,$$

where Q is a line bundle. If we use this to compute chern classes then we get

$$\begin{aligned} c_1(Q) &= c_1(\phi^* E) - c_1(\mathcal{O}_S(D)) \\ &= (k-u)H \end{aligned}$$

and so

$$\begin{aligned} c_2(\phi^* E) &= c_1(\mathcal{O}_S(D)) c_1(Q) \\ &= (k-u)u. \end{aligned}$$

But then $d = u(k-u)$, that is, $e(u) = 0$. $\square$

Corollary 15.5. *If none of the integers from* 0 *to* u *are roots of the polynomial* $e(t)$ *and* $u \leq m$ *then* Y *is not contained in a hypersurface of degree* k*.*

Proof. (15.4) implies that Σ_{u+1} is non-empty. But then Y is not contained in a hypersurface of degree k. $\square$

We now prove (15.3).

16. Stable vs unstable

Vector bundles are naturally divided into two quite distinct types, stable and unstable. We will only scrape the surface of this important topic.

Definition 16.1. *Let $\mathcal{E}$ be a coherent sheaf on a quasi-projective variety. We say that $\mathcal{E}$ is **reflexive** if it isomorphic to its double dual.*

It is not hard to see that the double dual of any coherent sheaf is reflexive. The main technical consequence of some results in homological algebra of local rings we will need is the following:

Lemma 16.2. *A rank one sheaf on a smooth variety is reflexive if and only if it is a line bundle.*

Using (16.2) we can define the first chern class of a torsion free sheaf $\mathcal{E}$ by the rule

$$c_1(\mathcal{E}) = c_1((\det \mathcal{E})^{**}),$$

where the determinant just means take the highest wedge.

Definition 16.3. *Let $\mathcal{E}$ a torsion free sheaf of rank r on a projective variety X and let H be an ample divisor. The **slope** of $\mathcal{E}$ with respect to H, denoted $\mu(\mathcal{E})$, is the ratio*

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{r}.$$

*We say that $\mathcal{E}$ is **semistable** if the slope of any coherent subsheaf $\mathcal{F}$ is at most the slope of $\mathcal{E}$,*

$$\mu(\mathcal{F}) \leq \mu(\mathcal{E}).$$

*We say that $\mathcal{E}$ is **stable** if we always have strict inequality, when the rank of $\mathcal{F}$ is neither zero nor r. We say that $\mathcal{E}$ is **unstable** if it is not stable.*

Note that the definition of stability might change if we change H but it won't change if we replace H by a multiple. In the case of $\mathbb{P}^n$ there is therefore no ambiguity in dropping the reference to H.

Theorem 16.4. *Let $\mathcal{E}$ be a torsion free sheaf on $\mathbb{P}^n$.*

TFAE

1. *$\mathcal{E}$ is stable (respectively semistable).*
2. *$\mu(\mathcal{F}) < \mu(\mathcal{E})$ (respectively $\leq$) for all coherent subsheaves (whose rank is neither zero nor r) such that $\mathcal{E}/\mathcal{F}$ is torsion free.*
3. *$\mu(\mathcal{Q}) > \mu(\mathcal{E})$ (respectively $\geq$) for all torsion free quotient sheaves (whose rank is neither zero nor r).*

Proof. (1) clearly implies (2). Suppose that $\mathcal{F} \subset \mathcal{E}$ is a subsheaf. Let

$$\mathcal{Q} = \frac{\mathcal{E}}{\mathcal{F}}.$$

and let

$$\mathcal{Q}'$$

be the free part of $\mathcal{Q}$. Then $\mathcal{Q}'$ is torsion free and there is a natural surjective map $\mathcal{E} \longrightarrow \mathcal{Q}'$. If $\mathcal{H}$ is the kernel then $\mathcal{F} \subset \mathcal{H}$ and both sheaves have the same rank. As $c_1(\mathcal{F}) \leq c_1(\mathcal{H})$ we have

$$\mu(\mathcal{F}) \leq \mu(\mathcal{H}).$$

Thus (2) implies (1).

Now suppose that

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{Q} \longrightarrow 0$$

is a short exact sequence of torsion free sheaves, or ranks s, r and t so that $r = s + t$. Note that

$$c_1(\mathcal{E}) = c_1(\mathcal{F}) + c_1(\mathcal{Q}).$$

We have

$$\mu(\mathcal{F}) < \mu(\mathcal{E})$$

if and only if

$$\frac{s+t}{s} c_1(\mathcal{F}) < c_1(\mathcal{F}) + c_1(\mathcal{Q}),$$

if and only if

$$c_1(\mathcal{F}) < \frac{s}{t} c_1(\mathcal{Q}),$$

if and only if

$$c_1(\mathcal{Q}) + c_1(\mathcal{F}) < \frac{s+t}{t} c_1(\mathcal{Q})$$

if and only if

$$\mu(\mathcal{Q}) > \mu(\mathcal{E}).$$

□

Lemma 16.5.

(1) Line bundles are stable.
(2) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are torsion free sheaves than $\mathcal{E}_1 \oplus \mathcal{E}_2$ is semistable if and only if $\mathcal{E}_1$ and $\mathcal{E}_2$ are semistable with the same slope.
(3) $\mathcal{E}$ is semistable if and only if $\mathcal{E}^*$ is semistable.
(4) If $\mathcal{E}$ is semistable then $\mathcal{E}(k)$ is semistable for all $k \in \mathbb{Z}$.

Proof. (1) is trivial.

Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are semistable with the same slope $\mu = \mu(\mathcal{E}_i)$. Then $\mu = \mu(\mathcal{E}_1 \oplus \mathcal{E}_2)$. Suppose that $\mathcal{F} \subset \mathcal{E}_1 \oplus \mathcal{E}_2$ is a coherent subsheaf. Then there is an induced commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{E}_1 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}_2 \arrow[r] & 0 \end{tikzcd}$$

where $\mathcal{F}_1 = \mathcal{F} \cap (\mathcal{E}_1 \oplus 0)$ and $\mathcal{F}_2 = \mathcal{F} \cap (0 \oplus \mathcal{E}_2)$. As $\mathcal{E}_i$ is semistable it follows that $c_1(\mathcal{F}_i) \leq \mu s_i$, where s_i is the rank of $\mathcal{F}_i$.

It follows that

$$\begin{aligned} \mu(\mathcal{F}) &= \frac{c_1(\mathcal{F}_1) + c_1(\mathcal{F}_2)}{s_1 + s_2} \\ &\leq \mu \frac{s_1 + s_2}{s_1 + s_2} \\ &= \mu. \end{aligned}$$

Thus $\mathcal{E}_1 \oplus \mathcal{E}_2$ is semistable.

Conversely if $\mathcal{E}_1 \oplus \mathcal{E}_2$ is semistable then $\mu(\mathcal{E}_i) = \mu$ as $\mathcal{E}_i$ is both a sub and a quotient sheaf. If $\mathcal{E}_1$ is not semistable then let $\mathcal{F}_1$ be a destabilising subsheaf of rank s. Consider

$$\mathcal{F}_1 \oplus \mathcal{E}_2 \subset \mathcal{E}_1 \oplus \mathcal{E}_2.$$

Then

$$\begin{aligned} \mu(\mathcal{F}_1 \oplus \mathcal{E}_2) &= \frac{c_1(\mathcal{F}_1) + c_1(\mathcal{E}_2)}{s + r_2} \\ &> \frac{\mu s + \mu r_2}{s + r_2} \\ &= \mu. \end{aligned}$$

Thus (2) holds.

Note that if $\mathcal{E}$ is semistable then $\mathcal{E}^*$ by (16.4). Thus (3) holds.

Note that if $\mathcal{F} \subset \mathcal{E}$ then $\mathcal{F}(k) \subset \mathcal{E}(k)$. As the slope of $\mathcal{E}$ and $\mathcal{E}(k)$ are the same, (4) is clear. □

Definition 16.6. *Let E be a vector bundle of rank r on $\mathbb{P}^n$.*

We say that E is ***normalised*** *if $-r < c_1(E) \leq 0$.*

It is clear that if E is a vector bundle then there is a unique integer k so that $E(k)$ is normalised.

Lemma 16.7. *Let E be a rank two normalised vector bundle on $\mathbb{P}^n$.*

Then E is stable if and only if $h^0(\mathbb{P}^n, E) = 0$.

If $c_1(E)$ is even then E is semistable if and only if $h^0(\mathbb{P}^n, E(-1)) = 0$.

Proof. We prove the first statement. One direction is clear; if $h^0(\mathbb{P}^n, E) \neq 0$ then $\mathcal{O}_{\mathbb{P}^n}$ is a torsion free subsheaf of E. The slope of both E and $\mathcal{O}_{\mathbb{P}^n}$ is zero and so E is not stable.

Now suppose that $h^0(\mathbb{P}^n, E) = 0$. Suppose that $\mathcal{F} \subset E$ is a torsion free subsheaf of rank one. If we replace $\mathcal{F}$ by its double dual then the slope only goes up. Thus we may assume that $\mathcal{F}$ is reflexive, that is, we may assume that $L = \mathcal{F}$ is a line bundle. If $L \simeq \mathcal{O}_{\mathbb{P}^n}(k)$ then $k < 0$ as $h^0(\mathbb{P}^n, E) = 0$.

But then

$$\begin{aligned} \mu(L) &\leq -1 \\ &< -\frac{1}{2} \\ &= \mu(E) = 0. \end{aligned}$$

Thus E is stable.

We now turn the second statement. One direction is again clear; if $h^0(\mathbb{P}^n, E(-1)) \neq 0$ then $\mathcal{O}_{\mathbb{P}^n}(1)$ is a torsion free subsheaf of E. The slope of E is zero and of $\mathcal{O}_{\mathbb{P}^n}(1)$ is one and so E is not semistable.

Now suppose that $h^0(\mathbb{P}^n, E(-1)) = 0$. Suppose that $\mathcal{F} \subset E$ is a torsion free of rank one. As before we may assume that $L = \mathcal{F}$ is a line bundle. If $L \simeq \mathcal{O}_{\mathbb{P}^n}(k)$ then $k < 1$ as $h^0(\mathbb{P}^n, E(-1)) = 0$.

But then

$$\mu(L) \leq 0 \leq \mu(E) = 0.$$

Thus E is semistable. $\square$

Lemma 16.8. *Let E be a rank two torsion free sheaf on $\mathbb{P}^2$ with chern classes c_1 and c_2.*

If E is stable then

$$c_1^2 - 4c_2 < 0.$$

If E is semistable then

$$c_1^2 - 4c_2 \leq 0.$$

Proof. The discriminant

$$\Delta = c_1^2 - 4c_2$$

is invariant under twisting as is stability. Thus we may assume that E is normalised.

Suppose that E is stable. Then

$$H^0(\mathbb{P}^2, E) = 0$$

and by duality

$$H^2(\mathbb{P}^2, E) = 0.$$

Hence
$$\chi(\mathbb{P}^2, E) = -h^1(\mathbb{P}^2, E) \leq 0.$$
Now Riemann-Roch for E on $\mathbb{P}^2$ reads
$$\chi(\mathbb{P}^2, E) = \frac{1}{2}\left(c_1^2 - 2c_2 + 3c_1 + 4\right).$$
Indeed, the Riemann-Roch formula is a rational polynomial in the chern classes of E. If we consider what happens for $E = \mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$ we get
$$\begin{aligned} \chi(\mathbb{P}^2, E) &= \binom{a+2}{2} + \binom{b+2}{2} \\ &= \frac{1}{2}\left((a+b)^2 - 2ab + 3(a+b) + 4\right) \\ &= \frac{1}{2}\left(c_1^2 - 2c_2 + 3c_1 + 4\right) \end{aligned}$$
and this determines the formula.

Thus we have
$$c_1^2 - 2c_2 + 3c_1 + 4 \leq 0.$$
There are two cases. If $c_1 = 0$ then $-2c_2 + 4 \leq 0$ so that $c_2 \geq 2$ and so $\Delta \leq 0 - 8 < 0$. If $c_1 = -1$ then $-2c_2 + 2 \leq 0$ so that $c_2 \geq 1$. In this case $\Delta \leq 1^2 - 4 < 0$.

Now suppose that E is semistable but not stable. Then c_1 is even so that we may assume that $c_1 = 0$. But then
$$\begin{aligned} 0 &\leq h^1(\mathbb{P}^2, E(-1)) \\ &= -\chi(\mathbb{P}^2, E(-1)) \\ &= -\frac{1}{2}\left(2^2 - 2(c_2 + 1) - 6 + 4\right) \\ &= c_2(E) \end{aligned}$$
Thus $c_2(E) \geq 0$ so that $\Delta \leq 0$. $\square$

We end with the connection between stability and simplicity.

Lemma 16.9. *Let $\phi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ be a non-trivial sheaf map between semistable sheaves of the same slope.*

If one of the sheaves is stable then ϕ is a monomorphism or generically an epimorphism.

Proof. Let $\mathcal{I} = \operatorname{Im}\phi$ be the image of ϕ. Then I is a torsion free subsheaf of rank at least one, as ϕ is non-trivial.

Suppose that
$$\operatorname{rk}(\mathcal{I}) < \operatorname{rk}(\mathcal{E}_1) \qquad \text{and} \qquad \operatorname{rk}(\mathcal{I}) < \operatorname{rk}(\mathcal{E}_2).$$

If $\mathcal{E}_1$ is stable then

$$\begin{aligned} \mu(\mathcal{I}) &\leq \mu(\mathcal{E}_2) \\ &< \mu(\mathcal{I}) \end{aligned}$$

and if $\mathcal{E}_2$ is stable then

$$\begin{aligned} \mu(\mathcal{I}) &< \mu(\mathcal{E}_2) \\ &\leq \mu(\mathcal{I}), \end{aligned}$$

which are both impossible.

Therefore, either $\operatorname{rk}(\mathcal{I}) = \operatorname{rk}(\mathcal{E}_1)$, in which case ϕ is a monomorphism or $\operatorname{rk}(\mathcal{I}) = \operatorname{rk}(\mathcal{E}_2)$, in which case ϕ is generically an epimorphism. $\square$

Corollary 16.10. *Let $\phi\colon \mathcal{E}_1 \longrightarrow \mathcal{E}_2$ be a non-trivial sheaf map between semistable sheaves with the same rank and first chern class.*

If one of the sheaves is stable then ϕ is an isomorphism.

Proof. By (16.9) ϕ is a monomorphism and so

$$\det\phi\colon\ \det\mathcal{E}_1 \longrightarrow \det\mathcal{E}_2$$

is also a monomorphism. As the first chern classes are the same, it follows that $\det\phi$ is an isomorphism so that ϕ is an isomorphism. $\square$

Theorem 16.11. *Stable bundles are simple.*

Proof. Let $\phi\colon E \longrightarrow E$ be an endomorphism of a stable bundle.

Pick a point $x \in \mathbb{P}^n$. Then $\phi_x\colon E_x \longrightarrow E_x$ is a linear endomorphism and so it has an eigenvalue λ. It follows that $\phi - \lambda \mathrm{id}_E$ is not an isomorphism so that is must be the zero map. But then ϕ is a homotherty so that E is simple. $\square$

Theorem 16.12. *Every simple rank two vector bundle on $\mathbb{P}^n$ is stable.*

Proof. We may assume that E is normalised. If E is not stable then

$$h^0(\mathbb{P}^n, E) \neq 0$$

so that

$$h^0(\mathbb{P}^n, E^*) \neq 0$$

as $E^* \simeq E \otimes \det E^*$. But then E is not simple. $\square$

17. Extremal Bundles

We look at various extremal cases of Hartshorne's conjecture. Let $Y \subset \mathbb{P}^{m+2}$ be a local complete intersection of codimension two of degree d such that $\det N_{Y/\mathbb{P}^{m+2}} = \mathcal{O}_Y(k)$. We have already seen that if

$$k \geq \frac{d}{\mu} + \mu,$$

for some $\mu \in (0, m]$ then Y is a complete intersection.

If E is the associated vector bundle then

$$c_1(E) = k$$
$$c_2(E) = d.$$

Suppose that α and β are the chern roots, so that

$$c(E) = (1 + \alpha)(1 + \beta)$$

and $k = c_1(E) = \alpha + \beta$ and $d = c_2(E) = \alpha\beta$.

Suppose that one of α and β lies in the interval $(0, m]$. Then (1) holds, with $\mu = \alpha$, so that Y is a complete intersection. Note that $\alpha > 0$ and $\beta > 0$ since $\alpha\beta = d > 0$ and $\alpha + \beta = k > 0$. Therefore if

$$k < 2m$$

then one of α and β belongs to $(0, m]$,

Suppose that $k > 2m$ and the first inequality does not hold. Then

$$\frac{d}{\mu} + \mu > 2m.$$

Thus

$$d > 2m\mu - \mu^2.$$

The RHS is maximised if we take $\mu = m$, in which case

$$d > m^2.$$

It follows that if $d \leq m^2$ then Y is a complete intersection.