18.781: LECTURE ON ZETA FUNCTIONS

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1. Functions in the Complex Plane

I start with two identities:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty,$$

$$1 + 2 + 3 + 4 + \cdots \stackrel{!}{=} -\frac{1}{12}.$$

The first appeared on Problem Set 1. The second is false, but not far from the truth.

In Anthony's lecture on April 24, he introduced meromorphic functions. A function f is meromorphic in the complex plane iff, for every $\alpha \in \mathbf{C}$ and s close enough to α, there is an expansion of the form

$$f(s) = \sum_{n \geq N} c_n (s - \alpha)^n, \qquad \text{where } c_n \in \mathbf{C} \text{ and } N \in \mathbf{Z}.$$

We say that f has a pole at α iff the integer N is negative. For more on meromorphic functions, see [SS] up through Chapter 3.

Many important meromorphic functions are defined by one formula in one region of the complex plane, and extended to the rest of the complex plane by another. For instance, in the half-plane $\Re(s) > 0$, we can define a function by the convergent integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt \tag{1.1}$$

It turns out that in this region,

$$\begin{aligned} \Gamma(s) &= \frac{1}{s}\Gamma(s+1) \\ &= \frac{1}{s(s+1)}\Gamma(s+2) \\ &= \cdots \end{aligned} \tag{1.2}$$

So we can <u>extend</u> the definition of Γ leftward by means of (1.2). It turns out that the resulting *gamma function* is the unique meromorphic function in the complex plane that agrees with the first formula when $\Re(s) > 0$.

Exercise 1. Prove (1.2) using integration by parts. Then use induction to prove that $\Gamma(n) = (n-1)!$ for every positive integer n.

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The mathematician Bernhard Riemann, also known for developing the geometric foundations of Einstein's general relativity, wrote a single, 8-page paper on number theory in 1859 [R], in which he proved:

Theorem 2 (Riemann)**.** *There is a unique function $\zeta(s)$ such that:*

(1) *$\zeta(s)$ is meromorphic in the complex plane.*
(2) *If $\Re(s) > 1$, then*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1.3}$$

Moreover, it satisfies

$$\Lambda(1-s) = \Lambda(s), \qquad \text{where } \Lambda(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s). \tag{1.4}$$

Proof sketch. Let

$$\vartheta(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t}.$$

In the half-plane $\Re(s) > 1$, show that

$$\Lambda(s) = \int_0^\infty \vartheta(t) t^{s/2}\,\frac{dt}{t} = -\frac{1}{s} - \frac{1}{1-s} + \int_1^\infty \vartheta(t)(t^{s/2} + t^{(1-s)/2})\,\frac{dt}{t},$$

where the second equality uses the Poisson summation identity of Fourier analysis. Using the fact that $\vartheta(t)$ decays exponentially as $t \to \infty$, show that the last expression converges everywhere except $s = 0, 1$. Hence it is meromorphic everywhere. It is manifestly invariant under $s \mapsto 1 - s$. □

Corollary 3. $\zeta(-1) = -\frac{1}{12}$.

Proof sketch. We compute

$$\Lambda(-1) = \pi^{1/2}\Gamma(-\tfrac{1}{2})\zeta(-1) = -2\pi^{1/2}\Gamma(\tfrac{1}{2})\zeta(-1),$$
$$\Lambda(2) = \pi^{-1}\Gamma(1)\zeta(2) = \pi^{-1}\zeta(2).$$

Setting these values equal,

$$\zeta(-1) = -\frac{\pi^{-3/2}\zeta(2)}{2\Gamma(\frac{1}{2})}.$$

The terms $\Gamma(\frac{1}{2})$ and $\zeta(2)$ are respectively given by (1.1) and (1.3), so they are explicit integrals, albeit difficult ones. Euler solved both: $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\zeta(2) = \frac{\pi^2}{6}$, which give the desired answer. □

Geometrically, (1.4) says that the function $\Lambda(s)$ has reflection symmetry across the line in the complex plane where $\Re(s) = \frac{1}{2}$. Going further, Riemann proposed the following conjecture about the zeros of $\Lambda(s)$.

Conjecture 4 (Riemann Hypothesis)**.** *If $s \in \mathbf{C}$ satisfies $\Lambda(s) = 0$, then $\Re(s) = \frac{1}{2}$.*

One can check that this statement is equivalent to the statement that if $\zeta(s) = 0$ and $0 \le \Re(s) \le 1$, then $\Re(s) = \frac{1}{2}$. The function $\zeta(s)$ also has "trivial zeros" at $s = -2, -4, -6, \ldots$ For more details, see [SS, Ch. 7].

2. Prime Numbers and Zeta Zeros

The function $\zeta(s)$ is now called the *Riemann zeta function*. Why do we care about its zeros? It turns out they are related to the prime numbers.

When $\Re(s) > 1$, we have the so-called Euler product identity

$$\begin{aligned}\zeta(s) = \sum_{n=1}^{\infty} n^{-s} &= \prod_{\text{prime } p > 0} (1 + p^{-s} + p^{-2s} + \cdots) \\ &= \prod_{\text{prime } p > 0} \frac{1}{1 - p^{-s}}.\end{aligned}$$

Taking logarithmic Taylor series,

$$\log \zeta(s) = \sum_p \log \frac{1}{1 - p^{-s}} = \sum_p \sum_{m=1}^{\infty} \frac{p^{-ms}}{m}.$$

Therefore,

$$\frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds} \log \zeta(s) = -\sum_p \sum_{m=1}^{\infty} p^{-ms} \log p.$$

So in the half-plane $\Re(s) > 1$, we find that $\zeta'(s)/\zeta(s)$ is a weighted sum over the prime numbers. At the same time, the Cauchy residue formula from Anthony's lecture shows that for a sufficiently nice loop γ in the complex plane,

$$\#\text{ of zeros of } \zeta(s) \text{ within } \gamma = \frac{1}{2\pi i} \int_\gamma \frac{\zeta'(s)}{\zeta(s)}\, ds.$$

By refining these ideas, one can show a close relationship between the distribution of prime numbers among the integers and the distribution of the zeros of $\zeta(s)$ in the region where $0 \leq \Re(s) \leq 1$. See equation (1) of [D, Ch. 17].

3. From $\mathbf{Q}$ to K

There are many generalizations of $\zeta(s)$, all believed or known to satisfy analogues of the Riemann Hypothesis.

We introduced number fields K and their rings of integers $\mathcal{O}_K \subseteq K$. The *Dedekind zeta function* of K is defined by

$$\zeta_K(s) = \sum_{\substack{\text{nonzero ideals} \\ I \subseteq \mathcal{O}_K}} \mathbf{N}(I)^{-s}, \qquad \text{where } \mathbf{N}(I) = |\mathcal{O}_K / I|.$$

It turns out that if $K = \mathbf{Q}(\sqrt{d})$ and $I = \alpha\mathcal{O}_K$, then $\mathbf{N}(I) = \mathbf{N}(\alpha)$ in the usual sense. Just like before,

$$\zeta_K(s) = \prod_{\substack{\text{nonzero prime ideals} \\ P \subseteq \mathcal{O}_K}} \frac{1}{1 - \mathbf{N}(P)^{-s}},$$

and $\zeta_K(s)$ has a meromorphic continuation to the complex plane that satisfies a certain symmetry.

Exercise 5. Check that $\zeta_{\mathbf{Q}}(s) = \zeta(s)$.

Exercise 6. Use Fermat's two-squares theorem to check that

$$\zeta_{\mathbf{Q}(i)}(s) = \zeta(s) \cdot \prod_{\substack{\text{prime } p \equiv 1 \\ \pmod 4}} \frac{1}{1-p^{-s}} \prod_{\substack{\text{prime } p \equiv 3 \\ \pmod 4}} \frac{1}{1+p^{-s}}.$$

How are the factors on the right related to the Legendre symbol $\left(\frac{-1}{p}\right)$?

For more on number fields and their Dedekind zeta functions, see [M] up through Chapter 7.

4. From Numbers to Polynomials

I have mentioned the dictionary of analogies between numbers and polynomials. In what follows, fix a prime $p > 0$, and write $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$ to emphasize that we view this ring as a field. Fix a nonzero, squarefree polynomial

$$f(x, y) \in \mathbf{F}_p[x, y].$$

The solutions of $f(x, y) = 0$ form a curve in the xy-plane.

Let $\mathcal{O}_f = \mathbf{F}_p[x, y]/(f(x, y))$. How can we get hold of the nonzero prime ideals of $\mathcal{O}_f$? Any solution $(a, b) \in \mathbf{F}_p^2$ to the equation $f(x, y) = 0$ provides a prime ideal: namely,

$$P = (x - a)\mathcal{O}_f + (y - b)\mathcal{O}_f.$$

However, there are others. It turns out that just as the field $\mathbf{R}$ has the algebraic closure $\mathbf{C}$, so the field $\mathbf{F}_p$ has an algebraic closure $\bar{\mathbf{F}}_p$: See [M, Appendix 3] or [IR, Ch. 7] for more information. In general, every nonzero prime ideal of $\mathcal{O}_f$ corresponds to some $\bar{\mathbf{F}}_p^2$-valued solution of $f(x, y) = 0$.

We say that f is *nonsingular* iff $\nabla f(a, b) \not\equiv (0, 0)$ for any solution $(a, b) \in \bar{\mathbf{F}}_p^2$. This means the curve $f(x, y) = 0$ has no singularities, *i.e.*, "sharp points". If f is nonsingular, then $\mathcal{O}_f$ behaves a lot like $\mathcal{O}_K$: For instance, any chain of prime ideals in $\mathcal{O}_f$ has length zero or one, and every ideal of $\mathcal{O}_f$ factors uniquely into prime ideals. In this case, the *Weil zeta function* of f is defined by

$$\zeta(s, f) = \sum_{\substack{\text{nonzero ideals} \\ I \subseteq \mathcal{O}_f}} \mathbf{N}(I)^{-s} = \prod_{\substack{\text{nonzero prime ideals} \\ P \subseteq \mathcal{O}_f}} \frac{1}{1 - \mathbf{N}(P)^{-s}}.$$

Remark 7. We have given the version of $\zeta(s, f)$ corresponding to what is called the affine curve defined by $f(x, y) = 0$. In the literature, it is more common to see the version corresponding to the projective curve, which includes further solutions "at infinity." This introduces a factor of $1 - p^{-s}$ into the denominator.

The mathematician André Weil announced the following theorem in 1948–1949, in a more general form: namely, one allowing curves not confined to the xy-plane, and over arbitrary finite fields, though still defined by polynomial equations [W].

Theorem 8 (Weil)**.** *Suppose f is nonsingular. Then we have*

$$\zeta(s, f) = \frac{A(p^{-s})}{B(p^{-s})} \qquad \textit{for some polynomials } A(t), B(t) \in \mathbf{F}_p[t].$$

Moreover, if $s \in \mathbf{C}$ satisfies $\zeta(s, f) = 0$, then $\Re(s) = \frac{1}{2}$.

Example 9. The polynomial $f(x, y) = y$ is always nonsingular. The corresponding curve in the xy-plane is the x-axis. We compute $\mathscr{O}_f = \mathbf{F}_p[x]$ and

$$\zeta(s, f) = \sum_{\substack{\text{nonzero ideals} \\ I \subseteq \mathbf{F}_p[x]}} \mathbf{N}(I)^{-s} = \sum_{\substack{\text{nonzero monic} \\ g(x) \in \mathbf{F}_p[x]}} (p^{\deg(g)})^{-s}.$$

Since there are p^d monic polynomials of degree d, we arrive at

$$\zeta(s, f) = \sum_{d \geq 0} p^d (p^{-ds}) = \frac{1}{1 - p^{1-s}}.$$

So here, the analogue of the Riemann Hypothesis is trivial!

Example 10. If $p \neq 2, 3$, then $f(x, y) = x^3 - y^2 + 1$ is nonsingular. Here it is more difficult to compute prime ideals of $\mathscr{O}_f$, but it can be shown that

$$\zeta(s, f) = \frac{1 - \chi(p)p^{-s} + p^{1-2s}}{1 - p^{1-s}}, \qquad \text{where } \chi(p) = -\sum_{x \in \mathbf{F}_p} \left(\frac{x^3 + 1}{p}\right).$$

(Above, $\left(\frac{-}{p}\right)$ denotes the Legendre symbol. By convention, we set $\left(\frac{0}{p}\right) = 0$. A miraculous theorem called the Hasse bound states that

$$|\chi(p)| \leq 2\sqrt{p}.$$

This bound ensures that the solutions of $1 - \chi(p)T + pT^2 = 0$ satisfy $|T| = \frac{1}{\sqrt{p}}$, whence $\Re(s) = \frac{1}{2}$ upon substituting $T = p^{-s}$.

Exercise 11. Check the bound above for $p = 7$.

Exercise 12. Check that $\chi(p) = p - |\{(a, b) \in \mathbf{F}_p^2 \mid a^3 - b^2 + 1 = 0\}|$.

For more details on and examples of Weil's theorem, see [IR, Ch. 7–8, 10–11].

Weil conjectured a generalization where curves were replaced by higher-dimensional geometric figures called smooth algebraic varieties. Deligne proved the generalization in 1974, using tools from the vast discipline of algebraic geometry [De].

References

[D] H. Davenport. *Multiplicative Number Theory*. 2nd Ed. Rev. H. L. Montgomery. Springer–Verlag (1980).

[De] P. Deligne. La conjecture de Weil : II. *Publ. math. de l'I.H.É.S.*, tome 52 (1980), 137–252.

[IR] K. Ireland & M. Rosen. A Classical Introduction to Modern Number Theory. 2nd Ed. Springer-Verlag (1990).

[M] D. A. Marcus. *Number Fields*. Springer-Verlag (1977).

[R] B. Riemann. On the Number of Prime Numbers less than a Given Quantity [Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse]. *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin* (Nov. 1859). Trans. D. R. Wilkins (1998). `https://www.claymath.org/sites/default/files/ezeta.pdf`

[SS] R. Shakarchi & E. Stein. *Complex Analysis*. Princeton Lectures in Analysis, Vol II. Princeton University Press (2003).

[W] A. Weil. Numbers of Solutions of Equations in Finite Fields. *Bull. AMS* (1949), 497–508.

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