PROBLEM SET 1

18.781 SPRING 2023

Due Monday, February 13. You may consult books, papers, and websites as long as you cite them and write up your solutions in your own words. Do not request answers on forums online. To get full points on a proof-based problem, *please write in complete sentences.*

Book. (Stillwell, *Elements of Number Theory*)

(1) 1.1.1–1.1.3
(2) 1.3.2–1.3.3 (read about McN*ggets first)
(3) 1.3.4–1.3.5
(4) 1.3.6
(5) 1.5.2–1.5.3 (read about Mersenne primes first)

Non-Book.

Problem 1. Use Eratosthenes's sieve to identify all prime numbers less than <u>120</u>. What is the largest prime whose multiples need to be sieved out?

Problem 2. Let $T_n = \frac{1}{2}n(n+1)$, the nth triangular number. Use induction to prove that for all $n \in \mathbf{N}$, we have

$$1^3 + 2^3 + \cdots + n^3 = T_n^2.$$

Problem 3. Prove that for all $n \in \mathbf{N}$, the number

$$2^{2n-1} - 9n^2 + 21n - 14$$

is divisible by 27.

Problem 4. Prove that the infinite series

$$\sum_{n=1}^{\infty} n^{-s}$$

converges for $s > 1$, and diverges for $s = 1$. (You may assume calculus.)

Problem 5. Assuming the unique prime factorization of positive integers, prove that for all real $s > 1$ and $N \in \mathbf{N}$, we have the inequality

$$\sum_{1 \leq n \leq N} n^{-s} \leq \prod_{\text{prime } p \,\leq\, N} \left(1 + p^{-s} + \cdots + p^{-Ns}\right).$$

Use the $N \to \infty$ limit of this inequality, together with Problem 4, to deduce another proof of the infinitude of the prime numbers.