# MATH 665 PROBLEM SET 1

FALL 2024

**Due Thursday, September 19.** You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words.

**Problem 1.** (1) Compute the sizes of the conjugacy classes of $S_4, S_5, S_6$.
(2) Use (1) to show that $A_4$ is not simple, but $A_5$ and $A_6$ are.

**Problem 2.** (1) Show that $|\mathrm{SL}_2(\mathbf{F}_7)| = 2 \cdot 3 \cdot 7 \cdot 8$.
(2) Find a reference listing the eleven conjugacy classes of $\mathrm{SL}_2(\mathbf{F}_7)$.
(3) Use (2) to compute the six conjugacy classes of

$$\mathrm{PSL}_2(\mathbf{F}_7) = \mathrm{SL}_2(\mathbf{F}_7)/\{\pm 1\}$$

and their sizes.
(4) Use (1) and (3) to show that $\mathrm{PSL}_2(\mathbf{F}_7)$ is simple.

**Problem 3.** Over $\bar{\mathbf{F}}_q$, for $q$ odd, let $G = \mathrm{SL}_2$. Let $B = TU$ be its upper-triangular subgroup, where $T$ is the diagonal torus and $U$ the unipotent radical of $B$. Let $F : G \to G$ correspond to the split $\mathbf{F}_q$-form (**Sep 3**), so that $B, T, U$ are $F$-stable. For any character $\chi$ of $T^F$, viewed as a character of $B^F$, let $I_\chi = \mathrm{Ind}_{B^F}^{G^F}(\chi)$.

(1) Taking $q = 3$:
  (a) Use Bruhat to find the number of double cosets of $U^F$ in $G^F$.
  (b) For all $\chi$, use Mackey (**Sep 5**) to decompose $I_\chi$ into its irreducible summands as a representation of $G^F$. The total number of summands, as we run over all $\chi$, should match your answer to (a).
(2) Repeat (2), now taking $q = 5$.

**Problem 4.** Keep the setup of the previous problem. Recall the Deligne–Lusztig variety (**Sep 10**)

$$\tilde{X}_s = \{gU \in G/U \mid g^{-1}F(g) \in U\dot{s}U\}, \quad \text{where } \dot{s} = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

The $G$-action on $\mathbf{A}^2$ induces an isomorphism $G/U \xrightarrow{\sim} \mathbf{A}^2 \setminus \{0\}$. Show that at the level of $\bar{\mathbf{F}}_q$-points, this isomorphism identifies $\tilde{X}_s$ with the plane curve $xy^q - x^q y = 1$, where $x, y$ are the standard coordinates on $\mathbf{A}^2$.

**Problem 5.** Let $q$ be any prime power. Over $\bar{\mathbf{F}}_q$, let $X$ be an algebraic variety with an action of a smooth algebraic group $H$. Suppose that there are Frobenius maps $F$ on $X$ and $H$ such that $F(h \cdot x) = F(h) \cdot F(x)$. Show that:

(1) If $H$ is *connected*, then every $F$-stable $H(\bar{\mathbf{F}}_q)$-orbit on $X(\bar{\mathbf{F}}_q)$ has an $F$-fixed point. *Hint:* Pick a point and apply Lang's theorem (**Sep 5**).
(2) In the setting of (1), deduce that there is a bijection $(X/H)^F \simeq X^F/H^F$.

(3) If $H$ is not connected, then the conclusions to (1)–(2) fail, even when $X = \mathbf{A}^1$.

**Problem 6.** Over any algebraically closed field $k$, let $Z \subseteq \mathrm{GL}_2$ be the subgroup of scalar matrices, acting on the larger group by multiplication.

(1) Compute the subring $k[\mathrm{GL}_2]^Z \subseteq k[\mathrm{GL}_2]$.
(2) Deduce that the embedding $\mathrm{SL}_2 \to \mathrm{GL}_2$ descends to an isomorphism

$$\mathrm{GL}_2 /\!\!/ Z \xrightarrow{\sim} \mathrm{SL}_2 /\!\!/ \{\pm 1\}.$$

Above, $k[X /\!\!/ H] := k[X]^H$ for any algebraic variety $X$ over $k$ with an action of an algebraic group $H$.

This problem suggests why we prefer not to define an algebraic group $\mathrm{PSL}_2$ distinct from $\mathrm{PGL}_2$.[1]

[1] See https://mathoverflow.net/a/16150 for further context.