1. Weil Divisors

Definition 1.1. *We say that a scheme X is* ***regular in codimension one*** *if every local ring of dimension one is regular, that is, the quotient $\mathfrak{m}/\mathfrak{m}^2$ is one dimensional, where $\mathfrak{m}$ is the unique maximal ideal of the corresponding local ring.*

Regular in codimension one often translates to smooth in codimension one.

When talking about Weil divisors, we will only consider schemes which are

$(*)$ noetherian, integral, separated, and regular in codimension one.

Definition 1.2. *Let X be a scheme satisfying $(*)$. A* ***prime divisor*** *Y on X is a closed integral subscheme of codimension one.*

A ***Weil divisor*** *D on X is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors.*

Thus a Weil divisor is a formal linear combination $D = \sum_Y n_Y Y$ of prime divisors, where all but finitely many $n_Y = 0$. We say that D is **effective** if $n_Y \geq 0$.

Definition 1.3. *Let X be a scheme satisfying $(*)$, and let Y be a prime divisor, with generic point η. Then $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field K.*

The ***valuation*** *ν_Y* ***associated to*** *Y is the corresponding valuation.*

Note that as X is separated, Y is determined by its valuation. If $f \in K = K(X)$ and $\nu_Y(f) > 0$ then we say that f has a **zero of order** $\nu_Y(f)$; if $\nu_Y(f) < 0$ then we say that f has a **pole of order** $-\nu_Y(f)$.

Definition-Lemma 1.4. *Let X be a scheme satisfying $(*)$, and let $f \in K^*$.*

$$(f) = \sum_Y \nu_Y(f) Y \in \operatorname{Div} X.$$

Proof. We have to show that $\nu_Y(f) = 0$ for all but finitely many Y. Let U be the open subset where f is regular. Then the only poles of f are along $Z = X - U$. As Z is a proper closed subset and X is noetherian, Z contains only finitely many prime divisors.

Similarly the zeroes of f only occur outside the open subset V where $g = f^{-1}$ is regular. □

Any divisor D of the form (f) will be called **principal**.

Lemma 1.5. *Let X be a scheme satisfying $(*)$.*

The principal divisors are a subgroup of $\operatorname{Div} X$*.*

Proof. The map

$$K^* \longrightarrow \operatorname{Div} X,$$

is easily seen to be a group homomorphism. □

Definition 1.6. *Two Weil divisors D and D' are called* ***linearly equivalent****, denoted $D \sim D'$, if and only if the difference is principal. The group of Weil divisors modulo linear equivalence is called the* ***divisor Class group****, denoted* $\operatorname{Cl} X$*.*

We will also denote the group of Weil divisors modulo linear equivalence as $A_{n-1}(X)$.

Proposition 1.7. *If k is a field then*

$$\operatorname{Cl}(\mathbb{P}^n_k) \simeq \mathbb{Z}.$$

Proof. Note that if Y is a prime divisor in $\mathbb{P}^n_k$ then Y is a hypersurface in $\mathbb{P}^n$, so that $I = \langle G \rangle$ and Y is defined by a single homogeneous polynomial G. The degree of G is called the degree of Y.

If $D = \sum n_Y Y$ is a Weil divisor then define the degree $\deg D$ of D to be the sum

$$\sum n_Y \deg Y,$$

where $\deg Y$ is the degree of Y.

Note that the degree of any rational function is zero. Thus there is a well-defined group homomorphism

$$\deg\colon \operatorname{Cl}(\mathbb{P}^r_k) \longrightarrow \mathbb{Z},$$

and it suffices to prove that this map is an isomorphism. Let H be defined by X_0. Then H is a hyperplane and H has degree one. The divisor $D = nH$ has degree n and so the degree map is surjective. One the other hand, if $D = \sum n_i Y_i$ is effective, and Y_i is defined by G_i,

$$(\prod_i G^{n_i}/X_0^d) = D - dH,$$

where d is the degree of D, so that $D \sim dH$. □

The next case, at least over an algebraically closed field is a smooth cubic curve in $\mathbb{P}^2_k$. We will need:

Theorem 1.8. *A smooth cubic curve is always irrational.*

Example 1.9. *Let C be a smooth cubic curve in $\mathbb{P}^2_k$. Suppose that the line $Z = 0$ is a flex line to the cubic at the point $P_0 = [0 : 1 : 0]$. If the equation of the cubic is $F(X, Y, Z)$ this says that $F(X, Y, 0) = X^3$. Therefore the cubic has the form $X^3 + ZG(X, Y, Z)$. If we work on the open subset $U_3 \simeq \mathbb{A}^2_k$, then we get*

$$x^3 + g(x, y) = 0,$$

where $g(x, y)$ has degree at most two. If we expand $g(x, y)$ as a polynomial in y,

$$g_0(x)y^2 + g_1(x)y + g_2(x),$$

then $g_0(x)$ must be a non-zero scalar, since otherwise C is singular (a nodal or cuspidal cubic). We may assume that $g_0 = 1$. If we assume that the characteristic is not two then we may complete the square to get

$$y^2 = x^3 + g(x),$$

for some quadratic polynomial $g(x)$. If we assume that the characteristic is not three then we may complete the cube to get

$$y^2 = x^3 + ax + b,$$

for some a and $b \in k$.

Now any two sets of three collinear points are linearly equivalent (since the equation of one line divided by another line is a rational function on the whole $\mathbb{P}^2_k$). In fact given any three points P, Q and P' we may find Q' such that $P + Q \sim P' + Q'$; indeed the line $l = \langle P, Q\rangle$ meets the cubic in one more point R. The line $l' = \langle R, P'\rangle$ then meets the cubic in yet another point Q'. We have

$$P + Q + R \sim P' + Q' + R'.$$

Cancelling we get

$$P + Q \sim P' + Q'.$$

It follows that if there are further linear equivalences then there are two points P and P' such that $P \sim P'$. This gives us a rational function f with a single zero P and a single pole P'; in turn this gives rise to a morphism $C \longrightarrow \mathbb{P}^1$ which is an isomorphism. It turns out that a smooth cubic is not isomorphic to $\mathbb{P}^1$, so that in fact the only relations are those generated by setting two sets of three collinear points to be linearly equivalent.

Put differently, the rational points of C form an abelian group, where three points sum to zero if and only if they are collinear, and P_0 is declared to be the identity. The divisors of degree zero modulo linear equivalence are equal to this group.

In particular, an elliptic curve is very far from being isomorphic to $\mathbb{P}^1_k$.

It is interesting to calculate the Class group of a toric variety X, which always satisfies $(*)$. By assumption there is a dense open subset $U \simeq \mathbb{G}_m^n$. The complement Z is a union of the invariant divisors.

Lemma 1.10. *Suppose that X satisfies $(*)$, let Z be a closed subset and let $U = X \setminus Z$.*

Then there is an exact sequence

$$\mathbb{Z}^k \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0,$$

where k is the number of components of Z which are prime divisors.

Proof. If Y is a prime divisor on X then $Y' = Y \cap U$ is either a prime divisor on U or empty. This defines a group homomorphism

$$\rho\colon \mathrm{Div}(X) \longrightarrow \mathrm{Div}(U).$$

If $Y' \subset U$ is a prime divisor then let Y be the closure of Y' in X. Then Y is a prime divisor and $Y' = Y \cap U$. Thus ρ is surjective. If f is a rational function on X and $Y = (f)$ then the image of Y in $\mathrm{Div}(U)$ is equal to $(f|_U)$. If $Z = Z' \cup \bigcup_{i=1}^k Z_i$ where Z' has codimension at least two then the map which sends $(m_1, m_2, \dots, m_k)$ to $\sum m_i Z_i$ generates the kernel. $\square$

Example 1.11. *Let $X = \mathbb{P}^2_k$ and C be an irreducible curve of degree d. Then $\mathrm{Cl}(\mathbb{P}^2 - C)$ is equal to $\mathbb{Z}_d$. Similarly $\mathrm{Cl}(\mathbb{A}^n_k) = 0$.*

It follows by (1.10) that there is an exact sequence

$$\mathbb{Z}^k \longrightarrow \mathrm{Cl}(X) \longrightarrow \mathrm{Cl}(U) \longrightarrow 0.$$

Applying this to $X = \mathbb{A}^n_k$ it follows that $\mathrm{Cl}(U) = 0$. So we get an exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

We want to identify the kernel. This is equal to the set of principal divisors which are supported on the invariant divisors. If f is a rational function such that (f) is supported on the invariant divisors then f has no zeroes or poles on the torus; it follows that $f = \lambda\chi^u$, where $\lambda \in k^*$ and $u \in M$.

It follows that there is an exact sequence

$$M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Lemma 1.12. *Let $u \in M$. Suppose that X is the affine toric variety associated to a cone σ, where σ spans $N_{\mathbb{R}}$. Let v be a primitive generator of a one dimensional ray τ of σ and let D be the corresponding invariant divisor.*

Then $\mathrm{ord}_D(\chi^u) = \langle u, v\rangle$. *In particular*

$$(\chi^u) = \sum_i \langle u, v_i\rangle D_i,$$

where the sum ranges over the invariant divisors.

Proof. We can calculate the order on the open set $U_\tau = \mathbb{A}^1_k \times \mathbb{G}_m^{n-1}$, where D corresponds to $\{0\} \times \mathbb{G}_m^{n-1}$. Using this, we are reduced to the one dimensional case. So $N = \mathbb{Z}$, $v = 1$ and $u \in M = \mathbb{Z}$. In this case χ^u is the monomial x^u and the order of vanishing at the origin is exactly u. □

It follows that if $X = X(F)$ is the toric variety associated to a fan F which spans $N_{\mathbb{R}}$ then we have a short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Example 1.13. *Let σ be the cone spanned by $2e_1 - e_2$ and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. There are two invariant divisors D_1 and D_2. The principal divisor associated to $u = f_1 = (1, 0)$ is $2D_1$ and the principal divisor associated to $u = f_2 = (0, 1)$ is $D_2 - D_1$. So the class group is $\mathbb{Z}_2$.*

Note that the dual $\check{\sigma}$ is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the monoid $S_\sigma = \check{\sigma} \cap M$ are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. So the group algebra

$$A_\sigma = k[x, xy, xy^2] = \frac{k[u, v, w]}{\langle v^2 - uw\rangle},$$

and $X = U_\sigma$ is the quadric cone.

Now suppose we take the standard fan associated to $\mathbb{P}^2$. The invariant divisors are the three coordinate lines, D_1, D_2 and D_3. If $f_1 = (1, 0)$ and $f_2 = (0, 1)$ then

$$(\chi^{f_1}) = D_1 - D_3 \qquad \text{and} \qquad (\chi^{f_2}) = D_2 - D_3.$$

So the class group is $\mathbb{Z}$.

2. Cartier Divisors

We now turn to the notion of a Cartier divisor.

Definition 2.1. *Given a ring A, let S be the multiplicative set of non-zero divisors of A. The localisation A_S of A at S is called the **total quotient ring** of A.*

*Given a scheme X, let $\mathcal{K}$ be the sheaf associated to the presheaf, which associates to every open subset $U \subset X$, the total quotient ring of $\Gamma(U, \mathcal{O}_X)$. $\mathcal{K}$ is called the **sheaf of total quotient rings**.*

Definition 2.2. *A **Cartier divisor** on a scheme X is any global section of $\mathcal{K}^*/\mathcal{O}_X^*$.*

In other words, a Cartier divisor is specified by an open cover U_i and a collection of rational functions f_i, such that f_i/f_j is a nowhere zero regular function.

A Cartier divisor is called **principal** if it is in the image of $\Gamma(X, \mathcal{K}^*)$. Two Cartier divisors D and D' are called **linearly equivalent**, denoted $D \sim D'$, if and only if the difference is principal.

Definition 2.3. *Let X be a scheme satisfying $(*)$. Then every Cartier divisor determines a Weil divisor.*

Informally a Cartier divisor is simply a Weil divisor defined locally by one equation. If every Weil divisor is Cartier then we say that X is **factorial**. This is equivalent to requiring that every local ring is a UFD; for example every smooth variety is factorial.

Example 2.4. *The quadric cone Q, given by $xy - z^2 = 0$ in $\mathbb{A}^3_k$ is not factorial. The line l, given by $x = z = 0$, is a Weil divisor which is not Cartier (one needs to check that the ideal $\langle x, z\rangle$ inside $\mathcal{O}_{Q,0}$ is not principal). The hyperplane $x = 0$ cuts out the double line $2l$.*

Definition-Lemma 2.5. *Let X be a scheme.*

The set of invertible sheaves forms an abelian group $\operatorname{Pic}(X)$, where multiplication corresponds to tensor product and the inverse to the dual.

Proof. It is clear that tensor product is commutative and associative and that $\mathcal{O}_X$ plays the role of the identity. But if $\mathcal{M} = \operatorname{Hom}(\mathcal{L}, \mathcal{O}_X)$ then

$$\mathcal{M} \underset{\mathcal{O}_X}{\otimes} \mathcal{L} \simeq \operatorname{Hom}(\mathcal{L}, \mathcal{L}) \simeq \mathcal{O}_X. \qquad \square$$

Definition 2.6. *Let D be a Cartier divisor, represented by $\{(U_i, f_i)\}$. Define a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}$ by taking the subsheaf generated by f_i^{-1} over the open set U_i.*

Proposition 2.7. *Let X be a scheme.*

(1) *The association $D \longrightarrow \mathcal{O}_X(D)$ defines a correspondence between Cartier divisors and invertible subsheaves of $\mathcal{K}$.*
(2) *If $\mathcal{O}_X(D_1 - D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$, as subsheaves of $\mathcal{K}$*
(3) *Two Cartier divisors D_1 and D_2 are linearly equivalent if and only if $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$ (not necessarily as subsheaves of $\mathcal{K}$.*

Let's consider which Weil divisors on a toric variety are Cartier. We classify all Cartier divisors whose underlying Weil divisor is invariant; we dub these Cartier divisors T-Cartier. We start with the case of the affine toric variety associated to a cone $\sigma \subset N_{\mathbb{R}}$. By (2.7) it suffices to classify all invertible subsheaves $\mathcal{O}_X(D) \subset \mathcal{K}$. Taking global sections, since we are on an affine variety, it suffices to classify all fractional ideals,

$$I = H^0(X, \mathcal{O}_X(D)).$$

Invariance of D implies that I is graded by M, that is, I is a direct sum of eigenspaces. As D is Cartier, I is principal at the distinguished point x_σ of U_σ, so that $I/\mathfrak{m}I$ is one dimensional, where

$$\mathfrak{m} = \sum k \cdot \chi^u.$$

It follows that $I = A_\sigma \chi^u$, so that $D = (\chi^u)$ is principal. In particular, the only Cartier divisors are the principal divisors and X is factorial if and only if the Class group is trivial.

Example 2.8. *The quadric cone Q, given by $xy - z^2 = 0$ in $\mathbb{A}^3_k$ is not factorial. We have already seen* (1.13) *that the class group is $\mathbb{Z}_2$.*

If $\sigma \subset N_{\mathbb{R}}$ is not maximal dimensional then every Cartier divisor on U_σ whose associated Weil divisor is invariant is of the form (χ^u) but

$$(\chi^u) = (\chi^{u'}) \qquad \text{if and only if} \qquad u - u' \in \sigma^\perp \cap M = M(\sigma).$$

So the T-Cartier divisors are in correspondence with $M/M(\sigma)$.

Now suppose that $X = X(F)$ is a general toric variety. Then a T-Cartier divisor is given by specifying an element $u(\sigma) \in M/M(\sigma)$, for every cone σ in F. This defines a divisor $(\chi^{-u(\sigma)})$; equivalently a fractional ideal

$$I = H^0(X, \mathcal{O}_X(D)) = A_\sigma \cdot \chi^{u(\sigma)}.$$

These maps must agree on overlaps; if τ is a face of σ then $u(\sigma) \in M/M(\sigma)$ must map to $u(\tau) \in M/M(\tau)$.

The data

$$\{\, u(\sigma) \in M/M(\sigma) \,|\, \sigma \in F \,\},$$

for a T-Cartier divisor D determines a continuous piecewise linear function ϕ_D on the support $|F|$ of F. If $v \in \sigma$ then let

$$\phi_D(v) = \langle u(\sigma), v \rangle.$$

Compatibility of the data implies that ϕ_D is well-defined and continuous. Conversely, given any continuous function ϕ, which is linear and integral (given by an element of M) on each cone, we can associate a unique T-Cartier divisor D. If $D = a_i D_i$ the function is given by $\phi_D(v_i) = -a_i$.

Note that

$$\phi_D + \phi_E = \phi_{D+E} \qquad \text{and} \qquad \phi_{mD} = m\phi_D.$$

Note also that $\phi_{(\chi^u)}$ is the linear function given by u. So D and E are linearly equivalent if and only if ϕ_D and ϕ_E differ by a linear function.

If X is any variety which satisfies $(*)$ then the natural map

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X),$$

is an embedding. It is an interesting to compare $\mathrm{Pic}(X)$ and $\mathrm{Cl}(X)$ on a toric variety. Denote by $\mathrm{Div}_T(X)$ the group of T-Cartier divisors.

Proposition 2.9. *Let $X = X(F)$ be the toric variety associated to a fan F which spans $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \mathrm{Div}_T(X) \arrow[r] \arrow[d, hook] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbb{Z}^s \arrow[r] & \mathrm{Cl}(X) \arrow[r] & 0
\end{tikzcd}$$

In particular

$$\rho(X) = \mathrm{rank}(\mathrm{Pic}(X)) \leq \mathrm{rank}(\mathrm{Cl}(X)) = s - n.$$

Further $\mathrm{Pic}(X)$ *is a free abelian group.*

Proof. We have already seen that the bottom row is exact. If $\mathcal{L}$ is an invertible sheaf then $\mathcal{L}|_U$ is trivial. Suppose that $\mathcal{L} = \mathcal{O}_X(E)$. Pick a rational function such that $(f)|_U = E|_U$. Let $D = E - (f)$. Then D is T-Cartier and exactness of the top row is easy.

Finally, $\mathrm{Pic}(X)$ is subgroup of the direct sum of $M/M(\sigma)$ and each of these is a lattice, whence $\mathrm{Pic}(X)$ is torsion free. $\square$

3. Smoothness and the Zariski tangent space

We want to give an algebraic notion of the tangent space. In differential geometry, tangent vectors are equivalence classes of maps of intervals in $\mathbb{R}$ into the manifold. This definition lifts to algebraic geometry over $\mathbb{C}$ but not over any other field (for example a field of characteristic p).

Classically tangent vectors are determined by taking derivatives, and the tangent space to a variety X at x is then the space of tangent directions, in the whole space, which are tangent to X. Even is this is how we will compute the tangent space in general, it is still desirable to have an intrinsic definition, that is, a definition which does not use the fact that X is embedded in $\mathbb{P}^n$.

Now note first that the notion of smoothness is surely local and that if we want an intrinsic definition, then we want a definition that only uses the functions on X. Putting this together, smoothness should be a property of the local ring of X at p. On the other hand taking derivatives is the same as linear approximation, which means dropping quadratic and higher terms.

Definition 3.1. *Let X be a variety and let $p \in X$ be a point of X. The **Zariski tangent space** of X at p, denoted T_pX, is equal to the dual of the quotient*

$$\mathfrak{m}/\mathfrak{m}^2,$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$.

Note that $\mathfrak{m}/\mathfrak{m}^2$ is a vector space. Suppose that we are given a morphism

$$f\colon X \longrightarrow Y,$$

which sends p to q. In this case there is a ring homomorphism

$$f^*\colon \mathcal{O}_{Y,q} \longrightarrow \mathcal{O}_{X,p}$$

which sends the maximal ideal $\mathfrak{n}$ into the maximal ideal $\mathfrak{m}$. Thus we get an induced map

$$df\colon \mathfrak{n}/\mathfrak{n}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2.$$

On the other hand, geometrically the map on tangent spaces obviously goes the other way. It follows that we really do want the dual of $\mathfrak{m}/\mathfrak{m}^2$. In fact $\mathfrak{m}/\mathfrak{m}^2$ is the dual of the Zariski tangent space, and is referred to as the *cotangent space*.

In particular, given a morphism $f\colon X \longrightarrow Y$ carrying p to q, then there is a linear map

$$df\colon T_pX \longrightarrow T_qY.$$

Definition 3.2. *Let X be a quasi-projective variety.*

*We say that X is **smooth** at p if the local dimension of X at p is equal to the dimension of the Zariski tangent space at p.*

Now the tangent space to $\mathbb{A}^n$ is canonically a copy of $\mathbb{A}^n$ itself, considered as a vector space based at the point in question. If $X \subset \mathbb{A}^n$, then the tangent space to X is included inside the tangent space to $\mathbb{A}^n$. The question is then how to describe this subspace.

Lemma 3.3. *Let $X \subset \mathbb{A}^n$ be an affine variety. Suppose that $f_1, f_2, \dots, f_k$ generate the ideal I of X. Then the tangent space of X at p, considered as a subspace of the tangent space to $\mathbb{A}^n$, via the inclusion of X in $\mathbb{A}^n$, is equal to the kernel of the Jacobian matrix.*

Proof. Clearly it is easier to give the dual description of the cotangent space.

If $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{\mathbb{A}^n,p}$ and $\mathfrak{n}$ is the maximal ideal of $\mathcal{O}_{X,p}$, then clearly the natural map $\mathfrak{m} \longrightarrow \mathfrak{n}$ is surjective, so that the induced map on contangent spaces is surjective. Dually, the induced map on the Zariski tangent space is injective, so that T_pX is indeed included in $T_p\mathbb{A}^n$.

We may as well choose coordinates $x_1, x_2, \dots, x_n$ so that p is the origin. In this case $\mathfrak{m} = \langle x_1, x_2, \dots, x_n \rangle$ and $\mathfrak{n} = \mathfrak{m}/I$. Moreover $\mathfrak{m}/\mathfrak{m}^2$ is the vector space spanned by $dx_1, dx_2, \dots, dx_n$, where dx_i denotes the equivalence class $x_i + \mathfrak{m}^2$, and $\mathfrak{n}/\mathfrak{n}^2$ is canonically isomorphic to $\mathfrak{m}/(\mathfrak{m}^2 + I)$. Now the transpose of the Jacobian matrix, defines a linear map

$$K^k \longrightarrow K^n = T_p^*\mathbb{A}^n,$$

and it suffices to prove that the image of this map is the kernel of the map

$$df \colon \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{n}/\mathfrak{n}^2.$$

Let $g \in \mathfrak{m}$. Then

$$g(x) = \sum a_i x_i + h(x),$$

where $h(x) \in \mathfrak{m}^2$. Thus the image of $g(x)$ in $\mathfrak{m}/\mathfrak{m}^2$ is equal to $\sum_i a_i dx_i$. Moreover, by standard calculus a_i is nothing more than

$$a_i = \left.\frac{\partial g}{\partial x_i}\right|_p .$$

Thus the kernel of the map df is generated by the image of f_i in $\mathfrak{m}/\mathfrak{m}^2$, which is

$$\sum_j \left.\frac{\partial f_i}{\partial x_j}\right|_p dx_j,$$

which is nothing more than the image of the Jacobian. □

Lemma 3.4. *Let X be a quasi-projective variety. Then the function*

$$\lambda\colon X \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\lambda(x)$ is the dimension of the Zariski tangent space at x.

Proof. Clearly this result is local on X so that we may assume that X is affine. In this case the Jacobian matrix defines a morphism π from X to the space of matrices and the locus where the Zariski tangent space has a fixed dimension is equal to the locus where this morphism lands in the space of matrices of fixed rank. Put differently the function λ is the composition of π and an affine linear function of the rank on the space of matrices. Since the rank function is upper semicontinuous, the result follows. □

Lemma 3.5. *Every irreducible quasi-projective variety is birational to a hypersurface.*

Proof. Let X be a quasi-projective variety of dimension k, with function field L/K. Let $L/M/K$ be an intermediary field, such that M/K is purely transcendental of transcendence degree, so that L/M is algebraic. As L/M is a finitely generated extension, it follows that L/M is finite. Suppose that L/M is not separable. Then there is an element $y \in L$ such that $y \notin M$ but $x_1 = y^p \in M$. We may extend x_1 to a transcendence basis $x_1, x_2, \dots, x_k$ of M/K. Let M' be the intermediary field generated by $y, x_2, x_3, \dots, x_k$. Then M'/K is a purely transcendental extension of K and

$$[L : M] = [L : M'][M' : M] = p[L : M'].$$

Repeatedly replacing M by M' we may assume that L/M is a separable extension.

By the primitive element Theorem, L/M is generated by one element, say α. It follows that there is polynomial $f(x) \in M[x]$ such that α is a root of $f(x)$. If $M = K(x_1, x_2, \dots, x_k)$, then clearing denominators, we may assume that $f(x) \in K[x_1, x_2, \dots, x_k][x] \simeq K[x_1, x_2, \dots, x_{k+1}]$. But then X is birational to the hypersurface defined by $F(X)$, where $F(X)$ is the homogenisation of $f(x)$. □

Proposition 3.6. *The set of smooth points of any variety is Zariski dense.*

Proof. Since the dimension of the Zariski tangent space is upper semicontinuous, and always at least the dimension of the variety, it suffices to prove that every irreducible variety contains at least one smooth point. By (3.5) we may assume that X is a hypersurface. Passing to an affine open subset, we may assume that X is an affine hypersurface. Let f be a definining equation, so that f is an irreducible polynomial. Then the set of singular points of X is equal to the locus of points where every partial derivative vanishes. If g is a non-zero partial derivative of f, then g is a non-zero polynomial of degree one less than f, and so cannot vanish on X.

If all the partial derivatives of f are the zero polynomial, then f is a pth power, where the characteristic is p, which contradicts the fact that f is irreducible. $\square$

Note that if we take a smooth variety X and blow up a point p, then the exceptional divisor E is canonically the projectivisation of the Zariski tangent space to X at p,

$$E = \mathbb{P}(T_pX).$$

Indeed the point is that E picks up the different tangent directions to X at p, and this is exactly the set of lines in T_pX.

One defines the Zariski tangent space to a scheme X, at a point x, using exactly the same definition, the dual of

$$\mathfrak{m}/\mathfrak{m}^2,$$

where $\mathfrak{m} \subset \mathcal{O}_{X,x}$ is the maximal ideal of the local ring. However in general, if we have the equality of dimensions of both the Zariski tangent space and the local dimension, we only call X **regular at** $x \in X$. Smoothness is a more restricted notion in general.

Having said this, if X is a quasi-projective variety over an algebraically closed field then X is smooth as a variety if and only if it is smooth as a scheme over $\operatorname{Spec} k$. In fact an abstract variety over $\operatorname{Spec} k$ is smooth if and only if it is regular. Note that if x is a specialisation of ξ and X is regular at x then X is regular at ξ, so it is enough to check that X is regular at the closed points.

One can sometimes use the Zariski tangent space to identify embedded points. If X is a scheme and $Y = X_{\text{red}}$ is the reduced subscheme then $x \in X$ is an embedded point if

$$\dim T_xX > \dim T_xY,$$

and X is reduced away from x. For example, if X is not regular at x but Y is regular at x then $x \in X$ is an embedded point.

It is interesting to see which toric varieties are smooth. The question is local, so we might as well assume that $X = U_\sigma$ is affine. If $\sigma \subset N_{\mathbb{R}}$ does not span $N_{\mathbb{R}}$, then $X \simeq U_{\sigma'} \times \mathbb{G}_m^l$, where σ' is the same cone as

σ embedded in the space it spans. So we might as well assume that σ spans $N_{\mathbb{R}}$. In this case X contains a unique fixed point x_σ which is in the closure of every orbit. Since X only contains finitely many orbits, it follows that X is smooth if and only if X is regular at x_σ. The maximal ideal of x_σ is generated by χ^u, where $u \in S_\sigma$. The square of the maximal is generated by χ^{u+v}, where u and v are two elements of S_σ. So a basis for $\mathfrak{m}/\mathfrak{m}^2$ is given by elements of S_σ that are not sums of two elements. Since the elements of S_σ generate the group M, the elements of S_σ which are not sums of two elements, must generate the group. Given an extremal ray of $\check{\sigma}$, a primitive generator of this ray is not the sum of two elements in S_σ. So $\check{\sigma}$ must have n edges and they must generate M. So these elements are a basis of the lattice and in fact $X \simeq \mathbb{A}^n_k$.

4. Sard's Theorem

A basic result in the theory of C^∞-maps is Sard's Theorem, which states that the set of points where a map is singular is a subset of measure zero (of the base). Since any holomorphic map between complex manifolds is automatically C^∞, and the derivative of a polynomial is the same as the derivative as a holomorphic function, it follows that any morphism between varieties, over $\mathbb{C}$, is smooth over an open subset. In fact by the Lefschetz principle, this result extends to any variety over $\mathbb{C}$.

Theorem 4.1. *Let $f\colon X \longrightarrow Y$ be a morphism of varieties over a field of characteristic zero.*

Then there is a dense open subset U of Y such that if $q \in U$ and $p \in f^{-1}(q) \cap X_{sm}$ then the differential $df_p\colon T_pX \longrightarrow T_qY$ is surjective. Further, if X is smooth, then the fibres $f^{-1}(q)$ are smooth if $q \in U$.

Let us recall the Lefschetz principle. First recall the notion of a first order theory of logic. Basically this means that one describes a theory of mathematics using a theory based on predicate calculus. For example, the following is a true statement from the first order theory of number theory,

$$\forall n \forall x \forall y \forall z \quad n \geq 3 \implies x^n + y^n \neq z^n.$$

One basic and desirable property of a first order theory of logic is that it is complete. In other words every possible statement (meaning anything that is well-formed) can be either proved or disproved. It is a very well-known result that the first order theory of number theory is not complete (Gödel's Incompleteness Theorem). What is perhaps more surprising is that there are interesting theories which are complete.

Theorem 4.2. *The first order logic of algebraically closed fields of characteristic zero is complete.*

Notice that a typical statement of the first order logic of fields is that a system of polynomial equations does or does not have solution. Since most statements in algebraic geometry turn on whether or not a system of polynomial equations have a solution, the following result is very useful.

Principle 4.3 (Lefschetz Principle)**.** *Every statement in the first order logic of algebraically closed fields of characteristic zero, which is true over $\mathbb{C}$, is in fact true over any algebraically closed field of characteristic.*

In fact this principle is immediate from (4.2). Suppose that p is a statement in the first order logic of algebraically closed fields of characteristic zero. By completeness, we can either prove p or not p. Since p holds over the complex numbers, there is no way we can prove not p. Therefore there must be a proof of p. But this proof is valid over any field of characteristic zero, so p holds over any algebraically closed field of characteristic zero.

A typical application of the Lefschetz principle is (4.1). By Sard's Theorem, we know that (4.1) holds over $\mathbb{C}$. On the other hand, given a variety X over an algebraically closed field, of characteristic zero, whether or not (4.1) holds for X, can be reformulated in the first order logic of algebraically closed fields of characteristic zero. Therefore, by the Lefschetz principle, (4.1) is true for X, so that (4.1) is true over an algebraically closed field of characteristic zero.

One way to think of the Lefschetz principle is as follows. Take a variety X defined over an algebraically closed field L of characteristic zero. Realise $X \subset \mathbb{P}^n$. Then X is defined by finitely many polynomials $F_1, F_2, \dots, F_k$. The coefficients of these polynomials define a finitely generated extension $L/K/\mathbb{Q}$. Therefore we can find a scheme X' over K such that X is obtained from X by the base change $\operatorname{Spec} L \longrightarrow \operatorname{Spec} K$. On the other hand, K can be embedded into $\mathbb{C}$. The base change $\operatorname{Spec} \mathbb{C} \longrightarrow \operatorname{Spec} K$ gives us a variety Y over $\mathbb{C}$. Most of the time, properties of X follow from properties of Y.

Perhaps even more interesting is that (4.1) fails in characteristic p. Let $f\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the morphism $t \longrightarrow t^p$. If we fix s, then the equation

$$x^p - s$$

is purely inseparable, that is, has only one root. Thus f is a bijection. However, df is the zero map, since $dz^p = pz^{p-1}dz = 0$. Thus df_p is nowhere surjective. Note that the fibres of this map, as schemes, are isomorphic to zero dimensional schemes of length p.

5. The inverse function theorem

We now want to aim for a version of the Inverse function Theorem. In differential geometry, the inverse function theorem states that if a function is an isomorphism on tangent spaces, then it is locally an isomorphism. Unfortunately this is too much to expect in algebraic geometry, since the Zariski topology is too weak for this to be true. For example consider a curve which double covers another curve. At any point where there are two points in the fibre, the map on tangent spaces is an isomorphism. But there is no Zariski neighbourhood of any point where the map is an isomorphism.

Thus a minimal requirement is that the morphism is a bijection. Note that this is not enough in general for a morphism between algebraic varieties to be an isomorphism. For example in characteristic p, Frobenius is nowhere smooth and even in characteristic zero, the parametrisation of the cuspidal cubic is a bijection but not an isomorphism.

Lemma 5.1. *If $f\colon X \longrightarrow Y$ is a projective morphism with finite fibres, then f is finite.*

Proof. Since the result is local on the base, we may assume that Y is affine. By assumption $X \subset Y \times \mathbb{P}^n$ and we are projecting onto the first factor. Possibly passing to a smaller open subset of Y, we may assume that there is a point $p \in \mathbb{P}^n$ such that X does not intersect $Y \times \{p\}$.

As the blow up of $\mathbb{P}^n$ at p, fibres over $\mathbb{P}^{n-1}$ with fibres isomorphic to $\mathbb{P}^1$, and the composition of finite morphisms is finite, we may assume that $n = 1$, by induction on n.

We may assume that p is the point at infinity, so that $X \subset Y \times \mathbb{A}^1$, and X is affine. Now X is defined by $f(x) \in A(Y)[x]$, where the coefficients of $f(x)$ lie in $A(Y)$. Suppose that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_0.$$

We may always assume that a_n does not vanish at y. Passing to the locus where a_n does not vanish, we may assume that a_n is a unit, so that dividing by a_n, we may assume that $a_n = 1$. In this case the ring B is a quotient of the ring

$$A[x]/\langle f \rangle.$$

But the latter is generated over A by 1, x, $\dots x^{n-1}$, and so is a finitely generated module over A. □

Theorem 5.2. *Let $f\colon X \longrightarrow Y$ be a projective morphism between quasi-projective varieties.*

Then f is an isomorphism if and only if it is a bijection and the differential df_p is injective.

Proof. One direction is clear. Otherwise assume that f is projective and a bijection on closed points. Then f is finite by (5.1). The result is local on the base, so we may assume that $Y = \operatorname{Spec} C$ is affine, in which case $X = \operatorname{Spec} D$ is affine, where C is a finitely generated D-module. Pick $x \in X$ and let $y = f(x)$. Then $x = \mathfrak{p}$ and $y = \mathfrak{q}$ are two prime ideals in C and D. Let A be the local ring of Y at y, B of X at x. Then A is the localisation of C at the multiplicative subset $S = C - \mathfrak{q}$ and as x is the unique point of the fibre, B is the localisation of D by the multiplicative subset $T = S \cdot D$, so that B is a finitely generated A-module.

Let $\phi\colon A \longrightarrow B$ be the induced ring homomorphism. Then B is a finitely generated A-module and we just need to show that ϕ is an isomorphism.

As f is a bijection on closed points, it follows that ϕ is injective. So we might as well suppose that ϕ is an inclusion. Let $\mathfrak{m}$ be the maximal ideal of A and let $\mathfrak{n}$ be the maximal ideal of B. By assumption

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow \frac{\mathfrak{n}}{\mathfrak{n}^2},$$

is surjective. But then

$$\mathfrak{m}B + \mathfrak{n}^2 = \mathfrak{n}.$$

By Nakayama's Lemma applied to the B-module $\mathfrak{n}/\mathfrak{m}B$, it follows that $\mathfrak{m}B = \mathfrak{n}$. But then

$$B/A \otimes A/\mathfrak{m} = B/(\mathfrak{m}B + A) = B/(\mathfrak{n} + A) = 0.$$

Nakayama's Lemma applied to the finitely generated A-module B/A implies that $B/A = 0$ so that ϕ is an isomorphism. □

Lemma 5.3. *Suppose that $X \subset \mathbb{P}^n$ is a quasi-projective variety and suppose that $\pi\colon X \longrightarrow Y$ is the morphism induced by projection from a linear subspace Λ.*

Let $y \in Y$. Then $\pi^{-1}(y) = \langle \Lambda, y \rangle \cap X$. If further this fibre consists of one point, then the map between Zariski tangent spaces is an isomorphism if the intersection of $\langle \Lambda, x \rangle$ with the Zariksi tangent space to X at x has dimension zero.

Proof. Easy. □

Proposition 5.4. *Let X be a smooth irreducible subset of $\mathbb{P}^n$ of dimension k. Consider the projection Y of X down to a smaller dimensional projective space $\mathbb{P}^m$, from a linear space Λ of dimension $n - m - 1$.*

If the dimension of $m \geq 2k+1$ and Λ is general (that is, belongs to an appropriate open subset of the Grassmannian) then π is an isomorphism.

Proof. Since projection from a general linear space is the same as a sequence of projections from general points, we may assume that Λ is in fact a point p, so that $m = n - 1$.

Now we know that π is a bijection provided that p does not lie on any secant line. Since the secant variety has dimension at most $2k+1$, it follows that we may certainly find a point away from the secant variety, provided that $n > 2k+1$. Now since a tangent line is a limit of secant lines, it follows that such a point will also not lie on any tangent lines.

But then π is then an isomorphism on tangent spaces, whence an isomorphism. $\square$

For example, it follows that any curve may be embedded in $\mathbb{P}^3$ and any surface in $\mathbb{P}^5$.

6. Tangent lines to plane curves

Question 6.1. *Let C be a curve in $\mathbb{P}^2$ and let $p \in \mathbb{P}^2$.*

How many tangent lines does p lie on?

The first thing that we will need is a natty way to describe the projective tangent space to a variety.

Definition 6.2. *Let $X \subset \mathbb{P}^n$.*

The ***projective tangent space to** X **at*** *p is the closure of the affine tangent space.*

(Note the difference between the projective tangent space and the projectivisation of the tangent space.) In other words the projective tangent space has the same dimension as the affine tangent space and is obtained by adding the suitable points at infinity. Suppose that the curve is defined by the polynomial $F(X, Y, Z)$. Then the tangent line to C at p, is

$$\left.\frac{\partial F}{\partial X}\right|_p X + \left.\frac{\partial F}{\partial Y}\right|_p Y + \left.\frac{\partial F}{\partial Z}\right|_p Z.$$

Of course it suffices to check that we get the right answer on an affine piece.

Lemma 6.3. *Let F be a homogeneous polynomial of degree d in $X_0, X_1, \dots, X_n$. Then*

$$dF = \sum X_i \frac{\partial F}{\partial X_i}$$

Proof. Both sides are linear in F. Thus it suffices to prove this for a monomial of degree d, when the result is clear. $\square$

It follows then that the tangent line above does indeed pass through p. The rest is easy.

Finally we will need Bézout's Theorem.

Theorem 6.4 (Bézout's Theorem)**.** *Let C and D be two curves defined by homogenous polynomials of degrees d and e. Suppose that $C \cap D$ does not contain a curve.*

Then $|C \cap D|$ is at most de, with equality iff the intersection of the two tangent spaces at $p \in C \cap D$ is equal to p.

We are now ready to answer (6.1).

Lemma 6.5. *Let $C \subset \mathbb{P}^n$ be a curve in $\mathbb{P}^2$ and let $p \in \mathbb{P}^2$ be a general point.*

Then p lies on $d(d-1)$ tangent lines.

Proof. Fix $p = [a : b : c]$ and let D be the curve defined by

$$G = a\frac{\partial F}{\partial X} + b\frac{\partial F}{\partial Y} + c\frac{\partial F}{\partial Z}.$$

Then G is a polynomial of degree $d-1$. Consider a point q where C intersects D. Then the tangent line to C at q is given by

$$\left.\frac{\partial F}{\partial X}\right|_q X + \left.\frac{\partial F}{\partial Y}\right|_q Y + \left.\frac{\partial F}{\partial Z}\right|_q Z.$$

But then since p satisfies this equation, as q lies on D, it follows that p lies on the tangent line of C at q. Similarly it is easy to check the converse, that if p lies on the tangent line to C at q, then q is an intersection point of C and D.

Now apply Bézout's Theorem. $\square$

There is an interesting way to look at all of this. In fact one may generalise the result above to the case of curves with nodes. Note that if you take a curve in $\mathbb{P}^3$ and take a general projection down to $\mathbb{P}^2$, then you get a nodal curve. Indeed it is easy to pick the point of projection not on a tangent line, since the space of tangent lines obviously sweeps out a surface; it is a little more involved to show that the space of three secant lines is a proper subvariety. (6.5) was then generalised to this case and it was shown that if δ is the number of nodes, then the number

$$\frac{(d-1(d-2)}{2} - \delta$$

is an invariant of the curve.

Here is another way to look at this. Suppose that we project our curve down to $\mathbb{P}^1$ from a point. Then we get a finite cover of $\mathbb{P}^1$, with d points in the general fibre. Lines tangent to C passing through p then count the number of branch points, that is, the number of points in the base where the fibre has fewer than d points. Since this tangent line is only tangent to p and is simply tangent (that is, there are no flex points) there are $d-1$ points in this fibre, and the ramification point corresponding to the branch point is where two sheets come together.

The modern approach to this invariant is quite different. If we are over the complex numbers $\mathbb{C}$, changing perspective, we may view the curve C as a Riemann surface covering another Riemann surface D. Now the basic topological invariant of a compact oriented Riemann surface is it's genus. In these terms there is a simple formula that connects the genus of C and B, in terms of the ramification data, known as Riemann-Hurwitz,

$$2g - 2 = d(2h-2) + b,$$

where g is the genus of C, h the genus of B, d the order of the cover and b the contribution from the ramification points. Indeed if locally on C, the map is given as $z \longrightarrow z^e$ so that e sheets come together, the contribution is $e-1$.

In our case, $B = \mathbb{P}^1$ which is of genus 0, for each branch point, we have simple ramification, so that $e = 2$ and the contribution is one, making a total $b = d(d-1)$. Thus

$$2g - 2 = -2d + d(d-1).$$

Solving for g we get

$$g = \frac{(d-1)(d-2)}{2}.$$

Note that if $d \leq 2$, then we get $g = 0$ as expected (that is $C \simeq \mathbb{P}^1$) and if $d = 3$ then we get an elliptic curve.

7. Ample invertible sheaves

Theorem 7.1. *Let X be a scheme over a ring A.*

(1) *If $\phi\colon X \longrightarrow \mathbb{P}^n_A$ is an A-morphism then $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ is an invertible sheaf on X, which is generated by the global sections $s_0, s_1, \dots, s_n$, where $s_i = \phi^* x_i$.*

(2) *If $\mathcal{L}$ is an invertible sheaf on X, which is generated by the global sections $s_0, s_1, \dots, s_n$, then there is a unique A-morphism $\phi\colon X \longrightarrow \mathbb{P}^n_A$ such that $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ and $s_i = \phi^* x_i$.*

Proof. It is clear that $\mathcal{L}$ is an invertible sheaf. Since $x_0, x_1, \dots, x_n$ generate the ring $A[x_0, x_1, \dots, x_n]$, it follows that $x_0, x_1, \dots, x_n$ generate the sheaf $\mathcal{O}_{\mathbb{P}^n_A}(1)$. Thus $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. Hence (1).

Now suppose that $\mathcal{L}$ is an invertible sheaf generated by $s_0, s_1, \dots, s_n$. Let

$$X_i = \{\, p \in X \mid s_i \notin \mathfrak{m}_p\mathcal{L}_p \,\}.$$

Then X_i is an open subset of X and the sets $X_0, X_1, \dots, X_n$ cover X. Define a morphism

$$\phi_i\colon X_i \longrightarrow U_i,$$

where U_i is the standard open subset of $\mathbb{P}^n_A$, as follows: Since

$$U_i = \operatorname{Spec} A[y_0, y_1, \dots, y_n],$$

where $y_j = x_j/x_i$, is affine, it suffices to give a ring homomorphism

$$A[y_0, y_1, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}).$$

We send y_j to s_j/s_i, and extend by linearity. The key observation is that the ratio is a well-defined element of $\mathcal{O}_{X_i}$, which does not depend on the choice of isomorphisms $\mathcal{L}|_V \simeq \mathcal{O}_V$, for open subsets $V \subset X_i$.

It is then straightforward to check that the set of morphisms $\{\phi_i\}$ glues to a morphism ϕ with the given properties. □

Example 7.2. *Let $X = \mathbb{P}^1_k$, $A = k$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_k}(2)$.*

In this case, global sections of $\mathcal{L}$ are generated by S^2, ST and T^2. This morphism is represented globally by

$$[S : T] \longrightarrow [S^2 : ST : T^2].$$

The image is the conic $XZ = Y^2$ inside $\mathbb{P}^2_k$.

More generally one can map $\mathbb{P}^1_k$ into $\mathbb{P}^n_k$ by the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_k}(n)$. More generally still, one can map $\mathbb{P}^m_k$ into $\mathbb{P}^n_k$ using the invertible sheaf $\mathcal{O}_{\mathbb{P}^m_k}(1)$.

Corollary 7.3.

$$\operatorname{Aut}(\mathbb{P}^n_k) \simeq \operatorname{PGL}(n+1, k).$$

Proof. First note that $\mathrm{PGL}(n+1,k)$ acts naturally on $\mathbb{P}^n_k$ and that this action is faithful.

Now suppose that $\phi \in \mathrm{Aut}(\mathbb{P}^n_k)$. Let $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_k}(1)$. Since $\mathrm{Pic}(\mathbb{P}^n_k) \simeq \mathbb{Z}$ is generated by $\mathcal{O}_{\mathbb{P}^n_k}(1)$, it follows that $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(\pm 1)$. As $\mathcal{L}$ is globally generated, we must have $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$. Let $s_i = \phi^* x_i$. Then $s_0, s_1, \dots, s_n$ is a basis for the k-vector space $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$. But then there is a matrix

$$A = (a_{ij}) \in \mathrm{GL}(n+1,k) \qquad \text{such that} \qquad s_i = \sum_{ij} a_{ij} x_j.$$

Since the morphism ϕ is determined by $s_0, s_1, \dots, s_n$, it follows that ϕ is determined by the class of A in $\mathrm{GL}(n+1,k)$. □

Lemma 7.4. *Let $\phi\colon X \longrightarrow \mathbb{P}^n_A$ be an A-morphism. Then ϕ is a closed immersion if and only if*

(1) $X_i = X_{s_i}$ is affine, and
(2) the natural map of rings

$$A[y_0, y_1, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}) \qquad \textit{which sends} \qquad y_j \longrightarrow \frac{\sigma_j}{\sigma_i},$$

is surjective.

Proof. Suppose that ϕ is a closed immersion. Then X_i is isomorphic to $\phi(X) \cap U_i$, a closed subscheme of affine space. Thus X_i is affine. Hence (1) and (2) follows as we have surjectivity on all of the localisations.

Now suppose that (1) and (2) hold. Then X_i is a closed subscheme of U_i and so X is a closed subscheme of $\mathbb{P}^n_A$. □

Theorem 7.5. *Let X be a projective scheme over an algebraically closed field k and let $\phi\colon X \longrightarrow \mathbb{P}^n_k$ be a morphism over k, which is given by an invertible sheaf $\mathcal{L}$ and global sections $s_0, s_1, \dots, s_n$ which generate $\mathcal{L}$. Let $V \subset \Gamma(X, \mathcal{L})$ be the space spanned by the sections.*

Then ϕ is a closed immersion if and only if

*(1) V **separates points**: that is, given p and $q \in X$ there is $\sigma \in V$ such that $\sigma \in \mathfrak{m}_P \mathcal{L}_p$ but $\sigma \notin \mathfrak{m}_q \mathcal{L}_q$.*
*(2) V **separates tangent vectors**: that is, given $p \in X$ the set*

$$\{\, \sigma \in V \mid \sigma \in \mathfrak{m}_p \mathcal{L}_p \,\},$$

spans $\mathfrak{m}_p \mathcal{L}_p / \mathfrak{m}_p^2 \mathcal{L}_p$.

Proof. Suppose that ϕ is a closed immersion. Then we might as well consider $X \subset \mathbb{P}^n_k$ as a closed subscheme. In this case (1) is clear. Just pick a linear function on the whole of $\mathbb{P}^n_k$ which vanishes at p but not at q (equivalently pick a hyperplane which contains p but not q).

Similarly linear functions on $\mathbb{P}^n_k$ separate tangent vectors on the whole of projective space, so they certainly separate on X.

Now suppose that (1) and (2) hold. Then ϕ is clearly injective. Since X is proper over $\operatorname{Spec} k$ and $\mathbb{P}^n_k$ is separated over $\operatorname{Spec} k$ it follows that ϕ is proper. In particular $\phi(X)$ and ϕ is a homeomorphism onto $\phi(X)$. It remains to show that the map on stalks

$$\mathcal{O}_{\mathbb{P}^n_k,p} \longrightarrow \mathcal{O}_{X,x},$$

is surjective. But the same piece of commutative algebra as we used in the proof of the inverse function theorem, works here. $\square$

Definition 7.6. *Let X be a noetherian scheme. We say that an invertible sheaf $\mathcal{L}$ is* ***ample*** *if for every coherent sheaf $\mathcal{F}$ there is an integer $n_0 > 0$ such that $\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^n$ is globally generated, for all $n \geq n_0$.*

Lemma 7.7. *Let $\mathcal{L}$ be an invertible sheaf on a Noetherian scheme. TFAE*

(1) $\mathcal{L}$ is ample.
(2) $\mathcal{L}^m$ is ample for all $m > 0$.
(3) $\mathcal{L}^m$ is ample for some $m > 0$.

Proof. (1) implies (2) implies (3) is clear.

So assume that $\mathcal{M} = \mathcal{L}^m$ is ample and let $\mathcal{F}$ be a coherent sheaf. For each $0 \leq i \leq m-1$, let $\mathcal{F}_i = \mathcal{F} \otimes \mathcal{L}^i$. By assumption there is an integer n_i such that $\mathcal{F}_i \otimes \mathcal{M}^n$ is globally generated for all $n \geq n_i$. Let n_0 be the maximum of the n_i. If $n \geq n_0 m$, then we may write $n = qm + i$, where $0 \leq i \leq m-1$ and $q \geq n_0 \geq n_i$.

But then

$$\mathcal{F} \otimes \mathcal{L}^m = \mathcal{F}_i \otimes \mathcal{M}^q,$$

which is globally generated. $\square$

Theorem 7.8. *Let X be a scheme of finite type over a Noetherian ring A and let $\mathcal{L}$ be an invertible sheaf on X.*

Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample for some $m > 0$.

Proof. Suppose that $\mathcal{L}^m$ is very ample. Then there is an immersion $X \subset \mathbb{P}^r_A$, for some positive integer r, and $\mathcal{L}^m = \mathcal{O}_X(1)$. Let $\bar{X}$ be the closure. If $\mathcal{F}$ is any coherent sheaf on X then there is a coherent sheaf $\overline{\mathcal{F}}$ on $\bar{X}$, such that $\mathcal{F} = \overline{\mathcal{F}}|_X$. By Serre's result, $\overline{\mathcal{F}}(k)$ is globally generated for all $k \geq k_0$, for some integer k_0. It follows that $\mathcal{F}(k)$ is globally generated, for all $k \geq k_0$, so that $\mathcal{L}^m$ is ample, and the result follows by (7.7).

Conversely, suppose that $\mathcal{L}$ is ample. Given $p \in X$, pick an open affine neighbourhood U of p so that $\mathcal{L}|_U$ is free. Let $Y = X - U$, give it the reduced induced strucure, with ideal sheaf $\mathcal{I}$. Then $\mathcal{I}$ is coherent. Pick $n > 0$ so that $\mathcal{I} \otimes \mathcal{L}^n$ is globally generated. Then we may find $s \in \mathcal{I} \otimes \mathcal{L}^n$ not vanishing at p. We may identify s with $s' \in \mathcal{O}_U$ and then $p \in U_s \subset U$, an affine subset of X.

By compactness, we may cover X by such open affines and we may assume that n is fixed. Replacing $\mathcal{L}$ by $\mathcal{L}^n$ we may assume that $n = 1$. Then there are global sections $s_1, s_2, \dots, s_k \in H^0(X, \mathcal{L})$ such that $U_i = U_{s_i}$ is an open affine cover.

Since X is of finite type, each $B_i = H^0(U_i, \mathcal{O}_{U_i})$ is a finitely generated A-algebra. Pick generators b_{ij}. Then $s^n b_{ij}$ lifts to $s_{ij} \in H^0(X, \mathcal{L}^n)$. Again we might as well assume $n = 1$.

Now let $\mathbb{P}^N_A$ be the projective space with coordinates $x_1, x_2, \dots, x_k$ and x_{ij}. Locally we can define a map on each U_i to the standard open affine, by the obvious rule, and it is standard to check that this glues to an immersion. $\square$

8. Linear systems

Definition 8.1. *Let $\mathcal{L}$ be an invertible sheaf on a smooth projective variety over an algebraically closed field. Let $s \in H^0(X, \mathcal{L})$. The divisor (s) of zeroes of s is defined as follows. By assumption we may cover X by open subsets U_i over which we may identify $s|_{U_i}$ with $f_i \in \mathcal{O}_{U_i}$. The defines a Cartier divisor $\{(U_i, f_i)\}$.*

It is a simple matter to check that the Cartier divisor does not depend on our choice of trivialisations. Note that as X is smooth the Cartier divisor may safely be identified with the corresponding Weil divisor.

Lemma 8.2. *Let X be a smooth projective variety over an algebraically closed field. Let D_0 be a divisor and let $\mathcal{L} = \mathcal{O}_X(D_0)$.*

(1) If $s \in H^0(X, \mathcal{L})$, $s \neq 0$ then $(s) \sim D_0$.
(2) If $D \geq 0$ and $D \sim D_0$ then there is a global section $s \in H^0(X, \mathcal{L})$ such that $D = (s)$.
(3) If $s_i \in H^0(X, \mathcal{L})$, $i = 1$ and 2, are two global sections then $(s_1) = (s_2)$ if and only if $s_2 = \lambda s_1$ where $\lambda \in k^*$.

Proof. As $\mathcal{O}_X(D_0) \subset \mathcal{K}$, the section s corresponds to a rational function f. If D_0 is the Cartier divisor $\{(U_i, f_i)\}$ then $\mathcal{O}_X(D_0)$ is locally generated by f_i^{-1} so that multiplication by f_i induces an isomorphism with $\mathcal{O}_{U_i}$. D is then locally defined by $f f_i$. But then

$$D = D_0 + (f).$$

Hence (1).

Now suppose that $D > 0$ and $D = D_0 + (f)$. Then $(f) \geq -D_0$. Hence

$$f \in H^0(X, \mathcal{O}_X(D_0)) \subset H^0(X, \mathcal{K}) = K(X),$$

and the divisor of zeroes of f is D. This is (2).

Now suppose that $(s_1) = (s_2)$. Then

$$D_0 + (f_1) = (s_1) = (s_2) = D_0 + (f_2).$$

Cancelling, we get that $(f_1) = (f_2)$ and the rational function f_1/f_2 has no zeroes nor poles. Since X is a projective variety, $f_1/f_2 = \lambda$, a constant. □

Definition 8.3. *Let D_0 be a divisor. The* ***complete linear system*** *associated to D_0 is the set*

$$|D_0| = \{\, D \in \mathrm{Div}(X) \mid D \geq 0, D \sim D_0 \,\}.$$

We have seen that

$$|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Thus $|D|$ is naturally a projective space.

Definition 8.4. *A* ***linear system*** *is any linear subspace of a complete linear system* $|D_0|$.

In other words, a linear system corresponds to a linear subspace, $V \subset H^0(X, \mathcal{O}_X(D_0))$. We will then write

$$|V| = \{\, D \in |D_0| \mid D = (s), s \in V \,\} \simeq \mathbb{P}(V) \subset \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Definition 8.5. *Let* $|V|$ *be a linear system. The* ***base locus*** *of* $|V|$ *is the intersection of the elements of* $|V|$.

Lemma 8.6. *Let* X *be a smooth projective variety over an algebraically closed field, and let* $|V| \subset |D_0|$ *be a linear system.*

V *generates* $\mathcal{O}_X(D_0)$ *if and only if* $|V|$ *is base point free.*

Proof. If V generates $\mathcal{O}_X(D_0)$ then for every point $x \in X$ we may find an element $\sigma \in V$ such that $\sigma(x) \neq 0$. But then $D = (\sigma)$ does not contain x, and so the base locus is empty.

Conversely suppose that the base locus is empty. The locus where V does not generate $\mathcal{O}_X(D_0)$ is a closed subset Z of X. Pick $x \in Z$ a closed point. By assumption we may find $D \in |V|$ such that $x \notin D$. But then if $D = (\sigma)$, $\sigma(x) \neq 0$ and σ generates the stalk $\mathcal{L}_x$, a contradiction. Thus Z is empty and $\mathcal{O}_X(D_0)$ is globally generated. □

Example 8.7. *Consider* $\mathcal{O}_{\mathbb{P}^1}(4)$. *The complete linear system* $|4p|$ *defines a morphism into* $\mathbb{P}^4$, *where* $p = [0 : 1]$ *and* $q = [1 : 0]$, *given by* $\mathbb{P}^1 \longrightarrow \mathbb{P}^4$, $[S : T] \longrightarrow [S^4 : ST^3 : S^2T^2 : ST^3 : T^4]$. *If we project from* $[0 : 0 : 1 : 0 : 0]$ *we will get a morphism into* $\mathbb{P}^3$, $[S : T] \longrightarrow [S^4 : ST^3 : ST^3 : T^4]$. *This corresponds to the sublinear system spanned by* $4p$, $3p + q$, $p + 3q$, $4q$.

Consider $\mathcal{O}_{\mathbb{P}^2}(2)$ *and the corresponding complete linear system. The map associated to this linear system is the Veronese embedding* $\mathbb{P}^2 \longrightarrow \mathbb{P}^5$, $[X : Y : Z] \longrightarrow [X^2 : Y^2 : Z^2 : YZ : XZ : XY]$.

Note also the notion of separating points and tangent directions becomes a little clearer in this more geometric setting. Separating points means that given x and $y \in X$, we can find $D \in |V|$ such that $x \in D$ and $y \notin D$. Separating tangent vectors means that given any irreducible length two zero dimensional scheme z, with support x, we can find $D \in |V|$ such that $x \in D$ but z is not contained in D. In fact the condition about separating tangent vectors is really the limiting case of separating points.

Thinking in terms of linear systems also presents an inductive approach to proving global generation. Suppose that we consider the complete linear system $|D|$. Suppose that we can find $Y \in |D|$. Then the base locus of $|D|$ is supported on Y. On the other hand suppose that $\mathcal{I}$ is the ideal sheaf of Y in X. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

As X is smooth D is Cartier and $\mathcal{O}_X(D)$ is an invertible sheaf. Tensoring by locally free preserves exactness, so there are short exact sequences,

$$0 \longrightarrow \mathcal{I}(mD) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_Y(mD) \longrightarrow 0.$$

Taking global sections, we get

$$0 \longrightarrow H^0(X, \mathcal{I}(mD) \longrightarrow H^0(X, \mathcal{O}_X(mD) \longrightarrow H^0(Y, \mathcal{O}_Y(mD).$$

At the level of linear systems there is therefore a linear map

$$|D| \longrightarrow |D|_Y|.$$

Consider another application of the ideas behind this section. Consider the problem of parametrising subvarieties or subschemes X of projective space $\mathbb{P}^r_k$. Any subscheme is determined by the homogeneous ideal $I(X)$ of polynomials vanishing on X. As in the case of zero dimensional schemes, we would like to reduce to the data of a vector subspace of fixed dimension in a fixed vector space. The obvious thing to consider is polynomials of degree d and the vector subspace of polynomials of polynomials of degree d vanishing on X. But how large should we take d to be?

The first observation is that if $\mathcal{I}$ is the ideal sheaf of X in $\mathbb{P}^r_k$ then

$$I_d = H^0(\mathbb{P}^r_k, \mathcal{I}(d)),$$

where $\mathcal{I}(d)$ is the Serre twist. To say that I_d determines X, is essentially equivalent to saying that $\mathcal{I}(d)$ is globally generated. Fixing some data about X (in the case of zero dimensional schemes this would be the length) we would then like a positive integer d_0 such that if $d \geq d_0$ then two things are true:

- $\mathcal{I}(d)$ is globally generated.
- $h^0(\mathbb{P}^r_k, \mathcal{I}(d))$, the dimension of the space of global sections, is independent of X.

Now there is a short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^r_k} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Twisting by d, we get

$$0 \longrightarrow \mathcal{I}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^r_k}(d) \longrightarrow \mathcal{O}_X(d) \longrightarrow 0.$$

Taking global sections gives another exact sequence.

$$0 \longrightarrow H^0(\mathbb{P}^r_k, \mathcal{I}(d)) \longrightarrow H^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(d)) \longrightarrow H^0(X, \mathcal{O}_X(d)).$$

Again, it would be really nice if this exact sequence were exact on the right. Then global generation of $\mathcal{I}(d)$ would be reduced to global generation of $\mathcal{O}_X(d)$ and one could read of $h^0(\mathbb{P}^r_k, \mathcal{I}(d))$ from $h^0(X, \mathcal{O}_X(d))$.

9. Ample divisors on toric varieties

It is interesting to see what happens for toric varieties. Suppose that $X = X(F)$ is the toric variety associated to the fan $F \subset N_{\mathbb{R}}$. Recall that we can associate to a T-Cartier divisor $D = \sum a_i D_i$, a continuous piecewise linear function

$$\phi_D \colon |F| \longrightarrow \mathbb{R},$$

where $|F| \subset N_{\mathbb{R}}$ is the support of the fan, which is specified by the rule that $\phi_D(v_i) = -a_i$.

We can also associate to D a rational polyhedron

$$\begin{aligned} P_D &= \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \quad \forall i \,\} \\ &= \{\, u \in M_{\mathbb{R}} \,|\, u \geq \phi_D \,\}. \end{aligned}$$

Lemma 9.1. *If X is a toric variety and D is T-Cartier then*

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{u \in P_D \cap M} k \cdot \chi^u.$$

Proof. Suppose that $\sigma \in F$ is a cone. Then, we have already seen that

$$H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D)) = \bigoplus_{u \in P_D(\sigma) \cap M} k \cdot \chi^u,$$

where

$$P_D(\sigma) = \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \quad \forall v_i \in \sigma \,\}.$$

These identifications are compatible on overlaps. Since

$$H^0(X, \mathcal{O}_X(D)) = \bigcap_{\sigma \in F} H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D))$$

and

$$P_D = \bigcap_{\sigma \in F} P_D(\sigma),$$

the result is clear. $\square$

It is interesting to compute some examples. First, consider $\mathbb{P}^1$. A T-Cartier divisor is a sum $ap + bq$ (p and q the fixed points). The corresponding function is

$$\phi(x) = \begin{cases} -ax & x > 0 \\ bx & x < 0. \end{cases}$$

The corresponding polytope is the interval

$$[-a, b] \subset \mathbb{R} = N_{\mathbb{R}}.$$

There are $a+b+1$ integral points, corresponding to the fact that there are $a+b+1$ monomials of degree $a+b$.

For $\mathbb{P}^2$ and dD_3, P_D is the convex hull of $(0,0)$, $(d,0)$ and $(0,d)$. The number of integral points is

$$\frac{(d+1)^2}{2} + \frac{d+1}{2} = \frac{(d+2)(d+1)}{2},$$

which is the usual formula.

Let D be a Cartier divisor on a toric variety $X = X(F)$ given by a fan F. It is interesting to consider when the complete linear system $|D|$ is base point free. Since any Cartier divisor is linearly equivalent to a T-Cartier divisor, we might as well suppose that $D = \sum a_i D_i$ is T-Cartier. Note that the base locus of the complete linear system of any Cartier divisor is invariant under the action of the torus. Since there are only finitely many orbits, it suffices to show that for each cone $\sigma \in F$ the point $x_\sigma \in U_\sigma$ is not in the base locus. It is also clear that if x_σ is not in the base locus of $|D|$ then in fact one can find a T-Cartier divisor $D' \in |D|$ which does not contain x_σ. Equivalently we can find $u \in M$ such that

$$\langle u, v_i \rangle \geq -a_i,$$

with strict equality if $v_i \in \sigma$. The interesting thing is that we can reinterpret this condition using ϕ_D.

Definition 9.2. *The function $\phi\colon V \longrightarrow \mathbb{R}$ is **upper convex** if*

$$\phi(\lambda v + (1-\lambda)w) \geq \lambda\phi(v) + (1-\lambda)w \qquad \forall v, w \in V.$$

*When we have a fan F and ϕ is linear on each cone σ, then ϕ is called **strictly upper convex** if the linear functions $u(\sigma)$ and $u(\sigma')$ are different, for different maximal cones σ and σ'.*

Theorem 9.3. *Let $X = X(F)$ be the toric variety associated to a T-Cartier divisor D.*

Then

1. *$|D|$ is base point free if and only if ϕ_D is upper convex.*
2. *D is very ample if and only if ϕ_D is strictly upper convex and the semigroup S_σ is generated by*

$$\{\, u - u(\sigma) \,|\, u \in P_D \cap M \,\}.$$

Proof. (1) follows from the remarks above. (2) is proved in Fulton's book. □

For example if $X = \mathbb{P}^1$ and

$$\phi(x) = \begin{cases} -ax & x > 0 \\ bx & x < 0. \end{cases}$$

so that $D = ap + bq$ then ϕ is upper convex if and only if $a + b \geq 0$ in which case D is base point free. D is very ample if and only if $a+b > 0$. When ϕ is continuous and linear on each cone σ, we may restate the upper convex condition as saying that the graph of ϕ lies under the graph of $u(\sigma)$. It is strictly upper convex if it lies strictly under the graph of $u(\sigma)$, for all n-dimensional cones σ.

Using this, it is altogether too easy to give an example of a smooth proper toric variety which is not projective. Let $F \subset N_{\mathbb{R}} = \mathbb{R}^3$ given by the edges $v_1 = -e_1$, $v_2 = -e_2$, $v_3 = -e_3$, $v_4 = e_1 + e_2 + e_3$, $v_5 = v_3 + v_4$, $v_6 = v_1 + v_4$ and $v_7 = v_2 + v_4$. Now connect v_1 to v_5, v_3 to v_7 and v_2 to v_6 and v_5 to v_6, v_6 to v_7 and v_7 to v_5.

It is not hard to check that X is smooth and proper (proper translates to the statement that the support $|F|$ of the fan is the whole of $N_{\mathbb{R}}$). Suppose that ϕ is strictly upper convex. Let w be the midpoint of the line connecting v_1 and v_5. Then

$$w = \frac{v_1 + v_5}{2} = \frac{v_3 + v_6}{2}.$$

Since v_1 and v_5 belong to the same maximal cone, ϕ is linear on the line connecting them. In particular

$$\phi(w) = \phi(\frac{v_1 + v_5}{2}) = \frac{1}{2}\phi(v_1) + \frac{1}{2}\phi(v_5).$$

Since v_1, v_5 and v_3 belong to the same cone and v_6 does not, by strict convexity,

$$\phi(w) = \phi(\frac{v_3 + v_6}{2}) > \frac{1}{2}\phi(v_3) + \frac{1}{2}\phi(v_6).$$

Putting all of this together, we get

$$\phi(v_1) + \phi(v_5) > \phi(v_3) + \phi(v_6).$$

By symmetry

$$\begin{aligned} \phi(v_1) + \phi(v_5) &> \phi(v_3) + \phi(v_6) \\ \phi(v_2) + \phi(v_6) &> \phi(v_1) + \phi(v_7) \\ \phi(v_3) + \phi(v_7) &> \phi(v_2) + \phi(v_5). \end{aligned}$$

But adding up these three inequalities gives a contradiction.

10. Relative proj and projective bundles

We want to define a relative version of Proj, in pretty much the same way we defined a relative version of Spec. We start with a scheme X and a quasi-coherent sheaf $\mathcal{S}$ sheaf of graded $\mathcal{O}_X$-algebras,

$$\mathcal{S} = \bigoplus_{d \in \mathbb{N}} \mathcal{S}_d,$$

where $\mathcal{S}_0 = \mathcal{O}_X$. It is convenient to make some simplifying assumptions:

(†) X is Noetherian, $\mathcal{S}_1$ is coherent, $\mathcal{S}$ is locally generated by $\mathcal{S}_1$.

To construct relative Proj, we cover X by open affines $U = \operatorname{Spec} A$. With a view towards what comes next, we denote global sections of $\mathcal{S}$ over U by $H^0(U, \mathcal{S})$. Then $\mathcal{S}(U) = H^0(U, \mathcal{S})$ is a graded A-algebra, and we get $\pi_U \colon \operatorname{Proj} \mathcal{S}(U) \longrightarrow U$ a projective morphism. If $f \in A$ then we get a commutative diagram

$$\begin{tikzcd}
\operatorname{Proj} \mathcal{S}(U_f) \arrow[r] \arrow[d, "\pi_{U_f}"'] & \operatorname{Proj} \mathcal{S}(U) \arrow[d, "\pi_U"'] \\
U_f \arrow[r] & U.
\end{tikzcd}$$

It is not hard to glue π_U together to get $\pi \colon \mathbf{Proj}\, \mathcal{S} \longrightarrow X$. We can also glue the invertible sheaves together to get an invertible sheaf $\mathcal{O}(1)$.

The relative consruction has some similarities to the old construction.

Example 10.1. *If X is Noetherian and*

$$\mathcal{S} = \mathcal{O}_X[T_0, T_1, \dots, T_n],$$

then satisfies (†) *and* $\mathbf{Proj}\, \mathcal{S} = \mathbb{P}^n_X$.

Given a sheaf $\mathcal{S}$ satisfying (†), and an invertible sheaf $\mathcal{L}$, it is easy to construct a quasi-coherent sheaf $\mathcal{S}' = \mathcal{S} \star \mathcal{L}$, which satisfies (†). The graded pieces of $\mathcal{S}'$ are $\mathcal{S}_d \otimes \mathcal{L}^d$ and the multiplication maps are the obvious ones. There is a natural isomorphism

$$\phi \colon P' = \mathbf{Proj}\, \mathcal{S}' \longrightarrow P = \mathbf{Proj}\, \mathcal{S},$$

which makes the diagram commute

$$\begin{tikzcd}
P' \arrow[rr, "\phi"] \arrow[dr, "\pi'"'] & & P \arrow[dl, "\pi"] \\
& X, &
\end{tikzcd}$$

and

$$\phi^* \mathcal{O}_P(1) \simeq \mathcal{O}_{P'}(1) \otimes \pi'^* \mathcal{L}.$$

Note that π is always proper; in fact π is projective over any open affine and properness is local on the base. Even better π is projective if X has an ample line bundle; see (II.7.10).

There are two very interesting families of examples of the construction of relative Proj. Suppose that we start with a locally free sheaf $\mathcal{E}$ of rank $r \geq 2$. Note that

$$\mathcal{S} = \bigoplus \operatorname{Sym}^d \mathcal{E},$$

satisfies (†). $\mathbb{P}(\mathcal{E}) = \mathbf{Proj}\, \mathcal{S}$ is the **projective bundle** over X associated to $\mathcal{E}$. The fibres of $\pi \colon \mathbb{P}(\mathcal{E}) \longrightarrow X$ are copies of $\mathbb{P}^n$, where $n = r - 1$. We have

$$\bigoplus_{l=0}^{\infty} \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) = \mathcal{S},$$

so that in particular

$$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}.$$

Also there is a natural surjection

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Indeed, it suffices to check both statements locally, so that we may assume that X is affine. The first statement is standard and the second statement reduces to the statement that the sections $x_0, x_1, \dots, x_n$ generate $\mathcal{O}_P(1)$.

The most interesting result is:

Proposition 10.2. *Let $g \colon Y \longrightarrow X$ be a morphism.*

Then a morphism $f \colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X is the same as giving an invertible sheaf $\mathcal{L}$ on Y and a surjection $g^* \mathcal{E} \longrightarrow \mathcal{L}$.

Proof. One direction is clear; if $f \colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ is a morphism over X, then the surjective morphism of sheaves

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1),$$

pulls back to a surjective morphism

$$g^* \mathcal{E} = f^*(\pi^* \mathcal{E}) \longrightarrow \mathcal{L} = f^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Conversely suppose we are given an invertible sheaf $\mathcal{L}$ and a surjective morphism of sheaves

$$g^* \mathcal{E} \longrightarrow \mathcal{L}.$$

I claim that there is then a unique morphism $f \colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X, which induces the given surjection. By uniqueness, it suffices to prove this result locally. So we may assume that $X = \operatorname{Spec} A$ is affine and

$$\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}_X,$$

is free. In this case surjectivity reduces to the statement that the images $s_0, s_1, \dots, s_n$ of the standard sections generate $\mathcal{L}$, and the result reduces to one we have already proved. $\square$

11. Blowing up schemes

We now consider how to define the blow up for an arbitrary scheme. Recall

(†) X is Noetherian, $\mathcal{S}_1$ is coherent, $\mathcal{S}$ is locally generated by $\mathcal{S}_1$.

Definition 11.1. *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Let*

$$\mathcal{S} = \bigoplus_{d=0}^{\infty} \mathcal{I}^d,$$

where $\mathcal{I}^0 = \mathcal{O}_X$ and $\mathcal{I}^d$ is the dth power of $\mathcal{I}$. Then $\mathcal{S}$ satisfies (†).

*$\pi \colon \mathbf{Proj}\, \mathcal{S} \longrightarrow X$ is called the **blow up** of $\mathcal{I}$ (or Y, if Y is the subscheme of X associated to $\mathcal{I}$).*

Example 11.2. *Let $X = \mathbb{A}^n_k$ and let P be the origin. We check that we get the usual blow up. Let*

$$A = k[x_1, x_2, \dots, x_n].$$

As $X = \operatorname{Spec} A$ is affine and the ideal sheaf $\mathcal{I}$ of P is the sheaf associated to $\langle x_1, x_2, \dots, x_n \rangle$,

$$Y = \mathbf{Proj}\, \mathcal{S} = \operatorname{Proj} S,$$

where

$$S = \bigoplus_{d=0}^{\infty} I^d.$$

There is a surjective map

$$A[y_1, y_2, \dots, y_n] \longrightarrow S,$$

of graded rings, where y_i is sent to x_i. $Y \subset \mathbb{P}^n_A$ is the closed subscheme corresponding to this morphism. The kernel of this morphism is

$$\langle y_i x_j - y_j x_i \rangle,$$

which are the usual equations of the blow up.

Definition 11.3. *Let $f \colon Y \longrightarrow X$ be a morphism of schemes. We are going to define the **inverse image ideal sheaf** $\mathcal{I}' \subset \mathcal{O}_Y$. First we take the inverse image of the sheaf $f^{-1}\mathcal{I}$, where we just think of f as being a continuous map. Then $f^{-1}\mathcal{I} \subset f^{-1}\mathcal{O}_X$. Let $\mathcal{I}' = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ be the ideal generated by the image of $f^{-1}\mathcal{I}$ under the natural morphism $f^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_Y$.*

Theorem 11.4 (Universal Property of the blow up)**.** *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent ideal sheaf.*

If $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ then $\pi^{-1}\mathcal{I}\cdot\mathcal{O}_Y$ is an invertible sheaf. Moreover π is universal amongst all such morphisms. If $f\colon Z \longrightarrow X$ is any morphism such that $f^{-1}\mathcal{I}\cdot\mathcal{O}_Z$ is invertible then there is a unique induced morphism $g\colon Z \longrightarrow Y$ which makes the diagram commute

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "f"'] & Y \arrow[d, "\pi"] \\ & X. \end{tikzcd}$$

Proof. By uniqueness, we can check this locally. So we may assume that $X = \operatorname{Spec} A$ is affine. As $\mathcal{I}$ is coherent, it corresponds to an ideal $I \subset A$ and

$$X = \operatorname{Proj} \bigoplus_{d=0}^{\infty} I^d.$$

Now $\mathcal{O}_Y(1)$ is an invertible sheaf on Y. It is not hard to check that $\pi^{-1}\mathcal{I}\cdot\mathcal{O}_Y = \mathcal{O}_Y(1)$.

Now show that we are given $f\colon Z \longrightarrow X$. We first construct g, then show that if g is any factorisation of f, the pullback ideal sheaf is invertible and then finally show that g is unique.

Pick generators $a_0, a_1, \dots, a_n$ for I. This gives rise to a surjective map of graded A-algebras

$$\phi\colon A[x_0, x_1, \dots, x_n] \longrightarrow \bigoplus_{d=0}^{\infty} I^d,$$

whence to a closed immersion $Y \subset \mathbb{P}^n_A$. The kernel of ϕ is generated by all homogeneous polynomials $F(x_0, x_1, \dots, x_n)$ such that $F(a_0, a_1, \dots, a_n) = 0$.

Now the elements $a_0, a_1, \dots, a_n$ pullback to global sections $s_0, s_1, \dots, s_n$ of the invertible sheaf $\mathcal{L} = f^{-1}\mathcal{I}\cdot\mathcal{O}_Y$ and $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. So we get a morphism

$$g\colon Z \longrightarrow \mathbb{P}^n_A,$$

over X, such that $g^*\mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{L}$ and $g^{-1}x_i = s_i$. Suppose that $F(x_0, x_1, \dots, x_n)$ is a homogeneous polynomial in the kernel of ϕ. Then $F(a_0, a_1, \dots, a_n) = 0$ so that $F(s_0, s_1, \dots, s_n) = 0$ in $H^0(Z, \mathcal{L}^d)$. It follows that g factors through Y.

Now suppose that $f\colon Z \longrightarrow X$ factors through $g\colon Z \longrightarrow Y$. Then

$$f^{-1}\mathcal{I}\cdot\mathcal{O}_Z = g^{-1}(\mathcal{I}\cdot\mathcal{O}_Y)\cdot\mathcal{O}_Z = g^{-1}\mathcal{O}_Y(1)\cdot\mathcal{O}_Z.$$

Thus $\mathcal{L} = f^{-1}\mathcal{I} \cdot \mathcal{O}_Z$ is an invertible sheaf.

Therefore there is a surjective map

$$g^*\mathcal{O}_Y(1) \longrightarrow \mathcal{L}.$$

But then this map must be an isomorphism and so $g^*\mathcal{O}_Y(1) = \mathcal{L}$. $s_i = g^*x_i$ and uniqueness follows. $\square$

Note that by the universal property, the morphism π is an isomorphism outside of the subscheme V defined by $\mathcal{I}$. We may put the universal property differently. The only subscheme with an invertible ideal sheaf is a Cartier divisor (local generators of the ideal, give local equations for the Cartier divisor). So the blow up is the smallest morphism which turns a subscheme into a Cartier divisor. Perhaps surprisingly, therefore, blowing up a Weil divisor might give a non-trivial birational map.

If X is a variety it is not hard to see that π is a projective, birational morphism. In particular if X is quasi-projective or projective then so is Y. We note that there is a converse to this:

Theorem 11.5. *Let X be a quasi-projective variety and let $f\colon Z \longrightarrow X$ be a birational projective morphism.*

Then there is a coherent ideal sheaf $\mathcal{I}$ and a commutative diagram

$$\begin{tikzcd}
Z \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "\pi"] \\
& X, &
\end{tikzcd}$$

where $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ and the top row is an isomorphism.

12. Flips and flops

It is interesting to figure out the geometry behind the example of a toric variety which is not projective. To warm up, suppose that we start with $\mathbb{A}^3_k$. This is the toric variety associated to the fan spanned by e_1, e_2, e_3. Imagine blowing up two of the axes. This corresponds to inserting two vectors, $e_1 + e_2$ and $e_1 + e_3$. However the order in which we blow up is significant. Let's introduce some notation. If we blow up the x-axis $\pi\colon Y \longrightarrow X$ and then the y-axis, $\psi\colon Z \longrightarrow Y$, let's call the exceptional divisors E_1 and E_2, and let E_1' denote the strict transform of E_1 on Z. E_1 is a $\mathbb{P}^1$-bundle over the x-axis. The strict transform of the y-axis in Y intersects E_1 in a point p. When we blow up this curve, $E_1' \longrightarrow E_1$ blows up the point p. The fibre of E_1' over the origin therefore consists of two copies Σ_1 and Σ_2 of $\mathbb{P}^1$. Σ_1 is the strict transform of the fibre of E_1 over the origin and Σ_2 is the exceptional divisor. The fibre Σ of E_2 over the origin is a copy of $\mathbb{P}^1$. Σ and Σ_2 are the same curve in Z.

The example of a toric variety which is not projective is obtained from $\mathbb{P}^3$ by blowing up three coordinate axes, which form a triangle. The twist is that we do something different at each of the three coordinate points. Suppose that $\pi\colon X \longrightarrow \mathbb{P}^3$ is the birational morphism down to $\mathbb{P}^3$, and let E_1, E_2 and E_3 be the three exceptional divisors. Over one point we extract E_1 first then E_2, over the second point we extract first E_2 then E_3 and over the last point we extract first E_3 then E_1.

To see what has gone wrong, we need to work in the homology and cohomology groups of X. Any curve C in X determines an element of $[C] \in H_2(X, \mathbb{Z})$. Any Cartier divisor D in X determines a class $[D] \in H^2(X, \mathbb{Z})$. We can pair these two classes to get an intersection number $D \cdot C \in \mathbb{Z}$. One way to compute this number is to consider the line bundle $\mathcal{L} = \mathcal{O}_X(D)$ associated to D. Then

$$D \cdot C = \deg \mathcal{L}|_C.$$

If D is ample then this intersection number is always positive. This implies that the class of every curve is non-trivial in homology.

Suppose the reducible fibres of E_1, E_2 and E_3 over their images are $A_1 + A_2$, $B_1 + B_2$ and $C_1 + C_3$. Suppose that the general fibres are A, B and C. We suppose that A_1 is attached to B, B_1 is attached to C

and C_1 is attached to A. We have

$$\begin{aligned}
[A] &= [A_1] + [A_2] \\
&= [B] + [A_2] \\
&= [B_1] + [B_2] + [A_2] \\
&= [C] + [B_2] + [A_2] \\
&= [C_1] + [C_2] + [B_2] + [A_2] \\
&= [A] + [C_2] + [B_2] + [A_2],
\end{aligned}$$

in $H_2(X, \mathbb{Z})$, so that

$$[A_2] + [B_2] + [C_2] = 0 \in H_2(X, \mathbb{Z}).$$

Suppose that D were an ample divisor on X. Then

$$0 = D \cdot ([A_2] + [B_2] + [C_2]) = D \cdot [A_2] + D \cdot [B_2] + D \cdot [C_2] > 0,$$

a contradiction.

There are a number of things to say about this way of looking at things, which lead in different directions. The first is that there is no particular reason to start with a triangle of curves. We could start with two conics intersecting transversally (so that they lie in different planes). We could even start with a nodal cubic, and just do something different over the two branches of the curve passing through the node. Neither of these examples are toric, of course. It is clear that in the first two examples, the morphism

$$\pi \colon X \longrightarrow \mathbb{P}^3,$$

is locally projective. It cannot be a projective morphism, since $\mathbb{P}^3$ is projective and the composition of projective is projective. It also follows that π is not the blow up of a coherent sheaf of ideals on $\mathbb{P}^3$. The third example is not even a variety. It is a complex manifold (and in fact it is something called an algebraic space). In particular the notion of the blow up in algebraic geometry is more delicate than it might first appear.

The second thing is to consider the difference between the order of blow ups of the two axes. Suppose we denote the composition of blowing up the x-axis and then the y-axis by $\pi_1 \colon X_1 \longrightarrow \mathbb{A}^3$ and the composition the other way by $\pi_2 \colon X_2 \longrightarrow \mathbb{A}^3$. Now X_1 and X_2 agree outside the origin. Let $\phi \colon X_1 \dashrightarrow X_2$ be the resulting birational map. If $A_1 + A_2$ is the fibre of π_1 over the origin and $B_1 + B_2$ is the fibre of π_2 over the origin, then ϕ is in fact an isomorphism outside A_2 and B_2. So ϕ is a birational map which is an isomorphism in codimension one, in fact an isomorphism outside a curve, isomorphic to $\mathbb{P}^1$. ϕ is an example of a *flop*. In terms of fans, we have four vectors v_1, v_2, v_3 and v_4, such that

$$v_1 + v_3 = v_2 + v_4,$$

and any three vectors span the lattice. If σ is the cone spanned by these four vectors, then $Q = U_\sigma$ is the cone over a quadric. There are two ways to subdivide σ into two cones. Insert the edge connecting v_1 to v_3 or the edge corresponding to $v_2 + v_4$. The corresponding morphisms extract a copy of $\mathbb{P}^1$ and the resulting birational map between the two toric varieties is a (simple) flop. One can also insert the vector $w = v_1 + v_3$, to get a toric variety Y. The corresponding exceptional divisor is $\mathbb{P}^1 \times \mathbb{P}^1$. These toric varieties fit into a picture

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr] & \\
X_1 \arrow[rr, dashed] \arrow[dr] & & X_2 \arrow[dl] \\
 & Q. &
\end{tikzcd}$$

The two maps $Y \longrightarrow X_i$ correspond to the two projections of $\mathbb{P}^1 \times \mathbb{P}^1$ down to $\mathbb{P}^1$. Now we know that $\pi_i \colon X_i \longrightarrow Q$ corresponds to blowing up a coherent ideal sheaf. In fact it corresponds to blowing up a Weil divisor (in fact this is a given, as π_i does not extract any divisors), the plane determined by either ruling.

Finally, it is interesting to wonder more about the original examples of varieties which are not projective. Note that in the case when we blow up either a triangle or a conic if we make one flop then we get a projective variety. Put differently, if we start with a projective variety then it is possible to get a non-projective variety by flopping a curve. When does flopping a curve mean that the variety is no longer projective? A variety is projective if it contains an ample divisor. Ample divisors intersect all curve positively. Note that any sum of ample divisors is ample.

Definition 12.1. *Let X be a proper variety. The* ***ample cone*** *is the cone in $H^2(X, \mathbb{R})$ spanned by the classes of the ample divisors.*

The ***Kleiman-Mori cone*** $\overline{\mathrm{NE}}(X)$ *in $H_2(X, \mathbb{R})$ is the closure of the cone spanned by the classes of curves.*

The significance of all of this is the following:

Theorem 12.2 (Kleiman's Criteria)**.** *Let X be a proper variety (or even algebraic space).*

A divisor D is ample if and only if the linear functional

$$\psi\colon H_2(X,\mathbb{R}) \longrightarrow \mathbb{R},$$

given by $\phi(\alpha) = [D]\cdot\alpha$ is strictly positive on $\overline{\mathrm{NE}}(X) - \{0\}$.

Using Kleiman's criteria, it is not hard to show that if $\phi\colon X \dashrightarrow Y$ is a flop of the curve C and X is projective then Y is projective if and only if the class of $[C]$ generates a one dimensional face of $\overline{\mathrm{NE}}(X)$.

13. Kähler differentials

Let A be a ring, B an A-algebra and M a B-module.

Definition 13.1. *An A-**derivation** of B into M is a map $\mathrm{d}\colon B \longrightarrow M$ such that*

(1) $\mathrm{d}(b_1 + b_2) = \mathrm{d}b_1 + \mathrm{d}b_2$.
(2) $\mathrm{d}(bb') = b'\mathrm{d}b + b\mathrm{d}b'$.
(3) $\mathrm{d}a = 0$.

Definition 13.2. *The module of **relative differentials**, denoted $\Omega_{B/A}$, is a B-module together with an A-derivation, $\mathrm{d}\colon B \longrightarrow \Omega_{B/A}$, which is universal with this property:*

If M is a B-module and $\mathrm{d}'\colon B \longrightarrow M$ is an A-derivation then there exists a unique B-module homomorphism $f\colon \Omega_{B/A} \longrightarrow M$ which makes the following diagram commute:

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_{B/A} \arrow[d, "f"] \\ & M. \end{tikzcd}$$

One can construct the module of relative differentials in the usual way; take the free B-module, with generators

$$\{\, \mathrm{d}b \mid b \in B \,\},$$

and quotient out by the three obvious sets of relations

(1) $\mathrm{d}(b_1 + b_2) - \mathrm{d}b_1 - \mathrm{d}b_2$,
(2) $\mathrm{d}(bb') - b'\mathrm{d}b - b\mathrm{d}b'$, and
(3) $\mathrm{d}a$.

The map $\mathrm{d}\colon B \longrightarrow M$ is the obvious one.

Example 13.3. *Let $B = A[x_1, x_2, \dots, x_n]$. Then $\Omega_{B/A}$ is the free B-module generated by $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_n$.*

Proposition 13.4. *Let A' and B be A-algebras and $B' = B \underset{A}{\otimes} A'$. Then*

$$\Omega_{B'/A'} = \Omega_{B'/A} \underset{B}{\otimes} B'$$

Furthermore, if S is a multiplicative system in B, then

$$\Omega_{S^{-1}B/A} = S^{-1}\Omega_{B/A}.$$

Suppose that $X = \operatorname{Spec} B \longrightarrow Y = \operatorname{Spec} A$ is a morphism of affine schemes. The **sheaf of relative differentials** $\Omega_{X/Y}$ is the quasi-coherent sheaf associated to the module of relative differentials $\Omega_{B/A}$.

Example 13.5. *Let* $X = \operatorname{Spec} \mathbb{R}$ *and* $Y = \operatorname{Spec} \mathbb{Q}$. *Then* $\mathrm{d}\pi \in \Omega_{X/Y} = \Omega_{\mathbb{R}/\mathbb{Q}}$ *is a non-zero differential.*

One could use the affine case to construct the sheaf of relative differentials globally. A better way to proceed is to use a little bit more algebra (and geometric intuition):

Proposition 13.6. *Let* B *be an* A*-algebra. Let*

$$B \underset{A}{\otimes} B \longrightarrow B,$$

be the diagonal morphism $b \otimes b' \longrightarrow bb'$ *and let* I *be the kernel. Consider* $B \underset{A}{\otimes} B$ *as a* B*-module by multiplication on the left. Then* I/I^2 *inherits the structure of a* B*-module. Define a map*

$$\mathrm{d}\colon B \longrightarrow \frac{I}{I^2},$$

by the rule

$$\mathrm{d}b = 1 \otimes b - b \otimes 1.$$

Then I/I^2 *is the module of differentials.*

Now suppose we are given a morphism of schemes $f\colon X \longrightarrow Y$. This induces the diagonal morphism

$$\Delta\colon X \longrightarrow X \underset{Y}{\times} X.$$

Then Δ defines an isomorphism of X with its image $\Delta(X)$ and this is locally closed in $X \underset{Y}{\times} X$, that is, there is an open subset $W \subset X \underset{Y}{\times} X$ and $\Delta(X)$ is a closed subset of W (it is closed in $X \underset{Y}{\times} X$ if and only if X is separated).

Definition 13.7. *Let* $\mathcal{I}$ *be the sheaf of ideals of* $\Delta(X)$ *inside* W. *The* ***sheaf of relative differentials***

$$\Omega_{X/Y} = \Delta^* \left(\frac{\mathcal{I}}{\mathcal{I}^2} \right).$$

Theorem 13.8 (Euler sequence)**.** *Let* A *be a ring, let* $Y = \operatorname{Spec} A$ *and* $X = \mathbb{P}^n_A$.

Then there is a short exact sequence of $\mathcal{O}_X$*-modules*

$$0 \longrightarrow \Omega_{X/Y} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Proof. Let $S = A[x_0, x_1, \dots, x_n]$ be the homogeneous coordinate ring of X. Let E be the graded S-module $S(-1)^{n+1}$, with basis $e_0, e_1, \dots, e_n$ in degree one. Define a (degree 0) homomorphism of graded S-modules

$E \longrightarrow S$ by sending $e_i \longrightarrow x_i$ and let M be the kernel. We have a left exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow S.$$

This gives rise to a short exact sequence of $\mathcal{O}_X$-modules,

$$0 \longrightarrow \tilde{M} \longrightarrow \mathcal{O}_X(-1)^{n+1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Note that even though $E \longrightarrow S$ is not surjective, it is surjective in all non-negative degrees, so that the map of sheaves is surjective.

It remains to show that $\tilde{M} \simeq \Omega_{X/Y}$. First note that if we localise at x_i, then $E_{x_i} \longrightarrow S_{x_i}$ is a surjective homomorphism of free S_{x_i}-modules, so that M_{x_i} is a free S_{x_i}-module of rank n, generated by

$$\{\, e_j - \frac{x_j}{x_i} e_i \,|\, j \neq i \,\}.$$

It follows that if U_i is the standard open affine subset of X defined by x_i then $\tilde{M}|_{U_i}$ is a free $\mathcal{O}_{U_i}$-module of rank n generated by the sections

$$\{\, \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i \,|\, j \neq i \,\}.$$

(We need the extra factor of x_i to get elements of degree zero.)

We define a map

$$\phi_i \colon \Omega_{X/Y}|_{U_i} \longrightarrow \tilde{M}|_{U_i},$$

as follows. As $U_i = \operatorname{Spec} k[\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_n}{x_i}]$, it follows that $\Omega_{X/Y}$ is the free $\mathcal{O}_{U_i}$-module generated by

$$\mathrm{d}\left(\frac{x_0}{x_i}\right), \mathrm{d}\left(\frac{x_1}{x_i}\right), \dots, \mathrm{d}\left(\frac{x_n}{x_i}\right).$$

So we define ϕ_i by the rule

$$\mathrm{d}\left(\frac{x_j}{x_i}\right) \longrightarrow \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i.$$

ϕ_i is clearly an isomorphism. We check that we can glue these maps to a global isomorphism,

$$\phi \colon \Omega_{X/Y} \longrightarrow \tilde{M}.$$

On $U_i \cap U_j$, we have

$$\left(\frac{x_k}{x_i}\right) = \left(\frac{x_k}{x_j}\right)\left(\frac{x_j}{x_i}\right).$$

Hence in $(\Omega_{X/Y})|_{U_i \cap U_j}$ we have

$$\mathrm{d}\left(\frac{x_k}{x_i}\right) - \frac{x_k}{x_j}\mathrm{d}\left(\frac{x_j}{x_i}\right) = \frac{x_j}{x_i}\mathrm{d}\left(\frac{x_k}{x_j}\right).$$

If we apply ϕ_i to the LHS and ϕ_j to the RHS, we get the same thing, namely

$$\frac{1}{x_i x_j}\left(x_j e_k - x_k e_j\right).$$

Thus the isomorphisms ϕ_i glue together. $\square$

14. Smoothness

Definition 14.1. *A variety is **smooth** (aka non-singular) if all of its local rings are regular local rings.*

Theorem 14.2. *The localisation of any regular local ring at a prime ideal is a regular local ring.*

Thus to check if a variety is smooth it is enough to consider only the closed points.

Theorem 14.3. *Let X be an irreducible separate scheme of finite type over an algebraically closed field k.*

Then $\Omega_{X/k}$ is locally free of rank $n = \dim X$ if and only if X is a smooth variety over k.

If $X \longrightarrow Z$ is a morphism of schemes and $Y \subset X$ is a closed subscheme, with ideal sheaf $\mathcal{I}$. Then there is an exact sequence of sheaves on Z,

$$\frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/Z} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/Z} \longrightarrow 0.$$

Theorem 14.4. *Let X be a smooth variety of dimension n. Let $Y \subset X$ be an irreducible closed subscheme with sheaf of ideals $\mathcal{I}$.*

Then Y is smooth if and only if

(1) $\Omega_{Y/k}$ is locally free, and
(2) the sequence above is also left exact:

$$0 \longrightarrow \frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0.$$

Furthermore, in this case, $\mathcal{I}$ is locally generated by $r = \operatorname{codim}(Y, X)$ elements and $\frac{\mathcal{I}}{\mathcal{I}^2}$ is locally free of rank r on Y.

Proof. Suppose (1) and (2) hold. Then $\Omega_{Y/k}$ is locally free and so we only have to check that its rank q is equal to the dimension of Y. Then $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $n-q$. Nakayama's lemma implies that $\mathcal{I}$ is locally generated by $n-q$ elements and so $\dim Y \geq n-(n-q) = q$. On the other hand, if $y \in Y$ is any closed point $q = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ and so $q \geq \dim Y$. Thus $q = \dim Y$. This also establishes the last statement.

Now suppose that Y is smooth. Then $\Omega_{Y/k}$ is locally free of rank $q = \dim Y$ and so (1) is immediate. On the other hand, there is an exact sequence

$$\frac{\mathcal{I}}{\mathcal{I}^2} \longrightarrow \Omega_{X/k} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y/k} \longrightarrow 0.$$

Pick a closed point $y \in Y$. As $\mathcal{I}/\mathcal{I}^2$ is locally free of rank $r = n - q$, we may pick sections $x_1, x_2, \dots, x_r$ of $\mathcal{I}$ such that $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_r$ generate the kernel of the second map.

Let $Y' \subset X$ be the corresponding closed subscheme. Then, by construction, $\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_r$ generate a free subsheaf of rank r of $\Omega_{X/k} \otimes \mathcal{O}_{Y'}$ in a neighbourhood of y. It follows that for the exact sequence for Y'

$$\frac{\mathcal{I}'}{\mathcal{I}'^2} \longrightarrow \Omega_{X/Z} \otimes \mathcal{O}_Y \longrightarrow \Omega_{Y'/Z} \longrightarrow 0,$$

the first map is injective and $\Omega_{Y'/k}$ is locally free of rank $n - r$. But then Y' is smooth and $\dim Y' = \dim Y$. As $Y \subset Y'$ and Y' is integral, we must have $Y = Y'$ and this gives (2). $\square$

Theorem 14.5 (Bertini's Theorem)**.** *Let $X \subset \mathbb{P}^n_k$ be a closed smooth projective variety. Then there is a hyperplane $H \subset \mathbb{P}^n_k$, not containing X, such that $H \cap X$ is regular at every point.*

Furthermore the set of such hyperplanes forms an open dense subset of the linear system $|H| \simeq \mathbb{P}^n_k$.

Proof. Let $x \in X$ be a closed point. Call a hyperplane H **bad** if either H contains X or H does not contain X but it does contain x and $X \cap H$ is not regular at x. Let B_x be the set of all bad hyperplanes at x. Fix a hyperplane H_0 not containing x, defined by $f_0 \in V = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Define a map

$$\phi_x \colon V \longrightarrow \mathcal{O}_{X,x}/\mathfrak{m}^2,$$

as follows. Given by $f \in V$, f/f_0 is a regular function on $X - X \cap X_0$. Send f to the image of f/f_0 to its class in the quotient $\mathcal{O}_{X,x}/\mathfrak{m}^2$. Now $x \in X \cap H$ if and only if $\phi_x(f) \in \mathfrak{m}$ and $x \in X \cap H$ is a regular point if and only if $\phi_x(f) \neq 0$.

Thus B_x is precisely the kernel of ϕ_x. Now as k is algebraically closed and x is a closed point, ϕ_x is surjective. If $\dim X = r$ then $\mathcal{O}_{X,x}/\mathfrak{m}^2$ has dimension $r + 1$ and so B_x is a linear subspace of $|H|$ of dimension $n - r - 1$.

Let $B \subset X \times |H|$ be the set of pairs (x, H) where $H \in B_x$. Then B is a closed subset. Let $p \colon B \longrightarrow X$ and $q \colon B \longrightarrow |H|$ denote projection onto either factor. p is surjective, with irreducible fibres of dimension $n - r - 1$. It follows that B is irreducible of dimension $r + (n - r - 1) = n - 1$. The image of q has dimension at most $n - 1$. Hence $q(B)$ is a proper closed subset of $|H|$. $\square$

Remark 14.6. *We will see later that $H \cap X$ is in fact connected, whence irreducible, so that in fact $Y = H \cap X$ is a smooth subvariety.*

Definition 14.7. *Let X be a smooth variety. The **tangent sheaf***

$$\mathcal{T}_X = \mathbf{Hom}_{\mathcal{O}_X}(\Omega_{X/k}, \mathcal{O}_X).$$

Note that the tangent sheaf is a locally free sheaf of rank equal to the dimension of X.

15. The canonical bundle and divisor

Definition 15.1. *Let X be a smooth variety of dimension n over a field k. The* ***canonical sheaf****, denoted ω_X, is the highest wedge of the sheaf of relative differentials,*

$$\omega_X = \bigwedge^n \Omega_{X/k}.$$

Note that ω_X is an invertible sheaf on X. We may write $\omega_X = \mathcal{O}_X(K_X)$, for some Cartier divisor K_X. The interesting thing is that we may generalise this:

Definition 15.2. *Let X be a normal variety over a field k. Let $U \subset X$ be the smooth locus, an open subset of X, whose complement has codimension at least two.*

The ***canonical divisor****, denoted K_X, is the Weil divisor obtained by picking a Weil divisor representing the invertible sheaf ω_U and then taking the closure.*

Note that the canonical divisor is only defined up to linear equivalence.

Definition 15.3. *Let X be a smooth projective variety over a field k. The* ***geometric genus*** *of X, denoted $p_g(X)$, is the dimension of the k-vector space $H^0(X, \omega_X)$. The m-th* ***plurigenus****, denoted $P_m(X)$, is the dimension of the k-vector space $H^0(X, \mathcal{O}_X(mK_X))$. The* ***irregularity*** *of X, denoted $q(X)$, is the dimension of the k-vector space $H^0(X, \Omega_{X/k})$.*

Note that $p_g(X) = P_1(X)$. If X is a curve, then $p_g(X) = P_1(X) = q(X)$.

Definition 15.4. *Let X be a scheme. We say that X satisfies* ***condition*** *S_2 if every regular function defined on an open subset U whose complement has codimension at least two, extends to the whole of X.*

Lemma 15.5 (Serre's criterion)**.** *Let X be an integral scheme.*

Then X is normal if and only if it is regular in codimension one (condition R_1) and satisfies condition S_2.

It is interesting to write down some example of varieties which are R_1 but not normal, that is, which are not S_2.

Example 15.6. *Let S be the union of two smooth surfaces S_1 and S_2 joined at a single point p. For example, two general planes in $\mathbb{A}^4$ which both contain the same point p. Let $U = S - \{p\}$. Then U is the disjoint union of $U_1 = S_1 - \{p\}$ and $S_2 - \{p\}$, so U is smooth and the* *codimension of the complement is two. Let $f\colon U \longrightarrow k$ be the function which takes the value 1 on U_1 and the value 0 on U_2. Then f is regular, but it does not even extend to a continuous function, let alone a regular function, on S.*

Example 15.7. *Let C be a projection of a rational normal quartic down to $\mathbb{P}^3$, for example the image of*

$$[S:T] \longrightarrow [S^4 : S^3T : ST^3 : T^4] = [A:B:C:D].$$

Let S be the cone over C. Then S is regular in codimension one, but it is not S_2. Indeed,

$$\frac{B^2}{A} = S^2T^2 = \frac{C^2}{D}.$$

is a regular function whose only pole is along $A = 0$ and $D = 0$, that is, only at $(0,0,0,0)$ of S.

Note that the coordinate ring

$$k[S^4, S^3T, ST^3, T^4] = \frac{k[A,B,C,D]}{\langle AD - BC, B^3 - A^2C, C^3 - BD^2\rangle},$$

is indeed not integrally closed in its field of fractions. Indeed,

$$\alpha = \frac{B^2}{A},$$

is a root of the monic polynomial $u^2 - BC$.

Theorem 15.8. *Let X and X' be two smooth projective varieties over a field k.*

If X and X' are birational then $p_g(X) = p_g(X')$, $P_n(X) = P_n(X')$ and $q(X) = q(X')$.

Proof. We will just prove that the geometric genus is a birational invariant. By symmetry, it suffices to show that $p_g(X') \leq p_g(X)$. By assumption there is a birational map $\phi\colon X \dashrightarrow X'$. Let $V \subset X$ be the largest open subset of X for which this map restricts to a morphism, $f\colon V \longrightarrow X'$. This induces a map of sheaves,

$$f^*\Omega_{X'/k} \longrightarrow \Omega_{V/k}.$$

Since these are both locally free of the same rank $n = \dim V$, taking the highest wedge, we get

$$f^*\omega_{X'} \longrightarrow \omega_V.$$

Since f is birational there is an open subset $U \subset V$ such that $f(U)$ is open in X' and f induces an isomorphism $U \longrightarrow f(U)$. Since a non-zero section of an invertible sheaf cannot vanish on any non-empty open subset, we have an injection on global sections

$$H^0(X', \omega_{X'}) \longrightarrow H^0(V, \omega_V).$$

So it suffices to show that the natural restriction map

$$H^0(X, \omega_X) \longrightarrow H^0(V, \omega_V),$$

is an isomorphism.

First off, we note that the codimension of the complement $X - V$ is at least two. Indeed, let P be a codimension one point. Then $\mathcal{O}_{X,P}$ is a DVR, as X is smooth. We already have a map of the generic point of X to X'. As X' is projective it is proper, so that there is a unique morphism $\operatorname{Spec} \mathcal{O}_{X,P} \longrightarrow X'$ compatible with ϕ. This morphism extends to a neighbourhood of P, so that f is defined in a neighbourhood of P, that is, $P \in V$.

To show that the restriction map is bijective, it suffices to show that if $U \subset X$ is an open subset for which $\omega_X|_U \simeq \mathcal{O}_U$, then the natural restriction map

$$H^0(U, \mathcal{O}_U) \longrightarrow H^0(U \cap V, \mathcal{O}_{U \cap V})$$

is an isomorphism. But this follows as $U - V$ has codimension at least two and X is normal; any function on X which is regular in codimension two is regular. □

Definition 15.9. *Let Y be a smooth subvariety of a smooth variety X over a field k, with ideal sheaf $\mathcal{I}$. The locally free sheaf $\mathcal{I}/\mathcal{I}^2$ is called the **conormal sheaf**. Its dual*

$$\mathcal{N}_{Y/X} = \mathbf{Hom}_{\mathcal{O}_Y}(\frac{\mathcal{I}}{\mathcal{I}^2}, \mathcal{O}_Y),$$

*is called the **normal sheaf** of Y in X.*

Note that by taking duals of the usual exact sequence on Y we get

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_X \longrightarrow \mathcal{N}_{Y/X} \longrightarrow 0.$$

Theorem 15.10 (Adjunction formula)**.** *Let Y be a smooth subvariety of codimension r of a smooth variety X over a field k. Then*

$$\omega_Y \simeq \omega_X \otimes \bigwedge^r \mathcal{N}_{Y/X}.$$

If $r = 1$ then if we consider Y as a divisor on X and put $\mathcal{L} = \mathcal{O}_X(Y)$, we get

$$\omega_Y \simeq \omega_X \otimes \mathcal{L} \otimes \mathcal{O}_Y.$$

In terms of divisors,

$$K_Y = (K_X + Y)|_Y.$$

Proof. Follows from the exact sequence above, after taking highest wedge and then the dual. □

It is interesting to calculate the canonical divisor in the case of a smooth toric variety. To calculate the canonical divisor, we need to write down a rational (or meromorphic in the case of $\mathbb{C}$) differential form. Note that if $z_1, z_2, \dots, z_n$ are coordinates on the torus then

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

is invariant under the action of the torus, so that the associated divisor is supported on the invariant divisor.

To calculate the zeroes and poles of this meromorphic differential, we may work locally about any invariant divisor. So we may assume that $X = U_\sigma$ is affine, isomorphic to $\mathbb{A}^1 \times \mathbb{G}_m^{n-1}$. As usual, we reduce to the case when $n = 1$, in which case we have

$$\frac{\mathrm{d}z}{z},$$

which has a simple pole at 0.

Thus this rational form has a simple pole along every invaraint divisor, that is

$$K_X + D \sim 0,$$

where D is a sum of the invariant divisors. For example,

$$-K_{\mathbb{P}^n} = H_0 + H_1 + \cdots + H_n \sim (n+1)H.$$

One can check this with the formula one gets using the Euler sequence.

16. Homological algebra and derived functors

Let X be any analytic space. Then there is an exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X^{\mathrm{an}} \longrightarrow \mathcal{O}_X^* \longrightarrow 0,$$

where $\mathcal{O}_X^*$ is the sheaf of nowhere zero holomorphic functions under multiplication. The map

$$\mathcal{O}_X^{\mathrm{an}} \longrightarrow \mathcal{O}_X^*,$$

sends a holomorphic function f to $e^{2\pi i f}$, so that the kernel is clearly $\mathbb{Z}$, the sheaf of locally constant, integer valued functions. Given a nowhere zero holomorphic function g, locally we can always find a function f mapping to g, since locally we can always take logs. Thus the map of sheaves is surjective as it is surjective on stalks.

Now suppose we take global sections. We get an exact sequence

$$0 \longrightarrow H^0(X, \mathbb{Z}) \longrightarrow H^0(X, \mathcal{O}_X^{\mathrm{an}}) \longrightarrow H^0(X, \mathcal{O}_X^*),$$

but in general the last map is not surjective.

For example, let $X = \mathbb{C} - \{0\}$. Then z is a nowhere zero function which is not the exponential of any holomorphic function; the logarithm is not a globally well-defined function on the whole punctured plane.

Sheaf cohomology is introduced exactly to fix lack of exactness on the left.

Definition 16.1. *An* ***abelian category*** $\mathcal{U}$ *is a category such that for every pair of objects A and B, $\mathrm{Hom}(A, B)$ has the structure of an abelian group and the composition law is linear; finite direct sums exist; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and every morphism can be factored into an epimorphism followed by a monomorphism.*

Example 16.2. *Here are some examples of abelian categories:*

1. *The category of abelian groups.*
2. *The category of modules over a ring A.*
3. *The category of sheaves of abelian groups on a topological space X.*
4. *The category of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.*
5. *The category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on a scheme X.*
6. *The category of coherent sheaves of $\mathcal{O}_X$-modules on a noetherian scheme X.*

A **complex** $A^\bullet$ of objects in an abelian category is a sequence of objects, indexed by $\mathbb{Z}$, together with coboundary maps

$$d^i\colon A^i \longrightarrow A^{i+1},$$

such that the composition of any two is zero.

The ith **cohomology** of the complex, is obtained in the usual way:

$$h^i(A^\bullet) = \operatorname{Ker} d^i / \operatorname{Im} d^{i+1}.$$

A morphism of complexes $f\colon A^\bullet \longrightarrow B^\bullet$ is simply a collection of morphisms $f^i\colon A^i \longrightarrow B^i$ compatible with the coboundary maps. They give rise to maps

$$h^i(f)\colon h^i(A^\bullet) \longrightarrow h^i(B^\bullet).$$

Two morphisms f and g are **homotopic** if there are maps $k^i\colon A^i \longrightarrow B^{i-1}$ such that

$$f - g = dk + kd.$$

If f and g are homotopic then $h^i(f) = h^i(g)$. Two complexes are **homotopic** if there are maps $f\colon A^\bullet \longrightarrow B^\bullet$ and $g\colon B^\bullet \longrightarrow A^\bullet$ whose composition either way is homotopic to the identity.

A functor F from one abelian category $\mathcal{U}$ to another $\mathcal{B}$ is called **additive** if for any two objects A and B in $\mathcal{U}$, the induced map

$$\operatorname{Hom}(A, B) \longrightarrow \operatorname{Hom}(FA, FB),$$

is a group homomorphism. F is **left exact** if in addition, given an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

we get an exact sequence,

$$0 \longrightarrow FA \longrightarrow FB \longrightarrow FC.$$

Example 16.3. *Fix an object T. The functor*

$$A \longrightarrow \operatorname{Hom}(A, T),$$

is (contravariant) left exact, so that we have an exact sequence

$$0 \longrightarrow \operatorname{Hom}(C, T) \longrightarrow \operatorname{Hom}(B, T) \longrightarrow \operatorname{Hom}(A, T).$$

An object I of $\mathcal{U}$ is **injective** if the functor

$$A \longrightarrow \operatorname{Hom}(A, I),$$

is exact. An **injective resolution** of an object A is a complex $I^\bullet$, such that I is zero in negative degrees, I^i is an injective object and there is a morphism $A \longrightarrow I^0$ such that the obvious complex is exact.

We say that $\mathcal{U}$ has **enough injectives** if every object embeds into an injective object. In this case, every object has an injective resolution and any two such are homotopic.

Given a category with enough injectives, define the **right derived functors** of a left exact functor F by fixing an injective resolution and then let
$$R^iF(A) = h^i(F(I^\bullet)).$$
Now it is a well-known result that the category of modules over a ring has enough injectives.

Proposition 16.4. *Let $(X, \mathcal{O}_X)$ be a ringed space.*
Then the category of $\mathcal{O}_X$-modules has enough injectives.

Proof. Let $\mathcal{F}$ be a sheaf. For every $x \in X$ embed the stalk into an injective $\mathcal{O}_{X,x}$-module, $\mathcal{F}_x \longrightarrow I_x$. Let j denote the inclusion of $\{x\}$ into X and let
$$\mathcal{I} = \prod_{x \in X} j_* I_x.$$
Now suppose that we have a sheaf $\mathcal{G}$ of $\mathcal{O}_X$-modules. Then
$$\begin{aligned}\mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) &= \prod_{x \in X} \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, j_* I_x) \\ &= \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x).\end{aligned}$$
In particular there is a natural map $\mathcal{F} \longrightarrow \mathcal{I}$, which is injective, as it is injective on stalks. Secondly the functor
$$\mathcal{G} \longrightarrow \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}),$$
is the direct product over all $x \in X$ of the functor which sends $\mathcal{G}$ to its stalk, which is exact, followed by the functor,
$$\mathcal{G}_x \longrightarrow \mathbf{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x),$$
which is exact, as I_x is an injective module. Thus
$$\mathcal{G} \longrightarrow \mathbf{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}),$$
is exact, which means that $\mathcal{I}$ is injective. □

Corollary 16.5. *Let X be a topological space.*
Then the category of sheaves on X has enough injectives.

Proof. The category of sheaves on X is equivalent to the category of $\mathcal{O}_X$-modules on the ringed space $(X, \mathbb{Z})$. □

If X is a topological space then we define $H^i(X, \mathcal{F})$ to be the right derived functor of
$$\mathcal{G} \longrightarrow \Gamma(X, \mathcal{G}).$$
Given an exact sequence
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$

we get a long exact sequence of cohomology, so that we indeed fix the lack of exactness.

It is interesting to examine why we don't work with projective sheaves instead of injective sheaves. After all, projective modules are much easier to understand than injective modules. A module is projective if and only if it is direct summand of a free module, so any free module is projective.

However if X is a topological space there are almost never enough projectives in the category of sheaves on X. For example, suppose we have a topological space with the following property. There is a closed point $x \in X$ such that for any neighbourhood V of x in X there is a smaller connected open neighbourhood U of x, that is,

$$x \in U \subset V \subset X,$$

where $U \neq V$ is connected. Let $\mathcal{F} = \mathbb{Z}_{\{x\}}$ be the extension by zero of the constant sheaf $\mathbb{Z}$ on x, so that

$$\mathcal{F}(W) = \begin{cases} \mathbb{Z} & \text{if } x \in W \\ 0 & \text{otherwise.} \end{cases}$$

I claim that $\mathcal{F}$ is not the quotient of a projective sheaf. Suppose that

$$\mathcal{P} \longrightarrow \mathcal{F},$$

is a morphism of sheaves, where $\mathcal{P}$ is projective. Let V be any open neighbourhood of x. Pick

$$x \in U \subset V \subset X,$$

where $U \neq V$ is open and connected. Let $\mathcal{G} = \mathbb{Z}_U$ be the extension by zero of the locally constant sheaf $\mathbb{Z}$ on U, so that

$$\mathcal{G}(W) = \begin{cases} \mathbb{Z} & \text{if } W \subset U \\ 0 & \text{otherwise.} \end{cases}$$

As $\mathcal{P}$ is projective we have a commutative diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, dashed] \arrow[dr] & \mathcal{G} \arrow[d] \\ & \mathcal{F}. \end{tikzcd}$$

But $\mathcal{G}(V) = \mathbb{Z}_U(V) = 0$, so that the map $\mathcal{P}(V) \longrightarrow \mathcal{F}(V)$ is the zero map. But then the map on stalks is zero, so that the map $\mathcal{P} \longrightarrow \mathcal{F}$ is not surjective and $\mathcal{F}$ is not the quotient of a projective sheaf.

Note that if $(X, \mathcal{O}_X)$ is a ringed space then there are potentially two different ways to take the right derived functors of $\Gamma(X, \mathcal{F})$, if $\mathcal{F}$ is an $\mathcal{O}_X$-module. We could forget that X is a ringed space or we could work in the smaller category of $\mathcal{O}_X$-modules. We check that it does not matter in which category we work.

Definition 16.6. *Let $\mathcal{F}$ be a sheaf. We say that $\mathcal{F}$ is **flasque** if for every pair of open subsets $V \subset U \subset X$ the natural map*

$$\mathcal{F}(U) \longrightarrow \mathcal{F}(V),$$

is surjective.

For any open subset $U \subset X$ let $\mathcal{O}_U$ be the extension by zero of the structure sheaf on U, $\mathcal{O}_X|_U$.

Lemma 16.7. *If $(X, \mathcal{O}_X)$ is a ringed space then every injective $\mathcal{O}_X$-module $\mathcal{I}$ is flasque.*

Proof. Let $V \subset U$ be open subsets. Then we have an inclusion $\mathcal{O}_V \longrightarrow \mathcal{O}_U$ of sheaves of $\mathcal{O}_X$-modules. As $\mathcal{I}$ is injective we get a surjection

$$\operatorname{Hom}(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U) \longrightarrow \operatorname{Hom}(\mathcal{O}_V, \mathcal{I}) = \mathcal{I}(V). \qquad \square$$

Lemma 16.8. *If $\mathcal{F}$ is a flasque sheaf on a topological space X then*

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > 0$.

Proof. Embed $\mathcal{F}$ into an injective sheaf and take the quotient to get a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{G} \longrightarrow 0.$$

As $\mathcal{I}$ is injective it is flasque and so $\mathcal{G}$ is flasque. As $\mathcal{F}$ is flasque there is an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{G}) \longrightarrow 0,$$

which taking the long exact sequence of cohomology, shows that

$$H^1(X, \mathcal{F}) = H^1(X, \mathcal{I}) = 0,$$

and

$$H^i(X, \mathcal{F}) = H^{i-1}(X, \mathcal{G}),$$

which is zero by induction on i. $\square$

Proposition 16.9. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the derived functors of*

$$\mathcal{F} \longrightarrow \operatorname{Hom}(X, \mathcal{F}),$$

for either the category of sheaves of $\mathcal{O}_X$-modules or simply the category of sheaves on X to the category of abelian groups coincide.

Proof. Take an injective resolution of $\mathcal{F}$ in the category of $\mathcal{O}_X$-modules. Injective is flasque and flasque is acyclic, so this gives us a resolution by acylics in the category of sheaves on X and this is enough to calculate the right derived functors. □

Suppose that X is scheme over an affine scheme

$$X \longrightarrow S = \operatorname{Spec} A.$$

Now

$$H^0(S, \mathcal{O}_S) = A,$$

and all higher cohomology of any $\mathcal{O}_X$-module $\mathcal{F}$ is naturally an A-module.

17. Higher vanishing

Theorem 17.1. *Let X be a noetherian topological space of dimension n.*

Then

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > n$ and any sheaf of abelian groups.

The basic idea is to reduce to the case of the quotient of $\mathbb{Z}_U$. The first thing is to reduce to the finitely generated case. Recall that any ring is the direct limit of its finitely generated subrings, so really all we need is a couple of standard results about direct limits.

Suppose A is a directed set and $(\mathcal{F}_\alpha)$ is a direct system of sheaves indexed by A. Then we may take the direct limit $\varinjlim \mathcal{F}_\alpha$.

Lemma 17.2. *On a noetherian topological space, the direct limit of flasque is flasque.*

Proof. Suppose $(\mathcal{F}_\alpha)$ is a direct system of flasque sheaves. Suppose that $V \subset U$ are open subsets. For each i we have a surjection

$$\mathcal{F}_\alpha(U) \longrightarrow \mathcal{F}_\alpha(V).$$

Now $\varinjlim$ is an exact functor, so

$$\varinjlim \mathcal{F}_\alpha(U) \longrightarrow \varinjlim \mathcal{F}_\alpha(V),$$

is surjective. But on a noetherian topological space we have

$$(\varinjlim \mathcal{F}_\alpha)(U) = \varinjlim \mathcal{F}_\alpha(U),$$

and so

$$(\varinjlim \mathcal{F}_\alpha)(U) \longrightarrow (\varinjlim \mathcal{F}_\alpha)(V),$$

is surjective, so that $\varinjlim \mathcal{F}_\alpha$ is flasque. $\square$

Proposition 17.3. *Let X be a noetherian topological space and let $(\mathcal{F}_\alpha)$ be a direct system of abelian sheaves indexed by A.*

Then there are natural isomorphisms

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha)$$

Proof. By definition of the limit, for each α, there are maps $\mathcal{F}_\alpha \longrightarrow \varinjlim \mathcal{F}_\alpha$. This gives a map on cohomology

$$H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha).$$

Taking the limit of these maps gives a morphism

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha).$$

It is easy to check that this is an isomorphism for $i = 0$.

The general case follows using the notion of a δ-functor; see (III.2.9). $\square$

Lemma 17.4. *Let $Y \subset X$ be a closed subset of a topological space, let $\mathcal{F}$ be a sheaf of abelian groups on Y and let $j\colon Y \hookrightarrow X$ be the natural inclusion.*

Then

$$H^i(Y, \mathcal{F}) \simeq H^i(X, j_*\mathcal{F}).$$

Proof. Suppose that $\mathcal{I}^\bullet$ is a flasque resolution of $\mathcal{F}$. Then $j_*\mathcal{I}^\bullet$ is a flasque resolution of $j_*\mathcal{F}$. $\square$

It is customary to abuse notation and consider $\mathcal{F}$ as a sheaf on X, without bothering to write $j_*\mathcal{F}$.

Proof of (17.1). We introduce some convenient notation. Let $\mathcal{F}$ be a sheaf on X. If $Y \subset X$ is a closed subset then $\mathcal{F}_Y$ denotes the extension by zero of the sheaf $\mathcal{F}|_Y$; similarly if $U \subset X$ is an open subset, then $\mathcal{F}_U$ denotes the extension by zero of the sheaf $\mathcal{F}|_U$. Note that if $U = X - Y$ then there is an exact sequence

$$0 \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0.$$

The proof proceeds by Noetherian induction and induction on $n = \dim X$.

Step 1: We reduce to the case X is irreducible. We proceed by induction on the number m of irreducible components of X. If $m = 1$ there is nothing to prove. Otherwise let Y be an irreducible component of X and let $U = X - Y$. Let Z be the closure of U. Then $\mathcal{F}_U$ can be considered as a sheaf on Z, which has $m - 1$ irreducible components. By induction on m,

$$H^i(Y, \mathcal{F}|_Y) = H^i(Z, \mathcal{F}_U) = 0,$$

for all $i > n$, and so

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > n$, by considering the long exact sequence of cohomology associated to the short exact sequence

$$0 \longrightarrow \mathcal{F}_U \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_Y \longrightarrow 0.$$

So we may assume that X is irreducible.

Step 2: Suppose that $n = 0$. Then X and the empty set are the only open subsets of X. In this case, to give a sheaf on X is the same as to give an abelian group, and it clear that taking global sections is an exact functor. But then

$$H^i(X, \mathcal{F}) = 0,$$

for all $i > 0$.

Thus we may assume that $n > 0$.

Step 3: Let

$$B = \bigcup_U \mathcal{F}(U),$$

and let A be the set of all finite subsets of B. Given $\alpha \in A$, let $\mathcal{F}_\alpha$ be the subsheaf of $\mathcal{F}$ generated by the elements of α. Then A is a directed set and $\mathcal{F} = \varinjlim \mathcal{F}_\alpha$. By (17.3) we may therefore assume that $\mathcal{F}$ is finitely generated.

Step 4: Suppose that $\beta \subset \alpha$ and let r be the cardinality of the difference. Then there is an exact sequence

$$0 \longrightarrow \mathcal{F}_\beta \longrightarrow \mathcal{F}_\alpha \longrightarrow \mathcal{G} \longrightarrow 0,$$

where $\mathcal{G}$ is generated by r elements. So by induction on r and the long exact sequence of cohomology associated to the short exact sequence above, we are reduced to the case when $\mathcal{F}$ is generated by a single element, so that $\mathcal{F}$ is a quotient of $\mathbb{Z}_U$ for some open subset U,

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathbb{Z}_U \longrightarrow \mathcal{F} \longrightarrow 0.$$

Step 5: We reduce to the case when $\mathcal{F} = \mathbb{Z}_U$.

For each $x \in U$, $\mathcal{R}_x \subset \mathbb{Z}_x = \mathbb{Z}$. If $\mathcal{R} = 0$ there is nothing to prove; otherwise let d be the smallest positive integer which appears in $\mathcal{R}_x$. Then

$$\mathcal{R}|_V = d \cdot \mathbb{Z}|_V,$$

for some non-empty open subset $V \subset U$. In this case, we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_V \longrightarrow \mathcal{R} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Now $\mathcal{Q}$ is supported on a smaller set and so

$$H^i(X, \mathcal{Q}) = 0,$$

for all $i \geq n$. Taking the long exact sequence of cohomology, we see that

$$H^i(X, \mathcal{R}) = H^i(X, \mathbb{Z}_V),$$

for all $i > n$.

Thus we may assume that $\mathcal{F} = \mathbb{Z}_U$.

Step 6: Consider the short exact sequence

$$0 \longrightarrow \mathbb{Z}_U \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_Y \longrightarrow 0,$$

where $Y = X - U$. By induction on the dimension,

$$H^i(X, \mathbb{Z}_Y) = 0,$$

for all $i \geq n$. Thus

$$H^i(X, \mathbb{Z}_U) = H^i(X, \mathbb{Z}),$$

for all $i > n$. But $\mathbb{Z}$ is a locally constant sheaf on an irreducible space, so that $\mathbb{Z}$ is flasque, and flasque is acyclic. $\square$

18. Cohomology of affine schemes

Proposition 18.1. *Let I be an injective module over a noetherian ring A.*

Then the sheaf $\tilde{I}$ on $X = \operatorname{Spec} A$ is flasque.

Corollary 18.2. *Let X be a noetherian scheme.*

Then ever quasi-coherent sheaf $\mathcal{F}$ on X can be embedded in a flasque, quasi-coherent sheaf $\mathcal{G}$.

Proof. Let $U_i = \operatorname{Spec} A_i$ be a finite open affine cover of X and let $\mathcal{F}|_{U_i} = \tilde{M}_i$ for each i. Pick an embedding of M_i into an injective A_i-module I_i. Let $f_i \colon U_i \longrightarrow X$ be the inclusion and let

$$\mathcal{G} = \bigoplus_i f_{i*}\tilde{I}_i.$$

Now for each i there is an injective map $\mathcal{F}|_{U_i} \longrightarrow \tilde{I}_i$, which induces a map $\mathcal{F} \longrightarrow f_{i*}\tilde{I}_i$. This induces a map $\mathcal{F} \longrightarrow \mathcal{G}$, which is clearly injective.

But $\tilde{I}_i$ is flasque and quasi-coherent on U_i, so that $f_{i*}\tilde{I}_i$ is flasque and quasi-coherent on X. But then $\mathcal{G}$ is flasque and quasi-coherent. □

Theorem 18.3 (Serre)**.** *Let X be a Noetherian scheme.*

TFAE

(1) X is affine,
(2) $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves,
(3) $H^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.

Proof. Suppose X is affine. Let $M = H^0(X, \mathcal{F})$ and take an injective resolution $I^\bullet$ of M in the category of A-modules. Then $\tilde{I}^\bullet$ is a flasque resolution of $\mathcal{F}$ on X. If we take global sections we get back the original injective resolution of M, so that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. Thus (1) implies (2).

(2) implies (3) is immediate. So suppose that $H^1(X, \mathcal{I}) = 0$ for all coherent sheaves of ideals $\mathcal{I}$.

Fix a closed point p of X together with an open affine neighbourhood U of p and let $Y = X - U$. Then there is a short exact sequence

$$0 \longrightarrow \mathcal{I}_{Y \cup \{p\}} \longrightarrow \mathcal{I}_Y \longrightarrow k(p) \longrightarrow 0.$$

This gives us an exact sequence

$$H^0(X, \mathcal{I}_Y) \longrightarrow H^0(X, k(p)) \longrightarrow H^1(X, \mathcal{I}_{Y \cup \{p\}}) \longrightarrow 0.$$

But then there is regular function $f \in H^0(X, \mathcal{O}_X)$ which is not zero at p and which does not vanish on U, so that $p \in X_f \subset U$ is an open neighbourhood of p. As $X_f = U_f$ (thinking of $f \in A = H^0(X, \mathcal{O}_X)$), it follows that X_f is affine.

As X is noetherian, it is compact, so that we can cover X by finitely many open affines, X_{f_i}, where $f_1, f_2, \dots, f_r \in A$.

Finally we check that $f_1, f_2, \dots, f_r$ generate the unit ideal. There is a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^r \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

The last map α sends $(a_1, a_2, \dots, a_r)$ to $\sum a_i f_i$. It is surjective as it is surjective on stalks. $\mathcal{F}$ is then the kernel of α.

There is a filtration of $\mathcal{F}$ as follows:

$$\mathcal{F} \cap \mathcal{O}_X \subset \mathcal{F} \cap \mathcal{O}_X^2 \subset \mathcal{F} \cap \mathcal{O}_X^3 \subset \mathcal{F} \cap \mathcal{O}_X^r = \mathcal{F}.$$

The quotients are naturally $\mathcal{O}_X$-submodules of $\mathcal{O}_X$, that is, the quotients are coherent sheaves of ideals. Taking the long exact sequence of cohomology (r times), we get that $H^1(X, \mathcal{F}) = 0$. Taking the long exact sequence of cohomology of the sequence above, we get that α is surjective on global sections. But then

$$1 = \alpha(a_1, a_2, \dots, a_r) = \sum a_i f_i,$$

in the ideal generated by $f_1, f_2, \dots, f_r$. (II.2.17) shows that X is affine. $\square$

19. Čech Cohomology

We would like to have a way to compute sheaf cohomology. Let X be a topological space and let $\mathcal{U} = \{U_i\}$ be an open cover, which is locally finite. The group of k-cochains is

$$C^k(\mathcal{U}, \mathcal{F}) = \bigoplus_I \Gamma(U_I, \mathcal{F}),$$

where I runs over all $(k+1)$-tuples of indices and

$$U_I = \bigcap_{i \in I} U_i,$$

denotes intersection. k-cochains are skew-commutative, so that if we switch two indices we get a sign change.

Define a coboundary map

$$\delta^k \colon C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^{k+1}(\mathcal{U}, \mathcal{F}).$$

Given $\sigma = (\sigma_I)$, we have to construct $\tau = \delta(\sigma) \in C^{k+1}(\mathcal{U}, \mathcal{F})$. We just need to determine the components τ_J of τ. Now $J = \{j_0, j_1, \dots, j_k\}$. If we drop an index, then we get a k-tuple. We define

$$\tau_J = \left. \left(\sum_{i=0}^{k} (-1)^i \sigma_{J - \{i_i\}} \right) \right|_{U_J}.$$

The key point is that $\delta^2 = 0$. So we can take cohomology

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = Z^i(\mathcal{U}, \mathcal{F}) / B^i(\mathcal{U}, \mathcal{F}).$$

Here Z^i denotes the group of i-cocycles, those elements killed by δ^i and B^i denotes the group of coboundaries, those cochains which are in the image of δ^{i-1}. Note that $\delta^i(B^i) = \delta^i \delta^{i-1}(C^{i-1}) = 0$, so that $B^i \subset Z^i$.

The problem is that this is not enough. Perhaps our open cover is not fine enough to capture all the interesting cohomology. A **refinement** of the open cover $\mathcal{U}$ is an open cover $\mathcal{V}$, together with a map h between the indexing sets, such that if V_j is an open subset of the refinement, then for the index $i = h(j)$, we have $V_j \subset U_i$. It is straightforward to check that there are maps,

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^i(\mathcal{V}, \mathcal{F}),$$

on cohomology. Taking the (direct) limit, we get the Čech cohomology groups,

$$\check{H}^i(X, \mathcal{F}).$$

For example, consider the case $i = 0$. Given a cover, a cochain is just a collection of sections, (σ_i), $\sigma_i \in \Gamma(U_i, \mathcal{F})$. This cochain is a cocycle if $(\sigma_i - \sigma_j)|_{U_{ij}} = 0$ for every i and j. By the sheaf axiom, this means that there is a global section $\sigma \in \Gamma(X, \mathcal{F})$, so that in fact

$$H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}).$$

It is also sometimes possible to untwist the definition of $\check{H}^1$. A 1-cocycle is precisely the data of a collection

$$(\sigma_{ij}) \in \Gamma(\mathcal{U}, \mathcal{F}),$$

such that

$$\sigma_{ij} - \sigma_{ik} + \sigma_{jk} = 0.$$

In general of course, one does not want to compute these things using limits. The question is how fine does the cover have to be to compute the cohomology? As a first guess one might require that

$$\check{H}^i(U_j, \mathcal{F}) = 0,$$

for all j, and $i > 0$. In other words there is no cohomology on each open subset. But this is not enough. One needs instead the slightly stronger condition that

$$\check{H}^i(U_I, \mathcal{F}) = 0.$$

Theorem 19.1 (Leray)**.** *If X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups and $\mathcal{U}$ is an open cover such that*

$$\check{H}^i(U_I, \mathcal{F}) = 0,$$

for all $i > 0$ and indices I, then in fact the natural map

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \simeq \check{H}^i(X, \mathcal{F}),$$

is an isomorphism.

Finally, we need to construct the coboundary maps. Suppose that we are given a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

We want to define

$$\check{H}^i(X, \mathcal{H}) \longrightarrow \check{H}^{i+1}(X, \mathcal{F}).$$

Cheating a little, we may assume that we have a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & C^i(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & C^i(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d] & C^i(\mathcal{U}, \mathcal{H}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{F}) \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{G}) \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{H}) \arrow[r] & 0.
\end{tikzcd}$$

Suppose we start with an element $t \in \check{H}^i(X, \mathcal{H})$. Then t is the image of $t' \in \check{H}^i(\mathcal{U}, \mathcal{H})$, for some open cover $\mathcal{U}$. In turn t' is represented by $\tau \in Z^i(\mathcal{U}, \mathcal{H})$. Now we may suppose our cover is sufficiently fine, so that $\tau_I \in \Gamma(U_I, \mathcal{H})$ is the image of $\sigma_I \in \Gamma(U_I, \mathcal{G})$ (and this fixes the cheat). Applying the boundary map, we get $\delta(\sigma) \in C^{i+1}(\mathcal{U}, \mathcal{G})$. Now the image of $\delta(\sigma)$ in $C^{i+1}(\mathcal{U}, \mathcal{H})$ is the same as $\delta(\tau)$, which is zero, as τ is a cocycle. But then by exactness of the bottom rows, we get $\rho \in C^{i+1}(\mathcal{U}, \mathcal{F})$. It is straightforward to check that ρ is a cocycle, so that we get an element $r' \in \check{H}^{i+1}(\mathcal{U}, \mathcal{F})$, whence an element r of $\check{H}^{i+1}(X, \mathcal{F})$, and that r does not depend on the choice of σ.

One can check that Čech Cohomology coincides with sheaf cohomology. In the case of a scheme, we already know that it suffices to work with any cover $\mathcal{U}$ such that U_I is affine. From now on, we won't bother to distinguish between sheaf cohomology and Čech Cohomology.

20. Cohomology of projective space

Let us calculate the cohomology of projective space.

Theorem 20.1. *Let A be a Noetherian ring. Let $X = \mathbb{P}^r_A$.*

(1) *The natural map $S \longrightarrow \Gamma_*(X, \mathcal{O}_X)$ is an isomorphism.*

(2)
$$H^i(X, \mathcal{O}_X(n)) = 0 \qquad \text{for all} \qquad 0 < i < r \text{ and } n.$$

(3)
$$H^r(X, \mathcal{O}_X(-r-1)) \simeq A.$$

(4) *The natural map*
$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X, \mathcal{O}_X(-r-1)) \simeq A,$$
is a perfect pairing of finitely generated free A-modules.

Proof. Let
$$\mathcal{F} = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n).$$
Then $\mathcal{F}$ is a quasi-coherent sheaf. Let $\mathcal{U}$ be the standard open affine cover. As every intersection is affine, it follows that we may compute sheaf cohomology using this cover. Now
$$\Gamma(U_I, \mathcal{F}) = S_{x_I},$$
where
$$x_I = \prod_{i \in I} x_i.$$
Thus Čech cohomology is the cohomology of the complex
$$\prod_{i=0}^{r} S_{x_i} \longrightarrow \prod_{i<j}^{r} S_{x_i x_j} \longrightarrow \dots \longrightarrow S_{x_0 x_1, \dots x_r}.$$
The kernel of the first map is just $H^0(X, \mathcal{F})$, which we already know is S. Now let us turn to $H^r(X, \mathcal{F})$. It is the cokernel of the map
$$\prod_i S_{x_0 x_1 \dots \hat{x}_i \dots x_r} \longrightarrow S_{x_0 x_1 \dots x_r}.$$
The last term is the free A-module with generators all monomials in the Laurent ring (that is, we allow both positive and negative powers).

The image is the set of monomials where x_i has non-negative exponent for at least one i. Thus the cokernel is naturally identified with the free A-module generated by arbitrary products of reciprocals x_i^{-1},
$$\{\, x_0^{l_0} x_1^{l_1} \dots x_r^{l_r} \mid l_i < 0 \,\}.$$

The grading is then given by

$$l = \sum_{i=0}^{r} l_i.$$

In particular

$$H^r(X, \mathcal{O}_X(-r-1)),$$

is the free A-module with generator $x_0^{-1}x_1^{-1}\dots x_r^{-1}$. Hence (3).

To define a pairing, we declare

$$x_0^{l_0}x_1^{l_1}\dots x_r^{l_r},$$

to be the dual of

$$x_0^{m_0}x_1^{m_1}\dots x_r^{m_r} = x_0^{-1-l_0}x_1^{-1-l_1}\dots x_r^{-1-l_r}.$$

As $m_i \geq 0$ if and only if $l_i < 0$ it is straightforward to check that this gives a perfect pairing. Hence (4).

It remains to prove (2). If we localise the complex above with respect to x_r, we get a complex which computes $\mathcal{F}|_{U_r}$, which is zero in positive degree, as U_r is affine. Thus

$$H^i(X, \mathcal{F})_{x_r} = 0,$$

for $i > 0$ so that every element of $H^i(X, \mathcal{F})$ is annihilated by some power of x_r.

To finish the proof, we will show that multiplication by x_r induces an inclusion of cohomology. We proceed by induction on the dimension. Suppose that $r > 1$ and let $Y \simeq \mathbb{P}^{r-1}_A$ be the hyperplane $x_r = 0$. Then

$$\mathcal{I}_Y = \mathcal{O}_X(-Y) = \mathcal{O}_X(-1).$$

Thus there are short exact sequences

$$0 \longrightarrow \mathcal{O}_X(n-1) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Now $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < r-1$, by induction, and the natural restriction map

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective (every polynomial of degree n on Y is the restriction of a polynomial of degree n on X). Thus

$$H^i(X, \mathcal{O}_X(n-1)) \simeq H^i(X, \mathcal{O}_X(n)),$$

for $0 < i < r-1$, and even if $i = r-1$, then we get an injective map. But this map is the one induced by multiplication by x_r. □

Theorem 20.2 (Serre vanishing). *Let X be a projective variety over a Noetherian ring and let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf.*

(1) $H^i(X,\mathcal{F})$ are finitely generated A-modules.
(2) There is an integer n_0 such that $H^i(X,\mathcal{F}(n)) = 0$ for all $n \geq n_0$ and $i > 0$.

Proof. By assumption there is an immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. As X is projective, it is proper and so i is a closed immersion. If $\mathcal{G} = i_*\mathcal{F}$ then

$$H^i(\mathbb{P}^r_A,\mathcal{G}) \simeq H^i(X,\mathcal{F}).$$

Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$ we may assume that $X = \mathbb{P}^r_A$.

If $\mathcal{F} = \mathcal{O}_X(q)$ then the result is given by (20.1). Thus the result also holds is $\mathcal{F}$ is a direct sum of invertible sheaves. The general case proceeds by descending induction on i. Now

$$H^i(X,\mathcal{F}) = 0,$$

if $i > r$, by Grothendieck's vanishing theorem. On the other hand, $\mathcal{F}$ is a quotient of a direct sum $\mathcal{E}$ of invertible sheaves. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{R}$ is coherent. Twisting by $\mathcal{O}_X(n)$ we get

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{F}(n) \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we get isomorphisms

$$H^i(X,\mathcal{F}(n)) \simeq H^{i+1}(X,\mathcal{R}(n)),$$

for n large enough, and we are done by descending induction on i. $\square$

Theorem 20.3. *Let A be a Noetherian ring and let X be a proper scheme over A. Let $\mathcal{L}$ be an invertible sheaf on X. TFAE*
(1) $\mathcal{L}$ is ample.
(2) For every coherent sheaf $\mathcal{F}$ on X there is an integer n_0 such that

$$H^i(X,\mathcal{F}\otimes\mathcal{L}^n) = 0,$$

for $n \geq n_0$.

Proof. Suppose that (1) holds. Pick a positive integer m such that $\mathcal{M} = L^{\otimes m}$ is very ample. Let $\mathcal{F}_r = \mathcal{F}\otimes\mathcal{L}^r$, for $0 \leq r \leq m-1$. By (20.2) we may find n_r depending on r such that $H^i(X,\mathcal{F}_r\otimes\mathcal{M}^n) = 0$ for all $n \geq n_r$ and $i > 0$. Let p be the maximum of the n_r. Given $n \geq n_0 = pm$, we may write $n = qm + r$, for some $0 \leq r \leq m-1$ and $q \geq p$. Then

$$H^i(X,\mathcal{F}\otimes\mathcal{L}^n) = H^i(X,\mathcal{F}_r\otimes\mathcal{M}^q) = 0,$$

for any $i > 0$. Hence (1) implies (2).

Now suppose that (2) holds. Let $\mathcal{F}$ be a coherent sheaf. Let $p \in X$ be a closed point. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \longrightarrow 0,$$

where $\mathcal{I}_p$ is the ideal sheaf of p. If we tensor this exact sequence with $\mathcal{L}^n$ we get an exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p \longrightarrow 0.$$

By hypotheses we can find n_0 such that

$$H^1(X, \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for all $n \geq n_0$. It follows that the natural map

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p),$$

is surjective, for all $n \geq n_0$. It follows by Nakayama's lemma applied to the local ring $\mathcal{O}_{X,p}$ that that the stalk of $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. As $\mathcal{F}$ is a coherent sheaf, for each integer $n \geq n_0$ there is an open neighbourhood U of p, depending on n, such that sections of $H^0(X, \mathcal{F} \otimes \mathcal{L}^n)$ generate the sheaf at every point of U.

If we take $\mathcal{L} = \mathcal{O}_X$ it follows that there is an integer n_1 such that $\mathcal{L}^{n_1}$ is generated by global sections over an open neighbourhood V of p. For each $0 \leq r \leq n_1 - 1$ we may find U_r such that $\mathcal{F} \otimes \mathcal{L}^{n_0+r}$ is generated by global sections over U_r. Now let

$$U_p = V \cap U_0 \cap U_1 \cap \cdots \cap U_{n_1-1}.$$

Then

$$\mathcal{F} \otimes \mathcal{L}^n = (\mathcal{F} \otimes \mathcal{L}^{n_0+r}) \otimes (\mathcal{L}^{n_1})^m,$$

is generated by global sections over the whole of U_p for all $n \geq n_0$.

Now use compactness of X to conclude that we can cover X by finitely many U_p. $\square$

One of the most interesting applications of sheaf cohomology are very useful formulae for the number of global sections.

Definition 21.1. *Let $P(z) \in \mathbb{Q}[z]$ be a polynomial. We say that $P(z)$ is **numerical** if $P(n) \in \mathbb{Z}$ for any sufficiently large integer n.*

Lemma 21.2.

(1) If $P(z)$ is a numerical polynomial then we may find integers $c_0, c_1, \dots, c_r$ such that

$$P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r.$$

In particular $P(n) \in \mathbb{Z}$ for every $n \in \mathbb{Z}$.

(2) If $f \colon \mathbb{Z} \longrightarrow \mathbb{Z}$ is any function and there is a numerical polynomial $Q(z)$ such that $\Delta(f) = f(n+1) - f(n) = Q(n)$ for n sufficiently large then there is a numerical polynomial $P(z)$ such that $f(n) = P(n)$ for n sufficiently large.

Proof. We prove (1) by induction on the degree r of P. Since

$$\binom{z}{r} = \frac{z(z-1)\cdots(z-r+1)}{r!} = \frac{z^r}{r!} + \dots,$$

is a polynomial of degree n, they form a basis for all polynomials and we may certainly find rationals $c_0, c_1, \dots, c_r$ such that

$$P(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r.$$

Note that

$$Q(z) = \Delta P(z) = P(z+1) - P(z) = c_0\binom{z}{r-1} + c_1\binom{z}{r-2} + \dots + c_{r-1},$$

is a numerical polynomial. By induction on the degree, $c_0, c_1, \dots, c_{r-1}$ are integers. It follows that c_r is an integer, as $P(n)$ is an integer for n large. This is (1).

For (2), suppose that

$$Q(z) = c_0\binom{z}{r} + c_1\binom{z}{r-1} + \dots + c_r,$$

for integers $c_0, c_1, \dots, c_r$. Let

$$P(z) = c_0\binom{z}{r+1} + c_1\binom{z}{r} + \dots + c_r\binom{z}{1}.$$

Then $\Delta P(z) = Q(z)$ so that $(f - P)(n)$ is a constant c_{r+1} for any n sufficiently large, so that $f(n) = P(n) + c_{r+1}$ for any n sufficiently large. □

Theorem 21.3 (Asymptotic Riemann-Roch)**.** *Let X be a normal projective variety of dimension n and let $\mathcal{O}_X(1)$ be a very ample line bundle. Suppose that $X \subset \mathbb{P}^k$ has degree d.*

Then

$$h^0(X, \mathcal{O}_X(m)) = \frac{dm^n}{n!} + ...,$$

is a polynomial of degree n, for m large enough, with the given leading term.

Proof. First suppose that X is smooth. Let Y be a general hyperplane section. Then Y is smooth by Bertini. The trick is to compute $\chi(X, \mathcal{O}_X(m))$ by looking at the exact sequence

$$0 \longrightarrow \mathcal{O}_X(m-1) \longrightarrow \mathcal{O}_X(m) \longrightarrow \mathcal{O}_Y(m) \longrightarrow 0.$$

The Euler characteristic is additive so that

$$\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m-1)) = \chi(Y, \mathcal{O}_Y(m)).$$

(21.2) implies that $\chi(X, \mathcal{O}_X(m))$ is a polynomial of degree n, with the given leading term. Now apply Serre vanishing.

For the general case we need that if X is normal and Y is a general hyperplane section, then Y is a normal projective variety of degree d. Y is regular in codimension one by a Bertini type argument and one can check that Y is S_2. □

It is fun to use similar arguments to prove special cases of Riemann-Roch.

Theorem 21.4 (Riemann-Roch for curves)**.** *Let C be a smooth projective curve of genus g and let D be a divisor of degree d.*

$$h^0(X, \mathcal{O}_C(D)) = d - g + 1 + h^0(C, \mathcal{O}_C(K_C - D)).$$

Proof. We first check that

$$\chi(C, \mathcal{O}_C(D)) = d - g + 1.$$

We may write

$$D = \sum a_i p_i.$$

We proceed by induction on $\sum |a_i|$. Let $p = p_1$. If $a_1 > 0$ then consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(D-p) \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

The Euler characteristic is additive, so that

$$\chi(C, \mathcal{O}_C(D)) = \chi(C, \mathcal{O}_C(D - p)) + 1.$$

The LHS is equal to $(d - 1) - g + 1 + 1 = d - g + 1$ by induction. If $a_1 < 0$ then consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_C(D + p) \longrightarrow \mathcal{O}_p \longrightarrow 0.$$

The Euler characteristic is additive, so that

$$\chi(C, \mathcal{O}_C(D)) = \chi(C, \mathcal{O}_C(D + p)) - 1.$$

The RHS is equal to $d - g + 1 - 1 = (d - 1) - g + 1$ by induction.

So we are reduced to the case when $d = 0$. Note that

$$h^1(C, \mathcal{O}_C(D)) = h^0(C, \mathcal{O}_C(K_C - D)),$$

by Serre duality. In particular

$$\chi(C, \mathcal{O}_C) = 1 - g,$$

which completes the induction. □

Theorem 21.5 (Riemann-Roch for surfaces)**.** *Let S be a smooth projective surface of irregularity q and geometric genus p_g over an algebraically closed field of characteristic zero. Let D be a divisor on S.*

$$\chi(S, \mathcal{O}_S(D)) = \frac{D^2}{2} - \frac{K_S \cdot D}{2} + 1 - q + p_g.$$

Proof. Pick a very ample divisor H such that $H + D$ is very ample. Let C and Σ be general elements of $|H|$ and $|H + D|$. Then C and Σ are smooth. There are two exact sequences

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(D + H) \longrightarrow \mathcal{O}_C(D + H) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(D + H) \longrightarrow \mathcal{O}_\Sigma(D + H) \longrightarrow 0.$$

As the Euler characteristic is additive we have

$$\begin{aligned} \chi(S, \mathcal{O}_S(D + H)) &= \chi(S, \mathcal{O}_S(D)) + \chi(C, \mathcal{O}_C(D + H)) \\ \chi(S, \mathcal{O}_S(D + H)) &= \chi(S, \mathcal{O}_S) + \chi(\Sigma, \mathcal{O}_\Sigma(D + H)). \end{aligned}$$

Subtracting we get

$$\chi(S, \mathcal{O}_S(D)) - \chi(S, \mathcal{O}_S) = \chi(\Sigma, \mathcal{O}_\Sigma(D + H)) - \chi(C, \mathcal{O}_C(D + H)).$$

Now

$$\begin{aligned} \chi(\Sigma, \mathcal{O}_\Sigma(D + H)) &= (D + H) \cdot \Sigma - \deg K_\Sigma/2 \\ \chi(C, \mathcal{O}_C(D + H)) &= (D + H) \cdot C - \deg K_C/2, \end{aligned}$$

applying Riemann-Roch for curves to both C and Σ. We have

$$(D+H)\cdot\Sigma = (D+H)\cdot H + (D+H)\cdot D,$$

and by adjunction

$$K_\Sigma = (K_S+\Sigma)\cdot\Sigma \qquad \text{and} \qquad K_C = (K_S+C)\cdot C.$$

So putting all of this together we get

$$\begin{aligned}\chi(S,\mathcal{O}_S(D)) - \chi(S,\mathcal{O}_S) &= (D+H)\cdot D + \frac{1}{2}((K_S+C)\cdot C - (K_S+\Sigma)\cdot\Sigma) \\ &= (D+H)\cdot D + \frac{1}{2}K_S\cdot(C-\Sigma) + \frac{1}{2}(H\cdot H - (H+D)\cdot(H+D)) \\ &= \frac{D\cdot D}{2} - \frac{1}{2}K_S\cdot D.\end{aligned}$$

We have

$$c = \chi(S,\mathcal{O}_S) = h^0(S,\mathcal{O}_S) - h^1(S,\mathcal{O}_S) + h^2(S,\mathcal{O}_S) = 1-q+p_g.$$

Here we used the highly non-trivial fact that

$$h^1(S,\mathcal{O}_S) = h^0(S,\Omega^1_S) = q,$$

from Hodge theory and Serre duality

$$h^2(S,\mathcal{O}_S) = h^0(S,\omega_S) = p_g. \qquad \square$$

Remark 21.6. *One can turn Riemann-Roch for surfaces around and use the arguments in the proof of* (21.5) *to prove basic properties of the intersection number.*

We will need a little bit of intersection theory.

Given any variety X we can define cycles of any dimension on X. A **cycle** α is a formal linear combination of closed subvarieties, $\sum n_V V$. If V all have the same dimension k then we say α is a k**-cycle**. Two k-cycles α and β are **rationally equivalent** if there is a $k+1$ dimensional subvariety W which contains the support of α and β and α and β are linearly equivalent divisors on the normalisation of W. If X has dimension n then an $(n-1)$-cycle is the same as a Weil divisor.

Note that we can pullback Cartier divisors. We can also pushforward Weil divisors, or more generally cycles. If $f\colon X \longrightarrow Y$ is a proper morphism and V is an irreducible closed subvariety with image W then

$$f_*V = \begin{cases} dW & \text{if } f|_V\colon V \longrightarrow W \text{ is generically finite of degree } d \\ 0 & \text{otherwise.} \end{cases}$$

In other words if the image of V is lower dimensional then $f_*V = 0$. If the image W of V has the same dimension then $f_*V = dW$ where d is the degree of V over W. We extend the pushforward by linearity to all cycles.

Note that we can intersect a cycle α with a Cartier divisor D, to get a cycle $\alpha \cdot D$. By linearity we may assume that $\alpha = V$ is a closed irreducible subvariety. In this case we can define a linear equivalence of Cartier divisors on V. If the support of D does not contain then simply restrict the equations of D to V. If the support of D does contain V then restrict the invertible sheaf $\mathcal{O}_X(D)$ to V to get an invertible sheaf on V. An invertible sheaf is the same as a linear equivalence class of Cartier divisors. Now pushforward the corresponding Weil divisor, via the natural inclusion $V \longrightarrow X$ to get a cycle on X.

Now pushforward is not a ring homomorphism, but it is almost is:

Theorem 22.1 (Push-pull)**.** *Let $f\colon X \longrightarrow Y$ be a proper morphism of varieties. Let α be a cycle on X and let D be a Cartier divisor on Y. Then*

$$f_*(\alpha \cdot f^*D) = f_*\alpha \cdot D.$$

0-cycles are formal sums of points $\sum n_p p$. The degree is the sum $\sum n_p$. Note that two rationally equivalent 0-cycles have the same degree.

If X is a smooth projective variety over $\mathbb{C}$ then we can associate to any cycle α a class in homology. As usual, by linearity it is enough to do this for irreducible subvarieties V. Take a simplicial decomposition of X which induces a simplicial decomposition of V. Then V defines a class $[V] \in H_*(X, \mathbb{Z})$. A divisor D determines a class in cohomology $[D] \in H^2(X, \mathbb{Z})$. We can pair this with a homology classes. This is compatible with the algebraic intersection product

$$[D \cdot \alpha] = [D] \cap [\alpha] \in H_*(X, \mathbb{Z}).$$

Note also that there is a topological push-pull formula.

Now let us consider what happens for surfaces. 1-cycles, or Weil divisors, are nothing more than formal sums of curves. If we intersect a Weil divisor with a Cartier divisor, we will get a rational equivalence class of 0-cycles. The intersection number is just the degree of the 0-cycles. From now on, the intersection product will denote the degree.

We can compute the degree locally.

Definition 22.2. *Let S be a smooth surface and let p be a point of S. Let D_1 and D_2 be two Cartier divisors on S.*

First suppose that $D_1 = C_1$ and $D_2 = C_2$ are prime divisors. The ***local intersection number*** *of D_1 and D_2 at p,*

$$i_p(D_1, D_2) = \dim_k \mathcal{O}_{S,p}/\langle f_1, f_2 \rangle,$$

where f_1 and f_2 are local generators of the ideals of C_1 and C_2.

Now extend this by linearity to any two divisors with no common components.

It is interesting to check that the local intersection number coincides with geometric intuition.

Example 22.3. *Let $S = \mathbb{A}^2$, let C_1 be the x-axis $= 0$ and let C_2 be the conic $y = x^2$. Then C_1 and C_2 are tangent. The local intersection number is the dimension of the k-vector space*

$$\frac{k[x, y]}{\langle y, y - x^2 \rangle} = \frac{k[x]}{\langle x^2 \rangle} = k\langle 1, x \rangle$$

which is two, as expected.

Proposition 22.4. *If S is a projective surface and D_1 and D_2 are two divisors with no common components*

$$D_1 \cdot D_2 = \sum_p i_p(D_1, D_2).$$

Here the sum is over all points p in the interesection.

Theorem 22.5 (Bézout's Theorem)**.** *Let C and D be two curves defined by homogenous polynomials of degrees d and e. Suppose that $C \cap D$ does not contain a curve.*

Then $|C \cap D|$ is at most de, with equality if and only if the intersection of the two tangent spaces at $p \in C \cap D$ is equal to p.

Proof. $C \sim dL$ and $D \sim eL$, where L is a line. In this case

$$|C \cap D| \leq \sum_p i_p(C, D) = C \cdot D = (dL) \cdot (eL) = (de)L^2 = de. \qquad \square$$

Definition 22.6. *Let $C \subset S$ be a curve in a smooth surface. Let $p \in S$ be a point of S.*

The ***multiplicity of*** C ***at*** $p \in S$ *is the largest μ such that $\mathcal{I}_p \subset \mathfrak{m}^\mu$ where $\mathfrak{m}$ is the maximal ideal of S at p in $\mathcal{O}_{S,p}$ and $\mathcal{I}$ is the ideal sheaf of C in S.*

Note that $\mathcal{I} = \langle f \rangle$, so we just want the largest μ such that $f \in \mathfrak{m}^\mu$. If we work over $\mathbb{C}$, then we can choose coordinates x and y. In this case $\mathfrak{m} = \langle x, y \rangle$ and f is a power series in x and y. If we expand f in powers of x and y,

$$f(x, y) = f_0 + f_1 + f_2 + \dots$$

where f_i is homogenous of degree i then the multiplicity μ is the smallest integer such that $f_\mu \neq 0$.

Lemma 22.7. *Let $C \subset S$ be a curve in a smooth surface. Let $p \in S$ be a point of S and let $\pi \colon T \longrightarrow S$ be the blow up of S at p. Let $\tilde{C}$ be the strict transform of C. Then*

$$\pi^* C = \tilde{C} + \mu E.$$

Proof. Pick coordinates so that $y = 0$ is not tangent to any branch of C. Then $T \subset S \times \mathbb{P}^1$ and local coordinates on T are given by (x, t), where $y = tx$. In this case

$$f(x, y) = f_\mu(x, xt) + f_{\mu+1}(x, xt) = x^\mu(f_\mu(1, t) + x f_\mu(1, t) + \dots).$$

As $x = 0$ is the equation of E the result is clear. $\square$

Proposition 22.8. *Let S be a smooth surface, let $p \in S$ be a point of S and let $\pi \colon T \longrightarrow S$ be the blow up of S at p, with exceptional divisor E.*

Then $E^2 = -1$.

Proof. We give two proofs of this result.

Here is the first. Note that this is a local computation. So we might as well assume that $S = \mathbb{P}^2$. Pick a line L passing through p. Then L is a Cartier divisor on S. We have

$$\pi^* L = M + E,$$

where M is the strict transform of L.

Let L_1 and L_2 be two general lines through p. Then

$$L^2 = L_1 \cdot L_2 = 1,$$

that is, two lines meet in one point. Let M_1 and M_2 be the strict transforms of L_1 and L_2. Then M_1 and M_2 don't meet, by definition of the blow up. Thus

$$M^2 = M_1 \cdot M_2 = 0.$$

To practice using push-pull, let us calculate $E \cdot \pi^* L$. By push-pull,

$$E \cdot \pi^* L = \pi_*(E \cdot \pi^* L) = \pi_* E \cdot L = 0.$$

Thus

$$1 = M \cdot \pi^* L = M \cdot (M + E) = M \cdot M + M \cdot E = 1,$$

which is consistent.

Now let us calculate $E \cdot \pi^* L$. By push-pull this is

$$\pi_*(E \cdot \pi^* L) = \pi_* E \cdot L = 0,$$

since $\pi_* E = 0$. On the other hand,

$$E \cdot \pi^* L = E \cdot (M + E) = E \cdot M + E^2 = 1 + E^2.$$

Thus $E^2 = -1$.

Here is the second method. The ideal sheaf of E in T is given by $\mathcal{O}_T(-E)$. By definition of the blow up, this restricts to $\mathcal{O}_E(1) = \mathcal{O}_{\mathbb{P}^1}(1)$. Thus $\mathcal{O}_T(E)$ restricts to $\mathcal{O}_E(-1)$, so that $E|_E$ has degree -1. □

Definition-Lemma 22.9. *Let S be a smooth surface and let $C \subset S$ be a proper irreducible curve.*

Any two of the following three properties implies the third:

(1) *$C \simeq \mathbb{P}^1$.*
(2) *$C^2 = -1$.*
(3) *$K_S \cdot C = -1$.*

In this case we call E a -1-curve.

Proof. By adjunction

$$(K_S + C)|_C = K_C.$$

Thus

$$2g - 2 = \deg K_C = (K_S + C) \cdot C.$$

Note that $C \simeq \mathbb{P}^1$ if and only if $g = 0$. The result is then clear. □

Lemma 22.10. *Let $\pi\colon T \longrightarrow S$ be the blow up of a smooth point of a smooth surface. Let E be the exceptional divisor.*

Then

$$K_T = \pi^* K_S + E.$$

Proof. Note that π is an isomorphism outside p, so that

$$K_T = \pi^* K_S + aE,$$

for some integer a. It suffices to check that $a = 1$; we will give two proofs of this result.

Here is the first. We have already seen that $E \simeq \mathbb{P}^1$ and $E^2 = -1$. So $K_T \cdot E = -1$ by (22.9). On the other hand,

$$-1 = K_T \cdot E = (\pi^* K_S + aE) \cdot E = K_S \pi_* E + aE^2 = -a.$$

Thus $a = 1$.

The second is by direct computation. Let (x, y) be local coordinates on S. Then

$$\omega = \mathrm{d}x \wedge \mathrm{d}y,$$

is a meromorphic differential with no poles or zeroes in a neighbourhood of p. Local coordinates upstairs are (x, t), where $y = xt$.

$$\begin{aligned}
\pi^*\omega &= \mathrm{d}x \wedge \mathrm{d}(xt) \\
&= \mathrm{d}x \wedge (t\mathrm{d}x + x\mathrm{d}(t) \\
&= t\mathrm{d}x \wedge \mathrm{d}x + x\mathrm{d}x \wedge \mathrm{d}(t) \\
&= x\mathrm{d}x \wedge \mathrm{d}t.
\end{aligned}$$

Thus the pullback of a meromorphic differential from S always has a simple zero along E. □

Lemma 22.11. *Let $\pi\colon T \longrightarrow S$ be the blow up of a smooth point of a smooth surface.*

Then

$$K_T^2 = K_S^2 - 1.$$

Proof.

$$K_T^2 = K_T \cdot (\pi^* K_S + E) = K_T \cdot \pi^* K_S + K_T \cdot E = K_S^2 - 1.$$ □

23. Fano varieties

Definition 23.1. *A smooth projective variety is **Fano** if $-K_X$ is ample.*

Example 23.2. *Let $X \subset \mathbb{P}^r$ be a smooth hypersurface of degree d. By adjunction*

$$K_X = (K_{\mathbb{P}^r} + X)|_X = (d - r - 1)H,$$

where H is the class of a hyperplane. Thus X is Fano if and only if $d \leq r$.

Note that the product of Fano varieties is Fano. If C is a smooth projective curve then C is Fano if and only if $C \simeq \mathbb{P}^1$. A Fano surface is called a del Pezzo surface. What are the del Pezzo surfaces? $\mathbb{P}^2$ is a del Pezzo surface. $\mathbb{P}^1 \times \mathbb{P}^1$ is a del Pezzo surface; either use the fact that it is product of Fanos or use the fact that it is isomorphic to a quadric in $\mathbb{P}^3$. A smooth cubic surface is a del Pezzo surface. Let S be the blow up of $\mathbb{P}^2$ at a point. Then there is a morphism $S \longrightarrow \mathbb{P}^1$, with fibres copies of $\mathbb{P}^1$.

The degree d of a del Pezzo surface is the self-interesection of $-K_S$.

Theorem 23.3. *Let $\pi\colon S \longrightarrow \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ along r general points.*

Then S is a del Pezzo surface if and only if $r \leq 8$.

Here general means that the points belong to a Zariski open subset of the product of r copies of $\mathbb{P}^2$.

Proof. Note that

$$d = K_S^2 = K_{\mathbb{P}^2}^2 - r = 9 - r.$$

If S is a del Pezzo surface then $d > 0$ and so $r \leq 8$.

Now suppose that $r \leq 8$. Suppose that we blow up $p_1, p_2, \dots, p_r$ with exceptional divisors $E_1, E_2, \dots, E_r$. Then

$$K_S = \pi^* K_{\mathbb{P}^2} + \sum E_i.$$

We have to show that $-K_S$ is ample. We will just prove a small part of this result. If $r \leq 6$ then we will show that $-K_S$ is very ample. We assume that no three points of $p_1, p_2, \dots, p_r$ are collinear and no six points lie on a conic. We have to show that the linear system $|-K_S|$ separates points and tangent vectors.

The linear system

$$|-K_{\mathbb{P}^2}| = |3L|,$$

is the linear system of all cubic curves. Thus divisors in the linear system

$$|-K_S| = |3\pi^*L - \sum E_i|,$$

correspond to cubics through $p_1, p_2, \dots, p_r$.

Suppose that we pick two points x and $y \in S$. This gives us two points p_{r+1} and p_{r+2} in $\mathbb{P}^2$. If x and y don't belong to the exceptional divisors, then we have $r+2 \leq 8$ distinct points in $\mathbb{P}^2$. We check that these impose independent conditions on cubics. We have to check that for all $i \leq r+1 \leq 7$ we can find a cubic through $p_1, p_2, \dots, p_i$ not containing p_{i+1}.

If $i \leq 6$ then pick three pairs of lines. As no three points of $p_1, p_2, \dots, p_r$ are collinear, then we can choose our lines not containing p_{i+1}. If $i = 7$ then pick a conic through five points and a line through the last point.

Thus if x and y don't lie on an exceptional divisor, then we may find a divisor in the linear system $|-K_S|$ through x not through y.

If x belongs to an exceptional divisor then $p_{r+1} = p_j$ for some j. Let's suppose $p_{r+1} = p_r$. Suppose that y does not belong to an exceptional divisor.

Then we need to find a cubic through $p_1, p_2, \dots, p_r$ with the tangent direction determined by x not passing through p_{r+2}.

If $r = 7$ then $|-K_S|$ defines a two to one map to $\mathbb{P}^2$, branched over a quartic curve C, $\pi \colon S \longrightarrow \mathbb{P}^2$. It follows, by Riemann-Hurwitz that

$$K_S = \pi^*(K_{\mathbb{P}^2} + 1/2C),$$

so that $-K_S$ is the pullback of an ample divisor L. Thus $-K_S$ is ample. In fact $-2K_S$ is very ample.

If $r = 8$ then $|-K_S|$ defines a pencil, $\pi \colon S \longrightarrow \mathbb{P}^1$. Two cubics intersect in 8 points, so that the pencil has a base point and π is not a morphism. $|-2K_S|$ defines a double cover of a quadric cone in $\mathbb{P}^3$. $-3K_S$ is very ample. $\square$

It is interesting to consider the image of the anticanonical linear system. If $r = 6$ so that $d = 3$ then $|-K_S|$ is the linear system of cubics through six points. The linear system of all cubics is 9 dimensional and we have already seen that six points impose independent conditions. Therefore $|-K_S|$ is a three dimensional linear system, and S is embedded as a smooth hypersurface of degree 3 in $\mathbb{P}^3$, a cubic surface. This raises the interesting question, is every smooth cubic the blow up of $\mathbb{P}^2$ in six points?

Let's count moduli. The space of six points on $\mathbb{P}^2$ has dimension $6 \cdot 2 = 12$. The automorphism group of $\mathbb{P}^2$ is PGL(3) which has dimension

$3\cdot 3-1=8$. Thus there is a four dimensional family of non-isomorphic del Pezzo surfaces obtained by blowing up 6 points.

The space of cubics is a linear system of dimension

$$\binom{3+3}{3}-1=\binom{6}{3}-1=19.$$

The automorphism group of $\mathbb{P}^3$ is $\mathrm{PGL}(4)$ which has dimension $4\cdot 4-1=15$. Thus there is a four dimensional family of non-isomorphic cubic surfaces.

This suggests that every cubic surface is the blow up of $\mathbb{P}^2$ at six points. Unfortunately, it is hard to prove this directly. The natural map

$$\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2 \dashrightarrow \mathbb{P}^{19}$$

which sends a six-tuple of points $(p_1,p_2,p_3,p_4,p_5,p_6)$ to the corresponding cubic surface is a rational map, not a morphism.

If $r=5$ then $d=4$. The linear system of cubics through five points is a copy of $\mathbb{P}^4$. Thus we get a surface $S\subset\mathbb{P}^4$ of degree 4. Consider the exact sequence

$$0\longrightarrow \mathcal{I}_S\longrightarrow \mathcal{O}_{\mathbb{P}^4}\longrightarrow \mathcal{O}_S\longrightarrow 0.$$

If we twist this by $\mathcal{O}_{\mathbb{P}^4}(2)$, we get

$$0\longrightarrow \mathcal{I}_S(2)\longrightarrow \mathcal{O}_{\mathbb{P}^4}(2)\longrightarrow \mathcal{O}_S(2)\longrightarrow 0.$$

Taking global sections we get a left exact sequence

$$0\longrightarrow H^0(\mathbb{P}^4,\mathcal{I}_S(2))\longrightarrow H^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(2))\longrightarrow H^0(S,\mathcal{O}_S(2)).$$

Now

$$h^0(\mathbb{P}^4,\mathcal{O}_{\mathbb{P}^4}(2))=\binom{4+2}{2}=\binom{6}{2}=15.$$

On the other hand the linear system $|\mathcal{O}_S(2))|$ corresponds to sextics in $\mathbb{P}^2$ which have multiplicity two at five points. It is 3 conditions to be singular at any given point (in local coordinates we need the constant term and the coefficient of both x and y to vanish). Thus

$$h^0(S,\mathcal{O}_S(2))=\binom{6+2}{2}-3\cdot 5=13.$$

Thus

$$h^0(\mathbb{P}^4,\mathcal{I}_S(2)\geq 2.$$

It follows that there are two quadratic polynomials which vanish on S, that is, there are two quadric hypersurfaces Q_1 and Q_2 which contain S. Now the intersection of two quadrics is a surface of degree 4, so that we must have $S=Q_1\cap Q_2$ (as schemes).

Conversely suppose we are given two quadrics Q_1 and Q_2 whose intersection is a smooth surface S.

$$K_{Q_1} = (K_{\mathbb{P}^4} + Q_1)|_{Q_1} = -3H.$$

Thus

$$K_S = (K_{Q_1} + S)|_S = -H,$$

so that S is anticanonically embedded.

Let's count moduli again. The moduli space of surfaces which are the blow up of five points has dimension 2. Picking Q_1 and Q_2 is like picking a pencil l of quadrics, that is, a line l in the space of all quadrics, that is a point l of a Grassmannian. The space of all quadrics is a copy of $\mathbb{P}^{14}$ and so we get a

$$2(14 - 1) = 26,$$

dimensional family. The automorphism group of $\mathbb{P}^4$ is PGL(5) which has dimension 24, so we get two dimensions of moduli.

If $r = 4$ then we blow up four points of $\mathbb{P}^2$. There is no moduli and our surface gets embedded as a degree five surface in $\mathbb{P}^5$. If $r \leq 3$ then S is toric, since three points in general points are the same as the three coordinate points.

Theorem 23.4. *Let S be a del Pezzo surface.*
Then S is isomorphic either to
(1) $\mathbb{P}^1 \times \mathbb{P}^1$, or
(2) $\mathbb{P}^2$ blown up in $r \leq 8$ points.

What can we say about lines on the cubic, from this point of view? Well, $C \subset S$ is a line if and only if

$$1 = H \cdot C = -K_S \cdot C.$$

Since a line is isomorphic to $\mathbb{P}^1$, we see that lines on the cubic surface are the same as -1-curves. What are the -1-curves on $\mathbb{P}^2$ blown up at $r \leq 8$ points?

We have

$$\operatorname{Pic}(S) = \mathbb{Z} \cdot H \bigoplus_{i=1}^{r} \mathbb{Z}E_i,$$

where $H = \pi^* L$ is the pullback of a line. Thus a general curve in S has class

$$aH - \sum a_i E_i,$$

where a, $a_1, a_2, \dots, a_r$ are integers. The class of K_S is

$$-3H + \sum E_i.$$

Note that

$$\begin{aligned} H^2 &= 1, \\ H \cdot E_i &= 0 \\ E_i \cdot E_j &= \begin{cases} -1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If C is a -1-curve then $K_S \cdot C = -1$ and $C^2 - 1$. This gives us two Diophantine equations:

$$-3a + \sum a_i = -1 \qquad \text{and} \qquad a^2 - \sum a_i^2 = -1.$$

Let us start by guessing some solutions. We already know that if we blow up a point then the exceptional divisor is a -1-curve. Thus $a = 0$,

$$a_i = \begin{cases} -1 & \text{if } i = k \\ 0 & \text{otherwise,} \end{cases}$$

is a solution, for any k. There are r such exceptional curves.

The self-intersection of a line is 1. If we blow up a point on the line then the self-intersection of the strict transform is 0. If we blow up two points then the self-intersection of the strict transform is -1. Thus the strict transform of a line through two points is a -1-curve. There are

$$\binom{r}{2},$$

such lines. These correspond to the solutions $a = 1$,

$$a_i = \begin{cases} 1 & \text{if } i = k \text{ or } i = l \\ 0 & \text{otherwise,} \end{cases}$$

for any pair $k \neq l$.

The self-intersection of a conic is 4. If we blow up a five point on the conic then the self-intersection of the strict transform is -1. Thus the strict transform of a conic through five points is a -1-curve. There are

$$\binom{r}{5},$$

such conics. These correspond to the solutions $a = 2$,

$$a_i = \begin{cases} 1 & \text{if } i \in I \\ 0 & \text{otherwise,} \end{cases}$$

for any set I of five indices.

If $r = 6$ this gives

$$6 + \binom{6}{2} + \binom{6}{5} = 6 + 15 + 6 = 27$$

-1-curves. As a cubic surface has no more than 27 lines, every smooth cubic surface contains 27 lines.

If $r = 5$ we get

$$5 + \binom{5}{2} + \binom{5}{5} = 5 + 10 + 1 = 16,$$

-1-curves. Thus there are 16 lines on the intersection of two quadrics in $\mathbb{P}^4$.

One can check that if $r \leq 7$ that there are no other solutions.

We end with one of the most impressive and important results in the classification of surfaces:

Theorem 23.5 (Castelnuovo)**.** *Let S be a smooth projective surface and let $C \subset S$ be a curve.*

Then C is a -1-curve if and only if there is birational morphism $\pi \colon S \longrightarrow T$, which blows up a smooth point $p \in T$, with exceptional divisor C.

Proof. One direction is clear; if π blows up a smooth point then E is a -1-curve.

We prove the converse. Let H be an ample divisor. Then $H \cdot E = k > 0$. Possibly replacing H by a multiple, we may assume that $k > 1$ and furthermore that $K_S + H$ is ample. Let

$$D = K_S + (k-1)E + H.$$

Then

$$D \cdot C = (K_S + (k-1)E + H) \cdot C = -1 - (k-1) + k = 0.$$

On the other hand, let m be a positive integer such that $A = m(K_S+H)$ is very ample. By Serre vanishing we may assume that

$$h^1(S, A) = 0.$$

Note that if $B \in |A|$ then $B + m(k-1)E \in |mD|$. Thus the linear system $|mD|$ separates points and tangent vectors outside of the support of E. In particular the base locus of $|mD|$ is contained in E.

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-E) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

Twisting by mD we get

$$0 \longrightarrow \mathcal{O}_S(mD - E) \longrightarrow \mathcal{O}_S(mD) \longrightarrow \mathcal{O}_E(mD) \longrightarrow 0.$$

Now
$$mD - E = A + lE,$$
where $l = m(k-1) - 1$. We show that
$$h^1(S, A + iE) = 0,$$
for $0 \leq i \leq l$. The case $i = 0$ follows by assumption. There is an exact sequence
$$0 \longrightarrow \mathcal{O}_S(A + (i-1)E) \longrightarrow \mathcal{O}_S(A + iE \longrightarrow \mathcal{O}_E(A + iE) \longrightarrow 0.$$
$(A + iE) \cdot E = mk - i > 0$. Thus
$$h^1(E, A + iE) = h^1(\mathbb{P}^1, (mk - i)p) = 0.$$
Taking the long exact sequence of cohomology we see that
$$h^1(S, A + iE) = h^1(S, A + (i-1)E) = 0,$$
by induction on i. Thus the linear system $|mD|$ is base point free. Consider the image T of the corresponding morphism $S \longrightarrow \mathbb{P}^N$. Since $|mD|$ separates points and tangent vectors outside E, T is a surface and the corresponding morphism $\pi\colon S \longrightarrow T$ is birational. Let G be the restriction of a hyperplane from $\mathbb{P}^N$. Then $\pi^*G = mD$. Let C' be the image of C. Then
$$0 = (mD) \cdot C = \pi^*G \cdot C = G \cdot C'.$$
As G is ample, C' is zero dimensional and so C' is a point. Thus π contracts C.

It remains to prove that T is smooth. Consider the divisor $mD - E$.
$$E \cdot (mD - E) = 1.$$
By the same argument as above, $mD - E$ is base point free. In fact it separates points and tangent vectors, so that it is very ample. We may find $\Sigma \in |mD - E|$ such that Σ is smooth. $\Sigma \cdot C = 1$. Thus $\Sigma' = \pi(\Sigma)$ is smooth. But $\Sigma + C \in |mD|$ and $\pi(\Sigma + C) = \Sigma'$ so that Σ' is Cartier. But then T is smooth. $\square$