Nordic Summer School/NAVF
Symposium in Mathematics
Oslo, August 5–25, 1976

IRREDUCIBILITY OF THE COMPACTIFIED JACOBIAN

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Introduction

Compactifications of jacobians of integral curves and their compatibility with specialization have been considered by Igusa [11], by Mayer and Mumford [13], and by D'Souza ([5], Bombay thesis, preliminary version, 1973, being improved for *Astérisque*; a copy of this improved manuscript was not available to us at the time of this writing). Seshadri and Oda [14] have studied the compactified jacobian for reducible curves. Nakamura and Namikawa, according to [14], have dealt with compactified jacobians in the complex analytic context.

Igusa worked with a Lefschetz pencil of hyperplane sections on a smooth surface (a general member is a smooth curve and finitely many members have a node as a singularity). He defined the compactification for a singular member as the limit of the jacobians of the non-singular members using Chow coordinates (and Chow's construction of the jacobian). Mayer and Mumford announced an intrinsic characterization of the compactified jacobian as the moduli space of torsion-free sheaves with rank one and Euler characteristic $1-p_a$. They said that it could be constructed for any integral curve using geometric invariant theory and that the jacobian is an open subscheme of it.

D'Souza studied the irreducibility of the compactified jacobian. The irreducibility is equivalent to the denseness of the jacobian because the jacobian is irreducible. He proved the irreducibility for an integral curve X with arithmetic genus p_a and with $\delta_x=1$ at each singularity x (so the arithmetic and geometric genera differ by the total number of singularities of X). Moreover, assuming X has only nodes as singularities, he showed that the completion of the local ring of a singularity of the compactified jacobian

[1] Partially supported by a special grant from Universidad Simón Bolívar.

[2] Supported by CNRS exchange fellowship in France.

[3] Partially supported by the NSF under grant no. MCS74-07275.

is isomorphic to $k[[x_0, \cdots, x_{p_a}]]/x_0x_1$. His proof uses the Abel map from the appropriate Hilbert scheme [see 7] to the compactified jacobian, and he showed the map was smooth when it should be (see (9) below) for a Gorenstein curve. (In his thesis, D'Souza dealt only with the case in which X has only nodes as singularities, but our impression is that he will handle the case in which X is Gorenstein in his article for *Astérisque.*)

Below, we show that the compactified jacobian P is irreducible for an integral curve X lying on a smooth surface Y. Since any integral curve with embedding dimension at most 2 at each point can be embedded in a smooth surface [3], this theorem imples D'Souza's result on the irreducibility. Our proof also uses the Abel map to transfer the problem to the Hilbert scheme B of X. The Hilbert scheme B is the scheme of zeroes of a section of a locally free sheaf on the Hilbert scheme of Y. This gives a lower bound on the dimension of B. On the other hand, B has a stratification into sets of low dimension. In addition, the Hilbert scheme of Y is smooth of dimension $2n$ as we prove below in (3). (A different proof is found in [6] for the case that Y is projective and the base is a field or a discrete valuation ring.) The combination of these results yields (5) that B is cut out by a regular section and that it is irreducible. We conclude as corollaries that P is (under the same hypotheses as in (5)) irreducible, reduced and locally a complete intersection. Thus, the jacobian is dense in P and every torsion-free sheaf of rank one on X is a limit of invertible sheaves. It would be interesting to know just when P is a strict complete intersection (that is, when every tangent cone $\mathrm{gr}_{m_p}(\mathcal{O}_{P,p})$ is also a complete intersection) for a curve on a smooth surface; D'Souza showed P is so when the curve has only nodes as singularities.

Our result is sharp. We construct a curve which is integral, smooth except at one point, and a complete intersection in $\mathbb{P}^3$, so Gorenstein in particular, for which the compactified jacobian is reducible, containing a component whose dimension is larger than that of the jacobian. In our example, there are many torsion-free sheaves of rank one which are not limits of invertible sheaves.

In our example, the subset $\mathrm{Sm}^n(X)$ of $\mathrm{Hilb}^n_{(X/S)}$ of smoothable 0-dimensional subschemes of length n is not all of $\mathrm{Hilb}^n_{(X/S)}$; in fact, there are many thick points on X not smoothable in X or even in the ambient $\mathbb{P}^3$. On the other hand, Theorem (8) states that for a reduced curve on a smooth ambient surface, $\mathrm{Sm}^n(X)$ is all of $\mathrm{Hilb}^n_{(X/S)}$.

The general theory of the compactified jacobian of a family X/S can be developed in the spirit of Grothendieck's theory of the Picard scheme, as announced in [2]. The heart of this development is a theory of linear systems, which allows us to represent the Abel map as the twisted family of projective spaces associated to a manageable sheaf. It leads naturally to the surprising conclusion that for a family of non-Gorenstein curves, the theory is more satisfactory with $\mathrm{Quot}_{(\omega/X/S)}$, where ω is the dualizing sheaf, in place of $\mathrm{Hilb}_{(X/S)}$ as the source of the Abel map. (Note that the two are isomorphic when the family of curves is Gorenstein.) For example, the Abel map from $\mathrm{Quot}^n_{(\omega/X/S)}$ is smooth for $n \geqq 2p_a - 2$ and surjective for $n \geqq p_a - 1$. Since here we deal exclusively with Gorenstein curves except for (10) and (11), we have chosen to work with $\mathrm{Hilb}^n_{(X/S)}$ (but note the awkward proof required for (10)), and we use both [2] and [5] as references.

The authors would like to thank Tor Gulliksen for a fruitful discussion about local complete intersections. We would also like to thank the Mathematics Institutes of the University of Aarhus and the University of Oslo for the generous hospitality extended to us.

Throughout, S will be a locally noetherian (base) scheme.

LEMMA (1). *Let Z/S be a flat quasi-projective family of schemes, and let n be a positive integer.*

(i) *There is an open subscheme of the Hilbert scheme* $\mathrm{Hilb}_{(Z/S)}$ *of Z/S* [*see* 7] *parametrizing those subschemes of Z/S that are regularly embedded.*

(ii) *Let V be the open subscheme of* $\mathrm{Hilb}^n_{(Z/S)}$ (*the Hilbert scheme of 0-dimensional subschemes W of Z/S with $\chi(\mathcal{O}_W) = n$*) *parametrizing those 0-dimensional subschemes which are regularly embedded* (*see* (i)), *and let w be a geometric point of V representing a subscheme W.*

(a) *At the point w of V, the scheme* $\mathrm{Hilb}^n_{(Z/S)}$ *is smooth over S with relative dimension*

$$\dim (H^0(W, N))$$

where N is the normal sheaf of W in the fiber of Z/S containing it.

(b) *If the fiber of Z/S containing W has dimension d at each point in the support of W, then the relative dimension of* $\mathrm{Hilb}^n_{(Z/S)}$ *at w is equal to dn.*

PROOF. Assertion (i) follows easily from the fact that the set of points of a base scheme over which a flat and proper subscheme is transversally regularly embedded is open [8; IV_4, 19.2.4]. Assertion (ii(a)) follows easily from the infinitesimal study of the Hilbert scheme [7; 221–23] and the fact that $H^1(W, N)$ is zero since W is zero dimensional. Assertion (ii(b)) follows from (ii(a)), from the fact that N is free with rank d, and from the equality $n = h^0(W, \mathcal{O}_W)$.

PROPOSITION (2). *Let Z/S be a quasi-projective family of schemes whose geometric fibers are connected. Then the geometric fibers of* $\mathrm{Hilb}^n_{(Z/S)}$ *are connected.*

PROOF. We may assume S is equal to the spectrum of an algebraically closed field. The norm map (see [7], 221–26),

$$\mathfrak{n}_{(Z/S)} : \mathrm{Hilb}^n_{(Z/S)} \to \mathrm{Symm}^n_S(Z),$$

has connected fibers ([6], Proposition 2.2). It is proper because Z may be embedded in a projective scheme Y, the map $\mathfrak{n}_{(Z/S)}$ is the restriction of $\mathfrak{n}_{(Y/S)}$ to $\mathrm{Symm}^n_S(Z)$, and $\mathrm{Hilb}^n_{(Y/S)}$ is proper. Therefore, since $\mathrm{Symm}^n_S(Z)$ is connected (because Z is), $\mathrm{Hilb}^n_{(Z/S)}$ is connected.

PROPOSITION (3). *Let Y/S be a smooth quasi-projective family of surfaces. For each positive integer n,* $\mathrm{Hilb}^n_{(Y/S)}$ *is smooth over S with relative dimension $2n$.*

PROOF. For the smoothness assertion, we use the infinitesimal criterion [8; IV_4, 17.5.4]. Take S to be the spectrum of an artinian local ring, take S_0 to be a subscheme of S, and take W_0 to be a 0-dimensional subscheme of $Y_0 = Y \times_S S_0$ that is flat over S_0. We must lift W_0 to a flat subscheme W of Y/S.

For any open subscheme U of Y that contains W_0, it is clearly sufficient to lift W_0 to a flat subscheme of U/S. Since W_0 is a finite set of (closed) points and since Y is quasi-projective, W_0 is contained in an affine open subset of Y. Hence we may assume Y is affine.

Consider a resolution of $\mathcal{O}_{W_0}$,

$$0 \longrightarrow K \xrightarrow{u_0} \mathcal{O}_{Y_0}^{\oplus(m+1)} \xrightarrow{v_0} \mathcal{O}_{Y_0} \longrightarrow \mathcal{O}_{W_0} \longrightarrow 0. \tag{3.1}$$

Since W_0 and Y_0 are S_0-flat, K is S_0-flat. Let s be the point of S. Then the restriction of (3.1) to the fiber over s is also exact. Moreover $K(s)$ is locally free because $Y(s)$ is regular [8; IV_4, 17.5.2] with dimension 2. Hence K is locally free. Moreover, the exactness of (3.1) implies that K has rank m off W_0, hence it has rank m everywhere. By shrinking Y, we may assume K is free.

By a theorem of Burch ([4], p. 944; see [12], p. 148 for an outline of a proof) the map v_0 is equal to $a(\bigwedge^m u_0)$ where a is a unit. Since the image and kernel of $a(\bigwedge^m u_0)$ are equal to the image and the kernel of $(\bigwedge^m u_0)$, we may assume a is equal to 1. Now, lift u_0 to a map $u : \mathcal{O}_Y^{\oplus m} \to \mathcal{O}_Y^{\oplus(m+1)}$. Then the sequence,

$$0 \longrightarrow \mathcal{O}_Y^{\oplus m} \xrightarrow{u} \mathcal{O}_Y^{\oplus(m+1)} \xrightarrow{\bigwedge^m u} \mathcal{O}_Y \longrightarrow \mathcal{O}_W \longrightarrow 0, \tag{3.2}$$

where $\mathcal{O}_W$ is the cokernel of $\bigwedge^m u$, restricts to (3.1), and its restriction to the fiber is exact. Hence (3.2) is exact and $\mathcal{O}_W$ is S-flat (see [8], IV_3, 11.3.7). Thus, there is a lifting W of W_0 to a flat subscheme of Y/S. So $\mathrm{Hilb}^n_{(Y/S)}$ is smooth over S.

To prove the dimension assertion, we may clearly assume that S is the spectrum of a field and that Y is connected. The scheme $\mathrm{Hilb}^n_{(Y/S)}$ is connected (2), so irreducible because $\mathrm{Hilb}^n_{(Y/S)}$ is regular. The subscheme V parametrizing regularly embedded subschemes is open (1, (ii)) and clearly non-empty, so it is dense in $\mathrm{Hilb}^n_{(Y/S)}$. Since V has dimension $2n$ by (1, (ii(b))), $\mathrm{Hilb}^n_{(Y/S)}/S$ has relative dimension $2n$.

PROPOSITION (4). *Let Y be a quasi-projective S-scheme, E a locally free $\mathcal{O}_Y$-Module, and $X \subset Y$ the subscheme of zeroes of a section of E. Set $A = \mathrm{Hilb}^n_{(Y/S)}$ and $B = \mathrm{Hilb}^n_{(X/S)}$. Let $W \subset Y \times A$ denote the universal 0-dimensional subscheme of Y/S with degree n, and let $p: W \to A$ denote the structure morphism. Then B is equal to the subscheme of A of zeroes of a section of the locally free $\mathcal{O}_Y$-Module $p_*(E \mid W)$, whose rank is equal to nr where r is the rank of E. Moreover, the formation of $p_*(E \mid W)$ and the formation of the section commute with base change.*

PROOF. Let I denote the Ideal of X. Then B is clearly equal to the subscheme of A of zeroes of the natural composition,

$$I_A \to \mathcal{O}_{Y \times A} \to \mathcal{O}_W.$$

The section of E defining X corresponds to a surjection $E^\vee \to I$, so B is also equal to the subscheme of A of zeroes of the induced composition,

$$(E \mid W)^\vee \to I_W \to \mathcal{O}_W. \tag{4.1}$$

Since E is locally free and p is finite, the formation of

$$\underline{\mathrm{Hom}}_W((E \mid W)^\vee, \mathcal{O}_W) = E \mid W$$

and the formation of

$$p_* \underline{\mathrm{Hom}}_W((E \mid W)^\vee, \mathcal{O}_W) = p_*(E \mid W)$$

commute with base change. Consequently, B is equal to the subscheme of A of zeroes of the section of $p_*(E \mid W)$ arising from (4.1). It is clear from the construction that the formation of the section commutes with base change. Finally, since p is finite and flat, with degree n, clearly $p_*(E \mid W)$ is locally free with rank rn.

THEOREM (5). *Let X/S be a flat family of geometrically reduced curves lying on a smooth quasi-projective family of surfaces Y/S. Set $A = \mathrm{Hilb}^n_{(Y/S)}$, and set $B = \mathrm{Hilb}^n_{(X/S)}$. Then B is equal to the subscheme of A of zeroes of a section σ of a locally free $\mathcal{O}_A$-Module F with rank n, and the formations of F and of σ commute with base change; moreover, σ is transversally regular over S. In addition, B is equal to the set-theoretic closure of $\mathrm{Div}^n_{(U/S)}$ in A where U denotes the smooth locus of X/S.*

PROOF. First, X is a relative effective divisor of Y/S because X/S is flat and its fibers are reduced, pure 1-codimensional subschemes of the locally factorial schemes $Y(s)$. Consequently, by (4), B is equal to the subscheme of A of zeroes of a section σ of a locally free $\mathcal{O}_A$-Module F with rank n, and the formations of F and of σ commute with base change.

To show that σ is transversally regular, it is enough to check on the geometric fibers of A/S [8; IV$_3$, 11.3.8]. Moreover, to check that $\mathrm{Div}^n_{(U/S)}$ is dense in B, it is clearly enough to check on the geometric fibers. Therefore, we may assume S is the spectrum of an algebraically closed field.

Let $\mathrm{Symm}^n_S(X)$ denote the n-fold symmetric product of X/S, and let

$$p: X^{\times n} \to \mathrm{Symm}^n_S(X)$$

denote the canonical projection; it is a finite morphism. Let r be an integer satisfying $0 \leqq r \leqq n$, let π denote a sequence of integers $(n_1, \cdots, n_r)$ satisfying $n_1 \geqq n_2 \geqq \cdots \geqq n_r$ and $n' = \sum_{i=1}^r n_i \leqq n$, and set $m = n - n'$. Let $U^{(n)}_{(r,\pi)}$ denote the (finite) union

$$U^{(n)}_{(r,\pi)} = \bigcup p\,(\{x_1\}^{n_1} \times \cdots \times \{x_r\}^{n_r} \times U^{\times m}),$$

where $(x_1, \cdots, x_r)$ runs through the set of r-tuples of distinct singular points of X. Each $U^{(n)}_{(r,\pi)}$ is clearly constructible, and its dimension is m because $U^{\times m}$ has dimension m and p is finite. Clearly the (finite) union of the sets $U^{(n)}_{(r,\pi)}$ over all pairs (r, π) covers $\mathrm{Symm}^n_S(X)$.

Let $w_n: B \to \mathrm{Symm}^n_S(X)$ denote the norm map $\mathfrak{n}_{(X/S)}$ [see 7; 221–26], and set $B^{(n)}_{(r,\pi)} = w_n^{-1}(U^{(n)}_{(r,\pi)})$. Clearly B is equal to the finite union of the constructible sets $B^{(n)}_{(r,\pi)}$.

Assume the estimate,

$$\dim (B^{(n)}_{(r,\pi)}) \leqq n - r. \tag{5.1}$$

Since B is equal to the subscheme of A of zeroes of a section of a locally free $\mathcal{O}_A$-Module F with rank n, every component of B has codimension less than or equal to n [7; 0_{IV}, 16.3.7]. Since A has dimension $2n$ by (2), every component of B has dimension greater than or equal to n. Therefore, (5.1) implies that B has pure dimension n and is the closure of $B^n_{(0,\varnothing)}$. Moreover, since A is regular (2), so Cohen-Macaulay, the section of F defining B is regular [7; 0_{IV}, 16.5.6]. Finally, $B^{(n)}_{(0,\varnothing)}$ is equal to the open set $\mathrm{Div}^n_{(U/S)}$ because a 0-cycle on a smooth curve is the norm of a unique 0-dimensional subscheme, which is a divisor; so $\mathrm{Div}^n_{(U/S)}$ is dense in B.

To complete the proof, we need only establish the estimate (5.1). The fiber of the norm map over a geometric point $n_1x_1 + \cdots + n_rx_r + y_1 + \cdots y_m$ is clearly equal to the product of the fibers of the maps $w_{n'}: B^{(n')}_{(r,\pi)} \to U^{(n')}_{(r,\pi)}$ and $w_m: B^{(m)}_{(0,\varnothing)} \to U^{(m)}_{(0,\varnothing)}$. The dimension of the latter fiber is clearly zero.

Hence the commutativity of the diagram,

$$\begin{array}{ccc} \mathrm{Hilb}^{n'}_{(Y/S)} & \xrightarrow{w} & \mathrm{Symm}^{n'}_S(Y) \\ \uparrow & & \uparrow \\ B^{n'}_{(r,\pi)} & \xrightarrow{w_{n'}} & \mathrm{Symm}^{n'}_S(X) \end{array}$$

and the estimate,

$$\dim(w^{-1}(x)) \leqq n'-r \quad \text{for} \quad x \in \mathrm{Symm}^{n'}_S(Y),$$

([9], Cor. 2, p. 820) imply the estimate,

$$\dim(w_{n'}^{-1}(x)) \leqq n'-r \quad \text{for} \quad x \in U^{(n)}_{(r,\pi)}.$$

Since the dimension of $U^{(n)}_{(r,\pi)}$ is equal to m, estimate (5.1) holds by [8; IV_2, 5.6.7].

COROLLARY (6). *The norm map,*

$$\mathfrak{n}_{(X/S)}: \mathrm{Hilb}^n_{(X/S)} \to \mathrm{Symm}^n_S(X)$$

is birational for all n.

PROOF. By [7; 221–27], the norm map,

$$\mathfrak{n}_{(U/S)}: \mathrm{Hilb}^n_{(U/S)} \to \mathrm{Symm}^n_S(U),$$

is an isomorphism. Obviously $\mathrm{Hilb}^n_{(U/S)}$ is equal to $\mathrm{Div}^n_{(U/S)}$. By (5), $\mathrm{Div}^n_{(U/S)}$ is dense in $\mathrm{Hilb}^n_{(X/S)}$. Since U is dense in X, clearly $\mathrm{Symm}^n_S(U)$ is dense in $\mathrm{Symm}^n_S(X)$. Thus $\mathfrak{n}_{(X/S)}$ is birational.

COROLLARY (7). *The scheme* $\mathrm{Hilb}^n_{(X/S)}$ *is flat over S and each geometric fiber is locally a complete intersection, Cohen-Macaulay, reduced, and n-dimensional. Moreover, if the geometric fibers of X/S are integral (resp. connected), then the geometric fibers of* $\mathrm{Hilb}^n_{(X/S)}$ *are integral (resp. connected).*

PROOF. Since the section σ is transversally regular, $\mathrm{Hilb}^n_{(X/S)}$ is flat over S [8; IV_4, 19.2.1]. Since $\mathrm{Hilb}^n_{(Y/S)}$ is smooth over S with relative dimension $2n$ by (3), and since F is locally free with rank n, the geometric fibers are local complete intersections with dimension n, so Cohen-Macaulay with dimension n [8; 0_{IV}, 16.5.6]. Since $\mathfrak{n}_{(X/S)}$ is birational, the geometric fibers of $\mathrm{Hilb}^n_{(X/S)}$ are generically regular, and they are irreducible (resp. connected) if X is irreducible (resp. connected). Since the geometric fibers of $\mathrm{Hilb}^n_{(X/S)}$ are Cohen-Macaulay, they are reduced, and they are integral if X is integral.

THEOREM (8). *Let X/S be a flat family of geometrically reduced curves lying on a smooth quasi-projective family of surfaces. Let V be the open subset of* $\mathrm{Symm}^n_S(X)$ *corresponding to 0-cycles* $\sum_{i=1}^n x_i$ *on the fibers of X/S where the* x_i

are distinct smooth points. Then $(\mathfrak{n}_{(X/S)})^{-1}(V)$ *and* $\mathrm{Div}^n_{(U/S)}$ *are scheme-theoretically dense in* $\mathrm{Hilb}^n_{(X/S)}$.

PROOF. Since the data commute with base change, we may assume S is the spectrum of an algebraically closed field [8; IV$_3$, 11.10.10]. The set V is obviously dense in $\mathrm{Symm}^n_S(U)$. So, since $\mathfrak{n}_{(U/S)}$ is an isomorphism [7; 221–27] and since $\mathrm{Div}^n_{(U/S)}$ is dense in $\mathrm{Hilb}^n_{(X/S)}$ by (5), the open set $\mathfrak{n}^{-1}_{(X/S)}(V)$ is dense in $\mathrm{Hilb}^n_{(X/S)}$. Finally, since $\mathrm{Hilb}^n_{(X/S)}$ is Cohen-Macaulay, so has no embedded components, $\mathfrak{n}^{-1}_{(X/S)}(V)$ and $\mathrm{Div}^n_{(U/S)}$ are scheme-theoretically dense in $\mathrm{Hilb}^n_{(X/S)}$ [8; IV$_4$, 20.2.13 (iv)].

THEOREM (9). *Let X/S be a flat projective family of geometrically integral curves with arithmetic genus p_a lying on a smooth quasi-projective family of surfaces parametrized by S. Let P denote the compactified jacobian of X/S. Then P/S is flat and its geometric fibers are integral, Cohen-Macaulay, local complete intersections with dimension p_a. Moreover, P contains the jacobian* $\mathrm{Pic}^0_{(X/S)}$ *as an open scheme-theoretically dense subscheme.*

PROOF. For each integer m, the S-scheme P_m is defined (see [2], Theorem 2) as the S-scheme which parametrizes the isomorphism classes of torsion-free $\mathcal{O}_X$-Modules of rank 1 with Euler characteristic m. Clearly P is equal to $P_{(1-p_a)}$. Since P_m is isomorphic to P_{m+rd} for any r where d is the degree of $\mathcal{O}_X(1)$, it suffices to prove the assertions for P_m with $m<3-3p_a$.

Set $n=1-p_a-m$. Then, since X is Gorenstein, and since $n>2p_a-2$ holds, the Abel map,

$$\mathscr{A}_{(X/S)}:\mathrm{Hilb}^n_{(X/S)}\rightarrow P_m,$$

is smooth with relative dimension $n-p_a$ and surjective (see [2], Theorem 4 or [5]). So, since $\mathrm{Hilb}^n_{(X/S)}$ is S-flat and its geometric fibers are integral and Cohen-Macaulay with dimension n by (7), the scheme P_m is flat over S ($\mathscr{A}_{(X/S)}$ is faithfully flat) and its geometric fibers are integral with dimension p_a and Cohen-Macaulay by [8; IV$_4$, 17.5.8]. Since $\mathrm{Div}^n_{(X/S)}$ is dense in $\mathrm{Hilb}^n_{(X/S)}$ by (5) and since P_m is Cohen-Macaulay, its image is scheme-theoretically dense in P_m.

Finally since the geometric fibers of $\mathrm{Hilb}^n_{(X/S)}$ are local complete intersections (7), the geometric fibers of P_m are local complete intersections by the following general result: If B/A is a formally smooth extension of noetherian local rings, then A is a complete intersection if and only if B is a complete intersection. This result is an easy consequence of the theory of homology of algebras. Alternately, a proof in our case may be based on [8; IV$_4$, 19.3.10, 19.3.4, and 17.5.3(d'')].

LEMMA. (10). *Assume S is a noetherian scheme, and let X/S be a flat projective family of geometrically integral curves with genus p_a. Then there* *exists an integer m such that the Abel map,*

$$\mathcal{A}_{(X/S)}:\mathrm{Hilb}^n_{(X/S)}\to P_m \qquad n=1-p_a-m,$$

is surjective and such that P is isomorphic to P_m.

PROOF. Let $(p_1,\cdots,p_t)$ be the generic points of the irreducible components of P, and, for each i, let g_i be a geometric point with image p_i. For each i, let I_i be a universal sheaf on the geometric fiber $X(g_i)\times P(g_i)$; one exists ([2], Theorem 3) because $X(g_i)$ has a smooth rational point. By Serre's theorem [8; III, 2.2.2 (iv)], there is an integer r such that for each i, the sheaf,

$$\underline{\mathrm{Hom}}_{X(g_i)}(I_i,\mathcal{O}_{X(g_i)})(r)=\underline{\mathrm{Hom}}_{X(g_i)}(I_i(-r),\mathcal{O}_{X(g_i)}), \tag{10.1}$$

is generated by its global sections. Set

$$m=\chi(I_i(-r));$$

clearly m is independent of i.

Twisting by $(-r)$ induces an isomorphism $d_m: P \xrightarrow{\sim} P_m$. Set $\bar{p}_i=d_m(p_i)$ and $\bar{g}_i=d_m(g_i)$. Then d_m clearly carries each $I_i(-r)$ to a universal sheaf, J_i, on $X(\bar{g}_i)\times P_m(\bar{g}_i)$. Since, for each i, the sheaf (10.1) is generated by its global sections, the isomorphic sheaf, $\underline{\mathrm{Hom}}_{X(\bar{g}_i)}(J_i,\mathcal{O}_{X(\bar{g}_i)})$, is also, so it has a non-zero global section σ_i. The image $\sigma_i(J_i)$ is an Ideal of $\mathcal{O}_{X(\bar{g}_i)}$ and defines a point of $\mathrm{Hilb}^n_{(X/S)}$ in the fiber over $\bar{p}_i$. Since $\mathrm{Hilb}^n_{(X/S)}$ is projective, $\mathcal{A}_{(X/S)}$ is proper, and so its image is closed. Since this image contains each generic point $\bar{p}_i$ of P_m, it is therefore all of P_m. Thus $\mathcal{A}_{(X/S)}$ is surjective.

PROPOSITION (11). *Let X/S be a flat projective family of geometrically integral curves. Then the compactified jacobian P of X/S has geometrically connected fibers and it is proper over S. If, in addition, S is connected, then P is connected.*

PROOF. First note that the second assertion follows from the first, and for the first we may assume S is noetherian. By (10), there are an integer n and a surjective map (isomorphic to the Abel map),

$$\mathrm{Hilb}^n_{(X/S)}\to P.$$

Since $\mathrm{Hilb}^n_{(X/S)}$ has geometrically connected fibers (2), P has geometrically connected fibers. Moreover, since $\mathrm{Hilb}^n_{(X/S)}$ is projective, P is proper over S.

REMARK (12). Theorem (9), Lemma (10) and Proposition (11) hold for each P_j as the proofs in fact really show. The various P_j need not be isomorphic if there is no section of the smooth locus of X/S although they become isomorphic after an étale base change that provides one.

EXAMPLE (13). The result in the theorem is sharp. It fails for a complete intersection X in the projective 3-space $\mathbb{P}^3$ over an algebraically closed field; X is integral with only one singularity.

In fact P, the compactified jacobian of X, is connected (11) and has a component with dimension strictly greater than p_a, while the closure of the jacobian, $\mathrm{Pic}^0_{(X/S)}$, has dimension p_a. So, P can be neither irreducible nor Cohen-Macaulay. In particular, X has torsion-free sheaves of rank 1 which are not limits of invertible sheaves.

Let x be a closed point of $\mathbb{P}^3$, let R denote the local ring of x, let $\mathfrak{m}$ denote the maximal ideal of R, and let d be a positive integer. The homogeneous ideal associated to $\mathfrak{m}^d$ is clearly generated by its homogeneous elements of degree d. So by a form of Bertini's Theorem ([3]), there are two surfaces of degree $(d+1)$ whose intersection contains the thick point, $Z=\mathrm{Spec}\,(R/\mathfrak{m}^d)$, and which is smooth except at x and irreducible. Take X to be this intersection.

For simplicity of computation, we take $d=4e$. We now establish a lower bound,

$$h=\dim\ (\mathrm{Hilb}^n_{(X/S)})\geqq\frac{d^4}{16}=16e^4, \tag{13.1}$$

for a certain integer n satisfying,

$$n\leqq\frac{d^3}{3}. \tag{13.2}$$

Each subspace V of $\mathfrak{m}^{d-1}/\mathfrak{m}^d$ is clearly an ideal of $A=R/\mathfrak{m}^d$; the length of A/V is equal to $n=\binom{d+1}{3}+s$ for $s=\dim(V)$. It follows that $\mathrm{Hilb}^n_{(Z/S)}$ contains $\mathrm{Grass}_s\,(\mathfrak{m}^{d-1}/\mathfrak{m}^d)$. The former is contained in $\mathrm{Hilb}^n_{(X/S)}$, and the latter has dimension

$$s(\dim\,(\mathfrak{m}^{d-1}/\mathfrak{m}^d)-s)=s(2e(4e+1)-s).$$

Take $s=e(4e+1)$. Then (13.1) and (13.2) follow. (This argument is the one found in ([10], p. 75).)

Note in passing that $\mathrm{Div}^n_{(X/S)}$ is open in $\mathrm{Hilb}^n_{(X/S)}$ and its dimension is equal to n by (1, (ii(b))). Therefore, for large d, the scheme $\mathrm{Hilb}^n_{(X/S)}$ is reducible with a component having dimension greater than n, and X contains 0-dimensional subschemes of degree n which are not smoothable or equivalently are not limits of divisors.

To verify the assertions about P, we first find p_a, which (an infinitesimal analysis shows) is equal to the dimension of $\mathrm{Pic}^0_{(X/S)}$. The dualizing sheaf ω_X of X is given by the formulas,

$$\omega_X\cong(\omega_{\mathbb{P}^3}\mid X)\otimes(\Lambda^2N(X,\mathbb{P}^3))^{-1}\cong\mathcal{O}_X(2d-2).$$

It follows that p_a is given by the formula,

$$p_a = (d-1)(d+1)^2 - 1. \tag{13.3}$$

Consider the Abel map,

$$\mathcal{A}_{(X/S)} : \mathrm{Hilb}^n_{(X/S)} \to P_m,$$

where $m = 1 - p_a - n$. Let δ denote the maximal fiber dimension of $\mathcal{A}_{(X/S)}$. Suppose we have proved the estimate

$$\delta \leqq n. \tag{13.4}$$

Then since P_m is isomorphic to P, the dimension h of $\mathrm{Hilb}^n_{(X/S)}$ is estimated by

$$h \leqq \dim(P) + n, \qquad [8; \mathrm{IV}_2, 5.6.7]. \tag{13.5}$$

If P were irreducible, then its dimension would be equal to p_a because $\mathrm{Pic}^0_{(X/S)}$ is open in P. If P had pure dimension p_a, then (13.3) and (13.5) would yield the inequality,

$$h \leqq (d-1)(d+1)^2 - 1 + n.$$

However, for large d, this would contradict (13.1) and (13.2). Therefore, for d large, P is not irreducible, and it contains a component whose dimension is greater than p_a.

It remains to establish (13.4). Let D represent a point of $\mathrm{Hilb}^n_{(X/S)}$, and let $\delta(D)$ be the dimension of the fiber of the Abel map containing this point. Let I denote the Ideal of D. There are formulas,

$$\begin{aligned}\delta(D) &= \dim(\mathrm{Hom}(I \otimes \omega_X, \omega_X)) - 1\\ &= h^1(X, I \otimes \omega_X) - 1;\end{aligned}$$

the first equality holds because ω_X is invertible and the second holds by duality ([1], VIII, 1.15, p. 167). Now, the exact sequence,

$$0 \to I \otimes \omega_X \to \omega_X \to \mathcal{O}_D \to 0,$$

obtained by tensoring with ω_X the sequence relating I and D, yields the estimate,

$$h^1(X, I \otimes \omega_X) \leqq n + h^1(X, \omega_X).$$

This estimate establishes the bound,

$$\delta(D) + 1 \leqq n + 1.$$

Thus, (13.4) holds.

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