INTERSECTION THEORY CLASS 1

RAVI VAKIL

CONTENTS

1. WELCOME!

Hi everyone — welcome to Math 245, Introduction to intersection theory in algebraic geometry. Today I'd like to give you a brief introduction to the subject, and then I'd like to hit the ground running.

Course webpage: http://math.stanford.edu/~vakil/245/. I intend to post notes for most lectures. I make no promises as to their quality; they are basically my notes to myself, slightly prettified when I have time.

Scheduling. We are tentatively considering switching the times of the course to something like Mondays and Wednesdays 9:30-10:50.

About the subject. We'll be using Fulton's book *Intersection Theory*. There are currently copies available at the bookstore, and it's about $45, which is cheap. I should point out that it's one of the few paperbacks you'll see in that Springer series; the author is one of the few people with both the stature and the interest to force Springer to put out a cheaper paperback version.

Intersection theory deals loosely with the following sort of problem: intersect two things of some codimension, get something of expected codimension. We'll basically construct something that looks like homology, and later something that looks like cohomology.

In algebraic geometry: can deal with singular things. Also over other fields. Even in holomorphic category, can get more refined information: 2 points on elliptic curve are the same in homology, but not in "Chow".

Before the 1970's, the field was a mess — there were a great deal of painful ad hoc constructions that people used. Fulton and MacPherson understood the right way to

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describe it, and the result is in this book. In particular, the first 8 chapters are the heart of the subject, and subsume volumes and volumes of earlier work. Each of the subsequent chapters is a different important application; although not every application is important to everyone, every application is important to someone.

I'd like to suggest two readings for you for next day, and they are both light. First, read the MathReview for this book, to get a sense of the context in which these ideas appeared. www.mathscinet.org. Second, take a look at the introduction.

What you need to know: ideally: a first course in algebraic geometry, such as much of Hartshorne Chapter II, plus flatness. However: we can make do, if you're willing to work. The reason he can move so quickly is because of the power embedded in certain algebro-geometric ideas. So we're not going to be able to avoid notions such as flatness, or Cartier divisors. It doesn't matter if you haven't seen Chern classes before; you'll get a definition here.

2. Examples

Before I dive into the subject, let me start with some examples.

The first will get across some notion of *intersection multiplicity*. Also, if you haven't seen schemes or varieties before, this will get your feet wet. Admittedly, I'll let you dip in your toe here, and throw you in the deep end. (Talk to me!)

Picture of parabola $x = y^2$. Project it to t-line. Call the plane $\mathbb{A}^2$. $(x, y) \to x = t$.

Algebra: $K[x, y]$ surjects into $K[x, y]/(x - y^2)$. Affine schemes correspond to rings (the categories are the "same" with the arrows reversed). I'm being agnostic about my field. (I'll try to call it K throughout the course.) Closed immersions correspond to surjections. Ideals correspond to closed subschemes. $K[t]$. Map of rings in the opposite direction $t \mapsto x$.

Let's intersect this with $t = 1$, or equivalently $x = 1$. $K[x, y]/(x - 1)$. Intersecting these schemes corresponds to taking the union of the deals. (Unions of schemes correspond to intersections of ideals.)

$$K[x, y]/(x - y^2, x - 1) \cong K[y]/(y^2 - 1) \cong (K[y]/(y + 1)) \oplus (K[y]/(y - 1))$$

(the latter by the Chinese remainder theorem; here I'm assuming $\operatorname{char} K \neq 2$). Notice how we can see this in the picture. Great, the line meets the parabola at 2 points. If I replace 1 by something else nonzero, then I still get 2 points (assuming K is algebraically closed!).

Complication 1 is not important: for most of this course I'll assume the field is algebraically closed. But for those of you willing to think about nonalgebraically closed fields, like $\mathbb{Q}$, consider

$$\mathbb{Q}[x, y]/(x - y^2, x - 2) \cong \mathbb{Q}[y]/(y^2 - 2).$$

But now $y^2 - 2$ is irreducible! So we get a *single* point, but we want it to count for 2. *What to do?*

Complication 2 is more serious, but you've seen it before: what happens when you intersect with $x = 0$? Then even if you only care about the complex numbers, you definitely only get 1 point. In this case, you also want to say that the multiplicity is 2. Here's the algebra:

$$\mathbb{Q}[x, y]/(x - y^2, x) \cong \mathbb{Q}[y]/(y^2).$$

(In complication 1, the ring is a domain; in the second case it isn't; the ring has a nilpotent y. Those of you who have seen the geometry will know how to draw this.)

Okay, how do we get a consistent answer of 2 no matter which value of x we pick? Answer: we measure the size of the fiber by counting the *dimension* of the ring as a vector space. (That was an important observation that will come up later!) Then in complication 1 we get 2, and in complication 2 we get 2 as well.

Interpretation of complication 2: we get 1 point of multiplicity 2.

Interpretation of complication 1: we get 1 point of multiplicity 1, but that point counts for 2. Bizarre, isn't it?!

Here's a picture that might (or might not) help you deal with complication 1. (Give a triple cover picture.)

So: we have a good way of intersecting things of complimentary codimension meeting at a bunch of points: we intersect the schemes, and count the *length* at those points, which in these cases is a *dimension*.

2.1. Fundamental theorem of algebra. Given a nonzero polynomial $f(x)$ of degree n, the number of zeros, counted appropriately, is n.

Idea of "proof": $K[x]/(f(x))$ is a dimension n vector space.

Solving things corresponds to breaking $f(x)$ into prime factors. Examples:

- $\mathbb{C}[x]/(x(x-1)(x+2)) \cong \mathbb{C}[x]/x \oplus \mathbb{C}[x]/(x-1) \oplus \mathbb{C}[x]/(x+2)$. (using Chinese remainder)
- $\mathbb{C}[x]/x^2(x+1) \cong \mathbb{C}[x]/x^2 \oplus \mathbb{C}[x]/(x+1)$. (One point with multiplicity 2, and one point with multiplicity 1.)
- $\mathbb{R}[x]/x^2(x^2+1) \cong \mathbb{R}[x]/x^2 \oplus \mathbb{R}[x]/(x^2+1)$.
- $\mathbb{Q}[x]/x^2(x^2-2) \cong \mathbb{R}[x]/x^2 \oplus \mathbb{R}[x]/(x^2-2)$.

2.2. Bezout's Theorem. Given two curves in $\mathbb{P}^2$ of degree d and e with no common components, they meet in de points, counted properly. More generally, given n curves in $\mathbb{P}^n$ of degree $d_1, \ldots, d_n$, such that their intersection is zero-dimensional, they meet in $d_1 \cdots d_n$ points, counted properly.

We haven't proved this, but these things were known long ago, certainly before the twentieth century. But here's an example of why people got nervous.

Let X_1 and X_2 be two random planes in $\mathbb{P}^4$. They meet in a point. We can even give them co-ordinates. Say the coordinates on $\mathbb{P}^4$ are $[v; w; x; y; z]$. We'll say that X_1 corresponds to $w = x = 0$ and X_2 corresponds to $y = z = 0$. Then $X_1 \cap X_2$ meet at the point $[1; 0; 0; 0; 0]$. (Remember how projective space works: $[1; 0; 0; 0; 0] = [17; 0; 0; 0; 0]$.)

Let X be the union $X_1 \cup X_2$ in $\mathbb{P}^4$. Then this reasonably has degree 2. Let P be third random plane. Then P meets X_1 in one point, X_2 in another, and misses their intersection, to it meets X in 2 points. So far so good. If we move P around we should always get 2 (so long as $\dim X \cap P = 0$).

But: we get something strange if we put P through $X_1 \cap X_2$: we'll get 3. Here's the calculation. We'll work locally on projective space, on the open set where $v \neq 0$, so we can set $v = 1$. Coordinates on this 4-space are given by w, x, y, z. The open set corresponds to the ring $K[w, x, y, z]$. (You can let $K = \mathbb{C}$.)

X_1 corresponds to the ideal (w, x), and X_2 corresponds to the ideal (y, z). So $X_1 \cup X_2$ corresponds to $(w, x) \cap (y, z)$ (Ask.) $= (wy, wz, xy, xz)$.

Let's say P is given by $w = y$, $x = z$. This indeed meets X in one point. (Where does it meet X_1? $w = y$, $x = z$, $w = x = 0$, so $w = x = y = z = 0$. Where does it meet X_2? similar.)

But let's put our machine to work, and work out the scheme-theoretic intersection. Ideal:

$$K[w, x, y, z]/(wy, wz, xy, xz, w - y, x - z) \cong K[y, z]/(y^2, yz, z^2, yz) \cong K[y, z]/(y^2, yz, z^2).$$

This is a dimension 3 vector space (with basis $1, y, z$).

This was very alarming; Serre figured out what to do. I want to write down the answer, but modern intersection theory bypasses this, so you shouldn't worry about it much — this may even begin to give you a warm feeling in your stomach for the new version of the subject.

Suppose you wanted to intersect two things of complementary dimension in $\mathbb{A}^n$, corresponding to $K[x_1, \ldots, x_n]/I_1$ and $K[x_1, \ldots, x_n]/I_2$ respectively. (Let $R = K[x_1, \ldots, x_n]$ for convenience.) Interpretation of old formula

$$\dim_k \frac{R}{I_1} \otimes \frac{R}{I_2}.$$

Now $\otimes$ is a slightly weird thing. For example, it is right exact. If $0 \to A \to B \to C \to 0$ is exact then we only know that $M \otimes A \to M \otimes B \to M \otimes C \to 0$ is exact.

There is something that could go to the left that would make this look like a long exact sequence in cohomology:

$$\begin{tikzcd}
\arrow[r] & \mathrm{Tor}^2(M,A) \arrow[r] & \mathrm{Tor}^2(M,B) \arrow[r] & \mathrm{Tor}^2(M,C) \arrow[r] & {} \\
\arrow[r] & \mathrm{Tor}^1(M,A) \arrow[r] & \mathrm{Tor}^1(M,B) \arrow[r] & \mathrm{Tor}^1(M,C) \arrow[r] & {} \\
\arrow[r] & M \otimes A \arrow[r] & M \otimes B \arrow[r] & M \otimes C \arrow[r] & 0
\end{tikzcd}$$

$\mathrm{Tor}^0(M, A)$ should be interpreted as $M \otimes A$.

General philosophy h^0 shouldn't behave well in families, but $\sum(-1)^i h^i = \chi$ should. Serre says that the right thing is:

$$\sum (-1)^i \dim_k \mathrm{Tor}^i(\frac{R}{I_1}, \frac{R}{I_2})$$

and he proved it. Magically, in all of our previous examples, all the "higher" Tor's vanished. ("Cohen-Macaulay".) But we were just lucky.

3. Strategy

Here's the strategy we're going to use. Here are things that homology satisfies in "usual" topology in "good circumstances". We have cycles in homology, and cycle classes, which are cycles modulo homotopy.

(1) Two points on a curve are homotopic.
(2) There sometimes a pullback on homology, when the map is a submersion. $\pi : X \to Y$, $\dim X = \dim Y + d$, then $\pi^* : H_n(Y) \to H_{n+d}(X)$.
(3) There is a pushforward in homology by proper morphisms. Proper: image of closed sets is closed. $X \to Y$, $\pi_* : H_n(X) \to H_n(Y)$.

Homology satisfies lots of other things. (In order to make this precise, Robert suggests: allow locally finite chains.)

We'll *define* our version of homology groups, which we'll call *Chow groups*, using this.

(1) Two points on $\mathbb{P}^1$ are defined to be *rationally equivalent*.
(2) If $X \to Y$ is *flat* then there is a pullback. $\pi : X \to Y$, $\dim X = \dim Y + d$, then $\pi^* : H_n(Y) \to H_{n+d}(X)$.
(3) If $X \to Y$ is *proper* (new definition!) then we have a pushforward: $X \to Y$, $\pi_* : H_n(X) \to H_n(Y)$.

We'll require that rational equivalences pullback and pushforward. This will turn out to give an amazing theory! (For example, it will give 2 in that $\mathbb{P}^4$ example without needing Tor's etc.)

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INTERSECTION THEORY CLASS 2

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CONTENTS

The webpage http://math.stanford.edu/~vakil/245/ is up, and has last day's notes.

The new times *starting next week* will be **Mondays 9–10:50** and **Wednesdays 10–10:50**. So there *will be* a class on Friday.

To do: read the summaries of Chapters 1 and 2.

Looking over today's notes, I realize that what will be newest and most disconcerting for those who haven't seen schemes is the fact that we can localize at the generic point of a subvariety X of a scheme Y. What this means is that we are considering the ring of rational functions defined in a neighborhood of the generic point of X in Y; in other words, they are defined on a dense open subset of X. This is indeed a ring (you can add and multiply). The dimension of this ring is the difference of the dimensions of X and Y (or more precisely dimensions of X and "Y near X"). Recall that the points of Y correspond to irreducible subvarieties of Y; the "old-fashioned" ("before schemes") points are the *closed* points in the Zariski topology. So what are the points of $\operatorname{Spec} \mathcal{O}_{X,Y}$, or equivalently, what are the prime ideals of the ring $\mathcal{O}_{X,Y}$? They are the irreducible subvarieties of Y *containing* X. The *maximal* ideal of this local ring corresponds to X itself.

1. LAST DAY

1.1. Examples. I showed you some examples. For example: Parabola $x = y^2$ projected to t-line. $\mathbb{Q}[t] \mapsto \mathbb{Q}[x, y]/(x - y^2)$ via $t \to x$. (I'm letting my field be $\mathbb{Q}$ for the moment.) Intersecting parabola with a vertical line $x = \alpha$. We get the scheme

$$\operatorname{Spec} \mathbb{Q}[x, y]/(x - y^2, x - \alpha) \cong \operatorname{Spec} \mathbb{Q}[y]/(y^2 - \alpha)$$

Date: Wednesday, September 29, 2004.

which is length 2 over the base field $\mathbb{Q}$. If $\alpha = 1$, we get 2 points:

$$\mathbb{Q}[y]/(y^2 - \alpha) \cong (K[y]/(y+1)) \oplus (K[y]/(y-1))$$

If $\alpha = 0$, we get 1 point, with multiplicity 2:

$$\mathbb{Q}[y]/(y^2)$$

has only one maximal ideal. If $\alpha = 2$, we get 1 point with multiplicity 1, but this point has "degree 2 over $\mathbb{Q}$"; the residue field is a degree 2 extension of $\mathbb{Q}$.

1.2. Strategy. We're going to define Chow groups of a variety X as cycles modulo "homotopy" (called *rational equivalence*). Dimension k cycles are easy: they are dimension k subvarieties of X. More subtle is rational equivalence.

(1) Two points on $\mathbb{P}^1$ are defined to be rationaly equivalent.
(2) If $X \to Y$ is *flat* then there is a pullback. $\pi : X \to Y$, $\dim X = \dim Y + d$, then $\pi^* : H_n(Y) \to H_{n+d}(X)$.
(3) If $X \to Y$ is *proper* (new definition!) then we have a pushforward: $X \to Y$, $\pi_* : H_n(X) \to H_n(Y)$.

Just to be clear before we start: throughout this course we'll work over a field, to be denoted K. We'll consider schemes X that are sometimes called *algebraic schemes over* K. They are *schemes of finite type over* K. This means that you get them by gluing together a finite number of affine schemes of the form $\operatorname{Spec} K[x_1, \dots, x_n]/I$. Mild generalization of algebraic variety. All morphisms between algebraic schemes are *separated* and *of finite type*. In this language, a variety is a reduced irreducible algebraic scheme. We'll end up localizing schemes: this leads to the notation of "essentially of finite type" = localizations of schemes/rings of finite type.

2. Zeros and poles

Given a *rational function* on an irreducible variety X, I'll define its order of pole or zero along a codimension 1 variety. (A *rational function* is a(n algebraic) function on some dense (Zariski-)open set.)

An irreducible codimension 1 variety is called a *Weil divisor*.

Example: $(x-1)^2(x^2-2)/(x-3)$ over $\mathbb{C}$. Over $\mathbb{Q}$. Weil divisors.

If X is generically nonsingular=smooth along Weil divisor, then "the same thing will work". More precicsely, in this case the local ring along the subvariety is dimension 1, with $\mathfrak{m}/\mathfrak{m}^2 = 1$, i.e. it is a *discrete valuation ring*, which I'll assume you've seen.

Discrete valuation rings are certain local rings $(A, \mathfrak{m})$. Here are some characterizations:

- an integral domain in which every ideal is principal over K
- a regular local ring of dimension 1
- a dimension 1 local ring that is integrally closed in its fraction field

- etc.

If generator of $\mathfrak{m}$ is π, then the ideals are all of the form (π^n) or 0. The corresponding scheme as 2 points; it is "the germ of a smooth curve".

Examples: $K[x, y]$, localized along divisor $x = 0$. We get rational functions of the form $f(x, y)/g(x, y)$ where x is not a factor of g. This is a local ring, and it is a DVR! Given any rational function, you can tell me the order of poles or zeros. (Ask: $(x^2 - 3y)/(x^2 + x^4 y)$?) Then this also works if x is replaced by some other irreducible polynomial, e.g. $x^2 - 3y$. This is nice and multiplicative.

So what if X is *singular* along that divisor ($\dim \mathfrak{m}/\mathfrak{m}^2 > 1$)? Example: $y^2 = x^3 - x^2$, the rational function y/x.

Exercise. Consider y/x on $y^2 = x^3$. What is the order of this pole/zero? (This will be homework, due date TBA.)

Patch 1: If V is a Weil divisor, and r is a rational function that gives an element of the local ring $\mathcal{O}_{V,X}$, then define

$$\boxed{\operatorname{ord}_V(r) = \dim_K \mathcal{O}_{V,X}/(r)}.$$

(What it means to be in the local ring, intuitively: at a general point of V it is defined. More precisely: there is an open set meeting V — not necessarily containing it — where the rational function is an actual function. For example, x/y on $\operatorname{Spec} K[x, y]$ is defined near the generic point of $x = 0$. Language of *generic points*.) Then given a general rational function, f, we can always write it as $f = r_1/r_2$, where r_1 and r_2 lie in the local ring.

(But we need to check that if we write f as a fraction in two different ways, then the answer is the same. That's true. More on that in a minute.)

Technical problem: If you have a dimension 1 local ring $(A, \mathfrak{m})$ with quotient field K, then A isn't necessarily a K-vector space. $\mathbb{Z}_{(p)}$, $p\mathbb{Z}_{(p)}$. (Exercise: Find an example in characteristic 0.)

Better:

$$\boxed{\operatorname{ord}_V(r) = l_{\mathcal{O}_{V,X}}(\mathcal{O}_{V,X}/(r))}.$$

Recall "length" is the one more than the length of the longest series of nested modules you can fit in a row. So the "length" of a vector space over K is its dimension.

Fact: ord **is well-defined (Appendix A.3):** If $ab = cd$ then $l(A/(a)) + l(A/(b)) = l(A/(c)) + l(A/(d))$. Hence this thing is well-defined.

Facts about facts. (I will pull facts out of Fulton's appendix as black boxes. But if you take a look at the appendix, you'll see that these results are very easy. The vast majority of proofs in A.1–A.5 are no longer than a few lines. With the exception of the section on determinantal identities — which we likely won't use in this course — I think almost no proof is longer than half a page. He even has a crash course in algebraic geometry in Appendix B.)

Fact: finiteness of zeros and poles (Appendix B.4.3). For a given r, there are only a finite number of Weil divisors V where $\operatorname{ord}_V(r) \neq 0$.

3. THE CHOW GROUP

Let X be an algebraic scheme (again: finite type over field K). Recall: A k-cycle is a *finite* formal sum $\sum n_i[V_i]$, $n_i \in \mathbb{Z}$. A cycle is *positive* if all $n_i \geq 0$, some $n_i > 0$. (I forgot to mention this.) Call this $Z_k[X]$, the group of k-cycles.

$$\boxed{Z_k[X] = \left\{ \sum n_i[V_i], \quad n_i \in \mathbb{Z} \right\}.}$$

For any $(k+1)$-dimensional subvariety W of X, and any nonzero rational function $r \in R(W)^*$, define a K-cycle on X by

$$[\operatorname{div}(r)] = \sum \operatorname{ord}_V(r)[V].$$

This generates a subgroup $\operatorname{Rat}_k X$, the subgroup of *cycles rationally equivalent to* 0.

(You can probably see where I'm going to go with this.) Define

$$\boxed{A_k(X) = Z_k[X] / \operatorname{Rat}_k[X]}$$

(Say visually.)

Note: this definition doesn't care about any nonreduced structure on X: $A_k[X] \equiv A_k[X^{\mathrm{red}}]$.

4. PROPER PUSHFORWARDS

Next day we'll see that rational equivalence pushes forward under proper maps. First:

4.1. Crash course in proper morphisms:. A morphism $f : X \to Y$ is said to be *proper* if it is separated (true in our case of algebraic schemes), of finite type (true in our case), and *universally closed*. (Closed: takes closed sets to closed sets. Universally closed: for any $Y' \to Y$, $X \times_Y Y' \to Y'$ is closed.) Key examples: *projective* morphisms are proper. A morphism $f : X \to Y$ is projective if Y can be covered by opens such that on each open U, $f^{-1}(X) \times_Y U \to U$ factors $f^{-1}(X) \times_Y U \hookrightarrow \mathbb{P}^k \times U \to U$ where the left morphism is a *closed immersion*.

Finite morphisms are projective, hence proper. A morphism is *finite* if for each affine open $U = \operatorname{Spec} S$, $f^{-1}(U)$ is affine $= \operatorname{Spec} R$, and the corresponding map of rings $S \to R$ is a finite ring extension, i.e. R is a finitely generated S-*module* (which is stronger than a finitely generated S-*algebra*!). Example: parabola double-covering line. (How to recognize: finite implies each point of target has finite number of preimages. Reverse implication isn't true. finite = proper plus this property.) Another example: closed immersion.

Finite, projective, and proper morphisms are preserved by base change: if f is one of them, then f' is too in the following fiber diagram:

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

(They are also preserved by composition: f, g proper etc. implies $g \circ f$ is too.)

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INTERSECTION THEORY CLASS 3

RAVI VAKIL

CONTENTS

The new times will be **Mondays 9–10:50** and **Wednesdays 10–10:50**. Because this is an advanced course, I won't have office hours; I'm happy to talk about it at any time. My 210 office hours are MW2:05–3 in case you want a specific time when I'll definitely be in my office.

1. LAST DAY

Some comments on last day:

I should have been clearer on what I meant by "numbers of zeros and poles of a rational function $r \in R(X)$ along a Weil divisor (codimension 1 subvariety)." I meant the function $\operatorname{ord}_V(r)$. I defined it as follows. If r is actually defined at the generic point of V, we have

$$\boxed{\operatorname{ord}_V(r) = l_{\mathcal{O}_{V,X}}(\mathcal{O}_{V,X}/(r))}$$

and then we define additively for quotients of two such: $\operatorname{ord}_V(r/s) = \operatorname{ord}_V(r) - \operatorname{ord}_V(s)$. Recall "length" is the one more than the length of the longest series of nested modules you can fit in a row, so the "length" of a vector space over K is its dimension.

The algebraic fact from Fulton shows that this function is well-defined. In doing the following exercise, *use* this definition.

Exercise. Consider y/x on $y^2 = x^3$. What is the order of this pole/zero?

I then defined the Chow group.

$$\boxed{Z_k X = \left\{ \sum n_i [V_i], \quad n_i \in \mathbb{Z} \right\}.}$$

is the group of k-cycles. A cycle is *positive* if all $n_i \geq 0$, some $n_i > 0$. (I may have forgotten to say this.)

Date: Friday, October 1, 2004.

The homotopies, or "rational equivalences" among k-cycles, were generated as follows. For any $(k+1)$-dimensional subvariety W of X, and any nonzero rational function $r \in R(W)^*$, define a K-cycle on X by

$$[\mathrm{div}(r)] = \sum \mathrm{ord}_V(r)[V].$$

This generates a subgroup $\mathrm{Rat}_k X$, the subgroup of *cycles rationally equivalent to* 0.

Then $A_k(X) = Z_k X / \mathrm{Rat}_k X$.

2. Proper, projective, finite

2.1. Proper, projective, finite morphisms. Crash course in proper morphisms: A morphism $f : X \to Y$ is said to be *proper* if it is separated (true in our case of algebraic schemes), of finite type (true in our case), and *universally closed*. (Closed: takes closed sets to closed sets. Universally closed: for any $Y' \to Y$, $X \times_Y Y' \to Y'$ is closed.)

Some pictures: $f : \mathbb{A}^1 \to \mathrm{Spec}\, K$ is not proper. f is certainly separated and of finite type and closed, so what's the problem? Consider the fibered diagram:

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{A}^1 \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d, "f"] \\ \mathbb{P}^1 \arrow[r] & \mathrm{Spec}\, K \end{tikzcd}$$

The projection on the left isn't closed: consider the graph of $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$.

First approximation of how to think of proper morphisms, if you are a complex geometer: fibers are compact in the analytic topology. Warning: $\mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ isn't proper (it isn't closed), so I need to say something a bit more refined.

Key examples: *projective* morphisms are proper. As I said last day, a morphism $f : X \to Y$ is projective if Y can be covered by opens such that on each open U, $f^{-1}(X) \times_Y U \to U$ factors $f^{-1}(X) \times_Y U \hookrightarrow \mathbb{P}^k \times U \to U$ where the left morphism is a *closed immersion*.

Finite morphisms are projective, hence proper. A morphism is *finite* if for each affine open $U = \mathrm{Spec}\, S$, $f^{-1}(U)$ is affine $= \mathrm{Spec}\, R$, and the corresponding map of rings $S \to R$ is a finite ring extension, i.e. R is a finitely generated S-*module* (which is stronger than a finitely generated S-*algebra*!). I'll repeat the example from last time: parabola double-covering line. (How to recognize: finite implies each point of target has finite number of preimages. Reverse implication isn't true. finite = proper plus this property.) Another example: closed immersion.

Third (important) example: normalization (in good cases, such as those we'll consider). This requires a theorem in algebra! Normalization of an affine algebraic scheme $\mathrm{Spec}\, R$ is $\mathrm{Spec}\, \tilde{R}$, where $\tilde{R}$ is the normalization of R in its function field. Normalization of an algebraic scheme X in general is obtained by gluing. (Theorem: this is possible, and also independent of what affine cover you take of X.)

Crash course in normalization: given a variety W, define its normalization as follows. If A is affine, let $\tilde{A}$ be its integral closure in its function field $R(A) = R(W)$. We have $\operatorname{Spec}\tilde{A} \to \operatorname{Spec} A$. Do this for every open affine set of W. Fact: they all glue together. The result is called the normalization. Fact: The normalization map $\tilde{W} \to W$ is finite (algebra fact, Hartshorne Thm I.3.9A), hence proper. Hence: normalizations are regular in codimension 1. (Proof: all local rings are integrally closed; in particular true for dimension 1 rings = codimension 1 subvarieties; hence any dimension 1 local rings $(A, \mathfrak{m})$ is a discrete valuation ring, which (thanks to an earlier crash course) satisfies $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = 1$, which is the definition of nonsingularity.

Finite, projective, and proper morphisms are preserved by base change: if f is one of them, then f' is too in the following fiber diagram:

$$\begin{tikzcd} W \arrow[r, "f'"] \arrow[d] & X \arrow[d] \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

(They are also preserved by composition: f, g proper implies $g \circ f$ is too. Ditto for projective and finite.)

3. Proper pushforwards

3.1. For any subvariety V of X, let $W = f(V)$ be the image; it is closed (image of closed is closed for proper morphisms). I want to define $f_* V$. If $\dim W < \dim V$, define $f_* V = 0$. Otherwise, $R(V)$ is a finite field extension of $R(W)$ (both are field extensions of K of transcendence dgree $\dim V$). Set

$$\boxed{\deg(V/W) = [R(V) : R(W)].}$$

Define $f_* Z_k X \to Z_k Y$ by

$$\boxed{f_*[V] = \deg(V/W)[W]}.$$

Note: If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)_* = g_* f_*$.

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INTERSECTION THEORY CLASS 4

RAVI VAKIL

CONTENTS

Homework due on Monday:

1. Find the order of y/x at origin in $y^2 = x^3$ using the length definition.

2. In no more than half a page, explain why Bezout's Theorem for plane curves is true (i.e. explicate Fulton's Example 1.4.1). Feel free to assume that F is irreducible.

You can also get a "bye" for two weeks of homework by (at some point in the future) explaining to me the "rational equivalence pushes forward under proper morphisms" result (Prop. 1.4).

We're in the process of seeing that cycles (proper) pushforward and (flat) pullback, and that rational equivalences do to.

We need a lot of algebra to set ourselves up. This will decrease in later chapters.

Also, Rob will give Wednesday's class; he'll end Chapter 1 and start Chapter 2.

1. PROPER PUSHFORWARDS

1.1. For any subvariety V of X, let $W = f(V)$ be the image; it is closed (image of closed is closed for proper morphisms). I want to define f_*V. If $\dim W < \dim V$, define $f_*V = 0$. Otherwise, $R(V)$ is a finite field extension of $R(W)$ (both are field extensions of K of transcendence degree $\dim V$). Set

$$\boxed{\deg(V/W) = [R(V) : R(W)].}$$

Date: Monday, October 4, 2004. (Typos corrected Oct. 16.)

In the complex case, this degree is what you think it is: it's the number of preimages of a general point. In positive characteristic, this needn't be true; $K[t^p] \to K[t]$ gives a map of schemes that is degree p but is one-to-one on points.

Define $f_* Z_k X \to Z_k Y$ by

$$\boxed{f_*[V] = \deg(V/W)[W]}.$$

Note: If $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(g \circ f)_* = g_* f_*$.

Example: the parabola example (what happens to points, and to the entire parabola). Normalization.

Big Theorem. If $f : X \to Y$ is a *proper* morphisms, and α is a k-cycle on X which is rationally equivalent to zero, then $f_*\alpha$ is rationally equivalent to zero on Y.

Hence there is a pushforward for Chow groups: $f_* : A_k X \to A_k Y$.

I'm not going to prove this; I'll only point out that we can reduce this statement to something simpler:

Littler theorem. Let $f : X \to Y$ be a proper surjective morphism of varieties, and let $r \in R(X)^*$. Then

(a) $f_*[\operatorname{div}(r)] = 0$ if $\dim Y < \dim X$
(b) $f_*[\operatorname{div}(r)] = [\operatorname{div} N(r)]$ if $\dim Y = \dim X$

In (b), $R(X)$ is a finite extension of $R(Y)$, and $N(r)$ is the *norm* of r.

This is a really natural reduction. We need only to prove it for a generator of rational equivalence, which involves $X' \hookrightarrow X$ of dimension $k+1$, and $\alpha = \operatorname{div}(r)$ for $r \in K(X')$. We can just work on X' instead. We can also replace Y by $f(X)$, because this construction doesn't care about anything else.

So we can now deal with two varieties. ...

Here are some consequences.

Bonus 1. We can now define the *degree* of a dimension 0 cycle class (= cycle mod rational equivalence) on something proper over K. (Definition: *complete = proper over* K. This is a common word, but I may try to avoid it.)

Definition. If $\alpha = \sum n_P P$ is a zero-cycle on X, define the *degree* of α to be $\sum n_P \deg[P/K]$ (the sum of the degree extensions). Example: $\operatorname{Spec} \mathbb{Q}[x]/(x^2+2)$ over $\operatorname{Spec} \mathbb{Q}$, there is one point, that *counts for* 2.

Definition/theorem. If X is a *complete scheme* then define the *degree* of an element of $A_0 X$ to be the degree of the pushforward to a point $\operatorname{Spec} K$. This makes sense by the big theorem.

Homework: As a corollary, "prove" Bezout's theorem for plane curves. Fulton essentially does this in Example 1.4.1, so read what he has to say, and write it up in your own words. (Less than a page is fine.)

Bonus 2. Recall that we were annoyed at having to working out $\operatorname{ord}_{\mathcal{O}_{V,X}}(r)$ for $r \in R(X)$, and needing to use lengths, and not what we know about DVR's. This theorem tells us we don't have to. We could pull r back to the normalization $\tilde{X}$ of X, which is regular in codimension 1. We work out how it vanishes on all the divisors mapping to V. (There are a finite number, by finiteness of normalization, which I said earlier.)

Exercise. Check your answer to $\operatorname{ord}(y/x)$ at $(0,0)$ on $y^2 = x^3$, i.e. $k[x,y]/(y^2 - x^3)$ by pulling it back to the normalization, which is $k[t]$, given by $t \mapsto (t^2, t^3)$.

2. Flat pullback

2.1. Crash course in flat morphisms. A morphism $f : X \to Y$ is *flat* if locally it can be written as $f : \operatorname{Spec} A \to \operatorname{Spec} B$ (so $B \to A$) where A is *flat* B-module. A B-module M is *flat* if for every exact sequence

$$0 \to P \to Q \to R \to 0,$$

the sequence

$$0 \to M \otimes P \to M \otimes Q \to M \otimes R \to 0$$

is also exact. (The only issue is the inclusion $M \otimes P \hookrightarrow M \otimes Q$.)

Idea "flat morphisms are nice". They are more general than fibrations, but have all the same properties.

Easy facts to know:

- flatness is preserved by base change
- anything is flat over a point (as all modules over a field are flat!)
- the composition of flat morphisms is again flat
- open immersions are flat. projections from a vector bundle or $\mathbb{A}^n$-bundles are flat ($R[x_1, \dots, x_n]$ is a flat R-module). The projection $Y \times Z \to Z$ is flat (using base change and the 2nd bullet point).

Harder facts to know:

- A dominant morphism $X \to Y$ from a variety to a smooth curve is flat.
- More generally, a morphism from a scheme to a smooth curve is flat iff all associated points of X map to the generic point of Y.
- Hence: If $X \to Y$ is a morphism of varieties, there is no "dimension-jumping". (Otherwise, if $X \to Y$ has dimension-jumping, basechange to a smooth curve that "sees" the dimension-jumping, and then use this fact.)
- More general fact still: If $X \to Y$ is a morphism of schemes, then associated points of X map to associated points of Y.

Joe asked about another useful fact: in the case of morphisms of finite type, *flat morphisms are open*, i.e. the image of an open set is an open set.

Examples of flat morphisms: the map of the parabola to the line is one. Reason: $k[x]$ is a flat $k[x^2]$-module, as it is a free $k[x^2]$-module.

(Draw also a family of nodal curves.)

Goal: flat pullbacks exist. In other words, we'll define out how cycles pullback, and then we'll check that rational equivalences pull back to rational equivalences.

You can see why we don't like dimensional jumping. But it's interesting that we don't mind degenerations as in the family of nodal curves, or in the parabola example.

Definition. Let Y be a pure k-dimensional scheme, with irreducible components $Y_1, \dots,$ Y_q. Then define the *fundamental cycle* $[Y]$ to be $\sum_1^q m_i[Y_i]$ in $Z_k(Y)$. where m_i is the length of $\mathcal{O}_{Y_i,Y}$. (The local rings $\mathcal{O}_{Y_i,Y}$ are "local Artin rings", corresponding to zero-dimensional local schemes.)

Example: $k[x,y]/(y^2(x+y)^3)$. The length of the local rings at the two generic points are 2 and 3 respectively.

(Note: if Y is a subscheme of X, then $[Y]$ is naturally in $Z_k(X)$ of course; $Z_k[Y] \hookrightarrow \mathbb{Z}_k[X]$ of course.)

Definition (pulling back cycles). Suppose $f : X \to Y$ is flat of relative dimension n. If V is an irreducible subvariety of Y, let $f^*[V] := [f^{-1}(V)]$. Then by linearity, I know how to pull back any linear combination of subvarieties. Hence I've defined $f^* : Z_k Y \to Z_{k+n} X$.

What's wrong with that? Well, we don't know that $(gf)^* = f^* g^*$. Example (picture omitted in notes): branched double cover of a branched double cover. Fortunately we get 4 both ways. But does this work in general?

Lemma (pulling back fundamental classes). If $f : X \to Y$ is flat, then for any equidimensional subscheme Z of Y, $f^*[Z] = [f^{-1}(Z)]$. In other words, the pullback of a fundamental class of a scheme is the fundamental class of the pullback of a scheme.

(Direct algebra from the appendix; omitted.)

This makes us happy, because schemes pullback nicely; $f^{-1}g^{-1}(Z) = (gf)^{-1}(Z)$. Thus pullbacks are functorial.

Proposition (Flat pullback commutes with proper pushforward). Let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

be a fibered square, with g flat and f proper (so g' flat and f' proper). Then $f'_* g'^* \alpha = g^* f_* \alpha$.

This is on the level of *cycles*. We don't yet know that we can flat-pullback cycle classes.

Proof also by direct algebra. Reduce first to the case where X and Y are varieties, and f is surjective. Here are the reductions: it suffices to do this for a generator of $Z_k X$, so $\alpha = [V]$ where V is a variety. $f(V)$ is also a variety (remember f is proper, hence $f(\text{closed})$ is closed). Base change the entire square by $f(V) \to Y$. Then we can assume $f(V) = Y$. Next base change the upper arrow by $V \to X$:

$$\begin{tikzcd} X' \times_X V \arrow[r] \arrow[d] & V \arrow[d] \\ X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Then turn this into a calculation involving local rings (omitted). □

We'll next show that rational equivalences flat-pullback to rational equivalences. Hence we'll have shown that we have flat pullback of Chow groups.

Preliminary Algebraic Lemma. Let X be a purely n-dimensional scheme, with irreducible components $X_1, \dots, X_r$, and geometric multipliciteis $m_1, \dots, m_r$. Let D be an effective Cartier divisor on X. Let $D_i = D \cap X_i$ be the restriction of D to X_i. Then $[D] = \sum m_i [D_i]$ in $Z_{n-1}(X)$.

(An *effective Cartier divisor* is a subscheme locally cut out by a single function that is not a zero-divisor.)

This is certainly reasonable! (Draw a picture, when D doesn't have a component along the intersection of two of the X_i's.) I omitted this explanation in class due to time.

Proof. One checks this along each Weil divisor V of X. Immediately reduces to algebra. Let me get us to the algebra. We'll check that each codimension one subvariety V of X appears with the same multiplicity on both sides of the equation. We reduce to the local situation: let A be the local ring of X along V, and $a \in A$ a local equation for D. The minimal prime ideals p_i in A correspond to the irreducible components X_i of X which contain V.

$m_i = l_{A_{p_i}}(A_{p_i})$. The multiplicity of $[V]$ in $[D]$ is $l_A(A/aA)$. The multiplicity of $[V]$ in $[D_i]$ is $l_{A/p_i}(A/(p_i + aA))$. So we want to show:

$$l_A(A/aA) = \sum m_i l_{A/p_i}(A/p_i + aA).$$

This is shown in the appendix. □ □

Preliminary Geometric Lemma. A cycle α in Z_kX is rationally equivalent to zero if and only if there are $(k+1)$-dimensional subvarieties $V_1, \dots, V_t$ of $X \times \mathbb{P}^1$, such that the projections from V_i to $\mathbb{P}^1$ are dominant, with

$$\alpha = \sum_{i=1}^{t} ([V_i(0)] - [V_i(\infty)])$$

(Draw picture.)

Before I get into it, notice that flatness is already in the picture here: each $V_i \to \mathbb{P}^1$ is flat. We'll see that $[V_i(0)]$ is the flat pullback of 0, and ditto for ∞.

Proof. (I only roughly sketched this proof in class.) If there are such subvarieties, then $\alpha \sim 0$: Certainly the classes on $X \times \mathbb{P}^1$ are each rationally equivalent to 0 by the definition of rational equivalence. The projection $X \times \mathbb{P}^1 \to X$ is proper (because $\mathbb{P}^1 \to pt$ is proper, and properness is preserved by base change).

Now for the other direction. We need to show this for a generator of rational equivalence on X, so there is a subvariety W of dimension $k+1$ in X, and a rational function on W $r \in R(W)^*$. This gives a rational map $W \dashrightarrow \mathbb{P}^1$. Let V be the closure of the graph of this rational map, so $V \subset W \times \mathbb{P}^1 \hookrightarrow X \times \mathbb{P}^1$. (The generic point of W maps to the generic point of $\mathbb{P}^1$, so the same is true of V.) V maps birationally and properly onto W. That morphism is degree 1. If f is the induced morphism to $\mathbb{P}^1$, then $\operatorname{div}(r) = p_*[\operatorname{div}(f)]$ by our big theorem on proper pushforwards, which in turn equals $[V(0)] - [V(\infty)]$. $\square$

Theorem. Let $f : X \to Y$ be flat of relative dimension n, and $\alpha \in Z_k(Y)$ which is rationally equivalent to 0. Then $f^*\alpha$ is rationall equivalent to 0 in $Z_{n+k}X$.

Thus we get *flat pullbacks* $f^* : A_kY \to A_{k+n}X$.

Proof. (I did not give this proof in class.) We may deal with a generator of rational equivalence. Thanks to the geometric lemma, we can take our generator to be of the form $\alpha = [V(0)] - [V(\infty)]$.

$$\begin{tikzcd}
W = \arrow[ddr, "h"'] & (f\times 1)^{-1}(V) \arrow[r, "\text{cl. imm.}"] \arrow[d, "\text{flat}"] & X \times \mathbb{P}^1 \arrow[r, "p", "\text{flat prop.}"'] \arrow[d, "\text{flat}"] & X \arrow[d, "f", "\text{flat}"'] \\
& V \arrow[r, "\text{cl. imm.}"] \arrow[d, "g"] & Y \times \mathbb{P}^1 \arrow[r, "q", "\text{flat prop.}"'] \arrow[dl] & Y \\
& \mathbb{P}^1 & &
\end{tikzcd}$$

We have a cycle α that is rationally equivalent to 0 on Y. It is the proper pushforward of $[g^{-1}(0)] - [g^{-1}(\infty)]$ from V. When we pull back this class from Y to X, we want to see that it is rationally equivalent to 0. But by our lemma showing that proper pushforwards and flat pullbacks commute, that's the same as pulling back to W, and pushing forward to X. The pullback to W is $[h^{-1}(0)] - [h^{-1}(\infty)]$.

We feel like we're done: we just push this forward to X, and that should be it. But: W may not be a variety (it may be reducible and nonreduced), so we don't (yet) know that this class is rationally equivalent to 0. This is why we need our algebraic lemma. Let $[W] = \sum m_i[W_i]$. Since

$$[h_i^{-1}(0)] - [h_i^{-1}(\infty)] = \operatorname{div}(h_i)]$$

is ratinoally equivalent to 0, it suffices to verify that $[h^{-1}(P)] = \sum m_i[h_i^{-1}(P)]$ (and then plug in $P = 0$ and ∞). And that's precisely what the algebraic lemma tells us. $\square$

3. Parsimonious definition of Chow groups

(I discussed this aside rather quickly.)

I'd promised earlier that Chow groups would satisfy 3 conditions: (a) 0 would be rationally equivalence to ∞ in $\mathbb{P}^1$. (b) They would satisfy flat pullbacks. (c) They would satisfy proper pushforward.

We've shown this. Now note that these three things define Chow groups. Translation: anything satisfying these three things is a quotient of the Chow group, so the Chow group is the "minimal" thing satisfying these three conditions. To prove this, all we have to do is show that if W is a $(k+1)$-dimensional subvariety of X, and r is a rational function on W, then $\operatorname{div}(r)$ is forced to be 0. $W \dashrightarrow^{r} \mathbb{P}^1$. Pullback $(0) - (\infty)$. Pushforward by closed immersion $W \to X$.

Something else to point out: the divisor of zeros and poles of a rational function r on a variety W is easy to understand if W is regular in codimension 1. It was a pain otherwise. Here's an alternate way of computing it.

Pull back the function r to $\tilde{W}$. Do the calculation there. Then take proper pushforward.

4. Things Rob will tell you about on Wednesday

4.1. **Excision exact sequence. Proposition.** Let Y be a closed subscheme of X, and let $U = X - Y$. Let $i : Y \hookrightarrow X$ be the closed immersion (proper!) and $j : U \to X$ be the open immersion (flat!). Then

$$A_k Y \xrightarrow{i_*} A_k X \xrightarrow{j^*} A_k U \longrightarrow 0$$

is exact for all k.

(Aside: you certainly expect more on the left!)

Proof. We quickly check that

$$Z_k Y \xrightarrow{i_*} Z_k X \xrightarrow{j^*} Z_k U \longrightarrow 0$$

is exact. (Do it!) Hence we get exactness on the right in our desired sequence. We also get the composition of the two left arrows in our sequence is zero.

Next suppose $\alpha \in Z_k X$ and $j^*\alpha = 0$. That means $j^*\alpha = \sum_i \operatorname{div} r_i$ where each $r_i \in R(W_i)^*$, where W_i are subvarieties of U. So r_i is also a rational function on $R(\overline{W_i})$ where $\overline{W_i}$ is the closure in X. To be clearer, call this rational function $\overline{r}_i$ Hence $j^*(\alpha - \sum[\operatorname{div}(\overline{r}_i)]) = 0$ in $Z_k U$, and hence $j^*(\alpha - \sum[\operatorname{div}(\overline{r}_i)]) \in Z_k Y$, and we're done. □

Rob will also state:

Definition. $Y \to X$ is an *affine bundle of rank* n over X if there is an open covering $\cup U_\alpha$ of X such that $f^{-1}(U_i) \cong U_i \times \mathbb{A}^n \to U_i$. This is a flat morphism.

Proposition. Let $p : E \to X$ be an affine bundle of rank n. Then the flat pullback $p^* : A_k X \to A_{k+n} E$ is surjective for all k.

Proof omitted.

Immediate corollary: $A_k \mathbb{A}^n = 0$ for $k \neq n$.

He may not state the rest:

Exercise: Example 1.9.3 (a). Show that $A_k(\mathbb{P}^n)$ is generated by the class of a k-dimensional linear space. (Hint: use the excision exact sequence.)

Example 1.9.4: Let H be a reduced irreducible hypersurface of degree d in $\mathbb{P}^n$. Then $[H] = d[L]$ for L a hyperplane, and $A_{n-1}(\mathbb{P}^n - H) = \mathbb{Z}/d/Z$. Thus the codimension 1 Chow group is torsion. (Caution: where are you using reduced and irreducible?)

$Z_k X \otimes Z_l Y \to Z_{k+l}(X \times Y)$ by $[V] \times [W] = [V \times W]$.

Proposition.

(a) if $\alpha \sim 0$ then $\alpha \times \beta \sim 0$. There are exterior products $A_k X \otimes A_l Y \to A_{k+l}(X \times Y)$.
(b) If f and g are proper, then so is $f \times g$, and $(f \times g)_*(\alpha \times \beta) = f_*\alpha \times g_*\beta$. Hence exterior product respects proper pushforward.
(c) If f and g are flat of relative dimensions m and n, (so $f \times g$ is flat of relative dimension $m + n$), then
$$(f \times g)^*(\alpha \times \beta) = f^*\alpha \times g^*\beta.$$
Hence exterior product respects flat pullback.

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INTERSECTION THEORY CLASS 6

RAVI VAKIL

CONTENTS

Where we are: proper pushforwards and flat pullbacks. We need a disproportionate amount of algebra to set ourselves up. This will decrease in later chapters.

Last day: Rob proved the *excision exact sequence*:

Proposition. Let Y be a closed subscheme of X, and let $U = X - Y$. Let $i : Y \hookrightarrow X$ be the closed immersion (proper!) and $j : U \to X$ be the open immersion (flat!). Then

$$A_k Y \xrightarrow{i_*} A_k X \xrightarrow{j^*} A_k U \longrightarrow 0$$

is exact for all k.

(Aside: you certainly expect more on the left!)

Proof. We quickly check that

$$Z_k Y \xrightarrow{i_*} Z_k X \xrightarrow{j^*} Z_k U \longrightarrow 0$$

is exact. Hence we get exactness on the right in our desired sequence. We also get the composition of the two left arrows in our sequence is zero.

Next suppose $\alpha \in Z_k X$ and $j^*\alpha = 0$. That means $j^*\alpha = \sum_i \operatorname{div} r_i$ where each $r_i \in R(W_i)^*$, where W_i are subvarieties of U. So r_i is also a rational function on $R(\overline{W_i})$ where $\overline{W_i}$ is the closure in X. To be clearer, call this rational function $\overline{r}_i$ Hence $j^*(\alpha - \sum[\operatorname{div}(\overline{r}_i)]) = 0$ in $Z_k U$, and hence $j^*(\alpha - \sum[\operatorname{div}(\overline{r}_i)]) \in Z_k Y$, and we're done. $\square$

Date: Monday, October 11, 2004.

Definition. $Y \to X$ is an *affine bundle of rank* n over X if there is an open covering $\cup U_\alpha$ of X such that $f^{-1}(U_i) \cong U_i \times \mathbb{A}^n \to U_i$. This is a flat morphism.

Proposition. Let $p : E \to X$ be an affine bundle of rank n. Then the flat pullback $p^* : A_k X \to A_{k+n} E$ is surjective for all k.

Immediate corollary: $A_k \mathbb{A}^n = 0$ for $k \neq n$.

Homework: Example 1.9.3 (a). Show that $A_k(\mathbb{P}^n)$ is generated by the class of a k-dimensional linear space. (Hint: use the excision exact sequence.)

Example 1.9.4: Let H be a reduced irreducible hypersurface of degree d in $\mathbb{P}^n$. Then $[H] = d[L]$ for L a hyperplane, and $A_{n-1}(\mathbb{P}^n - H) = \mathbb{Z}/d/Z$. Thus the codimension 1 Chow group is torsion. (Caution: where are you using reduced and irreducible?)

$Z_k X \otimes Z_l Y \to Z_{k+l}(X \times Y)$ by $[V] \times [W] = [V \times W]$.

Proposition.

(a) if $\alpha \sim 0$ then $\alpha \times \beta \sim 0$. There are exterior products $A_k X \otimes A_l Y \to A_{k+l}(X \times Y)$.
(b) If f and g are proper, then so is $f \times g$, and $(f \times g)_*(\alpha \times \beta) = f_*\alpha \times g_*\beta$. Hence exterior product respects proper pushforward.
(c) If f and g are flat of relative dimensions m and n, (so $f \times g$ is flat of relative dimension $m + n$), then
$$(f \times g)^*(\alpha \times \beta) = f^*\alpha \times g^*\beta.$$
Hence exterior product respects flat pullback.

1. Divisors

There are three related concepts of divisors: Weil divisors, Cartier divisors, and (a concept local to intersection theory) pseudodivisors.

A *Weil divisor* on a variety X is a formal sum of codimension 1 subvarieties.

The notion of Cartier divisor looks more unusual when you first see it. A *Cartier divisor* is defined by data (U_α, f_α) where the U_α form an open covering of X and f_α are non-zero functions in $R(U_\alpha) = R(X)$, subject to the condition that f_α/f_β is a unit (regular, nowhere vanishing function) on the intersection $U_\alpha \cap U_\beta$. There is an equivalence classes of Cartier divisors.

The rational functions are called *local equations*. Local equations are defined up to multiplication by a unit.

Baby Example: $X = U_\alpha = \mathbb{A}^1 - \{1\}$, local equation $1/t^2$. Another local equation for the same Cartier divisor: $(t-1)/t^2$.

1.1. **Crash course in Cartier divisors and invertible sheaves (aka line bundles).** (See Appendix B.4 for an even faster introduction!)

$$\begin{tikzcd}
\operatorname{Pic} X = \{\text{ Car. div.}\} \ / \ \text{lin. equiv.} \arrow[r, leftrightarrow] & \{\text{invertible sheaves}\} \\
= \{\text{Car. div. / princ. Car. div. }\} \arrow[d] & = \{\text{line bundles}\} \arrow[d] \\
\{\text{ Cartier divisors }\} \arrow[r, leftrightarrow] & \{\text{ inv. sheaves w. nonzero rat'l sec. }\}/\text{inv. funcs.}\Gamma(X, \mathcal{O}_X^*)
\end{tikzcd}$$

Given a Cartier divisor (U_α, f_α), here's how you produce an invertible sheaf $\mathcal{F}$. I need to tell you $\mathcal{F}(U)$. $\mathcal{F}(U) = \{(g_\alpha \in \mathcal{O}_X^*(U_\alpha \cap U))_\alpha = R(X) : g_\alpha f_\alpha \in \mathcal{O}_X(U_\alpha \cap U)), g_\alpha f_\alpha = g_\beta f_\beta \in \mathcal{O}_X(U \cap U_\alpha \cap U_\beta)\}$. You can check that this is indeed a sheaf, and it is locally trivial: check for $U = U_\alpha$ that $\mathcal{F}(U_\alpha)$ consists of rational functions g_α on U_α such that $g_\alpha f_\alpha$ is a regular function. Thus the g_α are all of the form $\mathcal{O}(U_\alpha)/f_\alpha$ (regular functions divided by f_α), and thus as a $\mathcal{O}(U_\alpha)$-module, it is isomorphic to $\mathcal{O}(U_\alpha)$ itself.

Baby Example: $X = U_\alpha = \mathbb{A}^1 - \{1\}$, $f_\alpha = 1/t^2$. The rational functions on X are $K[t, 1/(t-1)]$. The module corresponding to this Cartier divisor is $K[t, 1/(t-1)]t^2$, which is clearly isomorphic to $K[t, 1/(t-1)]$.

A Cartier divisor is *effective* if the f_α are all *regular functions* ("have no poles"). Thus we can add to that square above: { effective Cartier divisors } correspond to invertible sheaves with nonzero sections.

A Cartier divisor is *principal* if it is the divisor of a rational function i.e. $\operatorname{div}(r)$ where $r \in R(X)^*$. Two Cartier divisors differing by a principal Cartier divisor give rise to the same invertible sheaf.

Rob told you that the Cartier divisor form an abelian group $\operatorname{Div}(X)$. When you mod out by the subgroup of principal Cartier divisors, you get the group of invertible sheaves $\operatorname{Pic} X$.

The *support* of a Cartier divisor D, denoted $|D|$, is the union of all subvarieties Z of X such that the local equation for D in the ring $\mathcal{O}_{Z,X}$ is not a unit. This is a closed algebraic subset of X of pure codimension one.

Notice: invertible sheaves pull back, but Cartier divisors don't necessarily. (Give an example.)

We have a map from Cartier divisors to Weil divisors. Linear equivalence of Cartier divisors rational equivalence of Weil divisors, hence this map descends to $\operatorname{Pic} X \to A_{\dim X - 1} X$.

1.2. **Pseudo-divisors.** A *pseudo-divisor* on a scheme X is a triple where L is an invertible sheaf on X, Z is a closed subset, and s is a nowhere vanishing section of L on $X - Z$.

As of last day, you know: Pseudo-divisors pull back. And if X is a *variety*, any pseudo-divisor on X is represented by some Cartier divisor on X. (A Cartier divisor D represents a pseudo-divisor (L, Z, s) if $|D| \subset Z$, and there is an isomorphism $\mathcal{O}_X(D) \to L$ which away form Z takes s_D (the "canonical section") to s.) Furthermore, if $Z \neq X$, D is uniquely determined. If $Z = X$, then D is determined up to linear equivalence.

Hence given any pseudo-divisor D, we get a Weil divisor class in $A_{n-1}X$. But we can do better. Given a pseudo-divisor D, we get a Weil divisor class $[D] \in A_{n-1}(|D|)$.

2. Intersecting with divisors

We will now define our first intersections, that with Cartier divisors, or more generally pseudo-divisors. Let D be a pseudo-divisor on a scheme X. We define $D \cdot [V]$ where V is a k-dimensional subvariety. $D \cdot [V] := [j^*D]$ where j is the inclusion $V \hookrightarrow X$. This lies in $A_{k-1}V \cap |D|$. Hence we can do this with any finite combination of varieties. Note that we get a map $Z_kX \to A_{k-1}X$, but we're asserting more: we're getting classes not just on X, but on subsets smaller than X.

Proposition 2.3.

(a) (linearity in α) If D is a pseudo-divisor on X, and α and α' are k-cycles on X, then
$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha' \quad \text{in } A_{k-1}(|D| \cap (|\alpha| \cup |\alpha'|)).$$
(b) (linearity in D) If D and D' are pseudo-divisors on X, and α is a k-cycle on X, then
$$(D + D') \cdot \alpha = D \cdot \alpha + D' \cdot \alpha \quad \text{in } A_{k-1}((|D| \cup |D'|) \cap |\alpha|).$$
(c) (projection formula) Let D be a pseudo-divisor on X, $f : X' \to X$ a proper morphism, α a k-cycle on X', and g the morphism from $f^{-1}(|D|) \cap |\alpha|$ to $|D| \cap f(|\alpha|)$ induced by f. Then
$$g_*(f^*D \cdot \alpha) = D \cdot f_*(\alpha) \quad \text{in } A_{k-1}(|D| \cap f(|\alpha|)).$$
(d) (commutes with flat base change) Let D be a pseudo-divisor on X, $f : X' \to X$ a flat morphism of relative dimension n, α a k-cycle on X, and g the induced morphism from $f^{-1}(|D| \cap |\alpha|)$ to $|D| \cap |\alpha|$. Then
$$f^*D \cdot f^*\alpha = g^*(D \cdot \alpha) \quad \text{in } A_{k+n-1}(f^{-1}(|D| \cap |\alpha|).$$
(e) If D is a pseudodivisor on X whose line bundle $\mathcal{O}_X(D)$ is trivial, and α is a k-cycle on X, then
$$D \cdot \alpha = 0 \quad \text{in } A_{k-1}(|\alpha|).$$

Proof next day.

Example: Intersection of two curves in $\mathbb{P}^2$, C_1 and C_2. We get a number. Old-fashioned intersection theory (Hartshorne V): $\mathcal{O}(C_1)|_{C_2}$ gives you a number.

This tells you a bit more: the class has "local contributions" from each connected component of the intersection.

Excess intersection can happen! Example: A line meeting itself.

Remark: This proves some of the things Fulton said about Bezout in the first chapter.

Here's a natural question: if you intersect two effective Cartier divisors, then if you reverse the order of intersection, you had better get the same thing!

$D \cdot [D'] = D' \cdot [D]$?

Big Theorem 2.4 Let D and D' be Cartier divisors on an n-dimensional variety X. Then $D \cdot [D'] = D' \cdot [D]$ in $A_{n-2}(|D| \cap |D'|)$.

We'll prove this next day, or the day after.

Corollary. Let D be a pseudo-divisor on a scheme X, and α a k-cycle on X which is rationally equivalent to zero. Then $D \cdot \alpha = 0$ in $A_{k-1}|D|$.

Corollary. Let D and D' be pseudo-divisors on a scheme X. Then for any k-cycle α on X,

$$D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha)$$

in $A_{k-2}(|D| \cap |D'| \cap |\alpha|)$.

Hence we can make sense of phrases such as $D_1 \cdot D_2 \cdots D_n \cdot \alpha$.

2.1. **The first Chern class of a line bundle.** If L is a line bundle on X, we define "$c_1(L)\cap$". If V is a subvariety, then write the restriction of L to C as $\mathcal{O}_V(C)$ for some Cartier divisor C. Then define $c_1(L) \cap [V] = [C]$. (C is well-defined up to linear equivalence, so this makes sense in $A_{\dim V-1}V \hookrightarrow A_{\dim V-1}X$.) Extend this by linearity to define $c_1(L)\cap : Z_kX \to A_{k-1}X$.

Proposition 2.5.

(a) If α is rationally equivalent to 0 on X, then $c_1(L) \cap \alpha = 0$. There is therefore an induced homomorphism $c_1(L)\cap : A_kX \to A_{k-1}X$. (That's what we'll usually mean by $c_1(L) \cap \cdot$.)

(b) (commutativity) If L, L' are line bundles on X, α a k-cycle on X, then

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha) \quad \text{in } A_{k-2}X.$$

(c) (projection formula) If $f : X' \to X$ is a proper morphism, L a line bundle on X, α a k-cycle on X', then

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha) \quad \text{in } A_{k-1}X.$$

(d) (flat pullback) If $f : X' \to X$ is flat of relative dimension n, l a line bundle on X, α a k-cycle on X, then

$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha) \quad \text{in } A_{k+n-1}X'.$$

(e) (additivity) If L and L' are line bundles on X, α a k-cycle on X, then

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L' \cap \alpha) \quad \text{and}$$

$$c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha \quad \text{in } A_{k-1}X.$$

We'll prove this next day.

2.2. **Gysin pullback.** Define the *Gysin pullback* as follows. Suppose $i : D \to X$ is an inclusion of an effective Cartier divisor. Define $i^* : Z_k X \to A_{k-1} D$ by

$$i^* \alpha = D \cdot \alpha.$$

Proposition.

(a) If α is rationally equivalent to zero on X then $i^* \alpha = 0$. (Hence we get induced homomorphisms $i^* : A_k X \to A_{k-1} D$.)
(b) If α is a k-cycle on X, then $i_* i^* \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$ in $A_{k-1} X$.
(c) If α is a k-cycle on D, then $i^* i_* \alpha = c_1(N) \cap \alpha$ in $A_{k-1} D$, where $N = i^* \mathcal{O}_X(D)$. N is the normal (line) bundle. (Caution to differential geometers: D could be singular, and then you'll be confused as to why this should be called the normal bundle.)
(d) If X is purely n-dimensional, then $i^*[X] = [D]$ in $A_{n-1} D$.
(e) (Gysin pullback commutes with $c_1(L) \cap$) If L is a line bundle on X, then

$$i^*(c_1(L) \cap \alpha) = c_1(i^* L) \cap i^* \alpha$$

in $A_{k-2} D$ for any k-cycle α on X.

Proof next day; although in fact you may be able to see how all but (d) comes from what we've said earlier today. (Part (d) comes from something we discussed earlier, but I'll leave that for next time.)

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INTERSECTION THEORY CLASS 7

RAVI VAKIL

CONTENTS

1. INTERSECTING WITH A PSEUDODIVISOR

Here's where we are. We have defined divisors of 3 sorts: Weil divisors, Cartier divisors, and pseudo-divisors (L, Z, s).

I'd like to make something more explicit than I have. An *effective* Cartier divisor on a scheme is a closed subscheme locally cut out by *one* function, and that function is not a *zero-divisor*. (Translation: the zero-set does not contain any associated points.) $\operatorname{Pic} X =$ group of line bundles $=$ Cartier divisors modulo linear equivalence $=$ Cartier divisors modulo principal Cartier divisors. We get a map from Cartier divisors to Weil divisors that descends to $\operatorname{Pic} X \to A_{\dim X - 1} X$.

We defined intersection with pseudo-divisors by linearity starting with $D \cdot [V]$, where $j : V \hookrightarrow X$ is a variety, by $D \cdot [V] = [j^* D]$. We'll do three things with this. First (or more correctly, last), this will be leveraged to define more complicated intersections, and to show that they behave well. Second, we'll use this to define the first Chern class of a line bundle, denoted $c_1(L)\cap$. Third, we'll use it to define the Gysin pullback for a closed immersion of an effective Cartier divisor $j : D \hookrightarrow X$.

Proposition 2.3.

(a) (linearity in α) If D is a pseudo-divisor on X, and α and α' are k-cycles on X, then
$$D \cdot (\alpha + \alpha') = D \cdot \alpha + D \cdot \alpha' \quad \text{in } A_{k-1}(|D| \cap (|\alpha| \cup |\alpha'|)).$$
(b) (linearity in D) If D and D' are pseudo-divisors on X, and α is a k-cycle on X, then
$$(D + D') \cdot \alpha = D \cdot \alpha + D' \cdot \alpha \quad \text{in } A_{k-1}((|D| \cup |D'|) \cap |\alpha|).$$

Date: Wednesday, October 13, 2004.

(c) (projection formula) Let D be a pseudo-divisor on X, $f : X' \to X$ a proper morphism, α a k-cycle on X', and g the morphism from $f^{-1}(|D|) \cap |\alpha|$ to $|D| \cap f(|\alpha|)$ induced by f. Then

$$g_*(f^*D \cdot \alpha) = D \cdot f_*(\alpha) \quad \text{in } A_{k-1}(|D| \cap f(|\alpha|)).$$

(d) (commutes with flat base change) Let D be a pseudo-divisor on X, $f : X' \to X$ a flat morphism of relative dimension n, α a k-cycle on X, and g the induced morphism from $f^{-1}(|D| \cap |\alpha|)$ to $|D| \cap |\alpha|$. Then

$$f^*D \cdot f^*\alpha = g^*(D \cdot \alpha) \quad \text{in } A_{k+n-1}(f^{-1}(|D| \cap |\alpha|).$$

(e) If D is a pseudodivisor on X whose line bundle $\mathcal{O}_X(D)$ is trivial, and α is a k-cycle on X, then

$$D \cdot \alpha = 0 \quad \text{in } A_{k-1}(|\alpha|).$$

Proof. (a) This follows from the definition; $D \cdot \alpha$ is linear in the second argument because it was defined by linearity and $D \cdot [V]$ for V a subvariety! Hence for the rest of the proof we can assume $\alpha = [V]$.

(b) Recall the definition of $D \cdot [V]$: We pull the pseudo-Cartier divisor D back to V. We take any Cartier divisor giving that pseudo-divisor (let me sloppily call this D as well). We then take the Weil divisor corresponding to that Cartier divisor: $D \mapsto \sum_W \operatorname{ord}_W(D)$. This latter is a group homomorphism.

(c) It suffices to deal with the case $X' = V$ and $X = f(V)$:

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, "\text{proper}"] & X' \arrow[r, dash] \arrow[d, "\text{proper}"] & V \\ f(V) \arrow[r, hook] & X \arrow[r, dash] & D \end{tikzcd}$$

If we've proved the desired result for the left portion of the above diagram, then we've proved what we wanted.

D can be chosen to be some Cartier divisor on $X = f(V)$. Note that f^*D is *also* Cartier: the support of D doesn't contain the generic point of $F(V)$ hence f^*D doesn't contain the generic point of V. Then we want to prove: $f_*[f^*D] = \deg(X'/X)[D]$. This is a local question on X, so we can assume $D = \operatorname{div}(r)$ for some rational function on X. Then we have:

$$f_*[\operatorname{div}(f^*r)] = [\operatorname{div}(N(f^*r))]$$

(came up in discussion of why proper pushforwards exist)

$$= \operatorname{div}(r^{\deg(X'/X)})$$

(definition of norm)

$$= (\deg X'/X)[\operatorname{div} r].$$

(d) (Skipped for the sake of time) Again we can assume $V = X$, so D is represented by a Cartier divisor. We want to prove $[f^*D] = f^*[D]$ as cycles on X. Both sides are additive, so we need only prove it for the case where D is effective. But we've shown earlier (Lemma in Section 2 of Class 4 notes, Oct. 4) that fundamental classes of subschemes behave well with respective to flat pullbacks, so we're done.

(e) We may assume again that $\alpha = [V]$, $V = X$, and D is a Cartier divisor on X. We know that D is principal. Then we want to show that $[D] = 0$ in $A_{k-1}(X)$. This follows from the fact that there is a group homomorphism from the group of Cartier divisors modulo linear equivalence (i.e. modulo principal divisors) to the group of Weil divisors modulo linear equivalence (the latter is $A_{k-1}(X)$). $\square$

2. The first Chern class of a line bundle

The key result of this chapter is:

Big Theorem 2.4. Let D and D' be Cartier divisors on an n-dimensional variety X. Then $D \cdot [D'] = D' \cdot [D]$ in $A_{n-1}(|D| \cap |D'|)$.

Proof soon.

Given a line bundle L of a scheme X, for any sub*variety* V of X, $L|_V$ is isomorphic to $\mathcal{O}_V(C)$ for some Cartier divisor C on V (determined up to linear equivalence). The Weil divisor $[C]$ determines a well-defined element in $A_{k-1}(X)$, denoted by

$$c_1(L) \cap [V] := [C].$$

We extend this by linearity to get a map $c_1(L)\cap : Z_k X \to A_{k-1} X$.

Proposition 2.5.

(a) If α is rationally equivalent to 0 on X, then $c_1(L) \cap \alpha = 0$. There is therefore an induced homomorphism $c_1(L)\cap : A_k X \to A_{k-1} X$. (That's what we'll usually mean by $c_1(L) \cap \cdot$.)

(b) (commutativity) If L, L' are line bundles on X, α a k-cycle on X, then

$$c_1(L) \cap (c_1(L') \cap \alpha) = c_1(L') \cap (c_1(L) \cap \alpha) \quad \text{in } A_{k-2}X.$$

(c) (projection formula) If $f : X' \to X$ is a proper morphism, L a line bundle on X, α a k-cycle on X', then

$$f_*(c_1(f^*L) \cap \alpha) = c_1(L) \cap f_*(\alpha) \quad \text{in } A_{k-1}X.$$

(d) (flat pullback) If $f : X' \to X$ is flat of relative dimension n, l a line bundle on X, α a k-cycle on X, then

$$c_1(f^*L) \cap f^*\alpha = f^*(c_1(L) \cap \alpha) \quad \text{in } A_{k+n-1}X'.$$

(e) (additivity) If L and L' are line bundles on X, α a k-cycle on X, then

$$c_1(L \otimes L') \cap \alpha = c_1(L) \cap \alpha + c_1(L' \cap \alpha) \quad \text{and}$$

$$c_1(L^\vee) \cap \alpha = -c_1(L) \cap \alpha \quad \text{in } A_{k-1}X.$$

Proof. (a) follows from a corollary to our big theorem that I stated last time: If D is a pseudo-divisor on X, α a k-cycle on X which is rationally equivalent to 0. Then $D \cdot \alpha = 0$ in $A_{k-1}(|D|)$.

(b) follows from another corollary to the big theorem that I stated last day: If D and D' are pseudo-divisors on a scheme X. Then for any k-cycle α on X, $D \cdot (D' \cdot \alpha) = D' \cdot (D \cdot \alpha)$ in $A_{k-2}(|D| \cap |D'| \cap |\alpha|)$.

The remaining three follow immediately from Proposition 2.3 above.

3. Gysin pullback

We also defined the *Gysin pullback*: Suppose $i : D \to X$ is an inclusion of an effective Cartier divisor. Define $i^* : Z_k X \to A_{k-1} D$ by

$$i^* \alpha = D \cdot \alpha.$$

Proposition.

(a) If α is rationally equivalent to zero on X then $i^*\alpha = 0$. (Hence we get induced homomorphisms $i^* : A_k X \to A_{k-1} D$.)
(b) If α is a k-cycle on X, then $i_* i^* \alpha = c_1(\mathcal{O}_X(D)) \cap \alpha$ in $A_{k-1} X$.
(c) If α is a k-cycle on D, then $i^* i_* \alpha = c_1(N) \cap \alpha$ in $A_{k-1} D$, where $N = i^* \mathcal{O}_X(D)$. N is the normal (line) bundle. (Caution to differential geometers: D could be singular, and then you'll be confused as to why this should be called the normal bundle.)
(d) If X is purely n-dimensional, then $i^*[X] = [D]$ in $A_{n-1} D$.
(e) (Gysin pullback commutes with $c_1(L) \cap$) If L is a line bundle on X, then

$$i^*(c_1(L) \cap \alpha) = c_1(i^* L) \cap i^* \alpha$$

in $A_{k-2} D$ for any k-cycle α on X.

Proof. (a) follows from the first corollary last time.
(b) follows from the definition: both are $D \cap \alpha$, as a class on X.
(c) too: both are $D \cap \alpha$, but as a class on D.
(d) says that $[D] = D \cdot [X]$, which we proved earlier, although you may not remember it.
(e) follows from the second corollary from last time. □

4. Towards the proof of the big theorem

Big theorem. Let D and D' be Cartier divisors on an n-dimensional variety X. Then $D \cdot [D'] = D' \cap [D]$ in $A_{n-2}(|D| \cap |D'|)$.

The case where D and D' have no common components, so $|D| \cap |D'|$ is codimension 2, boils down to algebra, and the details are thus omitted here. Here's how it boils down to algebra: $A_{n-2}(|D| \cap |D'|) = Z_{n-w}(|D| \cap |D'|)$, so rational equivalence doesn't come into it. This is a local question, so we can consider a particular codimension 2 point, and then consider an affine neighborhood of that $\operatorname{Spec} A$. Upon localizing at that point, we have a question about a dimension 2 local ring A_P.

So the real problem is what to do if D and D' have a common component. We'll deal with this by induction on this:

$$\epsilon(D, D') := \max\{\operatorname{ord}_V(D) \operatorname{ord}_V(D') : \operatorname{codim}(V, X) = 1\}.$$

Note that we know the result when $\epsilon = 0$.

The proof involves an extremely clever use of blowing up.

4.1. **Crash course in blowing up.** I'm going to repeat this next time, in more detail. Let X be a scheme, and $\mathcal{I} \subset \mathcal{O}_X$ a sheaf of ideals on X. (Technical requirement automatically satisfied in our situation: $\mathcal{I}$ should be a coherent sheaf, i.e. finitely generated.) Here is the "universal property" definition of blowing-up. Then the blow-up of $\mathcal{O}_X$ along $\mathcal{I}$ is a morphism $\pi : \tilde{X} \to X$ satisfying the following universal property. $f^{-1}\mathcal{I}\mathcal{O}_{\tilde{X}}$ (the "inverse ideal sheaf") is an invertible sheaf of ideals, i.e. an effective Cartier divisor, called the *exceptional divisor*. (Alternatively: the scheme-theoretic pullback of the subscheme $\mathcal{O}/\mathcal{I}$ is a closed subscheme of $\tilde{X}$ which is (effective) Cartier, and this is called the exceptional (Cartier) divisor E.) If $f : Z \to X$ is any morphism such that $(f^{-1}\mathcal{I})\mathcal{O}_Z$ is an invertible sheaf of ideals on Z (i.e. the pullback of $\mathcal{O}/\mathcal{I}$ is an effective Cartier divisor), then there exists a unique morphism $g : Z \to \tilde{X}$ factoring f.

$$\begin{tikzcd} Z \arrow[r, dashed, "g"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

In other words, if you have a morphism to X, which, when you pull back the ideal I, you get an effective Cartier divisor, then this factors through $\tilde{X} \to X$.

As with all universal property statements, any two things satisfying the universal property are canonically isomorphic.

Theorem: Blow-ups exist. The proof is by construction: show that $\mathbf{Proj} \oplus_{d \geq 0} \mathcal{I}^d$ satisfies the universal property. (See Hartshorne II.7, although his presentation is opposite.)

This construction shows that in fact π is projective (hence proper).

I'm going to start next day by discussing three examples: (i) blowing up a point in the plane, (ii) blowing up along an effective Cartier divisor, and (iii) blowing up X along itself.

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INTERSECTION THEORY CLASS 8

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CONTENTS

1. PROOF OF KEY RESULT OF CHAPTER 2

Our goal now is to prove the key result of Chapter 2. It's not impressive in and of itself, but we used it to do a lot of other things.

Big Theorem 2.4. Let D and D' be Cartier divisors on an n-dimensional variety X. Then $D \cdot [D'] = D' \cdot [D]$ in $A_{n-1}(|D| \cap |D'|)$.

Last time, I discussed the case where D and D' have no common components, so $|D| \cap |D'|$ is codimension 2. I didn't prove it, but argued that it boils down to algebra. So the real problem is what to do if D and D' have a common component.

The proof involves an extremely clever use of blowing up. Given the background of the people in this class, I've had to make some decisions as to what arguments to include, and I think I'd most like to give you some feeling for blowing up, and then to outline the proof, rather than getting into the gory details.

1.1. **Crash course in blowing up.** Last time I began to talk about blowing up. Let X be a scheme, and $\mathcal{I} \subset \mathcal{O}_X$ a sheaf of ideals on X. (Technical requirement automatically satisfied in our situation: $\mathcal{I}$ should be a coherent sheaf, i.e. finitely generated.) Here is the "universal property" definition of blowing-up. Then the blow-up of $\mathcal{O}_X$ along $\mathcal{I}$ is a morphism $\pi : \tilde{X} \to X$ satisfying the following universal property. $f^{-1}\mathcal{I}\mathcal{O}_{\tilde{X}}$ (the "inverse ideal sheaf") is an invertible sheaf of ideals, i.e. an effective Cartier divisor, called the *exceptional divisor*. (Alternatively: the scheme-theoretic pullback of the subscheme $\mathcal{O}/\mathcal{I}$ is a closed subscheme of $\tilde{X}$ which is (effective) Cartier, and this is called the exceptional (Cartier) divisor E.) If $f : Z \to X$ is any morphism such that $(f^{-1}\mathcal{I})\mathcal{O}_Z$ is an invertible sheaf

Date: Monday, October 18, 2004.

of ideals on Z (i.e. the pullback of $\mathcal{O}/\mathcal{I}$ is an effective Cartier divisor), then there exists a unique morphism $g : Z \to \tilde{X}$ factoring f.

$$\begin{tikzcd} Z \arrow[r, dashed, "g"] \arrow[dr, "f"'] & \tilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

In other words, if you have a morphism to X, which, when you pull back the ideal I, you get an effective Cartier divisor, then this factors through $\tilde{X} \to X$.

As with all universal property statements, any two things satisfying the universal property are canonically isomorphic.

Theorem: Blow-ups exist. The proof is by construction: show that $\mathbf{Proj} \oplus_{d \geq 0} \mathcal{I}^d$ satisfies the universal property. (See Hartshorne II.7, although his presentation is opposite.)

This construction shows that in fact π is projective (hence proper).

Example 1. The "typical" first example is the blow-up of the plane at a point, $\mathrm{Bl}_0\mathbb{A}^2$. Let $X = \{(p \in \mathbb{A}^2, \ell$ line in plane through p and $0\,\}$. Note that (i) X is smooth (it is an $\mathbb{A}^1$-bundle = total space of a line bundle over the $\mathbb{P}^1$ parametrizing the possible ℓ), (ii) it has a map π to $\mathbb{A}^2$, (iii) π is an isomorphism away from p, and $\pi^{-1}p \cong \mathbb{P}^1$. This $\mathbb{P}^1$ is codimension 1 on a smooth space, hence an effective Cartier divisor. Fact: This satisfies the universal property, hence is a blow-up. More generally, if you blow up a point on a smooth surface, the same story happens. More generally still, if you blow up a smooth variety X along a smooth subvariety V of codimension k, you get something that is isomorphic away from V, and the preimage of V is a $\mathbb{P}^{k-1}$-bundle over V; it is the projectivized normal bundle (i.e. points of the exceptional divisor E correspond to points of V along with a line in the normal bundle to V in X.)

Weirder things can happen.

Example 2. If you blow up X along an effective Cartier divisor D, then nothing changes. $(X, D) \to X$ already satisfies the universal property, tautologically.

Example 3. If you blow up X along itself, it disappears. For example, consider $X = \mathbb{A}^1$, and $\mathcal{I} = 0$. Then there is *no way* to pullback this ideal sheaf and get a Cartier divisor, which is codimension 1. Well, there *is* one way: via the morphism $\emptyset \to X$.

Example 3a. If you blow up X along one of its components, the component is blown away (disappears), and the rest will be affected too (blown up along their intersection with the old component).

Fun Example 4. Consider the cone, and blow it up along a line. The line is not a Cartier divisor, as we showed last day. Hence the blow-up does *something*. Moreover, it does nothing away from cone point. It turns out that this does indeed smooth out the cone! (It does the same thing as blowing up the cone point by itself.)

Remark. If X is a variety and $Y \neq X$, then $\tilde{X} \to X$ is birational.

1.2. **Back to the proof.** D and D' are two Cartier divisors, cut out locally by a single equation. Let $D \cap D'$ be the intersection scheme of D and D'. Let $\pi : \tilde{X} \to X$ be the blow-up of X along $D \cap D'$, and let $E = \pi^{-1}(D \cap D')$ be the exceptional divisor. The local equations for $\pi^* D$ and $\pi^* D'$ are divisible by the local equation for E. Translation: D and D' both lie in the ideal sheaf of $D \cap D'$, hence their pullback lies in the (Cartier) ideal sheaf of E. Hence we can write equalities of Cartier divisors:

$\pi^* D = E + C, \pi^* D' = E + C'$.

Let

$$\epsilon(D, D') := \max\{\operatorname{ord}_V(D) \operatorname{ord}_V(D') : \operatorname{codim}(V, X) = 1\}.$$

Note that we know the result when $\epsilon = 0$. We're going to work by induction on ϵ.

Omitted Lemma. (a) C and C' are disjoint. (This is a special case of Hartshorne Exercise II.7.12.) (b) If $\epsilon(D, D') > 0$, then $\epsilon(C, E), \epsilon(C', E) < \epsilon(D, D')$.

Proof is omitted. But caution: something very interesting is going on here. I'll give three examples to show you this. First, suppose L_1, L_2, and L_3 are three general lines in $\mathbb{P}^2$. If $D = L_1$ and $D' = L_1 + L_3$, then $D \cap D' = L_1$, and the blow-up does nothing. However, $E = L_1$, and then $C = \emptyset$ and $C' = L_3$.

Next, suppose $D = L_1 + L_2$ and $D' = L_1 + L_3$. Then the trouble occurs because $D \cap D'$ includes L_1. But the blow-up does something else; it blows up $L_2 \cap L_3$. Let E_{23} be the exceptional divisor of the blow-up of $L_2 \cap L_3$. Then the exceptional divisor of the blow-up that *we* care about is $L_1 + E_{23}$. Then we get C is the proper transform of L_2 and C' is the proper transform of L_3.

Finally, suppose $D = 2L_1 + L_2$ and $D' = L_1 + L_3$. Then the scheme-theoretic intersection $D \cap D'$ consists of the point $L_2 \cap L_3$, as well as L_1, *but also* some additional "fuzz" where L_1 meets L_3! When you blow this up, what happens? (Well, I can tell you what happens in this case — it's the same as blowing up the two points $L_1 \cap L_3$ and $L_2 \cap L_3$ — but in general this is quite complicated. I find it fascinating that we don't ever have to know precisely what happens to prove this lemma.)

Lemma. If D, D' are Cartier divisors on X, $\pi : \tilde{X} \to X$ is a proper birational morphism of varieties, $\pi^* D = B \pm C$, $\pi^* D = B' \pm C'$, for Cartier divisors B, C, B', C' on $\tilde{X}$ with $|B| \cup |C| \subset \pi^{-1}(|D|)$, $|B'| \cup |C'| \subset \pi^{-1}(|D'|)$, and the theorem holds for each pair (B, B'), (B, C'), (C, B'), (C, C') on $\tilde{X}$, then the theorem holds for (D, D') on X.

Proof.

$$\begin{aligned} D \cdot [D'] &= \pi_*((B \pm C) \cdot [B' \pm C']) \quad \text{(projection formula, note } \pi_*([B' \pm C']) = [D']) \\ &= \pi_*(B \cdot [B'] \pm B \cdot [C'] \pm C \cdot [B'] \pm C \cdot [C']) \quad \text{(linearity)} \\ &= \pi_*(B' \cdot [B] \pm C' \cdot B \pm [B'] \cdot [C] \pm C' \cdot [C]) \quad \text{(hypothesis)} \\ &= \pi_*((B' \pm C') \cdot [B \pm C]) \quad \text{(linearity)} \\ &= D' \cdot [D] \quad \text{(projection formula)} \end{aligned}$$

□

Now let's finish off the proof of the big theorem.

Case D *and* D' *effective.* We do this by induction on $\epsilon(D, D')$. The case $\epsilon = 0$ is already done (or more precisely, assumed!), as described earlier. If $\epsilon(D, D') > 0$, then blow up X along $D \cap D'$. Then the omitted lemma asserts that the theorem holds for (E, C') and (C, E). The theorem also holds for (E, E) stupidly (clearly $E \cdot [E] = E \cdot [E]$), and also for (C, C') for different stupid reasons ($C \cdot [C'] = 0 = C' \cdot [C]$). So the above lemma completes this proof.

Case D' *effective.* Let $\mathcal{J}$ be the ideal sheaf of denominators of D. (Translation: locally, on an open set $\operatorname{Spec} A$, is consists of those functions who, when multiplied by the generator of D in $R(X)$, turn it into a regular function.) Blow up X along $\mathcal{J}$. Then $\pi^* D = C - E$ where E is the exceptional divisor, and C is an effective Cartier divisor. Then the previous case covers $(C, \pi^* D')$ and $(E, \pi^* D')$ on $\tilde{X}$, so we're done by the Lemma.

General case. Blow up X along the ideal sheaf of denominators of D'. Then the pairs $(\pi^* D, C)$ and $(\pi^* D, E)$ are covered by the previous case, so we're done by the Lemma. □

2. Vector bundles, and Segre and Chern classes

In the next chapter, we're going to generalize the notion of the first Chern class of a line bundle to the notion of an arbitrary Chern class on an arbitrary vector bundle. These Chern classes will have similar properties to those you may have seen elsewhere, but we get at them in a strangely backwards way, by defining Segre classes first. The generating function for Segre classes will be inverse to that of Chern classes.

When you look through this chapter, you'll note that only a very small portion of it consists of propositions and theorems. The rest is full of useful examples.

2.1. **Segre classes of vector bundles.** Let E be a vector bundle of rank $e+1$ on an algebraic scheme X. Let $P = \mathbb{P}E$ be the $\mathbb{P}^e$-bundle of lines on E, and let $p = p_E : P \to X$ be the projection. Note that it is both flat and proper (explain).

The line bundle $\mathcal{O}(1)$. On P there is a canonically defined line bundle, called the *tautological bundle*, denoted $\mathcal{O}(-1)$ or $\mathcal{O}_E(-1)$. For any point of P, I'll need to give you a one-dimensional vector space in some natural way. But each point of P corresponds to a line of E.

Define $\mathcal{O}(1)$ as the dual of $\mathcal{O}(-1)$, and let $\mathcal{O}(n)$ be $\mathcal{O}(1)^{\otimes n}$ (with the obvious convention if n is nonpositive).

Here's a second "definition" of $\mathcal{O}(1)$. This is somewhat informal; making it precise it a bit inefficient. Define the "projective completion" of E to be the projective bundle "compactifying" E. As sets, it is $E \coprod \mathbb{P}E$. It can also be described as $\mathbb{P}(E + \mathbf{1})$ where $\mathbf{1}$ is the trivial line bundle. ($\mathbf{1}$ is slightly unfortunate notation; but I'm following Fulton.) It is a $\mathbb{P}^{e+1}$-bundle. On it, $\mathbb{P}E$ is an effective Cartier divisor, and this divisor class is $\mathcal{O}_{\mathbb{P}(E+\mathbf{1})}(1)$. Restricting this divisor class to $\mathbb{P}E$ gives $\mathcal{O}_{\mathbb{P}E}(1)$. (Note that this is not automatically an effective Cartier divisor class on $\mathbb{P}E$.)

Remark. On $\mathbb{P}^e$, there is a line bundle / invertible sheaf $\mathcal{O}(1)$, and indeed $\mathcal{O}_E(1)$ restricts to each of the fibers to give $\mathcal{O}(1)$. But this doesn't determine the class $\mathcal{O}_E(1)$. Indeed, if I pull back any line bundle on X to P, I get a line bundle trivial on each of the fibers, so $\mathcal{O}_E(1) \otimes \mathcal{L}$ has this property for any invertible sheaf $\mathcal{L}$.

Definition. Define homomorphisms

$$s_i(E)\cap : A_k X \to A_{k-i} X$$

by $\alpha \mapsto p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^*\alpha)$. Note that this indeed maps from $A_k X \to A_{k-i} X$.

Warm-up proposition. (First Segre class of a line bundle) If E is a line bundle on X, $\alpha \in A_* X$, then

$$s_1(E) \cap \alpha = -c_1(E) \cap \alpha.$$

Proof. In this case $\mathbb{P}E = X$, and $\mathcal{O}_E(-1) = E$ so $\mathcal{O}_E(1) = E^\vee$, hence $s_1(E) \cap \alpha = c_1(\mathcal{O}_E(1) \cap \alpha) = -c_1(E) \cap \alpha$. ◻

Segre class Theorem. (a) for all $\alpha \in A_k X$, (i) $s_i(E) \cap \alpha = 0$ for $i < 0$, and (ii) $s_0(E) \cap \alpha = \alpha$.

(b) (commutativity) If E and F are vector bundles on X, and $\alpha \in A_k X$, then for all i, j,

$$s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha).$$

(c) (Segre classes behave well with respect to proper pushforward) If $f : X' \to X$ is proper, E a vector bundle on X, $\alpha \in A_* X'$, then for all i,

$$f_*(s_i(f^*E) \cap \alpha) = s_i(E) \cap f_*(\alpha).$$

(d) (Segre classes behave well with respect to flat pullback) If $f : X' \to X$ is flat, E a vector bundle on X, $\alpha \in A_* X$

$$s_i(f^*E) \cap f^*\alpha = f^*(s_i(E) \cap \alpha).$$

Corollary. The flat pullback $p^* : A_k X \to A_{k+e}(\mathbb{P}E)$ is a split monomorphism: by (a) (ii), an inverse is $\beta \mapsto p_*(c_1(\mathcal{O}_E(1))^e \cap \beta)$.

Corollary. It makes sense to multiply by various polynomials in Segre classes of various bundles, by the commutativity part (b).

Proof of theorem. I'll prove a smattering of these.

(c) Suppose $f : X' \to X$ is proper, E a vector bundle on X. There is a fibre square

$$\begin{tikzcd} \mathbb{P}(f^*E) \arrow[r, "f'"] \arrow[d, "p'"] & \mathbb{P}E \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

with $f'^*\mathcal{O}_E(1) = \mathcal{O}_{f^*E}(1)$. (All morphisms here are proper, the top one because proper morphisms are preserved by fibred squares.) Then

$$\begin{aligned} f_*(s_i(f^*E) \cap \alpha) &= f_* p'_*(c_1(\mathcal{O}_{\mathbb{P}f^*E}(1))^{e+i} \cap p'^*\alpha) \quad \text{(def'n of } s_i\cap) \\ &= p_* f'_*(c_1(f'^*\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap p'^*\alpha) \quad \text{(commutativity of proper pushforwards)} \\ &= p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap f'_* p'^*\alpha) \\ &\qquad \text{(proj. formula for } c_1\text{, i.e. behaves well w.r.t. pr. push.)} \\ &= p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap p^* f_*\alpha) \quad \text{(pr. push. and flat pull. commute)} \\ &= s_i(E) \cap f_*\alpha \quad \text{(def'n of } s_i\cap) \end{aligned}$$

(d) **Exercise.**

(a) We may assume that $\alpha = [V]$. Then by (c), using the (proper) closed immersion $V \hookrightarrow X$, we may assme $X = V$. Then for $i < 0$, $s_i(E) \cap [V] \in A_{\dim V - i} X = 0$, so (i) is done. Similarly,

$$s_0(E) \cap [V] = p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^e \cap [P]) = m[V]$$

for some m. We will show that $m = 1$. We can check this on an open set of V, so restrict to an open set where E is a trivial bundle. Then $P = \mathbb{P}E = X \times \mathbb{P}^e$, and $\mathcal{O}(1)$ has sections whose zero scheme is $X \times \mathbb{P}^{e-1}$. Then $c_1(\mathcal{O}(1)) \cap [X \times \mathbb{P}^e] = [X \times \mathbb{P}^{e-1}]$ (from earlier theorem on c_1 of an effective Cartier divisor). Repeat this e times to get the desired result.

(b) next day...

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INTERSECTION THEORY CLASS 9

RAVI VAKIL

CONTENTS

I have one update from last time, and this is aimed more at the experts. Rob pointed out that there was no reason that we know that a Cartier divisor can be expressed as a difference (or quotient) of effective Cartier divisors. More precisely, a Cartier divisor can be described cohomologically as follows. Let X be a scheme. We have a sheaf $\mathcal{O}^*$ of invertible functions. There is another sheaf $\mathcal{K}^*$ that are things that locally look like quotients of a function by a nonzerodivisor. (If X is a variety, then $\mathcal{K}^*$ is the constant sheaf with $\mathcal{K}^*(U) = R(X)$ for all U.) Then I informally described Cartier divisors of X as determined by certain data: there is an open cover of X by open sets U_i; we have an element of $\mathcal{K}^*$ for each U_i; and on $U_i \cap U_j$ the quotient of the two elements of $\mathcal{K}^*$ corresponding to i and j is an element of $\mathcal{O}_X^*$. We then mod out by an equivalence relation that I was careless about defining. This definition translates to the more compact notation: Cartier divisors are global sections of the (quotient) sheaf $\mathcal{K}^*/\mathcal{O}_X^*$. (More generally, we get a sheaf of Cartier divisors $\mathcal{K}^*/\mathcal{O}_X^*$.) The description I gave was the Cech description of a quotient sheaf. This drives home the point that any Cartier divisor is *locally* the quotient of two effective Cartier divisors, but not necessarily globally. I don't know of any specific examples of a Cartier divisor that is not the quotient/difference of two effective Cartier divisors, and I would like to see one.

Fulton is then proving that even though we don't know for sure that any Cartier divisor D on X is a difference of two effective divisors, we can construct a proper surjective morphism $\pi : \tilde{X} \to X$ such that $\pi^* D$ is Cartier, and a difference of effective Cartier divisors. Moreover, he tells us what to do: define a closed subscheme by taking the "ideal sheaf of denominators" of the Cartier divisor, and blow it up. The example I said I'd like to see corresponds to the question: find a scheme and a Cartier divisor where this ideal sheaf of denominators is not Cartier. I'm still a bit perplexed; it seems to me that it should always be Cartier, as "Cartier-ness" is a local condition, and locally every Cartier divisor is principal. (I'm assuming, as we are throughout this course, that all schemes are essentially of finite type, so the Cech description certainly holds.)

Date: Wednesday, October 20, 2004.

As an aside: when you see that Cartier divisors are global sections of a quotient sheaf, you should immediately be curious about the corresponding long exact sequence of cohomology.

$$0 \to \mathcal{O}^* \to \mathcal{K}^* \to \mathcal{K}^*/\mathcal{O}^* \to 0$$

gives us

$$0 \to H^0(X, \mathcal{O}^*) \to H^0(X, \mathcal{K}^*) \to H^0(X, \mathcal{K}^*/\mathcal{O}^*) \to H^1(X, \mathcal{O}^*) \to H^1(X, \mathcal{K}^*) = 0.$$

The right term is 0 because $\mathcal{K}$ is a flasque (=flabby) sheaf. All the other terms have obvious meanings too. The image of $H^0(X, \mathcal{K}^*)$ is the set of principal Cartier divisors. (An element of $H^0(X, \mathcal{O}^*)$ gives a trivial principal divisor.) $H^1(X, \mathcal{O}_X^*) = \operatorname{Pic} X$. So this shows that $\operatorname{Pic} X \cong$ Cartier divisors modulo principal divisors.

1. Vector bundles, and Segre and Chern classes

1.1. **Segre classes of vector bundles.** Let E be a vector bundle of rank $e+1$ on an algebraic scheme X. Let $P = \mathbb{P}E$ be the $\mathbb{P}^e$-bundle of lines on E, and let $p = p_E : P \to X$ be the projection. Note that it is both flat and proper (explain).

Define homomorphisms

$$s_i(E) \cap : A_k X \to A_{k-i} X$$

by $\alpha \mapsto p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^*\alpha)$. Note that this indeed maps from $A_k X \to A_{k-i} X$.

Segre class Theorem. (a) for all $\alpha \in A_k X$, (i) $s_i(E) \cap \alpha = 0$ for $i < 0$, and (ii) $s_0(E) \cap \alpha = \alpha$.

(b) (commutativity) If E and F are vector bundles on X, and $\alpha \in A_k X$, then for all i, j,

$$s_i(E) \cap (s_j(F) \cap \alpha) = s_j(F) \cap (s_i(E) \cap \alpha).$$

(c) (Segre classes behave well with respect to proper pushforward) If $f : X' \to X$ is proper, E a vector bundle on X, $\alpha \in A_* X'$, then for all i,

$$f_*(s_i(f^*E) \cap \alpha) = s_i(E) \cap f_*(\alpha).$$

(d) (Segre classes behave well with respect to flat pullback) If $f : X' \to X$ is flat, E a vector bundle on X, $\alpha \in A_* X$

$$s_i(f^*E) \cap f^*\alpha = f^*(s_i(E) \cap \alpha).$$

(a) and (c) proved last time. (d) **Exercise.** Before proving (b), let me mention some useful consequences.

Corollary. The flat pullback $p^* : A_k X \to A_{k+e}(\mathbb{P}E)$ is a split monomorphism: by (a) (ii), an inverse is $\beta \mapsto p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^e \cap \beta)$.

(We're going to use this soon in the proof of the splitting principle, so don't forget this. We'll also soon see that $A_m \mathbb{P}E \cong \oplus_{i=0}^e A_{m-i} X$. The inclusion $A_{m-i} X \hookrightarrow A_m \mathbb{P}E$ will be given by $c_1(\mathcal{O}_{\mathbb{P}E}(1))^i \cap \beta$. The projection will be given by: "cap with $c_1(\mathcal{O}_{\mathbb{P}E}^{e-i}$ and push forward.)

Corollary. It makes sense to multiply by various polynomials in Segre classes of various bundles, by the commutativity part (b).

Proof of (b). It won't be surprising how we get commutativity. Consider the fibered square:

$$\begin{tikzcd} & Q \arrow[dl, "q'"'] \arrow[dr, "p'"] & \\ \mathbb{P}E \arrow[dr, "p"'] & & \mathbb{P}F \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

where p and q are the projections; all morphisms are projective bundles. Let $f+1$ be the rank of F (and as usual $e+1$ is the rank of E). Then:

$$\begin{aligned} s_i(E) \cap (s_j(F) \cap \alpha) &= p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap p^*(q_*(c_1(\mathcal{O}_{\mathbb{P}F}(1))^{f+j} \cap q^*\alpha))) \\ &\qquad \text{(left side of desired equality)} \\ &= p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap q'_*(p'^*(c_1(\mathcal{O}_{\mathbb{P}F}(1))^{f+j} \cap q^*\alpha))) \\ &\qquad \text{(pr. pushforwards and fl. pullbacks "commute")} \\ &= p_* q'_*(c_1(q'^*\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap (p'^* c_1(\mathcal{O}_{\mathbb{P}F}(1))^{f+j} \cap p'^* q^*\alpha)) \\ &\qquad \text{(proj. form. and flat pull. behaves well w.r.t. } c_1\text{)} \\ &= q_* p'_*(p'^* c_1(\mathcal{O}_{\mathbb{P}F}(1))^{f+j} \cap (c_1(q'^*\mathcal{O}_{\mathbb{P}E}(1))^{e+i} \cap p'^* q^*\alpha)) \\ &\qquad \text{(prop. pushforwards commute, and } c_1\text{'s commute)} \\ &= \text{(then go backwards to get desired result)} \end{aligned}$$

□

Exercise. Let E be a vector bundle of rank $e+1$, L a line bundle. Show that

$$s_p(E \otimes L) = \sum_{i=0}^{p} (-1)^{p-1} \binom{e+p}{e+i} s_i(E) c_1(L)^{p-i}$$

(Hint: Identify $\mathbb{P}E$ with $\mathbb{P}(E \otimes L)$, with universal subbundle $\mathcal{O}_{\mathbb{P}E}(-1) \otimes p^*L$. Then $s_p(E \otimes L) \cap \alpha = p_*((c_1(\mathcal{O}_{\mathbb{P}E}(1)) - c_1(p^*L))^{e+p} \cap p^*\alpha)$.)

1.2. **Chern classes.** We now define Chern classes. Define the Segre power series $s_t(E)$ to be the generating function of the s_i:

$$s_t(E) = \sum_{i=0}^{\infty} s_i(E) t^i = 1 + s_1(E)t + s_2(E)t^2 + \cdots .$$

Define the *Chern power series* (soon to be Chern polynomial!) as the inverse of $s_t(E)$:

$$c_t(E) = \sum_{i=0}^{\infty} c_i(E) t^i = 1 + c_1(E)t + c_2(E)t^2 + \cdots .$$

$$c_t(E) s_t(E) = 1.$$

Hence $c_0(E) = 1$, $c_1(E) = -s_1(E)$, $c_2(E) = s_1(E)^2 - s_2(E)$, $\dots$,

$$c_n(E) = -s_1(E)c_{n-1}(E) - s_2(E)c_{n-2}(E) - \cdots - s_n(E).$$

Note: The old-fashioned definition of $c_1(L)$ agrees with the new definition of $c_1(L)$, by the last part of the previous Theorem.

Chern class Theorem. The Chern classes satisfy the following properties.

(a) (vanishing) For all bundles E on X, and all $i > \operatorname{rank} E$, $c_i(E) = 0$.

(b) (commutativity) For all bundles E, F on X, integers i and j, and cycles α on X,

$$c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha).$$

(c) (projection formula, i.e. Chern classes behave well with respect to proper pushforward) Let E be a vector bundle on X, $f : X' \to X$ a proper morphism. Then

$$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha)$$

for all cycles α on X' and all i.

(d) (Chern classes behave well with respect to flat pullback) Let E be a vector bundle on X, $f : X' \to X$ a flat morphism. Then

$$c_i(f^*E) \cap f^*\alpha = f^*(c_i(E) \cap \alpha)$$

for all cycles α on X, and all i.

(e) (Whitney sum) For any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

of vector bundles on X, then $c_t(E) = c_t(E') \cdot c_t(E'')$, i.e. $c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'')$.

(f) (Normalization) If E is a line bundle on a variety X, D a Cartier divisor on X with $\mathcal{O}(D) \cong E$, then $c_1(E) \cap [X] = [D]$.

(b), (c), and (d) follow from the Segre class theorem above. I explained (f) last time. Thus we have to show (a) and (e). I'll set up the right way of thinking about (a) and (e), and then prove them next day.

Splitting principle. This uses a very nice (and very important) construction, the splitting principle. It is *not* true that the every vector bundle splits into a direct sum of line bundles. However, the splitting principle in essence tells that we can pretend that every vector bundle splits, not into a direct sum, but into a nice filtration.

Given a vector bundle E on a scheme X, there is a flat morphism $f : X' \to X$ such that

(1) $f^* : A_*X \to A_*X'$ is injective, and
(2) f^*E has a filtration by subbundles

$$f^*E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0.$$

Injectivity shows that if we can show some equality involving Chern classes on the pull-back to X', then it will imply the same equality downstairs on X.

The construction is pretty simple: it will be a tower of projective bundles. Recall that we showed earlier today that if F is any vector bundle on Y, and $g : \mathbb{P}F \to Y$, then $g^* : A_k X \mapsto A_{k+e}\mathbb{P}E$ is an injection, so we'll get (1) immediately. We'll constructive the tower of projective bundles inductively on the rank of E. If $r = 1$, we're already done. Otherwise, let $g : \mathbb{P}E \to X$. On $\mathbb{P}E$, we can split off the tautological subline bundle.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}E}(-1) \arrow[r] & g^*E \arrow[r] \arrow[d, dash] & Q \arrow[r] & 0 \\ & & \mathbb{P}E \arrow[d, "g"] & & \\ & & X & & \end{tikzcd}$$

Here Q is the quotient bundle of rank one less than that of E.

Thus we've shown how to split a single vector bundle. Clearly we can split any finite number of vector bundles in this way as well.

I stated the following result, and will prove it next time.

Lemma. Assume that E is filtered with line bundle quotients $L_1, \dots, L_r$. Let s be a section of E, and let Z be the closed subset of X where s vanishes. Then for any k-cycle α on X, there is a $(k-r)$-cycle β on Z with

$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha = \beta$$

in $A_{k-r}X$. (Even better, we will see that we will get equality in $A_{k-r}(Z)$: we have pinned down (or "localized") this class even further.) In particular, if s is nowhere zero, then $\prod_{i=1}^{r} c_1(L_i) = 0$. (Recall $r = \operatorname{rank} E$.)

I suggested that people browse through the many examples in this chapter, including the Chern character and the Todd class.

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INTERSECTION THEORY CLASS 10

RAVI VAKIL

CONTENTS

1. LAST TIME

Let E be a vector bundle of rank $e+1$ on an algebraic scheme X. Let $P = \mathbb{P}E$ be the $\mathbb{P}^e$-bundle of lines on E, and let $p = p_E : P \to X$ be the projection. The Segre classes are defined by:

$$s_i(E) \cap : A_k X \to A_{k-i} X$$

by $\alpha \mapsto p_*(c_1(\mathcal{O}(1))^{e+i} \cap p^* \alpha)$.

Corollary to Segre class theorem. The flat pullback $p^* : A_k X \to A_{k+e}(\mathbb{P}E)$ is a split monomorphism: by (a) (ii), an inverse is $\beta \mapsto p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^e \cap \beta)$.

2. CHERN CLASSES

We then defined Chern classes. Define the Segre power series $s_t(E)$ to be the generating function of the s_i. Define the *Chern power series* (soon to be Chern polynomial!) as the inverse of $s_t(E)$.

We're in the process of proving parts of the Chern class theorem. Left to do:

Chern class Theorem. The Chern classes satisfy the following properties.

(a) (vanishing) For all bundles E on X, and all $i > \operatorname{rank} E$, $c_i(E) = 0$.

(e) (Whitney sum) For any exact sequence

$$0 \to E' \to E \to E'' \to 0$$

Date: Monday, October 25, 2004.

of vector bundles on X, then $c_t(E) = c_t(E') \cdot c_t(E'')$, i.e. $c_k(E) = \sum_{i+j=k} c_i(E')c_j(E'')$.

Notation. The Chern classes and Segre classes of all vector bundles determine a ring of operators on Chow groups. I won't give this ring a name (or I may tentatively call it the Segre-Chern ring); later we will define a ring A^*X of operators, in which these Chern and Segre classes will lie.

Splitting principle. I introduced the splitting principle, which tells that we can pretend that every vector bundle splits, not into a direct sum, but into a nice filtration.

Given a vector bundle E on a scheme X, there is a flat morphism $f : X' \to X$ such that

(1) $f^* : A_*X \to A_*X'$ is injective, and
(2) f^*E has a filtration by subbundles

$$f^*E = E_r \supset E_{r-1} \supset \cdots \supset E_1 \supset E_0 = 0.$$

Injectivity shows that if we can show some equality involving Chern classes on the pull-back to X', then it will imply the same equality downstairs on X.

The construction was pretty simple: we took a tower of projective bundles.

I should have said explicitly: we've shown how to split a single vector bundle. But clearly we can split any finite number of vector bundles in this way as well.

Lemma. Assume that E is filtered with line bundle quotients $L_1, \ldots, L_r$. Let s be a section of E, and let Z be the closed subset of X where s vanishes. Then for any k-cycle α on X, there is a $(k-r)$-cycle class β on Z (i.e. an element of $A_{k-r}Z$) with

$$\prod_{i=1}^{r} c_1(L_i) \cap \alpha = \beta$$

in $A_{k-r}X$. (Even better, we will see that we will get equality in $A_{k-r}(Z)$: we have pinned down (or "localized") this class even further.) In particular, if s is nowhere zero, then $\prod_{i=1}^{r} c_1(L_i) = 0$. (Recall $r = \operatorname{rank} E$.)

Proof. For simplicity of exposition, let me show you how this works for $r = 2$. We have $0 \to L_1 \to E \to L_2 = 0$. The section s of E induces a section $\bar{s}$ of L_2. If Y is the zero scheme of $\bar{s}$, then $(L_2, Y, \bar{s})$ is a pseudodivisor D_2 on X. Let $j : Y \hookrightarrow X$ be the closed immersion. Intersecting with D_2 gives a class $D_2 \cdot \alpha$ in $A_{k-1}Y$ such that $c_1(L_2) \cap \alpha = j_*(D_2 \cdot \alpha)$. By the projection formula ("proper pushforward behaves with respect to c_1"):

$$c_1(L_1) \cap c_1(L_2) \cap \alpha = j_*(c_1(j^*L_1) \cap (D_2 \cdot \alpha)).$$

The bundle $L_1Y = j^*E$ has a section, induced by s, whose zero set is Z. So $c_1(j^*L_1) \cap (D_2 \cdot \alpha) \in A_{k-2}Z$ as desired.

The general argument is just the same (an induction). $\square$

Lemma. Suppose E has a filtration by subbundles $E = E_r \supset E_{r-1} \supset \cdots \supset E_0 = 0$ with quotients $L_r, \ldots, L_1$. Then

$$c_t(E) = \prod_{i=1}^{r}(1 + c_1(L_i)t).$$

Proof. Let $p : \mathbb{P}E \to X$ be the associated projective bundle. We have a tautological subbundle $\mathcal{O}_{\mathbb{P}E}(-1) \to p^*E$ on $\mathbb{P}E$. Twisting (tensoring) this inclusion by the line bundle $\mathcal{O}_{\mathbb{P}E}(1)$, we get

$$\mathcal{O}_{\mathbb{P}E} \to (p^*E) \otimes \mathcal{O}_{\mathbb{P}E}(1).$$

In other words, we have a nowhere vanishing section of $(p^*E) \otimes \mathcal{O}_{\mathbb{P}E}(1)$. Note that $(p^*E) \otimes \mathcal{O}_{\mathbb{P}E}(1)$ has a filtration with quotient line bundles $p^*L_i \otimes \mathcal{O}_{\mathbb{P}E}(1)$. Thus our previous lemma implies that

$$\prod_{i=1}^{r} c_1(p^*L_i \otimes \mathcal{O}_{\mathbb{P}E}(1)) = 0.$$

We'll now unwind this to get the result. Let $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1))$ for convenience. Let σ_i be the ith symmetric function in $c_1(L_1), \ldots, c_1(L_r)$. Let $\tilde{\sigma}_i$ be the ith symmetric function in $c_1(p^*L_1), \ldots, c_1(p^*L_r)$.

We want to show that $(1 + \sigma_1 t + \sigma_2 t^2 + \cdots + \sigma_r t^r) = c_t(E)$.

We know that $c_1(p^*L_i \otimes \mathcal{O}_{\mathbb{P}E}(1)) = c_1(p^*L_i) + c_1(\mathcal{O}_{\mathbb{P}E}(1)) = c_1(p^*L_i) + \zeta$. Hence we know:

$$\zeta^r + \tilde{\sigma}_1 \zeta^{r-1} + \cdots + \tilde{\sigma}_r = 0.$$

(We feel like turning ζ into $1/t$ and using injectivity. That's in spirit what we'll do.) Multiply by ζ^{i-1} for some i. Pick any $\alpha \in A_*X$, and cap the equation with $p^*\alpha$. Then pushforward:

$$p_*(\zeta^{e+i} \cap p^*\alpha) + p_*(\tilde{\sigma}_1 \zeta^{e+i-1} \cap p^*\alpha) + \cdots + p_*(\tilde{\sigma}_r \zeta^{i-1} \cap p^*\alpha) = 0.$$

Thus these are Segre classes:

$$s_i(E) \cap \alpha + \sigma_1 s_{i-1}(E) \cap \alpha + \cdots + \sigma_r s_{i-r}(E) \cap \alpha = 0. \tag{1}$$

Multiply this by a formal variable t^i, and add up over all i to get:

$$(1 + \sigma_1 t + \cdots + \sigma_r t^r) s_t(E) = 0.$$

Oops, that wasn't quite right! Equation (1) holds for $i > 0$, so in fact

$$(1 + \sigma_1 t + \cdots + \sigma_r t^r) s_t(E) = \text{constant}.$$

But that constant is 1. Thus by the definition of $c_t(E)$, we get our desired result: $c_t(E) = 1 + \sigma_1 t + \cdots + \sigma_r t^r$. $\square$

I'm now finally ready to prove (a) and (e) of the Chern class theorem. It suffices to prove (a) assuming that E is filtered. But then $c_t(E) = \prod_{i=1}^{r}(1 + c_1(L_i)t)$ is clearly a polynomial of degree at most r — we've proved (a).

(e) is also easy. Given an exact sequence of vector bundles as in the statement, pullback to a flat $f : X' \to X$ so that both the (pullback of the) kernel E' and the (pullback of the) cokernel E'' split into line bundles. Then the pullback of E also splits. Thus by the lemma,

$$c_t(f^*E) = c_t(f^*E')c_t(f^*E'').$$

$\square$

Notation. If X is a pure-dimensional scheme, and P is a polynomial in Chern classes (or Segre classes) of various vector bundles of total codimension $\dim X$, then $\deg P \cap [X]$ is a number. This is denoted $\int_X P$. Example 1: Suppose X is a compact projective manifold (i.e. nonsingular complex projective variety) of dimension n, and T_X is the tangent bundle. Then $c_n(T_X)$ is a codimension n Chern class. Fact: $\int_X c_n(T_X) := c_n(T_X) \cap [X] = \chi(X)$, where $\chi(X)$ is the (topological) Euler characteristic. Example 2: Suppose $i : X \hookrightarrow \mathbb{P}^N$ is a projective variety of dimension n. Then $i^*\mathcal{O}_{\mathbb{P}^N}(1)$ is a line bundle on X. Then

$$\int_X c_1(i^*\mathcal{O}_{\mathbb{P}^N}(1))^d := c_1(i^*\mathcal{O}_{\mathbb{P}^N}(1))^d \cap X = \deg X.$$

(Reason: we can interpret each factor $c_1(i^*\mathcal{O}_{\mathbb{P}^N}(1))$ as intersecting with a randomly chosen hyperplane.)

2.1. **Fun with the splitting principle.** Thanks to the splitting principle, given the Chern classes of a vector bundle, you can find the Chern classes of other related vector bundles.

The way I think about it: imagine that the Chern polynomial factors (even though it doesn't!). Imagine that the bundle splits (even though it doesn't!).

Example 1: Dual bundle. Suppose E is a vector bundle, and $E^\vee$ is the dual bundle. Then $c_i(E^\vee) = (-1)^i c_i(E)$. (Reason: $c_t(E) = c_{-t}(E)$. The reason for this in turn is that if you assume that E is filtered (which we may do by the splitting principle) then $E^\vee$ is filtered too. Do you see why?

Example 2: Tensor products. I'll do a specific example, in the hope that you'll see the general pattern. Suppose E and F are rank 2 bundles. Then $E \otimes F$ is a rank 4 bundle. We can compute its Chern classes in terms of those of E and F. Suppose E has Chern roots e_1 and e_2, and suppose F has Chern roots f_1 and f_2. (Translation: assume that both E and F can be filtered. Let e_1 and e_2 be the line bundle quotients of the filtration of E, and similarly for f_1 and f_2.) Thus from

$$1 + c_1(E)t + c_2(E)t^2 = (1 + e_1 t)(1 + e_2 t)$$

we get $e_1 + e_2 = c_1(E)$ and $e_2 = c_2(E)$, and similarly for F. Then

$$\begin{aligned} c_t(E \otimes F) &= (1 + (e_1 + f_1)t)(1 + (e_1 + f_2)t)(1 + (e_2 + f_1)t)(1 + (e_2 + f_2)t) \\ &= 1 + (2e_1 + 2e_2 + 2f_1 + 2f_2)t + \cdots \\ &= 1 + (2c_1(E) + 2c_1(F))t + \cdots \end{aligned}$$

from which we get $c_1(E \otimes F) = 2c_1(E) + 2c_1(F)$, and similarly we can compute formulae for higher Chern classes of $E \otimes F$.

To justify that first equality for $c_t(E \otimes F)$, we need to give a filtration of $E \otimes F$ using the filtrations of E and F. I'll leave that for you.

Example 3: Exterior powers. I'll again do a specific example to illustrate a general principle. Suppose E is rank 3, with Chern roots e_1, e_2, e_3. In other words, as assume we have a specific filtration of E. The $\wedge^2 E$ is also rank 3, with Chern roots $e_1 + e_2$, $e_1 + e_3$, $e_2 + e_3$. Again, we do this by producing a filtration of $\wedge^2 E$ induced by that filtration on E.

Thus we can find the Chern classes of $\wedge^2 E$ in terms of those of E. We know $e_1+e_2+e_3 = c_1(E)$, $e_1e_2 + e_2e_3 + e_3e_1 = c_2(E)$, and $e_1e_2e_3 = c_3(E)$. Thus

$$\begin{aligned} c_t(\wedge^2(E)) &= (1 + (e_1 + e_2)t)(1 + (e_1 + e_3)t)(1 + (e_2 + e_3)t) \\ &= 1 + (2e_1 + 2e_2 + 2e_3)t + \cdots . \end{aligned}$$

In general, if E is rank n and we want to compute the Chern classes of $\wedge^k E$, the roots are sums of k distinct Chern roots of E.

Exercise: if E is rank n, then you can check that $\wedge^n E = \det E$. Show that $c_1(E) = c_1(\det E)$. This gives a different interpretation of c_1 of a vector bundle — as c_1 of the determinant bundle.

Exercise: what about symmetric powers? If E is rank 2, can you compute the Chern classes of $\operatorname{Sym}^4 E$?

Homework (due Nov. 1.) Suppose E is a bundle of rank r on a scheme X, p is the projection $\mathbb{P}E \to X$, and $\zeta = c_1(\mathcal{O}_{\mathbb{P}E}(1))$. Show that $\zeta^r + c_1(p^*E)\zeta^{r-1} + \cdots + c_r(p^*E) = 0$. (Hint: consider the exact sequence of vector bundles on $\mathbb{P}E$: $0 \to \mathcal{O}_{\mathbb{P}E}(-1) \to p^*E \to Q \to 0$.)

Example: Chern classes of the tangent bundle to projective space:

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \to 0.$$

For convenience let, $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. Hence $c_t(T_{\mathbb{P}^n}) = (1+Ht)^{n+1}$. (Note that $\deg c_n(T_{\mathbb{P}^n}) = n + 1$, which is indeed the topological Euler characteristic of $\mathbb{P}^n$.)

Example: Chern classes of the tangent bundle of a hypersurface in Y in X:

$$0 \to T_Y \to T_X|_Y \to N \to 0.$$

$(N \cong \mathcal{O}_X(Y))$.

Suppose next that $X = \mathbb{P}^n$, and Y is a degree d hypersurface. Let H denote the restriction of $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$ to Y. (Equivalently, it is c_1 of the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ to Y: we've shown that c_1 commutes with any pullback.) Then as operators on A_*Y, we get

$$c_t(T_Y) = (1 + Ht)^{n+1}(1 + dHt)^{-1} = (1 + Ht)^{n+1}\left(1 - dHt + (dHt)^2 - (dHt)^3 + \cdots\right)$$

You can use this to compute the topological Euler characteristic of a hypersurface, or inductively, of a complete intersection. (Fun exercise: use this to work out the genus of a degree d plane curve.)

2.2. **The Chern character and Todd class.** The Chern character ch is defined by $\mathrm{ch}(E) = \sum_{i=1}^r e^{\alpha_i}$. Then if $0 \to E' \to E \to E'' \to 0$ is a short exact sequence of vector bundles, $\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E'')$. (You should immediately see the corresponding long exact sequence!) Also, $\mathrm{ch}(E \otimes E') = \mathrm{ch}(E)\mathrm{ch}(E')$.

The Todd class is defined by $\mathrm{td}(E) = \prod_{i=1}^r Q(\alpha_i)$ where

$$Q(x) = \frac{x}{1-e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k}.$$

Again, $\mathrm{td}(E) = \mathrm{td}(E')\mathrm{td}(E'')$.

Sample application. Let X be an n-dimensional abelian variety lying in projective space $i : X \hookrightarrow \mathbb{P}^m$. Then $m \geq 2n$, and if equality holds, then $\deg X = \binom{2n+1}{n}$. Fact: for an abelian variety, T_X is a trivial bundle. (Reason over $\mathbb{C}$, $X = \mathbb{C}^n$ modulo a lattice.) Hence T_X has all Chern classes 0 (except c_0).

The first two cases are relative straightforward: if $n = 1$, then this corresponds to curves in planes; the only way for a genus 1 curve to lie in $\mathbb{P}^2$ is if it is degree 3.

If $n = 2$: there is no way for an abelian surface to be a hypersurface in $\mathbb{P}^3$. Reason: we've computed Chern classes of hypersurfaces.

It can sit in $\mathbb{P}^4$, but we'll see that it can only sit as a degree 10 hypersurface, and there is a famous such example called the Horrocks-Mumford abelian variety.

Here's the proof. $0 \to T_X \to i^* T_Y \to N \to 0$. $c_i(i^*T_Y) = c_i(N)$. Now the rank of N is $m - n$. $c_i(i^*T_Y) = \binom{m+1}{i} H^i$. If $i \leq n$, this is non-zero, as $H^n = \deg X[pt] \in A_0 X$. On the other hand, $c_i(N) = 0$ for $i > \operatorname{rank} N$, and $\operatorname{rank} N = n - m$. Thus $m > n$.

2.3. **Looking forward to next day: Rational equivalence on bundles.** I stated a couple of things that we'll do on Wednesday.

Theorem Let E be a vector bundle of rank $r = e + 1$ on a scheme X, with projection $\pi : E \to X$. Let $\mathbb{P}E$ be the associated projective bundle, with projection $p : \mathbb{P}E \to X$. Recall the definition of the line bundle $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E$.

(a) The flat pullback $\pi^* : A_{k-r}X \to A_k E$ is an isomorphism for all k.

(b) Each $\beta \in A_k \mathbb{P}E$ is uniquely expressible in the form

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^* \alpha_i,$$

for $\alpha \in A_{k-e+i}X$. Thus there are canonical isomorphisms

$$\theta_E : \oplus_{i=0}^{e} A_{k-e+i} X \xrightarrow{\sim} A_k \mathbb{P}E.$$

$\theta_E : \oplus \alpha_i \mapsto \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i p^* \alpha_i$.

Intersecting with the zero-section of a vector bundle. We can already intersect with the zero-section of a line bundle (i.e. an effective Cartier divisor); we get a map $A_k X \to A_{k-1} D$, which we've called the Gysin pullback.

Definition: Gysin pullback by zero section of a vector bundle. Let $s = s_E$ denote the zero section of a vector bundle E. s is a morphism from X to E with $\pi \circ s = id_X$. By part (a) of the Chern class theorem allows us to define *Gysin homomorphisms* $s^* : A_k E \to A_{k-r} X$, $r = \operatorname{rank} E$, by $s^*(\beta) := (\pi^*)^{-1}(\beta)$.

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INTERSECTION THEORY CLASS 11

RAVI VAKIL

CONTENTS

1. RATIONAL EQUIVALENCE ON BUNDLES

Last time I stated:

Theorem. Let E be a vector bundle of rank $r = e+1$ on a scheme X, with projection $\pi : E \to X$. Let $\mathbb{P}E$ be the associated projective bundle, with projection $p : \mathbb{P}E \to X$. Recall the definition of the line bundle $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E$.

(a) The flat pullback $\pi^* : A_{k-r}X \to A_kE$ is an isomorphism for all k.

(b) Each $\beta \in A_k\mathbb{P}E$ is uniquely expressible in the form

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_i,$$

for $\alpha \in A_{k-e+i}X$. Thus there are canonical isomorphisms

$$\theta_E : \oplus_{i=0}^{e} A_{k-e+i}X \xrightarrow{\sim} A_k\mathbb{P}E.$$

$\theta_E : \oplus \alpha_i \mapsto \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i p^*\alpha_i$.

Proof. Here's the plan: π^* surjective, θ_E surjective, θ_E injective, π^* injective. So the proof is a delicate interplay between E and $\mathbb{P}E$.

We'll make repeated use of something Rob stated, from the end of the first Chapter: the "excision exact sequence". Suppose X is a scheme, U an open set, and Z the complement (a closed subset). Then the following sequence is exact:

$$A_kZ \to A_kX \to A_kU \to 0.$$

I'll now show surjectivity of π^* and θ_E. First reduction: it suffices to deal with the case where E is the trivial bundle. Proof by the induction on the dimension of X. Here's the π^* argument:

Date: Wednesday, October 27, 2004.

Let U be a dense open set where E is trivial. Then its complement Y is of dimension strictly smaller than X.

$$\begin{tikzcd} A_*Y \arrow[r] \arrow[d] & A_*X \arrow[r] \arrow[d] & A_*U \arrow[r] \arrow[d] & 0 \\ A_*(\pi^{-1}Y) \arrow[r] \arrow[d] & A_*E \arrow[r] \arrow[d, "?"] & A_*(\pi^{-1}U) \arrow[r] \arrow[d] & 0 \\ 0 & 0 & 0 & \end{tikzcd}$$

The two horizontal rows are exact. By the inductive hypothesis, the left column is exact. We're assuming we know the result for trivial vector bundles, so the right column is also exact. Then the central vertical row is exact, by a quick diagram chase.

The same argument works for θ_E. Here's the exact sequence:

$$\begin{tikzcd} A_*Y \arrow[r] \arrow[d] & A_*X \arrow[r] \arrow[d] & A_*U \arrow[r] \arrow[d] & 0 \\ \oplus A_*(p^{-1}Y) \arrow[r] \arrow[d] & \oplus A_*\mathbb{P}E \arrow[r] \arrow[d, "?"] & \oplus A_*(p^{-1}U) \arrow[r] \arrow[d] & 0 \\ 0 & 0 & 0 & \end{tikzcd}$$

So let's show surjectivity of π^* and θ_E in the case where E is a trivial bundle. I'll show both by induction on the rank of E. In the case where the rank is 0, both are clearly surjective. (In fact, π^* is tautologically an isomorphism, and $\mathbb{P}E$ is the empty set, and the left side of θ_E is the empty direct sum!)

We assume the result for E and prove it for $E \oplus \mathbf{1}$.

The surjectivity of π^* in the trivial bundle was shown in Chapter 1, so for the sake of time I'll omit it. (The atomic statement that needs to be shown: $A_k X \to A_{k+1}(X \times \mathbb{A}^1)$ is surjective. Then by induction $A_k X \to A_{k+1}(X \times \mathbb{A}^n)$ is surjective.)

Recall that $\mathbb{P}(E \oplus \mathbf{1}) = \mathbb{P}E \coprod E$, where $\mathbb{P}E$ is a closed subset and E is an open subset; let $i : \mathbb{P}E \hookrightarrow \mathbb{P}(E \oplus \mathbf{1})$ be the closed immersion, and $j : E \hookrightarrow \mathbb{P}(E \oplus \mathbf{1})$ be the open immersion. (In fact $\mathbb{P}E$ is a Cartier divisor, in class $\mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1})}(1)$; this was one of my definitions of $\mathcal{O}(1)$.) Let q be the morphism $\mathbb{P}(E \oplus \mathbf{1}) \to X$. The excision exact sequence gives us:

$$\begin{tikzcd} A_k\mathbb{P}E \arrow[r, "i_*"] & A_k\mathbb{P}(E \oplus \mathbf{1}) \arrow[r] & A_kE \arrow[r] & 0 \\ & A_{k-r}X \arrow[u, "q^*"] \arrow[ur, "\pi^*"'] & & \end{tikzcd}$$

You may feel like drawing an arrow $A_{k-r}X \to A_kP$, but that's not right; the morphism is of course $A_{k-r}X \to A_{k-1}P$, as the fiber dimension of $A_k\mathbb{P}E \to A_{k-r}$ is $r - 1$.

Remark: For any $\alpha \in A_*X$, $c_1(\mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1})}(1)) \cap q^*\alpha = i_* p^*\alpha$. Reason: I'll show this for any cycle $\alpha \in Z_*X$. Then we can interpret the left side as pulling the cycle back to $\mathbb{P}(E \oplus \mathbf{1})$, and intersecting with the Cartier divisor $\mathbb{P}E$. But that's exactly the same as the right side. (That's basically how we defined c_1 of a line bundle!)

Suppose $\beta \in A_*\mathbb{P}(E\oplus\mathbf{1})$. Then we can write $j^*\beta = \pi^*\alpha$ for some $\alpha \in A_*X$ (by surjectivity of π^*). Then $\beta - q^*\alpha$ maps to 0 in A_kE, so it is in $A_k\mathbb{P}E$ by our excision exact sequence. Then by our inductive assumption that we already know surjectivity for smaller-dimensional schemes, we know:

$$\beta - q^*\alpha = i_*\left(\sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i \cap p^*\alpha_i\right)$$

for some $\alpha_i \in A_*X$. As $i^*\mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1})} = \mathcal{O}_{\mathbb{P}E}(1)$:

$$\cdots = \beta - q^*\alpha = i_*\left(\sum_{i=0}^{e} i^*c_1(\mathcal{O}_{\mathbb{P}E\oplus\mathbf{1}}(1))^i \cap p^*\alpha_i\right)$$

Then by the projection formula we get

$$\begin{aligned} \cdots &= \beta - q^*\alpha = \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E\oplus\mathbf{1}}(1))^i \cap i_*p^*\alpha_i \\ &= \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E\oplus\mathbf{1}}(1))^i \cap c_1(\mathcal{O}_{\mathbb{P}E\oplus\mathbf{1}}(1) \cap q^*\alpha_i \end{aligned}$$

(the last step by using the remark). Thus we see that $\theta_{E\oplus\mathbf{1}}$ is surjective.

We next show that θ_E is an *isomorphism*. Suppose we have a relation

$$\sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i \cap p^*\alpha_i = 0.$$

If the α_i are not all zero, then let k be the largest integers with $\alpha_k \neq 0$. Then

$$p_*(c_1(\mathcal{O}_{\mathbb{P}E}(1))^{e-k} \cap \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i \cap p^*\alpha_i) = \alpha_k$$

by our Segre class theorem, giving a contradiction.

Finally, we'll show that π^* is an isomorphism. I claim that as before, it suffices to do this for trivial bundles. The argument is by Noetherian induction again.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d, "?"] & \\
A_*Y \arrow[r] \arrow[d] & A_*X \arrow[r] \arrow[d] & A_*U \arrow[r] \arrow[d] & 0 \\
A_*(\pi^{-1}Y) \arrow[r] \arrow[d] & A_*E \arrow[r] \arrow[d] & A_*(\pi^{-1}U) \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}$$

Now we'll do it by induction on the rank. So we want to show that $A_k X \hookrightarrow A_{k+1} X \times \mathbb{A}^1 \hookrightarrow A_{k+2} X \times \mathbb{A}^2 \hookrightarrow \cdots$: we just need to show the rank 1 case. Suppose $\alpha \in A_k X$ and $\pi^* \alpha \in A_{k+1}(X \times \mathbb{A}^1) = 0$. Consider $q^* \alpha \in A_{k+1}(X \times \mathbb{P}^1)$. As θ_E is an isomorphism, we have

$$q^* \alpha = i_*(p^* \alpha_0 + c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \cap p^* \alpha_1).$$

(One nice thing about $\mathbb{P}^1$ is that $\mathcal{O}_{\mathbb{P}^1}(1)^2 = 0$: the intersection of two distinct points is empty!) Using our remark:

$$q^* \alpha = c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \cap q^* \alpha_0 + c_1(\mathcal{O}_{\mathbb{P}^1}(1))^2 \cap p^* \alpha_1.$$

(Thus by injectivity of θ_E (which is uniqueness of α_0 and α_1) we have $\alpha_1 = 0$.) But the first part of the Segre class theorem stated that if we take a class α downstairs, pull it back to a projective bundle, and cap it with the right number of $\mathcal{O}(1)$'s (corresponding to the projective bundle), and push it forward, we'll get α again. Hence

$$\begin{aligned} \alpha &= q_*(c_1(\mathcal{O}_{\mathbb{P}^1}(1)) \cap q^* \alpha) \\ &= c_1(\mathcal{O}_{\mathbb{P}^1}(1))^2 \cap q^* \alpha_0 + c_1(\mathcal{O}_{\mathbb{P}^1}(1))^3 \cap p^* \alpha_1 \\ &= 0 \end{aligned}$$

□

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INTERSECTION THEORY CLASS 12

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CONTENTS

1. RATIONAL EQUIVALENCE ON BUNDLES

Last time we mostly proved:

Theorem. Let E be a vector bundle of rank $r = e + 1$ on a scheme X, with projection $\pi : E \to X$. Let $\mathbb{P}E$ be the associated projective bundle, with projection $p : \mathbb{P}E \to X$. Recall the definition of the line bundle $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}E}(1)$ on $\mathbb{P}E$.

(a) The flat pullback $\pi^* : A_{k-r}X \to A_kE$ is an isomorphism for all k.

(b) Each $\beta \in A_k\mathbb{P}E$ is uniquely expressible in the form

$$\beta = \sum_{i=0}^{e} c_1(\mathcal{O}(1))^i \cap p^*\alpha_i,$$

for $\alpha \in A_{k-e+i}X$. Thus there are canonical isomorphisms

$$\theta_E : \oplus_{i=0}^{e} A_{k-e+i}X \xrightarrow{\sim} A_k\mathbb{P}E.$$

$\theta_E : \oplus\alpha_i \mapsto \sum_{i=0}^{e} c_1(\mathcal{O}_{\mathbb{P}E}(1))^i p^*\alpha_i$.

Proof. Our plan was to prove this in the following order: π^* surjective, θ_E surjective, θ_E injective, π^* injective. The proof is a delicate interplay between E and $\mathbb{P}E$. We had done all but the last step, and we had reduced the last step to the case where E is a trivial bundle, i.e. we wanted to show that $A_*X \hookrightarrow A_*(X \times \mathbb{A}^r)$. By induction, we needed to deal with the case where E had rank 1.

Date: Monday, November 1, 2004.

We repeatedly used the "excision exact sequence". Suppose X is a scheme, U an open set, and Z the complement (a closed subset). Then the following sequence is exact:

$$A_k Z \to A_k X \to A_k U \to 0.$$

A construction we used throughout the proof was the following: Note that $\mathbb{P}(E \oplus \mathbf{1}) = \mathbb{P}E \coprod E$, where $\mathbb{P}E$ is a closed subset and E is an open subset; let $i : \mathbb{P}E \hookrightarrow \mathbb{P}(E \oplus \mathbf{1})$ be the closed immersion, and $j : E \hookrightarrow \mathbb{P}(E \oplus \mathbf{1})$ be the open immersion. (In fact $\mathbb{P}E$ is a Cartier divisor, in class $\mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1})}(1)$; this was one of my definitions of $\mathcal{O}(1)$.) Let q be the morphism $\mathbb{P}(E \oplus \mathbf{1}) \to X$. The excision exact sequence gives us:

$$\begin{tikzcd}
A_k\mathbb{P}E \arrow[r, "i_*"] & A_k\mathbb{P}(E \oplus \mathbf{1}) \arrow[r] & A_k E \arrow[r] & 0 \\
& A_{k-r}X \arrow[u, "q^*"] \arrow[ur, "\pi^*"'] & &
\end{tikzcd}$$

We showed the following useful *Remark:* For any $\alpha \in A_*X$, $c_1(\mathcal{O}_{\mathbb{P}(E\oplus\mathbf{1})}(1)) \cap q^*\alpha = i_* p^* \alpha$.

So we want to show that $A_k X \hookrightarrow A_{k+1}(X \times \mathbb{A}^1) \hookrightarrow A_{k+2}(X \times \mathbb{A}^2) \hookrightarrow \cdots$. By induction we just need to show the rank 1 case: $A_k X \hookrightarrow A_{k+1}(X \times \mathbb{A}^1)$. Rather than starting this proof in the middle, I'll let you read it in the book; it is relative straightforward, compared to the rest of the argument.

1.1. **Intersecting with the zero-section of a vector bundle.** We can already intersect with the zero-section of a line bundle (i.e. an effective Cartier divisor); we get a map $A_k X \to A_{k-1} D$, which we've called the Gysin pullback.

Definition: Gysin pullback by zero section of a vector bundle. Let $s = s_E$ denote the zero section of a vector bundle E. s is a morphism from X to E with $\pi \circ s = id_X$. By part (a) of the Chern class theorem allows us to define *Gysin homomorphisms* $s^* : A_k E \to A_{k-r} X$, $r = \operatorname{rank} E$, by $s^*(\beta) := (\pi^*)^{-1}(\beta)$.

This ability to intersect with zero sections of vector bundles will be the basis for many important future constructions.

You should think of this as intersecting with the zero-section of a vector bundle. This should be a codimension r intersection. In fact there is "excess" intersection — the actual intersection is codimension 0 — but there is a class of the right dimension.

Proposition. Let $\beta \in A_k E$, and let $\overline{\beta}$ be any element of $A_k(\mathbb{P}(E \oplus \mathbf{1}))$ which restricts to β in $A_k E$. Then $s^*(\beta) = q_*(c_r(Q) \cap \overline{\beta})$ where q is the projection from $\mathbb{P}(E \oplus \mathbf{1})$ to X, and Q is the universal (rank r) quotient bundle of $q^*(E \oplus \mathbf{1})$.

Proof omitted (but is in book, and isn't too long). Note that c_r is the "top" Chern class.

Example If s is the zero section of a vector bundle E of rank r on X, then $s^* s_*(\alpha) = c_r(E) \cap \alpha$. This is a special case of the excess intersection formula.

2. Cones and Segre classes of subvarieties

2.1. **Introduction.** If X is a subvariety of a variety Y, the Segre class $s(X,Y)$ of X in Y is a class in A_*X defined as follows. $C = C_X Y$ is the normal cone to X in Y, $\mathbb{P}C$ is the projectivized normal cone, p the projection from $\mathbb{P}C$ to X. I'll define the normal cone soon. Then

$$s(X,Y) = \sum_{k \geq 0} p_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}C]).$$

Note that this is a class, *not* an operator.

In the case when X is a smooth subvariety of a smooth variety, C is the normal bundle. More generally, if Y is arbitrary, then X is a *local complete intersection* (hereafter *lci*) in Y (what Fulton calls a *regular imbedding*) if it is scheme-theoretically cut out by r equations, where r is the codimension of X in Y. (Example 1: X is a smooth subvariety of a smooth variety. Example 2: *any* Cartier divisor. Example 3: the union of the x and y axes in $\mathbb{A}^3$.) If X is a regular imbedding (=lci) in X, then X still has a normal bundle, defined as follows: if $\mathcal{I}$ is the ideal sheaf cutting out X, then $\mathcal{I}/\mathcal{I}^2$ is a vector bundle of rank r. This is the *conormal bundle*, and its dual is the normal bundle. (Warning: in differential geometry, if $X \hookrightarrow Y$, then X has a *tubular neighborhood* that looks like the normal bundle. In algebraic geometry, there are no such small neighborhoods, but in some sense it is even worse: in example 3, the total space $Y = \mathbb{A}^3$ is smooth, but the total space of the normal bundle — a vector bundle over a nodal curve — is singular.)

If X is regularly imbedded (=lci) in Y, then the definition of $s(X,Y)$ turns into

$$s(X,Y) = s(N) \cap [X] = c(N)^{-1} \cap [X].$$

More generally still, if X is arbitrarily horrible in arbitrarily horrible Y, it still has a *normal cone*. I'll define that shortly. Whatever it is, we'll have the same equation

$$s(X,Y) = s(N) \cap [X] = c(N)^{-1} \cap [X].$$

These Segre classes have a fundamental birational invariance: if $f : Y' \to Y$ is a birational proper morphism, and $X' = f^{-1}X$, then $s(X',Y')$ pushes forward to $s(X,Y)$. The coefficient of $[X]$ in $s(X,Y)$ is the multiplicity of Y along X. This magical invariance will be the main result of Chapter 4.

2.2. **Cones.** I'll now define *cone*. Let X be a scheme, and let $S^\cdot = \oplus_{i \geq 0} S^i$ be a sheaf of graded $\mathcal{O}_X$-algebras. Assume $\mathcal{O}_X \to S^0$ is surjective, S^1 is coherent, and $S^\cdot$ is generated (as an algebra) by S^1. This sounds complicated, but it isn't. It is defined so you can take $\underline{\mathrm{Proj}}(S^\cdot)$, and that this makes sense, and has a line bundle $\mathcal{O}(1)$.

Here's how it works: over any affine open set $\operatorname{Spec} R$ of X, $S^\cdot$ is a graded R-algebra, generated in degree 1. Then we can take $\underline{\mathrm{Proj}}$ of this graded R-algebra. The fact that the algebra is generated in degree 1 (by R_1 say) means that we have a surjective map of graded rings

$$\operatorname{Sym}^i R_1 \to \oplus_i R^i$$

which, upon applying $\underline{\mathrm{Proj}}$, becomes

$$X' \hookrightarrow X \times \mathbb{P}(R^1)^\vee$$

where $\mathbb{P}(R^1)^\vee$ is an honest projective bundle. So the morphism $X' \to X$ is projective and has a line bundle called $\mathcal{O}(1)$. You can do this over each affine, and glue the result together, and the $\mathcal{O}(1)$'s also glue together.

Example 1: say let E be a vector bundle, and $S^i = \mathrm{Sym}^i(E^\vee)$. Then $\underline{\mathrm{Proj}}\, S^\cdot = \mathbb{P}E$.

Example 2: Say $T^i = \mathrm{Sym}^i(E^\vee \oplus \mathbf{1}) = S^i \oplus S^{i-1}z$, so (better) $T^\cdot = S^\cdot[z]$. Then $\underline{\mathrm{Proj}}\, T^\cdot = \mathbb{P}E$.

Example 3: The blow-up can be described in this way, and it will be good to know this. Suppose X is a subscheme of Y, cut out by ideal sheaf $\mathcal{I}$. (In our situation where all schemes are finite type, $\mathcal{I}$ is a coherent sheaf.) Then let $S^\cdot = \oplus_i \mathcal{I}^i$, where $\mathcal{I}$ is the ith power of the ideal $\mathcal{I}$. ($\mathcal{I}^0$ is defined to be $\mathcal{O}_X$.) Then $\mathrm{Bl}_X Y \cong \underline{\mathrm{Proj}}\, S^\cdot$. A short calculation shows that the exceptional divisor class is $\mathcal{O}(-1)$. The *exceptional divisor* turns out to be $\underline{\mathrm{Proj}} \oplus \mathcal{I}^n/\mathcal{I}^{n+1}$. (Note that this is indeed a graded sheaf of algebras.) As $\oplus \mathcal{I}^n \to \oplus \mathcal{I}^n/\mathcal{I}^{n+1}$ is a surjective map of rings, this indeed describes a closed subscheme of the blow-up. (Remember this formula — it will come up again soon!)

Now I'll finally define *cone*. Let $S^\cdot$ be a sheaf of graded $\mathcal{O}_X$-algebras as before. Then $C = \underline{\mathrm{Spec}}(S^\cdot)$ is a *cone*. (We can construct $\underline{\mathrm{Spec}}(S^\cdot)$ of a sheaf of algebras in the same way as we can construct $\underline{\mathrm{Proj}}$; in fact it is a logically prior construction.)

Remember that $\mathbb{P}(E \oplus \mathbf{1}) = E \coprod \mathbb{P}E$. The direct generalization is: $\underline{\mathrm{Proj}}(S^\cdot[z]) = C \coprod \underline{\mathrm{Proj}}(S^\cdot) = \underline{\mathrm{Spec}}\, S^\cdot \coprod \underline{\mathrm{Proj}}(S^\cdot)$. The argument is just the same. The right term is a Cartier divisor in class $\mathcal{O}_{\underline{\mathrm{Proj}}(S^\cdot[z])}(1)$.

2.3. **Segre class of a cone.**

The *Segre class* of a cone C on X, denoted $s(C)$, is the class in A_*X defined by the formula

$$s(C) = q_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [\underline{\mathrm{Proj}}(C \oplus \mathbf{1})] \right).$$

This is very much the same definition as for vector bundles, *except* in the vector bundle case we get *operators* on Chow groups. In this case we get elements of Chow groups themselves: we are capping with a fundamental class!

Proposition (a) If E is a vector bundle on X, then $s(E) = c(E)^{-1} \cap [X]$, where $c(E)$ is the total Chern class of X, $r = \mathrm{rank}(E)$. $c(E) = 1 + c_1(E) + \cdots + c_r(E)$. (I would write $s(E) = s(E) \cap [X]$, but the two uses of $s(E)$ are confusing!) This is basically our definition of Segre/Chern classes.

(b) Let $C_1, \ldots, C_t$ be the irreducible components of C, m_i the geometric multiplicities of C_i in C. Then $s(C) = \sum_{i=1}^t m_i s(C_i)$. (Note that the C_i are cones as well, so $s(C_i)$ makes sense.) In other words, we can compute the Segre class piece by piece.

Sketch of proof of (b). This is because each of the C_i is a cone. $[\underline{\mathrm{Proj}}(C \oplus \mathbf{1})] = \cup m_i [\underline{\mathrm{Proj}}(C_i \oplus \mathbf{1})]$. $\square$

Example. For any cone C, $s(C \oplus \mathbf{1}) = s(C)$. (In the language of Dan's talk last week, the Segre class of a cone depends on its stable equivalence class.)

2.4. **The Segre class of a subscheme.** Let X be a closed subscheme of a scheme Y (not necessarily lci).

I told you that $\mathcal{I}/\mathcal{I}^2$ is the conormal bundle of a local complete intersection subscheme. In general, it is the conormal *sheaf*.

Consider $\sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}$. (Recall that $\underline{\mathrm{Proj}}$ of this sheaf gives us the exceptional divisor of the blow-up.) *Define* the normal cone $C = C_X Y$ by

$$C = \underline{\mathrm{Spec}} \sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

Define the *Segre class* of X in Y as the Segre class of the normal cone:

$$s(X, Y) = s(C_X Y) \in A_* X.$$

Proposition Let $f : Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, $g : X' \to X$ the induced morphism.

(a) If f is proper, Y irreducible, and f maps each irreducible component of Y' onto Y then

$$g_*(s(X', Y')) = \deg(Y'/Y) s(X, Y).$$

(b) If f is flat, then

$$g^*(s(X', Y')) = s(X, Y).$$

Let me point out why I find this a remarkable result. X' is a priori some nasty scheme; even if it is nice, its codimension in Y' isn't necessarily the same as the codimension of X in Y. The argument is quite short, and shows that what we've proved already is quite sophisticated.

I will give the proof next time. Today I gave most of the proof, by describing the diagram around which everything revolves.

Let me assume that Y' is irreducible. (It's true in general, and I may deal with the general case later.)

Let me first write the diagram on the board, and then explain it.

$$\begin{tikzcd}
& \mathcal{O}_{\underline{\mathrm{Proj}}(C'\oplus\mathbf{1})}(1) = G^*\mathcal{O}_{\underline{\mathrm{Proj}}(C\oplus\mathbf{1})}(1) \arrow[dr] & & \\
& \mathcal{O}_{\underline{\mathrm{Proj}}(C\oplus\mathbf{1})}(1) \arrow[dr] & \underline{\mathrm{Proj}}(C'\oplus\mathbf{1}) \arrow[r, "\text{Cartier div}"] \arrow[d, "G"] \arrow[ddll, "q'"'] & \mathrm{Bl}_{X'\times 0}(Y'\times\mathbb{A}^1) \arrow[d, "F"] \\
& & \underline{\mathrm{Proj}}(C\oplus\mathbf{1}) \arrow[r, "\text{Cartier div}"] \arrow[ddll, "q"] & \mathrm{Bl}_{X\times 0}(Y\times\mathbb{A}^1) \\
X' \arrow[d, "g"] & & & \\
X & & &
\end{tikzcd}$$

Explanation: We blow up $Y \times \mathbb{A}^1$ along $X \times 0$, and similarly for Y' and X'. The exceptional divisor of $\mathrm{Bl}_{X\times 0}(Y\times\mathbb{A}^1)$ is $\underline{\mathrm{Proj}}(C\oplus\mathbf{1})$, and similarly for Y' and X'. The universal property of blowing up $Y \times \mathbb{A}^1$ shows that there exists a unique morphism G from the top exceptional divisor to the bottom. Moreover, by construction, the exceptional divisor upstairs is the pullback of the exceptional divisor downstairs (that's the statement about the two $\mathcal{O}(1)$'s in the diagram). Let q be the morphism from the exceptional divisor $\underline{\mathrm{Proj}}(C \oplus \mathbf{1})$ to X, and similarly for q'. That square commutes: $q \circ G = g \circ q'$ (basically because that morphism G was defined by the universal property of blowing up).

We'll finish the proof next time (and I'll describe this diagram once again).

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INTERSECTION THEORY CLASS 13

RAVI VAKIL

CONTENTS

1. Where we are: Segre classes of vector bundles, and Segre classes of cones

We first defined *Segre class of vector bundles* over an arbitrary scheme X. If E is a vector bundle, we get an operator on class on X. We define it by projectivizing E, so we have a flat and proper morphism $\mathbb{P}E \to X$, pulling back α to $\mathbb{P}E$, capping with $\mathcal{O}(1)$ a certain number of times, and pushing forward.

Hence we get $s_i(E) \cap : A_k X \to A_{k-i} X$, and for example we checked the non-immediate fact that $s_0(E)$ is the identity. (Recall s_0 involved pulling back, capping with precisely $\operatorname{rank} E - 1$ copies of $\mathcal{O}(1)$, and then pushing forward.) Note that $s_k(E) = s_k(E \oplus \mathbf{1})$, as the Whitney product formula gives $s(E \oplus \mathbf{1}) = s(E)s(\mathbf{1}) = s(E)$.

We want to generalize this to cones. Here again is the definition of a *cone* on a scheme X. Let $S^\cdot = \oplus_{i \geq 0} S^i$ be a sheaf of graded $\mathcal{O}_X$-algebras. Assume $\mathcal{O}_X \to S^0$ is surjective, S^1 is coherent, and $S^\cdot$ is generated (as an algebra) by S^1. Then you can define $\underline{\operatorname{Proj}}(S^\cdot)$, which has a line bundle $\mathcal{O}(1)$. $\underline{\operatorname{Proj}}(S^\cdot) \to X$ is a projective (hence proper) morphism, but it isn't necessarily flat! (Draw a picture, where the cone has components of different dimension.) Flat morphisms have equidimensional fibers, and cones needn't have this.

A couple of important points, brought out by Joe and Soren. I've been imprecise with terminology. Although one often sees phrases such as "the cone is $C = \operatorname{Spec}(S^\cdot)$", we lose a little information this way; the cone should be defined to be the graded sheaf $S^\cdot$. The sheaf can be recovered from $C_X Y$ along with the action of the multiplicative group $\mathcal{O}_X^*$; the nth graded piece is the part of the algebra where the multiplicative group acts with weight n.

Example 1: say let E be a vector bundle, and $S^i = \operatorname{Sym}^i(E^\vee)$. Then $\underline{\operatorname{Proj}}\, S^\cdot = \mathbb{P}E$. *Example 2:* Say $T^i = \operatorname{Sym}^i(E^\vee \oplus \mathbf{1}) = S^i \oplus S^{i-1} z$, so (better) $T^\cdot = S^\cdot[z]$. Then $\underline{\operatorname{Proj}}\, T^\cdot = \mathbb{P}E$. *Example 3:*

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$\underline{\mathrm{Proj}}(S^\cdot[z]) = C \coprod \underline{\mathrm{Proj}}(S^\cdot) = \underline{\mathrm{Spec}}\, S^\cdot \coprod \underline{\mathrm{Proj}}(S^\cdot)$. The argument is just the same. The right term is a Cartier divisor in class $\mathcal{O}_{\underline{\mathrm{Proj}}(S^\cdot[z])}(1)$. *Example 4:* The blow-up can be described in this way, and it will be good to know this. Suppose X is a subscheme of Y, cut out by ideal sheaf $\mathcal{I}$. (In our situation where all schemes are finite type, $\mathcal{I}$ is a coherent sheaf.) Then let $S^\cdot = \oplus_i \mathcal{I}^i$, where $\mathcal{I}$ is the ith power of the ideal $\mathcal{I}$. ($\mathcal{I}^0$ is defined to be $\mathcal{O}_X$.) Then $\mathrm{Bl}_X Y \cong \underline{\mathrm{Proj}}\, S^\cdot$. A short calculation shows that the exceptional divisor class is $\mathcal{O}(-1)$. The *exceptional divisor* turns out to be $\underline{\mathrm{Proj}} \oplus \mathcal{I}^n/\mathcal{I}^{n+1}$. (Note that this is indeed a graded sheaf of algebras.) As $\oplus \mathcal{I}^n \to \oplus \mathcal{I}^n/\mathcal{I}^{n+1}$ is a surjective map of rings, this indeed describes a closed subscheme of the blow-up. (Remember this formula — it will come up again soon!)

So the same construction of Segre classes of vector bundles doesn't work: there is no flat pullback to $\underline{\mathrm{Proj}}(S^\cdot)$. So what do we do?

Idea (slightly wrong): We can't pull classes back to $\underline{\mathrm{Proj}}(S^\cdot)$. But there is a natural class up there already: the fundamental class. So we define

$$s(C) \stackrel{?}{=} q_*\left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [\underline{\mathrm{Proj}}\, C]\right)$$

where q is the morphism $\underline{\mathrm{Proj}}\, C \to X$. Instead, as Segre class of vector bundles are stable with respect to adding trivial bundles, we define

$$s(C) := q_*\left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [\underline{\mathrm{Proj}}(C \oplus \mathbf{1})]\right)$$

where q is the morphism $\underline{\mathrm{Proj}}(C \oplus \mathbf{1}) \to X$. Why is adding in this trivial factor the right thing to do? Partial reason: if C is the 0 cone, i.e. $S^i = 0$ for $i > 0$, then $\underline{\mathrm{Proj}}\, C$ is empty, but $\underline{\mathrm{Proj}}\, C \oplus \mathbf{1}$ is not; we get different answers. But if you add more $\mathbf{1}$'s, you will then get the same answer: $s(C \oplus \mathbf{1} \oplus \cdots \oplus \mathbf{1}) = s(C)$.

(Exercise: show that $s(C \oplus \mathbf{1}) = s(C)$.)

Note: s has pieces in various dimensions.

Last time I proved:

Proposition. (a) If E is a vector bundle on X, then $s(E) = c(E)^{-1} \cap [X]$, where $c(E)$ is the total Chern class of X, $r = \mathrm{rank}(E)$. $c(E) = 1 + c_1(E) + \cdots + c_r(E)$. (I would write $s(E) = s(E) \cap [X]$, but the two uses of $s(E)$ are confusing!) This is basically our definition of Segre/Chern classes.

(b) Let $C_1, \ldots, C_t$ be the irreducible components of C, m_i the geometric multiplicities of C_i in C. Then $s(C) = \sum_{i=1}^t m_i s(C_i)$. (Note that the C_i are cones as well, so $s(C_i)$ makes sense.) In other words, we can compute the Segre class piece by piece.

2. The normal cone, and the Segre class of a subvariety

Let X be a closed subscheme of a scheme Y (not necessarily lci = local complete intersection), cut out by ideal sheaf $\mathcal{I}$.

$\mathcal{I}/\mathcal{I}^2$ is the conormal sheaf to X; it is a sheaf on X. (Why is it a sheaf on X? Locally, say $Y = \operatorname{Spec} R$, and $X = \operatorname{Spec} R/I$. Then this is the R-module I/I^2. The fact that I said that it is an R-module makes it a priori a sheaf on Y. But note that it is also an R/I module; the action of I on I/I^2 is the zero action.) If X is a local complete intersection (regular imbedding), then this turns out to be a vector bundle.

Consider $\sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}$. (Recall that $\underline{\operatorname{Proj}}$ of this sheaf gives us the exceptional divisor of the blow-up.) *Define* the normal cone $C = C_X Y$ by

$$C = \underline{\operatorname{Spec}} \sum_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

Define the *Segre class* of X in Y as the Segre class of the normal cone:

$$s(X,Y) = s(C_X Y) \in A_* X.$$

If X is regularly imbedded (=lci) in Y, then the definition of $s(X,Y)$ is

$$s(X,Y) = s(N) \cap [X] = c(N)^{-1} \cap [X].$$

The following geometric picture will come up in the central construction in intersection (the deformation to the normal cone). $X \times \mathbb{A}^1 \hookrightarrow Y \times \mathbb{A}^1$. Then blow up $X \times 0$ in $Y \times \mathbb{A}^1$. The ideal sheaf of $X \times 0$ is $\mathcal{I}[t]$, where t is the coordinate on $\mathbb{A}^1$. Thus the normal cone to $X \times 0$ in $Y \times \mathbb{A}^1$ is $C_X Y[t]$. Hence the exceptional divisor is $\underline{\operatorname{Proj}}(C_X Y[t])$ (draw a picture). Inside it is the Cartier divisor $t = 0$, which is $\underline{\operatorname{Proj}}(C_X Y)$.

3. Segre classes behave well with respect to proper and flat morphisms

This is the key result of the chapter.

Proposition. Let $f : Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, $g : X' \to X$ the induced morphism.

(a) If f proper, Y irreducible, and f maps each irreducible component of Y' onto Y then

$$g_*(s(X',Y')) = \deg(Y'/Y) s(X,Y).$$

(b) If f flat, then

$$g^*(s(X',Y')) = s(X,Y).$$

Let me repeat why I find this a remarkable result. X' is a priori some nasty scheme; even if it is nice, its codimension in Y' isn't necessarily the same as the codimension of X in Y. The argument is quite short, and shows that what we've proved already is quite sophisticated.

As a special case, this result shows that Segre classes have a fundamental birational invariance: if $f : Y' \to Y$ is a birational proper morphism, and $X' = f^{-1}X$, then $s(X', Y')$ pushes forward to $s(X, Y)$.

Proof. Let me assume that Y' is irreducible. (It's true in general, and I may deal with the general case later.)

Let me first write the diagram on the board, and then explain it.

$$\begin{tikzcd}
 & \mathcal{O}_{\underline{\mathrm{Proj}}(C'\oplus\mathbf{1})}(1) = G^*\mathcal{O}_{\underline{\mathrm{Proj}}(C\oplus\mathbf{1})}(1) \arrow[dr] & & \\
 & \mathcal{O}_{\underline{\mathrm{Proj}}(C\oplus\mathbf{1})}(1) \arrow[dr] & \underline{\mathrm{Proj}}(C'\oplus\mathbf{1}) \arrow[r, "\text{Cartier div}"] \arrow[d, "G"] \arrow[ddll, "q'"'] & \mathrm{Bl}_{X'\times 0}(Y'\times\mathbb{A}^1) \arrow[d, "F"] \\
 & & \underline{\mathrm{Proj}}(C\oplus\mathbf{1}) \arrow[r, "\text{Cartier div}"] \arrow[ddll, "q"] & \mathrm{Bl}_{X\times 0}(Y\times\mathbb{A}^1) \\
X' \arrow[d, "g"] & & & \\
X & & &
\end{tikzcd}$$

We blow up $Y \times \mathbb{A}^1$ along $X \times 0$, and similarly for Y' and X'. The exceptional divisor of $\mathrm{Bl}_{X\times 0}(Y \times \mathbb{A}^1)$ is $\underline{\mathrm{Proj}}(C \oplus \mathbf{1})$, and similarly for Y' and X'. The universal property of blowing up $Y \times \mathbb{A}^1$ shows that there exists a unique morphism G from the top exceptional divisor to the bottom. Moreover, by construction, the exceptional divisor upstairs is the pullback of the exceptional divisor downstairs (that's the statement about the two $\mathcal{O}(1)$'s in the diagram). Let q be the morphism from the exceptional divisor $\underline{\mathrm{Proj}}(C \oplus \mathbf{1})$ to X, and similarly for q'. That square commutes: $q \circ G = g \circ q'$ (basically because that morphism G was defined by the universal property of blowing up).

Now $f_*[Y' \times \mathbb{A}^1] = d[Y \times \mathbb{A}^1]$ (where I am sloppily using the name f for the morphism $Y' \times \mathbb{A}^1 \to Y \times \mathbb{A}^1$). This is computed on a dense open set, so blow-up doesn't change this fact:

$$F_*[\mathrm{Bl}_{X'\times 0} Y' \times \mathbb{A}^1] = d[\mathrm{Bl}_{X\times 0} Y \times \mathbb{A}^1].$$

Now we've shown that proper pushforward commutes with intersecting with a (pseudo-)Cartier divisor. Hence

$$G_*[\underline{\mathrm{Proj}}(C' \oplus \mathbf{1})] = d[\underline{\mathrm{Proj}}(C \oplus \mathbf{1})].$$

Now I'm going to prove (a), and I'm going to ask you to prove (b) with me, so pay attention!

$$\begin{aligned}
g_*s(X',Y') &= g_*q'_*\left(\sum_i c_1(G^*(\mathcal{O}(1))^i \cap [\mathbb{P}(C' \oplus \mathbf{1})])\right) \quad \text{(by def'n)} \\
&= q_*G_*\left(\sum_i c_1(G^*(\mathcal{O}(1))^i \cap [\mathbb{P}(C' \oplus \mathbf{1})])\right) \quad \text{(prop. push. commute)} \\
&= q_*\left(\sum_i c_1((\mathcal{O}(1))^i \cap d[\mathbb{P}(C \oplus \mathbf{1})])\right) \quad \text{(proj. form.)} \\
&\qquad \text{(i.e. } c_1 \text{ commutes with prop. pushforward)} \\
&= ds(X,Y) \quad \text{(by def'n)}
\end{aligned}$$

Now (b) is similar:

$$\begin{aligned}
g^*s(X,Y) &= g^*q_*\left(\sum_i c_1((\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus \mathbf{1})])\right) \quad \text{(by def'n)} \\
&= q'_*G^*\left(\sum_i c_1((\mathcal{O}(1))^i \cap [\mathbb{P}(C \oplus \mathbf{1})])\right) \quad \text{(push/pull commute)} \\
&= q'_*\left(\sum_i c_1((G^*\mathcal{O}(1))^i \cap G^*[\mathbb{P}(C \oplus \mathbf{1})])\right) \\
&= s(X,Y) \quad \text{(by def'n)}
\end{aligned}$$

□

We immediately have:

Corollary. With the same assumptions as the proposition, if X' is *regular imbedded* (=lci) in Y' , with normal bundle N', then

$$g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)s(X,Y).$$

If $X \subset Y$ is also regularly imbedded, with normal bundle N, then

$$g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)(c(N)^{-1} \cap [X]).$$

To see why the first part might matter: Suppose $X \hookrightarrow Y$ is a very nasty closed immersion. Then blow up Y along X, to get Y' with exceptional divisor X'. Then X' *is* regularly imbedded (lci) in Y' — it is a Cartier divisor! This is the content of the next corollary.

Corollary. Let X be a open closed subscheme of a variety Y. Let $\tilde{Y}$ be the blow-up of Y along X, $\tilde{X} = \mathbb{P}C$ the exceptional divisor, $\eta : \tilde{X} \to X$ the projection. Then

$$\begin{aligned} s(X, Y) &= \sum_{k \geq 1} (-1)^{k-1} \eta_*(\tilde{X}^k) \\ &= \sum_{i \geq 0} \eta_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}C]) \end{aligned}$$

In that first equation, the term $\tilde{X}^k$ should be interpreted as the kth self intersection of the Cartier divisor $\tilde{X}$, also known as the exceptional divisor.

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INTERSECTION THEORY CLASS 14

RAVI VAKIL

CONTENTS

1. WHERE WE ARE: SEGRE CLASSES OF VECTOR BUNDLES, AND SEGRE CLASSES OF CONES

1.1. **Segre classes of cones.** Once again, the definition of a *cone* on a scheme X. Let $S^\cdot = \oplus_{i \geq 0} S^i$ be a sheaf of graded $\mathcal{O}_X$-algebras. Assume $\mathcal{O}_X \to S^0$ is surjective, S^1 is coherent, and $S^\cdot$ is generated (as an algebra) by S^1. I'm happy calling this the cone. $C = \underline{\mathrm{Spec}}\, S^\cdot$. $\underline{\mathrm{Proj}}(S^\cdot)$ has a line bundle $\mathcal{O}(1)$. (The "underline" under Spec and $\underline{\mathrm{Proj}}$ is meant to distinguish the "sheafy" version from the usual version of these constructions.) Define the *Segre class*

$$s(C) := q_*(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [\underline{\mathrm{Proj}}(C \oplus \mathbf{1})])$$

where q is the morphism $\underline{\mathrm{Proj}}(C \oplus \mathbf{1}) = \underline{\mathrm{Proj}}(S^\cdot[t]) \to X$.

If $X \hookrightarrow Y$ is a closed immersion of schemes, the *normal cone* is $\sum_{n=0}^{\infty} \mathcal{I}^n / \mathcal{I}^{n+1}$. The Segre class of X in Y is defined to be the Segre class of the normal cone. More on the normal cone shortly. Last day we finished proving:

Proposition ("functoriality of Segre classes of subschemes"). Let $f : Y' \to Y$ be a morphism of pure-dimensional schemes, $X \subset Y$ a closed subscheme, $X' = f^{-1}(X)$ the inverse image scheme, $g : X' \to X$ the induced morphism.

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(a) If f proper, Y irreducible, and f maps each irreducible component of Y' onto Y then
$$g_*(s(X', Y')) = \deg(Y'/Y)s(X, Y).$$

(b) If f flat, then
$$g^*(s(X', Y')) = s(X, Y).$$

2. What the "functoriality of Segre classes of subschemes" buys us

As a special case, this result shows that Segre classes have a fundamental birational invariance: if $f : Y' \to Y$ is a birational proper morphism, and $X' = f^{-1}X$, then $s(X', Y')$ pushes forward to $s(X, Y)$.

From (a), we immediately have:

Corollary. With the same assumptions as the proposition, if X' is *regular imbedded* (=lci) in Y' , with normal bundle N', then
$$g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)s(X, Y).$$
If $X \subset Y$ is also regularly imbedded, with normal bundle N, then
$$g_*(c(N')^{-1} \cap [X']) = \deg(Y'/Y)(c(N)^{-1} \cap [X]).$$

To see why the first part might matter: Suppose $X \hookrightarrow Y$ is a very nasty closed immersion. Then blow up Y along X, to get Y' with exceptional divisor X'. Then X' *is* regularly imbedded (lci) in Y' — it is a Cartier divisor! This is the content of the next corollary.

Corollary. Let X be a closed subscheme of a variety Y. Let $\tilde{Y}$ be the blow-up of Y along X, $\tilde{X} = \mathbb{P}C$ the exceptional divisor, $\eta : \tilde{Y} \to Y$ the projection. Then
$$\begin{aligned} s(X, Y) &= \sum_{k \geq 1} (-1)^{k-1} \eta_*(\tilde{X}^k) \\ &= \sum_{i \geq 0} \eta_*(c_1(\mathcal{O}(1))^i \cap [\mathbb{P}C]) \end{aligned}$$

In that first equation, the term $\tilde{X}^k$ should be interpreted as the kth self intersection of the Cartier divisor $\tilde{X}$, also known as the exceptional divisor. In other words, it should be interpreted as meaning the second line.

2.1. The multiplicity of a variety along a subvariety.

We'll now define them multiplicity of a scheme Y along a subvariety X. (As a special case, this will define the multiplicity of a variety at a closed (=old-fashioned) point. That special case is a fundamental commutative algebra notion due to Samuel.) If the general point of X is a smooth point of Y, we'll get 1. Definition: $s(X, Y) \in A_*X$. Then $s(X, Y) = e_X Y[X]$ + lower order terms. $e_X Y$ is the multiplicity.

Useful exercise: What is the multiplicity of $(0,0)$ in the cusp $y^2 - x^3$? Here the characteristic is not 2 or 3. (Answer: 2. Hint: blow this up. The blow-up is $\operatorname{Spec} k[t] \to \operatorname{Spec} k[x,y]/(y^2 - x^3)$ given by $t \mapsto (t^2, t^3)$.)

Example. If $\operatorname{codim}(X, Y) = n > 0$, define the multiplicity $e_X Y$ as follows. Let q be the projection $\underline{\operatorname{Proj}}(C \oplus \mathbf{1}) \to X$ and p be the projection $\underline{\operatorname{Proj}}\, C \to X$.

$$\begin{aligned} e_X Y[X] &= q_*(c_1(1))^n \cap [\underline{\operatorname{Proj}}(C \oplus \mathbf{1})] \\ &= p_*(c_1(\mathcal{O}(1))^{n-1} \cap [\underline{\operatorname{Proj}}\, C] \\ &= (-1)^{n-1} p_*(\tilde{X}^n) \end{aligned}$$

Here $\tilde{X}$ is the exceptional divisor of the blow-up.

Back to the multiplicity of a variety at a closed (=old-fashioned) point: Let A be the local ring of Y at our point, $\mathfrak{m}$ the maximal ideal of A, $A/\mathfrak{m} = k$. *Fact:*

$$\dim_k \left(\sum_{i=1}^{t} \mathfrak{m}^{i-1}/\mathfrak{m}^i \right) = l_A(A/\mathfrak{m}^t)$$

is a polynomial of degree $n = \dim Y$ in t for $t \gg 0$, whose leading term is $(e_X Y) t^n / n!$. This even works at a (non-closed) point; just take A to be the local ring of Y along X, and $n = \operatorname{codim}(X, Y)$.

Useful exercise: See that this works in for the cusp point (the previous useful exercise). Note that as a vector space $k[x,y]/(y^2 - x^3) = \oplus_{n \geq 0, n \neq 1} k t^n$; note that the $n = 2$ term is kx, the $n = 3$ term is ky, the $n = 4$ term is kx^2, the $n = 5$ term is kxy, and the $n = 6$ term is $kx^3 = ky^2$.

3. Deformation to the normal cone

We next come to the central construction. There's not much for us to do here, as we've built up all the necessary machinery, and even seen the construction.

Here is the main goal. Suppose $X \to Y$ is a closed immersion of schemes. The idea is that $C = C_X Y$ "looks like Y near X"; it "is like a tubular neighborhood". But it is nicer than Y near X; in particular it is a cone.

Goal: We will define a *specialization homomorphism* $\sigma : A_k Y \to A_k C$.

I'll try to give you an intuitive idea for what this means. (Try it.)

3.1. The construction.

Here's how we do it. Let me set some notation. If $W \hookrightarrow Z$ is a closed immersion, recall that $\operatorname{Bl}_W Z$ is the blow-up of Z along W. For the purposes of the next few lectures, let $E_W Z$ be the exceptional divisor, and let $\mathcal{I}_W Z$ be the ideal sheaf. Then recall:

- $\operatorname{Bl}_W Z = \underline{\operatorname{Proj}} \oplus (\mathcal{I}_W Z)^n$
- $E_W Z = \underline{\operatorname{Proj}} \oplus (\mathcal{I}_W Z)^n / (\mathcal{I}_W Z)^{n+1}$

- $E_W Z \hookrightarrow \mathrm{Bl}_W Z$ is a closed immersion, and describes $E_W Z$ as an effective Cartier divisor, in fact in class $\mathcal{O}_{\underline{\mathrm{Proj}}\oplus(\mathcal{I}_W Z)^n}(1)$. The closed immersion is visible at the level of graded algebras

Suppose now that $X \hookrightarrow Y$ is a closed immersion. (Other notation: when Fulton says "imbedding", we will say "closed immersion".) Let's blow up $Y \times \mathbb{P}^1$ along $X \times 0$ and see what we get. (Here let t be a coordinate on $\mathbb{P}^1$. Notational caution: Fulton prefers to blow up $X \times \infty$.) We certainly have a morphism to $\mathbb{P}^1$:

$$\mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1) \to Y \times \mathbb{P}^1 \to \mathbb{P}^1.$$

Away from $t = 0$, the blow-up doesn't do anything: $\mathrm{Bl}_{X\times 0} Y|_{t\neq 0} = Y \times (\mathbb{P}^1 - 0)$.

So what is the fiber over $t = 0$? I claim it is the union of two things, that we can identify. One "piece" is $\mathrm{Bl}_X Y$, with exceptional (Cartier) divisor $E_X Y$. The other piece is $\underline{\mathrm{Proj}}(C_X Y \oplus \mathbf{1})$; this has a Cartier divisor "$\mathbb{P}(C_X Y \oplus \mathbf{1})$" $= C_X Y \oplus \mathbb{P} C_X Y \cong E_X Y$. We glue these two pieces together along $E_X Y$.

I want to convince you that we really get these two pieces. If you've never seen this before, I want to convince you that we get those two pieces, and you can be happy with that. At the end I'll explain how to verify that we get nothing else.

Consider the morphism $\mathrm{Bl}_{X\times 0} Y \times \mathbb{P}^1 \to Y \times \mathbb{P}^1$. Away from the $X \times 0$ on the target, this is an isomorphism. The exceptional divisor is

$$\begin{aligned} E_{X\times 0}\left(Y \times \mathbb{P}^1\right) &= \underline{\mathrm{Proj}} \oplus \left((I_{X\times 0} Y \times \mathbb{P}^1)^n / (I_{X\times 0} Y \times \mathbb{P}^1)^{n+1}\right) \\ &\cong \underline{\mathrm{Proj}} \left(\oplus (I_X Y)^n / (I_X Y)^{n+1}\right)[t] \\ &\cong \mathbb{P}(C_X Y \oplus \mathbf{1}). \end{aligned}$$

So we see the projective completion of the normal cone in this blow-up.

Let's next see the piece $\mathrm{Bl}_X Y$. Translation: we want a morphism $\mathrm{Bl}_X Y$ to $\mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1)$ that lies in the scheme-theoretic fiber $t = 0$, and we want this to be a closed immersion. I will just show you that the morphism exists; as usual we use the universal property. Consider the map $\mathrm{Bl}_X Y \to Y \times \mathbb{P}^1$ obtained via $\mathrm{Bl}_X Y \to Y \times 0 \hookrightarrow Y \times \mathbb{P}^1$. The pullback of $X \times 0$ is an effective Cartier divisor $E_X Y$. Thus by the universal property of blowing-up, we get a morphism $\mathrm{Bl}_X Y \to \mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1)$.

So I've given you an indication that we see both the projective completion of the normal cone, and $\mathrm{Bl}_X Y$, in the central fiber ($t = 0$) of $\mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1)$. How would you show that this is all we get, and that they are glued together along $E_X Y$? This is a local question, so we can take $Y = \mathrm{Spec}\, A$, and $X = \mathrm{Spec}\, A/I$. Then the question becomes completely explicit: $Y \times \mathbb{A}^1 = \mathrm{Spec}\, A[t]$. (We can work locally in $\mathbb{P}^1$ as well.) Then $\mathrm{Bl}_{X\times 0}(Y \times \mathbb{P}^1)$ locally is $\mathrm{Proj} \oplus (I, t)^n$. We are interested in the fiber over $t = 0$, so we mod out by t:

$$\rho^{-1}(0) = \mathrm{Proj}\left(\left(\oplus (I, t)^n\right) / \left(t \left(\oplus (I, t)^n\right)\right)\right).$$

We want to show that this is the union of

$$E_{X\times 0}(Y \times \mathbb{P}^1) = \mathrm{Proj}\left(\oplus (I, t)^n / (I, t)^{n+1}\right)$$

and

$$\mathrm{Bl}_X Y = \operatorname{Proj} \oplus I^n$$

glued along

$$E_X Y = \operatorname{Proj} \oplus \left(I^n / I^{n+1} \right).$$

Consider ordered pairs of elements of the second and third graded rings, that are required to give the same element in the fourth graded ring. Show that this ring is the same as the first graded ring. Finally, realize that this algebraic statement is precisely the geometric statement you want to prove. (I'm not going to give the details.)

4. Specialization to the normal cone

Let $X \hookrightarrow Y$ be a closed subscheme of a scheme, and $C = C_X Y$ the normal cone to X in Y. Recall our goal: to define *specialization homomorphism* $\sigma : A_k Y \to A_k C$.

Let me now do it. Let $M^\circ = \mathrm{Bl}_{X \times 0}(Y \times \mathbb{P}^1) - \mathrm{Bl}_X Y$. A picture is helpful here. Away from 0, M° is still $Y \times \mathbb{A}^1$. Over 0, the big blow-up was the projectivized completion of the normal cone $C_X Y \coprod \mathbb{P} C_X Y$ glued to $\mathrm{Bl}_X Y$ along $E_X Y = \mathbb{P} C_X Y$. We're throwing out $\mathrm{Bl}_X Y$, so the central fiber is now just the normal cone $C_X Y$. So we have really deformed Y to the normal cone. Hence this scheme M° is often called the "deformation to the normal cone". Let $i : C \hookrightarrow M^\circ$ be the closed immersion of the normal cone, and let $j : Y \times (\mathbb{P}^1 - 0) \hookrightarrow M^\circ$ be the open immersion of the complement.

Consider the following diagram:

$$\begin{tikzcd}
A_{k+1} C \arrow[r, "i_*"] & A_{k+1} M^\circ \arrow[r, "j^*"] \arrow[d, "i^*", "\text{Gysin map for divisors}"'] & A_{k+1}(Y \times \mathbb{A}^1) \arrow[r] & 0 \\
& A_k C & A_k Y. \arrow[u, "\sim"] &
\end{tikzcd}$$

The top row is the excision exact sequence. The right column is flat pullback and is an isomorphism, as flat pullback to the total space of a line bundle is always an isomorphism. The left column is the Gysin pullback map to divisors.

Now we have shown $i^* i_* : A_{k+1} C \to A_k C$ is the same as capping with c_1 of the normal (line) bundle to the divisor C in M°. But in this case the normal line bundle is trivial: it is the pullback of the normal bundle to $t = 0$ in $\mathbb{P}^1$. Thus $i^* i_* = 0$. Hence $A_{k+1} M^\circ \to A_k C$ descends to a map $A_{k+1}(Y \times \mathbb{A}^1) \to A_k C$, and hence we get a map $\sigma : A_k Y \to A_k C$, which is what we wanted! Here's the final diagram:

$$\begin{tikzcd}
A_{k+1} C \arrow[r, "i_*"] \arrow[dr, "i^* i_* = 0"'] & A_{k+1} M^\circ \arrow[r, "j^*"] \arrow[d, "i^*"] & A_{k+1}(Y \times \mathbb{A}^1) \arrow[r] \arrow[dl, "\therefore"'] & 0 \\
& A_k C & A_k Y. \arrow[u, "\sim"] \arrow[l, "\therefore \sigma"] &
\end{tikzcd}$$

Remark We could define this morphism more explicitly as follows. Define $\sigma : Z_k Y \to Z_k C$ by $\sigma([V]) = [C_{V \cap X} V]$ where V is a subvariety of Y. (Extend this to $Z_k Y$ by linearity.) Note that $C_{V\cap X}V \hookrightarrow C_X Y$, so this makes sense. **Proposition.** This descends to the morphism $A_k Y \to A_k C$ I just defined. *Sketch of proof.* In the bottom row of that last big diagram, it suffices to verify that $[V] \mapsto [C_{V\cap X}V]$. Hence it suffices to show that in the "southwest" morphism in the big diagram (marked "$\therefore$"), $[V \times \mathbb{A}^1]$ maps to $[C_{V\cap X}V]$. We take the subvariety $V \times \mathbb{A}^1 \hookrightarrow Y \times (\mathbb{P}^1 - 0)$, take its closure in M°, and intersect with the Cartier divisor $(t = 0) = C$. We can do this explicitly locally on Y, using $Y = \operatorname{Spec} A$, $X = \operatorname{Spec} A/I$, etc.; I'll omit this since I don't think we'll need this fact.

Corollary. Suppose $i : X \hookrightarrow Y$ is a locally complete intersection (regular imbedding) of codimension d, with normal bundle N. Define the *Gysin homomorphism* or *Gysin pullback*

$$i^* : A_k Y \to A_{k-d} X$$

as the composition

$$A_k Y \xrightarrow{\sigma} A_k N \xrightarrow{s_N^*} A_{k-d} X.$$

4.1. **Gysin pullback for local complete intersections.** We already had defined the Gysin pullback or Gysin homomorphism in the case where Y is a vector bundle over X. This extends it to when "Y looks like a vector bundle over X". Notice that the two definitions agree; one needs to check that the normal cone to a the zero section of a vector bundle is the vector bundle itself (which is true). Also, we showed that the Gysin pullback for vector bundles satisfied all sorts of nice properties; if we show that σ satisfies these nice properties too, then we'll know it for Gysin pullbacks to local complete intersections.

Note: $i^* i_*(\alpha) = c_d(N) \cap \alpha$. Reason: we know this for vector bundles.

Note also: If Y is purely n-dimensional, notice that $i^*[Y] = [X]$. Because $\sigma[Y] = [C]$, and $s_N^*[C] = [X]$.

4.2. **Intersection products on smooth varieties!** If X is an n-dimensional variety which is smooth over the ground field, then the diagonal morphism $\Delta : X \to X \times X$ is a local complete intersection of codimension n. Then we get an intersection product on $A_* X$!

$$A_p X \otimes A_q X \xrightarrow{\times} A_{p+q}(X \times X) \xrightarrow{\Delta^*} A_{p+q-n} X.$$

(Notice that we don't need X to be proper!)

I should probably be a bit clearer about that first map, which might reasonably be called $\boxtimes$. (You can see a discussion in Chapter 1 if you want.) Here's what we need: consider the map

$$Z_p X \otimes Z_q Y \xrightarrow{\times} Z_{p+q}(X \times Y)$$

defined on varieties by $[V] \times [W] = [V \times W]$, and defined generally by linearity. (We'll take $X = Y$, but we might as well do this in some generality.)

Lemma. If $\alpha \sim 0$ (or, symmetrically, $\beta \sim 0$) then $\alpha \times \beta \sim 0$.

(This is Prop. 1.10 (a) in the book.)

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1. Linear Systems

General Setup: L is a line bundle on our space X. We choose some nontrivial subspace of sections $V \subset H^0(X, L)$ with basis $\{s_0, \dots, s_r\}$.

General Goal: To understand the map to $\mathbb{P}^r$ induced by the sections, which 'should' be $x \mapsto [s_0(x); \dots; s_r(x)]$.

Possible problems:

(1) Does a choice of coordinate on our line bundle L alter the map?
Nope, since projective space is nice under scalar action
(2) Projective space isn't supposed to have a point $[0; \dots; 0]$, so what do we do when $x \in \ker s_i$ for each s_i?
This is a serious issue.

The good news about our second problem is that the offending set is as nice as we could want: it is naturally the (closed) subscheme of X which is cut out by the section s_i. This subscheme is the base locus of V, which we'll write as B. So we at least have a rational map $\varphi : X - B \to \mathbb{P}^r$.

Example. Let $X = \mathbb{P}^2$, and $L = O_{\mathbb{P}^2}(2)$. Some sections: $J := \{x^2, xy, y^2, xz, yz, z^2\}$. What's the base locus? It's nothing! Yeah! So we get a bona fide map $\mathbb{P}^2 \to \mathbb{P}^5$.

Example. Let's stick with our scheme and line bundle, but pick out some new sections: $R := \{x^2, xy, y^2, xz, yz\}$. Will we get lucky and not have a base locus again? No. This time $B = [0; 0; 1]$. So we get a rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^4$ defined away from $[0; 0; 1]$.

We're not happy with just having a rational map, since we think we should be able to 'fill in' the missing information.

Example. Let's return to the previous example for a minute, and see what's happening near the base locus. We'll let $[\alpha t; \beta t; 1]$ be a point near $[0; 0; 1]$ (of course, we won't let $\alpha = \beta = 0$ yet). Where does this point go?

$$[\alpha^2 t^2; \alpha\beta t^2; \beta t^2; \alpha t; \beta t] = [\alpha^2 t; \alpha\beta t; \beta^2 t; \alpha; \beta].$$

It seems that as we approach $[0; 0; 1]$ along the line $[\alpha t; \beta t; 1]$, the map is taking us to $[0; 0; 0; \alpha; \beta]$. Hmm...seems that we're getting an idea of what happens to a tangent to our base locus...what does that make us think of?

In fact, there is a way to extend the map we've been thinking about to the blowup of X along B, $\tilde{X} = Bl_B X$.

$$\begin{tikzcd} E \arrow[r, hook] & \tilde{X} = Bl_B X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ B \arrow[r, hook] & X \arrow[r, dashed] & \mathbb{P}^r \end{tikzcd}$$

How can we get our hands on f? We're going to pull back the line bundle L to $Bl_B X$ and tweak it remove the base locus. Fact: the bundle $\pi^* L - E$ has no base locus. Now we'll use this new line bundle to map to $\mathbb{P}^r$, and all will be well.

Example. Suppose our variety is $\mathbb{P}^1$, and we've chosen the bundle $O_{\mathbb{P}^1}(2)$, with sections $\{x^2, xy\}$. This gives a map $\mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ away from $[0;1]$, though we 'should' know how to fix this map to make it a bona fide map. How do we resolve this? Let's follow the formula above: we'll blow up $\mathbb{P}^1$ along $[0;1]$ (this will just give us $[0;1] \hookrightarrow \mathbb{P}^1$ back again, since $[0;1]$ is cartier and cut out by $x = 0$), and twist our sections by $-E$, which in this case means divide by x.

$$\begin{tikzcd} {[0;1]} \arrow[r, hook] & \mathbb{P}^1 \arrow[d, "\pi"'] \arrow[dr, "{[x;y]}"] & \\ {[0;1]} \arrow[r, hook] & \mathbb{P}^1 \arrow[r, dashed, "{[x^2;xy]}"] & \mathbb{P}^2 \end{tikzcd}$$

Aren't we in intersection theory? So you might want to know why I'm talking about this in an intersection theory class. The answer is that we can use chern classes to say something about the degree of $f_*[\tilde{X}]$, which we will see is connected to the segre class of B in X and the degree of the map f. Yeah!

$$\begin{aligned} \deg_f \tilde{X} &:= \deg(\tilde{X}/f(\tilde{X})) \int_{\mathbb{P}^r} c_1(O_{\mathbb{P}^r}(1))^{\dim(X)} \cap [f(\tilde{X})] \\ &= \int_{\tilde{X}} c_1(f^*(O_{\mathbb{P}^r}(1)))^{\dim(X)}. \end{aligned}$$

Theorem (see Fulton, Prop 4.4)**.**

$$\deg_f \tilde{X} = \int_X c_1(L)^n - \int_B c(L)^n \cap s(B, X).$$

Sketch of proof. To make life easy, I'll write $\dim(X) = n$. We saw before that $f^*(O_{\mathbb{P}^r}(1)) = \pi^*(L) \otimes O(-E)$, so let's substitute:

$$\begin{aligned}
\deg_f \tilde{X} &= \int_{\tilde{X}} c_1(f^*(O_{\mathbb{P}^r}(1)))^n = \int_{\tilde{X}} (c_1(\pi^* L) - c_1(O(E)))^n \\
&= \sum_{i=0}^{n} (-1)^i \binom{n}{i} \int_X c_1(L)^{n-i} \pi_* \left(c_1(O(E))^i \cap [\tilde{X}] \right) \\
&= \int_X c_1(L)^n - \int_X \sum_{i=1}^{n} \binom{n}{i} c_1(L)^{n-i} \cap (-1)^{i-1} \pi_* \left(E^i \right) \\
&= \int_X c_1(L)^n - \int_X \sum_{i=0}^{n} \binom{n}{i} c_1(L)^{n-i} \cap \sum_{k \geq 1} (-1)^{k-1} \pi_* \left(E^k \right) \\
&= \int_X c_1(L)^n - \int_B (1 + c_1(L))^n \cap s(B, X).
\end{aligned}$$

$\square$

Useful reminder: We saw in class last day that, using our language of the day,

$$s(B, X) = \sum_{k \geq 1} (-1)^{k-1} \pi_*(E^k).$$

Useful reminder: The degree of $\alpha \in A_k X$ is 0 whenever $k > 0$.

Example. Let's return to the first example. According to the previous theorem we evaluate

$$\int_X c_1(O_{\mathbb{P}^2}(2))^2 = 4 \left(\int_X c_1(O_{\mathbb{P}^2}(1)) \right)^2 = 4.$$

Since this example has no base locus, we see that the degree of our map to $\mathbb{P}^5$ is 4.

Example. Let's return to the second example. Since the base locus is $[0;1]$ we see that $s(B, X) = [\mathrm{pt}]$. So our (extended) map has degree $4 - 1 = 3$.

Example. Cremona: Let's try $\mathbb{P}^2 \xrightarrow{[yz;xz;xy]} \mathbb{P}^2$. The base locus is the subscheme $yz = xz = xy = 0$, otherwise known as the three reduced points $[1;0;0]$, etc. Each has segre class of a point, so that we get the degree of the map $4 - 3 = 1$. It's a birational morphism! What's the inverse? Itself.

Example. Here's an example where the image variety has degree 1, so that we're left computing the degree of the map. We'll use $O_{\mathbb{P}^2}(2)$ again, with sections x^2, y^2, z^2. There's no base locus, so our theorem returns 4 for the degree of the map $\mathbb{P}^2 \xrightarrow{[x^2;y^2;z^2]} \mathbb{P}^2$.

Example. If we choose sections x^2, xy, y^2 of $O_{\mathbb{P}^2}(2)$ to map into $\mathbb{P}^2$, the base locus is the non-reduced point $[0;0;1]$. Since the image loses dimension (it sits on the conic $ac = b^2$), our theorem tells us that $4 = e_B X$.

2. Intersection Product

In this section I'm going to tie up some of the loose ends Ravi left for me on Monday. Recall that we're interested in finding a map $A_p(X) \otimes A_q(X) \xrightarrow{\times} A_{p+q-n}(X)$. The things left for me are to clear up that there's a map $Z_k(X) \otimes Z_l(Y) \xrightarrow{\times} Z_{k+l}(X \times Y)$ and that this map gives us a map on cycle classes.

For the first, we define the map by giving its action on subvarieties and extending by linearity. We take $[W] \times [V] \mapsto [W \times V]$. The studious student asks why this product lands in the right cycle class.

$$\begin{aligned} \dim W \times_k V &= \dim\left((W \times_k V) \times_k \bar{k}\right) = \dim\left((W \times_k \bar{k}) \times_{\bar{k}} (V \times_k \bar{k})\right) \\ &= \dim\left(W \times_k \bar{k}\right) + \dim\left(V \times_k \bar{k}\right) = \dim W + \dim V. \end{aligned}$$

Here we have used two exercises from Chap 3 of Hartshorne and the following diagram

$$\begin{tikzcd}
 & {} \arrow[rr, squiggly] \arrow[dl, squiggly] \arrow[dr, dotted] \arrow[dd, dotted] & & V \times_k \bar{k} \arrow[dl, squiggly] \arrow[dd] \\
W \times_k \bar{k} \arrow[rr, squiggly] \arrow[dd] & & \bar{k} \arrow[dd, dotted] & \\
 & W \times_k V \arrow[rr] \arrow[dl] \arrow[dr, dotted] & & V \arrow[dl] \\
W \arrow[rr] & & k &
\end{tikzcd}$$

So we have left to justify that this gives us a map on cycle classes, and on the way we probably expect we get some result about push forwards and pull backs.

Theorem. *Let $\alpha \in A_k(X)$ and $\beta \in A_l(Y)$.*

- *If $\alpha \sim 0$ or $\beta \sim 0$, then $\alpha \times \beta \sim 0$.*
- *The product $f \times g$ of proper (resp., flat) maps is again proper (resp., flat), and $(f \times g)_* \alpha \times \beta = f_* \alpha \times g_* \beta$ (resp., $(f \times g)^* \alpha \times \beta = f^* \alpha \times g^* \beta$).*

Proof. Part 2 will follow once we split up $f \times g$ into the composition of $f \times mboxid$ and $\mathrm{id} \times g$. For part 1, assume that $\alpha \sim 0$, and reduce to the case where $W = Y$. Then $\alpha \times \beta$ is the pull back of α under the projection $X \times W \to X$. Since we can pull back classes under flat morphisms, we win. $\square$

INTERSECTION THEORY CLASS 16

RAVI VAKIL

CONTENTS

1. WHERE WE ARE

We've covered a lot of ground so far. I want to remind you that we've essentially defined a very few things, and spent all our energy on showing that they behave well with respect to each other. In particular: proper pushforward, flat pullback, $c.$, $s.$, $s.(X,Y)$. Gysin pullback for divisors; intersecting with pseudo-divisors. Gysin pullback for 0-sections of vector bundles.

We know how to calculate the Segre class of a cone.

$$s(C) := q_* \left(\sum_{i \geq 0} c_1(\mathcal{O}(1))^i \cap [\underline{\mathrm{Proj}}(C \oplus \mathbf{1})] \right)$$

where q is the morphism $\underline{\mathrm{Proj}}(C \oplus \mathbf{1}) \to X$.

Last day, Andy talked about linear systems.

1.1. Deformation to the normal cone.

Deformation to the normal cone. This is the central construction. Suppose $X \to Y$ is a closed immersion of schemes.

Goal: We will define a *specialization homomorphism* $\sigma : A_k Y \to A_k C$ where C is the normal cone $\sum_{n=0}^{\infty} \mathcal{I}^n / \mathcal{I}^{n+1}$.

If $W \hookrightarrow Z$ is a closed immersion, recall that $\mathrm{Bl}_W Z$ is the blow-up of Z along W. For the purposes of the next few lectures, let $E_W Z$ be the exceptional divisor, and let $\mathcal{I}_W Z$ be the ideal sheaf. Then recall:

Date: Monday, November 15, 2004.

- $\mathrm{Bl}_W Z = \underline{\mathrm{Proj}} \oplus (\mathcal{I}_W Z)^n$
- $E_W Z = \underline{\mathrm{Proj}} \oplus (\mathcal{I}_W Z)^n / (\mathcal{I}_W Z)^{n+1}$
- $E_W Z \hookrightarrow \mathrm{Bl}_W Z$ is a closed immersion, and describes $E_W Z$ as an effective Cartier divisor, in fact in class $\mathcal{O}_{\underline{\mathrm{Proj}} \oplus (\mathcal{I}_W Z)^n}(1)$. The closed immersion is visible at the level of graded algebras.

Blow up $Y \times \mathbb{P}^1$ along $X \times 0$. The central fiber turns into $\mathrm{Bl}_X Y$, union (the exceptional divisor of the blow-up) $\underline{\mathrm{Proj}}(C_X Y \oplus \mathbf{1}) = C_X Y \coprod \mathbb{P} C_X Y \cong E_X Y$. We glue these two pieces together along $E_X Y$.

We throw out $\mathrm{Bl}_X Y$: let $M^\circ = \mathrm{Bl}_{X \times 0}(Y \times \mathbb{P}^1) - \mathrm{Bl}_X Y$. (A picture is helpful here.) Away from 0, M° is still $Y \times \mathbb{A}^1$. Over 0, we see the normal cone $C_X Y$. So we have really deformed Y to the normal cone. Let $i : C \hookrightarrow M^\circ$ be the closed immersion of the normal cone, and let $j : Y \times (\mathbb{P}^1 - 0) \hookrightarrow M^\circ$ be the open immersion of the complement.

The argument from last week was slick enough that I'm going to repeat it (quickly). Consider the following diagram:

$$\begin{tikzcd}
A_{k+1} C \arrow[r, "i_*"] & A_{k+1} M^\circ \arrow[r, "j^*"] \arrow[d, "i^*", "\text{Gysin map for divisors}"'] & A_{k+1}(Y \times \mathbb{A}^1) \arrow[r] & 0 \\
& A_k C & A_k Y. \arrow[u, "\sim"] &
\end{tikzcd}$$

The top row is the excision exact sequence. The right column is flat pullback and is an isomorphism, as flat pullback to the total space of a line bundle is always an isomorphism. The left column is the Gysin pullback map to divisors.

We have shown $i^* i_* : A_{k+1} C \to A_k C$ is the same as capping with c_1 of the normal (line) bundle to the divisor C in M°. (Reminder for future use: if $i : W \hookrightarrow Z$ is the closed immersion of W into a vector bundle over W, as the zero section, then the map $i_* i^* : A_* W \to A_* W$ is capping with the top Chern class of the vector bundle.) In this case the normal line bundle is trivial: it is the pullback of the normal bundle to $t = 0$ in $\mathbb{P}^1$. Thus $i^* i_* = 0$. Hence $A_{k+1} M^\circ \to A_k C$ descends to a map $A_{k+1}(Y \times \mathbb{A}^1) \to A_k C$, and hence we get a map $\sigma : A_k Y \to A_k C$, which is what we wanted! The final diagram:

$$\begin{tikzcd}
A_{k+1} C \arrow[r, "i_*"] \arrow[dr, "i^* i_* = 0"'] & A_{k+1} M^\circ \arrow[r, "j^*"] \arrow[d, "i^*"] & A_{k+1}(Y \times \mathbb{A}^1) \arrow[r] \arrow[dl, "\therefore"'] & 0 \\
& A_k C & A_k Y. \arrow[u, "\sim"] \arrow[l, "\therefore \sigma"] &
\end{tikzcd}$$

1.2. **Gysin pullback for local complete intersections.** We already had defined the Gysin pullback or Gysin homomorphism in the case where Y is a vector bundle over X: $A_k Y \to A_{k-d} X$. We now extend it to when "Y looks like a vector bundle over X": when X is a local complete intersection inside Y. Define the *Gysin pullback* $i^* : A_k Y \to A_{k-d}$ as the composition

$$A_k Y \xrightarrow{\sigma} A_k N \xrightarrow{s_N^*} A_{k-d} X$$

where s_N^* is the old Gysin morphism for vector bundles. We're going to generalize this further soon!

I showed that the two definitions agree, by observing that the normal cone to a the zero section of a vector bundle is the vector bundle itself (which is true). Also, we showed earlier that the Gysin pullback for vector bundles satisfied all sorts of nice properties; if we show that σ satisfies these nice properties too, then we'll know it for Gysin pullbacks to local complete intersections.

Note: $i^* i_*(\alpha) = c_d(N) \cap \alpha$. Reason: we know this for vector bundles.

Note also: If Y is purely n-dimensional, notice that $i^*[Y] = [X]$. Because $\sigma[Y] = [C]$, and $s_N^*[C] = [X]$.

I concluded with:

1.3. **Intersection products on smooth varieties.** If X is an n-dimensional variety which is smooth over the ground field, then the diagonal morphism $\Delta : X \to X \times X$ is a local complete intersection of codimension n. Then we get an intersection product on $A_* X$!

$$A_p X \otimes A_q X \xrightarrow{\times} A_{p+q}(X \times X) \xrightarrow{\Delta^*} A_{p+q-n} X.$$

(Notice that we don't need X to be proper!)

I should probably be a bit clearer about that first map, which might reasonably be called $\boxtimes$. (You can see a discussion in Chapter 1 if you want.) Here's what we need: consider the map

$$Z_p X \otimes Z_q Y \xrightarrow{\times} Z_{p+q}(X \times Y)$$

defined on varieties by $[V] \times [W] = [V \times W]$, and defined generally by linearity. (We'll take $X = Y$, but we might as well do this in some generality.) We want this to descend to the level of Chow classes:

Lemma. If $\alpha \sim 0$ (or, symmetrically, $\beta \sim 0$) then $\alpha \times \beta \sim 0$.

(This is Prop. 1.10 (a) in the book.) Likely exercise: finish this proof.

2. Intersection Products

We're now ready to discuss the last chapter in the core of the book, on intersection products. We'll define the intersection product, and then we'll verify that it has a host of properties. This verification will involve lots of diagram-chasing and symbol-pushing, so I'm going to try to concentrate on helping you keep your eye on the big picture.

What we know so far: proper pushforward, flat pullback, $s.$ of cones, e.g. $s.(X, Y)$, $c.$. Gysin pullbacks have gotten more and more complicated: i) $X \hookrightarrow Y$ as a divisor. More generally

$$\begin{tikzcd} & V \arrow[d] \\ X \arrow[r, "\text{eff. Car. div.}"] & Y \end{tikzcd}$$

Then $X \xrightarrow{\text{loc. com. int.}} Y$. Now we go to the logical extreme:

$$\begin{tikzcd} & V \arrow[d] \\ X \arrow[r, "\text{loc. comp. int.}"] & Y \end{tikzcd}.$$

Here's the context in which we'll work. $i : X \hookrightarrow Y$ will be a local complete intersection of codimension d. Y is arbitrarily horrible. Suppose V is a scheme of pure dimension k, with a map $f : V \to Y$. Here I am *not* assuming V is a closed subscheme of Y. Then define W to be the closed subscheme of V given by pulling back the equations of X in Y:

$$\begin{tikzcd} W \arrow[r, "\text{cl. imm.}"] \arrow[d, "g"] & V \arrow[d, "f"] \\ X \arrow[r, "\text{cl. imm.}"] & Y \end{tikzcd}$$

(notice definiton of g). We'll define the intersection product $X \cdot V \in A_{k-d}W$. (We'll most obviously care about the case where $V \hookrightarrow Y$, but you'll see that this more general case will be handy too!)

The cone of X in Y is in fact a vector bundle (as $X \hookrightarrow Y$ is a local complete intersection); call it $N_X Y$. The cone $C_W Y$ to W in Y may be quite nasty; but we'll see (in just a moment) that it lives in the pullback of the normal bundle: $C_W Y \hookrightarrow g^* N_X Y$. Then we can define

$$X \cdot V = s^*[C_W V]$$

where $s : W \to g^* N_X Y$ is the zero-section. (Recall that the Gysin pullback lets us map classes in a vector bundle to classes in the base, dropping the dimension by the rank.)

Let's check that $C_W V \hookrightarrow g^* N_X Y$: The ideal sheaf $\mathcal{I}$ of X in Y generates the ideal sheaf $\mathcal{J}$ of W in V, hence there is a surjection

$$\oplus_n f^*(\mathcal{I}^n/\mathcal{I}^{n+1}) \to \oplus_n \mathcal{J}^n/\mathcal{J}^{n+1}.$$

This determines a closed imbedding of the normal cone $C_W V$ into the vector bundle N.

Algebraic fact (black box from appendix): as V is purely k-dimensional scheme, $C_W V$ is also. Then we may define $X \cdot V$ as I said we would: $X \cdot V = s^* C_{W/V}$.

Proposition. If ξ is the universal quotient bundle of rank d on $\mathbb{P}(g^* N_{X/Y} \oplus 1)$, and $q : \mathbb{P}(g^* N_{X/Y} \oplus 1) \to W$ is the projection, then

$$X \cdot V = q_*(c_d(\xi) \cap [\mathbb{P}(C_{W/V} \oplus 1)]).$$

Proof. Let $C = C_{W/V}$.

$$\begin{tikzcd}
C \arrow[r, "\text{cl. imm.}"] \arrow[d, "\text{open imm.}"] & N \arrow[d, "\text{open imm.}"] \\
C \coprod \mathbb{P}C \arrow[r, "\text{cl. imm.}"] \arrow[d, "="] & N \coprod \mathbb{P}N \arrow[d, "="] \\
\mathbb{P}(C \oplus \mathbf{1}) \arrow[r, "\text{cl. imm.}"] \arrow[d, "q"] & \mathbb{P}(N \oplus \mathbf{1}) \arrow[dl] \\
W \arrow[uuur, "s"] &
\end{tikzcd}$$

We want to take the cone C and intersect it with the zero section s of the vector bundle N (the top row of this diagram). We we can do this on the second row of the diagram. Recall that we proved: if $\beta \in A_k N$ and $\overline{\beta} \in A_k(\mathbb{P}(N \oplus \mathbf{1}))$ which restricts to β. Then $s^*\beta = q_*(c_r(\xi) \cap \overline{\beta})$ where ξ is the universal (rank r) quotient bundle of $q^*(N \oplus \mathbf{1})$. Then we're done. $\square$

Proposition. $X \cdot V = \{c(g^* N_{X/Y}) \cap s(W, V)\}_{k-d}$. (Here $\{\cdot\}_{k-d}$ means "take the dimension $k - d$ piece of $\cdot$.)

Proof. Consider the universal (or tautological) exact sequence

$$0 \to \mathcal{O}(-1) \to q^*N \oplus \mathbf{1} \to \xi \to 0$$

on $\mathbb{P}(N \oplus \mathbf{1})$. By the Whitney sum formula, $c(\xi)c(\mathcal{O}(-1)) = c(q^*N)$. Hence

$$q_*(c_d(\xi) \cap [\mathbb{P}(C \oplus \mathbf{1})]) = \{q_*(c(\xi) \cap [\mathbb{P}(C \oplus \mathbf{1})])\}_{k-d}$$

(essentially the previous proposition, but note that we've replaced $c_d(\xi)$ with $c(\xi)$)

$$= \{q_*(c(q^*N)s(\mathcal{O}(-1)) \cap [\mathbb{P}(C \oplus \mathbf{1})])\}_{k-d}$$

(using Whitney sum formula)

$$= \{c(N) \cap q_*(s(\mathcal{O}(-1)) \cap [\mathbb{P}(C \oplus \mathbf{1})])\}_{k-d}$$

(projection formula)

$$= \{c(N) \cap s(C)\}_{k-d}$$

(definition of Segre class of a cone). $\square$

Proposition. If $d = 1$ (X is a Cartier divisor on Y), V is a variety, and f is a closed immersion, then $X \cdot V$ is the intersection class we defined earlier ("cutting with a pseudo-divisor g^*X").

Proof omitted.

3. Refined Gysin homomorphisms

We now come to the last fundamental construction of the subject.

Let $i : X \to Y$ be a local complete intersection of codimension d as before, and let $f : Y' \to Y$ be any morphism.

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

As before, the normal cone $C' = C_{X'}Y'$ is a closed subcone of g^*N_XY. Define the *refined Gysin homomorphism* $i^!$ (pronounced i shriek, which is what people sometimes do when they first hear about this) as the composition:

$$A_kY' \xrightarrow{\sigma} A_kC' \longrightarrow A_kN \xrightarrow{s^*} A_{k-d}X' .$$

Note what we can now do: we used to be able to intersect with a local complete intersection of codimension d. Now we can intersect in a more general setting.

We'll next show that these homomorphisms behave well with respect to everything we've done before.

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INTERSECTION THEORY CLASS 17

RAVI VAKIL

CONTENTS

Where we're going, by popular demand: Grothendieck Riemann-Roch (15); comparison to Borel-Moore homology (chapter 19).

1. WHERE WE ARE

We defined the Gysin pullback $i^!$ and a rather general intersection product. Let $i : X \hookrightarrow Y$ be a local complete intersection of codimension d. Y is arbitrarily horrible. Suppose V is a scheme of pure dimension k, with a map $f : V \to Y$. Here I am *not* assuming V is a closed subscheme of Y. Then define W to be the closed subscheme of V given by pulling back the equations of X in Y:

$$\begin{tikzcd} W \arrow[r, "\text{cl. imm.}"] \arrow[d, "g"] & V \arrow[d, "f"] \\ X \arrow[r, "\text{cl. imm.}"] & Y \end{tikzcd}$$

(notice definition of g).

The cone of X in Y is in fact a vector bundle (as $X \hookrightarrow Y$ is a local complete intersection); call it $N_X Y$. The cone $C_W Y$ to W in Y may be quite nasty; but we saw that $C_W Y \hookrightarrow g^* N_X Y$. Then we define

$$X \cdot V = s^*[C_W V]$$

where $s : W \to g^* N_X Y$ is the zero-section. (Recall that the Gysin pullback lets us map classes in a vector bundle to classes in the base, dropping the dimension by the rank. Algebraic black box from appendix: as V is purely k-dimensional scheme, $C_W V$ is also.)

Last time I proved:

Date: Wednesday, November 17, 2004.

Proposition. If ξ is the universal quotient bundle of rank d on $\mathbb{P}(g^*N_{X/Y} \oplus 1)$, and $q : \mathbb{P}(g^*N_{X/Y} \oplus 1) \to W$ is the projection, then

$$X \cdot V = q_*(c_d(\xi) \cap [\mathbb{P}(C_{W/V} \oplus 1)]).$$

and

Proposition. $X \cdot V = \{c(g^*N_{X/Y}) \cap s(W, V)\}_{k-d}$. (Here $\{\cdot\}_{k-d}$ means "take the dimension $k - d$ piece of $\cdot$.)

and stated (without proof):

Proposition. If $d = 1$ (X is a Cartier divisor on Y), V is a variety, and f is a closed immersion, then $X \cdot V$ is the intersection class we defined earlier ("cutting with a pseudo-divisor g^*X").

1.1. **Refined Gysin homomorphisms** $i^!$. Let $i : X \to Y$ be a local complete intersection of codimension d as before, and let $f : Y' \to Y$ be any morphism.

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

As before, $C' = C_{X'}Y' \hookrightarrow g^*N_XY$. Define the *refined Gysin homomorphism* $i^!$ as the composition:

$$A_kY' \xrightarrow{\sigma} A_kC' \longrightarrow A_kN \xrightarrow{s^*} A_{k-d}X' .$$

Note what we can now do: we used to be able to intersect with a local complete intersection of codimension d. Now we can intersect in a more general setting.

We'll next show that these homomorphisms behave well with respect to everything we've done before. These are all important, but similar to what we've done before, so I'll state the various results. I'll just sporadically give proofs.

Handy fact: Say we want to prove something about $i^![V]$. Consider

$$\begin{tikzcd} X' \cap V \arrow[r] \arrow[d, "h"] & V \arrow[d] \\ X' \arrow[r] \arrow[d, "g"] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Then

$$i^![V] = c(g^*N_{X/Y}) \cap h_*s(X' \cap V, V).$$

Reason we like this: we already know Chern and Segre classes behave well. So we can reduce calculations about $i^!$ to things we've already proved. Reason for fact: Calculate

$X \cdot V$ using

$$\begin{tikzcd} X' \cap V \arrow[r] \arrow[d, "g \circ h"] & V \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

We get $c(h^*g^*N) \cap s(X' \cap V, V)$. Push this forward to X':

$$h_*(c(h^*g^*N) \cap s(X' \cap V, V)) = c(g^*N) \cap h_*s(X' \cap V, V))$$

using the projection formula. We now have to show that this really gives $i^![V]$. (Fulton uses this second version as the original definition.) Omitted.

Refined Gysin commutes with proper pushforward and proper pullback. Consider the fiber diagram

$$\begin{tikzcd} X'' \arrow[r, "i''"] \arrow[d, "q"] & Y'' \arrow[d, "p"] \\ X' \arrow[r, "i'"] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

where i is a locally closed intersection of codimension d.

(a) If p is proper and $\alpha \in A_k Y''$, then $i^! p_*(\alpha) = q_*(i^!\alpha)$ in $A_{k-d}X'$. (Caution: $i^!$ means two different things here!)
(b) If p is flat of relative dimension n, and $\alpha \in A_k Y'$, then $i^! p^*(\alpha) = q^*(i^!\alpha)$ in $A_{k+n-d}X''$.

Proof. (a) We may assume $\alpha = [V']$ (on Y''). Let $V = p(V')$ (on Y').

$$\begin{aligned} i^! p_* \alpha &= \deg(V'/V)\{c(g^*N_{X/Y}) \cap s(X' \cap V, V)\}_{k-d} && \text{previous proposition} \\ &= \{c(g^*N_{X/Y}) \cap q_* s(X'' \cap V', V')\}_{k-d} && \text{Segre classes push forward well} \\ &= q_*\{c(q^*g^*N_{X/Y}) \cap s(X'' \cap V', V')\}_{k-d} && \text{projection formula} \\ &= q_* i^! [V'] \end{aligned}$$

Compatibility. If i' is also a local complete intersection of codimension d, and $\alpha \in A_k Y''$, then $i^!\alpha = (i')^!\alpha$ in $A_{k-d}X''$.

It suffices to verify that $g^*N_X Y \cong N_{X'}Y'$. Reason: If $\mathcal{I}$ and $\mathcal{I}'$ are the respective ideal sheaves, the canonical epimorphism $g^*(\mathcal{I}/\mathcal{I}^2) \to \mathcal{I}'/(\mathcal{I}')^2$ must be an isomorphism. (Details omitted. X is locally cut out in Y by d equations. X' is cut out in Y' by (the pullbacks of) the same d equations.)

1.2. **Excess intersection formula.** Consider the same fiber diagram as before

$$\begin{tikzcd} X'' \arrow[r, "i''"] \arrow[d, "q"] & Y'' \arrow[d, "p"] \\ X' \arrow[r, "i'"] \arrow[d, "g"] & Y' \arrow[d, "f"] \\ X \arrow[r, "i'"] & Y \end{tikzcd}$$

where now i is still a locally closed intersection of codimension d, and i' is *also* a locally closed intersection, of possibly different dimension d'. Let N and N' be the two normal bundles; as before we have a canonical closed immersion $N' \hookrightarrow g^*N$. Let $E = g^*N/N'$ be the quotient vector bundle, of rank $d = d - d'$.

For any $\alpha \in A_kY''$, note that $i^!(\alpha)$ and $(i')^!(\alpha)$ differ in dimension by e. What is their relationship? Answer:

Excess intersection formula. For any $\alpha \in A_kY''$, $i^!(\alpha) = c_e(q^*E) \cap (i')^!(\alpha)$ in $A_{k-d}X''$.

(Proof short but omitted.)

Immediate corollary. Specialize to the case where the top row is the same as the middle row, and i' is an isomorphism:

$$\begin{tikzcd} X' \arrow[r, "\sim", "i'"'] \arrow[d, "g"] & Y' \arrow[d] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Then $i^!\alpha = c_d(g^*N) \cap \alpha$. Specialize again to $X' = Y' = X$ to get the self-intersection formula: $i^*i_*\alpha = c_d(N) \cap \alpha$.

Intersection products commute with Chern classes. Let $i : X \to Y$ be a locally closed intersection of codimension d,

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "i"'] & Y \end{tikzcd}$$

a fiber square, and F a vector bundle on Y'. Then for all $\alpha \in A_kY'$ and all $m \geq 0$,

$$i^!(c_m(F) \cap \alpha) = c_m(i'^*F) \cap i^!\alpha$$

in $A_{k-d-m}(X')$

Proof omitted.

Refined Gysin homomorphisms commute with each other. Let $i : X \to Y$ be a locally closed intersection of codimension d, $j : S \to T$ a locally closed intersection of codimension e. Let Y' be a scheme, $f : Y' \to Y$, $g : Y' \to T$ two morphisms. Form the fiber diagram

$$\begin{tikzcd}
X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d, "j'"] & S \arrow[d] \\
X' \arrow[r, "i'"'] \arrow[d] & Y' \arrow[r, "g"'] \arrow[d, "f"] & T \\
X \arrow[r, "i"'] & Y &
\end{tikzcd}$$

Then for all $\alpha \in A_k Y'$, $j^! i^! \alpha = i^! j^! \alpha$ in $A_{k-d-e} X''$.

Proof (long!) omitted. Idea: by blowing up to reduce to the case of divisors, as we did when we showed that the intersection of two divisors was independent of the order of intersection, long ago.

Functoriality.

The refined Gysin homomorphisms for a composite of locally closed intersections is the composite of the refined Gysin homomorphisms of the factors.

Consider a fiber diagram

$$\begin{tikzcd}
X' \arrow[r, "i'"] \arrow[d, "h"] & Y' \arrow[r, "j'"] \arrow[d, "g"] & Z' \arrow[d, "f"] \\
X \arrow[r, "i"'] & Y \arrow[r, "j"'] & Z.
\end{tikzcd}$$

If i (resp. j) is a locally closed intersection of codimension d (resp. e), then ji is a locally closed intersection of codimension $d + e$, and for all $\alpha \in A_k Z'$, $(ji)^! \alpha = i^! j^! \alpha$ in $A_{k-d-e} X'$.

Proof omitted. Similarly:

Second functoriality proposition. Consider a fiber diagram

$$\begin{tikzcd}
X' \arrow[r, "i'"] \arrow[d, "h"] & Y' \arrow[r, "p'"] \arrow[d, "g"] & Z' \arrow[d, "f"] \\
X \arrow[r, "i"'] & Y \arrow[r, "p"'] & Z.
\end{tikzcd}$$

(a) Assume that i is a locally closed intersection of codimension d, and that p and pi are flat of relative dimensions n and $n - d$. Then i' is a locally closed intersection of codimension d, p' and $p'i'$ are flat, and for $\alpha \in A_k Z'$,

$$(p'i')^* \alpha = i^! p'^* \alpha$$

in $A_{k+n-d} X'$.

(b) Assume that i is a locally closed intersection of codimension d, p is smooth of relative dimension n, and pi is locally closed intersection of codimension $d - n$. Then for all $\alpha \in A_k Z'$,

$$(pi)^! \alpha = i^! (p'^* \alpha)$$

in $A_{k+n-d} X'$.

Short proof, omitted.

2. Local complete intersection morphisms

A morphism $f : X \to Y$ is called a lci morphism of codimension d if it factors into a locally closed intersection $X \to P$ followed by a smooth morphism $p : X \to Y$. Examples: families of nodal curves over an arbitrary base; families of surfaces with mild singularities. Reason we care: often we want to consider families of things degenerating. We won't need this in the next two weeks, but it's worth at least giving the definition.

For any lci morphism $f : X \to Y$ of codimension d, and any morphism $h : Y' \to Y$, we have the fiber square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "h'"] & Y' \arrow[d, "h"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We want to define a refined Gysin homomorphism

$$f^! : A_k Y' \to A_{k-d} X'.$$

Here's how. Factor f into $p \circ i$ where $p : P \to Y$ is a smooth morphism of relative dimension $d + e$ and $i : X \hookrightarrow P$ is a local complete intersection of codimension e. Then form the fiber diagram

$$\begin{tikzcd} X' \arrow[r, "i'"] \arrow[d, "h'"] & P' \arrow[r, "p'"] \arrow[d] & Y' \arrow[d, "h"'] \\ X \arrow[r, "i"'] & P \arrow[r, "p"'] & Y. \end{tikzcd}$$

Then p' is smooth (smooth morphisms behave well under base change), and we define $f^! \alpha = i^!((p')^* \alpha)$ (smooth morphisms are flat; this is part of the definition).

Proposition (a) The definition of $f^!$ is independent of the factorization of f. (!!!) (b) If f is both lci and flat, then $f^! = f'^*$. (c) The assertions earlier (pushforward and pullback compatibility; commutativity; functoriality) for locally closed intersections are valid for arbitrary lci morphisms. There is also an excess intersection formula, that I won't bother telling you precisely.

Because (a) seems surprising, and the roof is short, I'll give it to you. If $X \xrightarrow{i_1} P_1 \xrightarrow{p_1} Y$ is another factorization of f, compare them both with the diagonal:

$$\begin{tikzcd} & & P_1 \arrow[dr, "p_1"] & \\ X \arrow[r, "(i,i_1)"] & P \times_Y P_1 \arrow[ur] \arrow[rr] \arrow[dr] & & Y. \\ & & P \arrow[ur, "p"'] & \end{tikzcd}$$

Use the second functoriality proposition (b).

Then (b) follows from (a). (c) is omitted.

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INTERSECTION THEORY CLASS 18

RAVI VAKIL

CONTENTS

Where we're going, by popular demand: Grothendieck Riemann-Roch (chapter 15); bivariant intersection theory and A^* (chapter 17).

1. LAST DAY

We defined the Gysin pullback $i^!$ in a rather general circumstance. I have only a few additional comments to make. Recall that a morphism $f : X \to Y$ is a *local complete intersection morphism* if f can be factored as a local complete intersection followed by a smooth morphism.

I don't know why one wouldn't more generally think of factorizations into a local complete intersection followed by a *flat* morphism.

I gave you a few examples as to why you might care about such morphisms. Here is another. If X and Y are smooth then *any* morphism between them is an lci morphism. Reason: factor it into

$$X \hookrightarrow X \times Y \to Y.$$

Date: Monday, November 22, 2004.

2. TOWARDS GROTHENDIECK-RIEMANN-ROCH

I'm going to first explain the terminology behind the statement, then give the statement. I will then give some examples to show you that the statement is in fact very powerful. And finally, I hope to sketch a proof in an important special case,

2.1. **The Chern character and the Todd class.**

Suppose E is a rank n vector bundle. Let $\alpha_1, \dots, \alpha_n$ be the Chern roots of the vector bundle, so $\alpha_1 + \cdots + \alpha_n = c_1(E)$, etc. Define

$$\mathrm{ch}(E) = \sum_{i=1}^{r} \exp(\alpha_i)$$

When you expand this out, you get:

$$\begin{aligned} \mathrm{ch}(E) &= \mathrm{rk}(E) + c_1 + \frac{1}{2}(c_1^2 - c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + c_3) \\ &\quad + \frac{1}{24}\left(c_1^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4\right) + \cdots \end{aligned}$$

So this makes sense for any coherent sheaf, not just a vector bundle. In that case, rank refers to the rank at the generic point.

Exercise. For any exact sequence of vector bundles $0 \to E' \to E \to E'' \to 0$, $\mathrm{ch}(E) = \mathrm{ch}(E') + \mathrm{ch}(E'')$. (This is true for coherent sheaves in general.)

For comparison, the Chern polynomial is *multiplicative* in exact sequences; the Chern character is *additive*.

Exercise. For tensor products of vector bundles, $\mathrm{ch}(E \otimes E') = \mathrm{ch}(E) \cdot \mathrm{ch}(E')$. I don't think this is true for coherent sheaves in general, but haven't checked. (I would expect $\sum_{i \geq 0} \mathrm{ch}(\underline{\mathrm{Tor}}^i(E, E')) = \mathrm{ch}(E) \cdot \mathrm{ch}(E')$.)

The *Todd class* $\mathrm{td}(E)$ of a vector bundle is defined by

$$\mathrm{td}(E) = \prod_{i=1}^{r} Q(\alpha_i)$$

where

$$Q(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k}.$$

The first few terms are

$$\begin{aligned} \mathrm{td}(E) &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \\ &\quad + \frac{1}{720}\left(-c_1^4 + 4c_1 2c_2 + 3c_2^2 + c_1c_3 - c_4\right) + \cdots \end{aligned}$$

If $0 \to E' \to E \to E'' \to 0$ is exact, then

$$\mathrm{td}(E) = \mathrm{td}(E')\,\mathrm{td}(E'').$$

Like the Chern polynomial, it is multiplicative in exact sequences.

2.2. **The Grothendieck groups K^0X and K_0X.** If you went to Dan Ramras' K-theory talks, you will have seen these.

The *Grothendieck group of vector bundles* K^0X on X is the group generated by vector bundles, modulo the relations on exact sequences $[E] = [E'] + [E'']$. Vector bundles pull back to vector bundles, and exact sequences of vector bundles pull back to exact sequences of vector bundles, so if a morphism $f : X \to Y$ induces a homomorphism $f^* : K^0X \to K^0Y$. However, vector bundles seldom pushforward to vector bundles.

K^0X is a *ring*: $[E] \cdot [F] = [E \otimes F]$.

The *Grothendieck group of coherent sheaves* K_0X on X is the group generated by coherent sheaves, modulo the same relations on exact sequences. Bad news: coherent sheaves pull back to coherent sheaves, but exact sequences of coherent sheaves don't pull back to exact sequences of coherent sheaves. So we don't have a pullback map $f^* : K_0X \to K_0Y$. Good news: we can make sense of pushforwards; if $f : X \to Y$ is a proper morphism, then coherent sheaves pushforward to coherent sheaves (see Hartshorne). Bad news: exact sequences don't push forward to exact sequences: If

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is an exact sequence on X, then we only get left exactness of pushforwards:

$$0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}''.$$

Good news: we can extend this to a long exact sequence:

$$\begin{array}{ccccccc} 0 \longrightarrow & R^0f_*\mathcal{F}' & \longrightarrow & R^0f_*\mathcal{F} & \longrightarrow & R^0f_*\mathcal{F}'' & \longrightarrow \\ & R^1f_*\mathcal{F}' & \longrightarrow & R^1f_*\mathcal{F} & \longrightarrow & R^1f_*\mathcal{F}'' & \longrightarrow \\ & R^2f_*\mathcal{F}' & \longrightarrow & R^2f_*\mathcal{F} & \longrightarrow & R^2f_*\mathcal{F}'' & \longrightarrow \cdots \end{array}$$

So this tell us how to define $f_* : K_0X \to K_0Y$, by

$$f_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}].$$

(See Hartshorne for more on these "higher direct image sheaves. They can be defined as follows: $R^if_*\mathcal{F}$ is the sheaf associated to the presheaf $U \to H^i(f^{-1}(U), \mathcal{F})$.

We obviously have a homomorphism $K^0X \to K_0X$.

K_0X is a K^0X-module: $K^0X \otimes K_0X \to X$ is given by $[E] \cdot [\mathcal{F}] = [E \otimes \mathcal{F}]$. (Exercise: this is well-defined. Key fact: if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of coherent sheaves, and E is a vector bundle, then $0 \to E \otimes \mathcal{F}' \to E \otimes \mathcal{F} \to E \otimes cF'' \to 0$ is exact. (Explain. Tensoring with locally frees is exact.)

Lemma. If $\alpha \in K^0Y$ and $\beta \in K_0X$, and $f : X \to Y$, then $f_*(f^*\alpha \cdot \beta) = \alpha f_* \beta$.

Proof. The projection formula $R^i f_*(f^*E \otimes \mathcal{F}) = E \otimes R^i f_* \mathcal{F}$, shown in Hartshorne. $\square$

Fact. If X is nonsingular, the map $K^0 X \to K_0 X$ is an isomorphism.

Reason: If X is nonsingular, then $\mathcal{F}$ has a finite resolution by locally free sheaves:

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to \mathcal{F} \to 0,$$

where the $n \leq \dim X$. Hence the inverse map is $[\mathcal{F}] = \sum_{i=0}^{n} (-1)^i [E_i]$. A sketch of the reason: show that there is a vector bundle surjecting onto $\mathcal{F}$. ("There are enough locally free's.") Build the sequence from right to left. By the time you reach E_n, you will run out of steam — the kernel at some point will already be locally free. How do you show this? You have a cohomological measure of the "non local freeness" of a coherent sheaf. If the measure is 0, the sheaf is the 0 sheaf. If $0 \to \mathcal{F}' \to E \to \mathcal{F} \to 0$, then you show that the cohomological measure of $\mathcal{F}'$ is one less than that of $\mathcal{F}$. (Hence if the cohomological measure is 1, then the sheaf is locally free.)

From now on, X will be smooth, so $K^0 X = K_0 X$, so I'll just call this group $K(X)$.

The Chern character map descends to $K(X)$:

$$ch : K(X) \to A(X)_{\mathbb{Q}}.$$

3. Statement of the theorem

Grothendieck-Riemann-Roch Theorem. For any $\alpha \in K(X)$,

$$\mathrm{ch}(f_* \alpha) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha) \cdot td(T_X)).$$

Here $f : X \to Y$ is a proper morphism of smooth varieties.

(I should point out where all the intersections take place, and where the pushforwards take place!)

This can be generalized further to singular schemes, but this is enough generality for now.

3.1. **Why you should care.** Before we get into proving it, let me first try to convince you how powerful it is. I'll first show that it gives you old-fashioned Riemann-Roch. (I won't try to convince you why you should care about Riemann-Roch for curves — that is a whole lecture in itself, or more!)

Let's apply this to Y a point, X a smooth curve, and α a line bundle L. Then we get

$$h^0(X, L) - h^1(X, L) = \cdots$$

On the right side, we have

$$f_*((1 + c_1(L))(1 + \frac{1}{2} c_1(T)) = f_*(1 + c_1(L) - \frac{1}{2} c_1(K)) = \deg(c_1(L) - \frac{1}{2} c_1(K)).$$

Recall that $c_1(K) = -c_1(T) = 2g - 2$. Thus the right side is $d - g + 1$.

That's the baby-est case. Let's make things more interesting. We'll keep Y a point and X a nonsingular curve, and now α is the class of a vector bundle V of rank r. Then we get

$$h^0(X,V) - h^1(X,V) = f_*((r + c_1(V))(1 + \frac{1}{2}c_1(T)) = f_*(c_1(V) + \frac{r}{2}c_1(T)) = d + r(1-g).$$

Let's generalize further; now V is a coherent sheaf, of "rank" r (rank at the generic point). The same formula holds!

Next let's go to the case where X is now a smooth surface, Y a point, and to keep things calm, let's make α the class of a line bundle L. Then the left side is

$$h^0(X,L) - h^1(X,L) + h^2(X,L).$$

The right side is

$$\begin{aligned} & f_*((1 + c_1(L) + \frac{1}{2}(c_1^2(L) - c_2(L))) \left(1 + \frac{c_1(T)}{2} + \frac{c_1^2(T) + c_2(T)}{12}\right) \\ = \; & \deg\left(\frac{c_1^2(L)}{2} - c_1(L)\cdot K/2 + \frac{K^2 + c_2(T)}{12}\right) \end{aligned}$$

which is Riemann-Roch for surfaces, which you can read about in Hartshorne chapter V.

More generally still, if X is a smooth surface, and E is a vector bundle, and Y is still a point, we get

$$\chi(X,E) = \int_X \mathrm{ch}(E)\cdot \mathrm{td}(T_X).$$

We have reproved the *Hirzebruch-Riemann-Roch* theorem. And this also works for coherent sheaves.

What about if Y is *not* a point? I'll describe why you care somewhat philosophically. Suppose you have a nice morphism $X \to Y$, interpreted as "nice family" (say of smooth surfaces). Say you have a vector bundle on the family. On each of the elements of the family (the fibers of the morphism), you have a vector bundle; let's say to make things nice that for every element of the family, this vector bundle has vanishing higher cohomology. Then $h^0(V)$ is constant, as $h^0(V) = \chi(V)$, and $\chi(V)$ is constant on connected families. Thus for each point of the base Y, you have a vector space of some rank $h^0(V)$. You should expect this to glue together into a vector bundle, and indeed it does: f_*V. (Again, to make this precise requires Hartshorne chapter III or its equivalent.) *Which vector bundle do you get?* For example, what are its Chern classes? Grothendieck-Riemann-Roch will answer this for you!

So let me emphasize: you're going to see a proof of GRR that will not be too bad; and as a special case you'll get old-fashioned Riemann-Roch for curves. I think the difficulty of this proof is comparable to the difficulty of building up the machinery behind the "usual" proof of Riemann-Roch in the algebraic category; so you may as well get a much more powerful result for the same price.

4. TOWARD A PROOF

I'll prove this in the case where $X \to Y$ factors through $X \hookrightarrow \mathbb{P}^n \times Y \to Y$ where the first is a closed immersion. (This is a projective morphism in the sense of Hartshorne, and a special case of a projective morphism according to other people, such as EGA. I don't want to get into this.) This isn't such an outrageous assumption; for example, if X is projective, then $X \hookrightarrow \mathbb{P}^n$, and then $X \hookrightarrow \mathbb{P}^n \times Y$.

Lemma. Given $X \xrightarrow{f} Z \xrightarrow{g} Y$. Suppose GRR holds for f and g. Then it holds for $g \circ f$.

Proof. This has been cooked up to be easy! (That Grothendieck is quite a tricky guy!) The pushforward of $\mathrm{ch}(\alpha)\,\mathrm{td}(T_X)$ by f is $\mathrm{ch}(f_*\alpha)\,\mathrm{td}(T_Z)$, by GRR for f. The pushforward of this in turn is $\mathrm{ch}(g_* f_* \alpha)\,\mathrm{td}(T_Y)$, by GRR for g. But then we have GRR for $g \circ f$: $(g \circ f)_*(\mathrm{ch}(\alpha)\,\mathrm{td}(T_X)) = \mathrm{ch}(g_* f_* \alpha)\,\mathrm{td}(T_Y)$. □

So our strategy is clear. We're going to prove GRR for closed immersions $X \hookrightarrow Y$, and we'll prove it for $\mathbb{P}^n \times Y \to Y$.

5. GROTHENDIECK-RIEMANN-ROCH FOR $\mathbb{P}^n \to pt$

Let me first work out $K(\mathbb{P}^n)$.

Theorem. The group $K_0(\mathbb{P}^m)$ is generated by the classes $[\mathcal{O}(n)]$, with $0 \leq n \leq m$.

First we show:

Lemma. $K_0(\mathbb{P}^m)$ is generated by the classes of line bundles $[\mathcal{O}(n)]$, without any restriction on n.

Proof. I will need some machinery we have not developed. How much extra you will need to consider as a "black box" will depend on how much you already know. Let $\mathcal{F}$ be any coherent sheaf. Our goal is to get a resolution of $\mathcal{F}$ by direct sums of line bundles:

$$0 \to \oplus\mathcal{O}(?) \to \oplus\mathcal{O}(?) \to \cdots \to \oplus\mathcal{O}(?) \to \mathcal{F} \to 0.$$

By an earlier statement, we need only show that for any coherent sheaf $\mathcal{F}$, we can find a surjection $\oplus_{i=1}^{j}\mathcal{O}(n) \to \mathcal{F}$, because then we can iterate this, and at some point we will get a 0.

By a property of ample line bundles, for $N \gg 0$, $\mathcal{F} \otimes \mathcal{O}(N)$ is generated by global sections. (It is then generated by a finite number of global sections, by a Noetherian argument.) That means that there is a surjection $\oplus_{i=1}^{j}\mathcal{O} \to \mathcal{F}(N)$. Twisting by $\mathcal{O}(-N)$, we get our desired surjection $\oplus_{i=1}^{j}\mathcal{O}(-N) \to \mathcal{F}$. □

The theorem is then proved once we know the next lemma:

Lemma. There is an exact sequence on $\mathbb{P}^m$

$$0 \to \mathcal{O} \to \oplus^{m+1}\mathcal{O}(1) \to \oplus^{\binom{m+1}{2}}\mathcal{O}(2) \to \cdots \oplus^{\binom{m+1}{j}}\mathcal{O}(j) \to \cdots \oplus^{m+1}\mathcal{O}(m) \to \mathcal{O}(m+1) \to 0.$$

Here's how this implies the theorem. Twisting this by $\mathcal{O}(N)$ we get:

$$0 \to \mathcal{O}(N) \to \oplus^{m+1}\mathcal{O}(N+1) \to \oplus^{\binom{m+1}{2}}\mathcal{O}(N+2) \to \cdots \oplus^{m+1}\mathcal{O}(N+m) \to \mathcal{O}(N+m+1) \to 0.$$

This expresses $[\mathcal{O}(N+m)]$ in terms of the classes of the $m+1$ smaller line bundles. Similarly, it expresses $[\mathcal{O}(N)]$ in terms of the classes of the $m+1$ larger line bundles. Thus by using this repeatedly, any line bundle can be expressed in terms of $\mathcal{O}$, $\mathcal{O}(1), \dots, \mathcal{O}(m)$.

Aside: you also get some interesting algebra out of this. Apply the Chern polynomial to this exact sequence. You get

$$\prod_{i=0}^{m+1}(1+iH)^{(-1)^i\binom{m+1}{i}} \equiv 1 \pmod{H^{m+1}}$$

Example $m=1$: $(1+H)^{-2}(1+2H)^1 \equiv 1 \pmod{H^2}$. Joe Rabinoff gave me a nice explanation of why this is true; I'll give it next day.

Sketch of proof of Lemma. We'll prove instead an exact sequence

$$0 \to \mathcal{O}(-m-1) \to \oplus^{m+1}\mathcal{O}(-m) \to \cdots \oplus^{m+1}\mathcal{O}(-1) \to \mathcal{O} \to 0$$

which is the dual (or alternatively, a twist) of the one we want. Let $V = \oplus^{m+1}\mathcal{O}(-1)$. Then this sequence is

$$0 \to \wedge^{m+1}V \to \wedge^m V \to \cdots \to \wedge^1 V \to \wedge^0 V \to 0.$$

You can check this on the level of graded modules. Let $S = k[x_0, \dots, x_m]$, with the usual grading. Let $\oplus^{m+1}S[-1]$. ($S[-1]$ is the same as S, except the grading is shifted by 1, so $S[-1]_1$ has dimension 1.) Define the map $V \to S$ by multiplication by $(x_0, \dots, x_m)$. This induces maps $\wedge^{j+1}V \to \wedge^j V$. Then you can check by hand that this is exact everywhere.; $\square$

Theorem. GRR is true for $\mathbb{P}^m \to \mathrm{pt}$ for the line bundles $\mathcal{O}(n)$ ($0 \le n \le m$). Hence GRR is true for $\mathbb{P}^m \to \mathrm{pt}$.

Proof. Now $p_*[\mathcal{O}(n)] = \chi(\mathbb{P}^m, \mathcal{O}(n))$. Now we can compute the cohomology groups of $\mathcal{O}(n)$ by hand, and we find that $h^i(\mathcal{O}(n)) = 0$ for $n \ge 0$. Thus $\chi(\mathbb{P}^m, \mathcal{O}(n)) = h^0(\mathbb{P}^m, \mathcal{O}(n))$. And this corresponds to the vector space of degree n polynomials with $m+1$ variables. This turns out to be $\binom{n+m}{m}$.

Hence we wish to prove that

$$\int_{\mathbb{P}^m} \mathrm{ch}(\mathcal{O}(n))\,\mathrm{td}(T_{\mathbb{P}^m}) = \binom{n+m}{m}.$$

Let's do this.

Let's first calculate $\mathrm{td}(T_{\mathbb{P}^m})$. The "Euler exact sequence" for the tangent bundle of projective space is

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus m+1} \to T_{\mathbb{P}^m} \to 0.$$

(Aside: notice that this is the beginning of that big exact sequence of direct sums of line bundles in the proof of the previous lemma (that was omitted in class)! This shouldn't be a coincidence, but I'm not precisely sure why not.) The Todd class is *multiplicative* for exact sequences, so we get

$$\mathrm{td}(T_{\mathbb{P}^m}) = \left(\frac{x}{1-e^{-x}}\right)^{m+1}$$

where $x = c_1(\mathcal{O}(1))$. We also have $\mathrm{ch}(\mathcal{O}(n)) = e^{nx}$. Thus we want to show that

$$\int_{\mathbb{P}^m} \frac{e^{nx}x^{m+1}}{(1-e^{-x})^{m+1}} = \binom{n+m}{m}.$$

The thing on the left says: "extract the x^m term the power series". So we want to prove

$$[x^m]\frac{e^{nx}x^{m+1}}{(1-e^{-x})^{m+1}} = \binom{n+m}{m}.$$

Now the left side

$$= [x^{-1}]\frac{e^{nx}}{(1-e^{-x})^{m+1}}$$

so we've turned this into a residue calculation, which is a reasonable quals problem. $\square$

Next, we'll show that knowing the result for $\mathbb{P}^m \to pt$ will imply the result for $\mathbb{P}^m \times Y \to Y$.

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INTERSECTION THEORY CLASS 19

RAVI VAKIL

CONTENTS

Today I'm going to try to finish the proof of Grothendieck-Riemann-Roch in the case of projective morphisms from smooth varieties to smooth varieties. We'll see that we're essentially going to prove it more generally for projective lci morphisms.

1. Recap of Last day

Recall the definition of the Chern character and Todd class. Suppose $\mathcal{F}$ is a coherent sheaf. Let $\alpha_1, \dots, \alpha_n$ be the Chern roots of the vector bundle, so $\alpha_1 + \cdots + \alpha_n = c_1(\mathcal{F})$, etc. Define $ch(\mathcal{F}) = \sum_{i=1}^r \exp(\alpha_i)$ This is *additive* on exact sequences. For vector bundles, we have $\mathrm{ch}(E \otimes E') = \mathrm{ch}(E) \cdot \mathrm{ch}(E')$.

The *Todd class* $\mathrm{td}(E)$ of a vector bundle is defined by $\mathrm{td}(E) = \prod_{i=1}^r Q(\alpha_i)$ where

$$Q(x) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} x^{2k}.$$

It is multiplicative in exact sequences.

We defined the Grothendieck groups K^0X and K_0X. They are vector bundles, respectively coherent sheaves, modulo the relation $[E] = [E'] + [E'']$. We have a pullback on K^0: $f^* : K^0X \to K^0Y$. K^0X is a *ring*: $[E] \cdot [F] = [E \otimes F]$. We have a pushforward on K_0: $f_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}]$.

We obviously have a homomorphism $K^0X \to K_0X$. K_0X is a K^0X-module: $K^0X \otimes K_0X \to X$ is given by $[E] \cdot [\mathcal{F}] = [E \otimes \mathcal{F}]$. Unproved fact: If X is nonsingular and projective, the map $K^0X \to K_0X$ is an isomorphism. (Reason: If X is nonsingular, then $\mathcal{F}$ has a finite resolution by locally free sheaves.)

Date: Wednesday, November 24, 2004.

The Chern character map descends to $K(X)$: $ch : K(X) \to A(X)_{\mathbb{Q}}$. This does not commute with proper pushforward; Grothendieck-Riemann-Roch explains how to fix this.

1.1. **New facts.** Here are some useful facts, that I didn't mention last time. We have an excision exact sequence for K_0: If $Z \hookrightarrow X$ is a closed immersion, and U is the open complement, we have an excision exact sequence

$$K_0(Z) \to K_0(X) \to K_0(U) \to 0.$$

The proof is similar to our proof for Chow; this is Hartshorne Exercise II.6.10(c).

Similarly, we have $K_0(\mathbb{A}^1 \times Y) \cong K(Y)$.

Last time I showed: **Lemma.** The group $K_0(\mathbb{P}^m)$ is generated by the classes $[\mathcal{O}_{\mathbb{P}^m}(n)]$, with $0 \leq n \leq m$.

(Incidentally, I mentioned an interesting algebraic problem coming out of my previous proof. Joe gave a nice proof of it. If I have time, I'll type it up and put it in the posted notes.)

I'd like to do it differently today. Instead, I'll show it is generated by the classes $[\mathcal{O}_{\mathbb{P}^m}(-n)]$, with $0 \leq n \leq m$.

Using the excision exact sequence for K-theory, and $\mathbb{P}^m = \mathbb{A}^0 \coprod \mathbb{A}^1 \coprod \cdots \coprod \mathbb{A}^m$, we get inductively: $K_0(\mathbb{P}^m)$ is generated by $n+1$ things: $[\mathcal{O}_{\mathbb{P}^0}]$, $[\mathcal{O}_{\mathbb{P}^1}]$, ..., $[\mathcal{O}_{\mathbb{P}^m}]$.

I'll now express these in terms of $\mathcal{O}_{\mathbb{P}^m}(n)$'s. From

$$0 \to \mathcal{O}_{\mathbb{P}^m}(-1) \to \mathcal{O}_{\mathbb{P}^m} \to \mathcal{O}_{\mathbb{P}^{m-1}} \to 0$$

shows $[\mathcal{O}_{\mathbb{P}^{m-1}}] = [\mathcal{O}_{\mathbb{P}^m}] - [\mathcal{O}_{\mathbb{P}^m}(-1)]$. Similarly,

$$\begin{aligned} [\mathcal{O}_{\mathbb{P}^{m-2}}] &= [\mathcal{O}_{\mathbb{P}^{m-1}}] - [\mathcal{O}_{\mathbb{P}^{m-1}}(-1)] \\ &= ([\mathcal{O}_{\mathbb{P}^m}] - [\mathcal{O}_{\mathbb{P}^m}(-1)]) - ([\mathcal{O}_{\mathbb{P}^m}(-1)] - [\mathcal{O}_{\mathbb{P}^m}(-2)]) \\ &= [\mathcal{O}_{\mathbb{P}^m}] - 2[\mathcal{O}_{\mathbb{P}^m}(-1)] - [\mathcal{O}_{\mathbb{P}^m}(-2)] \end{aligned}$$

and you see the pattern (established by the obvious induction). □

Important philosophy behind Riemann-Roch: $K(\mathbb{P}^m)$ and $A_*(\mathbb{P}^m)$ are both m-dimensional vector spaces; Chern character provides an isomorphism between them. Multiplying by the Todd class provides a "better" isomorphism between them.

More generally, the identical proof shows that for any Y, $K(Y) \otimes K(\mathbb{P}^m) \to K(\mathbb{P}^m \times Y)$ is surjective: cut up $\mathbb{P}^m \times Y$ into $Y \coprod \mathbb{A}^1 \times Y \coprod \cdots \coprod \mathbb{A}^m \times Y$, and proceed as before.

2. Statement of the theorem

Grothendieck-Riemann-Roch Theorem. Suppose $f : X \to Y$ is a proper morphism of smooth varieties. Then for any $\alpha \in K(X)$,

$$\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}(T_Y) = f_*(\mathrm{ch}(\alpha) \cdot \mathtt{td}(T_X)).$$

Interesting exercise: how do you make sense of this when X and Y are singular? For example, what if $X \to Y$ is a smooth morphism, we get $\mathrm{ch}(f_*\alpha)\cdot = f_*(\mathrm{ch}(\alpha) \cdot \mathtt{td}(T_{X/Y}))$ where X/Y is the *relative tangent bundle*. As another example, what if $X \to Y$ is a complete intersection? Then T_X and T_Y don't make sense, but $N_{X/Y}$ *is* a vector bundle, and then $\mathrm{ch}(f_*\alpha) \cdot \mathrm{td}(N_{X/Y}) = f_*(\mathrm{ch}(\alpha))$. Combining these two, you can now make sense of GRR in the case when f is an lci morphism (i.e. closed immersion followed by a smooth morphism).

The theorem may be interpreted to say that the homomorphism

$$\tau_X : K(X) \to A(X)_{\mathbb{Q}}$$

given by $\tau_X(\alpha) = \mathrm{ch}(\alpha) \cdot \mathrm{td}(T_X)$ commutes with proper pushforward: $f_* \circ \tau_X = \tau_Y \circ f_*$. Last time we showed that this implies **Lemma.** Given $X \xrightarrow{f} Z \xrightarrow{g} Y$. Suppose GRR holds for f and g. Then it holds for $g \circ f$.

Hence the strategy is now to show GRR for $Y \times \mathbb{P}^m \to Y$, and for closed immersions.

We'll use this interpretation of the theorem to show

Theorem. GRR is true for $\mathbb{P}^m \times Y \to Y$.

Proof. We showed last time that this is true in the case where Y is a point. Consider the following diagram.

$$\begin{tikzcd}
K(Y) \otimes K(\mathbb{P}^m) \arrow[r, "\tau_Y \otimes \tau_{\mathbb{P}^m}"] \arrow[d, "\times"] & A(Y)_{\mathbb{Q}} \otimes A(\mathbb{P}^m)_{\mathbb{Q}} \arrow[d, "\times"] \\
K(Y \times \mathbb{P}^m) \arrow[r, "\tau_Y \times \tau_{\mathbb{P}^m}"] \arrow[d, "f_*"] & A(Y \times \mathbb{P}^m)_{\mathbb{Q}} \arrow[d, "f_*"] \\
K(Y) \arrow[r, "\tau_Y"] & A(Y)_{\mathbb{Q}}
\end{tikzcd}$$

(I won't be using anything special about $\mathbb{P}^m$ now.) We want to show that the bottom square commutes.

Note that the top square commutes. Reason: $T_{Y \times \mathbb{P}^m} = p_1^* T_Y \oplus p_2^* T_{\mathbb{P}^m}$ (where p_1 and p_2 are the projections) from which $\mathrm{td}(T_{Y \times \mathbb{P}^m}) = \mathrm{td}(p_1^* T_Y) \times \mathrm{td}(p_2^* T_{\mathbb{P}^m})$.

Moreover the upper left vertical arrow is surjective.

So it suffices to show that the big rectangle commutes. But it does because we've already shown that GRR holds for $\mathbb{P}^m \to \mathrm{pt}$. $\square$

2.1. **GRR for a special case of closed immersions** $f : X \to Y = \mathbb{P}(N \oplus \mathbf{1})$**.** Suppose f is a closed immersion into a projective completion of a normal bundle. Let $d = \operatorname{rank} N$. We want to prove GRR for a vector bundle E. As the vector bundles generate $K(X)$, this will suffice.

This example comes the closest to telling me why the Todd class wants to be what it is. Let $p : Y = \mathbb{P}(N \oplus \mathbf{1}) \to X$ be the projection. Let $\mathcal{O}_Y(-1)$ be the tautological line bundle on $Y = \mathbb{P}^(N \oplus \mathbf{1})$. Then as in previous lectures we have a tautological exact sequence of vector bundles on Y:

$$0 \to \mathcal{O}_Y(-1) \to p^*(N \oplus \mathbf{1}) \to Q \to 0$$

where Q is the universal quotient bundle. (Recall that $f^*Q = N_{X/Y}$.) Here is something you have to think through, although we've implicitly used it before. We have a natural section of $p^*(Q \oplus \mathbf{1})$, the $\mathbf{1}$. This gives a section s of Q. This section vanishes precisely (scheme-theoretically) along X. In particular, for any $\alpha \in A(Y)$, $\boxed{f_*(f^*\alpha) = c_d(Q) \cdot \alpha}$. (This was one of our results about the top Chern class. $f_* f^*$ will knock the degree down by d, and we found that this operator was the same as capping with the top Chern class.)

Lemma. We can resolve the sheaf $f_*\mathcal{O}_X$ on Y by

$$0 \longrightarrow \wedge^d Q^\vee \longrightarrow \cdots \longrightarrow \wedge^2 Q^\vee \longrightarrow Q^\vee \xrightarrow{s^\vee} \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X \longrightarrow 0. \tag{1}$$

Note that everything except $f_*\mathcal{O}_X$ is a vector bundle on Y.

Proof. Rather than proving this precisely, I'll do a special case, to get across the main idea. This in fact becomes a proof, once the "naturality" of my argument is established. Suppose $Y = \operatorname{Spec} k[x_1, \dots, x_n]$, so $\mathcal{O}_Y = k[x_1, \dots, x_n]$ (a bit sloppily) and $X = \vec{0} \subset Y$. Then let's build a resolution of $\mathcal{O}_X$. We start with

$$\mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

We have a big kernel obviously: the ideal sheaf of $\mathcal{O}_X$. So our next step is:

$$\mathcal{O}_Y x_1 \oplus \mathcal{O}_Y x_2 \cdots \oplus \mathcal{O}_Y x_n \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

We still have a kernel; $(-x_2 x_1, x_1 x_2, 0, \cdots, 0)$ is in the kernel, for example. So our next step is:

$$\mathcal{O}_Y x_1 x_2 \oplus \cdots \oplus \mathcal{O}_Y x_{n-1} x_n \to$$

(We need to check that we've surjected onto the kernel! But that's not hard; you can try to prove that yourself.) And the pattern continues. We get:

$$0 \to \mathcal{O}_Y x_1 \cdots x_n \to \oplus_{i=1}^n \mathcal{O}_Y x_1 \cdots \hat{x_i} \cdots x_n \to \cdots \to \oplus_{i=1}^n \mathcal{O}_Y x_i \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

And this is what we wanted (in this special case).

(All that is missing for this to be a proof is to realize that $\oplus_{i=1}^n \mathcal{O}_Y x_i \to \mathcal{O}_Y$ is canonically $Q^\vee$.) ☐

If E is a vector bundle on X, then we have an explicit resolution of f_*E, by tensoring (1) with p^*E:

$$0 \longrightarrow \wedge^d Q^\vee \otimes p^*E \longrightarrow \cdots \longrightarrow Q^\vee \otimes p^*E \xrightarrow{s^\vee} p^*E \longrightarrow f_*E \longrightarrow 0.$$

(Tensoring with a vector bundle is exact, and $(p^*E) \otimes \mathcal{O}_X \cong f_*E$.)

Therefore

$$\boxed{\mathrm{ch}\, f_*[E] = \sum_{p=0}^{d} (-1)^p \,\mathrm{ch}(\wedge^p Q^\vee) \cdot \mathrm{ch}(p^*E).}$$

Lemma.

$$\sum_{p=0}^{d} (-1)^p \,\mathrm{ch}(\wedge^p Q^\vee) = c_d(Q) \cdot \mathrm{td}(Q)^{-1}.$$

This tells you why the Todd class is what it is!

Proof. This is remarkably easy. Let $\alpha_1, \ldots, \alpha_d$ be the Chern roots of Q. Then the Chern roots of $\wedge^p Q^\vee$ are $-\sum \alpha_{i_1} \cdots \alpha_{i_p}$. Hence $\mathrm{ch}(\wedge^p Q^\vee) = \sum e^{-\sum \alpha_{i_1} \cdots \alpha_{i_p}}$ from which

$$\begin{aligned}
\sum_{p=0}^{d} (-1)^p \,\mathrm{ch}(\wedge^p Q^\vee) &= \sum_{p=0}^{d} (-1)^p \sum e^{-\alpha_{i_1}} \cdots e^{-\alpha_{i_p}} \\
&= \prod_{i=1}^{d} (1 - e^{-\alpha_i}) \\
&= (\alpha_1 \cdots \alpha_d) \prod_{i=1}^{d} \frac{1 - e^{-\alpha_i}}{\alpha_i} \\
&= c_d(Q) \cdots \mathrm{td}(Q)^{-1}.
\end{aligned}$$

$\square$

Hence

$$\begin{aligned}
\mathrm{ch}\, f_*[E] &= c_d(Q)\, \mathrm{td}(Q)^{-1} \cdot \mathrm{ch}(p^*E) \\
&= f_*(f^* \mathrm{td}(Q)^{-1} \cdot f^* \mathrm{ch}(p^*E)) \quad \text{(using } c_d(Q) \cap \beta = f_*(f^*\beta)\text{, see 1st par of Section 2)} \\
&= f_*(\mathrm{td}(N_{X/Y})^{-1} \mathrm{ch}(E)) \quad \text{(using } f^*Q = N_{X/Y},\ f^*p^*E = E\text{)} \\
&= f_*(\mathrm{td}(T_X) f^* \mathrm{td}(T_Y)^{-1} \mathrm{ch}(E)) \\
&= \mathrm{td}(T_Y) f_*(\mathrm{td}(T_X)\, \mathrm{ch}(E)) \quad \text{(projection formula)}
\end{aligned}$$

as desired!

This ends the proof of GRR for a closed immersion of X into the projective completion of a normal bundle. $\square$

2.2. **GRR for closed immersions in general.** Suppose $f : X \to Y$ is a closed immersion. We'll prove GRR in this case; again, we need only to consider a generator of $K(X)$, a vector bundle E on X.

We'll show GRR by deformation to the normal cone.

Let $M = \mathrm{Bl}_{X \times \{\infty\}} Y \times \mathbb{P}^1$. (Draw picture.) Recall that the fiber over ∞ is $M_\infty = \mathrm{Bl}_X Y \coprod \mathbb{P}(N \oplus \mathbf{1})$.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & X \times \mathbb{P}^1 \arrow[d, "F"] & X \arrow[l] \arrow[d] \\
Y = M_0 \arrow[r] \arrow[d] & M = \mathrm{Bl}_{X \times 0} Y \times \mathbb{P}^1 \arrow[d] & M_\infty = \mathrm{Bl}_X Y \coprod \mathbb{P}(N \oplus \mathbf{1}) \arrow[l] \arrow[d] \\
\{0\} \arrow[r] & \mathbb{P}^1 & \{\infty\} \arrow[l]
\end{tikzcd}$$

Define F (above), $p : X \times \mathbb{P}^1 \to X$. Resolve $p^* E$ on M:

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to F_*(p^* E) \to 0.$$

Both $X \times \mathbb{P}^1$ and M are flat over $\mathbb{P}^1$ (recall that dominant morphisms from irreducible varieties to a smooth curve are always flat), so restriction of these exact sequences to the fibers M_0 and M_∞ (also known as tensoring with the structure sheaves of the fibers) preserves exactness.

Let $j_0 : Y \cong M_0 \hookrightarrow M$, $j_\infty : \mathrm{Bl}_X Y \cup \mathbb{P}(N \oplus \mathbf{1}) = M_\infty \hookrightarrow M$, $k : \mathbb{P}(N \oplus \mathbf{1}) \hookrightarrow M$, $l : \mathrm{Bl}_X Y \hookrightarrow M$.

Now $j_0^* G.$ resolves f_* on $Y = M_0$. So

$$\begin{aligned}
j_0 * (\mathrm{ch}(f_* E)) &= j_{0*} \mathrm{ch}(j_0^* G\cdot) \\
&= \mathrm{ch}(G.) \cap j_{0*}[Y] \quad \text{(proj. formula)} \\
&= \mathrm{ch}(G.) \cap j_{\infty *}[M_\infty] \quad \text{(pulling back rat'l equivalence } 0 \sim \infty \in \mathbb{P}^1\text{)} \\
&= \mathrm{ch}(G.) \cap (k_*[\mathbb{P}(N \oplus \mathbf{1})] + l_*[\mathrm{Bl}_X Y])
\end{aligned}$$

Now $G.$ is exact away from $X \times \mathbb{P}^1$, so it is exact on $\mathrm{Bl}_X Y$, so the Chern character of the complex (the alternating sums of the Chern characters of the terms) is 0. Hence:

$$= \mathrm{ch}(G.) \cap (k_*[\mathbb{P}(N \oplus \mathbf{1})])$$

Using the projection formula again:

$$= k_*(\mathrm{ch}(\overline{f}_* E) \cap [\mathbb{P}(N \oplus \mathbf{1})])$$

(where $\overline{f}$ is the map $X \hookrightarrow \mathbb{P}(N \oplus \mathbf{1})$). (We're writing this as $k_*(\mathrm{ch}(\overline{f}_* E)$.) So now we're dealing with the case $X \hookrightarrow \mathbb{P}(N \oplus \mathbf{1})$! We've already calculated that this is $f_*(\mathrm{td}(N)^{-1} \cdot \mathrm{ch}(E))$. As $[N] = [f^* T_Y] - [T_X]$:

$$\mathrm{ch}(f_* E)\, \mathrm{td}(T_Y) = f_*(\mathrm{ch}(E)\, \mathrm{td}(T_X))$$

and we're done! $\square$

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INTERSECTION THEORY CLASSES 20 AND 21: BIVARIANT INTERSECTION THEORY

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CONTENTS

1. WHAT WE'RE DOING THIS WEEK

In this final week of class, I'll describe bivariant intersection theory, covering much of Chapter 20. Again, you should notice that given chapters 1 through 6, we can comfortably jump into chapter 20.

Suppose $f : X \to Y$ is any morphism. Throughout today and Wednesday's lectures, we'll use the following notation. Suppose we are given any $Y' \to Y$. Define $X' = X \times_Y Y$, so we have a fiber square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

Recall that the final fundamental intersection construction we came up with was the following. Suppose f is a local complete intersection of codimension d (or more generally a local complete intersection morphism). Then we defined

$$f^! : A_k Y' \to A_{k-d} X'$$

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for all $Y' \to Y$. These Gysin pullbacks were well-behaved in all ways, and in particular compatible with proper pushforward, flat pullback, and intersection products.

An earlier example was that of a flat pullback; if f if flat of relative dimension n, then f' is too, and we got $f^* : A_k Y' \to A_{k-n} X'$, which again behaves well with respect to everything else.

We'll now generalize this notion. Define a *bivariant class* for *any* f (not just lci) as follows. It is a collection of homomorphisms $A_k Y' \to A_{k-p} X'$ for all $Y' \to Y$, all k, again compatible with pushforward, pullback, and intersection products. We'll call the group of such things $A^p(f : X \to Y)$.

We'll see that the group $A^{-k}(X \to pt)$ will be (canonically) isomorphic to $\boxed{A_k X}$. We'll see that $A^k(id : X \to X)$ is a ring, which Fulton calls the *cohomology* group; I might call the resulting ring the Chow ring. We'll denote this by $\boxed{A^k X}$.

The ring structure is a product of the form $A^p X \otimes A^q X \to A^{p+q} X$. We'll define more generally

$$\boxed{A^p(f : X \to Y) \otimes A^q(g : Y \to Z) \to A^{p+q}(g \circ f : X \to Z)}.$$

We'll prove *Poincare duality* when X is *smooth*: $\boxed{A^* X \cong A_* X}$ (as rings — recall we defined a ring structure on the latter). We'll define proper pushforward and pullback operations for bivariant groups. Basically, they'll behave the way you'd expect from homology and cohomology. This will give a cap product $A^* X \times A_* X \to A_* X$. **Alarming fact:** This ring is apparently not known to be commutative in general, because the argument requires resolution of singularities. (It is known to be commutative in characteristic 0, and for smooth things in positive characteristic, and a few more things.) I think it should be possible to show that the ring is commutative in general using technology not available when this theory was first developed, using Johan de Jong's "alteration theorem" in positive characteristic. If you would like to patch this hole, then come talk to me.

Okay, let's get started. Today I'll outline the results, and prove a few things; Wednesday I'll prove some more things.

2. Precise statements

Let $f : X \to Y$ be a morphism. For each $g : Y' \to Y$, form the fiber square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

A bivariant class c in $A^p(f : X \to Y)$ is a collection of homomorphisms

$$c_g^{(k)} : A_k Y' \to A_{k-p} X'$$

for all $g : Y' \to Y$, and all k, compatible with proper pushforwards, flat pullbacks, and intersection products. I'll make that precise in a moment, by stating 3 conditions explicitly. But first I want to show you that you've seen this before in several circumstances.

Example 1. If f is a local complete intersection, or more generally an lci morphism, we've defined $f^!$. This gives some inkling as to why we want to deal with maps $X \to Y$. We could have just had a class on Y', but we have more refined information; we have a class on X', that pushes forward to the more refined class on Y.

Example 2. If $f : X \to Y$ is the identity, and V is a vector bundle on Y, then the Chern classes are of this form: $\alpha \mapsto (g^* c_k(V)) \cap \alpha$.

Example 3 (which generalizes further): pseudodivisors. Let L be a line bundle on Y, and X the zero-scheme of a section s of L. (s might cut out a Cartier divisor, i.e. X will contain no associated points of Y; at the other extreme, s might be 0 everywhere.) Then we defined "capping with a pseudo-divisor": $f^* A_k Y \to A_{k-1} X$. Because pseudodivisors "pull back", X' is a pseudodivisor on Y' (with corresponding line bundle g^*L, and corresponding section g^*s), so we get a map $f^* A_k Y' \to A_{k-1} X'$, and this behaves well respect to everything else.

So we're creating a machine that in some sense incorporates most things we've done so far.

Here are the conditions.

(C_1): If $h : Y'' \to Y$ is proper, $g : Y' \to Y$ is arbitrary, and one forms the fiber diagram

(1)
$$\begin{tikzcd}
X'' \arrow[r, "f''"] \arrow[d, "h'"'] & Y'' \arrow[d, "h"] \\
X' \arrow[r, "f'"'] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd},$$

then for all $\alpha \in A_k Y''$,

$$c_g^{(k)}(h_* \alpha) = h'_* c_{gh}^{(k)} \alpha$$

in $A_{k-p} X'$.

(C_2): If $h : Y'' \to Y$ is flat of relative dimension n, and $g : Y' \to Y$ is arbitrary, and one forms the fiber diagram (1), then, for all $\alpha \in A_k Y'$,

$$c_{gh}^{(k+n)}(h^* \alpha) = h'^* c_g^{(k)} \alpha$$

in $A_{k+n-p} X''$.

(C_3): If $g : Y' \to Y$, $h : Y' \to Z'$ are morphisms, and $i : Z'' \to Z'$ is a local complete intersection of codimension e, and one forms the diagram

(2)
$$\begin{tikzcd} X'' \arrow[r, "f''"'] \arrow[d, "i''"'] & Y'' \arrow[r, "h'"'] \arrow[d, "i'"] & Z'' \arrow[d, "i"] \\ X' \arrow[r, "f'"'] \arrow[d, "g'"'] & Y' \arrow[r, "h"'] \arrow[d, "g"] & Z' \\ X \arrow[r, "f"'] & Y & \end{tikzcd}$$

then, for all $\alpha \in A_k Y'$,

$$i^! c_g^{(k)}(\alpha) = c_{gi'}^{(k-e)}(i^! \alpha)$$

in $A_{k-p-e} X''$.

2.1. **Basic operations and properties.** Here are some basic operations on bivariant Chow groups $A^*(X \to Y)$.

(P_1) *Product*: For all $f : X \to Y$, $g : Y \to Z$, we have

$$\cdot : A^p(f : X \to Y) \otimes A^q(g : Y \to Z) \to A^{p+q}(gf : X \to Z).$$

It is pretty immediate to show this: given any $Z' \to Z$, form the fiber diagram

(3)
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[r, "g'"] \arrow[d] & Z' \arrow[d] \\ X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z. \end{tikzcd}$$

If $\alpha \in A_k Z$, then $d(\alpha) \in A_{k-1} Y'$ and $c(d(\alpha)) \in A_{k-q-p} X'$, so we define $c \cdot d$ by

$$c \cdot d(\alpha) := c(d(\alpha)).$$

(P_2) *Pushforward*: Let $f : X \to Y$ be proper, $g : Y \to Z$ any morphism. Then there is a homomorphism ("proper pushforward"):

$$f_* : A^p(gf : X \to Z) \to A^p(g : Y \to Z).$$

Again, it's straightforward: given $Z' \to Z$, form the fiber diagram (3). If $c \in A^p gf)$, and $\alpha \in A_k(Z')$, then $c(\alpha) \in A_{k-p}(X')$. Since f' is proper, $f'_*(c(\alpha)) \in A_{k-p}(Y')$. Define $f_*(c)$ by

$$f_*(c)(\alpha) = f'_*(c(\alpha)).$$

(P_3): *Pullback* (not necessarily flat!!): Given $f : X \to Y$, $g : Y_1 \to Y$, form the fiber square

(4)
$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

For each p there is a homomorphism

$$g^* : A^p(f : X \to Y) \to A^p(f_1 : X_1 \to Y_1).$$

Again, we just follow our nose. Given $c \in A^p(f)$, $Y' \to Y_1$, then composing with g gives a morphism $Y' \to$. Therefor $c(\alpha) \in A_{k-p}(X')$, $X' = X \times_Y Y' = X_1 \times_{Y_1} Y'$. Set

$$g * (c)(\alpha) = c(\alpha).$$

Here are seven more axioms, which can also be easily verified.

(A_{pr}) *Associativity of products.* If $c \in A(X \to Y)$, $d \in A(Y \to Z)$, $e \in A(Z \to W)$, then

$$(c \cdot d) \cdot e = c \cdot (d \cdot e) \in A(X \to W).$$

(A_{pf}) *Functoriality of proper pushforward.* If $f : X \to Y$ and $g : Y \to Z$ are proper, $Z \to W$ arbitrary, and $c \in A(X \to W)$, then

$$(gf)_*(c) = g_*(f_*c) \in A(Z \to W).$$

(A_{pb}) *Functoriality of pullbacks.* If $c \in A(X \to Y)$, $g : Y_1 \to Y$, $h : Y_2 \to Y_1$, then

$$(gh)^*(c) = h^* g^*(c) \in A(X \times_Y Y_2 \to Y_2).$$

(A_{prpf}) *Product and pushforward commute.* If $f : X \to Y$ is proper, $Y \to Z$ and $Z \to W$ are arbitrary and $c \in A(X \to Z)$, $d \in A(Z \to W)$, then

$$f_*(c) \cdot d = f_*(c \cdot d) \in A(Y \to W).$$

(A_{prpb}) *Product and pullback commute.* If $c \in A(f : X \to Y)$, $d \in A(Y \to Z)$, and $g : Z_1 \to Z$ is a morphism, form the fiber diagram

$$\begin{tikzcd} X_1 \arrow[r, "f'"] \arrow[d] & Y_1 \arrow[r] \arrow[d, "g'"] & Z_1 \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \arrow[r] & Z. \end{tikzcd}$$

Then

$$g^*(c \cdot d) = g'^*(c) \cdot^* (d) \in A(X_1 \to Z_1).$$

(A_{pfpb}) *Proper pushforward and pullback commute.* If $f : X \to Y$ is proper, $Y \to Z$, $g : Z_1 \to Z$, and $c \in A(X \to Z)$ are given, then, with notation as in the preceding diagram

$$g^* f_* c = f'_*(g^* c) \in A(Y_1 \to Z_1).$$

$(A_?)$: *Projection formula.* Given a diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] & \\ X \arrow[r, "f"] & Y \arrow[r, "h"] & Z. \end{tikzcd}$$

with g proper, the square a fiber square, and $c \in A(X \to Y)$, $d \in A(Y' \to Z)$, then

$$c \cdot g_*(d) = g'_*(g^*(c) \cdot d) \in A(X \to Z).$$

3. Proving things

Let $S = \mathrm{Spec}\, K$, where K is some base field. There is a canonical homomorphism

$$\phi : A^{-p}(X \to S) \to A_p(X)$$

given by $c \mapsto c([S])$.

Proposition. This is an isomorphism.

Proof. We will define the inverse morphism. Given $a \in A_p(X)$, define a bivariant class $\psi(a) \in A^{-p}(X \to S)$ as follows: for any $Y \to S$, and any $\alpha \in A_k Y$, define

$$\psi(a)(\alpha) = a \times \alpha \in A_{p+k}(X \times_S Y).$$

(Here $a \times \alpha$ is the exterior product.) Since exterior products are compatible with proper pushforward, flat pullback, and intersections, $\psi(a)$ is a bivariant class.

Let's check that this really is an inverse to ϕ. $\psi(a)([S]) = a$ immediately, so $\phi \circ \psi$ is the identity. To show that $\psi \circ \phi$ is the identity, we have to show that $c(\alpha) = \phi(c) \times \alpha \in A_{k+p}(X \times_S Y)$ for all $\alpha \in A_k Y$. By compatibility with pushforward, we can assume $\alpha = [V]$, and $V = Y$ a variety of dimension k:

$$\begin{tikzcd}
X \times_S V \arrow[r] \arrow[d] & V \arrow[d, "\text{cl. imm.}"] \\
X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

Then $\alpha = p^*[S]$, where $p : V \to S$ is the morphism from V to S. Since c commutes with flat pullback,

$$c(\alpha) = c(p^*[S]) = p^* c([S]) = \phi(c) \times [V]$$

as desired. $\square$

3.1. The Chow ("cohomology") ring.

The Chow ("cohomology") ring. Define $A^p X := A^p(\mathrm{id} : X \to X)$. We have a cup product. We also have an element $1 \in A^0 X$, which acts as the identity. We have a cap product

$$\cap : A^p X \times A_q X \to A_{q-p} X$$

determined by

$$A^p(X \to X) \times A^{-q}(X \to S) \to A^{-(q-p)}(X \to S)$$

which makes $A_* X$ into a left $A^* X$-module. All of this follows formally from our axioms.

4.1. **Poincare duality. Theorem.** Let Y be a smooth, purely n-dimensional scheme (variety) — not necessarily proper (compact).
(a) The canonical homomorphism $\cap[Y] : A^p Y \to A_{n-p} Y$ is an isomorphism.
(b) The ring structure on $A^* Y$ is compatible with that defined on $A_* Y$ earlier. More generally, if $f : X \to Y$ is a morphism, $\beta \in A^* Y$, $\alpha \in A_* X$, then the class $f^*(\beta) \cap \alpha \in A_* X$ coincides with that constructed earlier.

We'll show something more general.

Theorem. Let $g : Y \to Z$ be a smooth morphism of relative dimension n, and let $[g] \in A^{-n}(g : Y \to Z)$ be the bivariant class corresponding to "flat pullback". Then for any morphism $f : X \to Y$ and any p,

$$\cdot[g] : A^p(f : X \to Y) \to A^{p-n}(gf : X \to Z)$$

is an isomorphism.

In general, if $f : X \to Y$ is a flat morphism, or a local complete intersection, or a local complete intersection morphism, the (flat or Gysin) pullback we've defined earlier defines a bivariant class, which we'll denote $[f]$. ($[f^*]$ might be better.) Fulton calls this bivariant class a *canonical orientation*. I'm not sure of the motivation for this terminology, so I'll avoid it.

Proof. We'll define the inverse homomorphism

$$A^{p-n}(gf : X \to Z) \to A^p(f : X \to Y).$$

Consider the fiber diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\gamma"'] & Y \arrow[d, "\delta"] & \\
X \times_Z Y \arrow[r, "f'"] \arrow[d, "p'"'] & Y \times_Z Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

where δ is the diagonal map, and p and q are the first and second projections. Here γ is the "graph" of the morphism $X \to Y$ over Z. Define

$$L : A^{p-n}(gf : X \to Z) \to A^P(X \to Y)$$

by $L(c) = [\gamma] \cdot g^*(c)$. Notice that γ and δ is a local complete intersection of codimension n, with $f'^*[\delta] = [\gamma]$. (This requires a check in the case of γ.)

Notice that $p' \circ \gamma : X \to X$ and $q \circ \delta : Y \to Y$ are both the identity morphisms (on X and Y respectively).

Let's verify that L and "multiplication by $[g]$" are inverse homomorphisms. (This is easier to understand if you see someone pointing at diagrams!) First,

$$\begin{aligned} L(c)\cdot[g] &= [\gamma]\cdot(g^*[c]\cdot[g]) \quad \text{(axiom } (A_{pr})) \\ &= [\gamma]\cdot[p']\cdot c \quad \text{(axiom } (C_2)) \\ &= [p'\circ\gamma]\cdot c = 1\cdot c = c \quad \text{(axiom } (A_{pr})) \end{aligned}$$

Second,

$$\begin{aligned} L(c\cdot[g]) &= f'^*[\delta]\cdot p^*(c)\cdot g^*[g] \quad \text{(axioms } (A_{prpb}), (A_{pr})) \\ &= (p\circ\delta)^*(c)\cdot[\delta][q] \quad \text{(axiom } (C_2)) \\ &= c\cdot[\delta\circ q] = c\cdot 1 = c \quad \text{(axiom } (A_{pr})). \end{aligned}$$

□

4.2. **Chern classes commute with all bivariant classes.** Put another way, any operation which commutes with proper pushforward, pullback, and intersections, automatically commutes with Chern classes. Precisely:

Proposition. Let $c \in A^q(f : X \to Y)$, $Y' \to Y$, $\alpha \in A_k(Y')$, E a vector bundle on Y'. Then

$$c(c_p(E)\cap\alpha) = c_p(f'^*E)\cap c(\alpha) \in A_{k-q-p}X'$$

where $f' : X' = X \times_Y Y' \to Y$.

Proof. Recall our definition of Chern classes. They are certain polynomials in Segre classes. Segre classes are defined using operations of the form $\alpha \mapsto p_*(c_1(\mathcal{O}(1))^i \cap p^*\alpha)$, and since c commutes with p_* and p^*, we just have to show that c commutes with $c_1(L)\cap$, where L is a line bundle on Y'. We may assume $\alpha = [V]$. Because c commutes with proper pushforward, we may assume $V = Y'$. Let $L = \mathcal{O}(D)$, D a Cartier divisor on V.

We can replace V by V', where $V' \to V$ is proper and birational, so we may assume $D = D_1 - D_2$, where D_1 and D_2 are effective. (Recall our trick in chapter 2: it isn't true that a Cartier divisor is a difference of two effective Cartier divisors, but we can do a clever blow-up and turn it into a difference of two effective Cartier divisors.) Let $i : D \hookrightarrow V$ be the inclusion. Then we've shown that $c_1(L)\cap\alpha = i_* i^! \alpha$, and since c commutes with i_* and $i^!$, c commutes with $c(L)$. □

4.3. **Bivariant classes vanish in dimensions that you'd expect them to. Proposition.** Let $f : X \to Y$ be a morphism. Let $m = \dim Y$, and let n be the largest dimension of any fiber $f^{-1}y$, $y \in Y$.

$$A^p(f : X \to Y) = 0 \quad \text{if } p < -n \text{ or } p > m.$$

(Think about why this is what you'd expect!) In particular, for any X, $A^pX = 0$ unless $0 \le p \le \dim X$.

(Proof omitted.)

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