Basic functions and the arc space of L-monoids

These are notes for two talks in the Beyond Endoscopy Learning Seminar at Columbia, Spring 2018. Our main references are [1] and [2].

Recall that two key constructions are required in the Braverman-Kazhdan program for proving analytic continuation and functional equations for general Langlands L-function $L(s,\pi,\rho)$. One is a suitable space of *Schwartz functions* $\mathcal{S}^\rho(G)$ at each local place, containing a distinguished function encoding the unramified local L-factor (known as the *basic function* $\mathcal{C}_\rho$ after Sakellaridis). The other is a generalized Fourier transform (known as the *Hankel transform* after Ngo) preserving the Schwartz space and the basic function. With a global Poisson summation formula, one should be able to establish the desired analytic properties of $L(s,\pi,\rho)$ in a way analogous to Godement-Jacquet theory for standard L-function on GL_n. Our goal today is to discuss the basic function $\mathcal{C}_\rho$ and to explain its an algebro-geometric interpretation due to Bouthier-Ngo-Sakellaridis, using the L-monoid $\bar{G}_\rho$ appeared in previous talks and its arc space.

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Satake/Langlands parameters

Let F be a non-archimedean local field. Let G be a split reductive group over F. Let $\hat{G}=\hat{G}(\mathbb{C})$ be its dual group. Let $\mathcal{H}=C_c^\infty(G(\mathcal{O})\backslash G(F)/G(\mathcal{O}))$ be the spherical Hecke algebra. Recall that the classical Satake transform

$$\mathrm{Sat}:\mathcal{H}\to\mathbb{C}[X_*(T)],\quad f\mapsto\left(t\mapsto\delta_B(t)^{1/2}\int_{N(F)}f(tn)dn\right),$$

induces an algebra isomorphism onto the W-invariants

$$\mathcal{H}\xrightarrow{\sim}\mathbb{C}[X_*(T)]^W.$$

An unramified representation π of $G(F)$ corresponds to a 1-dimensional character of $\mathcal{H}$, given by its action on the spherical vector

$$\pi(f)v=\int_{G(F)}f(g)\pi(g)vdg,\quad v\in\pi^{G(\mathcal{O})}.$$

Langlands noticed that $\mathbb{C}[X_*(T)]^W$ is the coordinate ring of the variety $\hat{T}/W$, so a 1-dimensional character $\mathbb{C}[X_*(T)]^W$ corresponds to a point $\alpha_\pi\in\hat{T}/W$, i.e., a semisimple conjugacy class in $\hat{G}$. In this we obtain a bijection $\pi\mapsto\alpha_\pi$ between unramified representations of $G(F)$ and the Satake (or rather, Langlands) parameters. The Satake transform is then characterized by the identity

$$\mathrm{tr}\,\pi(f)=\mathrm{Sat}(f)(\alpha_\pi),\quad f\in\mathcal{H}.$$

Also notice that the target of the Satake isomorphism can be identified with the representation ring of $\hat{G}$, and thus with the $\hat{G}$-invariant regular functions $\mathcal{O}(\hat{G})^{\hat{G}}$ on $\hat{G}$ (via the trace map).

Remark 1 Notice the Satake isomorphism is of combinatorial nature: both the source and the target of depends only on the root datum of G and the size of the residue field q. In fact, the Satake isomorphism can be defined over

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$\mathbb{Z}[q^{\pm 1/2}]$.

Basic functions

The importance of the Satake parameter is due to its key role in defining the unramified local L-factor $L(s,\pi,\rho)$. Let $\rho:\hat G\to \mathrm{GL}(V)$ be an irreducible representation. Recall by definition

$$L(s,\pi,\rho)=\det(1-\rho(\alpha_\pi)q^{-s})^{-1}.$$

Now if we have a diagonal matrix $A=\mathrm{diag}(\alpha_1,\dots\alpha_k)$, then

$$\det(1-At)^{-1}=\prod_{i=1}^k(1-\alpha_it)^{-1}=\prod_{i=1}^k(1+\alpha_it+\alpha_i^2t^2+\cdots)=1+(\mathrm{tr}\,A)t+(\mathrm{tr}\,\mathrm{Sym}^2 A)t$$

Therefore

$$L(s,\pi,\rho)=\sum_{d\ge 0}\mathrm{tr}(\mathrm{Sym}^d\rho)(\alpha_\pi)q^{-ds}.$$

To remove the dependence on π, we are motivated to introduce the following definition.

Definition 1 We define $\mathcal{C}_\rho^d(s)$ to be the inverse under the Satake transform of the function $\mathrm{tr}\,\mathrm{Sym}^d\rho\cdot q^{-ds}$ (so $\mathcal{C}_\rho^d(s_0)\in\mathcal{H}$ for any given $s=s_0$. Define the *basic function* to be

$$\mathcal{C}_\rho(s)=\sum_{d\ge0}\mathcal{C}_\rho^d(s).$$

When $\mathrm{Re}(s)\gg 0$, the sum is locally finite and makes sense as a function on $G(F)$.

Even though each $\mathcal{C}_\rho^d$ is compactly supported (with support lies in the K-double cosets indexed by dominant coweights of G corresponding to weights of $\mathrm{Sym}^d\rho$), the support gets larger when d increases and $\mathcal{C}_d$ is not longer compactly supported. Moreover, the values of $\mathcal{C}_\rho$ on each K-double cosets can be written down in terms of representation theory (related to Kazhdan-Lusztig polynomials) and thus involve quite complicated combinatorial quantities.

Example 1 Take $G=\mathbb{G}_m$, and $\rho=\mathrm{Std}$. Since $G=T$, both the source and target of the Satake isomorphism are identified as functions on $\mathbb{Z}$. The Satake transform sends the characteristic function $\mathbf{1}_{\mathrm{val}=d}$ to $\mathrm{tr}\,\mathrm{Sym}^d\rho: t\mapsto t^d$. So $\mathcal{C}_\rho^d=\mathbf{1}_{\mathrm{val}=d}$ and the basic function is given by $\mathcal{C}_\rho=\mathbf{1}_{\mathcal{O}}$ (always viewed as a function on $G(F)$). This generalizes to the standard representation of $G=\mathrm{GL}_n$, in which case $\mathcal{C}_\rho^d=\mathbf{1}_{\mathrm{M}_n(\mathcal{O})_{\mathrm{val}(\det)=d}}$ (this is already a nontrivial computation) and hence $\mathcal{C}_\rho=\mathbf{1}_{\mathrm{M}_n(\mathcal{O})}$.

Example 2 Take $G=\mathbb{G}_m$, and $\rho:\mathbb{G}_m\to\mathrm{GL}_2(\mathbb{C}), t\mapsto\mathrm{diag}(t,t)$ (i.e., $\rho=\mathrm{Std}\oplus\mathrm{Std}$). Then $\mathrm{Sym}^d\rho$ has dimension $d+1$ given by $t\mapsto(t^d,\dots,t^d)$, whose trace is $t\mapsto(d+1)t^d$. So the corresponding basic function is given by

$$\mathcal{C}_\rho=\sum_{d\ge0}(d+1)\mathbf{1}_{\mathrm{val}=d}=\mathrm{val}(\cdot)+1.$$

This is no longer the characteristic function of any set. More generally, take $G=\mathbb{G}_m$ and $\rho=\chi_1\oplus\cdots\oplus\chi_n$ ($\chi_i\ge0$). Then

$$\mathcal{C}_\rho=\sum_{d\ge0}\#\{(a_1,\dots,a_n):\sum a_i\chi_i=d, a_i\ge0\}\cdot\mathbf{1}_{\mathrm{val}=d}.$$

So the value of $\mathcal{C}_\rho$ encodes partition numbers, and can not have simple formula. We also see that the support of $\mathcal{C}_\rho$ is contained in the cone generated by the weights of ρ.

Example 3 Take $G=\mathrm{GL}_2$ and $\rho=\mathrm{Sym}^k\mathrm{Std}$. Then computing $\mathcal{C}_\rho$ amounts to decomposing $\mathrm{Sym}^d(\mathrm{Sym}^k\mathrm{Std})$ into irreducibles, again this is a difficult combinatoric problem. In fact, we have

$$\mathrm{Sym}^d\,\mathrm{Sym}^k\cong\bigoplus_{i=0}^{[dk/2]}(\mathrm{Sym}^{dk-2i}\otimes\det{}^{dk-i})^{\oplus N(d,k,i)}.$$

Here the multiplicity $N(d,k,i)=p(d,k,i)-p(d,k,i-1)$, and $p(d,k,i)$ is the number of partitions of i into at most k parts, having largest part at most d.

I hope these examples illustrate that writing down an explicit formula for the basic function is quite hopeless in general (but see Wen-Wei Li's paper). Instead we would like to focus on finding some natural algebro-geometric object which encodes these combinatoric information. This is the main motivation to introduce the L-monoid.

Vinberg's universal monoids

Let G be a split reductive group over a field k (later k will be the residue field of the local field F). Assume G has a nontrivial map to $\mathbb{G}_m$, denoted by $\det: G \to \mathbb{G}_m$. Assume $G' = \ker(\det)$ is semisimple and simply-connected. Our first goal is to construct Vinberg's *universal monoid* $\bar G$. It is a normal affine variety $\bar G$ fitting into a commutative diagram

$$\begin{tikzcd} G^+ \arrow[r, hook] \arrow[d] & \bar G \arrow[d] \\ \mathbb{G}_m^r \arrow[r, hook] & \mathbb{A}^r. \end{tikzcd}$$

This monoid is universal in the sense that every reductive monoid with derived group equal to G' can be obtained by base change from $\bar G$ (in fact the construction of $\bar G$ will only depend on G').

Let $T' \subseteq G'$ be a maximal torus of G' . Let $G^+ = (T' \times G')/\Delta(Z(G'))$. Let r be the semisimple rank of G' . Let $\{\omega_1, \ldots, \omega_r\}$ be the set of fundamental weights of G' (dual to the coroots). Let ρ_i be the fundamental representation of G' associated to ω_i . We extend ρ_i from G' to G^+ by

$$\rho_i^+ : G^+ = (T' \times G')/\Delta(Z(G')) \to \mathrm{GL}(V_i), \quad (t, g) \mapsto \omega_i(w(t^{-1}))\rho_i(g).$$

Here $w \in W$ is the longest element in the Weyl group. We also extend the simple roots α_i from T' to G^+ by

$$\alpha_i^+ : G^+ \to \mathbb{G}_m, \quad (t, g) \mapsto \alpha_i(t).$$

These extensions together give a homomorphism

$$(\alpha^+, \rho^+) : G^+ \to \mathbb{G}_m^r \times \prod_{i=1}^r \mathrm{GL}(V_i).$$

Definition 2 We define $\bar G$ to be the closure of the image of G^+ in

$$\mathbb{A}^r \times \prod_{i=1}^r \mathrm{End}(V_i).$$

Example 4 Consider $G = \mathrm{GL}_2$. Then $G' = \mathrm{SL}_2$, $T' \cong \mathbb{G}_m$, $G^+ = (T' \times \mathrm{SL}_2)/\mu_2$. We have $r = 1$, $\rho_1 = \mathrm{Std}$, $w(\mathrm{diag}(t, t^{-1})) = \mathrm{diag}(t^{-1}, t)$, $\omega_1(\mathrm{diag}(t, t^{-1})) = t$, $\alpha_1(\mathrm{diag}(t, t^{-1})) = t^2$. So

$$(\alpha^+, \rho^+) : (\mathrm{diag}(t, t^{-1}), g) \mapsto (t^2, \mathrm{diag}(t, t) \cdot g).$$

So $\bar G = \{(t, g) \in \mathbb{A}^1 \times \mathrm{M}_2 : t = \det g\}$. In other words, this is a monoid in $\mathbb{A}^5$ defined by the equation $t = ac - bd$ (which is smooth).

Ngo's L-monoids

Now let $\rho: \hat G \to \mathrm{GL}(V)$ be an irreducible representation. Let $T^{\mathrm{ad}} = T'/Z(G')$ be a maximal torus in the adjoint group of G' . The highest weight of ρ defines a cocharacter $\lambda_\rho : \mathbb{G}_m \to T$, hence a cocharacter of $\lambda_{\rho,\mathrm{ad}} : \mathbb{G}_m \to T^{\mathrm{ad}}$. We identify

$$T^{\mathrm{ad}} \cong \mathbb{G}_m^r, \quad t \mapsto (\alpha_1(t), \ldots, \alpha_r(t)),$$

using a choice of simple roots. Then

$$\lambda_{\rho,\mathrm{ad}} : \mathbb{G}_m \to \mathbb{G}_m^r$$

can be extended to a morphism of monoids

$$\bar\lambda_{\rho,\mathrm{ad}} : \mathbb{A}^1 \to \mathbb{A}^r.$$

Definition 3 The L *-monoid* $\bar G_\rho$ is defined by base changing the universal monoid $\bar G \to \mathbb{A}^r$ along $\bar\lambda_{\rho,\mathrm{ad}}$. So we have a commutative diagram

$$\begin{tikzcd} \bar G_\rho^\times \arrow[r, hook] \arrow[d] & \bar G_\rho \arrow[d] \\ \mathbb{G}_m \arrow[r, hook] & \mathbb{A}^1. \end{tikzcd}$$

Example 5 Again take $G = \mathrm{GL}_2$, $\rho = \mathrm{Sym}^n(\mathrm{Std})$. Then $\lambda_{\rho,\mathrm{ad}} : \mathbb{G}_m \to \mathbb{G}_m, t \mapsto t^n$. So we have

$$\bar G_\rho = \{(t, g) \in \mathbb{A}^1 \times \mathrm{M}_2 : t^n = \det g\}.$$

Notice that the unit group is GL_2 when F is odd and $\mathbb{G}_m \times \mathrm{SL}_2$ when F is even (the derived group is SL_2 in both cases). Notice that this monoid is singular at the origin when $n > 1$, which reflects the fact that the basic function are more complicated than the $n = 1$ case.

Remark 2 Assuming that $\mathbb{G}_m \xrightarrow{\hat{\det}} \hat G \xrightarrow{\rho} \mathrm{GL}(V)$ is the identity map (e.g., $n = 1$ in the previous example) ensures that the unit group of $\bar G_\rho$ is G and we obtain a commutative diagram

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d, "\det"] & \bar{G}_\rho \arrow[d] \\ \mathbb{G}_m \arrow[r, hook] & \mathbb{A}^1. \end{tikzcd}$$

The construction of $\bar{G}_\rho$ can be characterized in terms of toric varieties: it is the unique reductive monoid with unit group G such that the closure of any maximal torus T in $\bar{G}_\rho$ is the toric variety associated to T and the cone generated by the weights of ρ .

Arc spaces

Directly comes from the construction of $\bar{G}_\rho$ one sees that $\bar{G}_\rho(\mathcal{O}) \cap G(F)$ is exactly supported on the K -double cosets associated dominant weights generated by the weights of ρ . So the basic function $\mathcal{C}_\rho$ can be viewed as a function on $\bar{G}_\rho(\mathcal{O}) \cap G(F)$. Now take $F = k((t))$. Then we have the advantage of endowing $\bar{G}_\rho(\mathcal{O})$ an algebro-geometric structure over the residue field k .

Definition 4 Let X be an algebraic variety over a field k . We define its n *-th jet space* $\mathcal{L}_n(X)$ to be the functor sending a k -algebra R to the set $X(R[t]/t^{n+1})$. If X is affine, then $\mathcal{L}_n(X)$ is also representable by an affine X -scheme of finite type. In particular, $\mathcal{L}_n(X)(k) = X(k[t]/t^{n+1}) = \mathrm{Hom}(k[t]/t^{n+1}, X)$ consists of order- F arcs in X . When $n = 1$ we exactly recover the tangent bundle of X . For more general F , $\mathcal{L}_n(X)$ contains information about the singularities of X .

Example 6 $\mathcal{L}_n(\mathbb{A}^1) = \mathbb{A}^{n+1}$.

Example 7 Notice that if X is defined by $f(x,y) = 0$, then $\mathcal{L}_n(X)$ is defined by the equation $f(x + a_1 t + \cdots a_n t^n, y + b_1 t + \cdots b_n t^n) = 0 \pmod{t^n}$ with extra variables a_i, b_i . Take $X = \{xy = 0\} \subseteq \mathbb{A}^2$. Then $\mathcal{L}_n(X)$ is given by

$$(x_0 + x_1 t + \cdots x_n t^n)(y_0 + y_1 t + \cdots y_n t^n) = 0 \pmod{t^{n+1}}.$$

One can find $\mathcal{L}_n(X)$ exactly has $n+2$ irreducible components, each isomorphic to $\mathbb{A}^{n+1}$ given by the first k of the x -coordinates are 0 and first ℓ of the y -coordinates equal to zero, where $k + \ell = n+1$. The component with $k = 0$ maps to the line $x = 0$, and the component with $k = n+1$ maps to the other line $y = 0$. All the rest $n - 1$ components maps to the singularity (the origin).

Example 8 Take $X = \{x^3 + y^3 + z^3 = 0\} \subseteq \mathbb{A}^3$. Then $\mathcal{L}_n(X)$ has one irreducible component of dimension $2(m+1)$ which dominates X , and has one extra component of the same dimension mapping to the origin when $m \equiv 2 \pmod 3$.

If X is smooth, then the natural map $\mathcal{L}_n(X) \to X$ is smooth and surjective. In general, if X is not smooth, then $\mathcal{L}_n(X) \to X$ may fail to be surjective, and the transition maps can be rather complicated.

Definition 5 We define the (formal) *arc space* to be $\mathcal{L}(X) = \varprojlim_n \mathcal{L}_n(X)$. In particular, $\mathcal{L}(X)(k) = X(k[[t]]) = X(\mathcal{O})$, which consists of (formal) arcs $\mathbb{D} \to X$ of X (here $\mathbb{D} = \mathrm{Spf}\, k[[t]]$ is the formal disc).

Again if X is smooth then $\mathcal{L}(X) \to X$ is formally smooth and surjective. A theorem of John Nash says that the inverse image of X_{sing} in $\mathcal{L}(X)$ has only finitely many irreducible components, each corresponds to a component in the inverse image of X_{sing} in any resolution of singularities of $Y \to X$.

Definition 6 Let $X^\circ \subseteq X$ be a smooth open dense subvariety. We define $\mathcal{L}^\circ(X) \subseteq \mathcal{L}(X)$ to be the space of *non-degenerate arcs* in X° . Namely for a k -algebra R , $\mathcal{L}^\circ(X)(R)$ consists of arcs $\phi : \mathbb{D}_R \to X$ such that inverse image $\phi^{-1}(X^\circ)$ is open in $\mathbb{D}_R$ and surjects to $\mathrm{Spec}\, R$. In particular, we have

$$\mathcal{L}^\circ(X)(k) = X(\mathcal{O}) \cap X^\circ(F).$$

If one has a ℓ -adic sheaf $\mathcal{F}$ on $\mathcal{L}(X)$, then taking the Frobenius trace gives us a function

$$\mathcal{C}_\mathcal{F} : \mathcal{L}(X)(k) = X(\mathcal{O}) \to \overline{\mathbb{Q}_\ell}, \quad x \mapsto \mathrm{Tr}(\mathrm{Frob}_x : \mathcal{F}_x).$$

(if $\mathcal{F}$ is a complex, then take alternating trace on the cohomology groups). Similarly, if we only have a sheaf on $\mathcal{L}^\circ(X)$, we can still obtain a function on $X(\mathcal{O}) \cap X^\circ(F)$. When specializing to $X = \bar{G}_\rho$ and $X^\circ = G$, we can obtain a function on $\bar{G}_\rho(\mathcal{O}) \cap G(F)$ as desired. Our next goal is then to construct a canonical sheaf on $\mathcal{L}^\circ(\bar{G}_\rho)$, whose associated function gives the basic function $\mathcal{C}_\rho$.

IC sheaves and functions

If X is a variety over k, there is a canonical sheaf associated to X, i.e., its IC sheaf which generalizes the constant sheaf and encodes the singularities of X.

Definition 7 Let $j: X^\circ \hookrightarrow X$ be a smooth open dense subvariety. We define
$$\mathrm{IC}_X := j_{!*}\mathbb{Q}_\ell = \mathrm{im}(j_!\overline{\mathbb{Q}_\ell} \to Rj_*\overline{\mathbb{Q}_\ell}),$$
to be the middle extension of the constant sheaf on X° (so IC_X a complex of sheaves in the derived category of X). It is independent of the choice of X° and measures the singularities of X along the boundary. The shift $\mathrm{IC}_X[\dim X]$ serves as the dualizing sheaf for the Poincare(-Verdier) duality for singular varieties, and is a basic example of a perverse sheaf.

However, because the arc space $\mathcal{L}(X)$ is infinite type over k, there is no good theory of IC sheaves/perverse sheaves on $\mathcal{L}(X)$. Fortunately, the singularities of $\mathcal{L}(X)$ have a finite dimensional model.

Definition 8 A *finite dimensional formal model* of $\mathcal{L}(X)(k)$ at $x \in \mathcal{L}(X)(k)$ is a formal scheme Y_y (the subscript means taking formal completion), where Y is a finite type k-scheme and $y \in Y(k)$ a point such that
$$\mathcal{L}(X)_x \cong Y_y \times \mathbb{D}^\infty.$$

Theorem 1 (Drinfeld (2002), generalizing Grinberg—Kazhdan (2000) for $\mathrm{char}\, k = 0$) Finite dimensional model exists at each point $x \in \mathcal{L}^\circ(X)(k)$.

Bouthier-Ngo-Sakellaridis [1] show that the stalk $\mathrm{IC}_{Y,y}$ of the IC sheaf of Y does not depend on the choice of the finite dimensional formal model Y_y. It now makes sense to define the IC function on the non-degenerate arcs by
$$\mathrm{IC}_{\mathcal{L}(X)}: \mathcal{L}^\circ(X)(k) \to \overline{\mathbb{Q}_\ell}, \quad x \mapsto \mathrm{tr}(\mathrm{Frob}_y : \mathrm{IC}_{Y,y}).$$
It is a numerical invariant encoding the singularities of X. By taking $X^\circ = G$ and $X = \bar{G}_\rho$, we obtain
$$\mathrm{IC}_\rho : \bar{G}_\rho(\mathcal{O}) \cap G(F) \to \overline{\mathbb{Q}_\ell}.$$

Now we can state the main theorem of [1].

Theorem 2 (Bouthier-Ngo-Sakellaridis (2016)) Let ν_G be the half sum of all positive roots. Then
$$\mathrm{IC}_\rho = \mathcal{C}_\rho(-\langle \nu_G, \lambda_\rho\rangle).$$

Example 9 When $G = \mathrm{GL}_n$ and $\rho = \mathrm{Std}$, we have $\langle \nu_G, \lambda_\rho\rangle = (n-1)/2$. So we recover
$$\mathrm{tr}(\pi \otimes |\det|^s|)(\mathrm{IC}_\rho) = L(s-(n-1)/2, \pi).$$
The shift $(n-1)/2$ matches with the Godement-Jacquet zeta integral as well.

A global model

To prove the main theorem, we need a concrete construction of the finite dimensional model of $\mathcal{L}(X)$ at non-degenerate arcs. To do so we make use of a global smooth projective curve C/k. From now on, let $X = \bar{G}_\rho$ (with left and right G-actions).

Definition 9 Recall that an S-point of the quotient stack $[X/G]$ consists of a principal G-bundle $\mathcal{E}$ over S together with a G-equivariant map $\phi: \mathcal{E} \to X$. Consider the stack $\mathrm{Map}(C, [X/G])$, whose k-points consists of maps $\phi: C \to [X/G]$, namely a principal G-bundle $\mathcal{E}$ over k together a G-equivariant homomorphism $\mathcal{E} \to X$. We now add the non-degeneracy and define M to be the open substack of $\mathrm{Map}(C, [X/G])$ such that $\phi: \mathcal{E} \to X$ factors through $\phi: \mathcal{E}|_U \to G$ for a open subset $U \subseteq C$. Then one can show that M is an algebraic space locally of finite type.

Definition 10 To relate M to $\mathcal{L}(X)$, we fix a k-point $v \in C(k)$. Define $\tilde{M}$ to be the stack classifying a point $(\mathcal{E}, \phi) \in M$ together with $\theta: \mathbb{D}_v \times G \cong \mathcal{E}_v$, a trivialization of $\mathcal{E}$ on the formal disc $\mathbb{D}_v$. Then we have a canonical projection
$$\tilde{M} \to M,$$
which is a torsor under $\mathcal{L}(G)$, hence is formally smooth. On the other hand, given a point $(\mathcal{E}, \phi, \theta) \in \tilde{M}$, we obtain an arc by the composite map
$$\mathbb{D}_v \to \mathbb{D}_v \times G \xrightarrow{\theta} \mathcal{E}_v \hookrightarrow \mathcal{E} \xrightarrow{\phi} X.$$
Moreover, this arc is non-degenerate (by the non-degenerate requirement when defining M). Thus we obtain a morphism
$$\tilde{M} \to \mathcal{L}^\circ(X).$$

The following essentially says that there is no obstruction for deforming G-bundles while fixing the induced formal arc.

Proposition 1 Let $x \in \mathcal{L}^\circ(X)(k)$. Let $y \in \tilde{M}(k)$ be a point such that $\phi|_{C\setminus v}$ lies in the smooth locus of X, and such that its image in $\mathcal{L}^\circ(X)(k)$ is x (such y always exists by Beauville-Laszlo patching the trivial G-bundle). Then $\tilde{M}_y \to (\mathcal{L}^\circ X)_x$ is formally smooth.

It follows that

$$M_y \times \mathbb{D}^\infty \cong \tilde{M}_y \times \mathbb{D}^\infty \cong (\mathcal{L}^\circ X)_x \times \mathbb{D}^\infty,$$

and hence M_y can serve as a finite dimensional formal model of $\mathcal{L}^\circ(X)$ at x. In particular, we obtain

$$\mathrm{IC}_M(y) \cong \mathrm{IC}_\rho(x)$$

Geometric Satake

Let $y \in M(k)$. From the fixed map $\det : G \to \mathbb{G}_m$ one naturally associates to y a line bundle on k. Using the trivialization of $\mathcal{E}|_U$ induced from $\phi : \mathcal{E}|_U \to G$, we also obtain a generic section of this line bundle, hence a divisor D on k.

Let $D = \sum n_i v_i$ and $M_D \subseteq M$ be the substack whose associated divisor is D. By the Beauville-Laszlo patching, the data of a G-bundle $\mathcal{E}$ and a trivialization away from D is the same as giving G-bundles $\mathcal{E}_i$ on the formal disc $\mathbb{D}_{v_i}$ together with a trivialization on the punctured formal disc $\mathbb{D}_{v_i}^*$. Then we obtain a map into the affine Grassmannians Gr (whose k-points are $G(F)/G(\mathcal{O})$) at v_i's,

$$M_D \to \prod_{i=1}^m \mathrm{Gr}_{v_i}.$$

Moreover, a trivialization of $\mathcal{E}|_U$ actually comes from a G-equivariant map $\mathcal{E} \to X = \bar{G}_\rho$ if and only if $\mathcal{E}_i$ has invariant $\le n_i\lambda_\rho$ for each i. Thus we obtain an isomorphism

$$M_D \cong \prod_{i=1}^m \mathrm{Gr}_{v_i, \le n_i\lambda_\rho}.$$

Notice each term on the right is indeed a projective variety (a Schubert variety), which models singularity of $\mathcal{L}^\circ(X)(k)$ when $n_i \to \infty$. Varying D, we obtain an isomorphism

$$M(k) \cong \prod_{v\in|C|}{}' (\bar{G}_\rho(\mathcal{O}_v) \cap G(F_v))/G(\mathcal{O}_v).$$

Using this isomorphism and a fixed $v \in C(k)$, we can choose the point $y \in M(k)$ explicitly corresponding to a point $x \in \mathcal{L}^\circ(X)(k)$ such that $\mathrm{IC}_\rho(x)$ is the v-component of $\mathrm{IC}_M(y)$.

Now recall the geometric Satake correspondence.

Theorem 3 (Mirkovic-Vilonen (2007)) Let K_ρ be the IC sheaf of the Schubert variety $\mathrm{Gr}_{\le\lambda_\rho}$ shifted by its dimension $\langle 2\nu_G, \lambda_\rho\rangle$. Then the map $\rho \mapsto K_\rho$ gives an equivalence of tensor categories between the finite dimensional representations of $\hat{G}$ and $\mathcal{L}(G)$-equivariant perverse sheaves on Gr (the tensor structure given the convolution product).

Bouthier-Ngo-Sakellaridis show that

$$\mathrm{IC}_{M_D} \cong \boxtimes_{i=1}^m K_{v_i, \mathrm{Sym}^{n_i}(\rho)}[-n_i\langle 2\nu_G, \lambda_\rho\rangle](-n_i\langle \nu_G, \lambda_\rho\rangle).$$

(The symmetric power essentially comes from looking at the map $C^{n_i} \to \mathrm{Sym}^{n_i} C$). Hence by the geometric Satake we have

$$\mathrm{IC}_M = \prod_{v\in|C|} \sum_{d\ge 0} \mathcal{C}^d_{\rho,v}(-\langle \nu_G, \lambda_\rho\rangle).$$

The main theorem now follows by taking the v-component.

References

[1] Bouthier, A. and Ngô, B. C. and Sakellaridis, Y., *On the formal arc space of a reductive monoid,* Amer. J. Math. **138** (2016), no.1, 81--108.

[2] Ngo, Bao Chau, *Hankel transform, Langlands functoriality and functional equation of automorphic L-functions,* http://math.uchicago.edu/~ngo/takagi.pdf.