Geometric aspects of p-adic Hodge theory

These are my live-TeXed notes for Professor Bhargav Bhatt 's Eilenberg lectures *Geometric aspects of p-adic Hodge theory* at Columbia, Fall 2018.

Notice that Bhargav has written up his own notes (read those!) I keep the live-TeXed notes here for the sole purpose of reinforcing my understanding.

[-] **Contents**

Links

Chao Li's Homepage
Columbia University
Math Department

Introduction

The goal of this series to explain prismatic cohomology, which unifies various cohomology theories in p -adic geometry.

Motivation

The motivation comes from the classical de Rham comparison theorem.

Theorem 1 (de Rham, Serre) Let X be a compact complex manifold. Then

$$H^i(X, \mathbb{C}) \cong H^i_{\mathrm{dR}}(X).$$

Notice that RHS is the cohomology of the holomorphic (or algebraic, hence the name "Serre") de Rham complex of X, which depends on the geometry of X, while LHS only depends on the topology of X. Explicitly, the isomorphism is given by integration along cycles and each homology class $\gamma \in H_i(X, \mathbb{C})$ defines an obstruction to integrating A-forms on X.

RHS is rather computable (e.g., by Macaulay) using the defining equation of X, which helps one to obtain topological information about X. Conversely, the comparison shows that deforming the complex structure X without changing the underlying topology still keeps holomorphic invariants.

Question How to see mod p cohomology classes (or equivalently, p-torsion classes) on X "geometrically"?

Answer At least when X is algebraic, these give obstructions to integrating A-forms on "$X \bmod p$".

Global statements

Let $X \subseteq \mathbb{P}^n$ be a smooth closed subvariety, whose defining equations live in $\mathbb{Z}[1/N]$. Assume that X is still smooth mod $p \nmid N$ (i.e., good reduction outside N).

Theorem 2 (Bhatt-Morrow-Scholze) In this set-up, for any $p \nmid N$. We have

$$\dim_{\mathbb{F}_p} H^i(X^{\mathrm{an}}, \mathbb{Z}/p) \leq \dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(X_p).$$

Here X^{an} is the associated complex analytic manifold, X_p is the reduction of X mod p, and H^i_{dR} is the algebraic de Rham cohomology.

Remark 1

a. The inequality in Theorem 2 can be *strict* (examples can be constructed using fibrations with no sections but admitting sections mod p). For p large enough, both sides are equal to the dimension of $H^i(X^{\mathrm{an}}, \mathbb{C})$. For a fixed p, the Euler characteristics of both sides are the same.

b. The inequality *cannot* be upgraded to a "naturally defined" subquotient relation: there exists a version of the inequality for mod p^n coefficients, and there is an example such that LHS = $\mathbb{Z}/p^2$ while RHS = $\mathbb{Z}/p \times \mathbb{Z}/p$.

c. The inequality is also valid for any proper smooth formal scheme $X \to \mathrm{Spf}(V)$, where V is a p-adic valuation ring (after replacing singular cohomology by etale cohomology). In this case, if $i \cdot \mathrm{ram}(V) < p - 1$, then the inequality is actually an equality (this recovers earlier results of Faltings, Caruso).

d. The inequality also holds true for semistable reduction (Cesnavicus, Koshikawa). They basically carried out what we did for the good reduction case.

Corollary 1 If $\dim H^i_{\mathrm{dR}}(X^{\mathrm{an}}) = \dim H^i_{\mathrm{dR}}(X_p)$. Then $H^{i+1}(X^{\mathrm{an}}, \mathbb{Z})$ has no p-torsion.

This gives an *algebraic* way to control the torsion in singular cohomology.

Conversely, torsion in singular cohomology forces de Rham cohomology to be larger, as in the following example.

Example 1 Suppose Y is an Enriques surface in characteristic 2. Then Y has the form X_2 for some Enriques surface X in characteristic 0. X is always a quotient of K3 surface by a fixed point-free involution, so has fundamental group $\mathbb{Z}/2$. So the inequality in Theorem 2 implies that $H^1_{\mathrm{dR}}(Y) \neq 0$ even though $H^1_{\mathrm{dR}}(X) = 0$ (this recovers an example of Illusie, W. Lang).

The strategy of the proof of Theorem 2 is to consider a cohomology theory $H^*_A(X)$ valued in $A := \mathbb{F}_p[[u]]$-modules (an example of *prismatic cohomology*), such that

a. There is an identification $H^i_A(X)[1/u] = H^i(X^{\mathrm{an}}, \mathbb{F}_p) \otimes \mathbb{F}_p((t))$.

b. There is an injective map

$$H^i_A(X)/u \hookrightarrow H^i_{\mathrm{dR}}(X_p).$$

Notice that a) + b) clearly implies Theorem 2.

Local structure of prismatic cohomology

Fix a prime p for the rest of the semester ($p=2$ is a very good prime for computation in prismatic cohomology!) Set $A=\mathbb{Z}_p[[u]]$. Let

$$\phi: A\to A, \quad \phi(u)=u^p,$$

a "Frobenius lift" on A from A/p. Let $I=(u-p)\subseteq A$ be the ideal defining the "diagonal" of A. Let R be (the p-adic completion of) a smooth $\mathbb{Z}_p$-algebra. Write $R_{\mathbb{F}_p}=R/p$, and $R_C=R\otimes_{\mathbb{Z}_p}C$, where C is an algebraic closure of $\mathbb{Q}_p$. Let $\Omega^*_{R_{\mathbb{F}_p}}$ be the algebraic de Rham complex of $R_{\mathbb{F}_p}$.

Example 2 Let R be the p-adic completion of $\mathbb{Z}_p[x,x^{-1}]$. Then $\Omega^1_{R_{\mathbb{F}_p}}$ is a free R-module of rank 1 generated by dx/x, and higher exterior powers vanish. So we find

$$\Omega^*_{R_{\mathbb{F}_p}}=\bigoplus_{i\in\mathbb{Z}}(\mathbb{F}_px^i\xrightarrow{\times i}\mathbb{F}_px^i dx/x).$$

Theorem 3 For any such R, one can attach an A-complex (*prismatic complex*) $\Delta_{R/A}$ (canonical only in the derived category) and a map

$$\phi_{R/A}:\phi^*\Delta_{R/A}\to\Delta_{R/A}$$

such that

a. (de Rham comparison) $\phi^*\Delta_{R/A}\otimes_A A/I\cong\Omega^*_{R/\mathbb{Z}_p}$ (the diagonal computes the de Rham cohomology of $R/\mathbb{Z}_p$). In particular, $\phi^*\Delta_{R/A}\otimes_A\mathbb{F}_p\cong\Omega^*_{R_{\mathbb{F}_p}}$ (the origin gives the de Rham cohomology of $R_{\mathbb{F}_p}$).
b. (etale comparison) $\Delta_{R/A}\otimes_A\mathbb{F}_p((u))$ (the u-axis) recovers the $\mathbb{F}_p$-etale cohomology of R_C. In between the u-axis and the diagonal we obtain the $\mathbb{Z}_p$-etale cohomology of R_C.
c. (Hodge-Tate comparison) $H^i(\Delta_{R/A}\otimes_A A/I)=\Omega^i_{R/\mathbb{Z}_p}$.

Remark 2

a. Combining a) and c) recovers the *Cartier isomorphism*
$$H^i(\Omega^*_{R_{\mathbb{F}_p}})\cong\Omega^i_{R_{\mathbb{F}_p}}.$$
b. This theory globalizes and for X proper, we have the hypercohomology of the resulting complex gives the cohomology theory $H^*_A(X)$ mentioned before. The (etale comparison) is literally true in this proper case.
c. The triple $(A,\phi,I=(p-u))$ is an example of a *prism* (a commutative ring with a Frobenius lift and a principal ideal which behaves in a particular way with respect to the Frobenius lift). This theory actually works over any prism. For example, one can take $A=\mathbb{Z}_p[[q-1]]$ with $\phi:q\mapsto q^p$, and $I=[p]_q$, where $[p]_q=\frac{q^p-1}{q-1}$ is the q-analogue of p (which recovers p when $q=1$). Let R be the p-adic completion of the Laurent polynomial ring over A/I. Then
$$\Omega^*_{R/(A/I)}=\bigoplus_{i\in\mathbb{Z}}(A/Ix^i\xrightarrow{i}A/Ix^i dx/x).$$
In this case the prismatic complex is given by p-adic completion of the q*-deformation* (q*-de Rham complex*)
$$\Delta_{R/A}=\bigoplus_{i\in\mathbb{Z}}(Ax^i\xrightarrow{[i]_q}Ax^i dx/x).$$
It will take us a while to obtain this explicit formula, as the construction of the prismatic complex is not directly related to q-deformations.

δ-rings

Last time we tried to make the point that prismatic cohomology is a deformation of de Rham cohomology (traditionally crystalline cohomology is an example of such a deformation). We would like to stress that it is a good idea to also carry along a lift of the Frobenius with the deformation. This leads to the notion of δ-rings (Joyal, Buium, Borger...)

Let A be a (commutative) ring with a map $\phi : A \to A$ such that $\phi \equiv \mathrm{Fr} \pmod p$. Then for each $f \in A$, we have $\phi(f) = f^p + p\delta$. The notion of δ-ring is trying to remember δ (rather than ϕ). Notice that if A is p-torsion-free, then $\delta = \delta(f)$ is a function of f.

Definition 1 A δ-*ring* is a pair (A, δ) where A is a ring, $\delta : A \to A$ is a map of sets satisfying

a. $\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p} = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$,
b. $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(y)\delta(x)$,
c. $\delta(0) = \delta(1) = 0$.

Lemma 1

a. If (A, δ) is a δ-ring, then $\phi(f) = f^p + p\delta(f)$ gives a ring map $\phi : A \to A$ lifting $\mathrm{Fr} : A/p \to A/p$.
b. If A is p-torsion-free, then a) gives a bijection between the δ-structures on A and endomorphisms $\phi : A \to A$ lifting Fr on A/p.

Proof For example, given (A, δ), let us check $\phi : A \to A$ is additive:
$$\phi(x+y) = (x+y)^p + p\delta(x+y) = (x+y)^p + x^p + y^p - (x+y)^p + p\delta(x) + p\delta(y)$$
which is equal to
$$x^p + p\delta(x) + y^p + p\delta(y) = \phi(x) + \phi(y)$$
as desired. □

Remark 3 We also have an asymmetric expression
$$\delta(xy) = \phi(x)\delta(y) + y^p\delta(x),$$
which is sometime useful.

Example 3

a. There is a unique δ-ring structure on $\mathbb{Z}$, given by ϕ = identity. Then $\delta(n) = \frac{n - n^p}{p}$. One can check this is the initial object in δ-rings.
b. Let $A = \mathbb{Z}[x]$. Then for any polynomial $g(x) \in \mathbb{Z}[x]$, the map
$$\phi : A \to A, \quad \phi(x) = x^p + pg(x)$$
gives a δ-structure on A.
c. Let k be a perfect field of characteristic p. Then the ring of Witt vectors $W(k)$ (the unique p-adically complete and p-torsion-free ring lifting k) with $\phi : W(k) \to W(k)$ (induced by Fr on k), is a δ-ring.
d. Let $A = \mathbb{Z}[1/p]$-algebra. The any ring map $\phi : A \to A$ gives a δ-ring on A (as we can write down δ by dividing p).
e. Let $A = \mathbb{Z}[x]/(px, x^p)$ (not p-torsion-free). Then there exists a unique δ-structure on A with $\delta(x) = 0$ (the existence would not be true without quotienting x^p). So there can be p-torsion in δ-ring. However, p cannot kill 1 as the following lemma shows.

Lemma 2 There is no nonzero δ-ring A where $p^n = 0$ for some $n \geq 0$ (think: δ is a "p-derivation", which lowers the order of p-adic vanishing by 1).

Proof Say $n = 1$, and A is a $\mathbb{Z}/p$-algebra with a δ-structure. Apply δ to $p = 0$ we obtain $\delta(x) = 0$. On the other hand
$$\delta(p) = \frac{\phi(p) - p^p}{p} = \frac{p - p^p}{p} = 1 - p^{p-1} = 1.$$
Hence $0 = 1$ and A is the zero ring. □

Definition 2 An element x in a δ-ring A has *rank 1* if $\delta(x) = 0$. This terminology is motivated by K-theory where ϕ comes from the Adams operator and when $\delta(x) = 0$, $\phi(x) = x^p$ is what the Adams operator does on line bundles.

The category of δ-rings

Reference: C. Rezk, *Etale extensions of λ-rings*.

Definition 3 (Truncated Witt Vectors) For any ring A, we define a new ring $W_2(A)$ as follows: as a set $W_2(A) = A \times A$, with addition

$$(x, y) + (z, w) = (x + z, y + w + \frac{x^p + z^p - (x+z)^p}{p}),$$

and multiplication

$$(x, y) \cdot (z, w) = (xz, x^p w + z^p y + pyw).$$

Using these formulas, it is easy to check the following lemma.

Lemma 3 Let (A, δ) be a δ-ring. Then the map

$$A \to W_2(A), \quad a \mapsto (a, \delta(a))$$

is a ring map lifting the identity after the restriction map

$$\mathrm{res} : W_2(A) \to A, \quad (x, y) \mapsto x.$$

And conversely, any section of the restriction map $\mathrm{res} : W_2(A) \to A$ gives a δ-structure on A.

Remark 4 If A is p-torsion-free, then $W_2(A)$ sits in in the following pullback diagram

$$\begin{tikzcd} W_2(A) \arrow[r] \arrow[d] & A \arrow[d, "\mathrm{can}"] \\ A \arrow[r, "\mathrm{Fr} \circ \mathrm{can}"] & A/p. \end{tikzcd}$$

Lemma 4 The category of δ-rings has all limits and colimits, and they are computed on the underlying rings (i.e., limits and colimits commutes with forgetful functor to rings).

Proof It is easy to check for limits (embed the limit into a product and check component-wise). For colimits, say $\{A_i\}_{i \in I}$ is a diagram of δ-rings. They give maps $A_i \to W_2(A_i)$ compatible in $i \in I$. We obtain a map $\mathrm{colim}\, A_i \to \mathrm{colim}\, W_2(A_i)$. There is also a canonical map $\mathrm{colim}\, W_2(A) \to W_2(\mathrm{colim}\, A_i)$ (as W_2 is a functor). One can check the composition gives a δ-structure on $\mathrm{colim}\, A_i$. □

Remark 5 The lemma implies that the forgetful functor has both left and right adjoints (adjoint functor theorem). The right adjoint is nothing but the Witt vector functor $W(-)$ (Joyal). The left adjoint is the free object functor described as follows.

Lemma 5 The free δ-ring $\mathbb{Z}\{x\}$ on a variable x is given by $\mathbb{Z}\{x\} = \mathbb{Z}[x_0, x_1, x_2, \dots]$ and $\delta(x_i) = x_{i+1}$.

Remark 6 Notice that $\mathbb{Z}\{x\}$ is a δ-ring: the map $\phi(x_i) = x_i^p + px_{i+1}$ extends uniquely to a Frobenius lift on $\mathbb{Z}\{x\}$. This construction works for the universal property by looking at what δ^i does on the generator.

Remark 7 The free δ-ring $\mathbb{Z}\{x\}$ is p-torsion-free!

Example 4 Consider $\mathbb{Z}\{z\} \to \mathbb{Z}\{x, y\}$ say sending $z \mapsto x^2 + y^3 + xy$ (or any other polynomial). Pushing-out along $\mathbb{Z}\{z\} \to \mathbb{Z}, z \mapsto 0$ we obtain a δ-ring $\mathbb{Z}\{x, y\}/(x^2 + y^3 + xy)_\delta$ in which $x^2 + y^3 + xy = 0$. However, it is hard to explicitly compute $\mathbb{Z}\{x, y\}/(x^2 + y^3 + xy)_\delta$, as free δ-rings are of infinite type.

Lemma 6 Let A be a δ-ring. Let $S \subseteq A$ be a multiplicative subset such that $\phi(S) \subseteq S$. Then $S^{-1}A$ has a unique δ-structure compatible with that of A.

Proof First we assume that A is p-torsion-free. Then there exists a unique map $\phi_{s^{-1}A} : S^{-1}A \to S^{-1}A$ compatible with ϕ on A. Since $S^{-1}A$ is also p-torsion-free, and $\phi_{S^{-1}A}$ is a lift of Frobenius on $S^{-1}A$, we obtain a unique δ-structure on $S^{-1}A$.

In general A not p-torsion-free but we can reduce to the p-torsion-free case (this is a trick we will often use). One can find a free δ-ring F (p-torsion-free) and a multiplicative set $T \subseteq F$ such that $\phi(T) \subseteq T$ and a surjection $F \to A$ which takes T to S. We then obtain a pushout diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & T^{-1}F \arrow[d] \\ A \arrow[r] & S^{-1}A \end{tikzcd}$$

Now we use a colimit of δ-rings is a δ-ring to see that $S^{-1}A$ is also a δ-ring. And one can check the δ-structure is independent of the choice of F and T. □

Remark 8 Similar results hold for etale extensions of δ-rings and quotients of δ-rings.

Perfect δ-rings

Now we discuss a property that is special to the category of δ-rings.

Definition 4 A δ-ring A is *perfect* if $\phi : A \to A$ is an isomorphism.

Perfect δ-rings are essentially perfect algebras in characteristic p:

Theorem 4 The following categories are equivalent:

a. Perfect and p-adically complete δ-rings.
b. Perfect $\mathbb{F}_p$-algebras.

The equivalence is via $A \mapsto A/p$, and $R \mapsto W(R)$.

The key lemma to prove this theorem is the following:

Lemma 7 If A is a p-adically complete δ-ring and $x \in A$ is a p-torsion element, then $\phi(x) = 0$. In particular, perfect δ-rings are p-torsion-free.

Proof Apply δ to $px = 0$, we obtain

$$0 = \delta(px) = \phi(x)\delta(p) + p^p\delta(x).$$

Since A is p-adically complete, we know that $\delta(p) = 1 - p^{p-1}$ is a unit. To prove $\phi(x) = 0$ if suffice to show that $p^p\delta(x) = 0$. But

$$p^p\delta(x) = p^{p-1}(p\delta(x)) = p^{p-1}(\phi(x) - x^p) = p^{p-2}(\phi(px) - (px)x^{p-1}) = 0.$$

Here the last equality is due to $px = 0$. □

We recall the following standard construction.

Definition 5 Let R be a perfect $\mathbb{F}_p$-algebra. There exists a unique multiplicative map, the *Teichmuller lift*, $[\cdot] : R \to W(R)$ splitting the projection $W(R) \to R$. Given $x \in R$, we define $[x] \in W(R)/p^{n+1}$ to be $\tilde{x}_n^{p^n}$, where $\tilde{x}_n \in W(R)/p^{n+1}$ is some lift of $x^{1/p^n} \in R$. One can check this is well-defined using essentially the binomial identity:

Lemma 8 If $a, b \in A$ such that $a \equiv b \pmod{p^k A}$. Then $a^p \equiv b^p \pmod{p^{k+1} A}$.

Corollary 2 Any $f \in W(R)$ can be written uniquely as a power series

$$f = \sum_{i \geq 0} [a_i] p^i, \quad a_i \in R.$$

Proof One simply write a_0 to be the image of f in R, a_1 to be the image of $(f - [a_0])/p$ and so on. □

These a_i's are called the *Teichmuller coordinates* of f (though these coordinates does not respect the ring structure).

Distinguished elements

For a commutative ring A, we write $\mathrm{Rad}(A) = \cap_{\mathfrak{m} \subseteq A} \mathfrak{m}$ to be the Jacobson radical of A. A more useful way to think about it is that it consists of "small" elements

$$\mathrm{Rad}(A) = \{x \in A : 1 + xy \in A^\times, \forall y \in A\}.$$

We will *always* assume that $p \in \mathrm{Rad}(A)$ (so all other prime numbers are invertible in A).

Definition 6 An element d in a δ-ring A is *distinguished* (or *primitive*) if $\delta(d) \in A^\times$ (this terminology dates back to Fontaine). Since ϕ is a ring map, we know that if d is distinguished, then $\phi(d)$ is also distinguished.

A distinguished element can be thought of as a "deformation" of p.

Example 5

a. (crystalline cohomology) $A = \mathbb{Z}_{(p)}$. Then $d = p$ is distinguished as $\delta(p) = 1 - p^{p-1} \in A^\times$. In fact, $d = p$ is distinguished in any δ-ring A.
b. (q-de Rham cohomology) $A = \mathbb{Z}_p[[q-1]]$ ($\phi(q) = q^p$). Then $d = [p]_q$ is distinguished. In fact, consider the map $F : A \to A/(q-1) = \mathbb{Z}_p$. This a δ-map. Moreover, $x \in A$ is a unit if and only if $F(x)$ is a unit (as $q - 1$ is topologically nilpotent). Now $F(d) = 1 + 1 + \cdots + 1 = p$, so $F(\delta(d)) = \delta(F(d)) = \delta(p)$ is a unit, hence $\delta(d)$ is a unit. The intuition here is that specializing to $q = 1$ allows one check to an element is distinguished more easily.

c. (Breuil-Kisin cohomology) $A = \mathbb{Z}_p[[u]]$ ($\phi(u) = u^p$). Then $d = u - p \in A$ is distinguished. One can check this using the specialization $A \to A/u = \mathbb{Z}_p$. (here u is even "smaller" than p as ϕ contracts u).

d. ($A_{\inf}$ -cohomology) $A = (p, q-1)$-adic completion of $\mathbb{Z}_p[q, q^{1/p}, q^{1/p^2}, \ldots,]$ ($\phi(q) = q^p$). Then $d = [p]_q$ is distinguished (from b)).

In all these examples we know how to deform the de Rham cohomology from A/d to A, but the constructions are different (at least three constructions). Our goal is to uniforms these different constructions.

Lemma 9 Let $d \in A$ be a distinguished element in a δ-ring. Assume $p, d \in \mathrm{Rad}(A)$. Then $d \cdot u$ is also distinguished for any unit u.

Proof Notice that

$$\delta(du) = d^p\delta(u) + u^p\delta(d) + p\delta(u)\delta(d),$$

the first and third terms are in $\mathrm{Rad}(A)$ (as d and p are), and the second term is a unit (as d is distinguished), we know that $\delta(du)$ is also a unit. □

Lemma 10 (Irreducibility) Assume $d = fg$ is distinguished. Assume $p, f \in \mathrm{Rad}(A)$. Then f is distinguished and g is unit. (In particular, in a δ-ring there can not be a square root of p).

Proof Notice

$$\delta(d) = f^p\delta(g) + g^p\delta(f) + p\delta(f)\delta(g).$$

Again the first and third terms are in $\mathrm{Rad}(A)$. So the second term $g^p\delta(f)$ is a unit, and hence g is a unit and f is distinguished. □

Lemma 11 Assume $p, d \in \mathrm{Rad}(A)$. Then d is distinguished if and only if $p \in (d, \phi(d))$. In particular, the condition that d is distinguished only depends on the ideal (d).

Remark 9 Geometrically, the condition $p \in (d, \phi(d))$ says that the intersection of the two closed subschemes defined by d and $\phi(d)$ is completely supported in characteristic p.

Remark 10 This lemma will be the basis of the definition of a prism, as the condition $p \in (d, \phi(d))$ depends only of ϕ (which uses less information than remembering δ).

Proof Notice $\phi(d) = d^p + p(\mathrm{unit})$. So if d is distinguished, then $p \in (d, \phi(d))$. Conversely, write $p = ad + b\phi(d)$. We need to show that $\delta(d)$ is a unit. This is equivalent to that $\delta(d)$ is a unit in $A/(p, d)$, or equivalently $A/(p, d, \delta(d)) = 0$. We will proceed by contradiction. We may assume that $\delta(d) \in \mathrm{Rad}(A)$ (after localizing along the locus where $\delta(d)$ is not a unit). Then

$$p = ad + b(d^p + p\delta(d)) = cd + bp\delta(d)$$

for some $c \in A$. So

$$cd = p(1 - b\delta(d))).$$

Notice that RHS is distinguished as $\delta(d) \in \mathrm{Rad}(A)$ and so $1 - b\delta(d)$ is a unit. Thus the irreducibility lemma (Lemma 10) implies that d is distinguished, a contradiction. □

Lemma 12 Let R be a perfect $\mathbb{F}_p$-algebra.

a. An element $d \in W(R)$ is distinguished if and only if $a_1 \in R$ is a unit, where $d = \sum_{i \ge 0}[a_i]p^i$ is the Teichmuller expansion.

b. Any distinguished element of $W(R)$ is a non zero divisor.

c. If $d \in W(R)$ is distinguished, then $W(R)/(d)$ has bounded p-power-torsion. In fact, $W(R)/(d)[p^\infty] = W(R)/(d)[p]$.

Digression: derived completions

A prism will be a δ-ring together with an ideal that is locally cut out by distinguished elements (but not necessarily globally). It is also more convenient to assume that it is "complete" along the ideal. We thus need a good notion of completion (for non-noetherian rings).

Definition 7 Let $f_1, \ldots, f_r \in A$ and $I = (f_1, \ldots, f_r)$. An A-complex $M \in D(A)$ is *derived* I*-complete* if for any $f \in I$,

$$T(M; f) := \mathrm{Rlim}(\ldots \to M \xrightarrow{f} M \xrightarrow{f} M)$$

is 0 in $D(A)$ (here Rlim is the right derived functor of the inverse limit functor). This is equivalent to

$$M \cong \hat{M} := \mathrm{Rlim}_n M \otimes^L_{\mathbb{Z}[x_1,\ldots,x_r]} \otimes \mathbb{Z}[x_1,\ldots,x_r]/(x_1^n,\ldots,x_r^n),$$

where x_i acts on M by f_i. Notice that the completion here is less naive (using the noetherian ring $\mathbb{Z}[x_1,\ldots,x_r]$ instead of A). See notes for the rest of the assertions.

Proposition 1

a. All derived I-complete A-complexes form a triangulated subcategory closed under product. It has a left adjoint $M \mapsto \hat{M}$.
b. $M \in D(A)$ is derived I-complete if and only if each $H^i(M)$ (put in degree 0) is derived I-complete.
c. All derived I-complete A-modules form an abelian subcategory of all A-modules.
d. (Derived Nakayama) Assume $M \in D(A)$ is derived I-complete. Then $M = 0$ if and only if $M \otimes^L_A A/I = 0$.

Prisms

Recall (Lemma 11):

Corollary 3 Let $I \subseteq A$ be a locally principal ideal. Assume $I \subseteq \mathrm{Rad}(A)$. Then the following are equivalent:

a. $p \in (I, \phi(I))$.
b. I is (Zariski) locally generated by distinguished elements.
c. $p \in (I^p, \phi(I))$ (use $\phi(d) = d^p + p\delta(d)$).

Definition 8 A δ-*pair* (A, I) consists of a δ-ring A and an ideal $I \subseteq A$.

Definition 9 A *prism* is a δ-pair (A, I) satisfying

a. $I \subseteq A$ defines a Cartier divisor on $\mathrm{Spec}\, A$.
b. A is (derived) (p, I)-complete.
c. $p \in (I, \phi(I))$.

Prisms will be the objects of the prism site.

Definition 10 A map $(A, I) \to (B, J)$ of prisms (a ring map $A \to B$ sending I into J) is *(faithfully) flat* if $A/(p, I) \to B \otimes^L_A A/(p, I)$ is (faithfully) flat. Here flat means that the target complex has cohomology only in degree 0 and this cohomology is flat over the source.

Definition 11 A prism (A, I) is called *perfect* if A is perfect; *crystalline* if $I = (p)$; *bounded* if A/I has bounded p^∞-torsion. Every prism we will encounter will be bounded.

Example 6

a. Any p-torsionfree and p-adically complete δ-ring A gives a prism $(A, (p))$.
b. Perfect prisms = perfectoid rings.

We don't know a natural example of a prism of where I is locally principal but not principal (though abstract examples exist). The following lemma shows that I is not far from principal.

Lemma 13 Let (A, I) be a prism. Then $\phi(I)A$ is principal and any generator is a distinguished element. In particular, $I \in \mathrm{Pic}(A)[p]$ (exercise).

Proof Write $p = a + b$, where $a \in I^p, b \in \phi(I)$. Then one can check that b is a generator of $\phi(I)$. The key is to use the irreducibility lemma (Lemma 10) for distinguished elements (see notes). □

Lemma 14 Let $(A, I) \to (B, J)$ be a map of prisms. Then $I \otimes_A B \to J$ is an isomorphism (so a map of prisms is determined on the underlying δ-rings).

Proof Both sides are locally generated by distinguished elements. Use the irreducibility of distinguished elements locally on $\mathrm{Spec}\, B$. □

Perfect prisms

Definition 12 A commutative ring R is *perfectoid* if $R \cong A/I$ for a perfect prism (A, I).

Example 7

a. Let A be a perfect and p -adically complete δ -ring, and $I = (p)$. Then $A \cong W(R)$ for a perfect $\mathbb{F}_p$ -algebra R and $A/I = R$ (Theorem 4). So any perfect $\mathbb{F}_p$ -algebra is perfectoid.

b. $A = \mathbb{Z}_p[X^{1/p^\infty}]^\wedge_{(p,X)}$ with $\phi(X^{1/p^n}) = X^{1/p^{n-1}}$ is a p -torsionfree δ -ring, and $I = (X - p)$ is generated by a distinguished element. So $R = A/I = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge_{(p)}$ is a perfectoid ring.

Lemma 15 Let R be a perfect $\mathbb{F}_p$ -algebra. Let $f \in R$. Then $R[f^\infty]$ (f^∞ -torsion in R) is equal to $R[f^{1/p^n}]$ for any n. (The latter is killed by a small power of f, known as an *almost zero* module).

Proof Suppose $x \in R$ such that $f^m x = 0$ for some $m \geq 0$. So $f^m \cdot x^{p^n} = 0$ for any $n \geq 0$. By reducedness we obtain $f^{m/p^n} x = 0$ for any $n \geq 0$. □

Corollary 4 Let (A, I) be a perfect prism. Then $A/I[p^\infty] = A/I[p]$. In particular, perfect prisms are bounded.

Proof See notes. □

Theorem 5 The functor $(A, I) \mapsto R = A/I$ gives an equivalence between the category of perfect prisms and the category of perfectoid rings.

Proof The key is to recover (A, I) form $R = A/I$. We claim that $A = W(R^\flat)$, where $R^\flat = \varprojlim_\phi R/p$ (a perfect $\mathbb{F}_p$ -algebra) is Fontaine's tilting functor.

Choose $d \in I$ a generator. Then $R = A/(d)$ and $R/p = A/(p, d)$. As A/p is a perfect $\mathbb{F}_p$ -algebra, we have an identification of the tower

$$\cdots R/p \xrightarrow{\phi} R/p \xrightarrow{\phi} R/p$$

with the tower

$$\cdots A/(p, d^{p^2}) \xrightarrow{\text{can}} A/(p, d^p) \xrightarrow{\text{can}} A/(p, d).$$

Taking inverse limits we obtain

$$(A/p)^\wedge_{(d)} \cong R^\flat.$$

But $(A/p)^\wedge_{(d)}$ is isomorphic to A/p because A/p is d -complete (derived completion and usual completion are the same since d -torsion is bounded). Therefore $A \cong W(R^\flat)$. See the notes for the identification of I as the kernel of $W(R^\flat) \to R$. □

Definition 13 Let R be a perfectoid ring.

a. The *tilt* $R^\flat$ of R is $\varprojlim_\phi R/p$.

b. Write $A_{\text{inf}}(R) := W(R^\flat)$, and write $\theta_R : A_{\text{inf}}(R) \to R$.

c. The *special fiber* $\bar{R} := \varinjlim_\phi R/p$.

Example 8 $R = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge_{(p)}$. Then $A_{\text{inf}}(R) = \mathbb{Z}_p[X^{1/p^\infty}]^\wedge_{(p,X)}$ (a "2-dimensional" object), and $\bar{R} = \mathbb{F}_p$ (all nilpotents in R/p are killed when passing to the perfection).

Lemma 16 Let R be a perfectoid ring. Then

a. The Frobenius $\phi : R/p \to R/p$ is surjective.

b. There exists $\pi \in R$ such that $\pi^p = p \cdot \text{unit}$ and $\ker(\phi : R/p \to R/p) = (\pi)$.

c. The radical $\sqrt{pR}$ is a flat ideal and $(\sqrt{pR})^2 = \sqrt{pR}$ (the beginning of *almost math*).

d. $R[p^\infty] = R[p] = R[\sqrt{pR}]$.

Proof Write $R = A/(d)$ and $A = W(R^\flat)$. (a) is clear since Frobenius on A/p is surjective and R/p is a quotient of A/p. For (b), write $d = [a_0] + [a_1]p + \cdots$, where $a_1 \in (A/p)^\times$. So we may write $d = [a_0] - pu$, where $u \in A^\times$. Thus in $R = A/(d)$, we have $[a_0] = pu$. We may then take $\pi = [a_0^{1/p}]$ (the Teichmuller map is multiplicative). For (c), we claim that $([a_0^{1/p^n}]_{n \geq 1}) = \sqrt{pR}$. It suffices to show that $R/([a_0^{1/p^n}]_{n\geq 1})$ is reduced (equivalently, perfect). This follows from the general fact that if S is a perfect $\mathbb{F}_p$ -algebra, and $I = (f_1, \ldots, f_r) \subseteq S$ an ideal. Then $\sqrt{I} = (f_1^{1/p^\infty}, \ldots, f_r^{1/p^\infty})$. □

Proposition 2

a. If $R \to S$, $R \to T$ are maps of perfectoid rings. Then $S \otimes_R^{\mathrm{L},\wedge} T$ is perfectoid (no Tor even without flatness assumptions in the perfectoid world!)

b. If R is perfectoid, then $S = R/R[\sqrt{pR}]$ is perfectoid (and p-torsionfree), and

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ \bar{R} \arrow[r] & \bar{S} \end{tikzcd}$$

is a pullback diagram. So one can build any perfectoid ring using a p-torsionfree perfectoid ring S and a perfect $\mathbb{F}_p$-algebra $\bar{R}$. In particular, R is *reduced*.

The prismatic site and the prismatic cohomology

We will let (A, I) be a "base" prism. Assume $I = (d)$ is generated by a distinguished element and $A/I[p^\infty]$ is bounded.

Example 9

a. (crystalline) Let A be any p-torsionfree, p-complete δ-ring and $I = (p)$.

b. (Breuil-Kisin) Let $A = \mathbb{Z}_p[[u]]$ with $\phi(u) = u^p$ and $I = E(u)$, where $E(u)$ is any Eisenstein polynomial (e.g. $E(u) = u^p - p$).

c. ($A_{\inf}$) Let R be a perfectoid ring and $(A, I) = (W_{\inf}(R), \ker \theta_R)$.

d. (q-de Rham) Let $A = \mathbb{Z}_p[[q-1]]$ with $\phi(q) = q^p$, and $I = [p]_q$.

Let R be a formally smooth A/I-algebra (e.g., the p-adic completion of a smooth A/I-algebra). Our goal is to construct an object $\mathbb{\Delta}_{R/A} \in D(A)$ such that

a. $\mathbb{\Delta}_{R/A}/I$ gives differential forms on R relative to A/I.

b. $\mathbb{\Delta}_{R/A}[1/p]$ is related to etale cohomology of $R[1/p]$.

Definition 14 The *prismatic site* $(R/A)_{\mathbb{\Delta}}$ of R relative to A consists of prisms (B, IB) over A together with a map $R \to B/IB$ over A/I. (Notice the direction of the map is different from the crystalline site). Pictorially we have

$$\begin{tikzcd} A \arrow[rr] \arrow[d] & & B \arrow[d] \\ A/I \arrow[r] & R \arrow[r] & B/IB. \end{tikzcd}$$

We will write it as $(R \to R/IB \leftarrow B) \in (R/A)_{\mathbb{\Delta}}$.

Definition 15 Define functors $\mathcal{O}_{\mathbb{\Delta}}$ and $\bar{\mathcal{O}}_{\mathbb{\Delta}}$ on $(R/A)_{\mathbb{\Delta}}$ given by sending $(R \to B/IB \leftarrow B)$ to B (an δ-algebra over A) and B/IB (an R-algebra) respectively. (In the perfectoid case $\mathcal{O}_{\mathbb{\Delta}} = \mathbb{A}_{\inf}$ and $\bar{\mathcal{O}}_{\mathbb{\Delta}} = \mathcal{O}^+$).

Remark 11

a. One should really define $(R/A)_{\mathbb{\Delta}}$ as the opposite of the above definition, so $\mathcal{O}_{\mathbb{\Delta}}$ and $\bar{\mathcal{O}}_{\mathbb{\Delta}}$ are presheaves (we will focus on the affine case as the gluing will be easy).

b. Even though the definition of $(R/A)_{\mathbb{\Delta}}$ makes sense for any R, the theory works best for R formally smooth (or locally complete intersection). The same comment applies to the crystalline site.

Example 10

a. Let $R = A/I$. Then $(R/A)_{\mathbb{\Delta}}$ is the category of prisms over (A, I). In particular, it has a initial object and hence the prismatic cohomology will be simply A.

b. Let $R = A/I\langle x \rangle$ (the p-adic completion of $A/I[x]$). Then there is no initial object. There exists $\tilde{R}/A$ formally smooth lift of $R/(A/I)$ together with a δ-structure on $\tilde{R}$ (e.g., $\tilde{R} = A[x]^\wedge$ and $\delta(x) = 0$. Then $(A/I \to R \cong \tilde{R}/I\tilde{R}) \in (R/A)_{\mathbb{\Delta}}$. (One can also do the same for any R).

Definition 16 The *prismatic cohomology* of R is defined to be

$$\mathbb{\Delta}_{R/A} = R\Gamma((R/A)_{\mathbb{\Delta}}, \mathcal{O}_{\mathbb{\Delta}}) \in D(A).$$

It admits a Frobenius action $\phi_{R/A}$ (as the category $(R/A)_{\mathbb{\Delta}}$ admits a Frobenius action).

Definition 17 The *Hodge-Tate cohomology* of R is defined to be

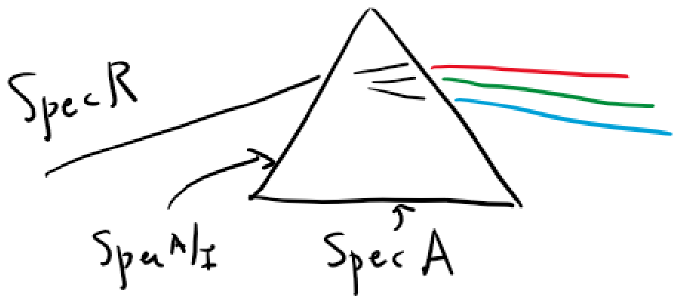
$$\bar{\Delta}_{R/A} = R\Gamma((R/A)_{\Delta}, \bar{\mathcal{O}}_{\Delta}) \in D(R).$$

It no longer has the Frobenius action (but is linear over the larger ring R).

These both are commutative algebra objects.

Example 11 Let $R = A/I$. Then $\Delta_{R/A} \cong A$.

Remark 12 One visualizes the prismatic site as a "prism", where $\operatorname{Spec} R$ goes through $\operatorname{Spec} A/I$ and outputs various different cohomology theory:



Remark 13 Since $\bar{\mathcal{O}}_{\Delta} = \mathcal{O}_{\Delta}/I\mathcal{O}_{\Delta}$, we know that we have (a quasi-isomorphism) $\bar{\Delta}_{R/A} = \Delta_{R/A} \otimes^{\mathbb{L}}_{A} A/I$. Since everything is complete, we can detect many things (e.g. vanishing) of $\Delta_{R/A}$ using $\bar{\Delta}_{R/A}$.

The Hodge-Tate comparison

Let $B \to C$ be a map of commutative rings. We have an *algebraic de Rham complex*

$$\Omega^*_{C/B} = (C \to \Omega^1_{C/B} \to \Omega^2_{C/B} \to \cdots).$$

We view it as a strictly commutative differential graded B-algebra:

a. (graded commutative) $a.b = (-1)^{\deg a \deg b} b.a$.
b. (strictly graded commutative) $a^2 = 0$ if $\deg a$ is odd (which only matters in characteristic 2).

Lemma 17 (Universal property of the de Rham complex) Let (E^*, d) be a graded commutative B-dga. Assume we have a map

$$\eta : C \to E^0$$

of B-algebras. Further assume that for any $x \in C$, $d(\eta(x))$ squares to 0. Then there exists a unique extension of η to graded commutative B-dgas,

$$\eta^* : (\Omega^*_{C/B}, d) \to (E^*, d).$$

We have a Bockstein exact sequence of sheaves on $(R/A)_{\Delta}$,

$$0 \to \mathcal{O}_{\Delta}/(d) \to \mathcal{O}_{\Delta}/(d^2) \to \mathcal{O}_{\Delta}/(d) \to 0.$$

Thus we obtain a *Bockstein differential*

$$\beta_d : H^*(\bar{\Delta}_{R/A}) \to H^{*+1}(\bar{\Delta}_{R/A}).$$

In this way we obtain a graded commutative A/I-dga $(H^*(\bar{\Delta}_{R/A}), \beta_d)$. We also have $\eta : R \to H^0(\bar{\Delta}_{R/A})$ (by R-algebra structure). One can also show that for any $x \in R$, $\beta_d(\eta(x))$ squares to 0. Thus by the universal property of the de Rham complex, we obtain a map from the de Rham complex to the Hodge-Tate complex:

$$\eta^*_R : (\Omega^*_R/(A/I), d_{\mathrm{dR}}) \to (H^*(\bar{\Delta}_{R/A}), \beta_d).$$

Remark 14 The differential β_d depends on the choice of b. There is a way to avoid this dependence (which introduces certain Breuil-Kisin twists).

Theorem 6 (Hodge-Tate comparison) η^*_R is an isomorphism (as genuine complexes). In particular,

$$H^i(\bar{\Delta}_{R/A}) \cong \Omega^i_{R/(A/I)}.$$

And so $\bar{\Delta}_{R/A}$ can be represented by a perfect complex (this is the source of finiteness in the global situation).

Remark 15

a. The Hodge-Tate comparison gives the Cartier isomorphism when $I=(p)$ (without the extra Frobenius twist).

b. One can work out the Hodge-Tate comparison explicitly for the q-de Rham complex (see notes).

c. One can deduce from the Hodge-Tate comparison theorem the Hodge-Tate decomposition for the etale cohomology of smooth projective variety X over a p-adic field K:

$$H^n(X,\mathbb{Q}_p)\otimes\mathbb{C}_p\cong\bigoplus_{i+j=n}H^i(X,\Omega^j)\otimes_K\mathbb{C}_p(-j).$$

How to compute cohomology of categories? It turns out to be extremely simple.

Definition 18 Let $\mathcal{C}$ be a small category. Let $\mathrm{PShv}(\mathcal{C})$ be the category of presheaves on $\mathcal{C}$. Then $R\Gamma(\mathcal{C},-)$ is the derived functor $D(\mathbf{Ab}(\mathcal{C}))\to D(\mathbf{Ab})$ of

$$F\mapsto H^0(\mathcal{C},F)=\varprojlim_{X\in\mathcal{C}}F(X).$$

Lemma 18 Assume there exists $X\in\mathcal{C}$ that is *weakly final* (i.e., $\mathrm{Hom}(Y,X)\neq\varnothing$ for any $Y\in\mathcal{C}$). Assume that $\mathcal{C}$ has finite nonempty product. Then $R\Gamma(\mathcal{C},F)$ is calculated by

$$F(X)\rightrightarrows F(X\times X)\,\substack{\rightarrow\\\rightarrow\\\rightarrow}\,F(X\times X\times X)\cdots$$

So in order to compute the cohomology of the prismatic complex, it suffices to find a weakly final object and compute its self-products.

Lemma 19 Let (B,J) be a δ-pair over (A,I). Then there exists a universal map $(B,J)\to(C,IC)$ to a prism over (A,I).

Proof Pure category theory. □

We denote this universal (C,IC) by $B\{\frac{J}{I}\}^\wedge$ (think: the universal prism where J becomes divisible by I).

Corollary 5 $(R/A)_{\Delta}$ has finite nonempty coproducts.

Proof Let $(R\to B/IB\leftarrow B)$ and $(R\to C/IC\leftarrow C)$ be two objects in $(R/A)_{\Delta}$. Set $D_0=B\otimes_A C$. We have two maps $R\to D_0/ID_0$ via $R\to B/IB\to D_0/ID_0$ (and similarly for C), which are not necessarily the same. ; Let

$$J=\ker(D_0\to B/IB\otimes_{A/I}C/IC\twoheadrightarrow B/IB\otimes_R C/IC),$$

and let $D=D_0\{\frac{J}{I}\}^\wedge$, then the two maps becomes the same in D. Then (D,ID) works. □

The goal today is to sketch a proof of the Hodge-Tate comparison theorem, which is divided into two steps.

a. If $(d)=(p)$ (characteristic p case), prove the crystalline comparison for $\Delta_{R/A}$ (before reduction).

b. Use the Cartier isomorphism when $(d)=(p)$ to deduce the Hodge-Tate comparison theorem in general.

Reminder on crystalline cohomology

Fix a p-torsionfree ring A, and a smooth A/p-algebra R.

Definition 19 (Divided power envelope) Let P be a (ind)-smooth A-algebra. Fix a surjection $P\to R$ over A. (think: embed $\mathrm{Spec}\,R$ into a smooth affine space.) Let $J=\ker(P\to R)$. The *divided power envelope of* (P,J) is defined to be

$$D_J(P)=P[\{\frac{x^n}{n!}\}_{n\geq 1,x\in J}]^\wedge$$

(p-adic completion), where $P[\{\frac{x^n}{n!}\}_{n\geq1,x\in J}]\subseteq P[1/p]$. (This is the characteristic p analogue of allowing integration of power series in J.)

One can check that there exists an induced surjection $D_J(P)\to R$ with kernel having divided powers. The map $P\to D_JP$ is universal with this property (hence the name envelope).

Example 12 Let $R=A/(p)$, $P=A[x]$, and $P\to R,\ x\mapsto 0$. Then

$$D_J(P)=A\langle x\rangle^\wedge=(\oplus_{n\geq1}Ax^i/i!)^\wedge$$

is the divided power polynomial ring. One can check that $d:D_J(P)\to D_J(P)dx$ has the cohomology of a point ($H^0=A$, $H^1=0$). This is the characteristic p analogue of the Poincare lemma.

Definition 20 Choose a surjection $P \to R$ with P (ind-)smooth over A. Let
$$P^\bullet = (P \rightrightarrows P \otimes_A P \overset{\to}{\rightrightarrows} \cdots)$$
a cosimplicial A-algebra. We have an induced surjection
$$P^n := P^{\otimes_A(n+1)} \xrightarrow{m} P \to R$$
with kernel $J^n \subseteq P^n$ for any n. In this way we obtain an ideal $J^\bullet \subseteq P^\bullet$. We define
$$C^\bullet_{\text{crys}}(R/A) = (D_{J^0}(P^0) \rightrightarrows D_{J^1}(P^1) \cdots).$$
(think: geometrically embed a singular space into powers of a smooth affine space and look at its tubular neighborhoods). Define
$$R\Gamma_{\text{crys}}(R/A) = \mathrm{Tot}(C^\bullet_{\text{crys}}(R/A)) \in D(A).$$

Theorem 7 (de Rham-crystalline comparison) If P is a smooth lift of R to A. Then
$$R\Gamma_{\text{crys}}(R/A) \cong (\Omega^\bullet_{P/A})^\wedge.$$
In particular, the de Rham complex on LHS is independent of the lift P.

Remark 16 In the non-affine situation, the lift may not exists but RHS still makes sense.

Corollary 6 There exists a canonical quasi-isomorphism
$$R\Gamma_{\text{crys}}(R/A) \otimes^{\mathbb{L}}_A A/p \cong \Omega^\bullet_{R/(A/p)}.$$

Remark 17 The advantage of this comparison is that the definition of LHS does not involve differential forms directly.

Definition 21 (Cartier isomorphism) Let $R^{(1)} = R \otimes_{A/p,\phi} A/p$ be the Frobenius twist of R. So the absolute Frobenius $R \to R$ factors as $R \to R^{(1)} \to R$, where the relative Frobenius $R^{(1)} \to R$ is A/p-linear. Notice that $\Omega^\bullet_{R/(A/p)}$ is an $R^{(1)}$-linear complex (by the Leibniz rule we have $d(\phi(x)) = 0$ for $x \in R^{(1)}$). We have a map
$$\eta_R : R^{(1)} \to H^0(\Omega^\bullet_{R/(A/P)}),$$
and also have a Bockstein differential
$$\beta_p : H^*(\Omega^\bullet_{R/(A/P)}) \to H^{*+1}(\Omega^\bullet_{R/(A/P)}).$$
So by the universal property (Lemma 17), we obtain a map of strictly graded commutative dgas,
$$(\Omega^\bullet_{R^{(1)}/(A/p)}, d_{\text{dR}}) \to (H^*(\Omega^\bullet_{R/(A/p)}), \beta_p).$$

Theorem 8 (Cartier) This map is an isomorphism.

Remark 18 See N. Katz, *Nilpotent connections and the monodromy theorem* for a good exposition of the Cartier isomorphism.

Relating divided powers to δ-structures

Lemma 20 Let A be a p-torsionfree δ-rings. Let P be an (ind-)smooth δ-algebra over A. Let $x \in P$ such that x is a non-zero-divisor on P/p. Then
$$P\left\{\frac{\phi(x)}{p}\right\}^\wedge \cong D_{(x)}(P).$$
Here LHS means formally adjoining in δ-rings, and the RHS is the divided power envelope.

Proof First consider the free case $P = \mathbb{Z}_p\{x\}$ over $A = \mathbb{Z}_p$. We have a pushout diagram
$$\begin{tikzcd} \mathbb{Z}_p\{z\} \arrow[r] \arrow[d, "z \mapsto \phi(x)"] & \mathbb{Z}_p\{z,y\}/(py - z)_\delta = \mathbb{Z}_p\{y\} \arrow[d] \\ \mathbb{Z}_p\{x\} \arrow[r] & P\{\phi(x)/p\}. \end{tikzcd}$$
Notice that the left vertical arrow is faithfully flat, and hence the right vertical arrow is faithfully flat, thus $P\{\phi(x)/p\}$ is p-torsionfree.

The next goal is to identify $C = P\{\phi(x)/p\} \subseteq P[1/p]$ with $D = P[\{x^n/n!\}] \subseteq P[1/p]$. First, by
$$\phi(x) = x^p + p\delta(x),$$
we obtain that
$$C = P\{x^p/p\} \subseteq P[1/p],$$
hence $x^p/p! \in C$. To check that $\gamma_n(x) = x^n/n! \in C$, by induction it suffices to show that if C is any p-torsionfree δ-ring, and $z \in C$ such that $\gamma_p(z) \in C$, then $\gamma_{p^2}(z) \in C$. In fact,

$$\delta(z^p/p) = \frac{\phi(z^p/p) - (z^p/p)^p}{p} = \frac{(z^p + p\delta(z))^p}{p^2} - \frac{z^{p^2}}{p^{p+1}}.$$

The second term has the same p-adic valuation as $\gamma_{p^2}(z)$, so it remains to show that the first term is in C. Indeed, by assumption that $\gamma_p(z) \in C$ we have

$$z^p + p\delta(z) \in pC,$$

and thus

$$(z^p + p\delta(z))^p \in p^p C \subseteq p^2 C.$$

Finally, one can show that $D \subseteq C$ (key fact: $\gamma_p(\gamma_n(x)) = \gamma_{np}(x) \cdot u$, where u is a p-adic unit). □

Corollary 7 Let $(A, (d))$ be a bounded prism. Let P be a free δ-algebra over A. Let $x_1, \dots, x_r \in P$ that form a regular sequence mod (p, d). Let $J = (d, x_1, \dots, x_r)$.

a. The derived (p, d)-completion E of $P\{x_1/d, \dots, x_r/d\}$ is (p, d)-completely flat over A.
b. $E \cong P\{J/d\}^\wedge$ (the prismatic envelope; so in this case one can avoid transfinite construction in Lemma 19).

Proof When $(d) = (p)$, reduce to the previous lemma. In general, do a base change argument. □

Crystalline comparison for prismatic cohomology

Let A be a p-torsionfree p complete δ-ring. Then $(A, (p))$ is a bounded prism. Let R be a smooth A/p-algebra.

Theorem 9 There exists a canonical isomorphism

$$\phi_A^* \Delta_{R/A} \cong R\Gamma_{\mathrm{crys}}(R/A)$$

of commutative algebra objects in $D(A)$, compatible with the Frobenius action on both sides.

Let us explain why there is a canonical morphism from LHS to RHS.

Choose a free δ-ring P/A together with a surjection $P \to A$. Recall that by definition $R\Gamma_{\mathrm{crys}}(R/A) = D_{J^\bullet}(P^\bullet)$. Since P is now chosen to have a δ-structure, we know that $D_{J^\bullet}(P^\bullet)$ is a cosimplicial δ-algebra over A. On the other hand, the kernel $K^\bullet$ of $D_{J^\bullet}(R^\bullet) \to R$ has divided powers. Consequently, we have

$$\phi(K^\bullet) \subseteq pD_{J^\bullet}(P^\bullet).$$

This the *key* relation between the δ-structure and divided power structure and gives a commutative diagram

$$\begin{tikzcd} D_{J^\bullet}(P^\bullet) \arrow[r, "\phi"] \arrow[d] & D_{J^\bullet}(P^\bullet) \arrow[d] \\ R = D_{J^\bullet}(P^\bullet)/K^\bullet \arrow[r] & D_{J^\bullet}(P^\bullet)/p \end{tikzcd}$$

Notice that the lower right part of the diagram gives a cosimplicial object in the prismatic site $(R/A)_\Delta$. Thus we obtain a canonical map

$$\Delta_{R/A} \to \mathrm{Tot}(\phi_* D_{J^\bullet}(P^\bullet)) = \phi_* R\Gamma_{\mathrm{crys}}(R/A)).$$

So by adjunction we obtain a map

$$\phi^* \Delta_{R/A} \to R\Gamma_{\mathrm{crys}}(R/A).$$

One can check this is an equivalence by explicitly computing both sides using Cech-Alexander complexes and using Lemma 20 to identify δ-structures and divided powers structures.

Hodge-Tate comparison

Let $(A, (d))$ be a bounded prism. Let R be a formally smooth A/d-algebra. Now let us sketch a proof of the Hodge-Tate comparison (Theorem 6).

a. Deduce the Hodge-Tate comparison from the crystalline comparison (Theorem 9) and the Cartier isomorphism (Theorem 8) when $(d) = (p)$ (and worry about the Frobenius twist when A/p is not perfect).

b. In general, set $D = A\{\frac{d}{p}\}^\wedge$. Then the irreducibility lemma for distinguished elements (Lemma 10) implies that $(d) = (p)$ in D. We look at the base change along $A \to D$. Using Lemma 20, we may deduce the theorem from the previous case $(d) = (p)$.

Corollary 8 For any formally smooth A/d-scheme X, one can define its *prismatic cohomology* $\Delta_{X/A} \in D(X, A)$, compatible with passing to affine open subsets $U = \mathrm{Spf}(R) \subseteq X$ (i.e., $R\Gamma(U, \Delta_{X/A}) = \Delta_{R/A}$).

Remark 19 To check $\Delta_{X/A}$ is a sheaf, it suffices to check it is a sheaf modulo d, which is true by identifying with differential forms via the Hodge-Tate comparison.

Extension to the singular case

Non-abelian derived functors

Our next goal is to explain how to extend the prismatic cohomology to the singular case. We will need to derive the functor of Kahler differentials, which is a functor from rings to modules. Since the category of rings is not abelian, we need some formalisms on non-abelian derived functors.

Let A be a commutative ring. Let $\mathbf{Poly}_A$ be the category of finitely generated polynomial A-algebras. Consider a functor

$$F : \mathbf{Poly}_A \to \mathbf{Ab} \subseteq D(\mathbf{Ab})$$

e.g., $F(B) = \Omega^1_{B/A}$. To derive F, we view $\mathbf{Poly}_A$ as "projective" objects in $\mathbf{CAlg}_A$.

Proposition 3 (non-abelian derived functors) There exists a unique extension

$$LF : \mathbf{CAlg}_A \to D(\mathbf{Ab})$$

such that

a. LF commutes with filtered colimits.

b. LF commutes with geometric realizations, i.e., if $P_\bullet \to B$ is a simplicial resolution of B in $\mathbf{CAlg}_A$, then $|LF(P_\bullet)| \cong LF(B)$.

Remark 20

a. Here $D(\mathbf{Ab})$ is treated as an ∞-category.

b. $|LF(P_\bullet)|$ is the homotopy colimit of $LF(P_\bullet)$, i.e., the direct sum totalization of the associated bicomplex.

c. There exists a canonical choice of a simplicial resolution $P_\bullet \to B$ such that each P_i is a (possibly not finitely generated) polynomial A-algebra:

$$P_0 = A[B], P_1 = A[A[B]], \dots,$$

(this is good for functoriality issues, but not so useful for explicit computation).

Cotangent complexes

Definition 22 (Cotangent complexes) Define the cotangent complex $L_{\cdot/A} : \mathbf{CAlg}_A \to D(A)$ to be $L\Omega^1_{\cdot/A}$.

Remark 21

a. $H^0(L_{B/A}) = \Omega^1_{B/A}$.

b. If $A \to B$ is smooth, then $L_{B/A} = \Omega^1_{B/A}[0]$.

c. (Transitivity triangle). For a composition $A \to B \to C$, we have an extra triangle

$$L_{B/A} \otimes^{\mathbb{L}}_B C \to L_{C/A} \to L_{C/B}.$$

d. If $A \to B$ is surjective with kernel I, then $H^0(L_{B/A}) = 0$ and $H^{-1}(L_{B/A}) \cong I/I^2$. Moreover, the other H^i vanishes if I is generated by a regular sequence. This will be sufficient for most explicit computation we will encounter.

e. If $A \to B$ is a local complete intersection, then $L_{B/A}$ is concentrated in two degrees given by $H^0 = \Omega^1_{B/A}$ and $H^{-1} = I/I^2$. If $A \to B$ is not a local complete intersection, then $L_{B/A}$ is unbounded (a conjecture of Quillen, proved by Avramov).

Derived de Rham cohomology

Fix a ground ring k of characteristic p.

Definition 23 Define the derived de Rham cohomology $\mathrm{dR}_{\cdot/k} : \mathbf{CAlg}_k \to D(k)$ to be $L\Omega^*_{\cdot/k}$.

In char 0, for an affine space the functor Ω^* is a constant functor so there is no higher derived de Rham cohomology. But in char p, even for an affine space Ω^* has a lot of cohomologies (given by the Cartier isomorphism). The following property helps us to control the derived de Rham cohomology.

Proposition 4 (Derived Cartier isomorphism) For any $A \in \mathbf{CAlg}_k$, there exists an increasing exhaustive filtration (*conjugate filtration*) $\mathrm{Fil}_k^{\mathrm{conj}}$ on $\mathrm{dR}_{A/k}$ equipped with canonical isomorphisms

$$\mathrm{gr}_i^{\mathrm{conj}}(\mathrm{dR}_{A/k}) \cong \wedge^i L_{A^{(i)}/k}[-i],$$

where $A^{(i)} = A \otimes^{\mathbb{L}}_{k,\phi} k$. More precisely, we have a lift of $\mathrm{dR}_{A/k}$ into the filtered derived category $\mathbf{CAlg}_k \to \mathrm{DF}(k) = \mathrm{Fun}(\mathbb{N}, D(k))$.

Corollary 9 If A/k is smooth, then $\mathrm{dR}_{A/k} \cong \Omega^*_{A/k}$.

Proof For A/k smooth, we have $\wedge^i L_{A^{(i)}/k} \cong \Omega^i_{A^{(i)}/k}$. □

The moral here is that if we have a functor whose cohomology can be described in terms of differential forms, then its derived functor should also has a description in terms of differential forms. We will realize this idea for prismatic cohomology as well.

Derived prismatic cohomology

Let (A, I) be a bounded prism. Let R be a formally smooth A/I-algebra. We constructed the prismatic cohomology $\Delta_{R/A} \in D(A)$ together with an action of the Frobenius ϕ. Moreover, we have the Hodge-Tate comparison

$$H^i(\bar{\Delta}_{R/A}) \cong \Omega^i_{R/(A/I)}.$$

Definition 24 The *derived prismatic cohomology*

$$L\Delta_{\cdot/A} : \mathbf{CAlg}_{A/I} \to D_{\mathrm{comp}}(A)$$

is obtained by deriving

$$\mathbf{Poly}_{A/I} \to D(A), \quad R \mapsto \Delta_{\hat{R}/A},$$

where $\hat{R}$ is the p-adic completion of R, and $D_{\mathrm{comp}}(A)$ is the derived category of (p, I)-complete A-complexes. Define

$$L\bar{\Delta}_{R/A} = L\Delta_{R/A} \otimes^{\mathbb{L}}_A A/I$$

which is the same as the derived functor of $R \mapsto \bar{\Delta}_{\hat{R}/A}$ (as the non-abelian derived functor commutes with filtered colimits, in particular with $\otimes^{\mathbb{L}}_A A/I$).

One can check that $L\Delta_{R/A} \cong \Delta_{R/A}$ if R is the p-adic completion of a polynomial A/I-algebra. We have the following derived version of the Hodge-Tate comparison.

Proposition 5 For any $R \in \mathbf{CAlg}_{A/I}$, we have an increasing exhaustive filtration $\mathrm{Fil}^{\mathrm{HT}}_\bullet$ on $L\bar{\Delta}_{R/A}$ such that

$$\mathrm{gr}_i^{\mathrm{HT}}(L\bar{\Delta}_{R/A}) \cong \wedge^i L_{R/(A/I)}[-i]$$

in $D_{\mathrm{comp}}(A)$.

From now on by abuse of notation we will write $\Delta_{R/A} := L\Delta_{R/A}$.

Proposition 6

a. $\Delta_{R/A} \in D_{\mathrm{comp}}(A)$ is a commutative algebra object.
b. $\bar{\Delta}_{R/A} \in D_{\mathrm{comp}}(R)$ is a commutative algebra object.

Perfections in mixed characteristic

Let (A, I) be a perfect prism (e.g., $(\mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}, ([p]_q))$, where $\phi(q) = q^p$). Let R be a p -complete A/I -algebra. Our next goal is to construct a "perfectoidization" R_{perfd} of R . It may not be perfectoid, but better be thought of as a "derived perfectoid ring". We will realize this using the prismatic cohomology of R .

First let us look at the case of *characteristic* p , where we already know what R_{perfd} should be.

Definition 25 Let k be a perfect field of characteristic p . For any k -algebra R , define $R_{\mathrm{perf}} := \varinjlim(R \xrightarrow{\phi} R \xrightarrow{\phi} R \cdots)$. The map $R \to R_{\mathrm{perf}}$ is the universal map from R to a perfect k -algebra.

Example 13 Let $R = k[X] = \oplus_{i \in \mathbb{N}} kx^i$. Then $R_{\mathrm{perf}} = k[X^{1/p^\infty}] = \otimes_{i \in \mathbb{N}[1/p]} kx^i$.

Proposition 7 Let R be a k -algebra, the prefection of the derived de Rham cohomology

$$\mathrm{dR}_{R/k,\mathrm{perf}} := \varinjlim(\mathrm{dR}_{R/k} \xrightarrow{\phi} \mathrm{dR}_{R/k} \cdots)$$

(where ϕ is induced by the Frobenius on R) identifies with R_{perf} via the projection $\mathrm{dR}_{R/k} \to R$.

Proof Reduce to R = a polynomial k -algebra, and use $\phi : \Omega^i_{R/k} \to \Omega^i_{R/k}$ is zero for any $i > 0$. □

Now we can do the same thing for prismatic cohomology.

Proposition 8 Let $A = W(k)$, $I = (p)$. Let R be a k -algebra. The perfection

$$\bar{\Delta}_{R/A,\mathrm{perf}} := \varinjlim(\bar{\Delta}_{R/A} \xrightarrow{\phi} \bar{\Delta}_{R/A} \cdots)$$

identifies with R_{perf} via the natural map $R \cong \mathrm{gr}_0^{\mathrm{HT}}(\bar{\Delta}_{R/A}) \to \bar{\Delta}_{R/A}$.

Proof Use the fact that ϕ kills $\mathrm{gr}_i^{\mathrm{HT}}(\bar{\Delta}_{R/A}) = \wedge^i L_{R/k}[-i]$. There is an extra subtlety: the Frobenius for the prismatic cohomology comes from the objects on $(R/A)_\Delta$, while the Frobenius on differential forms comes from the ring R in characteristic p . Nevertheless one can check they are the same. □

Corollary 10 We have a natural identification of

$$\Delta_{R/A,\mathrm{perf}} = \varinjlim(\Delta_{R/A} \xrightarrow{\phi} \Delta_{R/A} \cdots)^\wedge$$

(p -completion) and $W(R_{\mathrm{perf}})$.

Now let us come to the *mixed characteristic* case. Let (A, I) be a perfect prism. Let R be an A/I -algebra. We use the prismatic cohomology to define the perfectoidization of R .

Definition 26

a. $\Delta_{R/A,\mathrm{perf}} := \varinjlim(\Delta_{R/A} \xrightarrow{\phi} \Delta_{R/A} \cdots)^\wedge \in D_{\mathrm{comp}}(A)$ ((p, I) -completion).
b. $R_{\mathrm{perfd}} := \Delta_{R/A,\mathrm{perf}} \otimes^{\mathbb{L}}_A A/I \in D_{\mathrm{comp}}(R)$ (p -completion).

Proposition 9

a. $\Delta_{R/A,\mathrm{perf}} \in D_{\mathrm{comp}}(A)$ is a commutative algebra object, and ϕ is an isomorphism on it.
b. $R_{\mathrm{perfd}} \in D_{\mathrm{comp}}(R)$ is a commutative algebra object.

Example 14

a. If $I = (p)$, then $R_{\mathrm{perfd}} \cong R_{\mathrm{perf}}$.
b. If R is already perfectoid (e.g., $R = A/I$), then $R \cong R_{\mathrm{perfd}}$ (check: $\Delta_{R/A} \cong A_{\inf}(R)$, the unique lift of R from A/I to A).
c. Let $(A, I) = (\mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}, ([p]_q))$. Let $R = A/I[X^{\pm 1}]^\wedge$ (a torus). We use the fact that (more on this later)

$$\Delta_{R/A} = A[X^{\pm 1}]^\wedge \xrightarrow{\nabla_q} A[X^{\pm 1}]^\wedge dx/x, \quad \nabla_q(f(x)) = \frac{f(qx) - f(x)}{q-1} dx/x.$$

Then one can compute that

$$\Delta_{R/A,\mathrm{perf}} = A[X^{\pm 1/p^\infty}]^\wedge \xrightarrow{\gamma - 1} J \cdot A[X^{\pm 1/p^\infty}]^\wedge,$$

where

$$J = \ker(A \xrightarrow{q \mapsto 1} \mathbb{F}_p) = (\cup_n (p, q^{1/p^n} - 1))^\wedge \subseteq A$$

and $\gamma(x^i) = q^i x^i$ for any $i \in \mathbb{Z}[1/p]$. Notice that when specializing to $q = 1$ the term in degree one of this two-term complex is zero and we obtain a complex with one term. However, note that the element $(q-1) \cdot 1$ (= the image of dx/x) in degree one is not a co-boundary, and also nonzero in R_{perfd} (= nonzero mod $[p]_q$). Therefore R_{perfd} is genuinely derived (with nontrivial higher cohomology).

Example 15 Two examples of perfect prisms to keep in mind today:

a. $(A, I) = (\mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p,q-1)}, ([p]_q))$.
b. $(A, I) = (\mathbb{Z}_p[u^{1/p^\infty}]^\wedge_{(p,u)}, (p-u))$ (especially for etale comparison).

Proposition 10

a. $\Delta_{R/A,\mathrm{perf}} \in D^{\geq 0}(A)$ (there is no cohomology in negative degree; reason: power operation in homotopy theory).
b. If $\Delta_{R/A,\mathrm{perf}} \in D^{\leq 0}(A)$ (i.e., in degree 0), then $(\Delta_{R/A,\mathrm{perf}}, I\Delta_{R/A,\mathrm{perf}})$ is a perfect prism, and R_{perfd} is a perfectoid ring.
c. R_{perfd} is independent of the choice of (A, I).
d. The functor $R \mapsto R_{\mathrm{perfd}}$ commutes with faithfully flat base change $(A, I) \to (A', I')$.

As an application, we now reprove one of the key lemmas in André's recent breakthrough on the direct summand conjecture.

Theorem 10 (André) Let R be a perfectoid ring. Let $g \in R$. Then there exists p-completely faithfully flat map $R \to R_\infty$ of perfectoid rings such that g has a compatible system of p-power roots $\{g^{1/p^\infty}\}$ in R_∞.

Remark 22 This theorem is highly nontrivial. For example, take $R = \mathbb{Z}_p[p^{1/p^\infty}]^\wedge$, $g = p - 1$. Notice that there are no p-power roots of g in R. Andre's proof used the full power of the theory of perfectoid spaces and in particular the tilting equivalence.

Proof Let $(A, I) = (A_{\inf}(R), \ker(\theta))$ be the perfect prism corresponds to R. Let S be the p-adic completion of $R[X^{1/p^\infty}]/(x-g)$. Then $R \to S$ is p-completely faithfully flat. We claim that taking $R_\infty = S_{\mathrm{perfd}}$ solves the problem: i.e., S_{perfd} is perfectoid and $R \to S_{\mathrm{perfd}}$ is p-completely faithfully flat. It suffices to show the latter is true (as faithfully flat implies living in only degree 0). Equivalently, we need to show that $A \to \Delta_{S/A,\mathrm{perf}}$ is (p, I)-completely faithfully flat.

Now it suffices to show that each individual term $\Delta_{S/A}$ in the perfection is (p, I)-completely faithfully flat over A (use the stability of faithfully flatness for $\varinjlim$, and use A is perfect to change the twisted A-module to A-module structure), which is equivalent to showing that $R \to (\Delta_{S/A}/I)$ is p-completely faithfully flat.

By the Hodge-Tate comparison, it suffices to show that each graded piece $\wedge^i L_{S/R}[-i]$ is p-completely faithfully flat. Since $L_{R[X^{1/p^\infty}]/R}$ is p-adically zero (divisible by any p-power), we know that

$$L_{S/R} \cong L_{S/R[X^{1/p^\infty}]} = S[1],$$

as $x - g$ is a nonzerodivisor. So

$$\wedge^0 L_{S/R}[0] = S, \wedge^1 L_{S/R}[-1] \cong S, \cdots, \wedge^n L_{S/R}[-n] \cong \Gamma^n_S(S) \cong S.$$

($\Gamma^n_S(S)$ is the degree F divided power polynomials of S). Each graded piece is a free module of rank 1 in degree zero, hence p-complete faithfully flat. □

Corollary 11 Let R be a perfectoid ring and $I \subseteq R$ an ideal. Let $S = R/I$. Then S_{perfd} lives in degree 0, and $S \to S_{\mathrm{perfd}}$ is surjective.

Remark 23 The corollary can be rephrased in Scholze's language (see his torsion Galois representations paper): the notions of Zariski closed and strongly Zariski closed (cut out by ideals coming from characteristic p) are the same.

Example 16 Let $R_0 = \mathbb{Z}_p^{\mathrm{cyc}} = \mathbb{Z}_p[q^{1/p^\infty}]/([p]_q)^\wedge$. Let $R = R_0[X^{1/p^\infty}]^\wedge$ and $I = (X-1) \subseteq R$. I don't know how to describe $\ker(R/(X-1) \to (R/(X-1))_{\mathrm{perfd}})$ explicitly (or even write down a nontrivial element in the kernel explicitly).

The etale comparison

Let (A, I) be a perfect prism. Let R be a p-complete A/I-algebra. Assume R is finitely generated over A/I and $R[p^\infty]$ is bounded.

Theorem 11 There exists a canonical isomorphism

$$R\Gamma_{\mathrm{et}}(\mathrm{Spec}\, R[1/p], \mathbb{Z}/p^n) \cong (\Delta_{R/A}[1/d]/p^n)^{\phi=1}.$$

Remark 24 This etale comparison can be viewed as a description of the nearby cycles in terms of differential forms.

Our next goal is to prove Theorem 11. To simplify notation, assume $n=1$. The main steps are:

a. Reduce from $\mathbb{\Delta}_{R/A}$ to $\mathbb{\Delta}_{R/A,\mathrm{perf}}$ (most interesting step).

b. Reduce to R semiperfectoid (quotient of perfectoid; its prismatic cohomology lives in degree 0).

c. Reduce to R perfectoid (for which the theorem was known before: dates back to e.g. Fontaine-Wintenberger in 70's).

$\boxed{\text{Step (a)}}$ A *continuity property* inspired by topological cyclic homology.

Definition 27 Let B be an $\mathbb{F}_p$-algebra and $t\in B$. Define the category of Frobenius modules

$$D(B[F])=\{(M,\phi_M): M\in D(B), \phi_M: M\to\phi_*M\},$$

and a subcategory $D_{\mathrm{comp}}(B[F])$ such that M is t-complete. The colimits in $D_{\mathrm{comp}}(B[F])$ are computed by the t-completion of the usual colimit.

Proposition 11

a. The fixed point functor

$$(\cdot)^{\phi=1}: D(B[F])\to D(\mathbb{F}_p),\quad (M,\phi)\mapsto M^{\phi=1}:=\mathrm{fib}(M\xrightarrow{\phi-1}M)$$

commutes with colimits.

b. The fixed point functor

$$D_{\mathrm{comp}}(B[F])\to D(\mathbb{F}_p),\quad (M,\phi)\mapsto M[1/t]^{\phi=1}$$

commutes with colimits.

c. Let $(M,\phi)\in D_{\mathrm{comp}}(B[F])$, then the perfection $(M,\phi)\mapsto(M,\phi)_{\mathrm{perf}}$ induces an isomorphism on $(\cdot)^{\phi=1}$.

Proof Let $\{(M_i,\phi_i)\}$ be a diagram in $D_{\mathrm{comp}}(B[F])$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{colim}\, M_i \arrow[r, "a"] \arrow[d] & (\mathrm{colim}\, M_i)^\wedge \arrow[d] \\
\mathrm{colim}\, M_i[1/t] \arrow[r, "b"] & (\mathrm{colim}\, M_i)^\wedge[1/t].
\end{tikzcd}$$

Then (a) is equivalent to $\mathrm{fib}(a)^{\phi=1}=0$, and (b) is equivalent to $\mathrm{fib}(b)^{\phi=1}=0$. As $\mathrm{fib}: N\to N^\wedge$ is uniquely t-divisible, we know that $\mathrm{fib}(a)\cong\mathrm{fib}(b)$. Therefore it suffices to show that $\mathrm{fib}(a)^{\phi=1}=0$, or equivalently, $M\mapsto M^{\phi=1}$ commutes with colimits.

We claim that for any $(N,\phi)\in D_{\mathrm{comp}}(B[F])$, $N\mapsto N/t$ induces an isomorphism on $(\cdot)^{\phi=1}$ (then the result follows since $(\cdot)^{\phi=1}$ commutes with colimits before completion). To prove the claim, notice that $\mathrm{fib}(N\to N/t)$ has a complete descending filtration $\{t^iN\}_{i>0}$ such that ϕ is *topological nilpotent* on the filtration: $\phi(t^i)=t^{ip}\subsetneq t^{i+1}B$ (think: Frobenius is *contracting*). Thus $\phi-1$ is an isomorphism on $\mathrm{fib}(N\to N/t)$. □

The upshot is that to prove the Theorem 11, it suffices to show the same thing for $(\mathbb{\Delta}_{R/A,\mathrm{perf}}[1/d]/p^n)^{\phi=1}$ (after the perfection).

$\boxed{\text{Step (b)}}$ Let $T=A/I[X_1,\dots,X_n]^\wedge$ be a polynomial algebra. We take the perfectoid ring $T_\infty=A/I[X_1^{1/p^\infty},\dots,X_n^{1/p^\infty}]^\wedge$. We have a faithfully flat map $T\to T_\infty$. Let T_∞^* be the Cech nerve of $T\to T_\infty$ (i.e., $T_\infty\twoheadrightarrow T_\infty\hat{\otimes}_T T_\infty\cdots$). Then each T_∞^i is *semiperfectoid*.

For more general R, choose topological generators $f_1,\dots,f_n\in R$ to get a surjective map $T\to R, X_i\mapsto f_i$. Repeating the previous construction we obtain $R\to R_\infty^*$, where each R_∞^i is semiperfectoid.

Now the strategy is to reduce Theorem 11 for R to the same thing for R_∞^*.

Lemma 21

a. (descent for etale cohomology)

$$R\Gamma(\mathrm{Spec}\, R[1/p],\mathbb{F}_p)\cong\varprojlim R\Gamma(\mathrm{Spec}\, R_\infty^*[1/p],\mathbb{F}_p).$$

(true for any torsion contractible sheaves).

b. (descent for prismatic cohomology)

$$(\mathbb{A}_{R/A}[1/d]/p)^{\phi=1} \cong \varprojlim (\mathbb{A}_{R^*_\infty/A}[1/d]/p)^{\phi=1}$$

and the same for $\mathbb{A}_{\cdot/A}/p$ and $\mathbb{A}_{\cdot/A,\mathrm{perf}}[1/d]/p$ (the proof uses tools developed in Akhil Mathew's undergraduate thesis).

$\boxed{\text{Step (c)}}$ Let R be semiperfectoid and R_{perfd} be its perfectoidization. By Corollary 11, we know that R_{perfd} lives in degree 0 and hence is perfectoid. We claim that both sides of the desired isomorphism

$$R\Gamma_{\mathrm{et}}(\mathrm{Spec}\, R[1/p], \mathbb{Z}/p^n) \cong (\mathbb{A}_{R/A,\mathrm{perf}}[1/d]/p^n)^{\phi=1}$$

do not change when replacing R by R_{perfd}. For the RHS, we claim that

$$\mathbb{A}_{R/A,\mathrm{perf}} \cong \mathbb{A}_{R_{\mathrm{perfd}}/A}.$$

In fact both reduce to R_{perfd} mod d (by the Hodge-Tate comparison and the definition of R_{perfd}), and hence the claim is true by the derived completeness of both. For the LHS, one shows that $R \to R_{\mathrm{perfd}}$ induces an isomorphism of associated arc_p -sheaves (which we have not covered) and one gets (by a theorem) that they have the same etale cohomology for the generic fiber.

Therefore it remains to prove the etale comparison when R is perfectoid. In this case, by a (classical) theorem, we have

$$R\Gamma_{\mathrm{et}}(\mathrm{Spec}\, R[1/p], \mathbb{F}_p) \cong R^\flat[1/d]^{\phi=1}.$$

(in modern terminology, this can be proved using the pro-etale site of perfectoid spaces and the exactness of the Artin-Schreier sequence). Since R is perfectoid, we have $\mathbb{A}_{R/A} \cong A_{\mathrm{inf}}(R)$ and hence $\mathbb{A}_{R/A}[1/d]/p \cong R^\flat[1/d]$. Thus we have proved the etale comparison when R is perfectoid.

In particular, the $\mathbb{F}_p$ -etale cohomology of $\mathrm{Spec}\, R[1/p]$ lives in only two degrees (in fact, by the same etale comparison theorem with nontrivial coefficient systems, one can show that $\mathrm{Spec}\, R[1/p]$ is a $K(\pi_1)$).

Now let us come back to prove our main application.

Theorem 12 Let $C/\mathbb{Q}_p$ be complete and algebraically closed. Let $\mathcal{O}_C$ be the valuation ring (which is perfectoid) with residue field k. Let $X/\mathcal{O}_C$ be a proper smooth formal scheme. Then

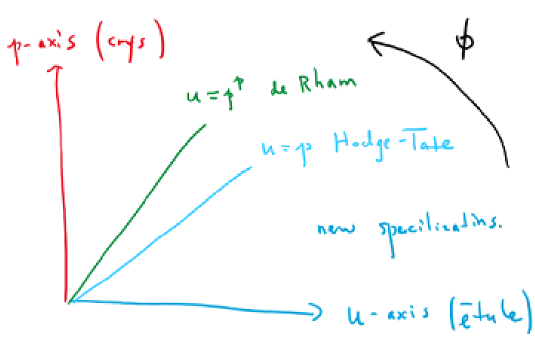
$$\dim_{\mathbb{F}_p} H^i_{\mathrm{et}}(X_C, \mathbb{F}_p) \le \dim_k H^i_{\mathrm{dR}}(X_k).$$

Proof Let $(A, I) = (A_{\mathrm{inf}}(\mathcal{O}_C), \ker\theta = (d))$ be the perfect prism associated to $\mathcal{O}_C$. So $A/(d) \cong \mathcal{O}_C$. The map $\mathcal{O}_C \to k$ induces a map $A \to W = W(k)$, and a map of perfect prisms $(A, I) \to (W, (p))$. The prismatic complex $\mathbb{A}_{X/A} \in D(X, A)$ is obtained by gluing $\mathbb{A}_{R/A}$ for all open $\mathrm{Spf}(R) \subseteq X$. Similarly, we have the prismatic complex $\mathbb{A}_{X_k/W} \in D(X_k, W)$. We have an base change isomorphism (by the Hodge-Tate comparison)

$$\mathbb{A}_{X/A} \otimes_A^{\mathbb{L},\wedge} W \cong \mathbb{A}_{X_k/W}.$$

(Notice that since X is a formal scheme, $X = X_k$ as topological spaces and we can identify sheaves on them).

Let $R\Gamma_A(X) = R\Gamma(X, \mathbb{A}_{X/A}) \in D(A)$, and similarly define $R\Gamma_W(X_k) \in D(W)$. We claim that $R\Gamma_A(X) \in D_{\mathrm{perf}}(A)$ is a *perfect complex*, i.e., represented by a finite complex of finite free modules. To prove the claim, it suffices to show that $R\Gamma_A(X)/d \in D_{\mathrm{perf}}(\mathcal{O}_C)$. This is true by the Hodge-Tate comparison: $R\Gamma_A(X)/d$ has a filtration whose graded pieces are given by $R\Gamma(X, \Omega^i_{X/\mathcal{O}_C})[-i]$, which is perfect since X is *proper* and *smooth*.



Let $V = A/p = \mathcal{O}_C^\flat$ with fraction field $C^\flat$ and residue field k. Then by the semicontinuity for finitely presented V -modules, we obtain

$$\dim_{C^\flat} H^i(R\Gamma_A(X) \otimes_A C^\flat) \le \dim_k H^i(R\Gamma_A(X) \otimes_A k).$$

The RHS is given by

$$R\Gamma_A(X) \otimes_A W \otimes_W k = R\Gamma_W(X_k) \otimes_W k,$$

which is given by (the Frobenius of twist) $R\Gamma_{\mathrm{dR}}(X_k)$ by the crystalline comparison for prismatic cohomology.

For the LHS: by the etale comparison we obtain

$$(R\Gamma_A(X) \otimes_A C^\flat)^{\phi=1} \cong R\Gamma_{\mathrm{et}}(X_C, \mathbb{F}_p).$$

We then apply the following linear algebra lemma to $M = R\Gamma_A(X) \otimes_A C^\flat$ to get the desired inequality. (In fact, using the q -de Rham complex, we will see that ϕ is an isomorphism for $M = R\Gamma_A(X) \otimes_A C^\flat$ and hence the dimension of the LHS is *equal* to the dimension of $\mathbb{F}_p$ -etale cohomology of the generic fiber X_C). □

Lemma 22 Let K be an algebraically closed field of characteristic p . Let $M \in D_{\mathrm{perf}}(K)$ and $\phi : M \to \phi_* M$. Then

$$H^i(M^{\phi=1}) \otimes_{\mathbb{F}_p} K \to H^i(M)$$

is injective, and moreover an isomorphism when $\phi : M \to \phi_* M$ is an isomorphism.

The q -de Rham complex

As an analogue of crystalline cohomology being the de Rham cohomology of a lift, we would like to compute $\Delta_{R/A}$ on the nose (as a genuine complex).

We will work with $\mathbb{Z}_p[[q-1]]$ with $\phi(q) = q^p$, and $d = [p]_q = \frac{q^p-1}{q-1}$. Let $\mathbb{Z}_p[\varepsilon] = \mathbb{Z}_p[[q-1]]/[p]_q$.

Definition 28 (Aomoto, Jackson) Let $R = \mathbb{Z}_p[x]^\wedge$. We define its q *-de Rham complex* (depending on the choice of x)

$$q\Omega^*_{R,\square} := R[[q-1]] \xrightarrow{\nabla_q} R[[q-1]]dx,$$

where we define the $\mathbb{Z}_p[[q-1]]$ -linear map

$$\nabla_q(f(x)) = \frac{f(qx) - f(x)}{qx - x} dx.$$

Notice that the definition makes sense since $f(qx) \equiv f(x) \pmod{qx - x}$. Notice that $\nabla_q(x^n) = [n]_q x^{n-1} dx$. Therefore

$$q\Omega^*_{R,\square}/(q-1) \cong \Omega^*_{R/\mathbb{Z}_p}$$

on the nose. We have a similar construction for $R = \mathbb{Z}_p[x_1, \ldots, x_n]^\wedge$.

Remark 25 One can check that $q\Omega^*_{R,\square}$ is not quasi-isomorphic to the constant deformation $\Omega^*_{R/\mathbb{Z}_p} \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p[[q-1]]$.

Remark 26 ∇_q satisfies the q *-Leibniz rule*

$$\nabla_q(f(x)g(x)) = f(x)\nabla_q(g(x)) + g(qx)\nabla_q(f(x)).$$

This means that we may make $q\Omega^*_{R,\square}$ a dga by making $q\Omega^1_{R,\square}$ a $q\Omega^0_{R,\square}$ -bimodule via

$$a(x) \cdot w \cdot w(x) = a(x)b(qx)w.$$

Notice that this dga is not commutative. Nevertheless, it turns out to be commutative up to all possible homotopies (an E_∞ -algebra).

Definition 29 Let S be a formally smooth $\mathbb{Z}_p$ -algebra. A *framing* of S is a formally etale map

$$\square : \mathbb{Z}_p[x_1, \ldots, x_n]^\wedge \to S.$$

We call such $(S, \square)$ a *framed pair*.

Definition 30 (q -de Rham complex for framed pairs) Let $(S, \square)$ be a framed pair. We obtain a formally etale map

$$\tilde{\square} : \mathbb{Z}[q-1, x_1, \ldots, x_n]^\wedge \to S[[q-1]]$$

of $\mathbb{Z}_p[[q-1]]$ -algebras. For $i \in \{1, \ldots, n\}$, we have an automorphism γ_i of $\mathbb{Z}_p[q-1, x_1, \ldots, x_n]^\wedge$ such that $\gamma_i(x_j) = x_j$ if $j \neq i$ and qx_j if $j = i$. Since $\gamma_i \equiv \mathrm{Id} \pmod{qx_i - x_i}$, we obtain a unique automorphism γ_i of $S[[q-1]]$ extending it (by the formal smoothness of S). We define

$$\nabla_{q,i}(f) = \frac{\gamma_i(f) - f}{qx_i - x_i} dx_i,$$

and the q *-de Rham complex* to be the Koszul complex

$$q\Omega^*_{S,\square} = \mathrm{Kos}(S[[q-1]]; \nabla_{q,1}, \ldots, \nabla_{q,n}) = (S[[q-1]] \xrightarrow{\nabla_q} \oplus_{i=1}^n S[[q-1]]dx_i \to \cdots).$$

Lemma 23

$$q\Omega^*_{S,\square}/(q-1) \cong \Omega^*_{S/\mathbb{Z}_p}.$$

Namely, $\nabla_{q,i}(f) = df/dx \pmod{q-1}$.

Proof Calculus. □

Conjecture 1 (Scholze) There is a symmetric monoidal functor $S \mapsto q\Omega_S$ from formally smooth $\mathbb{Z}_p$-algebras to $D_{\mathrm{comp}}(\mathbb{Z}_p[[q-1]])$, equipped with natural isomorphisms

$$q\Omega_S \cong q\Omega^*_{S,\square}$$

for each choice $\square$ of framing. In particular, $q\Omega_S$ is a commutative algebra in $D_{\mathrm{comp}}(\mathbb{Z}_p[[q-1]]$, and each $q\Omega^*_{S,\square}$ is an E_∞-algebra.

Our next goal is to prove Conjecture 1. It is hard to prove from first principle: it is already not clear how to write down the endomorphism of the q-de Rham complex for the simple change of variable $x \mapsto x+1$.

Remark 27

a. Conjecture 1 is closely related to integral p-adic Hodge theory, which proves the conjecture after base change along

$$\mathbb{Z}_p[[q-1]] \to \mathbb{Z}_p[[q^{1/p^\infty}]]^\wedge_{(p,q-1)}.$$

Thus Conjecture 1 can be thought of as a "de-perfection" of this statement.

b. Conjecture 1 is easy after base change along

$$\mathbb{Z}_p[[q-1]] \to \mathbb{Q}_p[[q-1]],$$

as the q-de Rham complex is indeed isomorphic to the de Rham complex over $\mathbb{Q}_p[[q-1]]$ (by Taylor expansion).

c. Conjecture 1 still makes sense if we replace $\mathbb{Z}_p$ with any p-complete ring, but it is not true in this generality. For example, it fails for $\mathbb{F}_p$ (as one cannot compute the $\mathbb{F}_p$-etale cohomology of the generic fiber from the special fiber). We will see that the natural general context Conjecture 1 holds is provided by q-PD thickenings.

The q-crystalline cohomology

Our goal is to construct a *q-crystalline site* $(R/\mathbb{Z}_p[[q-1]])_{\mathrm{qcrys}}$ whose cohomology is computed by q-de Rham complexes.

Write $A = \mathbb{Z}_p[[q-1]]$ with $\phi(q) = q^p$. Let $A \to \mathbb{Z}_p, q \mapsto 1$. Let $\mathbb{Z}_p[\varepsilon_p] := A/([p]_q)$. Observe that the ideals $(p, q-1)$ and $(p, [p]_q)$ define the same topology (one is contained in the power of the other and vice versa).

The main new ingredient is the notion of *q-PD thickenings*.

Definition 31 A *q-PD pair* (p, I) is a pair where D is a δ-algebra over A, and $I \subseteq D$ is an ideal such that

a. D and D/I are $(p, [p]_q)$-complete.

b. D is $[p]_q$-torsionfree.

c. I contains $q-1$ and $\phi(I) \subseteq [p]_q D$ (the analogue of the PD-structure, see Lemma 20).

We call $D \to D/I$ the corresponding *q-PD thickening*. Conjecture 1 will be proved in this generality.

Example 17

a. $(A, (q-1))$ is a q-PD pairing, and is the initial such pair. More generally for any $(p, [p]_q)$-completely flat δ-algebra D over A, we obtain a q-PD pair $(D, (q-1))$.

b. If $q = 1$ in D, then condition (c) is equivalent to that for any $x \in I$, $\frac{x^n}{n!} \in D$ for any $n \geq 1$ (the usual PD-structure requires $\frac{x^n}{n!} \in I$, but it turns out to not affect anything).

Lemma 24 Let D be a $[p]_q$-torsionfree δ-algebra over A. Let $f \in D$ such that $\phi(f) \in [p]_q D$. Then

$$\phi\left(\frac{\phi(f)}{[p]_q} - \delta(f)\right) \in [p]_q D.$$

Example 18 If $q = 1$, this lemma saying that if $f^p \in pD$, then $f^{p^2} \in p^{p+1} D$.

Proof Reduce to the universal case, where D is A-flat. We would like to show that

$$\frac{\phi^2(f)}{\phi([p]_q)} \equiv \phi(\delta(f)) \pmod{[p]_q D}.$$

In $D/[p]_q D$, we have $\phi([p]_q) = \frac{q^{p^2}-1}{q^p-1} = p$ (think: $[p]_q$ and $\phi([p]_q)$ are "transverse" to each other), so it suffices to show that

$$\phi^2(f) \equiv p\phi(\delta(f)) \pmod{[p]_q D}.$$

By definition,

$$\phi(f) = f^p + p\delta(f),$$

applying ϕ we obtain

$$\phi^2(f) = \phi(f)^p + p\phi(\delta(f)),$$

and $\phi(f)^p \in [p]_q D$ by assumption. □

Proposition 12 (existence of q-PD envelopes) Let R be a formally smooth $\mathbb{Z}_p$-algebra. Let P be formally smooth δ-algebra over A, equipped with a surjection $P \to R$ with kernel J. There is a universal map $(P, J) \to (D, I)$ to a q-PD pair. It has the following properties

a. D is $(p, [p]_q)$-completely flat over A.
b. The map $(P, J) \to (D, I)$ induces an isomorphism $R = P/J \cong D/I$.
c. The map $D/(q-1) \to R$ is the usual p-completed PD-envelope of $D/(q-1) \to R$.

We denote $D_{J,q}(P) := D$.

Proof Take $D = P\{\phi(x_1)/[p]_q, \ldots, \phi(x_n)/[p]_q\}^\wedge_{(p,q-1)}$, where $J = (q-1, x_1, \ldots, x_n)$, and $\{x_i\}$ form a regular sequence. □

Example 19 Let $R = \mathbb{Z}_p[t]^\wedge$, $P = A[x, y]^\wedge$, and $P \to R,\ x, y \mapsto t$ with $\delta(x) = \delta(y) = 0$, $J = (q-1, x, y)$. Then one can compute that $D_{J,q}(P)$ contains

$$\gamma_{k,q}(x-y) = \frac{(x-y)(x-qy)\cdots(x-q^{k-1}y)}{[k]_q[k-1]_q\cdots[1]_q}.$$

And they form a topological basis over $A[x]^\wedge$. (This was the motivating example for inventing the q-crystalline site).

Definition 32 (q-crystalline site) Let R be a formally smooth $\mathbb{Z}_p$-algebra. Define $(R/A)_{\mathrm{qcrys}}$ to be the category of q-PD pairs (D, I) such that $D/I = R$, and make it into a site via indiscrete topology (so presheaves=sheaves). Let $\mathcal{O}_{\mathrm{qcrys}}$ be the sheaf $(D, I) \mapsto D$. Define the q*-crystalline cohomology* to be

$$q\Omega_R = R\Gamma((R/A)_{\mathrm{qcrys}}, \mathcal{O}_{\mathrm{qcrys}}) \in D_{\mathrm{comp}}(A).$$

To compute $q\Omega_R$, we use the Cech-Alexander complexes for $q\Omega_R$ (compare Theorem 7). Choose a surjection $P \to R$ with kernel J such that x is a $(p, [p]_q)$-completion of a free δ-algebra over A. We obtain a cosimplicial δ-algebra over A

$$P^\bullet = (P \rightrightarrows P\hat{\otimes}_A P \cdots),$$

with ideal $J^\bullet \subseteq P^\bullet$ such that $P^\bullet/J^\bullet = R$. Taking q-PD envelopes gives a cosimplicial object of $(R/A)_{\mathrm{qcrys}}$,

$$D_{J^\bullet,q}(P^\bullet) = (D_{J^0,q}(P^0) \twoheadrightarrow D_{J^1,q}(P^1)\cdots).$$

Category theory (Lemma 18) then implies that $q\Omega_R$ is given by $D_{J^\bullet,q}(P^\bullet)$.

Theorem 13 ($q = 1$ specialization) There exists a canonical isomorphism

$$q\Omega_R \otimes^{\mathbb{L}}_A \mathbb{Z}_p \cong \Omega^*_{R/\mathbb{Z}_p}.$$

Proof

a. $\Omega^*_{R/\mathbb{Z}_p} \cong R\Gamma_{\mathrm{crys}}(\bar{R}/\mathbb{Z}_p)$ where $\bar{R} = R/p$ (Theorem 7).
b. $R\Gamma_{\mathrm{crys}}(\bar{R}/\mathbb{Z}_p)$ is computed by $D_{J^\bullet,q}(P^\bullet)/(q-1)$ (by Proposition 12 (c)), which also computes LHS by the what we just said. □

q-crystalline comparison and q-de Rham comparison

Let R be formally smooth over $\mathbb{Z}_p$. Let $R^{(1)} := R \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\varepsilon_p]$. Notice that we have the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\phi_A"] \arrow[d] & A \arrow[r, equal] \arrow[d] & A \arrow[d] \\
\mathbb{Z}_p = A/(q-1) \arrow[r] \arrow[d] & A/(q^p-1) \arrow[r] & A/([p]_q) \arrow[d] \\
R \arrow[rr] & & R^{(1)}.
\end{tikzcd}$$

The left vertical column corresponds to the q-crystalline cohomology, and the right vertical column corresponds to the prismatic cohomology. The twist $R^{(1)}$ is needed as $q-1$ is not a distinguished element but $[p]_q$ is.

Theorem 14 (q-crystalline comparison for prismatic cohomology) There exists a natural isomorphism
$$q\Omega_R \cong \Delta_{R^{(1)}/A}.$$

Proof To get a map $\Delta_{R^{(1)}/A} \to q\Omega_R$, we need to show that for each $(D \twoheadrightarrow R) \in (R/A)_{\mathrm{qcrys}}$, we get an object $(R^{(1)} \to D/[p]_q \leftarrow D) \in (R^{(1)}/A)_{\Delta}$. By the definition of q-PD thickening, we have a commutative diagram

$$\begin{tikzcd}
D \arrow[r, "\phi"] \arrow[d] & D \arrow[d] \\
R \arrow[r] & D/[p]_q D.
\end{tikzcd}$$

Linearizing along $A \xrightarrow{\phi} A$ we obtain the desired object in the prismatic site. To check this map is an isomorphism, we check after $\otimes^{\mathbb{L}}_A A/(q-1)$ and use the crystalline comparison for prismatic cohomology. □

Our next goal is to relate $q\Omega_R$ to $q\Omega^*_{R,\square}$. In the classical characteristic p setting this relation is saying that the crystalline-de Rham comparison does not depend the choice of the lift. One can basically add q everywhere in the proof for the classical case, and it works.

Definition 33 (q-de Rham complexes for q-PD envelopes) Let P be a formally smooth A-algebra, formally etale over $A[x_1, \ldots, x_n]^\wedge$ (which is given a unique δ-structure satisfying $\delta(x_i) = 0$). Assume there is a surjection $P \to R$ and D is the q-PD envelope of $P \to R$. By the following lemma, we can extend the q-de Rham complex $q\Omega^*_{R,\square}$ to $q\Omega^*_{D,\square}$ (from R to D).

Lemma 25 Each q-derivative $\nabla_{q,i} : P \to P$ extends uniquely to D.

Proof Recall that $\nabla_{q,i}(f) = \frac{\gamma_i(f)-f}{qx_i - x_i}$. It suffices to show that each γ_i extends uniquely to an automorphism of D congruent to $\mathrm{Id}_D \bmod qx_i - x_i$. Using the universal property of D, we need to show that for each $f \in J = \ker(P \to R)$, we have $\phi(\gamma_i(f)) \subseteq [p]_q D$ and
$$\frac{\phi(\gamma_i(f))}{[p]_q} \equiv \frac{\phi(f)}{[p]_q} \pmod{(qx_i - x_i)D}.$$
In fact, $\gamma_i(f) = f + (qx_i - x_i)g$ for some g, and so
$$\phi(\gamma_i(f)) = \phi(f) + (q^p - 1)x_i^p \phi(g).$$
Dividing by $[p]_q$, we obtain the desired result. □

Theorem 15 (q-crystalline-de Rham comparison) Let $(R, \square)$ be a framed $\mathbb{Z}_p$-algebra. Then there exists a natural isomorphism
$$q\Omega_R \cong q\Omega^*_{R,\square}.$$

Proof Let P be the unique lift of R to A with coordinates $x_1, \ldots, x_n$. Let $P^\bullet$ be the Cech nerve of $A \to P$. We have a surjection $P^n \to P \to R$ with kernel J^n, and $D_{J^\bullet}(P^\bullet)$ is a cosimplicial δ-algebra over A computing $q\Omega_R$. To relate it to $q\Omega^*_{R,\square}$, we use the bicomplex $q\Omega^*_{D_{J^\bullet},q}(P^\bullet)$. Notice that $q\Omega_R$ is computed by the first row of this bicomplex and $q\Omega^*_{R,\square}$ is computed by the first column of this bicomplex. To show they are quasi-isomorphic, we combine:

a. All horizontal maps give quasi-isomorphisms of the columns (reduce mod $q-1$ and use the Poincare lemma: adding free variables does not change the cohomology). So the bicomplex totalizes to the first column.

b. All rows except first column are acyclic (write down explicit combinatorial contracting homotopy). So the bicomplex also totalizes to the first row. □

Remark 28 One may use the q-de Rham complex (with the explicit description of the Frobenius in coordinates) to show that $\phi_A^* \Delta_{R/A} \to \Delta_{R/A}$ is an isogeny (i.e., it has an inverse up to multiplication by I^d, where $d = \dim(R/A)$).

Prismatic cohomology via topological Hochschild homology

Finally, let us mention how to recover $\Delta_{R/A}$ from the topological Homology homology $\mathrm{THH}(R)$. The characteristic 0 story is classical (Quillen, Connes, Tsygan...).

Definition 34 Let $A \to B$ be a map of (commutative) rings. Define the *Hochschild homology* to be $\mathrm{HH}(B/A) = B \otimes^{\mathbb{L}}_{B \otimes^{\mathbb{L}}_A B} B$ (i.e., derived self-intersection of the diagonal in $\mathrm{Spec}\, B \times_{\mathrm{Spec}\, A} \mathrm{Spec}\, B$).

We observe:

a. $\mathrm{HH}_0(B/A) = B$.
b. $\mathrm{HH}_1(B/A) \cong \Omega^1_{B/A}$

Theorem 16 (Hochschild-Kostant-Rosenberg) If $A \to B$ is smooth, then
$$\mathrm{HH}_*(B/A) \cong \Omega^*_{B/A}$$
as graded B-algebras. Moreover, if $\mathbb{Q} \subseteq A$ (characteristic 0), then we have a natural decomposition (a lift to the derived category)
$$\mathrm{HH}(B/A) \cong \bigoplus_i \Omega^i_{B/A}[i].$$

To bring the de Rham differential in, one uses the following observation of Connes: $\mathrm{HH}(B/A)$ has an S^1-action. To see this use
$$\mathrm{HH}(B/A) = B \otimes \mathrm{colim}\left(\begin{array}{ccc} \text{two pts} & \longrightarrow & \text{pt} \\ \downarrow & & \\ \text{pt} & & \end{array} \right) = B \otimes S^1.$$
(S^1 is obtained by gluing two $\mathbb{A}^1$'s at two points). Then $[S^1] \in H_1(S^1)$ induces a map
$$\mathrm{HH}_*(R) \to \mathrm{HH}_{*+1}(R)$$
which is the de Rham differential under the previous theorem.

Definition 35 Define the *periodic homology* to be the Tate cohomology of HH (a periodic version of HH),
$$\mathrm{HP}(B/A) = \mathrm{H}^*_{\mathrm{Tate}}(S^1, \mathrm{HH}(B/A)) := \mathrm{Cone}(\mathrm{HH}(B/A)_{hS^1}[1] \xrightarrow{\mathrm{Nm}} \mathrm{HH}(B/A)^{hS^1}).$$

Theorem 17 Assume $A \to B$ is smooth, and $\mathbb{Q} \subseteq A$. Then
$$\mathrm{HP}(B/A) \cong \Omega^*_{B/A} \otimes_A A[u, u^{-1}],$$
where u has degree 2.

Now let us come to the prismatic story. Let (A, I) be a perfect prism. Let S be a smooth algebra over $R = A/I$.

Definition 36 For any ring B, define the *topological Hochschild homology* $\mathrm{THH}(B) := B \otimes^{\mathbb{L}}_{B \otimes^{\mathbb{L}}_{\mathbb{S}} B} B$ (point: tensoring over the sphere spectrum $\mathbb{S}$ gets rid of the factorials in the denominators). It also has an S^1-action, and we similarly define the *topological periodic homology* $\mathrm{TP}(B)$ to be the Tate cohomology of $\mathrm{THH}(B)$.

Example 20 (Bokstedt, Hesselholt) Let R be perfectoid. Then $\mathrm{THH}(R)_* = R[u]$ where u has degree 2 (if tensor over $\mathbb{Z}$ instead of the sphere spectrum, one gets the divided power algebra), and $\mathrm{TP}_*(R) = A_{\inf}(R)[u, u^{-1}]$.

Theorem 18 (Bhatt-Morrow-Scholze) Let S be smooth over a perfectoid R. Then there is a filtration $\mathrm{Fil}^*_{\mathrm{mot}}$ on $\mathrm{TP}(S)$ with
$$\mathrm{gr}^*_{\mathrm{mot}} \mathrm{TP}(S) \cong \Delta_{S/A} \otimes_A A[u, u^{-1}].$$