MATH 665 PROBLEM SET 0

FALL 2024

Due Monday, September 19. You may consult books, papers, and websites as long as you cite all sources and write up your solutions in your own words. **This problem set is only assigned to the undergraduates in the course.**

Problem 1. A crash course in modern algebraic geometry.

(1) Read or review II.1–II.3 in Hartshorne's *Algebraic Geometry*.
(2) Do Exercises 1–6 from II.1.
(3) Do Exercises 1–4, 7, 8 from II.2.
(4) Read Chapters 5–6 in Milne, *Lectures on Étale Cohomology*.[1] You may find the Wikipedia article on "Grothendieck topology" helpful.
(5) Determine the finite étale covers of $\operatorname{Spec} \mathbf{F}_p$ and of $\operatorname{Spec} \bar{\mathbf{F}}_p(\!(t)\!)$ for prime p.

Problem 2. A crash course in Lie theory. You may find the following helpful:

- The section "Roots" in the Wikipedia article "Reductive group".
- The sections "Structure" and "Example root space decomposition..." in the Wikipedia article "Semisimple Lie algebra".

Over an algebraically closed field of characteristic not 2, let

$$\mathfrak{g} = \{\gamma \in \mathfrak{gl}_4 \mid \gamma^t J + J\gamma = 0\} \quad \text{and} \quad G = \{g \in \mathrm{GL}_4 \mid g^t J g = J\},$$

where J is the symplectic form

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Thus $\mathfrak{g}$ is the Lie algebra of G. Let $T \subseteq G$ be the subgroup of diagonal elements.

(1) Determine the diagonal and strictly upper-triangular elements of $\mathfrak{g}$. They should form subspaces of dimensions 2 and 4, respectively.
(2) Assuming that T is a maximal torus, use (1) to list the upper-triangular root subgroups $U_\alpha \subseteq G$. Use the fact that $tut^{-1} = \alpha(t)u$ for all $t \in T$ and $u \in U_\alpha$ to find the corresponding roots $\alpha : T \to \mathbf{G}_m$.
(3) Draw the character lattice $X(T) := \mathrm{Hom}(T, \mathbf{G}_m)$, and plot the roots in (2). Recall that the Weyl group of (G, T) is generated by the reflections that send $\alpha \mapsto -\alpha$. Which group is it?

In the literature, $\mathfrak{g}$ is known as the *symplectic Lie algebra* $\mathfrak{sp}_4$ and G is known as the *symplectic linear group* Sp_4.

[1] Available at https://www.jmilne.org/math/CourseNotes/LEC.pdf for free.