THE TRUNCATED INDUCTION OF $\mathfrak{S}_4$

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We would like to consider how the truncated induction, (for a definition see [Lus84, 4.1.7]), works for the Weyl group of type A_3. In other words the Symmetric group $\mathfrak{S}_4$. We start by recalling the character table of $\mathfrak{S}_4$.

	$d_\lambda(u)$	a_λ	(1^4)	$(2,1^2)$	(2^2)	$(3,1)$	(4)
$\chi_{(1^4)}$	u^6	6	1	-1	1	1	-1
$\chi_{(2,1^2)}$	$u^3(u^2+u+1)$	3	3	1	-1	0	-1
$\chi_{(2^2)}$	$u^2(u^2+1)$	2	2	0	2	-1	-1
$\chi_{(3,1)}$	$u(u^2+u+1)$	1	3	-1	1	0	1
$\chi_{(4)}$	1	0	1	1	1	1	1

We have given the conjugacy classes by partitions of 4 and have indexed the characters by these partitions. They have been ordered by the **dominance ordering**, (for a definition see [JK09, 1.4.5]), which coincides with the lexicographical ordering in this case. The character table of $\mathfrak{S}_4$ can be obtained in a number of ways but can also be found in [JL01, Section 18.1].

The polynomials $d_\lambda(u)$ are the **generic degree polynomials** associated to the irreducible representations of the Weyl group. In type A_{n-1} these can be calculated quite easily using a generalisation of the hook formula, (see [JK09, Theorem 2.3.21]). We start by recalling that the **Gaussian binomial coefficients** are defined to be

$$\binom{m}{r}_u = \begin{cases} 1 & r = 0, \\ \dfrac{\prod_{i=m}^{m-r+1}(1-u^m)}{\prod_{i=1}^{r}(1-u^i)} & 1 \leqslant r \leqslant m, \\ 0 & r > m. \end{cases}$$

In particular we define the following term for $1 \leqslant k \leqslant n$

$$[k] = \binom{k}{1}_u = \frac{(1-u^k)}{(1-u)} = \frac{u^k-1}{u-1} = u^{k-1} + u^{k-2} + \cdots + u + 1.$$

Now let $\lambda \vdash n$ be a partition of n such that $\lambda = (\lambda_1, \ldots, \lambda_t)$ and consider the irreducible character χ_λ of $\mathfrak{S}_n$ associated to λ. We have the generic degree polynomial associated to χ_λ is given by the polynomial

$$d_\lambda(u) = q^{\sum_{i=1}^{t}(i-1)\lambda_i}\frac{[n]!}{\prod_{ij}[h_{ij}^\lambda]},$$

where h_{ij}^λ are the **hook lengths** of all possible hooks in the tableaux associated to λ.

Example. We consider the case $\lambda = (2^2)$. Then we have the young tableaux associated to λ and all its hook lengths are

$$\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 2 & 1 \\ \hline \end{array}$$

Therefore using this information we have that

$$d_{(2^2)}(u) = u^{0+2}\frac{[4]!}{[3][2]^2[1]} = u^2\frac{[1][2][3][4]}{[1][2]^2[3]} = u^2\frac{[4]}{[2]} = u^2\left(\frac{u^4-1}{u^2-1}\right) = u^2(u^2+1).$$

Note that the value a_λ is just the highest power of u dividing the generic degree polynomial and hence is just $\sum_{i=0}^{t}(i-1)\lambda_i$.

We now wish to consider the **Young subgroups** or **parabolic subgroups** of $\mathfrak{S}_4$. These will all be isomorphic to either $\mathfrak{S}_4$, $\mathfrak{S}_3 \times \mathfrak{S}_1$, $\mathfrak{S}_2 \times \mathfrak{S}_2$, $\mathfrak{S}_2 \times \mathfrak{S}_1^2$, $\mathfrak{S}_1^4$. Hence these subgroups are in bijection with the partitions of 4. We will choose these subgroups by taking the most obvious subsets of the Coxeter generators and considering the subgroup generated by these elements. For example for $\mathfrak{S}_2 \times \mathfrak{S}_2$ we choose the subgroup $\langle(12),(34)\rangle$ of $\mathfrak{S}_4$.

We recall that every irreducible representation of $\mathfrak{S}_4$ occurs as a component of $1_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_4$ for some parabolic subgroup $\mathfrak{S}_\lambda$. In fact χ_λ is the unique irreducible representation which is a component of $1_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_4$ and $\mathrm{sgn}_{\mathfrak{S}_{\lambda'}} \uparrow \mathfrak{S}_4$, where λ' is the **dual partition** to λ, (for a definition see [JK09, 1.4.3]). We will now use the techniques of the symmetric group to decompose $1_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_4$ into irreducible representations for each partition λ.

Recall that $\langle 1_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_4, \chi_\mu\rangle \neq 0$ if and only if μ dominates λ in the dominance ordering. In fact, by [JK09, Theorem 2.8.5], the multiplicity of $\langle 1_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_4, \chi_\mu\rangle$ is given by the number of **semistandard tableaux** of shape λ and content μ.

Definition. Let $\lambda, \mu \vdash n$ be partitions of n. A **generalised Young tableaux** of shape λ and content μ is a Young tableaux of shape λ filled by the integers i exactly μ_i times. This tableaux is called **semistandard** if the rows are weakly increasing and the columns are strictly increasing.

Example. Consider $\lambda = (2, 1^2)$ and $\mu = (2^2)$. Then some examples of generalised Young tableaux of shape λ and content μ are

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & 2 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & 3 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 1 \\ \hline \end{array}$$

There is only one semistandard tableaux of shape λ and content μ and this is

1	1
2	3

Therefore we have $\langle 1 \uparrow \mathfrak{S}_{(2,1^2)}, \chi_{(2^2)}\rangle = 1$.

We can now create a matrix M which expresses the decomposition of the characters $1_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_4$ into irreducible characters, (see [JK09, 2.2.4]). Recall that the top row of this matrix is given by the dimensions of the irreducible characters of $\mathfrak{S}_4$ as $1_{\mathfrak{S}_1^4} \uparrow \mathfrak{S}_4$ is the character of the regular representation. Also every entry in the right hand side column of M is equal to 1 as the trivial character appears with multiplicity 1 in each $1_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_4$ because by Frobenius reciprocity

$$1 = \langle 1_{\mathfrak{S}_\lambda}, 1_{\mathfrak{S}_\lambda}\rangle = \langle 1_{\mathfrak{S}_\lambda}, 1_{\mathfrak{S}_4} \downarrow \mathfrak{S}_\lambda\rangle = \langle 1_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_4, 1_{\mathfrak{S}_4}\rangle.$$

We now calculate this matrix. On the left hand side of the matrix we have indicated the parabolic subgroup that we are inducing the trivial representation from and along the top we indicate the partitions relating to the irreducible characters of $\mathfrak{S}_4$.

$$\begin{array}{c c}
 & \begin{array}{ccccc} (1^4) & (2,1^2) & (2^2) & (3,1) & (4) \end{array} \\
\begin{array}{c} \mathfrak{S}_1^4 \\ \mathfrak{S}_2\times\mathfrak{S}_1^2 \\ \mathfrak{S}_2^2 \\ \mathfrak{S}_3\times\mathfrak{S}_1 \\ \mathfrak{S}_4 \end{array} &
\begin{bmatrix} 1 & 3 & 2 & 3 & 1 \\ & 1 & 1 & 2 & 1 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}
\end{array}.$$

Now we wish to verify Lusztig's statement [Lus84, 4.4.1] that every irreducible character of $\mathfrak{S}_n$ is equal to $\mathrm{J}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}(\mathrm{sgn})$, where $\mathrm{J}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n}$ denotes Lusztig's truncated induction. To do this we will need to know the decomposition of $\mathrm{sgn}_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_4$ for each partition $\lambda \vdash n$ of n. Now by [JL01, Exercise 4 - Chapter 21] we have that if G is a finite group, $H \leqslant G$ a subgroup of G, ψ a character of H and χ a character of G then

$$(\psi(\chi \downarrow H)) \uparrow G = (\psi \uparrow G)\chi.$$

Applying this to our situation we have

$$\mathrm{sgn}_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_n = (\mathrm{sgn}_{\mathfrak{S}_n} \downarrow \mathfrak{S}_\lambda) \uparrow \mathfrak{S}_n = (1_{\mathfrak{S}_\lambda}(\mathrm{sgn}_{\mathfrak{S}_n} \downarrow \mathfrak{S}_\lambda)) \uparrow \mathfrak{S}_n = (1_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_n)\, \mathrm{sgn}_{\mathfrak{S}_n}.$$

Therefore to decompose $\mathrm{sgn}_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_n$ for each partition $\lambda \vdash n$ we only have to multiply the decomposition of $1_{\mathfrak{S}_\lambda} \uparrow \mathfrak{S}_n$ by $\mathrm{sgn}_{\mathfrak{S}_n}$. However doing so will only permute the columns of the matrix M. This is because every irreducible representation χ_λ of $\mathfrak{S}_n$ is equal to $\mathrm{sgn}_{\mathfrak{S}_n} \chi_{\lambda'}$ where λ' is the dual partition to λ. Therefore we get our new decomposition matrix M' to be

$$
\begin{array}{c c}
 & \begin{array}{ccccc} (1^4) & (2,1^2) & (2^2) & (3,1) & (4) \end{array} \\
\begin{array}{c} \mathfrak{S}_1^4 \\ \mathfrak{S}_2\times\mathfrak{S}_1^2 \\ \mathfrak{S}_2^2 \\ \mathfrak{S}_3\times\mathfrak{S}_1 \\ \mathfrak{S}_4 \end{array} &
\left[\begin{array}{ccccc} 1 & 3 & 2 & 3 & 1 \\ 1 & 2 & 1 & 1 & \\ 1 & 1 & 1 & & \\ 1 & 1 & & & \\ 1 & & & & \end{array}\right]
\end{array}.
$$

Now we have the decomposition of the induced characters we need to consider which characters appear in the truncated induced character. To do this we will need to know the number a_{sgn} for the sgn representation of $\mathfrak{S}_\lambda$. Recall that the generic degree polynomial for a representation of a direct product of subgroups is given by the product of the generic degree polynomials for each component. We calculate the values a_{sgn} in the following table

$\mathfrak{S}_\lambda$	a_{sgn}
$\mathfrak{S}_1^4$	$0+0+0+0=0$
$\mathfrak{S}_2\times\mathfrak{S}_1^2$	$1+0+0=1$
$\mathfrak{S}_2^2$	$1+1=2$
$\mathfrak{S}_3\times\mathfrak{S}_1$	$3+0=3$
$\mathfrak{S}_4$	6

Therefore from this information it is clear to see that

$$
\mathrm{J}_{\mathfrak{S}_4}^{\mathfrak{S}_4}(\mathrm{sgn}) = \chi_{(1^4)} \quad \mathrm{J}_{\mathfrak{S}_3\times\mathfrak{S}_1}^{\mathfrak{S}_4}(\mathrm{sgn}) = \chi_{(2,1^2)} \quad \mathrm{J}_{\mathfrak{S}_2^2}^{\mathfrak{S}_4}(\mathrm{sgn}) = \chi_{(2^2)},
$$
$$
\mathrm{J}_{\mathfrak{S}_2\times\mathfrak{S}_1^2}^{\mathfrak{S}_4}(\mathrm{sgn}) = \chi_{(3,1)} \quad \mathrm{J}_{\mathfrak{S}_1^4}^{\mathfrak{S}_4}(\mathrm{sgn}) = \chi_{(4)}.
$$

Hence we have verified Lusztig's remark for $\mathfrak{S}_4$ and so every irreducible character lies in its own family in $\mathfrak{S}_4$.

References

[JK09] Gordon James and Adalbert Kerber. *The Representation Theory of The Symmetric Group*, volume 16 of *Encylopedia of Mathematics and its Applications*. Cambridge University Press, paperback reissue edition, 2009.

[JL01] Gordon James and Martin Liebeck. *Representations and Characters of Groups*. Cambridge University Press, second edition, 2001.

[Lus84] George Lusztig. *Characters of Reductive Groups Over a Finite Field*. Number 107 in Annals of Mathematics Studies. Princeton University Press, 1984.