1. Toric varieties

First some stuff about algebraic groups:

Definition 1.1. *Let G be a group. We say that G is an* ***algebraic group*** *if G is a quasi-projective variety and the two maps $m\colon G\times G \longrightarrow G$ and $i\colon G \longrightarrow G$, where m is multiplication and i is the inverse map, are both morphisms.*

It is easy to give examples of algebraic groups. Consider the group $G = \mathrm{GL}_n(K)$. In this case G is an open subset of $\mathbb{A}^{n^2}$, the complement of the zero locus of the determinant, which expands to a polynomial. Matrix multiplication is obviously a morphism, and the inverse map is a morphism by Cramer's rule. Note that there are then many obvious algebraic subgroups; the orthogonal groups, special linear group and so on. Clearly $\mathrm{PGL}_n(K)$ is also an algebraic group; indeed the quotient of an algebraic group by a closed normal subgroup is an algebraic group. All of these are affine algebraic groups.

Definition 1.2. *Let G be an algebraic group. If G is affine then we say that G is a* ***linear algebraic group****. If G is projective and connected then we say that G is an* ***abelian variety****.*

Note that any finite group is an algebraic group (both affine and projective). It turns out that any affine group is always a subgroup of a matrix group, so that the notation makes sense.

Definition 1.3. *The group $\mathbb{G}_m$ is $\mathrm{GL}_1(K)$. The group $\mathbb{G}_a$ is the subgroup of $\mathrm{GL}_2(K)$ of upper triangular matrices with one's on the diagonal.*

Note that as a group $\mathbb{G}_m$ is the set of units in K under multiplication and $\mathbb{G}_a$ is equal to K under addition, and that both groups are affine of dimension 1; in fact they are the only linear algebraic groups of dimension one, up to isomorphism.

Note that if we are given a linear algebraic group G, we get a Hopf algebra A. Indeed if A is the coordinate ring of G, then A is a K-algebra and there are maps

$$A \longrightarrow A\otimes A \qquad \text{and} \qquad A \longrightarrow A,$$

induced by the multiplication and inverse map for G.

It is not hard to see that the product of two algebraic groups is an algebraic group.

Definition 1.4. *The algebaic group $\mathbb{G}_m^k$ is called a* ***torus****.*

So a torus in algebraic geometry is just a product of copies of $\mathbb{G}_m$.

In fact one can define what it means to be a group scheme:

Definition 1.5. *Let $\pi\colon X \longrightarrow S$ be a morphism of schemes. We say that X is a **group scheme** over S, if there are three morphisms, $e\colon S \longrightarrow X$, $\mu\colon X \underset{S}{\times} X \longrightarrow X$ and $i\colon X \longrightarrow X$ over S which satisfy the obvious axioms.*

We can define a group scheme $\mathbb{G}_{m,\operatorname{Spec}\mathbb{Z}}$ over $\operatorname{Spec}\mathbb{Z}$, by taking

$$\operatorname{Spec}\mathbb{Z}[x,x^{-1}].$$

Given any scheme S, this gives us a group scheme $\mathbb{G}_{m,S}$ over S, by taking the fibre square. In the case when $S=\operatorname{Spec}k$, k an algebraically closed field, then $\mathbb{G}_{m,\operatorname{Spec}k}$ is $t(\mathbb{G}_m)$, the scheme associated to the quasi-projective variety $\mathbb{G}_m$. We will be somewhat sloppy and not be too careful to distinguish the two cases.

Similarly we can take

$$\mathbb{G}_{a,\operatorname{Spec}\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[x].$$

Definition 1.6. *Let G be an algebraic group and let X be a variety acted on by G, $\pi\colon G\times X \longrightarrow X$. We say that the action is **algebraic** if π is a morphism.*

For example the natural action of $\mathrm{PGL}_n(K)$ on $\mathbb{P}^n$ is algebraic, and all the natural actions of an algebraic group on itself are algebraic.

Definition 1.7. *We say that a quasi-projective variety X is a **toric variety** if X is irreducible and normal and there is a dense open subset U isomorphic to a torus, such that the natural action of U on itself extends to an action on the whole of X.*

For example, any torus is a toric variety. $\mathbb{A}^n_k$ is a toric variety. The natural torus is the complement of the coordinate hyperplanes and the natural action is as follows

$$((t_1,t_2,\dots,t_n),(a_1,a_2,\dots,a_n)) \longrightarrow (t_1a_1,t_2a_2,\dots,t_na_n).$$

More generally, $\mathbb{P}^n$ is a toric variety. The action is just the natural action induced from the action above. A product of toric varieties is toric.

One thing to keep track of are the closures of the orbits. For the torus there is one orbit. For affine space and projective space the closure of the orbits are the coordinate subspaces.

Definition 1.8. *Let M be a lattice and let $N=\operatorname{Hom}(M,\mathbb{Z})$ be the dual lattice.*

A ***strongly convex rational polyhedral cone*** $\sigma \subset N_{\mathbb{R}} = N \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ *is*

- *a cone, that is if* $v \in \sigma$ *and* $\lambda \in \mathbb{R}$, $\lambda \geq 0$ *then* $\lambda v \in \sigma$*;*
- *polyhedral, that is* σ *is the intersection of finitely many half spaces;*
- *rational, that is the half spaces are defined by equations with rational coefficients;*
- *strongly convex, that is* σ *contains no linear spaces other than the origin.*

A ***fan*** *in* N *is a set* F *of finitely many strongly convex rational polyhedra, such that*

- *every face of a cone in* F *is a cone in* F*, and*
- *the intersection of any two cones in* F *is a face of each cone.*

One can reformulate some of the parts of the definition of a strongly rational polyhedral cone. For example, σ is a polyhedral cone if and only if σ is the intersection of finitely many half spaces which are defined by homogeneous linear polynomials. σ is a strongly convex polyhedral cone if and only if σ is a cone over finitely many vectors (in other words a strongly convex polyhedral cone is the same as a cone over a polytope). And so on.

We will see that the set of toric varieties, up to isomorphism, are in bijection with fans, up to the action of $\mathrm{SL}(n, \mathbb{Z})$.

We first give the recipe of how to go from a fan to a toric variety. Suppose we start with σ. Form the dual cone

$$\check{\sigma} = \{\, v \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0, u \in \sigma \,\}.$$

Now take the integral points,

$$S_\sigma = \check{\sigma} \cap M.$$

Then form the (semi)group algebra,

$$A_\sigma = K[S_\sigma].$$

Finally form the affine variety,

$$U_\sigma = \operatorname{Spec} A_\sigma.$$

Given a semigroup S, to form the semigroup algebra $K[S]$, start with a K-vector space with basis χ^u, as u ranges over the elements of S. Given u and $v \in S$ define the product

$$\chi^u \cdot \chi^v = \chi^{u+v},$$

and extend this linearly to the whole of $K[S]$.

Example 1.9. *For example, suppose we start with $M = \mathbb{Z}^2$, σ the cone spanned by $(1,0)$ and $(0,1)$, inside $N_{\mathbb{R}} = \mathbb{R}^2$. Then $\check{\sigma}$ is spanned by the same vectors in $M_{\mathbb{R}}$. Therefore $S_\sigma = \mathbb{N}^2$, the group algebra is $\mathbb{C}[x,y]$ and so we get $\mathbb{A}^2$. Similarly if we start with the cone spanned by $e_1, e_2, \dots, e_n$ inside $N_{\mathbb{R}} = \mathbb{R}^n$ then we get $\mathbb{A}^n$.*

Now suppose we start with $\sigma = \{0\}$ in $\mathbb{R}$. Then $\check{\sigma}$ is the whole of $M_{\mathbb{R}}$, S_σ is the whole of $M = \mathbb{Z}^2$ and so $\mathbb{C}[M] = \mathbb{C}[x, x^{-1}]$. Taking Spec *we get the torus $\mathbb{G}_m$.*

More generally we always get a torus of dimension n if we take the origin in $\mathbb{R}^n$. Note that if $\tau \subset \sigma$ is a face then $\check{\sigma} \subset \check{\tau}$ is also a face so that $U_\sigma \subset U_\tau$ is an open subset. In fact

Lemma 1.10. *Let $\tau \subset \sigma \subset N_{\mathbb{R}}$ be a face of the cone σ.*

Then we may find $u \in S_\sigma$ such that

(1) $\tau = \sigma \cap u^\perp$,
(2)
$$S_\tau = S_\sigma + \mathbb{Z}^+(-u),$$
(3) *A_τ is a localisation of A_σ, and*
(4) *U_τ is a principal open subset of U_σ.*

Proof. The fact that every face of a cone is cut out by a hyperplane is a standard fact in convex geometry and this is (1). For (2) note that the RHS is contained in the LHS by definition of a cone. If $w \in S_\tau$ then $w + p \cdot u$ is in $\check{\sigma}$ for any p sufficiently large. If we take p to be also an integer this shows that w belongs to the RHS.

Let χ^u be the monomial corresponding to u. (2) implies that A_τ is the localisation of A_σ along χ^u. This is (3) and (4) is immediate from (3). □

Since the cone $\{0\}$ is a face of every cone, the affine scheme associated to a cone always contains a torus, which is then dense. In particular the affine scheme associated to a cone is always irreducible.

Definition 1.11. *Let $S \subset T$ be a subsemigroup of the semigroup T. We say that S is **saturated** in T if whenever $u \in T$ and $p \cdot u \in S$ for some positive integer p, then $u \in S$.*

Given a subsemigroup $S \subset M$ saturation is always with respect to M.

Lemma 1.12. *Suppose that $S \subset M$.*

Then $K[S]$ is integrally closed if and only if S is saturated.

Proof. Suppose that $K[S]$ is integrally closed.

Pick $u \in M$ such that $v = p \cdot u \in S$ for some positive integer p. Let $b = \chi^u$ and $a = \chi^v \in K[S]$. Then

$$b^p = \chi^{pu} = \chi^v = a,$$

so that b is a root of the monic polynomial $x^p - a \in K[S][x]$. As we are assuming that $K[S]$ is integrally closed this implies that $x \in K[S]$ which implies that $u \in S$. Thus S is saturated.

Now suppose that S is saturated. As $K[S] \subset K[M]$ and the latter is integrally closed, the integral closure L of $K[S]$ sits in the middle, $K[S] \subset L \subset K[M]$. The torus acts naturally on $K[M]$ and this action fixes $K[S]$, so that it also fixes L. L is therefore a direct sum of eigenspaces, which are all one dimensional (a set of commuting diagonalisable matrices are simultaneously diagonalisable) that is L has a basis of the form χ^u, as u ranges over some subset of M. It suffices to prove that $u \in S$.

Since $b = \chi^u$ is integral over $K[S]$, we may find $k_1, k_2, \dots, k_p \in K[S]$ such that

$$b^p + k_1 b^{p-1} + \cdots + k_p = 0.$$

We may assume that every term is in the same eigenspace as b^p. We may also assume that $k_p \neq 0$. As b^p and $k_0 \neq 0$ belong to the same eigenspace, which is one dimensional, we get $b^p \in K[S]$. Thus $pu \in S$ and so $u \in S$ as S is saturated. Thus $b \in K[S]$ and $K[S]$ is integrally closed. $\square$

Note that S_σ is automatically saturated, as $\check{\sigma}$ is a rational polyhedral cone. In particular U_σ is normal.

Example 1.13. *Suppose that we start with the semigroup S generated by 2 and 3 inside $M = \mathbb{Z}$. Then*

$$K[S] = K[t^2, t^3] = K[x, y]/\langle y^2 - x^3 \rangle.$$

Note that this does come with an action of $\mathbb{G}_m$;

$$(t, x, y) \longrightarrow (t^2 x, t^3 y).$$

However the curve $V(y^2 - x^3) \subset \mathbb{A}^2$ is not normal.

In fact some authorities drop the requirement that a toric variety is normal.

An action of the torus corresponds to a map of algebras

$$A_\sigma \longrightarrow A_\sigma \underset{K}{\otimes} A_0,$$

which is naturally the restriction of

$$A_0 \longrightarrow A_0 \underset{K}{\otimes} A_0.$$

It is straightforward to check that the restricted map does land in $A_\sigma \underset{K}{\otimes} A_0$.

Lemma 1.14 (Gordan's Lemma)**.** *Let $\sigma \subset \mathbb{M}_{\mathbb{R}}$ be a strongly convex rational cone.*

Then S_σ is a finitely generated semigroup.

Proof. Pick vectors $v_1, v_2, \dots, v_n \in S_\sigma$ which generate the cone $\check{\sigma}$. Consider the set

$$K = \{\, v \in M \mid v = \sum t_i v_i, t_i \in [0,1] \,\}.$$

Then K is compact. As M is discrete $K \cap M$ is finite. I claim that the elements of $K \cap M$ generate the semigroup S_σ. Pick $u \in S_\sigma$. Since $u \in \check{\sigma}$ and $v_1, v_2, \dots, v_n$ generate the cone, we may write

$$u = \sum \lambda_i v_i,$$

where $\lambda_i \in \mathbb{Q}$. Let $n_i = \llcorner \lambda_i \lrcorner$. Then

$$u - \sum n_i v_i = \sum (\lambda_i - \llcorner \lambda_i \lrcorner) v_i \in K \cap M.$$

As $v_1, v_2, \dots, v_n \in K \cap M$ the result follows. $\square$

Gordan's lemma (1.14) implies that U_σ is of finite type over K. So U_σ is an affine toric variety.

Example 1.15. *Suppose we start with the cone spanned by e_2 and $2e_1 - e_2$. The dual cone is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the semigroup are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. The corresponding group algebra is $A_\sigma = K[x, xy, xy^2]$. Suppose we call $u = x$, $v = xy$ and $w = xy^2$. Then $v^2 = x^2y^2 = x(xy^2) = uw$. Therefore the corresponding affine toric variety is given as the zero locus of $v^2 - uw$ in $\mathbb{A}^3$.*

Given a fan F, we get a collection of affine toric varieties, one for every cone of F. It remains to check how to glue these together to get a toric variety. Suppose we are given two cones σ and τ belonging to F. The intersection is a cone ρ which is also a cone belonging to F. Since ρ is a face of both σ and τ there are natural inclusions

$$U_\rho \subset U_\sigma \qquad \text{and} \qquad U_\rho \subset U_\tau.$$

We glue U_σ to U_τ using the natural identification of the common open subset U_ρ. Compatibility of gluing follows automatically from the fact that the identification is natural and from the combinatorics of the fan.

It is clear that the resulting scheme is of finite type over the groundfield. Separatedness follows from:

Lemma 1.16. *Let σ and τ be two cones whose intersection is the cone ρ.*

If ρ is a face of each then the diagonal map

$$U_\rho \longrightarrow U_\sigma \times U_\tau,$$

is a closed embedding.

Proof. This is equivalent to the statement that the natural map

$$A_\sigma \otimes A_\tau \longrightarrow A_\rho,$$

is surjective. For this, one just needs to check that

$$S_\rho = S_\sigma + S_\tau.$$

One inclusion is easy; the RHS is contained in the LHS. For the other inclusion, one needs a standard fact from convex geometry, which is called the separation lemma: there is a vector $u \in S_\tau \cap S_{-\tau}$ such that simultaneously

$$\rho = \sigma \cap u^\perp \qquad \text{and} \qquad \rho = \tau \cap u^\perp.$$

By the first equality $S_\rho = S_\sigma + \mathbb{Z}(-u)$. As $u \in S_{-\tau}$ we have $-u \in S_\tau$ and so the LHS is contained in the RHS. □

So we have shown that given a fan F we can construct a normal variety $X = X(F)$. It is not hard to see that the natural action of the torus corresponding to the zero cone extends to an action on the whole of X. Therefore $X(F)$ is indeed a toric variety.

Let us look at some examples.

Example 1.17. *Suppose that we start with $M = \mathbb{Z}$ and we let F be the fan given by the three cones $\{0\}$, the cone spanned by e_1 and the cone spanned by $-e_1$ inside $N_\mathbb{R} = \mathbb{R}$. The two big cones correspond to $\mathbb{A}^1$. We identify the two $\mathbb{A}^1$'s along the common open subset isomorphic to K^*. Now the first $\mathbb{A}^1 = \operatorname{Spec} K[x]$ and the second is $\mathbb{A}^1 = \operatorname{Spec} K[x^{-1}]$. So the corresponding toric variety is $\mathbb{P}^1$ (if we have homogeneous coordinates $[X : Y]$ on $\mathbb{P}^1$ coordinates on U_0 are $x = X/Y$ and $y = Y/X = 1/x$).*

Now suppose that we start with three cones in $N_\mathbb{R} = \mathbb{R}^2$, σ_1, σ_2 and σ_3. We let σ_1 be the cone spanned by e_1 and e_2, σ_2 be the cone spanned by e_2 and $-e_1 - e_2$ and σ be the cone spanned by $-e_1 - e_2$ and e_1. Let F be the fan given as the faces of these three cones. Note that the three affine varieties corresponding to these three cones are all copies of $\mathbb{A}^2$. Indeed, any two of the vectors, e_1, e_2 and $-e_1 - e_2$ are a basis not *only of the underlying vector space but they also generate the standard lattice. We check how to glue two such copies of* $\mathbb{A}^2$.

The dual cone of σ_1 *is the cone spanned by* f_1 *and* f_2 *in* $M_{\mathbb{R}} = \mathbb{R}^2$. *The dual cone of* σ_2 *is the cone spanned by* $-f_1$ *and* $-f_1 + f_2$. *So we have* $U_1 = \operatorname{Spec} K[x, y]$ *and* $U_2 = \operatorname{Spec} K[x^{-1}, x^{-1}y]$. *On the other hand, if we start with* $\mathbb{P}^2$ *with homogeneous coordinates* $[X : Y : Z]$ *and the two basic open subsets* $U_0 = \operatorname{Spec} K[Y/X, Z/X]$ *and* $U_1 = \operatorname{Spec} K[X/Y, Z/Y]$, *then we get the same picture, if we set* $x = Y/X$, $y = Z/X$ *(since then* $X/Y = x^{-1}$ *and* $Z/Y = Z/X \cdot X/Y = yx^{-1}$*). With a little more work one can check that we have* $\mathbb{P}^2$.

More generally, suppose we start with $n+1$ *vectors* $v_1, v_2, \dots, v_{n+1}$ *in* $N_{\mathbb{R}} = \mathbb{R}^n$ *which sum to zero such that the first* n *vectors* $v_1, v_2, \dots, v_n$ *span the standard lattice. Let* F *be the fan obtained by taking all the cones spanned by all subsets of at most* n *vectors. One can check that the resulting toric variety is* $\mathbb{P}^n$.

Now suppose that we take the four vectors e_1, e_2, $-e_1$ *and* $-e_2$ *in* $N_{\mathbb{R}} = \mathbb{R}^2$ *and let* F *be the fan consisting of all cones spanned by at most two vectors. Then we get four copies of* $\mathbb{A}^2$. *It is easy to check that the resulting toric variety is* $\mathbb{P}^1 \times \mathbb{P}^1$. *Indeed the top two fans glue together to get* $\mathbb{P}^1 \times \mathbb{A}^1$ *and so on.*

We have already seen that cones correspond to open subsets. In fact cones also correspond (in some sort of dual sense) to closed subsets, the closure of the orbits. First observe that given a fan F, we can associate a closed point x_σ to any cone σ. To see this, observe that one can spot the closed points of U_σ using semigroups:

Lemma 1.18. *Let* $S \subset M$ *be a semigroup. Then there is a natural bijection,*

$$\operatorname{Hom}(K[S], K) \simeq \operatorname{Hom}(S, K).$$

Here the RHS is the set of semigroup homomorphisms, where $K = \{0\} \cup K^*$ *is the multiplicative subsemigroup of* K *(and not the additive).*

Proof. Suppose we are given a ring homomorphism

$$f\colon K[S] \longrightarrow K.$$

Define

$$g\colon S \longrightarrow K,$$

by sending u to $f(\chi^u)$. Conversely, given g, define $f(\chi^u) = g(u)$ and extend linearly. □

Consider the semigroup homomorphism:

$$S_\sigma \longrightarrow \{0, 1\},$$

where $\{0,1\} \subset \{0\} \cup K^*$ inherits the obvious semigroup structure. We send $u \in S_\sigma$ to 1 if $u \in \sigma^\perp$ and send it 0 otherwise. Note that as $\sigma^\perp$ is a face of $\check{\sigma}$ we do indeed get a homomorphism of semigroups. By (1.18) we get a surjective ring homomorphism

$$K[S_\sigma] \longrightarrow K.$$

The kernel is a maximal ideal of $K[S_\sigma]$, that is a closed point x_σ of U_σ, with residue field K.

To get the orbits, take the orbits of these points. It follows that the orbits are in correspondence with the cones in F. In fact the correspondence is inclusion reversing.

Example 1.19. *For the fan corresponding to $\mathbb{P}^1$, the point corresponding to $\{0\}$ is the identity, and the points corresponding to e_1 and $-e_1$ are 0 and ∞. For the fan corresponding to $\mathbb{P}^2$ the three maximal cones give the three coordinate points, the three one dimensional cones give the three coordinate lines (in fact the lines spanned by the points corresponding to the two maximal cones which contain them). As before the zero cone corresponds to the identity point. The orbit is the whole torus and the closure is the whole of $\mathbb{P}^2$.*

Suppose that we start with the cone σ spanned by e_1 and e_2 inside $N_\mathbb{R} = \mathbb{R}^2$. We have already seen that this gives the affine toric variety $\mathbb{A}^2$. Now suppose we insert the vector $e_1 + e_2$. We now get two cones σ_1 and σ_2, the first spanned by e_1 and $e_1 + e_2$ and the second spanned by $e_1 + e_2$ and e_2. Individually each is a copy of $\mathbb{A}^2$. The dual cones are spanned by f_2, $f_1 - f_2$ and f_1 and $f_2 - f_1$. So we get $\operatorname{Spec} K[y, x/y]$ and $\operatorname{Spec} K[x, x/y]$.

Suppose that we blow up $\mathbb{A}^2$ at the origin. The blow up sits inside $\mathbb{A}^2 \times \mathbb{P}^1$ with coordinates (x, y) and $[S : T]$ subject to the equations $xT = yS$. On the open subset $T \neq 0$ we have coordinates s and y and $x = sy$ so that $s = x/y$. On the open subset $S \neq 0$ we have coordinates x and t and $y = xt$ so that $t = y/x$. So the toric variety above is nothing more than the blow up of $\mathbb{A}^2$ at the origin. The central ray corresponds to the exceptional divisor E, a copy of $\mathbb{P}^1$.

Definition 1.20. *Let G and H be algebraic groups which act on quasi-projective varieties X and Y. Suppose we are given an algebraic group homomorphism, $\rho\colon G \longrightarrow H$. We say that a morphism $f\colon X \longrightarrow Y$ is ρ-**equivariant** if f commutes with the action of G and H. If X and Y are toric varieties and G and H are the tori contained in X and Y then we say that f is a **toric morphism**.*

It is easy to see that the morphism defined above is toric. We can extend this picture to other toric surfaces. First a more intrinsic description of the blow up. Suppose we are given a ray, that is a one dimensional cone σ. Then we can describe σ by specifying the unique integral vector $v \in \sigma$ which is closest to the origin. Note that every other integral vector belonging to σ is a positive integral multiple of v, which we call the **primitive generator** v. Suppose we are given a toric surface and a two dimensional cone σ such that the primitive generators v and w of the two one dimensional faces of σ generate the lattice (so that up the action of $\mathrm{SL}(2, \mathbb{Z})$, σ is the cone spanned by e_1 and e_2). Then the blow up of the point corresponding to σ is a toric surface, which is obtained by inserting the sum $v + w$ of the two primtive generators and subdividing σ in the obvious way (somewhat like the barycentric subdivision in simplicial topology).

Example 1.21. *Suppose we start with $\mathbb{P}^2$ and the standard fan. If we insert the two vectors $-e_1$ and $-e_2$ this corresponds to blowing up two invariant points, say $[0 : 1 : 0]$ and $[0 : 0 : 1]$. Note that now $-e_1 - e_2$ is the sum of $-e_1$ and $-e_2$. So if we remove this vector this is like blowing down a copy of $\mathbb{P}^1$. The resulting fan is the fan for $\mathbb{P}^1 \times \mathbb{P}^1$.*

We can generalise this to higher dimensions. For example suppose we start with the standard cone for $\mathbb{A}^3$ spanned by e_1 and e_2 and e_3. If we insert the vector $e_1 + e_2 + e_3$ (thereby creating three maximal cones) this corresponds to blowing up the origin. (In fact there is a simple recipe for calculating the exceptional divisor; mod out by the central $e_1 + e_2 + e_3$; the quotient vector space is two dimensional and the three cones map to the three cones in the quotient two dimensional vector space which correspond to the fan for $\mathbb{P}^2$). Suppose we insert the vector $e_1 + e_2$. Then the exceptional locus is $\mathbb{P}^1 \times \mathbb{A}^1$. In fact this corresponds to blowing up one of the axes (the axis is a copy of $\mathbb{A}^1$ and over every point of the axis there is a copy of $\mathbb{P}^1$).

2. Some naive enumerative geometry

Question 2.1. *How many lines meet four fixed lines in $\mathbb{P}^3$?*

Let us first check that this question makes sense, that is let us first check that the answer is finite.

Definition 2.2. $\mathbb{G}(k, n)$ *denotes the space of r-dimensional linear subspaces of $\mathbb{P}^n$.*

We will assume that we have constructed the **Grassmannian** as a variety. The first natural question then is to determine the dimension of $\mathbb{G}(1, 3)$. We do so in an ad hoc manner. A line l in $\mathbb{P}^3$ is specified by picking two points p and q. Now the set of choices for two points p and q is equal to $\mathbb{P}^3 \times \mathbb{P}^3 - \Delta$, where Δ is the diagonal. Thus the set of choices of pairs of distint points is six dimensional.

Fix a line l. Then if we pick any two points p and q of this line, they give us the same line l. Thus the the Grassmannian of lines in $\mathbb{P}^3$ is $6 = 2 = 4$-dimensional.

It might help to look at this differently. Let

$$\Sigma = \{\, (P, l) \mid P \in l \,\} \subset \mathbb{G}(1, 3) \times \mathbb{P}^3.$$

Then Σ is a closed subset of the product $\mathbb{G}(1, 3) \times \mathbb{P}^3$. There are two natural projection maps.

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^3 \\ \mathbb{G}(1,3). & \end{tikzcd}$$

In fact Σ (together with this diagram) is called an **incidence correspondence**. It is interesting to consider the two morphisms p and q. First q. Pick a line $l \in \mathbb{G}(1, 3)$. Then the fibre of p over l consists of all points P that are contained in l, so that the fibres of p are all isomorphic to $\mathbb{P}^1$. Now consider the morphism q. Fix a point P. Then the fibre of q over P consists of all lines that contain P. Again the fibres of q are isomorphic.

To compute the dimension of $\mathbb{G}(1, 3)$, we compute the dimension of Σ in two ways, borrowing the following result from later in the class.

Theorem 2.3. *Let $\pi \colon X \longrightarrow Y$ be a morphism of varieties.*

Then there is an open subset U of Y, such that for every $y \in U$, the dimension of the fibre of π over is equal to k, a constant. Moreover the dimension of X is equal to the dimension of Y plus k.

We first apply (2.3) to q. The dimension of the base is 3. As every fibre is isomorphic, to compute k, we can consider any fibre. Pick any point P. Pick an auxiliary plane, not passing through P. Then the set of lines containing P is in bijection with the points of this auxiliary plane, so that the dimension of a fibre is two. Thus the dimension of $\Sigma = 3 + 2 = 5$.

Now we apply (2.3) to p. The dimension of any fibre is one. Thus the dimension of the Grassmannian is $5 - 1 = 4$, as before.

The next question is to determine how many conditions it is to meet a fixed line l_1. I claim it is one condition. The easiest way to see this, is to just to imagine swinging a sword around in space. This will cut any line into two. Thus any one dimensional family of lines meets a given line in finitely many points.

More formally, carry out the computation above, replacing Σ with Σ_1, the space of lines which meet l_1. The fibre of q over a point P is now a copy of $\mathbb{P}^1$ (parametrised by l_1). Thus Σ_1 has dimension 4 and the space of lines which meets l_1 has dimension $4 - 1 = 3$.

Thus we have a threefold in the fourfold $\mathbb{G}(1,3)$. Clearly we expect that four of them will intersect in a finite set of points. There are two ways to proceed. Here is one which uses the Segre variety:

Lemma 2.4. *Let l_1, l_2 and l_3 and m_1, m_2 and m_3 be two sequences of skew lines in $\mathbb{P}^3$.*

Then there is an element of $\mathrm{PGL}_4(K)$ carrying the first sequence to the second.

Proof. We may as well assume that the first set is given as

$$X = Y = 0 \qquad\qquad Z = W = 0 \qquad \text{and} \qquad X - Z = Y - W = 0.$$

Clearly we may find a transformation carrying m_1 to l_1 and m_2 to l_2. For example, pick four points on both sets of lines, and use the fact that any four sets of points in linear general position are projectively equivalent.

Consider the two planes $X = 0$ and $Y = 0$. m_3 cannot lie in either of these planes, else either the lines m_1 and m_3 or the lines m_1 and m_3 would not be skew. Consider the two points $[0 : a : b : c]$ and $[d : e : f : 0]$ where m_3 intersects the planes $X = 0$ and $Y = 0$.

Clearly m_3 is determined by these points, and in the case of l_3, we may take $a = c = d = f = 1$, $b = e = 0$. Pick an element $\phi \in \mathrm{PGL}_4(K)$ and represent it as a 4×4 matrix. If we decompose this 4×4 matrix in block form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where each block is a 2×2 matrix, the condition that ϕ fix l_1 and l_2 is equivalent to the condition that $B = C = 0$. By choosing A and D appropriately, we reduce to the case that $b = d = 0$. In this case the matrix

$$\begin{pmatrix} 1/d & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \\ 0 & 0 & 1/f & 0 \\ 0 & 0 & 0 & 1/c \end{pmatrix}$$

carries the two points to the standard two points, so that it carries m_3 to l_3. □

This result has the following surprising consequence.

Lemma 2.5. *Let l_1, l_2 and l_3 be three skew lines in $\mathbb{P}^3$.*

Then the family of lines that meets all three lines sweeps out a quadric surface in $\mathbb{P}^3$.

Proof. By (2.4) we may assume that the three lines are any set of three skew lines in $\mathbb{P}^3$. Now the Segre variety V in $\mathbb{P}^1 \times \mathbb{P}^1$ contains three skew lines (just choose any three lines of one of the rulings). Moreover any line of the other ruling certainly meets all three lines. So the set of lines meeting all three lines, certainly sweeps out at least a quadric surface.

To finish, suppose we are given a line l that meets l_1, l_2 and l_3. Then l meets V in three points. As V is defined by a quadratic polynomial, it follows that l is contained in V. Thus any line that meets all three lines, is contained in V. □

Theorem 2.6. *There are two lines that meet four general skew lines in $\mathbb{P}^3$.*

Proof. Fix the first three lines l_1, l_2 and l_3. We have already seen that the set l of lines that meets all three of these lines is precisely the set of lines of one ruling of the Segre variety (up to choice of coordinates).

Pick a line l_4 that meets V transversally in two points. Now for a line l of one ruling to meet the fourth line l_4, it must meet l_4 at a point $P = l \cap l_4$ of V. Moreover this point determines the line l. □

Here is an entirely different way to answer (2.1). Consider using the principle of continuity. Take two of the four lines and deform them so they become concurrent (or what comes to exactly the same thing, coplanar). Similarly take the other pair of lines and degenerate them until they also become concurrent.

Now consider how a line l can meet the four given lines.

Lemma 2.7. *Let l be a line that meets two concurrent lines l_1 and l_2 in $\mathbb{P}^3$.*

Then either l contains $l_1 \cap l_2$ or l is contained in the plane $\langle l_1, l_2 \rangle$.

Proof. Suppose that l does not contain the point $l_1 \cap l_2$. Then l meets l_i, $i = 1$, 2 at two points p_i contained in the plane $\langle l_1, l_2 \rangle$. □

Thus if l meets all four lines, there are three possibilties.

(1) l contains both points of intersection.
(2) l is contained in both planes.
(3) l contains one point and is contained in the other plane.

Clearly there is only one line that satisfies (1). It is not so hard to see that there is only one line that satisfies (2), it is the intersection of the two planes. Finally it is not so hard to see that (3) is impossible. Just choose the point outside of the plane.

Thus the answer is two. It is convenient to introduce some notation to compute these numbers, which is known as Schubert calculus. Let l denote the condition that we meet a fixed line. We want to compute l^4. We proceed formally. We have already seen that

$$l^2 = l_p + l_\pi$$

where l_p denotes the condition that a line contains a point, and l_π is the condition that l is contained in π.

Thus

$$\begin{aligned} l^4 &= (l^2)^2 \\ &= (l_p + l_\pi)^2 \\ &= l_p^2 + 2l_p l_\pi + l_\pi \\ &= 1 + 2 \cdot 0 + 1 = 2, \end{aligned}$$

where the last line is computed as before.

3. Grasmannians

We first treat Grassmanians classically. Fix an algebraically closed field K. We want to parametrise the space of k-planes W in a vector space V. The obvious way to parametrise k-planes is to pick a basis $v_1, v_2, \dots, v_k$ for W. Unfortunately this does not specify W uniquely, as the same vector space has many different bases. However, the line spanned by the vector

$$\omega = v_1 \wedge v_2 \wedge \dots \wedge v_k \in \bigwedge^k V,$$

is invariant under re-choosing a basis.

Definition 3.1. *The* ***Grassmannian*** $G(k, V)$ ***of*** k***-planes in*** V *is the set of rank one vectors in* $\mathbb{P}(\bigwedge^k V)$.

We set $G(k, n) = G(k, K^n)$ and $\mathbb{G}(k, n) = G(k+1, n+1)$. The latter may be thought of as the set of k-planes in $\mathbb{P}^n$.

Lemma 3.2. *The Grassmannian is a closed subset of* $\mathbb{P}^N$.

Proof. Consider the rational map

$$\prod^k \mathbb{P}(V) \dashrightarrow \mathbb{P}(\bigwedge^k V),$$

which sends $([v_1], [v_2], \dots [v_k])$ to $[v_1 \wedge v_2 \wedge \dots \wedge v_k]$. The image (that is, take the image of the graph) is the Grassmannian and the image under a rational map is closed. $\square$

The embedding of the Grassmannian inside $\mathbb{P}(\bigwedge^k V)$ is known as the Plücker embedding. If we choose a basis $e_1, e_2, \dots, e_n$ for V, then a general element of $\bigwedge^k V$ is given by

$$\sum_I p_I e_I,$$

where I ranges over all collections of increasing sequences of integers between 1 and n,

$$i_1 < i_2 < \dots < i_k,$$

and e_I is shorthand for the wedge of the corresponding vectors,

$$e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}.$$

The coefficients p_I are naturally coordinates on $\mathbb{P}(\bigwedge^k V)$, which are known as the Plücker coordinates.

There is another way to look at the construction of the Grassmannian which is very instructive. If we pick a basis $e_1, e_2, \dots, e_n$ for V, then let A be the $k \times n$ matrix whose rows are $v_1, v_2, \dots, v_k$, in this basis.

As before, this matrix does not uniquely specify $W \subset V$, since we could pick a new basis for W. However the operation of picking a new basis corresponds to taking linear combinations of the rows of our matrix, which in turn is the same as multiplying our matrix by a $k \times k$ invertible matrix on the left. In other words the Grassmannian is the set of equivalence classes of $k \times n$ matrices under the action of $\mathrm{GL}_k(K)$ by multiplication on the left.

It is not hard to connect the two constructions. Given the matrix A, then form all possible $k \times k$ determinants. Any such determinant is determined by specifying the columns to pick, which we indicate by a multindex I. In terms of $\bigwedge^k V$, this is the same as picking a basis and expanding our vector as a sum

$$\sum_I p_I e_I,$$

where, as before, e_I is the wedge of the corresponding vectors. For example consider the case $k = 2$, $n = 4$ (lines in $\mathbb{P}^3$). We have a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

The corresponding plane is given as the span of the rows. We can form six two by two determinants. Clearly these are invariant, up to scalars, under the action of $\mathrm{GL}_2(K)$.

The Grassmannian has a natural cover by open affine subsets, isomorphic to affine space, in much the same way that projective space has a cover by open affines, isomorphic to affine space. Pick a linear space U of dimension $n - k$, and consider the set of linear spaces W of dimension k which are complementary to U, that is, which meet U only at the origin. Identify V with the sum $V/U + U$. Then a linear space W complementary to U can be identified with the graph of a linear map

$$V/U \longrightarrow U.$$

It follows that the subset of all linear spaces W complementary to U is equal to

$$\mathrm{Hom}(V/U, U) \simeq K^{k(n-k)} \simeq \mathbb{A}_K^{k(n-k)}.$$

Another way to see this is as follows. Consider the first $k \times k$ minor. Suppose that the corresponding determinant is non-zero, that is the corresponding vectors are independent. In this case the $k \times k$ minor is equivalent to the identity matrix, and the only element of $\mathrm{GL}_k(K)$ which fixes the identity, is the identity itself. Thus we have a canonical representative of the matrix A for the linear space W. We are free to choose the other $k \times (n - k)$ block of the matrix, which gives us an affine space of dimension $k(n-k)$. The condition that the first $k \times k$ minor has non-zero determinant is an open condition, and this gives us an open affine cover by affine spaces of dimension $k(n-k)$. Note that the condition that the first $k \times k$ minor is invertible is equivalent to the condition that we do not meet the space given by the vanishing of the first k coordinates, which is indeed a linear space of dimension $n-k$.

It is interesting to write down the equations cutting out the image of the Grassmannian under the Plücker embedding, although this turns out to involve some non-trivial multilinear algebra. The problem is to characterise the set of rank one vectors ω in $\bigwedge^k V$.

Definition 3.3. *Let $\omega \in \bigwedge^k V$. We say that ω is* ***divisible*** *by $v \in V$ if there is an element $\phi \in \bigwedge^k V$ such that $\omega = \phi \wedge v$.*

Lemma 3.4. *Let $\omega \in \bigwedge^k V$.*

Then ω is divisible by v iff $\omega \wedge v = 0$.

Proof. This is easy. If $\omega = \phi \wedge v$, then

$$\begin{aligned} \omega \wedge v &= \phi \wedge v \wedge v \\ &= 0. \end{aligned}$$

To see the other direction, extend v to a basis $v_1 = e_1, e_2, \dots, e_n$ of V. Then we may expand ω in this basis.

$$\omega = \sum p_I e_I.$$

On the other hand

$$e_I \wedge v = \begin{cases} e_J & \text{if } 1 \notin I \text{, where } J = \{1\} \cup I \\ 0 & \text{if } 1 \in I. \end{cases}$$

Thus $\omega \wedge v = 0$ iff $p_I \neq 0$ implies $1 \in J$ iff v divides ω. □

Lemma 3.5. *Let $\omega \in \bigwedge^k V$.*

Then ω has rank one iff the linear map

$$\phi(\omega)\colon V \longrightarrow \bigwedge^{k+1} V \qquad\qquad v \longrightarrow \omega \wedge v,$$

has rank at most $n-k$.

Proof. Indeed $\phi(\omega)$ has rank at most $n-k$ iff the linear subspace of vectors dividing ω has dimension at least k iff ω has rank one. □

Now the map

$$\phi\colon \bigwedge^k V \longrightarrow \mathrm{Hom}(V, \bigwedge^{k+1} V),$$

is clearly linear. Thus the map ϕ can be interpreted as a matrix whose entries are linear coordinates of $\bigwedge^k V$ and the locus we want is given by the vanishing of the $(n-k+1) \times (n-k+1)$ minors.

Unfortunately the equations we get in this way won't be best possible. In particular they won't generate the ideal of the Grassmannian (they only cut out the Grassmannian set theoretically). To find equations that generate the ideal, we have to work quite a bit harder.

Lemma 3.6. *There is a natural pairing between $\bigwedge^k V$ and $\bigwedge^{n-k} V^*$. This pairing is well-defined up to scalars and preserves the rank.*

Proof. There is a natural pairing

$$\bigwedge^{k} V \times \bigwedge^{n-k} V \longrightarrow \bigwedge^{n} V,$$

which sends

$$(\omega, \eta) \longrightarrow \omega \wedge \eta.$$

On the other hand, $\bigwedge^n V$ is one dimensional so that it is non-canonically isomorphic to K and $(\bigwedge^{n-k} V)^*$ is isomorphic to $\bigwedge^{n-k} V^*$. $\square$

Given ω, let ω^* be the corresponding element of $\bigwedge^{n-k} V^*$. Now there is a natural map

$$\psi(\omega^*) \colon V^* \longrightarrow \bigwedge^{n-k+1} V^*$$

which sends

$$v^* \longrightarrow \omega^* \wedge v^*.$$

Further ω has rank one iff ω^* has rank one, which occurs if and only if $\psi(\omega^*)$ has rank k.

Moreover the kernel of $\phi(\omega)$, namely W, is precisely the annihilator of the kernel of $\psi(\omega^*)$. Dualising, we get maps

$$\phi^*(\omega) \colon \bigwedge^{k+1} V^* \longrightarrow V^* \qquad \text{and} \qquad \psi^*(\omega) \colon \bigwedge^{n-k+1} V \longrightarrow V,$$

whose images annihilate each other.

Thus ω has rank one iff for every $\alpha \in \bigwedge^{k+1} V^*$ and $\beta \in \bigwedge^{n-k+1} V^*$,

$$\Xi_{\alpha,\beta}(\omega) = \langle \phi^*(\omega)(\alpha), \psi^*(\omega)(\beta) \rangle = 0.$$

Now $\Xi_{\alpha,\beta}$ are quadratic polynomials, which are known as the Plücker relations. It turns out that they do indeed generate the ideal of the Grassmannian.

It is interesting to see what happens when $k = 2$:

Lemma 3.7. *Let $\omega \in \bigwedge^2 V$.*

Then ω has rank one iff $\omega \wedge \omega = 0$.

Proof. One direction is clear, in fact for every k, if ω has rank one then $\omega \wedge \omega = 0$.

To see the other direction, we need to prove that if ω has rank at least two, then $\omega \wedge \omega \neq 0$. First observe that if ω has rank at least two, then we may find a projection down to a vector space of dimension four, such that the image has rank two. Thus we may assume that V has dimension four and ω has rank two. In this case, up to change of coordinates,

$$\omega = e_1 \wedge e_2 + e_3 \wedge e_4,$$

and by direct computation, $\omega \wedge \omega$ is not zero. $\square$

Now

$$\omega = \sum_{i,j} p_{i,j} e_i \wedge e_j.$$

Suppose that $n = 4$. If we expand

$$\omega \wedge \omega,$$

then everything is a multiple of $e_1 \wedge e_2 \wedge e_3 \wedge e_4$. We need to pick a term from each bracket, so that the union is $\{1, 2, 3, 4\}$. In other words, the coefficient of the expansion is a sum over all partitions of $\{1, 2, 3, 4\}$ into two equal parts. By direct computation, we get

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23}.$$

In particular, $\mathbb{G}(1, 3)$ is a quadric in $\mathbb{P}^5$, of maximal rank. Unfortunately this also makes it clear that the Grassmannian is not a toric variety (if it were, it would be defined by a binomial, not a trinomial). It turns out that the Grassmannian is close to a toric variety (it is a spherical variety). In fact the algebraic group $\mathrm{GL}_n(V)$ acts transitively on $G(k, V)$. The stabiliser subgroup H of the k-plane $W \subset V$ spanned by the first k vectors is given by invertible matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

So

$$G(k, V) = \mathrm{GL}_n(V)/H.$$

As with the space of conics in $\mathbb{P}^2$, the main point of the Grassmannian, is that it comes with a universal family. We first investigate what this means in the baby case of quasi-projective varieties before we move on to the more interesting case of schemes.

Definition 3.8. *A* ***family of*** k***-planes in*** $\mathbb{P}^n$ ***over*** B *is a closed subset* $\Sigma \subset B \times \mathbb{P}^n$ *such that the fibres, under projection to the first factor, are identified with* k*-planes in* $\mathbb{P}^n$*.*

Definition 3.9. *Let F be the functor from the category of varieties to the category of sets, which assigns to every variety, the set of all (flat) families of k-planes in $\mathbb{P}^n$, up to isomorphism.*

Theorem 3.10. *The Grassmannian $\mathbb{G}(k,n)$ represents the functor F.*

It might help to unravel some of the definitions. Suppose that we are given a variety B. Essentially we have to show that there is a natural bijection of sets,

$$F(B) = \operatorname{Hom}(B, \mathbb{G}(k,n)).$$

The set on the left is nothing more than the set of all families of k-planes in $\mathbb{P}^n$, with base B. In particular given a morphism $f\colon B \longrightarrow \mathbb{G}(k,n)$, we are supposed to produce a family of k-planes over B. Here is how we do this. Suppose that we have constructed the natural family of k-planes over $\mathbb{G}(k,n)$,

$$\begin{tikzcd} \Sigma \arrow[r, hook] & \mathbb{G}(k,n) \times \mathbb{P}^n \arrow[d] \\ & \mathbb{G}(k,n), \end{tikzcd}$$

so that the fibre over $[\Lambda] \in \mathbb{G}(k,n)$ is exactly the set,

$$\{[\Lambda]\} \times \Lambda \subset \{[\Lambda]\} \times \mathbb{P}^n$$

that is, the k-plane Λ sitting inside $\mathbb{P}^n$. Then we obtain a family of k-planes over B, simply by taking the fibre square,

$$\begin{tikzcd} \Sigma' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma \arrow[d] \\ B \arrow[r, "f"] & \mathbb{G}(k,n). \end{tikzcd}$$

For this reason, we call the family $\Sigma \longrightarrow \mathbb{G}(k,n)$ the universal family. Note that we can reverse this process. Suppose that $\mathbb{G}(k,n)$ represents the functor F. By considering the identity morphism $\mathbb{G}(k,n) \longrightarrow \mathbb{G}(k,n)$, we get a family $\Sigma \longrightarrow \mathbb{G}(k,n)$, which is universal, in the sense that to obtain any other family, over any other base, we simply pullback Σ along the morphism $f\colon B \longrightarrow \mathbb{G}(k,n)$, whose existence is guaranteed by the universal property of $\mathbb{G}(k,n)$ (that is, that it represents the functor). To summarise the previous discussion: to prove (3.10) it suffices to construct the natural family over $\mathbb{G}(k,n)$ and prove that it is the universal family.

We won't prove (3.10) here. We will simply observe that the natural family exists, without proving that it is in fact also universal. Recall that the Grasmmannian is by definition the set of all rank one elements ω of $\bigwedge^{k+1} K^{n+1}$. The universal family is then the set

$$\{\,(\omega, v) \in \bigwedge^{k+1} V \times V \,|\, \omega \wedge v = 0\,\},$$

which is easily seen to be algebraic.

Before we go deeper into the geometry of the Grassmannian, it is interesting to note that the space of conics satisfies the same universal property. Suppose $\mathbb{P}^2 = \mathbb{P}(V)$. Then $\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2(V^*))$ represents the functor G which assigns to every variety B, the set of all (flat) families of conics in $\mathbb{P}^2$, over B. As before the key thing is to show that the natural family of conics in $\mathbb{P}^2$ over $\mathbb{P}^5$, is in fact a universal family. As before we won't show that the natural family is universal, but we observe that the natural family does exist. Indeed,

$$aX^2 + bY^2 + cZ^2 + dYZ + eXZ + fYZ,$$

is bihomogeneous of degree $(1, 2)$ and cuts out the natural family.

Using the diagram,

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^n \\ \mathbb{G}(k,n), & \end{tikzcd}$$

one can make some interesting constructions. For example, suppose we are given a closed subset $X \subset \mathbb{P}^n$. Then $p(q^{-1}(X))$ is a closed subvariety of $\mathbb{G}(k, n)$, consisting of all k-planes in $\mathbb{P}^n$ which intersect X. The first interesting case is that of a curve C in $\mathbb{P}^3$. In this case the general line does not meet the curve C. In fact we get a codimension one subvariety of $\mathbb{G}(1, 3)$. Conversely suppose we are given a closed subvariety Φ of $\mathbb{G}(k, n)$. Then $q(p^{-1}(\Phi))$ is a closed subvariety of $\mathbb{P}^n$, equal to

$$X = \bigcup_{\Lambda \in \Phi} \Lambda.$$

Note that X has the interesting property that through every point of X there passes a k-plane. Classically such varieties are called **scrolls**. Perhaps the first interesting example of a scroll is the quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$.

Let us give some more constructions of scrolls. Suppose that we are given two subvarieties X and Y of $\mathbb{P}^n$. Define a rational map

$$\phi\colon X \times Y \dashrightarrow \mathbb{G}(1, n),$$

by sending

$$([v], [w]) \longrightarrow [v \wedge w].$$

The subvariety in $\mathbb{P}^n$, corresponding to the image, is called the **join**. It is the closure of the union of all lines obtained by taking the span of a point of X and a point of Y. Note that ϕ is a morphism if X and Y are disjoint and in this case we don't need to take the closure. If we take $X = Y$, then we get the **secant variety of** X, which is the closure of all the lines which join two points of X.

Suppose that we are given a morphism $f\colon X \longrightarrow Y$, with the property that there is a point $x \in X$ such that $f(x) \neq x$. Consider the morphism

$$\psi\colon X \longrightarrow \mathbb{G}(1,n),$$

which is the composition of

$$X \longrightarrow X \times Y \qquad \text{given by} \qquad x \longrightarrow (x, f(x)),$$

and the morphism ϕ above. As before this gives us a scroll in $\mathbb{P}^n$, by taking the image. Note that all of this generalises to products of k varieties.

Definition 3.11. *Pick complimentary linear spaces $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ of dimensions $n_1, n_2, \dots, n_k$ in $\mathbb{P}^n$, where*

$$n + 1 = \sum_i (n_i + 1).$$

Pick rational normal curves $C_i \subset \Lambda_i$ in and pick identifications

$$\phi_i\colon \mathbb{P}^1 \longrightarrow C_i.$$

Let

$$X = \bigcup_{p \in \mathbb{P}^1} \langle \phi_1(p), \phi_2(p), \dots, \phi_k(p) \rangle.$$

Then X is called a ***rational normal scroll****.*

It is interesting to give some examples. Suppose that we pick two skew lines l and m in $\mathbb{P}^3$. Then we get a surface in $\mathbb{P}^3$, swept out by lines, meeting l and m. Suppose we pick coordinates such that $l = V(X,Y)$ and $m = V(Z,W)$. Identify $(0,0,a,b)$ with $(a,b,0,0)$. Then it is not hard to see that we get the surface $V(XW - YZ)$.

The next case is when we take a line and a complimentary plane in $\mathbb{P}^4$. The resulting surface in $\mathbb{P}^4$ is called the cubic scroll.

Let us now investigate how to work with the Grassmannian in the case of schemes. As in the case of affine and projective space we can define a scheme over $\operatorname{Spec}\mathbb{Z}$ and use this scheme to define the Grassmannian over any base scheme. In fact the equations defining the Grassmannian over an algebraically closed field have integral coefficients (better still, presumably 0 and ± 1) and this defines the Grassmannian as a closed subscheme of $\mathbb{P}^N_{\mathbb{Z}}$. However this somehow begs the question; what role does the Grassmannian play over an arbitrary base scheme S?. We want to extend the functor F, which is a priori defined only as a functor from varieties over K to (<u>Sets</u>), to a functor from the category of schemes over S to the category (<u>Sets</u>). To answer this question, we need to decide what we mean by a family of k-planes in $\mathbb{P}^n_S$. It turns out to be easier to answer what it means to have a family of vector subspaces of dimension $k+1$.

4. Coherent Sheaves

Definition 4.1. *If $(X, \mathcal{O}_X)$ is a locally ringed space, then we say that an $\mathcal{O}_X$-module $\mathcal{F}$ is **locally free** if there is an open affine cover $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is isomorphic to a direct sum of copies of $\mathcal{O}_{U_i}$. If the number of copies r is finite and constant, then $\mathcal{F}$ is called **locally free of rank** r (aka a **vector bundle**).*

*If $\mathcal{F}$ is locally free of rank one then we way say that $\mathcal{F}$ is **invertible** (aka **a line bundle**). The group of all invertible sheaves under tensor product, denoted $\operatorname{Pic}(X)$, is called the **Picard group** of X.*

*A **sheaf of ideals** $\mathcal{I}$ is any $\mathcal{O}_X$-submodule of $\mathcal{O}_X$.*

Definition 4.2. *Let $X = \operatorname{Spec} A$ be an affine scheme and let M be an A-module. $\tilde{M}$ is the sheaf which assigns to every open subset $U \subset X$, the set of functions*

$$s\colon U \longrightarrow \coprod_{\mathfrak{p}\in U} M_{\mathfrak{p}},$$

which can be locally represented at $\mathfrak{p}$ as a/g, $a \in M$, $g \in R$, $\mathfrak{p} \notin U_g \subset U$.

Lemma 4.3. *Let A be a ring and let M be an A-module. Let $X = \operatorname{Spec} A$.*

1. *$\tilde{M}$ is a $\mathcal{O}_X$-module.*
2. *If $\mathfrak{p} \in X$ then $\tilde{M}_{\mathfrak{p}}$ is isomorphic to $M_{\mathfrak{p}}$.*
3. *If $f \in A$ then $\tilde{M}(U_f)$ is isomorphic to M_f.*

Proof. (1) is clear and the rest is proved mutatis mutandis as for the structure sheaf. □

Definition 4.4. *An $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is called **quasi-coherent** if there is an open cover $\{U_i = \operatorname{Spec} A_i\}$ by affines and isomorphisms $\mathcal{F}|_{U_i} \simeq \tilde{M}_i$, where M_i is an A_i-module. If in addition M_i is a finitely generated A_i-module then we say that $\mathcal{F}$ is **coherent**.*

Proposition 4.5. *Let X be a scheme. Then an $\mathcal{O}_X$-module $\mathcal{F}$ is quasi-coherent if and only if for every open affine $U = \operatorname{Spec} A \subset X$, $\mathcal{F}|_U = \tilde{M}$. If in addition X is Noetherian then $\mathcal{F}$ is coherent if and only if M is a finitely generated A-module.*

This is proved using almost the same techniques as the proof for the structure sheaf; the key point is that if a collection of sections of $\mathcal{O}_X$ don't vanish simultaneously then we can write 1 as a linear combination of these sections.

Theorem 4.6. *Let $X = \operatorname{Spec} A$ be an affine scheme.*

The assignment $M \longrightarrow \tilde{M}$ defines an equivalence of categories between the category of A-modules to the category of quasi-coherent sheaves on X, which respects exact sequences, direct sum and tensor product, and which is functorial with respect to morphisms of affine schemes, $f\colon X = \operatorname{Spec} A \longrightarrow Y = \operatorname{Spec} B$. If in addition A is Noetherian, this functor restricts to an equivalence of categories between the category of finitely generated A-modules to the category of coherent sheaves on X.

Theorem 4.7. *Let X be a scheme.*

The kernel and cokernel of a morphism between two quasi-coherent sheaves is quasi-coherent. An extension of quasi-coherent sheaves is quasi-coherent, that is, if the two outer terms of a short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$

are quasi-coherent then so is middle.

If X is Noetherian then one can replace quasi-coherent by coherent.

Proof. Since this result is local, we may assume that $X = \operatorname{Spec} A$ is affine. The only non-trivial thing is to show that if $\mathcal{F}$ and $\mathcal{H}$ are quasi-coherent then so is $\mathcal{G}$. By (II.5.6) of Hartshorne, there is an exact sequence on global sections,

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0.$$

It follows that there is a commutative diagram,

$$\begin{tikzcd}
0 \arrow[r] & \tilde{F} \arrow[r] \arrow[d] & \tilde{G} \arrow[r] \arrow[d] & \tilde{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{H} \arrow[r] & 0,
\end{tikzcd}$$

whose rows are exact. By assumption, the first and third vertical arrow are isomorphisms, and the 5-lemma implies that the middle arrow is an isomorphism. $\square$

Lemma 4.8. *Let $f\colon X \longrightarrow Y$ be a scheme.*

1. *If $\mathcal{G}$ is a quasi-coherent sheaf (respectively X and Y are Noetherian and $\mathcal{G}$ is coherent) on Y then $f^*\mathcal{G}$ is quasi-coherent (respectively coherent).*
2. *If $\mathcal{F}$ is a quasi-coherent sheaf on X and either f is compact and separated or X is Noetherian then $f_*\mathcal{F}$ is quasi-coherent.*

Proof. (1) is local on both X and Y and so follows easily from the affine case.

(2) is local on Y, so we may assume that Y is affine. By assumption (either way) X is compact and so we may cover X by finitely many

open affines $\{U_i\}$. If f is separated then $U_i \cap U_j$ is affine. Otherwise X is Noetherian and we can cover $U_i \cap U_j$ by finitely many open affines U_{ijk}. If $V \subset Y$ is open then a section s of $\mathcal{F}$ on the open set $f^{-1}(V)$ is the same as to give sections of $\mathcal{F}$ on the open cover $\{f^{-1}(V) \cap U_i\}$ which agree on overlaps $\{f^{-1}(V) \cap U_{ijk}\}$ (this is simply a restatement of the sheaf axiom). It follows that there is an exact sequence of sheaves

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \bigoplus_i f_*(\mathcal{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} f_*(\mathcal{F}|_{U_{ijk}}).$$

The last two sheaves are quasi-coherent, since U_i and U_{ijk} are coherent and a direct sum of quasi-coherent sheaves is quasi-coherent. But then the first term is quasi-coherent, by (4.7). □

Definition-Lemma 4.9. *Let X be a scheme. If $Y \subset X$ is a closed subscheme, then the kernel of the morphism of sheaves*

$$\mathcal{O}_X \longrightarrow \mathcal{O}_Y,$$

*defines a quasi-coherent ideal sheaf $\mathcal{I}_Y$, called the **ideal sheaf of** Y **in** X, which is coherent if X is Noetherian.*

Conversely if $\mathcal{I} \subset \mathcal{O}_X$ is a quasi-coherent sheaf of ideals then there is a closed subscheme Y of X such that $\mathcal{I}$ is the ideal sheaf of Y in X.

Proof. If $Y \subset X$ is a closed subscheme then $\mathcal{I}_Y$ is a quasi-coherent sheaf, by (4.7), which is coherent if X is Noetherian.

Now suppose that $\mathcal{I}$ is quasi-coherent. Let Y be the support of the quotient sheaf $\mathcal{O}_X/\mathcal{I}$. Then Y is a closed subset. Uniqueness is clear. We have to check that $(Y, \mathcal{O}_X/\mathcal{F})$ is a closed subscheme. We may check this locally so that we may assume that $X = \operatorname{Spec} A$ is affine. Then $\mathcal{I} = \tilde{I}$, where $I = \Gamma(X, \mathcal{I}) \subset A$ is an A-submodule, that is an ideal. $(Y, \mathcal{O}_X/\mathcal{I})$ is then the closed affine subscheme corresponding to I. □

Remark 4.10. *Let $i\colon Y \longrightarrow X$ be a closed subscheme. If $\mathcal{F}$ is a sheaf on Y, then $\mathcal{G} = i_*\mathcal{F}$ is a sheaf on X, whose support is contained in Y. Conversely, given any sheaf $\mathcal{G}$ on X, whose support is contained in Y, then there is a unique sheaf $\mathcal{F}$ on Y such that $i_*\mathcal{F} = \mathcal{G}$.*

For this reason, it is customary, as in (4.9), to abuse notation, and to not distinguish between sheaves on Y and sheaves on X, whose support is contained in Y.

Note also that (4.9) implies that the closed subschemes $Y \subset X = \operatorname{Spec} A$ of an affine scheme are in bijection with the ideals $I \trianglelefteq A$.

Definition 4.11. *Let $f\colon X \longrightarrow S$ be a morphism of schemes. We say that f is **affine** if there is an open affine cover $\{S_i\}$ of S such that $f^{-1}(S_i)$ is an affine open subset of X.*

Remark 4.12. *It is straightforward to show that f is affine if and only if for every open affine subset $V \subset S$, $f^{-1}(V)$ is affine. Note that if $f\colon X \longrightarrow S$ is affine then $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras.*

Let S be a scheme and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Take an open affine cover $\{S_i = \operatorname{Spec} R_i\}$ of S. As $\mathcal{A}$ is quasi-coherent $\mathcal{A}|_{S_i} \simeq \tilde{A}_i$, for some R_i-algebra A_i. This gives a morphism or affine schemes $f_i\colon X_i = \operatorname{Spec} A_i \longrightarrow S_i$. By composition this gives a morphism $X_i \longrightarrow S$. It is straightforward to check that we can glue these morphisms together to get a scheme $X = \mathbf{Spec}\, \mathcal{A}$ and an affine morphism $f\colon X \longrightarrow S$.

Theorem 4.13. *Fix a scheme S. There is an equivalence of categories between affine morphisms $f\colon X \longrightarrow S$ and quasi-coherent sheaves of $\mathcal{A} = \mathcal{O}_S$-algebras.*

Let $\mathcal{Q}$ be a locally free sheaf of rank r on a scheme S. We can construct the symmetric algebra $\operatorname{Sym} \check{\mathcal{Q}}$. This is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. Let $X = \mathbf{Spec}(\operatorname{Sym} \check{\mathcal{Q}})$. The fibres of the affine morphism $f\colon X \longrightarrow S$ are affine spaces of dimension r. In fact, if $\mathcal{Q}$ is the trivial sheaf of rank r then $X = \mathbb{A}^r_S$, so that if $\{S_i\}$ is an open affine cover of S such that $\mathcal{Q}_i$ is the trivial sheaf of rank r then $X_i = \mathbb{A}^r_{S_i}$. Intuitively f is a fibre bundle, with fibres isomorphic to affine space. In fact f comes with a distinguished section and in fact X is (what is known as) a vector bundle of rank r over S. All of this discussion motivates the following:

Definition 4.14. *Let B a scheme. A* ***family of k-planes over B in an n-dimensional vector space*** *is a morphism of locally free sheaves*

$$\bigoplus_{i=1}^{n} \mathcal{O}_B \longrightarrow \mathcal{Q},$$

where $\mathcal{Q}$ has rank k.

Note that if we take global spec then we get a map of vector bundles, from the trivial vector bundle of rank n to a vector bundle of rank k. Note also that we need to work with quotient sheaves. The problem is that if we take the vector bundles associated to an inclusion of locally free sheaves this need not gives a map of vector bundles (recall that $V \subset E$ is a sub-vector bundle if the quotient vector bundle exists).

Definition 4.15. *Fix a scheme S. Let F be the functor from the category of schemes over S to (Sets) which assigns to every scheme B*

over S the set of all isomorphism classes of families of k-planes over B in an n-dimensional space.

Theorem 4.16. *There is a integral projective scheme $G_S(k, n)$ which represents the functor F.*

Note that the Grassmanian comes equipped with a locally free sheaf $\mathcal{Q}$ of rank k, which is a quotient of the trivial locally free sheaf of rank n. This sheaf is called the **universal quotient sheaf**.

Note also that the definition of the Grassmanian is inconsistent with the classical definition of projective space over an algebraically closed field K. If one follows the definition given above, the closed points of $\mathbb{P}^n_K$ with residue field K are surely the one dimensional quotients of K^{n+1} and not the one dimensional subspaces. If one adopts the functorial approach of schemes we have no choice but to define everything in terms of quotient spaces. For example, suppose we consider $\mathbb{P}^1_{\mathbb{Z}}$. Then we want to look at one dimensional objects attached to $\mathbb{Z}^2$. If we try to work with subgroups of rank one then we run into trouble,

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^2 \longrightarrow Q \longrightarrow 0.$$

If the inclusion map is given by $1 \longrightarrow (1, 0)$ then $Q \simeq \mathbb{Z}$ as expected. But if we consider something like $1 \longrightarrow (2, 0)$ then $Q \simeq \mathbb{Z}_2 \oplus \mathbb{Z}$, which is not correct. However if we consider sequences of the form

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^2 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

then K is always isomorphic to $\mathbb{Z}$. On the other hand, it would seem impractical to change the classical definition of projective space in anticipation of this problem.

Most of what we have done with algebras and modules, makes sense for graded algebras and graded modules, in which case we get sheaves on proj of the graded ring.

Definition 4.17. *Let S be a graded ring and let M be a graded S-module. If $\mathfrak{p} \triangleleft S$ is a homogeneous ideal, then $M_{(\mathfrak{p})}$ denotes those elements of the localisation $M_{\mathfrak{p}}$ of degree zero.*

$\tilde{M}$ is the sheaf on $\operatorname{Proj} S$, which given an open subset $U \subset \operatorname{Proj} S$, assigns the set $\tilde{M}(U)$ of those functions

$$s \colon U \longrightarrow \coprod_{\mathfrak{p} \in U} M_{(\mathfrak{p})},$$

which are locally fractions of degree zero.

Proposition 4.18. *Let S be a graded ring, let M be a graded S-module and let $X = \operatorname{Proj} S$.*

(1) *For any* $\mathfrak{p} \in X$, $(\tilde{M})_{\mathfrak{p}} \simeq M_{(\mathfrak{p})}$.
(2) *If* $f \in S$ *is homogeneous,*

$$(\tilde{M})_{U_f} \simeq \tilde{M}_{(f)}.$$

(3) $\tilde{M}$ *is a quasi-coherent sheaf. If* S *is Noetherian and* M *is finitely generated then* $\tilde{M}$ *is a coherent sheaf.*

Definition 4.19. *Let* $X = \operatorname{Proj} S$, *where* S *is a graded ring. If* n *is any integer, then set*

$$\mathcal{O}_X(n) = S(n)\tilde{}.$$

If $\mathcal{F}$ *is any sheaf of* $\mathcal{O}_X$*-modules,*

$$\mathcal{F}(n) = \mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X(n).$$

Let

$$\Gamma_*(X, \mathcal{F}) = \bigoplus_{m \in \mathbb{N}} \Gamma(X, \mathcal{F}(n)).$$

Lemma 4.20. *Let* S *be a graded ring,* $X = \operatorname{Proj} S$ *and let* M *be a graded* S*-module.*

(1) $\mathcal{O}_X(n)$ *is an invertible sheaf.*
(2) $\tilde{M}(n) \simeq M(n)\tilde{}$. *In particular* $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \simeq \mathcal{O}_X(m+n)$.
(3) *Formation of the twisting sheaf* $\mathcal{O}_X(1)$ *is functorial with respect to morphisms of graded rings.*

Proposition 4.21. *Let* A *be a ring, let* $S = A[x_0, x_1, \dots, x_r]$ *and let* $X = \mathbb{P}^r_A = \operatorname{Proj} A[x_0, x_1, \dots, x_r]$.

Then

$$\Gamma_*(X, \mathcal{O}_X) \simeq S.$$

Lemma 4.22. *Let* S *be a graded ring, generated as an* S_0*-algebra by* S_1.

If $X = \operatorname{Proj} S$ *and* $\mathcal{F}$ *is a quasi-coherent sheaf on* X, *then*

$$\Gamma_*(X, \mathcal{F})\tilde{} = \mathcal{F}.$$

Theorem 4.23. *Let* A *be a ring.*

(1) *If* $Y \subset \mathbb{P}^n_A$ *is a closed subscheme then* $Y = \operatorname{Proj} S/I$, *for some homogeneous ideal* $I \subset S = A[x_1, x_2, \dots, x_n]$.
(2) Y *is projective over* $\operatorname{Spec} A$ *if and only if it is isomorphic to* $\operatorname{Proj} T$ *for some graded ring* T, *for which there are finitely many elements of* T_1 *which generate* T *as a* $T_0 = A$*-algebra.*

Proof. Let $\mathcal{I}_Y$ the ideal sheaf of Y in X. Then there is an exact sequence,

$$0 \longrightarrow \mathcal{I}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Twisting by $\mathcal{O}_X(n)$ is exact (in fact $\mathcal{O}_X(n)$ is an invertible sheaf), so we get an exact sequence

$$0 \longrightarrow \mathcal{I}_Y(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Taking global sections is left exact, so we get an exact sequence

$$0 \longrightarrow \mathcal{I}_Y(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n).$$

Taking the direct sum, there is therefore an injective map

$$I = \Gamma_*(X, \mathcal{I}_Y) = \Gamma_*(X, \mathcal{O}_X) \simeq S.$$

It follows that $I \triangleleft S$ is a homogeneous ideal. Let $\tilde{I}$ be the associated sheaf. Since $\mathcal{I}_Y$ is quasi-coherent, (4.22) implies that $\tilde{I} = \mathcal{I}_Y$. But then the subscheme determined by I is equal to Y. Hence (1).

If Y is projective over $\operatorname{Spec} A$ then we may assume that $Y \subset \mathbb{P}^n_A$. By (1) $Y \simeq \operatorname{Proj} S/I$, and if $T = S/I$, then $T_0 \simeq A$ and the images of $x_0, x_1, \dots, x_n \in T_1$ generate T. Conversely, any such algebra is the quotient of S. The kernel I is a homogeneous ideal and $Y \simeq \operatorname{Proj} S/I$. □

Definition 4.24. *Let Y be a scheme. $\mathcal{O}_{\mathbb{P}^r_Y}(1) = g^*\mathcal{O}_{\mathbb{P}^r}(1)$ is the sheaf on $\mathbb{P}^r_Y$, where $g \colon \mathbb{P}^r_Y \longrightarrow \mathbb{P}^r_{\operatorname{Spec}\mathbb{Z}}$ is the natural morphism.*

We say that a morphism $i \colon X \longrightarrow Z$ is an ***immersion*** *if i induces an isomorphism of X with a locally closed subset of Y.*

We say that an invertible sheaf $\mathcal{L}$ on a scheme X over Y is ***very ample*** *if there is an immersion $i \colon X \longrightarrow \mathbb{P}^r_Y$ over Y, such that $\mathcal{L} \simeq i^*\mathcal{O}_{\mathbb{P}^r_Y}(1)$.*

Lemma 4.25. *Let X be a scheme over Y.*

Then X is projective over Y if and only if X is proper over Y and there is a very ample sheaf on X.

Proof. One direction is clear; if X is projective over Y, then it is proper and we just pullback $\mathcal{O}_{\mathbb{P}^r_Y}(1)$.

If X is proper over Y then the image of X in $\mathbb{P}^r_Y$ is closed, and so X is projective over Y. □

Definition 4.26. *Let X be a scheme and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is* ***globally generated*** *if there are elements $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that for every point $x \in X$, the images of s_i in the stalk $\mathcal{F}_x$, generate the stalk as an $\mathcal{O}_{X,x}$-module.*

Lemma 4.27. *Let X be a scheme and*
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(1) *$\mathcal{F}$ is globally generated.*
(2) *The natural map*

$$H^0(X, \mathcal{F}) \otimes \mathcal{O}_X \longrightarrow \mathcal{F},$$

is surjective.
(3) *$\mathcal{F}$ is a quotient of a free sheaf.*

Proof. Clear. □

Lemma 4.28 (Push-pull)**.** *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{G}$ be a locally free $\mathcal{O}_Y$-module.*

$$f_*(\mathcal{F} \underset{\mathcal{O}_X}{\otimes} f^*\mathcal{G}) = f_*\mathcal{F} \underset{\mathcal{O}_Y}{\otimes} \mathcal{G}.$$

Theorem 4.29 (Serre)**.** *Let X be a projective scheme over a Noetherian ring A, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.*

Then there is a positive integer $n_0 \geq 0$ such that $\mathcal{F}(n)$ is globally generated for all $n \geq n_0$.

Proof. By assumption there is a closed immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. Let $\mathcal{G} = i_*\mathcal{F}$. Then (4.28) implies that

$$\mathcal{G}(n) = i_*\mathcal{F}(n).$$

Then $\mathcal{F}(n)$ is globally generated if and only if $\mathcal{G}(n)$ is globally generated. As i is a closed immersion it is a proper morphism; as $\mathcal{F}$ is coherent, i is proper, and X and $\mathbb{P}^r_A$ are Noetherian, $\mathcal{G}$ is coherent. Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$, we may assume that $X = \mathbb{P}^r_A$.

Consider the standard open affine cover U_i, $0 \leq i \leq r$ of $\mathbb{P}^r_A$. Since $\mathcal{F}$ is coherent, $\mathcal{F}_i = \mathcal{F}|_{U_i} = \tilde{F}_i$, for some finitely generated $A[X_0/X_i, X_1/X_i, \dots, X_r/X_i]$-module F_i. Pick generators s_{ij} of F_i. For each j, we may lift $X_i^{n_{ij}} s_{ij}$ to t_{ij}, for some n_{ij} (see (II.5.14)). By finiteness, we may assume that $n = n_{ij}$ does not depend on i and j. Now the natural map

$$X_i^n\colon \mathcal{F} \longrightarrow \mathcal{F}(n),$$

is an isomorphism over U_i. Thus t_{ij} generate the stalks of $\mathcal{F}$. □

Corollary 4.30. *Let X be a scheme projective over a Noetherian ring A and let $\mathcal{F}$ be a coherent sheaf.*

Then $\mathcal{F}$ is a quotient of a direct sum of line bundles of the form $\mathcal{O}_X(n_i)$.

Proof. Pick $n > 0$ such that $\mathcal{F}(n)$ is globally generated. Then

$$\bigoplus_{i=1}^{k} \mathcal{O}_X \longrightarrow \mathcal{F}(n),$$

is surjective. Now just tensor by $\mathcal{O}_X(-n)$. $\square$

5. Dimension of schemes

Our aim in this section is give a formal definition of the dimension of a variety, to compute the dimension in specific examples and to prove some of the interesting properties of the dimension.

Definition 5.1. *Let X be a topological space.*

The ***dimension of*** *X is equal to the supremum of the length n of strictly increasing sequences of irreducible closed subsets of X,*

$$\varnothing \neq Z_0 \subset Z_1 \subset \cdots \subset Z_n.$$

We will call a chain ***maximal*** *if it cannot be extended a longer chain.*

Note that if X is Noetherian then the dimension of X is, by definition, equal to the maximal dimension of an irreducible component. Note that also that the dimension of X is equal to the dimension of any dense open subset, and that the dimension of any subset is at most the dimension of X.

In general this notion of dimension is a little unwieldy, even for Noetherian topological spaces (in fact, it is pretty clear that this definition is useless for any topological space that is not Noetherian or at least close to Noetherian).

For quasi-projective varieties it is much better behaved. For example,

Theorem 5.2. *Let X be a quasi-projective variety.*

Then the dimension of X is equal to the length of any maximal chain of irreducible subvarieties.

Definition 5.3. *Let $f\colon X \longrightarrow I$ be a map from a topological space to an ordered set I. We say that f is* ***upper semi-continuous****, if for every $a \in I$, the set*

$$\{\, x \in X \mid f(x) \geq a \,\},$$

is closed in X.

The key result is:

Theorem 5.4. *Let $\pi\colon X \longrightarrow Y$ be a dominant morphism of quasi-projective varieties. Then the function*

$$\mu\colon X \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\mu(p)$ is the local dimension of the fibre $X_p = \pi^{-1}(\pi(p))$ at p. Moreover if X_0 is any irreducible component of X, Y_0 the closure of the image, we have

$$\dim(X_0) = \dim(Y_0) + \mu_0,$$

where μ_0 is the minimum value of μ on X_0.

Note that semi-continuity of μ is equivalent to saying that the dimension can jump up on closed subsets, but not down. For example, consider what happens for the blow up of a point. In this case, μ is equal to zero outside of the exceptional divisor and it jumps up to one on the exceptional divisor.

We will prove these two results in tandem. Let $d = \dim X$. We will need an intermediary result, which is of independent interest:

Lemma 5.5. *Assume* $(5.2)_d$.

If $X \subset \mathbb{P}^n$ is a closed subset of dimension d and $H \subset \mathbb{P}^n$ is a hypersurface then

$$\dim(X \cap H) \geq \dim(X) - 1,$$

with equality if and only if $H \cap X$ does not contain a component of X of maximal dimension.

Proof. We might as well assume that X is irreducible and that H does not contain a component of X of maximal dimension. Pick a maximal chain of irreducible subvarieties of X which contains a component Y of $X \cap H$,

$$\varnothing \neq Z_0 \subset Z_1 \subset \cdots \subset Z_e.$$

Then $X = Z_e$ and $Y = Z_i$, some i. As we are assuming $(5.2)_d$, $d = e$ and $\dim Y = i$.

Suppose $Z \neq X$ is irreducible and

$$Y \subset Z \subset X.$$

I claim that $Z = Y$. To see this, if we pass to an open affine subset then Z and Y are defined by ideals $J \subset I \subset A$, where A is the coordinate ring, $I = \langle f \rangle$ is principal and J is a prime ideal. Pick $g \in J$, $g \neq 0$. Write $g = g_1 g_2 \dots g_k$ as a product of irreducibles. As J is a prime ideal, $g_i \in J$ for some i. As $g_i \in I$, $g_i = uf$, and u must be a unit as g_i is irreducible. But then $I = J$ and $Z = Y$.

It follows that $i = d - 1$ and so $\dim Y = d - 1$. □

Lemma 5.6. $(5.2)_{d-1}$ *implies* $(5.4)_d$.

Proof. The result is local on X, so we might as well assume that X and Y are irreducible and affine. We first show that

$$\mu(p) \geq \dim(X) - \dim(Y),$$

for every point of $p \in X$. If $e = \dim(Y) = \dim(X) = d$ there is nothing to prove. So we may assume that $e = \dim(Y) < d = \dim(X)$. Let $q = \pi(p)$. By (5.5) we may embed $Y \subset \mathbb{A}^n$ and pick a hyperplane $q \in H \subset Y$ such that $\dim(H \cap Y) = \dim(Y) - 1$. By an obvious induction, we may pick $\dim(Y)$ hyperplanes $H_1, H_2, \dots, H_e$, whose intersection is a finite set containing q. Working locally about q, we may assume that q is the only point in the intersection. Let $f_1, f_2, \dots, f_e$ be the corresponding polynomials. Then the fibre X_p is is defined by the polynomials $g_1, g_2, \dots, g_e$, where $g_i = \pi^* f_i$. So

$$\dim(X_p) \geq \dim(X) - \dim(Y),$$

as required.

To finish the proof, by Noetherian induction applied to X, it suffices to prove that there is an open subset U of X such that

$$\mu(p) \leq \dim(X) - \dim(Y),$$

for every $p \in U$. As usual, we may assume that $X \subset Y \times \mathbb{A}^n$ and that π is projection onto the second factor. Factoring π into the product of n projections, we may assume that $n = 1$, by induction on n. We may assume that $X \subset Y \times \mathbb{A}^1$ is closed. If $X = Y \times \mathbb{A}^1$ then $\mu_0 = 1$ and it is clear that $\dim X \geq \dim Y + 1$. As we have already proved the reverse inequality, $\dim X = \dim Y + 1$.

Otherwise there is a fibre of dimension zero. As X is a proper subset of Y, $\dim X = \dim Y$ and $\mu_0 = 0$. Working locally, we may assume that X is defined by polynomials of the form $F \in A(Y)[S, T]$. Further there is a polynomial $F \in A(Y)[S, T]$ vanishing on X, such that F_y is not the zero polynomial, for at least one $y \in Y$. In this case, the set of points where F_y is not the zero polynomial, is an open subset of Y, and for any point in this open subset, the fibre has dimension zero. □

Lemma 5.7. $(5.4)_d$ *implies* $(5.2)_d$.

Proof. We may assume that X is affine. Pick a finite projection down to $\mathbb{A}^n$. As we are assuming $(5.4)_d$, $n = d$. It clearly suffices to prove the result for $X = \mathbb{A}^d$. Consider projection down to $\mathbb{A}^{d-1}$. Given a maximal chain of irreducible subsets

$$\varnothing \neq Z_0 \subset Z_1 \subset \dots \subset Z_n = \mathbb{A}^d,$$

let

$$\varnothing \neq Y_0 \subset Y_1 \subset \dots \subset Y_n = \mathbb{A}^{d-1},$$

be the image in $\mathbb{A}^{d-1}$. Then there is an index i such that Z_i contains the general fibre and Z_{i-1} does not contain the general fibre. Other than that, Y_j determines Z_j and the result follows by induction on d. □

Proof of (5.2) *and* (5.4). Immediate from (5.6) and (5.7). □

Corollary 5.8. *Let $\pi\colon X \longrightarrow Y$ be a surjective and projective morphism of quasi-projective varieties. Then the function*

$$\lambda\colon Y \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\lambda(p)$ is the dimension of the fibre $X_p = \pi^{-1}(X)$ at p. Moreover if X_0 is any irreducible component of X, with image Y_0, then we have

$$\dim(X_0) = \dim(Y_0) + \lambda_0,$$

where λ_0 is the minimum value of λ on Y_0.

Proof. π is proper as it is projective. Therefore the set

$$\{\, y \in Y \mid \lambda(y) \geq k \,\},$$

is closed as it is the image of the set

$$\{\, x \in X \mid \mu(x) \geq k \,\},$$

which is closed by (5.4). □

Note that we cannot discard the hypothesis that π is projective in (5.8). For example, let X be the disjoint union of $\mathbb{A}^2$ minus the y-axis and a single point p. Define a morphism $\pi\colon X \longrightarrow Y = \mathbb{A}^1$ by sending the extra point to the origin and otherwise taking the projection onto the x-axis. Then the fibre dimension is one at every point of Y, other than at the origin, where it is zero. In particular λ is not upper semi-continuous in this example. On the other hand, μ is upper semi-continuous, by virtue of the fact that the extra point is isolated in X.

One rather beautiful consequence of (5.4) is the following:

Corollary 5.9. *Let $\pi\colon X \longrightarrow Y$ be a morphism of projective varieties.*

If Y is irreducible and every fibre of π is irreducible and of the same dimension, then X is irreducible.

Proof. Let $X = X_1 \cup X_2 \cup \dots \cup X_k$ be the decomposition of X into its irreducible components. Let $\pi_i = \pi|_{X_i}\colon X_i \longrightarrow Y_i$, where Y_i is the image of X_i and let $\lambda_i\colon X_i \longrightarrow \mathbb{N}$ be the function associated to π_i, as in (5.8). Let

$$Z_i = \{\, y \in Y_i \mid \lambda_i(y) \geq \lambda_0 \,\}.$$

(5.8) implies that the closed sets $Z_1, Z_2, \dots, Z_k$ cover Y. As Y is irreducible it follows that there is an index i, say $i = 1$, such that $Z_1 = Y_1 = Y$. But then the fibres of π_1 and π are equal, as they are of the same dimension and the fibres of π are irreducible. This is only possible if $X = X_1$. □

Example 5.10. *$\mathbb{P}^n$ has dimension n. More generally a toric variety containing a torus $\mathbb{G}_m^n$ has dimension n. In particular the toric variety corresponding to a fan F in N is equal to the rank of the free abelian group N.*

Consider $\mathbb{G}(k,n)$. Then this contains an open subset U isomorphic to $\mathbb{A}^{(k+1)(n-k)}$. So $\mathbb{G}(k,n)$ has dimension $(k+1)(n-k)$. For example, $\mathbb{G}(1,3)$ has dimension $2 \cdot 2 = 4$.

Suppose that X and Y are quasi-projective varieties. Then the dimension of $X \times Y$ is the sum of the dimensions.

We can use (5.4) to calculate the dimension using different methods. One way is to project onto a linear subspace. If we repeatedly project from a point (which is the same as projecting once from a linear space of positive dimension) then the induced morphism $X \longrightarrow \mathbb{P}^k$ will eventually become dominant. At this point the morphism is finite over an open subset and the dimension of X is then k. Note that if we go back one step, then the closure of the image of X will be a hypersurface in $\mathbb{P}^{k+1}$.

Equivalently, if $X \subset \mathbb{P}^n$ and X has dimension d then a general linear space of dimension $n-d-1$ is disjoint from X and a general linear space of dimension $n-d$ meets X in a finite set of points. Note that general means that the linear space belongs to an open set of the corresponding Grasmannian. If X is closed, we can do slightly better, since if X is closed of dimension d, then every linear space of dimension $n-d$ must intersect X.

To calculate the dimension of an algebraic variety one can also use:

Definition 5.11. *Let L/K be a field extension. The* ***transcendence degree*** *of L/K is equal to the supremum of the length $x_1, x_2, \dots, x_k$ of algebraically independent elements of L/K.*

It is easy to prove:

Theorem 5.12. *Let X be an irreducible quasi-projective variety.*

Then the dimension of X is equal to the transcendence degree of $K(X)/K$.

One trick to calculate dimensions is to use the generic point of a variety. If we have a morphism $\pi \colon X \longrightarrow Y$ of irreducible varieties then μ_0 is actually the dimension of the generic fibre X_η, over the residue field of the generic point η of Y. Indeed the generic point ξ of X maps to the generic point of Y and so ξ is also the generic point of the generic fibre. The dimension of the generic fibre is the transcendence degree of the residue field of ξ over the residue field of η. The dimension of X

is the transcendence degree of the residue field of ξ over K. But the transcendence degree is additive on extensions.

Perhaps an easy example will make all of this clear. Consider $\mathbb{A}^2_K$. Suppose the generic point is ξ, with residue field $K(x, y)$. This has transcendence degree two over K. If we take a projection down to $\mathbb{A}^1_K$, with generic point η and residue field $K(y)$ then the transcendence degree of $K(x, y)/K(y)$ is one, the dimension of the generic fibre. $K(y)/K$ also has transcendence degree one and $\mathbb{A}^1_K$ has dimension one, as expected.

Now let's turn to calculating the dimension of some more examples, using these new techniques. Let us first calculate the dimension of the universal family over the Grassmannian.

$$\begin{tikzcd} \Sigma \arrow[r, "q"] \arrow[d, "p"'] & \mathbb{P}^n \\ \mathbb{G}(k, n). & \end{tikzcd}$$

Note that there are two ways to proceed; we can either use the morphism p or q.

First we use the morphism p. If we fix an element $[\Lambda] \in \mathbb{G}(k, n)$ then the fibre of p will be a copy of the k-plane Λ. Thus every fibre of p is isomorphic to $\mathbb{P}^k$. It follows that Σ has dimension $k + (k + 1)(n - k)$.

Now let us use the morphism q. If we fix point $x \in \mathbb{P}^n$, then the fibre of q is equal to the set of k-planes in $\mathbb{P}^n$, containing x. This is isomorphic to a Grassmannian $\mathbb{G}(k - 1, n - 1)$. Thus the dimension of Σ is equal to $n + k(n - k)$, which is easily seen to be equal to the previous expression.

Note that also we can prove that Σ is irreducible. Either way, it fibres over an irreducible base, with irreducible fibres of the same dimension.

Similarly the universal family of conics has dimension six (=five+one=two+four) and this space is irreducible. It is perhaps more interesting to figure out the dimension of the secant variety and the space of incident l-planes to an irreducible projective variety $X \subset \mathbb{P}^n$.

First the space $\mathcal{C}_l(X)$ of l-planes which meets a closed subset X of $\mathbb{P}^n$. In this case the universal family over $\mathcal{C}_l(X)$ has dimension equal to

$$\dim X + l(n - l),$$

where the second factor is equal to the dimension of the space of l-planes which contains a point. Since we have already seen that this is a variety isomorphic to $\mathbb{G}(l - 1, n)$, it follows that the universal family is irreducible, provided X is irreducible.

In particular suppose that X has dimension k, and suppose that $l \leq n - k - 1$. Then a typical l-plane which meets X, will only meet X in one point. Thus the map from the universal family to $\mathbb{G}(l, n)$ is in fact birational, and the dimension of $\mathcal{C}_l(X)$ is

$$k + l(n - 1).$$

In other words the codimension of $\mathcal{C}_l(X)$ is

$$n - l - k.$$

Thus if $l = n - k - 1$, $\mathcal{C}_l(X)$ is a hypersurface in $\mathbb{G}(l, n)$.

Question 5.13. *Fix d. What is the smallest positive integer k such that any polynomial $f(x)$ of degree d over the field $\mathbb{C}$ is a sum of k dth powers of linear forms?*

One way to answer this problem is to use the secant variety to the rational normal curve of degree d. Let V be a two dimensional complex vector space. Then $\mathbb{P}^1 = \mathbb{P}(V)$ and the rational normal curve is the set of pure dth powers in the vector space $\mathbb{P}^d = \mathbb{P}(\mathrm{Sym}^d V)$ A polynomial $f(x)$ of degree d corresponds to a point of $\mathbb{P}^d$ and it is a sum of k dth powers if and only if belongs to the locus of $k-1$-planes which intersect C in k points. We want to know when this locus is the whole of $\mathbb{P}^n$. In this case its dimension is n.

It turns out that even when look at the locus of secant lines that this problem is very hard for a general variety X. In general, we have a rational morphism

$$X \times X \dashrightarrow \mathbb{G}(1, n)$$

Now note that if $l \subset \mathbb{G}(1, n)$ is a point of the image, then this map is not finite over l iff l is contained in X. Since the only subvariety with the property that the line through every two points is contained in the subvariety, is a linear space, we may assume that this map has finite fibres over an open set of the image. Then the image has dimension $2k$, where k is the dimension of X. Then the universal family over the image, has dimension $2k+1$ and the dimension of the image in $\mathbb{P}^n$ then has dimension $2k + 1$ as well, provided that through a general point of the secant variety (the closure of the set of lines that meet X in at least two points), there passes only finitely secant lines.

Thus the expected dimension of the secant variety is $2k+1$, provided this dimension is at most n. For example, the secant variety to a space curve is expected to be the whole of $\mathbb{P}^3$ and the secant variety to a surface in $\mathbb{P}^5$ is expected to be the whole of $\mathbb{P}^5$.

Definition 5.14. *Let X be a closed irreducible non-degenerate (that is X is not contained in a proper linear subspace) subvariety of $\mathbb{P}^n$.*

The ***deficiency of*** X*, denoted* $\delta(X)$*, is equal to the dimension of the family of secant lines passing through a general point of the secant variety.*

We have already seen then that the dimension of the secant variety is equal to $2k + 1 - \delta(X)$.

Let us calculate the secant variety to the d-uple embedding, at least in characteristic zero. Recall the if $X = \mathbb{P}(V) = \mathbb{P}^k$ then X is embedded in $\mathbb{P}(\operatorname{Sym}^d(V))$, as the space of rank one symmetric tensors (the pure powers). The secant variety then consists of all rank at most two symmetric tensors, that is anything which is a sum of two rank one symmetric tensors.

In the case of the Veronese, we get the space of rank two quadratic forms. As there are quadratic forms of rank three, it follows than the secant variety to the Veronese is a proper subset of $\mathbb{P}^5$. In fact the space of rank two symmetric tensors is a hypersurface in $\mathbb{P}^5$, given as the vanishing of a determinant. Expanding it follows that the secant variety is defined by a cubic polynomial. Note that the deficiency is equal to 1 in this case.

It is interesting to look at the dimension of some more exotic schemes. $\operatorname{Spec}\mathbb{Z}$ has dimension. Consider $\mathbb{A}^1_{\mathbb{Z}}$. This has dimension one over $\operatorname{Spec}\mathbb{Z}$ and absolute dimension two. Consider $\mathbb{A}^2_{\mathbb{Z}}$. This has absolute dimension two over $\operatorname{Spec}\mathbb{Z}$ and so it has absolute dimension three.

6. Non-reduced schemes and flat limits of zero dimensional schemes

6.1. **Some examples of zero dimensional schemes.** We would like to understand the geometric content behind non-reduced schemes. Let us start with a simple example. Let k be a field and let

$$A = \frac{k[\epsilon]}{\langle \epsilon^2 \rangle}.$$

Consider $T = \operatorname{Spec} A$. Clearly A contains only one prime ideal, namely $\langle \epsilon \rangle$. Thus T has only one point. However the stalk of the structure sheaf is not a field. To get a picture of T, we can embed this scheme in $\mathbb{A}^1_k$,

$$k[x] \longrightarrow A \qquad \text{to get} \qquad T \longrightarrow \mathbb{A}^1_k,$$

where $x \longrightarrow \epsilon$. In fact we can think of two points in $\mathbb{A}^1_k$, p_t and q_t and think about what happens when p_t approaches q_t. We might as well suppose that q_t is fixed, $q_t = \langle x \rangle$. Let $p_t = \langle x - t \rangle$. The ideal of the union is then

$$\langle x(x-t) \rangle.$$

As t approaches 0, it is natural to identify the limit as

$$\langle x^2 \rangle,$$

which is the ideal of $T \subset \mathbb{A}^1_k$. With this picture, it is natural to think of T as being a point, together with a tangent direction. Abstractly, T is a point together with a disembodied tangent vector. Note that this is a very natural way to think of tangent vectors algebraically; if we want to differentiate, then we want to expand in powers of ϵ and ignore all terms of degree two and higher. In fact

Definition 6.1. *Let $x \in X$ be a point of a scheme, with residue field k. The **Zariski tangent space** T_xX to X at x is the k-vector space of all morphisms over* $\operatorname{Spec} k$*,*

$$T_xX = \operatorname{Hom}(\operatorname{Spec} k[\epsilon]/\langle \epsilon^2 \rangle, X),$$

where the image of the unique point of $\operatorname{Spec} k[\epsilon]/\langle \epsilon^2 \rangle$ *is x.*

Note that T has many embeddings into $\mathbb{A}^2_k$. Indeed, think of two points approaching each other. If one of the points is fixed to be the origin and the other approaches along a smooth curve, then the limiting subscheme (we will make the naive notion of the limit more formal very shortly) is a copy of T. However, the two points remember the tangent direction of their approach. Thus the set of all embeddings of T into $\mathbb{A}^2$, is given by

$$\langle ax + by \rangle + \mathfrak{m}^2,$$

where $\mathfrak{m}$ is the maximal ideal.

Definition 6.2. *Let X be a zero dimensional scheme over a field k. The* ***length of*** *X is simply the dimension of the k-vector space $\mathcal{O}_X(X)$.*

Here is another, more direct way, to classify all zero dimensional subschemes of $\mathbb{A}^2$ of length two supported at the origin. Any such scheme is a closed subscheme. It follows that it is given by a surjective morphism,

$$k[x, y] \longrightarrow k[\epsilon]/\langle \epsilon^2 \rangle$$

Let $\mathfrak{a}$ be the kernel of this map. Then the radical of $\mathfrak{a}$ must be the ideal $m = \langle x, y \rangle$. Since the square of any element of the image is zero, it follows that

$$m^2 = \langle x^2, xy, y^2 \rangle,$$

must be contained in the kernel. In other words we have the inclusion of subschemes

$$T \subset Z \subset \mathbb{A}^2_k \qquad \text{where} \qquad Z = \operatorname{Spec} k[x, y]/\langle x^2, xy, y^2 \rangle.$$

Now

$$k[x, y]/\langle x^2, xy, y^2 \rangle,$$

is a vector space of dimension three over k and

$$k[\epsilon]/\langle \epsilon^2 \rangle,$$

has dimension two. It follows that there must be a linear form in the kernel, say $ax + by$. The quotient then has the right dimension, and this gives us the classification.

Another way to think of this is as follows. Let $f \in k[x, y] = \mathcal{O}_{\mathbb{A}^2_k}(\mathbb{A}^2_k)$. Then we may restrict f to T. Suppose that T is given by $\langle y, x^2 \rangle$. Let g be the restriction of f to $k[x, y]/\langle y, x^2 \rangle$. Then g picks out both the value of f at the origin and the coefficient of the x-term of the Taylor series of f . In other words, the restriction is determined by

$$f(0, 0) \qquad \text{and} \qquad \left.\frac{\partial f}{\partial x}\right|_{(0,0)}.$$

If we again think of a family of two distinct points approaching each other, then instead of evaluating f at the two distinct points, we evaluate f at the point and in the tangent direction of their approach.

Note yet another way to think of an embedding of T into a smooth variety X. Let C be a smooth curve, and let $x \in C$ be a point of C. Then if we truncate C to order two, then we get a copy of $T \subset X$.

In other words, we look at the subscheme defined by $I_C + \mathfrak{m}^2 \subset A$, where $U = \operatorname{Spec} A$ is an open affine neighbourhood of x, I_C is the ideal of C and $\mathfrak{m}$ is the ideal corresponding to x. In other words, a closed embedding of T inside X is an equivalence class of smooth curves, just like tangent vectors in classical differential geometry.

Consider the intersection of a line with a conic. If the line is not tangent to the conic, then the line intersects the conic in two points and these points span the line. But if the line is tangent to the conic we only get one point. Nevertheless, as a scheme we get a double point. Moreover, we can recover the line from the scheme as the smallest linear space which contains the scheme (aka the span).

Armed with this example, the geometric meaning of other non-reduced schemes becomes a little more clear. For example, consider the scheme

$$Z = \operatorname{Spec} \frac{k[\epsilon]}{\langle \epsilon^3 \rangle}.$$

Consider how to embed Z inside $\mathbb{A}^2_k$. Perhaps the easiest way to proceed is to pick a smooth curve and truncate to order three, rather than two. If the curve is $y = x^2$, then we look at

$$\langle y - x^2, x^3 \rangle.$$

This is obviously different from looking at

$$\langle y, x^3 \rangle.$$

Here, we have a tangent direction together with a second order tangent direction. We can think of this scheme as the limit as three points coming together. However in this case, the points need to approach each other along the same smooth curve. Indeed, note that there is in fact another irreducible zero dimensional scheme, up to isomorphism. Consider

$$\operatorname{Spec} k[x, y]/\langle x^2, xy, y^2 \rangle.$$

Note that this contains all length two subschemes of the origin with the same support. In other words, if we look at a function, then its restriction is determined by its value at the orgin and two independent derivatives. It is not hard to see that these are the only two possibilities for length three schemes of irreducible schemes.

6.2. **Flat limits.** It is interesting to see how one gets these examples as limits. First suppose that three points approach each other. Fixing one as the origin, we suppose that the other two approach along smooth curves. For example, suppose the points are

$$(0, 0) \qquad (t, 0) \qquad \text{and} \qquad (0, t).$$

The corresponding ideals are

$$\langle x, y\rangle \qquad \langle x-t, y\rangle \qquad \text{and} \qquad \langle x, y-t\rangle.$$

The product of these ideals is then

$$\langle x^3-x^2t, x^2t+xyt-xt^2, x^2y, xyt, x^2y-xyt, xyt+y^2t-yt^2, xy^2, y^3-y^2t\rangle,$$

which is the ideal of the union. Consider the limit, as t goes to zero. If we naively set $t=0$ then we get

$$\langle x^3, x^2y, x^2y, xy^2, y^3\rangle = \mathfrak{m}^3.$$

But this cannot be the limit since the corresponding scheme has length 6 and the limit should be a scheme of length 3. Howerver, for $t \neq 0$, we always get xy in the ideal. Thus the limit must also contain xy. From there it is easy to see that the limit must also contain x^2 and y^2, so that we get,

$$\langle x^2, xy, y^2\rangle = \mathfrak{m}^2,$$

which does indeed correspond to a scheme of length 3. So the flat limit of the three points is the scheme corresponding to $\mathfrak{m}^2$.

Now consider the case when we have three points approaching each other along a smooth curve. For example, take the three points

$$(0,0) \qquad (t, t^2) \qquad \text{and} \qquad (-t, t^2).$$

Since these points lie in the smooth curve $y-x^2$, which is isomorphic to $\mathbb{A}^1$, it follows that the flat limit is the unique length three scheme supported at the origin and contained in this curve, namely

$$\langle y-x^2, x^3\rangle.$$

Definition 6.3. *We say that a zero dimensional scheme z is **curvilinear** if it can be embedded in a smooth curve C.*

Note that a zero dimensional scheme is curvilinear if and only if its irreducible components are curvlinear. On the other hand if z is irreducible then it is curvilinear if and only if it is isomorphic to

$$\operatorname{Spec} \frac{k[\epsilon]}{\langle \epsilon^l\rangle},$$

for some positive integer l.

6.3. **Punctual Hilbert scheme.** It is interesting to look at this from a different perspective. One way to think about taking limits, is to think of all our schemes as definining points of a Grassmannian. Indeed, all of our schemes are defined by ideals, such that the quotient is of finite dimension. Now suppose that the support is a fixed point. Then it is not hard to see that if one fixes the length, then some fixed power of the maximal ideal, will be contained in our ideal (in fact the length itself will do). Taking quotients, we get a subvector space of a fixed vector space. Thus the locus of all length l zero dimensional subschemes is naturally a subset of a Grassmannian. In fact this locus is algebraic. A one parameter family of ideals is then the same as a curve in the Grassmannian. As the Grassmannian is projective the closure of this curve is projective and this defines the flat limit.

The resulting locus is called the punctual Hilbert scheme, and is denoted $\mathcal{H}^l_0$. As the name might suggest, this subset of the Grassmannian corresponding to ideals of length l is not only closed, but it actually inherits the structure of a closed subscheme. The locus of curvilinear schemes is easily seen to be an open subscheme, and it is denoted $\mathcal{C}^l_0$.

Let

$$R = \frac{k[x,y]}{\mathfrak{m}^l}.$$

Then R is a finite dimensional vector space over k and ideals I in $k[x,y]$ such that

$$\frac{k[x,y]}{I},$$

has length l are the same as ideals J in R such that the quotient

$$\frac{R}{J},$$

has length l. In particular $[J] \in \mathbb{G}(M,N)$ for appropriate (and easily computable) integers M and N (indeed, $M = N - l$). Since x and y generate the ring R it follows that J is an ideal if and only if

$$xJ \subset J \qquad \text{and} \qquad yJ \subset J.$$

These give equations defining the punctual Hilbert scheme as a closed subscheme of the Grassmannian.

It is interesting to see what happens when we consider all length three schemes supported at a point. In this case, every scheme is certainly a subscheme of

$$\operatorname{Spec} k[x,y]/\mathfrak{m}^3.$$

Now

$$k[x,y]/\mathfrak{m}^3,$$

is a vector space of dimension six, and so we are looking at the Grassmannian of three planes in a vector space of dimension six. The scheme corresponding to $\mathfrak{m}^2$ is one point in this space. Given any other length three scheme, there is a unique length two subscheme contained in this scheme. It is obtained by truncating the ideal in the obvious way. Now the space of length two schemes is nothing more than the space of all tangent directions at the point, so that this space is isomorphic to $\mathbb{P}^1$. Consider the fibre over a point of $\mathcal{C}_0^2$. We are looking at all ideals of the form

$$I = \langle y + q \rangle + \mathfrak{m}^3,$$

where q is a quadratic form. Now $y(y+q) = y^2 + yq \in I$. As $yq \in \mathfrak{m}^3$ it follows that $y^2 \in I$. Similarly $xy \in I$. But then we may choose $q = ax^2$. It is not hard to check that

$$\langle y + ax^2 \rangle + \mathfrak{m}^3 = \langle y + bx^2 \rangle + \mathfrak{m}^3,$$

iff $a = b$. Thus there is a morphism from the space of curvilinear schemes of length three down to the space of length two schemes. The fibres of this morphism are $\mathbb{A}^1$ and the base is $\mathbb{P}^1$. Now the unique non-curvilinear scheme is a limit of curvilinear schemes with fixed tangent direction;

$$\lim_{t \to 0} \langle ty + x^2 \rangle + \mathfrak{m}^3 = \mathfrak{m}^2.$$

Thus the unique non-curvilinear schemes is in the closure of every fibre. It follows that this space is a cone over $\mathbb{P}^1$. In fact it is the usual quadric cone in $\mathbb{A}^3$.

Note that $\mathcal{C}_0^3$ is an open subset of the projective scheme $\mathcal{H}_0^3$, and more generally, for any l, the Hilbert scheme gives a projective compactification of the curvilinear locus. It is also interesting to consider different compactifications of the curvilinear locus. For example, we have already seen that there is a closed embedding,

$$\mathcal{C}_0^3 \subset \mathcal{C}_0^3 \times \mathcal{C}_0^2,$$

where we send

$$z \longrightarrow (z, z_2),$$

and z_2 is the unique length two subscheme contained in z (here we work with irreducible schemes). The closure of the image inside $\mathcal{H}_0^3 \times \mathcal{H}_0^2$ defines another compactification $\mathcal{B}^3$ of $\mathcal{C}_0^3$. In fact this compactification is smooth; there is an obvious morphism $\mathcal{B}^3 \longrightarrow \mathcal{H}_0^3$ which just forgets the length two scheme. This morphism is an isomorphism over the curvilinear locus and over the point corresponding to the unique length three scheme which is not curvilinear, we get a whole copy of $\mathbb{P}^1$, as any length two scheme is contained in this length three scheme.

6.4. **Hilbert scheme.** In some ways it is more natural to consider all zero dimensional schemes inside $\mathbb{P}^n$ and not just the punctual ones. This is called the Hilbert scheme and is denoted $\mathcal{H}^l$. The construction given above does not work, since there is no obvious fixed finite dimensional vector space in which to work. We will see later that one can construct the full Hilbert scheme using a similar but more sophisticated argument. Assuming the existence of the Hilbert scheme, we can ask what does it look like. Once again the curvilinear locus $\mathcal{C}^l$ is an open subset.

It is also an interesting question to ask which zero dimensional schemes are limits of curvilinear schemes.

Theorem 6.4. *Let S be a smooth surface. Fix a positive integer l.*

Then the Hilbert scheme of zero dimensional schemes of length l is irreducible and smooth.

One obvious component of the Hilbert scheme is the closure of the space of curvilinear schemes, which is obviously irreducible (an open subset is simply the product of the surface with itself l times, minus the diagonals). Thus (6.4) really answers our question for surfaces; every zero dimensional scheme is a limit of curvilinear schemes.

It is interesting to look at the symetric product. One way to compactify the space of l distinct unordered points is to consider

$$S^{(l)} = S^l/\Sigma_l$$

where S^l is the l-fold product of S with itself and Σ_l is the symmetric group, acting on the obvious way on S^l. It turns out that there is a natural morphism

$$\mathcal{H}^l \longrightarrow S^{(l)},$$

which just assigns to a scheme its support (in fact the space on the right is the Chow scheme). This map is birational and in fact the Hilbert scheme gives a desingularisation of the symmetric product, which is in fact highly singular.

Perhaps surpisingly most of this fails in higher dimensions.

6.5. **Non-reduced curves and embedded components.** We can also look at doubled curves. For example, consider

$$\langle x^2 \rangle \subset k[x, y].$$

Then we get a double line in $\mathbb{A}^2_k$. Just as in the case of a double point, we can think of the non-reduced structure as being the data of some sort of tangent directions (or better normal directions). Note that abstractly we have a product

$$\mathbb{A}^1_k \times T,$$

since

$$k[x,y]/\langle x^2\rangle \simeq k[y] \otimes k[\epsilon]/\langle \epsilon^2\rangle.$$

Clearly this structure becomes quite rich if we consider double structures on copies of $\mathbb{P}^1$, for example double lines in $\mathbb{P}^3$. It turns out that there are continuous non-isomorphic families of double structure on a copy of $\mathbb{P}^1$.

It is also interesting to consider embedded components.

Definition 6.5. *Let X be a scheme and let Z be a locally closed subscheme. The* ***closure of*** *Z is the smallest closed subscheme of X which contains Z.*

Of course the closure of Z is the intersection of all the closed subschemes of X which contain it. One can also define the closure in terms of the induced immersion

$$Z \longrightarrow X.$$

It is the induced subscheme of X.

Definition 6.6. *If X is a scheme, then we say that X has an* ***embedded component*** *if there is a dense open subset of X whose closure is not equal to X.*

For example, if there is a dense open set U which is reduced then the closure of U is reduced, so X has an embedded component iff X is not reduced.

In terms of examples, we will only consider non-reduced scheme structures on $\mathbb{A}^1_k$. Perhaps the simplest example is to consider the subscheme of $\mathbb{A}^2_k$ defined by the ideal $\langle y^2, xy\rangle$. The support of this closed subscheme is the x-axis. The open subscheme $U = U_x$ is a reduced subscheme; on the other hand, this scheme is not reduced as $y \neq 0 \in k[x,y]/\langle y^2, xy\rangle$, and $y^2 = 0$. Thus the origin is an embedded component.

Note that the ideal of functions vanishing on this scheme is equal to the ideal of functions vanishing along the x-axis, which also vanish to order two at the origin. Algebraically

$$\langle y^2, xy\rangle = \langle y\rangle \cap \langle x^2, xy, y^2\rangle.$$

Put differently, the restriction of a function $f(x,y) \in k[x,y]$ to this scheme is determined by the function $g(x) = f(x,0)$ and the value of the partial derivative

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)}.$$

Note that it is convenient to think of this scheme as the union of two schemes, the line given by $\langle y\rangle$ and the 2nd infinitessimal neighbourhood of the origin,

$$\langle x^2, xy, y^2\rangle.$$

To go further into the theory of embedded components, we need to recall some facts from algebra.

Definition 6.7. *Let M be an R-module. The* ***primes associated to*** *M are simply the annihilators of any element of M. The* ***primes associated to an ideal*** *$I \subset R$ are then the primes associated to the quotient R/I.*

An ideal $\mathfrak{q} \subset \mathfrak{p}$ is called ***primary to*** *$\mathfrak{p}$ if $\mathfrak{p}$ is the radical of $\mathfrak{q}$ and for every pair of elements f and g of R if $fg \in \mathfrak{q}$ and $f \notin \mathfrak{p}$ then $g \in \mathfrak{q}$.*

For example $\langle x^2, xy, y^2\rangle$ is primary to $\langle x, y\rangle$ in the polynomial ring $k[x, y]$. Another way to restate the second condition is that the localisation map

$$R/\mathfrak{q} \longrightarrow R_{\mathfrak{q}}/\mathfrak{q}R_{\mathfrak{q}},$$

is injective.

The key point is that every ideal is the intersection of primary ideals (one should think of this as a factorisation, for example $\langle 6\rangle = \langle 2\rangle \cap \langle 3\rangle$). Unfortunately the elements of the intersection are not unique.

There are ways to eliminate some of the redundancy however. We may assume that no ideal of the intersection can be removed. We may also assume that the primes associated to the primary ideals are distinct (indeed the intersection of two primary ideals with the same prime ideal is primary to this prime ideal). We call this a **primary decomposition** of I and we call the ideals of the intersection the **primary components** of I. Now it is true that the set of prime ideals, for which an ideal in the intersection is primary, is unique. In fact these ideals are nothing but the prime ideals associated to I. The primary ideals of the intersection are not unique. It does turn out that a primary ideal of the intersection is unique, however, if the corresponding prime ideal is minimal.

For example,

$$\langle y^2, xy\rangle = \langle y\rangle \cap \langle x^2, xy, y^2\rangle,$$

is a primary decomposition of $I = \langle y^2, xy\rangle$. The associated primes are $\langle y\rangle$ and $\langle x, y\rangle$. Since $\langle y\rangle$ is minimal, $\langle y\rangle$ appears in every primary decomposition of I. However we could choose $\langle x, y^2\rangle$ or $\langle x + y, y^2\rangle$ or indeed $\langle x + ay, y^2 \rangle$ for the other ideal. Indeed, $\langle x^n, xy, y^2 \rangle$ will also do, for any $n \geq 1$.

Recall the definition of the length.

Definition 6.8. *Let M be an R-module. The **length** of M is the maximal length of a chain*

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{l-i} \supset M_l.$$

It turns out that the length of the primary ideal is fixed however; it may be computed as the maximal length of an ideal of finite length in $R_{\mathfrak{p}}/IR_{\mathfrak{p}}$.

For example it turns out that for our favourite example, the length is one.

7. Divisors

Definition 7.1. *We say that a scheme X is **regular in codimension one** if every local ring of dimension one is regular, that is, the quotient $\mathfrak{m}/\mathfrak{m}^2$ is one dimensional, where $\mathfrak{m}$ is the unique maximal ideal of the corresponding local ring.*

Regular in codimension one often translates to smooth in codimension one.

When talking about Weil divisors, we will only consider schemes which are

(∗) noetherian, integral, separated, and regular in codimension one.

Definition 7.2. *Let X be a scheme satisfying (∗). A **prime divisor** Y on X is a closed integral subscheme of codimension one.*

*A **Weil divisor** D on X is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors.*

Thus a Weil divisor is a formal linear combination $D = \sum_Y n_Y Y$ of prime divisors, where all but finitely many $n_Y = 0$. We say that D is **effective** if $n_Y \geq 0$.

Definition 7.3. *Let X be a scheme satisfying (∗), and let Y be a prime divisor, with generic point η. Then $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field K.*

*The **valuation** ν_Y **associated to** Y is the corresponding valuation.*

Note that as X is separated, Y is determined by its valuation. If $f \in K$ and $\nu_Y(f) > 0$ then we say that f has a **zero of order** $\nu_Y(f)$; if $\nu_Y(f) < 0$ then we say that f has a **pole of order** $-\nu_Y(f)$.

Definition-Lemma 7.4. *Let X be a scheme satisfying (∗), and let $f \in K^*$.*

$$(f) = \sum_Y \nu_Y(f) Y \in \operatorname{Div} X.$$

Proof. We have to show that $\nu_Y(f) = 0$ for all but finitely many Y. Let U be the open subset where f is regular. Then the only poles of f are along $Z = X - U$. As Z is a proper closed subset and X is noetherian, Z contains only finitely many prime divisors.

Similarly the zeroes of f only occur outside the open subset V where $g = f^{-1}$ is regular. □

Any divisor D of the form (f) will be called **principal**.

Lemma 7.5. *Let X be a scheme satisfying (∗).*

The principal divisors are a subgroup of $\operatorname{Div} X$.

Proof. The map
$$K^* \longrightarrow \operatorname{Div} X,$$
is easily seen to be a group homomorphism. $\square$

Definition 7.6. *Two Weil divisors D and D' are called* ***linearly equivalent****, denoted $D \sim D'$, if and only if the difference is principal. The group of Weil divisors modulo linear equivalence is called the* ***divisor Class group****, denoted $\operatorname{Cl} X$.*

We will also denote the group of Weil divisors modulo linear equivalence as $A_{n-1}(X)$.

Proposition 7.7. *If k is a field then*
$$\operatorname{Cl}(\mathbb{P}^r_n) \simeq \mathbb{Z}.$$

Proof. Note that if Y is a prime divisor in $\mathbb{P}^n_k$ then Y is a hypersurface in $\mathbb{P}^n$, so that $I = \langle G \rangle$ and Y is defined by a single homogeneous polynomial G. The degree of G is called the degree of Y.

If $D = \sum n_Y Y$ is a Weil divisor then define the degree $\deg D$ of D to be the sum
$$\sum n_Y \deg Y,$$
where $\deg Y$ is the degree of Y.

Note that the degree of any rational function is zero. Thus there is a well-defined group homomorphism
$$\deg\colon \operatorname{Cl}(\mathbb{P}^r_k) \longrightarrow \mathbb{Z},$$
and it suffices to prove that this map is an isomorphism. Let H be defined by X_0. Then H is a hyperplane and H has degree one. The divisor $D = nH$ has degree n and so the degree map is surjective. One the other hand, if $D = \sum n_i Y_i$ is effective, and Y_i is defined by G_i,
$$(\prod_i G^{n_i}/X_0^d) = D - dH,$$
where d is the degree of D. $\square$

Example 7.8. *Let C be a smooth cubic curve in $\mathbb{P}^2_k$. Suppose that the line $Z = 0$ is a flex line to the cubic at the point $P_0 = [0 : 1 : 0]$. If the equation of the cubic is $F(X, Y, Z)$ this says that $F(X, Y, 0) = X^3$. Therefore the cubic has the form $X^3 + ZG(X, Y, Z)$. If we work on the open subset $U_3 \simeq \mathbb{A}^2_k$, then we get*
$$x^3 + g(x, y) = 0,$$
where $g(x, y)$ has degree at most two. If we expand $g(x, y)$ as a polynomial in y,
$$g_0(x)y^2 + g_1(x)y + g_2(x),$$

then $g_0(x)$ must be a non-zero scalar, since otherwise C is singular (a nodal or cuspidal cubic). We may assume that $g_0 = 1$. If we assume that the characteristic is not two then we may complete the square to get

$$y^2 = x^3 + g(x),$$

for some quadratic polynomial $g(x)$. If we assume that the characteristic is not three then we may complete the cube to get

$$y^2 = x^3 + ax + ab,$$

for some a and $b \in k$.

Now any two sets of three collinear points are linearly equivalent (since the equation of one line divided by another line is a rational function on the whole $\mathbb{P}^2_k$). In fact given any three points P, Q and P' we may find Q' such that $P + Q \sim P' + Q'$; indeed the line $l = \langle P, Q\rangle$ meets the cubic in one more point R. The line $l' = \langle R, P'\rangle$ then meets the cubic in yet another point Q'. We have

$$P + Q + R \sim P' + Q' + R'.$$

Cancelling we get

$$P + Q \sim P' + Q'.$$

It follows that if there are further linear equivalences then there are two points P and P' such that $P \sim P'$. This gives us a rational function f with a single zero P and a single pole P'; in turn this gives rise to a morphism $C \longrightarrow \mathbb{P}^1$ which is an isomorphism. It turns out that a smooth cubic is not isomorphic to $\mathbb{P}^1$, so that in fact the only relations are those generated by setting two sets of three collinear points to be linearly equivalent.

Put differently, the rational points of C form an abelian group, where three points sum to zero if and only if they are collinear, and P_0 is declared to be the identity. The divisors of degree zero modulo linear equivalence are equal to this group.

In particular, an elliptic curve is very far from being isomorphic to $\mathbb{P}^1_k$.

It is interesting to calculate the Class group of a toric variety X, which always satisfies $(*)$. By assumption there is a dense open subset $U \simeq \mathbb{G}_m^n$. The complement Z is a union of the invariant divisors.

Lemma 7.9. *Suppose that X satisfies $(*)$, let Z be a closed subset and let $U = X \setminus Z$.*

Then there is an exact sequence

$$\mathbb{Z}^k \longrightarrow \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(U) \longrightarrow 0,$$

where k is the number of components of Z which are prime divisors.

Proof. If Y is a prime divisor on X then $Y' = Y \cap U$ is either a prime divisor on U or empty. This defines a group homomorphism

$$\rho\colon \operatorname{Div}(X) \longrightarrow \operatorname{Div}(U).$$

If $Y' \subset U$ is a prime divisor then let Y be the closure of Y' in X. Then Y is a prime divisor and $Y' = Y \cap U$. Thus ρ is surjective. If f is a rational function on X and $Y = (f)$ then the image of Y in $\operatorname{Div}(U)$ is equal to $(f|_U)$. If $Z = Z' \cup \bigcup_{i=1}^k Z_i$ where Z' has codimension at least two then the map which sends $(m_1, m_2, \dots, m_k)$ to $\sum m_i Z_i$ generates the kernel. $\square$

Example 7.10. *Let $X = \mathbb{P}^2_k$ and C be an irreducible curve of degree d. Then $\operatorname{Cl}(\mathbb{P}^2 - C)$ is equal to $\mathbb{Z}_d$. Similarly $\operatorname{Cl}(\mathbb{A}^n_k) = 0$.*

It follows by (7.9) that there is an exact sequence

$$\mathbb{Z}^k \longrightarrow \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(U) \longrightarrow 0.$$

Applying this to $X = \mathbb{A}^n_k$ it follows that $\operatorname{Cl}(U) = 0$. So we get an exact sequence

$$0 \longrightarrow K \longrightarrow \mathbb{Z}^s \longrightarrow \operatorname{Cl}(X) \longrightarrow 0.$$

We want to identify the kernel. This is equal to the set of principal divisors which are supported on the invariant divisors. If f is a rational function such that (f) is supported on the invariant divisors then f has no zeroes or poles on the torus; it follows that $f = \lambda\chi^u$, where $\lambda \in k^*$ and $u \in M$.

It follows that there is an exact sequence

$$M \longrightarrow \mathbb{Z}^s \longrightarrow \operatorname{Cl}(X) \longrightarrow 0.$$

Lemma 7.11. *Let $u \in M$. Suppose that X is the affine toric variety associated to a cone σ, where σ spans $N_{\mathbb{R}}$. Let v be a primitive generator of a one dimensional ray τ of σ and let D be the corresponding invariant divisor.*

Then $\operatorname{ord}_D(\chi^u) = \langle u, v\rangle$. In particular

$$(\chi^u) = \sum_i \langle u, v_i\rangle D_i,$$

where the sum ranges over the invariant divisors.

Proof. We can calculate the order on the open set $U_\tau = \mathbb{A}^1_k \times \mathbb{G}_m^{n-1}$, where D corresponds to $\{0\} \times \mathbb{G}_m^{n-1}$. Using this, we are reduced to the one dimensional case. So $N = \mathbb{Z}$, $v = 1$ and $u \in M = \mathbb{Z}$. In this case χ^u is the monomial x^u and the order of vanishing at the origin is exactly u. $\square$

It follows that if $X = X(F)$ is the toric variety associated to a fan F which spans $N_{\mathbb{R}}$ then we have short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^s \longrightarrow \mathrm{Cl}(X) \longrightarrow 0.$$

Example 7.12. *Let σ be the cone spanned by $2e_1 - e_2$ and e_2 inside $N_{\mathbb{R}} = \mathbb{R}^2$. There are two invariant divisors D_1 and D_2. The principal divisor associated to $u = f_1 = (1, 0)$ is $2D_1$ and the principal divisor associated to $u = f_2 = (0, 1)$ is $D_2 - D_1$. So the class group is $\mathbb{Z}_2$.*

Note that the dual $\check{\sigma}$ is the cone spanned by f_1 and $f_1 + 2f_2$. Generators for the monoid $S_\sigma = \check{\sigma} \cap M$ are f_1, $f_1 + f_2$ and $f_1 + 2f_2$. So the group algebra

$$A_\sigma = k[x, xy, xy^2] = \frac{k[u, v, w]}{\langle v^2 - uw \rangle},$$

and $X = U_\sigma$ is the quadric cone.

Now suppose we take the standard fan associated to $\mathbb{P}^2$. The invariant divisors are the three coordinate lines, D_1, D_2 and D_3. If $f_1 = (1, 0)$ and $f_2 = (0, 1)$ then

$$(\chi^{f_1}) = D_1 - D_3 \qquad \text{and} \qquad (\chi^{f_2}) = D_2 - D_3.$$

So the class group is $\mathbb{Z}$.

We now turn to the notion of a Cartier divisor.

Definition 7.13. *Given a ring A, let S be the multiplicative set of non-zero divisors of A. The localisation A_S of A at S is called the* ***total quotient ring*** *of A.*

Given a scheme X, let $\mathcal{K}$ be the sheaf associated to the presheaf, which associates to every open subset $U \subset X$, the total quotient ring of $\Gamma(U, \mathcal{O}_X)$. $\mathcal{K}$ is called the ***sheaf of total quotient rings****.*

Definition 7.14. *A* ***Cartier divisor*** *on a scheme X is any global section of $\mathcal{K}^*/\mathcal{O}_X^*$.*

In other words, a Cartier divisor is specified by an open cover U_i and a collection of rational functions f_i, such that f_i/f_j is a nowhere zero regular function.

A Cartier divisor is called **principal** if it is in the image of $\Gamma(X, \mathcal{K}^*)$. Two Cartier divisors D and D' are called **linearly equivalent**, denoted $D \sim D'$, if and only if the difference is principal.

Definition 7.15. *Let X be a scheme satisfying $(*)$. Then every Cartier divisor determines a Weil divisor.*

Informally a Cartier divisor is simply a Weil divisor defined locally by one equation. If every Weil divisor is Cartier then we say that X

is **factorial**. This is equivalent to requiring that every local ring is a UFD; for example every smooth variety is factorial.

Definition-Lemma 7.16. *Let X be a scheme.*

The set of invertible sheaves forms an abelian group $\mathrm{Pic}\,(X)$*, where multiplication corresponds to tensor product and the inverse to the dual.*

Definition 7.17. *Let D be a Cartier divisor, represented by $\{(U_i, f_i)\}$. Define a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}$ by taking the subsheaf generated by f_i^{-1} over the open set U_i.*

Proposition 7.18. *Let X be a scheme.*

1. *The association $D \longrightarrow \mathcal{O}_X(D)$ defines a correspondence between Cartier divisors and invertible subsheaves of $\mathcal{K}$.*
2. *If $\mathcal{O}_X(D_1 - D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$.*
3. *Two Cartier divisors D_1 and D_2 are linearly equivalent if and only if $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$.*

Let's consider which Weil divisors on a toric variety are Cartier. We classify all Cartier divisors whose underlying Weil divisor is invariant; we dub these Cartier divisors T-Cartier. We start with the case of the affine toric variety associated to a cone $\sigma \subset N_{\mathbb{R}}$. By (7.18) it suffices to classify all invertible subsheaves $\mathcal{O}_X(D) \subset \mathcal{K}$. Taking global sections, since we are on an affine variety, it suffices to classify all fractional ideals,

$$I = H^0(X, \mathcal{O}_X(D)).$$

Invariance of D implies that I is graded by M, that is, I is a direct sum of eigenspaces. As D is Cartier, I is principal at the distinguished point x_σ of U_σ, so that $I/\mathfrak{m}I$ is one dimensional, where

$$\mathfrak{m} = \sum k \cdot \chi^u.$$

It follows that $I = A_\sigma \chi^u$, so that $D = (\chi^u)$ is principal. In particular, the only Cartier divisors are the principal divisors and X is factorial if and only if the Class group is trivial.

Example 7.19. *The quadric cone Q, given by $xy - z^2 = 0$ in $\mathbb{A}^3_k$ is not factorial. We have already seen* (7.12) *that the class group is $\mathbb{Z}_2$.*

If $\sigma \subset N_{\mathbb{R}}$ is not maximal dimensional then every Cartier divisor on U_σ whose associated Weil divisor is invariant is of the form (χ^u) but

$$(\chi^u) = (\chi^{u'}) \qquad \text{if and only if} \qquad u - u' \in \sigma^\perp \cap M = M(\sigma).$$

So the T-Cartier divisors are in correspondence with $M/M(\sigma)$.

Now suppose that $X = X(F)$ is a general toric variety. Then a T-Cartier divisor is given by specifying an element $u(\sigma) \in M/M(\sigma)$, for every cone σ in F. This defines a divisor $(\chi^{-u(\sigma)})$; equivalently a fractional ideal

$$I = H^0(X, \mathcal{O}_X(D)) = A_\sigma \cdot \chi^{u(\sigma)}.$$

These maps must agree on overlaps; if τ is a face of σ then $u(\sigma) \in M/M(\sigma)$ must map to $u(\tau) \in M/M(\tau)$.

The data

$$\{\, u(\sigma) \in M/M(\sigma) \,|\, \sigma \in F \,\},$$

for a T-Cartier divisor D determines a continuous piecewise linear function ϕ_D on the support $|F|$ of F. If $v \in \sigma$ then let

$$\phi_D(v) = \langle u(\sigma), v \rangle.$$

Compatibility of the data implies that ϕ_D is well-defined and continuous. Conversely, given any continuous function ϕ, which is linear and integral (given by an element of M) on each cone, we can associate a unique T-Cartier divisor D. If $D = a_i D_i$ the function is given by $\phi_D(v_i) = -a_i$.

Note that

$$\phi_D + \phi_E = \phi_{D+E} \qquad \text{and} \qquad \phi_{mD} = m\phi_D.$$

Note also that $\phi_{(\chi^u)}$ is the linear function given by u. So D and E are linearly equivalent if and only if ϕ_D and ϕ_E differ by a linear function.

If X is any variety which satisfies $(*)$ then the natural map

$$\operatorname{Pic}(X) \longrightarrow \operatorname{Cl}(X),$$

is an embedding. It is an interesting to compare $\operatorname{Pic}(X)$ and $\operatorname{Cl}(X)$ on a toric variety. Denote by $\operatorname{Div}_T(X)$ the group of T-Cartier divisors.

Proposition 7.20. *Let $X = X(F)$ be the toric variety associated to a fan F which spans $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows:*

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & \operatorname{Div}_T(X) \arrow[r] \arrow[d, hook] & \operatorname{Pic}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbb{Z}^s \arrow[r] & \operatorname{Cl}(X) \arrow[r] & 0
\end{tikzcd}$$

In particular

$$\rho(X) = \operatorname{rank}(\operatorname{Pic}(X)) \leq \operatorname{rank}(\operatorname{Cl}(X)) = s - n.$$

Further $\operatorname{Pic}(X)$ *is a free abelian group.*

Proof. We have already seen that the bottom row is exact. If $\mathcal{L}$ is an invertible sheaf then $\mathcal{L}|_U$ is trivial. Suppose that $L = \mathcal{O}_X(E)$. Pick a rational function such that $(f)|_U = E|_U$. Let $D = E - (f)$. Then D is T-Cartier and exactness of the top row is easy.

Finally, $\operatorname{Pic}(X)$ is subgroup of the direct sum of $M/M(\sigma)$ and each of these is a lattice, whence $\operatorname{Pic}(X)$ is torsion free. $\square$

Example 7.21. *Let $C \subset \mathbb{P}^2_k$ be the nodal cubic $ZY^2 = X^3 + X^2Z$, so that in the affine piece $U_3 \simeq \mathbb{A}^2_k$, $C \cap U_3$ is given by $y^2 = x^2 + x^3$. Let N be the node. Note that if D is a Weil divisor whose support does not contain N then D is automatically a Cartier divisor. As in the case of the smooth cubic, if P, Q, R and P', Q' and R' are two triples of collinear points on C (none of which are N), then $P + Q + R \sim P' + Q' + R'$.*

Now we already know that the nodal cubic is not isomorphic to $\mathbb{P}^1$. This implies that if P and P' are two smooth points of C then P and P' are not linearly equivalent. It follows, with a little bit of work, that all linear equivalences on C are generated by the linear equivalences above.

The normalisation of C is isomorphic to $\mathbb{P}^1$; on the affine piece where $Z \neq 0$ the normalisation morphism is given as $t \longrightarrow (t^2 - 1, t(t^2 - 1))$. The inverse image of the node N contains two points of $\mathbb{P}^1$ and it follows that $C - \{N\}$ is isomorphic to $\mathbb{G}_m$. In fact one can check that this is an isomorphism of algebraic groups, where the group law on $C - \{N\}$ is given by declaring three collinear points to sum to zero.

There is an exact sequence of groups,

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \operatorname{Pic}(C) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the first map sends P to $P - P_0 = [0 : 1 : 0]$, and the second map is the degree map which sends $D = \sum n_i P_i$ to $\sum n_i$.

Note that even though we can talk about Weil divisors on C, it only makes sense to talk about linear equivalences of Weil divisors supported away from N. Indeed, the problem is that any line through N cuts out $2N + R$, where R is another point of C. Varying the line varies R but fixes $2N$. In terms of Cartier divisors, a line through N (and not tangent to a branch) is equivalent to a length two scheme contained in the line. As we vary the line, both R and the length two scheme vary.

It is interesting to consider what happens at the level of invertible sheaves. Consider an invertible sheaf $\mathcal{L}$ on C which is of degree zero, that is, conside an invertible sheaf which corresponds to a Cartier divisor D of degree zero. If $\pi\colon \mathbb{P}^1 \longrightarrow C$ is the normalisation map then

$$\pi^*\mathcal{L} = \pi^*\mathcal{O}_C(D) = \mathcal{O}_{\mathbb{P}^1}(\pi^* D),$$

has degree zero (to pullback a Cartier divisor, just pullback the defining equations. It is easy to check that this commutes with pullback of the sheaf). Since $\mathrm{Pic}(\mathbb{P}^1) \simeq \mathbb{Z}$*,* $\pi^* L \simeq \mathcal{O}_{\mathbb{P}^1}$*, the trivial sheaf. Now to get a sheaf on* C *we have to glue the two local rings over the inverse image* N_1 *and* N_2 *of* N*. The only isomorphisms of two such local rings are* $\mathbb{G}_m$ *acting by scalar multiplication (this is particularly tranparent if one thinks of a invertible sheaf as a line bundle; in this case we are just identifying two copies of a one dimensional vector space) and this is precisely the kernel of the degree map on* $\mathbb{P}^1$*).*

There is a similar picture for the cuspidal cubic, given as $Y^2Z = X^3$*. The only twist is that* $C - \{N\}$*, where* N *is the cusp, is now a copy of* $\mathbb{G}_a$*.*

8. Smoothness and the Zariski tangent space

We want to give an algebraic notion of the tangent space. In differential geometry, tangent vectors are equivalence classes of maps of intervals in $\mathbb{R}$ into the manifold. This definition lifts to algebraic geometry over $\mathbb{C}$ but not over any other field (for example a field of characteristic p).

Classically tangent vectors are determined by taking derivatives, and the tangent space to a variety X at x is then the space of tangent directions, in the whole space, which are tangent to X. Even is this is how we will compute the tangent space in general, it is still desirable to have an intrinsic definition, that is a definition which does not use the fact that X is embedded in $\mathbb{P}^n$.

Now note first that the notion of smoothness is surely local and that if we want an intrinsic definition, then we want a definition that only uses the functions on X. Putting this together, smoothness should be a property of the local ring of X at p. On the other hand taking derivatives is the same as linear approximation, which means dropping quadratic and higher terms.

Definition 8.1. *Let X be a variety and let $p \in X$ be a point of X. The **Zariski tangent space** of X at p, denoted T_pX, is equal to the dual of the quotient*

$$\mathfrak{m}/\mathfrak{m}^2,$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$.

Note that $\mathfrak{m}/\mathfrak{m}^2$ is a vector space. Suppose that we are given a morphism

$$f\colon X \longrightarrow Y,$$

which sends p to q. In this case there is a ring homomorphism

$$f^*\colon \mathcal{O}_{Y,q} \longrightarrow \mathcal{O}_{X,p}$$

which sends the maximal ideal $\mathfrak{n}$ into the maximal ideal $\mathfrak{m}$. Thus we get an induced map

$$df\colon \mathfrak{n}/\mathfrak{n}^2 \longrightarrow \mathfrak{m}/\mathfrak{m}^2$$

On the other hand, geometrically the map on tangent spaces obviously goes the other way. Therefore it follows that we really do want the dual of $\mathfrak{m}/\mathfrak{m}^2$. In fact $\mathfrak{m}/\mathfrak{m}^2$ is the dual of the Zariski tangent space, and is referred to as the *cotangent space.*

In particular, given a morphism $f\colon X \longrightarrow Y$ carrying p to q, then there is a linear map

$$df : T_pX \longrightarrow T_qY$$

Definition 8.2. *Let X be a quasi-projective variety.*

*We say that X is **smooth** at p if the local dimension of X at p is equal to the dimension of the Zariski tangent space at p.*

Now the tangent space to $\mathbb{A}^n$ is canonically a copy of $\mathbb{A}^n$ itself, considered as a vector space based at the point in question. If $X \subset \mathbb{A}^n$, then the tangent space to X is included inside the tangent space to $\mathbb{A}^n$. The question is then how to describe this subspace.

Lemma 8.3. *Let $X \subset \mathbb{A}^n$ be an affine variety, of dimension k, and suppose that $f_1, f_2, \dots, f_k$ generates the ideal I of X. Then the tangent space of X at p, considered as a subspace of the tangent space to $\mathbb{A}^n$, via the inclusion of X in $\mathbb{A}^n$, is equal to the kernel of the Jacobian matrix.*

Proof. Clearly it is easier to give the dual description of the cotangent space.

If $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{\mathbb{A}^n,p}$ and $\mathfrak{n}$ is the maximal ideal of $\mathcal{O}_{X,p}$, then clearly the natural map $\mathfrak{m} \longrightarrow \mathfrak{n}$ is surjective, so that the induced map on contangent spaces is surjective. Dually, the induced map on the Zariski tangent space is injective, so that T_pX is indeed included in $T_p\mathbb{A}^n$.

We may as well choose coordinates $x_1, x_2, \dots, x_n$ so that p is the origin. In this case $\mathfrak{m} = \langle x_1, x_2, \dots, x_n \rangle$ and $\mathfrak{n} = \mathfrak{m}/I$. Moreover $\mathfrak{m}/\mathfrak{m}^2$ is the vector space spanned by $dx_1, dx_2, \dots, dx_n$, where dx_i denotes the equivalence class $x_i + \mathfrak{m}^2$, and $\mathfrak{n}/\mathfrak{n}^2$ is canonically isomorphic to $\mathfrak{m}/(\mathfrak{m}^2 + I)$. Now the transpose of the Jacobian matrix, defines a linear map

$$K^k \longrightarrow K^n = T_p^*\mathbb{A}^n,$$

and it suffices to prove that the image of this map is the kernel of the map

$$df \colon \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathfrak{n}/\mathfrak{n}^2.$$

Let $g \in \mathfrak{m}$. Then

$$g(x) = \sum a_i x_i + h(x),$$

where $h(x) \in \mathfrak{m}^2$. Thus the image of $g(x)$ in $\mathfrak{m}/\mathfrak{m}^2$ is equal to $\sum_i a_i dx_i$. Moreover, by standard calculus a_i is nothing more than

$$a_i = \left.\frac{\partial g}{\partial x_i}\right|_p .$$

Thus the kernel of the map df is generated by the image of f_i in $\mathfrak{m}/\mathfrak{m}^2$, which is

$$\sum_j \left.\frac{\partial f_i}{\partial x_j}\right|_p dx_j,$$

which is nothing more than the image of the Jacobian. ☐

Lemma 8.4. *Let X be a quasi-projective variety. Then the function*

$$\lambda\colon X \longrightarrow \mathbb{N},$$

is upper semi-continuous, where $\lambda(x)$ is the dimension of the Zariski tangent space at x.

Proof. Clearly this result is local on X so that we may assume that X is affine. In this case the Jacobian matrix defines a morphism π from X to the space of matrices and the locus where the Zariski tangent space has a fixed dimension is equal to the locus where this morphism lands in the space of matrices of fixed rank. Put differently the function λ is the composition of π and an affine linear function of the rank on the space of matrices. Since the rank function is upper semicontinuous, the result follows. ☐

Lemma 8.5. *Every irreducible quasi-projective variety is birational to a hypersurface.*

Proof. Let X be a quasi-projective variety of dimension k, with function field L/K. By assumption there is an intermediary field $L/M/K$ such that M/K is purely transcendental of transcendence degree k. It follows that the extension L/M is algebraic and finitely generated, whence it is finite. By the primitive element Theorem, L/M is generated by one element, say α. It follows that there is polynomial $f(x) \in M[x]$ such that α is a root of $f(x)$. If $M = K(x_1, x_2, \dots, x_k)$, then clearing denominators, we may assume that $f(x) \in K[x_1, x_2, \dots, x_k][x] \simeq K[x_1, x_2, \dots, x_{k+1}]$. But then X is birational to the hypersurface defined by $F(X)$, where $F(X)$ is the homgenisation of $f(x)$. ☐

Proposition 8.6. *The set of smooth points of any variety is Zariski dense.*

Proof. Since the dimension of the Zariski tangent space is upper semicontinuous, and always at least the dimension of the variety, it suffices to prove that every irreducible variety contains at least one smooth point. By (8.5) we may assume that X is a hypersurface. Passing to an affine open subset, we may assume that X is an affine hypersurface. Let f be a definining equation, so that f is an irreducible polynomial. Then the set of singular points of X is equal to the locus of points where every partial derivative vanishes. But any partial derivative of f is a non-zero polynomial of degree one less than f, and so cannot vanish on X. ☐

A basic result in the theory of C^∞-maps is Sard's Theorem, which states that the set of points where a map is singular is a subset of measure zero (of the base). Since any holomorphic map between complex manifolds is automatically C^∞, and the derivative of a polynomial is the same as the derivative as a holomorphic function, it follows that any morphism between varieties, over $\mathbb{C}$, is smooth over an open subset. In fact by the Lefschetz principle, this result extends to any variety over $\mathbb{C}$.

Theorem 8.7. *Let $f \colon X \longrightarrow Y$ be a morphism of varieties over a field of characteristic zero.*

Then there is a dense open subset U of Y such that if $q \in U$ and $p \in f^{-1}(q) \cap X_{sm}$ then the differential $df_p \colon T_pX \longrightarrow T_qY$ is surjective. Further, if X is smooth, then the fibres $f^{-1}(q)$ are smooth if $q \in U$.

Let us recall the Lefschetz principle. First recall the notion of a first order theory of logic. Basically this means that one describes a theory of mathematics using a theory based on predicate calculus. For example, the following is a true statement from the first order theory of number theory,

$$\forall n \forall x \forall y \forall z \quad n \geq 3 \implies x^n + y^n \neq z^n.$$

One basic and desirable property of a first order theory of logic is that it is complete. In other words every possible statement (meaning anything that is well-formed) can be either proved or disproved. It is a very well-known result that the first order theory of number theory is not complete (Gödel's Incompleteness Theorem). What is perhaps more surprising is that there are interesting theories which are complete.

Theorem 8.8. *The first order logic of algebraically closed fields of characteristic zero is complete.*

Notice that a typical statement of the first order logic of fields is that a system of polynomial equations does or does not have solution. Since most statements in algebraic geometry turn on whether or not a system of polynomial equations have a solution, the following result is very useful.

Principle 8.9 (Lefschetz Principle)**.** *Every statement in the first order logic of algebraically closed fields of characteristic zero, which is true over $\mathbb{C}$, is in fact true over any algebraically closed field of characteristic.*

In fact this principle is immediate from (8.8). Suppose that p is a statement in the first order logic of algebraically closed fields of characteristic zero. By completeness, we can either prove p or not p. Since p holds over the complex numbers, there is no way we can prove not p. Therefore there must be a proof of p. But this proof is valid over any field of characteristic zero, so p holds over any algebraically closed field of characteristic zero.

A typical application of the Lefschetz principle is (8.7). By Sard's Theorem, we know that (8.7) holds over $\mathbb{C}$. On the other hand, (8.7), can be reformulated in the first order logic of algebraically closed fields of characteristic zero. Therefore by the Lefschetz principle, (8.7) is true over algebraically closed field of characteristic zero.

Perhaps even more interesting, is that (8.7) fails in characteristic p. Let $f\colon \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ be the morphism $t \longrightarrow t^p$. If we fix s, then the equation

$$x^p - s$$

is purely inseparable, that is, has only one root. Thus f is a bijection. However, df is the zero map, since $dz^p = pz^{p-1}dz = 0$. Thus df_p is nowhere surjective. Note that the fibres of this map, as schemes, are isomorphic to zero dimensional schemes of length p.

We now want to aim for a version of the Inverse function Theorem. In differential geometry, the inverse function theorem states that if a function is an isomorphism on tangent spaces, then it is locally an isomorphism. Unfortunately this is too much to expect in algebraic geometry, since the Zariski topology is too weak for this to be true. For example consider a curve which double covers another curve. At any point where there are two points in the fibre, the map on tangent spaces is an isomorphism. But there is no Zariski neighbourhood of any point where the map is an isomorphism.

Thus a minimal requirement is that the morphism is a bijection. Note that this is not enough in general for a morphism between algebraic varieties to be an isomorphism. For example in characteristic p, Frobenius is nowhere smooth and even in characteristic zero, the parametrisation of the cuspidal cubic is a bijection but not an isomorphism.

Lemma 8.10. *If $f\colon X \longrightarrow Y$ is a projective morphism with finite fibres, then f is finite.*

Proof. Since the result is local on the base, we may assume that Y is affine. By assumption $X \subset Y \times \mathbb{P}^n$ and we are projecting onto the first factor. Blowing up a point of $\mathbb{P}^n$, possibly passing to a smaller open affine subset of Y, we may assume that $n = 1$, by induction on n.

Pick $y \in Y$. Since X/Y has finite fibres, it follows that there is a point of the fibre $\{y\} \times \mathbb{P}^1$ not contained in X. Possibly passing to a smaller open subset, we may assume that $X \subset Y \times \mathbb{A}^1$, so that X is affine. Now X is defined by $f(x) \in A(Y)[x]$, where the coefficients of $f(x)$ lie in $A(Y)$. Suppose that

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0.$$

We may always assume that a_n does not vanish at y. Passing to the locus where a_n does not vanish, we may assume that a_n is a unit, so that dividing by a_n, we may assume that $a_n = 1$. In this case the ring B is a quotient of the ring

$$A[x]/\langle f \rangle.$$

But the latter is generated over A by 1, x, $\dots x^{n-1}$, and so is a finitely generated module over A. $\square$

Theorem 8.11. *Let $f\colon X \longrightarrow Y$ be a projective morphism between quasi-projective variety varieties.*

Then f is an isomorphism iff it is a bijection and the differential df_p is injective.

Proof. One direction is clear. Otherwise assume that f is projective and a bijection on closed points. Then f is finite by (8.10). Pick $x \in X$ and let $y = f(x)$. Let A be the local ring of Y at y, B of X at x. Let $\phi\colon A \longrightarrow B$ be the induced ring homomorphism. Then B is a finitely generated A-module, by assumption, and we just need to show that ϕ is an isomorphism.

As f is a bijection on closed points, it follows that ϕ is injective. So we might as well suppose that ϕ is an inclusion. Let $\mathfrak{m}$ be the maximal ideal of A and let $\mathfrak{n}$ be the maximal ideal of B. By assumption

$$\frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow \frac{\mathfrak{n}}{\mathfrak{n}^2},$$

is surjective. But then

$$\mathfrak{m}B + \mathfrak{n}^2 = \mathfrak{n}.$$

Nakayama's Lemma applied to the B-module $\mathfrak{n}/\mathfrak{m}B$ it follows that $\mathfrak{m}B = \mathfrak{n}$. But then

$$B/A \otimes A/\mathfrak{m} = B/(\mathfrak{m}B + A) = B/(\mathfrak{n} + A) = 0.$$

Nakayama's Lemma applied to the finitely generated A-module B/A implies that $B/A = 0$ so that ϕ is an isomorphism. $\square$

Lemma 8.12. *Suppose that $X \subset \mathbb{P}^n$ is a quasi-projective variety and suppose that $\pi\colon X \longrightarrow Y$ is the morphism induced by projection from a linear subspace.*

Let $y \in Y$. Then $\pi^{-1}(y) = \langle \Lambda, y\rangle \cap X$. If further this fibre consists of one point, then the map between Zariski tangent spaces is an isomorphism if the intersection of $\langle \Lambda, x\rangle$ with the Zariksi tangent space to X at x has dimension zero.

Proof. Easy. □

Lemma 8.13. *Let X be a smooth irreducible subset of $\mathbb{P}^n$ of dimension k. Consider the projection Y of X down to a smaller dimensional projective space $\mathbb{P}^m$, from a linear space Λ of dimension $n - m - 1$.*

If the dimension of $m \geq 2k + 1$ and Λ is general (that is belongs to an appropriate open subset of the Grassmannian) then π is an isomorphism.

Proof. Since projection from a general linear space is the same as a sequence of projections from general points, we may assume that Λ is in fact a point p, so that $m = n - 1$.

Now we know that π is a bijection provided that p does not lie on any secant line. Since the secant variety has dimension at most $2k + 1$, it follows that we may certainly find a point away from the secant variety, provided that $n > 2k + 1$. Now since a tangent line is a limit of secant lines, it follows that such a point will also not lie on any tangent lines.

But then π is then an isomorphism on tangent spaces, whence an isomorphism. □

For example, it follows that any curve may be embedded in $\mathbb{P}^3$ and any surface in $\mathbb{P}^5$. Now let us turn to the following classical problem in enumerative geometry.

Question 8.14. *Let $C \subset \mathbb{P}^n$ be a curve in $\mathbb{P}^2$ and let $p \in \mathbb{P}^2$.*

How many tangent lines does p lie on?

The first thing that we will need is a natty way to describe the projective tangent space to a variety.

Definition 8.15. *Let $X \subset \mathbb{P}^n$.*

The ***projective tangent space to** X **at*** *p is the closure of the affine tangent space.*

In other words the projective tangent space has the same dimension as the affine tangent space and is obtained by adding the suitable points at infinity. Suppose that the curve is defined by the polynomial $F(X, Y, Z)$. Then the tangent line to C at p, is

$$\left.\frac{\partial F}{\partial X}\right|_p X + \left.\frac{\partial F}{\partial Y}\right|_p Y + \left.\frac{\partial F}{\partial Z}\right|_p Z.$$

Of course it suffices to check that we get the right answer on an affine piece.

Lemma 8.16. *Let F be a homogeneous polynomial of degree d in $X_0, X_1, \dots, X_n$. Then*
$$dF = \sum X_i \frac{\partial F}{\partial X_i}$$

Proof. Both sides are linear in F. Thus it suffices to prove this for a monomial of degree d, when the result is clear. $\square$

It follows then that the tangent line above does indeed pass through p. The rest is easy.

Finally we will need Bézout's Theorem.

Theorem 8.17 (Bézout's Theorem)**.** *Let C and D be two curves defined by homogenous polynomials of degrees d and e. Suppose that $C \cap D$ does not contain a curve.*

Then $|C \cap D|$ is at most de, with equality iff the intersection of the two tangent spaces at $p \in C \cap D$ is equal to p.

We are now ready to answer (8.14).

Lemma 8.18. *Let $C \subset \mathbb{P}^n$ be a curve in $\mathbb{P}^2$ and let $p \in \mathbb{P}^2$ be a general point.*

Then p lies on $d(d-1)$ tangent lines.

Proof. Fix $p = [a : b : c]$ and let D be the curve defined by
$$G = a\frac{\partial F}{\partial X} + b\frac{\partial F}{\partial Y} + c\frac{\partial F}{\partial Z}.$$
Then G is a polynomial of degree $d-1$. Consider a point q where C intersects D. Then the tangent line to C at q is given by
$$\left.\frac{\partial F}{\partial X}\right|_q X + \left.\frac{\partial F}{\partial Y}\right|_q Y + \left.\frac{\partial F}{\partial Z}\right|_q Z.$$
But then since p satisfies this equation, as q lies on D, it follows that p lies on the tangent line of C at q. Similarly it is easy to check the converse, that if p lies on the tangent line to C at q, then q is an intersection point of C and D.

Now apply Bézout's Theorem. $\square$

There is an interesting way to look at all of this. In fact one may generalise the result above to the case of curves with nodes. Note that if you take a curve in $\mathbb{P}^3$ and take a general projection down to $\mathbb{P}^2$, then you get a nodal curve. Indeed it is easy to pick the point of projection not on a tangent line, since the space of tangent lines obviously sweeps out a surface; it is a little more involved to show that the space of three secant lines is a proper subvariety. (8.18) was then generalised to this case and it was shown that if δ is the number of nodes, then the number

$$\frac{d(d-1)}{2} - \delta$$

is an invariant of the curve.

Here is another way to look at this. Suppose that we project our curve down to $\mathbb{P}^1$ from a point. Then we get a finite cover of $\mathbb{P}^1$, with d points in the general fibre. Lines tangent to C passing through p then count the number of branch points, that is, the number of points in the base where the fibre has fewer than d points. Since this tangent line is only tangent to p and is simply tangent (that is, there are no flex points) there are $d-1$ points in this fibre, and the ramification point corresponding to the branch point is where two sheets come together.

The modern approach to this invariant is quite different. If we are over the complex numbers $\mathbb{C}$, changing perspective, we may view the curve C as a Riemann surface covering another Riemann surface D. Now the basic topological invariant of a compact oriented Riemann surface is it's genus. In these terms there is a simple formula that connects the genus of C and B, in terms of the ramification data, known as Riemann-Hurwitz,

$$2g - 2 = d(2h - 2) + b,$$

where g is the genus of C, h the genus of B, d the order of the cover and b the contribution from the ramification points. Indeed if locally on C, the map is given as $z \longrightarrow z^e$ so that e sheets come together, the contribution is $e-1$.

In our case, $B = \mathbb{P}^1$ which is of genus 0, for each branch point, we have simple ramification, so that $e = 2$ and the contribution is one, making a total $b = d(d-1)$. Thus

$$2g - 2 = -2d + d(d-1).$$

Solving for g we get

$$g = \frac{(d-1)(d-2)}{2}.$$

Note that if $d \leq 2$, then we get $g = 0$ as expected (that is $C \simeq \mathbb{P}^1$) and if $d = 3$ then we get an elliptic curve.

It also seems worth pointing out that if we take a smooth variety X and blow up a point p, then the exceptional divisor E is canonically the projectivisation of the Zariski tangent space to X at p,

$$E = \mathbb{P}(T_pX).$$

Indeed the point is that E picks up the different tangent directions to X at p, and this is exactly the set of lines in T_pX. Note the difference between the projective tangent space and the projectivisation of the tangent space.

It also seems worth pointing out that one define the Zariski tangent space to a scheme X, at a point x, using exactly the same definition, the dual of

$$\mathfrak{m}/\mathfrak{m}^2,$$

where $\mathfrak{m} \subset \mathcal{O}_{X,x}$ is the maximal ideal of the local ring. We have already seen that this the same as looking at the space of maps

$$\mathrm{Hom}(\operatorname{Spec} k[\epsilon]/\langle \epsilon^2 \rangle, X),$$

which sends $\langle \epsilon \rangle$ to x and where k is the residue field of $x \in X$. However in general, if we have the equality of dimensions of both the Zariski tangent space and the local dimension, we only call X **regular at** $x \in X$. Smoothness is a more restricted notion in general.

Having said this, if X is a quasi-projective variety over an algebraically closed field then X is smooth as a variety if and only if it is smooth as a scheme over $\operatorname{Spec} k$. In fact an abstract variety over $\operatorname{Spec} k$ is smooth if and only if it is regular. Note that if x is a specialisation of ξ and X is regular at x then X is regular at ξ, so it is enough to check that X is regular at the closed points.

Note that one can sometimes use the Zariski tangent space to identify embedded points. If X is a scheme and $Y = X_{\mathrm{red}}$ is the reduced subscheme then $x \in X$ is an embedded point if

$$\dim T_x X > \dim T_x Y,$$

and X is reduced out of x. For example, if X is not regular at x but Y is regular at x then $x \in X$ is an embedded point. Note however that it is possible that $x \in X$ is an embedded point but the Zariski tangent space is no bigger than it should be; for example if $\mathcal{H}_0^3$ is the punctual Hilbert scheme of a smooth surface then the underlying variety is a quadric cone. The vertex of the cone corresponds to the unique zero dimensional length three scheme which is not curivilinear and this is an embedded point, even though the Zariski tangent space is three dimensional.

It is interesting to see which toric varieties are smooth. The question is local, so we might as well assume that $X = U_\sigma$ is affine. If $\sigma \subset N_{\mathbb{R}}$ does not span $N_{\mathbb{R}}$, then $X \simeq U_{\sigma'} \times \mathbb{G}_m^l$, where σ' is the same cone as σ embedded in the space it spans. So we might as well assume that σ spans $N_{\mathbb{R}}$. In this case X contains a unique fixed point x_σ which is in the closure of every orbit. Since X only contains finitely many orbits, it follows that X is smooth if and only if X is regular at x_σ. The maximal ideal of x_σ is generated by χ^u, where $u \in S_\sigma$. The square of the maximal is generated by χ^{u+v}, where u and v are two elements of S_σ. So a basis for $\mathfrak{m}/\mathfrak{m}^2$ is given by elements of S_σ that are not sums of two elements. Since the elements of S_σ generate the group M, the elements of S_σ which are not sums of two elements, must generate the group. Given an extremal ray of $\check{\sigma}$, a primitive generator of this ray is not the sum of two elements in S_σ. So $\check{\sigma}$ must have n edges and they must generate M. So these elements are a basis of the lattice and in fact $X \simeq \mathbb{A}^n_k$.

9. Linear systems

Theorem 9.1. *Let X be a scheme over a ring A.*

(1) *If $\phi\colon X \longrightarrow \mathbb{P}^n_A$ is an A-morphism then $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ is an invertible sheaf on X, which is generated by the global sections $s_0, s_1, \dots, s_n$, where $s_i = \phi^* x_i$.*
(2) *If $\mathcal{L}$ is an invertible sheaf on X, which is generated by the global sections $s_0, s_1, \dots, s_n$, then there is a unique A-morphism $\phi\colon X \longrightarrow \mathbb{P}^n_A$ such that $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_A}(1)$ and $s_i = \phi^* x_i$.*

Proof. It is clear that $\mathcal{L}$ is an invertible sheaf. Since $x_0, x_1, \dots, x_n$ generate the ring $A[x_0, x_1, \dots, x_n]$, it follows that $x_0, x_1, \dots, x_n$ generate the sheaf $\mathcal{O}_{\mathbb{P}^n_A}(1)$. Thus $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. Hence (1).

Now suppose that $\mathcal{L}$ is an invertible sheaf generated by $s_0, s_1, \dots, s_n$. Let

$$X_i = \{\, p \in X \mid s_i \notin \mathfrak{m}_p \mathcal{L}_p \,\}.$$

Then X_i is an open subset of X and the sets $X_0, X_1, \dots, X_n$ cover X. Define a morphism

$$\phi_i\colon X_i \longrightarrow U_i,$$

where U_i is the standard open subset of $\mathbb{P}^n_A$, as follows: Since

$$U_i = \operatorname{Spec} A[y_0, y_1, \dots, y_n],$$

where $y_j = x_j/x_i$, is affine, it suffices to give a ring homomorphism

$$A[y_0, y_1, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}).$$

We send y_j to s_j/s_i, and extend by linearity. The key observation is that the ratio is a well-defined element of $\mathcal{O}_{X_i}$, which does not depend on the choice of isomorphism $\mathcal{L}|_{X_i} \simeq \mathcal{O}_{X_i}$.

It is then straightforward to check that the set of morphisms $\{\phi_i\}$ glues to a morphism ϕ with the given properties. □

Example 9.2. *Let $X = \mathbb{P}^1_k$, $A = k$, $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1_k}(2)$.*

In this case, global sections of $\mathcal{L}$ are generated by S^2, ST and T^2. This morphism is represented globally by

$$[S : T] \longrightarrow [S^2 : ST : T^2].$$

The image is the conic $XZ = Y^2$ inside $\mathbb{P}^2_k$.

More generally one can map $\mathbb{P}^1_k$ into $\mathbb{P}^n_k$ by the invertible sheaf $\mathcal{O}_{\mathbb{P}^1_k}(n)$. More generally still, one can map $\mathbb{P}^m_k$ into $\mathbb{P}^n_k$ using the invertible sheaf $\mathcal{O}_{\mathbb{P}^m_k}(1)$.

Corollary 9.3.

$$\operatorname{Aut}(\mathbb{P}^n_k) \simeq \operatorname{PGL}(n+1, k).$$

Proof. First note that $\mathrm{PGL}(n+1,k)$ acts naturally on $\mathbb{P}^n_k$ and that this action is faithful.

Now suppose that $\phi \in \mathrm{Aut}(\mathbb{P}^n_k)$. Let $\mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^n_k}(1)$. Since $\mathrm{Pic}(\mathbb{P}^n_k) \simeq \mathbb{Z}$ is generated by $\mathcal{O}_{\mathbb{P}^n_k}(1)$, it follows that $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(\pm 1)$. As $\mathcal{L}$ is globally generated, we must have $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^n_k}(1)$. Let $s_i = \phi^* x_i$. Then $s_0, s_1, \dots, s_n$ is a basis for the k-vector space $H^0(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1))$. But then there is a matrix

$$A = (a_{ij}) \in \mathrm{GL}(n+1,k) \qquad \text{such that} \qquad s_i = \sum_{ij} a_{ij} x_j.$$

Since the morphism ϕ is determined by $s_0, s_1, \dots, s_n$, it follows that ϕ is determined by the class of A in $\mathrm{GL}(n+1,k)$. □

Lemma 9.4. *Let $\phi\colon X \longrightarrow \mathbb{P}^n_A$ be an A-morphism. Then ϕ is a closed immersion if and only if*

(1) *$X_i = X_{s_i}$ is affine, and*
(2) *the natural map of rings*

$$A[y_0, y_1, \dots, y_n] \longrightarrow \Gamma(X_i, \mathcal{O}_{X_i}) \qquad \textit{which sends} \qquad y_i \longrightarrow \frac{\sigma_i}{\sigma_j},$$

is surjective.

Proof. Suppose that ϕ is a closed immersion. Then X_i is isomorphic to $\phi(X) \cap U_i$, a closed subscheme of affine space. Thus X_i is affine. Hence (1) and (2) follows as we have surjectivity on all of the localisations.

Now suppose that (1) and (2) hold. Then X_i is a closed subscheme of U_i and so X is a closed subscheme of $\mathbb{P}^n_A$. □

Theorem 9.5. *Let X be a projective scheme over an algebraically closed field k and let $\phi\colon X \longrightarrow \mathbb{P}^n_k$ be a morphism over k, which is given by an invertible sheaf $\mathcal{L}$ and global sections $s_0, s_1, \dots, s_n$ which generate $\mathcal{L}$. Let $V \subset \Gamma(X, \mathcal{L})$ be the space spanned by the sections.*

Then ϕ is a closed immersion if and only if

(1) *V **separates points**: that is, given p and $q \in X$ there is $\sigma \in V$ such that $\sigma \in \mathfrak{m}_P\mathcal{L}_p$ but $\sigma \notin \mathfrak{m}_q\mathcal{L}_q$.*
(2) *V **separates tangent vectors**: that is, given $p \in X$ the set*

$$\{\, \sigma \in V \mid \sigma \in \mathfrak{m}_p\mathcal{L}_p \,\},$$

spans $\mathfrak{m}_p\mathcal{L}_p/\mathfrak{m}_p^2\mathcal{L}_p$.

Proof. Suppose that ϕ is a closed immersion. Then we might as well consider $X \subset \mathbb{P}^n_k$ as a closed subscheme. In this case (1) is clear. Just pick a linear function on the whole of $\mathbb{P}^n_k$ which vanishes at p but not at q (equivalently pick a hyperplane which contains p but not q).

Similarly linear functions on $\mathbb{P}^n_k$ separate tangent vectors on the whole of projective space, so they certainly separate on X.

Now suppose that (1) and (2) hold. Then ϕ is clearly injective. Since X is proper over $\operatorname{Spec} k$ and $\mathbb{P}^n_k$ is separated over $\operatorname{Spec} k$ it follows that ϕ is proper. In particular $\phi(X)$ and ϕ is a homeomorphism onto $\phi(X)$. It remains to show that the map on stalks

$$\mathcal{O}_{\mathbb{P}^n_k,p} \longrightarrow \mathcal{O}_{X,x},$$

is surjective. But the same piece of commutative algebra as we used in the proof of the inverse function theorem, works here. $\square$

Definition 9.6. *Let X be a noetherian scheme. We say that an invertible sheaf $\mathcal{L}$ is **ample** if for every coherent sheaf $\mathcal{F}$ there is an integer $n_0 > 0$ such that $\mathcal{F} \underset{\mathcal{O}_X}{\otimes} \mathcal{L}^n$ is globally generated, for all $n \geq n_0$.*

Lemma 9.7. *Let $\mathcal{L}$ be an invertible sheaf on a Noetherian scheme. TFAE*

(1) *$\mathcal{L}$ is ample.*
(2) *$\mathcal{L}^m$ is ample for all $m > 0$.*
(3) *$\mathcal{L}^m$ is ample for some $m > 0$.*

Proof. (1) implies (2) implies (3) is clear.

So assume that $\mathcal{M} = \mathcal{L}^m$ is ample and let $\mathcal{F}$ be a coherent sheaf. For each $0 \leq i \leq m-1$, let $\mathcal{F}_i = \mathcal{F} \otimes \mathcal{L}^i$. By assumption there is an integer n_i such that $\mathcal{F}_i \otimes \mathcal{M}^n$ is globally generated for all $n \geq n_i$. Let n_0 be the maximum of the n_i. If $n \geq (n_0+1)m$, then we may write $n = qm + i$, where $0 \leq i \leq m-1$ and $q \geq n_0 \geq n_i$.

But then

$$\mathcal{F} \otimes \mathcal{L}^m = \mathcal{F}_i \otimes \mathcal{M}^q,$$

which is globally generated. $\square$

Theorem 9.8. *Let X be a scheme of finite type over a Noetherian ring A and let $\mathcal{L}$ be an invertible sheaf on X.*

Then $\mathcal{L}$ is ample if and only if $\mathcal{L}^m$ is very ample for some $m > 0$.

Proof. Suppose that $\mathcal{L}^m$ is very ample. Then there is an immersion $X \subset \mathbb{P}^r_A$, for some positive integer r, and $\mathcal{L}^m = \mathcal{O}_X(1)$. Let $\bar{X}$ be the closure. If $\mathcal{F}$ is any coherent sheaf on X then there is a coherent sheaf $\overline{\mathcal{F}}$ on $\bar{X}$, such that $\mathcal{F} = \overline{\mathcal{F}}|_X$. By Serre's result, $\overline{\mathcal{F}}(k)$ is globally generated for some positive integer k. It follows that $\mathcal{F}(k)$ is globally generated, so that $\mathcal{L}^m$ is ample, and the result follows by (9.7).

Conversely, suppose that $\mathcal{L}$ is ample. Given $p \in X$, pick an open affine neighbourhood U of p so that $\mathcal{L}|_U$ is free. Let $Y = X - U$, give it the reduced induced strucure, with ideal sheaf $\mathcal{I}$. Then $\mathcal{I}$ is coherent.

Pick $n > 0$ so that $\mathcal{I} \otimes \mathcal{L}^n$ is globally generated. Then we may find $s \in \mathcal{I} \otimes \mathcal{L}^n$ not vanishing at p. We may identify s with $s' \in \mathcal{O}_U$ and then $p \in U_s \subset U$, an affine subset of X.

By compactness, we may cover X by such open affines and we may assume that n is fixed. Replacing $\mathcal{L}$ by $\mathcal{L}^n$ we may assume that $n = 1$. Then there are global sections $s_1, s_2, \dots, s_k \in H^0(X, \mathcal{L})$ such that $U_i = U_{s_i}$ is an open affine cover.

Since X is of finite type, each $B_i = H^0(U_i, \mathcal{O}_{U_i})$ is a finitely generated A-algebra. Pick generators b_{ij}. Then $s^n b_{ij}$ lifts to $s_{ij} \in H^0(X, \mathcal{L}^n)$. Again we might as well assume $n = 1$.

Now let $\mathbb{P}^N_A$ be the projective space with coordinates $x_1, x_2, \dots, x_k$ and x_{ij}. Locally we can define a map on each U_i to the standard open affine, by the obvious rule, and it is standard to check that this glues to an immersion. $\square$

Definition 9.9. *Let $\mathcal{L}$ be an invertible sheaf on a smooth projective variety over an algebraically closed field. Let $s \in H^0(X, \mathcal{L})$. The divisor (s) of zeroes of s is defined as follows. By assumption we may cover X by open subsets U_i over which we may identify $s|_{U_i}$ with $f_i \in \mathcal{O}_{U_i}$. The defines a Cartier divisor $\{(U_i, f_i)\}$.*

It is a simple matter to check that the Cartier divisor does not depend on our choice of trivialisations. Note that as X is smooth the Cartier divisor may safely be identified with the corresponding Weil divisor.

Lemma 9.10. *Let X be a smooth projective variety over an algebraically closed field. Let D_0 be a divisor and let $\mathcal{L} = \mathcal{O}_X(D_0)$.*

(1) *If $s \in H^0(X, \mathcal{L})$, $s \neq 0$ then $(s) \sim D_0$.*
(2) *If $D \geq 0$ and $D \sim D_0$ then there is a global section $s \in H^0(X, \mathcal{L})$ such that $D = (s)$.*
(3) *If $s_i \in H^0(X, \mathcal{L})$, $i = 1$ and 2, are two global sections then $(s_1) = (s_2)$ if and only if $s_2 = \lambda s_1$ where $\lambda \in k^*$.*

Proof. As $\mathcal{O}_X(D_0) \subset \mathcal{K}$, the section s corresponds to a rational function f. If D_0 is the Cartier divisor $\{(U_i, f_i)\}$ then $\mathcal{O}_X(D_0)$ is locally generated by f_i^{-1} so that multiplication by f_i induces an isomorphism with $\mathcal{O}_{U_i}$. D is then locally defined by $f f_i$. But then

$$D = D_0 + (f).$$

Hence (1).

Now suppose that $D > 0$ and $D = D_0 + (f)$. Then $(f) \geq -D_0$. Hence

$$f \in H^0(X, \mathcal{O}_X(D_0)) \subset H^0(X, \mathcal{K}) = K(X),$$

and the divisor of zeroes of f is D. This is (2).

Now suppose that $(s_1) = (s_2)$. Then

$$D_0 + (f_1) = (s_1) = (s_2) = D_0 + (f_2).$$

Cancelling, we get that $(f_1) = (f_2)$ and the rational function f_1/f_2 has no zeroes nor poles. Since X is a projective variety, $f_1/f_2 = \lambda$, a constant. $\square$

Definition 9.11. *Let D_0 be a divisor. The* ***complete linear system*** *associated to D_0 is the set*

$$|D_0| = \{\, D \in \operatorname{Div}(X) \mid D \geq 0, D \sim D_0 \,\}.$$

We have seen that

$$|D| = \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Thus $|D|$ is naturally a projective space.

Definition 9.12. *A* ***linear system*** *is any linear subspace of a complete linear system $|D_0|$.*

In other words, a linear system corresponds to a linear subspace, $V \subset H^0(X, \mathcal{O}_X(D_0))$. We will then write

$$|V| = \{\, D \in |D_0| \mid D = (s), s \in V \,\} \simeq \mathbb{P}(V) \subset \mathbb{P}(H^0(X, \mathcal{O}_X(D_0))).$$

Definition 9.13. *Let $|V|$ be a linear system. The* ***base locus*** *of $|V|$ is the intersection of the elements of $|V|$.*

Lemma 9.14. *Let X be a smooth projective variety over an algebraically closed field, and let $|V| \subset |D_0|$ be a linear system.*

V generates $\mathcal{O}_X(D_0)$ if and only if $|V|$ is base point free.

Proof. If V generates $\mathcal{O}_X(D_0)$ then for every point $x \in X$ we may find an element $\sigma \in V$ such that $\sigma(x) \neq 0$. But then $D = (\sigma)$ does not contain x, and so the base locus is empty.

Conversely suppose that the base locus is empty. The locus where V does not generated $\mathcal{O}_X(D_0)$ is a closed subset Z of X. Pick $x \in Z$ a closed point. By assumption we may find $D \in |V|$ such that $x \notin D$. But then if $D = (\sigma)$, $\sigma(x) \neq 0$ and σ generates the stalk $\mathcal{L}_x$, a contradiction. Thus Z is empty and $\mathcal{O}_X(D_0)$ is globally generated. $\square$

Example 9.15. *Consider $\mathcal{O}_{\mathbb{P}^1}(4)$. The complete linear system $|4p|$ defines a morphism into $\mathbb{P}^4$, where $p = [0:1]$ and $q = [1:0]$, given by $\mathbb{P}^1 \longrightarrow \mathbb{P}^4$, $[S:T] \longrightarrow [S^4 : ST^3 : S^2T^2 : ST^3 : T^4]$. If we project from $[0:0:1:0:0]$ we will get a morphism into $\mathbb{P}^3$, $[S:T] \longrightarrow [S^4 : ST^3 : ST^3 : T^4]$. This corresponds to the sublinear system spanned by $4p$, $3p+q$, $p+3q$, $4q$.*

Consider $\mathcal{O}_{\mathbb{P}^2}(2)$ and the corresponding complete linear system. The map associated to this linear system is the Veronese embedding $\mathbb{P}^2 \longrightarrow \mathbb{P}^5$, $[X : Y : Z] \longrightarrow [X^2 : Y^2 : Z^2 : YZ : XZ : XY]$.

Note also the notion of separating points and tangent directions becomes a little clearer in this more geometric setting. Separating points means that given x and $y \in X$, we can find $D \in |V|$ such that $x \in D$ and $y \notin D$. Separating tangent vectors means that given any irreducible length two zero dimensional scheme z, with support x, we can find $D \in |V|$ such that $x \in D$ but z is not contained in D. In fact the condition about separating tangent vectors is really the limiting case of separating points.

Thinking in terms of linear systems also presents an inductive approach to proving global generation. Suppose that we consider the complete linear system $|D|$. Suppose that we can find $Y \in |D|$. Then the base locus of $|D|$ is supported on Y. On the other hand suppose that $\mathcal{I}$ is the ideal sheaf of Y in X. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

As X is smooth D is Cartier and $\mathcal{O}_X(D)$ is an invertible sheaf. Tensoring by locally free preserves exactness, so there are short exact sequences,

$$0 \longrightarrow \mathcal{I}(mD) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_Y(mD) \longrightarrow 0.$$

Taking global sections, we get

$$0 \longrightarrow H^0(X, \mathcal{I}(mD) \longrightarrow H^0(X, \mathcal{O}_X(mD) \longrightarrow H^0(Y, \mathcal{O}_Y(mD).$$

At the level of linear systems there is therefore a linear map

$$|D| \longrightarrow |D|_Y|.$$

It is interesting to see what happens for toric varieties. Suppose that $X = X(F)$ is the toric variety associated to the fan $F \subset N_{\mathbb{R}}$. Recall that we can associate to a T-Cartier divisor $D = \sum a_i D_i$, a continuous piecewise linear function

$$\phi_D \colon |F| \longrightarrow \mathbb{R},$$

where $|F| \subset N_{\mathbb{R}}$ is the support of the fan, which is specified by the rule that $\phi_D(v_i) = -a_i$.

We can also associate to D a rational polyhedron

$$\begin{aligned} P_D &= \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \quad \forall i \,\} \\ &= \{\, u \in M_{\mathbb{R}} \,|\, u \geq \phi_D \,\}. \end{aligned}$$

Lemma 9.16. *If X is a toric variety and D is T-Cartier then*

$$H^0(X, \mathcal{O}_X(D)) = \bigoplus_{u \in P_D \cap M} k \cdot \chi^u.$$

Proof. Suppose that $\sigma \in F$ is a cone. Then, we have already seen that

$$H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D)) = \bigoplus_{u \in P_D(\sigma) \cap M} k \cdot \chi^u,$$

where

$$P_D(\sigma) = \{\, u \in M_{\mathbb{R}} \,|\, \langle u, v_i \rangle \geq -a_i \quad \forall v_i \in \sigma \,\}.$$

These identifications are compatible on overlaps. Since

$$H^0(X, \mathcal{O}_X(D)) = \cap_{\sigma \in F} H^0(U_\sigma, \mathcal{O}_{U_\sigma}(D))$$

and

$$P_D = \cap_{\sigma \in F} P_D(\sigma),$$

the result is clear. □

It is interesting to compute some examples. First, suppose we consider $\mathbb{P}^1$. A T-Cartier divisor is a sum $ap + bq$ (p and q fixed points). The corresponding function is

$$\phi(x) = \begin{cases} -ax & x > 0 \\ -bx & x < 0. \end{cases}$$

The corresponding polytope is the interval

$$[-a, b] \subset \mathbb{R} = N_{\mathbb{R}}.$$

There are $a+b+1$ integral points, corresponding to the fact that there are $a+b+1$ monomials of degree $a+b$. For $\mathbb{P}^2$ and dD_3, P_D is the convex hull of $(0,0)$, $(d,0)$ and $(0,d)$. The number of integral points is

$$\frac{(d+1)^2}{2} + \frac{d+1}{2} = \frac{(d+2)(d+1)}{2},$$

which is the usual formula.

Let D be a Cartier divisor on a toric variety $X = X(F)$ given by a fan F. It is interesting to consider when the complete linear system $|D|$ is base point free. Since any Cartier divisor is linearly equivalent to a T-Cartier divisor, we might as well suppose that $D = \sum a_i D_i$ is T-Cartier. Note that the base locus of the complete linear system of any Cartier divisor is invariant under the action of the torus. Since there are only finitely many orbits, it suffices to show that for each cone $\sigma \in F$ the point $x_\sigma \in U_\sigma$ is not in the base locus. It is also clear that if x_σ is not in the base locus of $|D|$ then in fact one can find a

T-Cartier divisor $D' \in |D|$ which does not contain x_σ. Equivalently we can find $u \in M$ such that

$$\langle u, v_i \rangle \geq -a_i,$$

with strict equality if $v_i \in \sigma$. The interesting thing is that we can reinterpret this condition using ϕ_D.

Definition 9.17. *The function $\phi\colon V \longrightarrow \mathbb{R}$ is* ***upper convex*** *if*

$$\phi(\lambda v + (1-\lambda)w) \geq \lambda\phi(v) + (1-\lambda)w \qquad \forall v, w \in V.$$

When we have a fan F and ϕ is linear on each cone σ, then ϕ is called ***strictly upper convex*** *if the linear functions $u(\sigma)$ and $u(\sigma')$ are different, for different maximal cones σ and σ'.*

Theorem 9.18. *Let $X = X(F)$ be the toric variety associated to a T-Cartier divisor D.*

Then

1. *$|D|$ is base point free if and only if ψ_D is upper convex.*
2. *D is very ample if and only if ψ_D is strictly upper convex and the semigroup S_σ is generated by*

$$\{\, u - u(\sigma) \,|\, u \in P_D \cap M \,\}.$$

Proof. (1) follows from the remarks above. (2) is proved in Fulton's book. ☐

For example if $X = \mathbb{P}^1$ and

$$\phi(x) = \begin{cases} -ax & x > 0 \\ -bx & x < 0. \end{cases}$$

so that $D = ap + bq$ then ϕ is upper convex if and only if $a + b \geq 0$ in which case D is base point free. D is very ample if and only if $a+b > 0$. When ϕ is continuous and linear on each cone σ, we may restate the upper convex as saying that the graph of ϕ lies under the graph of $u(\sigma)$. It is strictly upper convex if it lies strictly under the graph of $u(\sigma)$, for all n-dimensional cones σ.

Using this, it is altogether too easy to give an example of a smooth proper toric variety which is not projective. Let $F \subset N_{\mathbb{R}} = \mathbb{R}^3$ given by the edges $v_1 = -e_1$, $v_2 = -e_2$, $v_3 = -e_3$, $v_4 = e_1 + e_2 + e_3$, $v_5 = v_3 + v_4$, $v_6 = v_1 + v_4$ and $v_7 = v_2 + v_4$. Now connect v_1 to v_5, v_3 to v_7 and v_2 to v_6 and v_5 to v_6, v_6 to v_7 and v_7 to v_5.

It is not hard to check that X is smooth and proper (proper translates to the statement that the support $|F|$ of the fan is the whole of $N_{\mathbb{R}}$).

Suppose that ψ is strictly upper convex. Let w be the midpoint of the line connecting v_1 and v_5. Then

$$w = \frac{v_1 + v_5}{2} = \frac{v_3 + v_6}{2}.$$

Since v_1 and v_5 belong to the same maximal cone, ψ is linear on the line connecting them. In particular

$$\psi(w) = \psi(\frac{v_1 + v_5}{2}) = \frac{1}{2}\psi(v_1) + \frac{1}{2}\psi(v_5).$$

Since v_1, v_5 and v_3 belong to the same cone and v_6 does not, by strict convexity,

$$\psi(w) = \psi(\frac{v_3 + v_6}{2}) > \frac{1}{2}\psi(v_3) + \frac{1}{2}\psi(v_6).$$

Putting all of this together, we get

$$\psi(v_1) + \psi(v_5) > \psi(v_2) + \psi(v_6).$$

By symmetry

$$\begin{aligned}
\psi(v_1) + \psi(v_5) &> \psi(v_3) + \psi(v_6) \\
\psi(v_2) + \psi(v_6) &> \psi(v_1) + \psi(v_7) \\
\psi(v_3) + \psi(v_7) &> \psi(v_2) + \psi(v_5).
\end{aligned}$$

But adding up these three inequalities gives a contradiction.

Consider another application of the ideas behind this section. Consider the problem of parametrising subvarieties or subschemes X of projective space $\mathbb{P}^r_k$. Any subscheme is determined by the homogeneous ideal $I(X)$ of polynomials vanishing on X. As in the case of zero dimensional schemes, we would like to reduce to the data of a vector subspace of fixed dimension in a fixed vector space. The obvious thing to consider is polynomials of degree d and the vector subspace of polynomials of polynomials of degree d vanishing on X. But how large should we take d to be?

The first observation is that if $\mathcal{I}$ is the ideal sheaf of X in $\mathbb{P}^r_k$ then

$$I_d = H^0(\mathbb{P}^r_k, \mathcal{I}(d)),$$

where $\mathcal{I}(d)$ is the Serre twist. To say that I_d determines X, is essentially equivalent to saying that $\mathcal{I}(d)$ is globally generated. Fixing some data about X (in the case of zero dimensional schemes this would be the length) we would then like a positive integer d_0 such that if $d \geq d_0$ then two things are true:

- $\mathcal{I}(d)$ is globally generated.
- $h^0(\mathbb{P}^r_k, \mathcal{I}(d))$, the dimension of the space of global sections, is independent of X.

Now there is a short exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_{\mathbb{P}^r_k} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Twisting by d, we get

$$0 \longrightarrow \mathcal{I}(d) \longrightarrow \mathcal{O}_{\mathbb{P}^r_k}(d) \longrightarrow \mathcal{O}_X(d) \longrightarrow 0.$$

Taking global sections gives another exact sequence.

$$0 \longrightarrow H^0(\mathbb{P}^r_k, \mathcal{I}(d)) \longrightarrow H^0(\mathbb{P}^r_k, \mathcal{O}_{\mathbb{P}^r_k}(d)) \longrightarrow H^0(X, \mathcal{O}_X(d)).$$

Again, it would be really nice if this exact sequence were exact on the right. Then global generation of $\mathcal{I}(d)$ would be reduced to global generation of $\mathcal{O}_X(d)$ and one could read of $h^0(\mathbb{P}^r_k, \mathcal{I}(d))$ from $h^0(X, \mathcal{O}_X(d))$.

10. Relative proj and the blow up

We want to define a relative version of Proj, in pretty much the same way we defined a relative version of Spec. We start with a scheme X and a quasi-coherent sheaf $\mathcal{S}$ sheaf of graded $\mathcal{O}_X$-algebras,

$$\mathcal{S} = \bigoplus_{d \in \mathbb{N}} \mathcal{S}_d,$$

where $\mathcal{S}_0 = \mathcal{O}_X$. It is convenient to make some simplifying assumptions:

(†) $\quad X$ is Noetherian, $\mathcal{S}_1$ is coherent, $\mathcal{S}$ is locally generated by $\mathcal{S}_1$.

To construct relative Proj, we cover X by open affines $U = \operatorname{Spec} A$. $\mathcal{S}(U) = H^0(U, \mathcal{S})$ is a graded A-algebra, and we get $\pi_U \colon \operatorname{Proj} \mathcal{S}(U) \longrightarrow U$ a projective morphism. If $f \in A$ then we get a commutative diagram

$$\begin{tikzcd}
\operatorname{Proj} \mathcal{S}(U_f) \arrow[r] \arrow[d, "\pi_{U_f}"'] & \operatorname{Proj} \mathcal{S}(U) \arrow[d, "U"'] \\
U_f \arrow[r] & U.
\end{tikzcd}$$

It is not hard to glue π_U together to get $\pi \colon \mathbf{Proj}\, \mathcal{S} \longrightarrow X$. We can also glue the invertible sheaves together to get an invertible sheaf $\mathcal{O}(1)$.

The relative consruction is very similar to the old construction.

Example 10.1. *If X is Noetherian and*

$$\mathcal{S} = \mathcal{O}_X[T_0, T_1, \dots, T_n],$$

then satisfies (†) *and* $\mathbf{Proj}\, \mathcal{S} = \mathbb{P}^n_X$.

Given a sheaf $\mathcal{S}$ satisfying (†), and an invertible sheaf $\mathcal{L}$, it is easy to construct a quasi-coherent sheaf $\mathcal{S}' = \mathcal{S} \star \mathcal{L}$, which satisfies (†). The graded pieces of $\mathcal{S}'$ are $\mathcal{S}_d \otimes \mathcal{L}^d$ and the multiplication maps are the obvious ones. There is a natural isomorphism

$$\phi \colon P' = \mathbf{Proj}\, \mathcal{S}' \longrightarrow P = \mathbf{Proj}\, \mathcal{S},$$

which makes the digram commute

$$\begin{tikzcd}
P' \arrow[rr, "\phi"] \arrow[dr, "\pi'"'] & & P \arrow[dl, "\pi"] \\
& S, &
\end{tikzcd}$$

and

$$\phi^* \mathcal{O}_{P'}(1) \simeq \mathcal{O}_P(1) \otimes \pi'^* \mathcal{L}.$$

Note that π is always proper; in fact π is projective over any open affine and properness is local on the base. π need not be projective, but if $\mathcal{L}$ is very ample then $\mathcal{O}_P(1) \otimes \mathcal{L}^d$ is very ample, if d is sufficiently large.

There are two very interesting family of examples of the construction of relative Proj. Suppose that we start with a locally free sheaf $\mathcal{E}$ of rank $r \geq 2$. Note that

$$\mathcal{S} = \bigoplus \operatorname{Sym}^d \mathcal{E},$$

satisfies (†). $\mathbb{P}(\mathcal{E}) = \mathbf{Proj}\, \mathcal{S}$ is the **projective bundle** over X associated to $\mathcal{E}$. The fibres of $\pi\colon \mathbb{P}(\mathcal{E}) \longrightarrow X$ are copies of $\mathbb{P}^n$, where $n = r - 1$. We have

$$\bigoplus_{l=0}^{\infty} \pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l) = \mathcal{S},$$

so that in particular

$$\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) = \mathcal{E}.$$

Also there is a natural surjection

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Indeed, it suffices to check both statements locally, so that we may assume that X is affine. The first statement is then (4.21) and the second statement reduces to the statement that the sections $x_0, x_1, \dots, x_n$ generate $\mathcal{O}_P(1)$. The most interesting result is:

Proposition 10.2. *Let $g\colon Y \longrightarrow X$ be a morphism.*

Then a morphism $f\colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X is the same as giving an invertible sheaf $\mathcal{L}$ on Y and a surjection $g^* \mathcal{E} \longrightarrow \mathcal{L}$.

Proof. One direction is clear; if $f\colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ is a morphism over X, then the surjective morphism of sheaves

$$\pi^* \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1),$$

pullsback to a surjective morphism

$$g^* \mathcal{E} = f^*(\pi^* \mathcal{E}) \longrightarrow \mathcal{L} = f^* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1).$$

Conversely suppose we are given an invertible sheaf $\mathcal{L}$ and a surjective morphism of sheaves

$$g^* \mathcal{E} \longrightarrow \mathcal{L}.$$

I claim that there is then a unique morphism $f\colon Y \longrightarrow \mathbb{P}(\mathcal{E})$ over X, which induces the given surjection. By uniquness, it suffices to prove this result locally. So we may assume that $X = \operatorname{Spec} A$ is affine and

$$\mathcal{E} = \bigoplus_{i=0}^{n} \mathcal{O}_X,$$

is free. In this case surjectivity reduces to the statement that the images $s_0, s_1, \dots, s_n$ of the standard sections generate $\mathcal{L}$, and the result reduces to one we have already proved. $\square$

Definition 10.3. *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent sheaf of ideals on X. Let*

$$\mathcal{S} = \bigoplus_{d=0}^{\infty} \mathcal{I}^d,$$

where $\mathcal{I}^0 = \mathcal{O}_X$ and $\mathcal{I}^d$ is the dth power of $\mathcal{I}$. Then $\mathcal{S}$ satisfies (†).

*$\pi\colon \mathbf{Proj}\,\mathcal{S} \longrightarrow X$ is called the **blow up** of $\mathcal{I}$ (or Y, if Y is the subscheme of X associated to $\mathcal{I}$).*

Example 10.4. *Let $X = \mathbb{A}^n_k$ and let P be the origin. We check that we get the usual blow up. Let*

$$A = k[x_1, x_2, \dots, x_n].$$

As $X = \operatorname{Spec} A$ is affine and the ideal sheaf $\mathcal{I}$ of P is the sheaf associated to $\langle x_1, x_2, \dots, x_n\rangle$,

$$Y = \mathbf{Proj}\,\mathcal{S} = \operatorname{Proj} S,$$

where

$$S = \bigoplus_{d=0}^{\infty} I^d.$$

There is a surjective map

$$A[y_1, y_2, \dots, y_n] \longrightarrow S,$$

of graded rings, where y_i is sent to x_i. $Y \subset \mathbb{P}^n_A$ is the closed subscheme corresponding to this morphism. The kernel of this morphism is

$$\langle y_i x_j - y_j x_i\rangle,$$

which are the usual equations of the blow up.

Definition 10.5. *Let $f\colon X \longrightarrow Y$ be a morphism of schemes. We are going to define the **inverse image ideal sheaf** $\mathcal{I}' \subset \mathcal{O}_Y$. First we take the inverse image of the sheaf $f^{-1}\mathcal{I}$, where we just think of f as being a continuous map. Then $f^{-1}\mathcal{I} \subset f^{-1}\mathcal{O}_Y$. Let $\mathcal{I}' = f^{-1}\mathcal{I} \cdot \mathcal{O}_Y$ be the ideal generated by the image of $f^{-1}\mathcal{I}$ under the natural morphism $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$.*

Theorem 10.6 (Universal Property of the blow up)**.** *Let X be a Noetherian scheme and let $\mathcal{I}$ be a coherent ideal sheaf.*

If $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ then $\pi^{-1}\mathcal{I}\cdot\mathcal{O}_Y$ is an invertible sheaf. Moreover π is universal amongst all such morphisms. If $f\colon Z \longrightarrow X$ is any morphism such that $f^{-1}\mathcal{I}\cdot\mathcal{O}_Z$ is invertible then there is a unique induced morphism $g\colon Z \longrightarrow Y$ which makes the diagram commute

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[dr, "f"'] & Y \arrow[d, "\pi"] \\ & X. \end{tikzcd}$$

Proof. By uniqueness, we can check this locally. So we may assume that $X = \operatorname{Spec} A$ is affine. As $\mathcal{I}$ is coherent, it corresponds to an ideal $I \subset A$ and

$$X = \operatorname{Proj}\bigoplus_{d=0}^{\infty} I^d.$$

Now $\mathcal{O}_Y(1)$ is an invertible sheaf on Y. It is not hard to check that $\pi^{-1}\mathcal{I}\cdot\mathcal{O}_Y = \mathcal{O}_Y(1)$.

Pick generators $a_0, a_1, \dots, a_n$ for I. This gives rise to a surjective map of rings

$$\phi\colon A[x_0, x_1, \dots, x_n] \longrightarrow I,$$

whence to a closed immersion $Y \subset \mathbb{P}^n_A$. The kernel of ϕ is generated by all homogeneous polynomials $F(x_0, x_1, \dots, x_n)$ such that $F(a_0, a_1, \dots, a_n) = 0$.

Now the elements $a_0, a_1, \dots, a_n$ pullback to global sections $s_0, s_1, \dots, s_n$ of the invertible sheaf $\mathcal{L} = f^{-1}\mathcal{L}\cdot\mathcal{O}_Y$ and $s_0, s_1, \dots, s_n$ generate $\mathcal{L}$. So we get a morphism

$$g\colon Z \longrightarrow \mathbb{P}^n_X,$$

over X, such that $g^*\mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{L}$ and $g^{-1}x_i = s_i$. Suppose that $F(x_0, x_1, \dots, x_n)$ is a homogeneous polynomial in the kernel of ϕ. Then $F(a_0, a_1, \dots, a_n) = 0$ so that $F(s_0, s_1, \dots, s_n) = 0$ in $H^0(X, \mathcal{L}^d)$. It follows that g factors through Y.

Now suppose that $f\colon Z \longrightarrow X$ factors through $g\colon Z \longrightarrow Y$. Then

$$f^{-1}\mathcal{I}\cdot\mathcal{O}_Z = g^{-1}(\mathcal{I}\cdot\mathcal{O}_Y)\cdot\mathcal{O}_Z = g^{-1}\mathcal{O}_Y(1)\cdot\mathcal{O}_Z.$$

Therefore there is a surjective map

$$g^*\mathcal{O}_Y(1) \longrightarrow \mathcal{L}.$$

But then this map must be an isomorphism and so $g^*\mathcal{O}_Y(1) = \mathcal{L}$. $s_i = g^*x_i$ and uniqueness follows. $\square$

Note that by the universal property, the morphism π is an isomorphism outside of the subscheme V defined by $\mathcal{I}$. We may put the universal property differently. The only subscheme with an invertible ideal sheaf is a Cartier divisor (local generators of the ideal, give local equations for the Cartier divisor). So the blow up is the smallest morphism which turns a subscheme into a Cartier divisor. Perhaps surprisingly, therefore, blowing up a Weil divisor might give a non-trivial birational map.

If X is a variety it is not hard to see that π is a projective, birational morphism. In particular if X is quasi-projective or projective then so is Y. We note that there is a converse to this:

Theorem 10.7. *Let X be a quasi-projective variety and let $f\colon Z \longrightarrow X$ be a birational projective morphism.*

Then there is an coherent ideal sheaf $\mathcal{I}$ and a commutative diagram

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "\pi"] \\ & X, & \end{tikzcd}$$

where $\pi\colon Y \longrightarrow X$ is the blow up of $\mathcal{I}$ and the top row is an isomorphism.

It is interesting to figure out the geometry behind the example of a toric variety which is not projective. To warm up, suppose that we start with $\mathbb{A}^3_k$. This is the toric variety associated to the fan spanned by e_1, e_2, e_3. Imagine blowing up two of the axes. This corresponds to inserting two vectors, $e_1 + e_2$ and $e_1 + e_3$. However the order in which we blow up is significant. Let's introduce some notation. If we blow up the x-axis $\pi\colon Y \longrightarrow X$ and then the y-axis, $\psi\colon Z \longrightarrow Y$, let's call the exceptional divisors E_1 and E_2, and let E_1' denote the strict transform of E_1 on Z. E_1 is a $\mathbb{P}^1$-bundle over the x-axis. The strict transform of the y-axis in Y intersects E_1 in a point p. When we blow up this curve, $E_1' \longrightarrow E_1$ blows up the point p. The fibre of E_1' over the origin therefore consists of two copies Σ_1 and Σ_2 of $\mathbb{P}^1$. Σ_1 is the strict transform of the fibre of E_1 over the origin and Σ_2 is the exceptional divisor. The fibre Σ of E_2 over the origin is a copy of $\mathbb{P}^1$. Σ and Σ_2 are the same curve in Z.

The example of a toric variety which is not projective is obtained from $\mathbb{P}^3$ by blowing up three coordinate axes, which form a triangle. The twist is that we do something different at each of the three coordinate points. Suppose that $\pi\colon X \longrightarrow \mathbb{P}^3$ is the birational morphism down to $\mathbb{P}^3$, and let E_1, E_2 and E_3 be the three exceptional divisors.

Over one point we extract E_1 first then E_2, over the second point we extract first E_2 then E_3 and over the last point we extract first E_3 then E_1.

To see what has gone wrong, we need to work in the homology and cohomology groups of X. Any curve C in X determines an element of $[C] \in H_2(X, \mathbb{Z})$. Any Cartier divisor D in X determines a class $[D] \in H^2(X, \mathbb{Z})$. We can pair these two classes to get an intersection number $D \cdot C \in \mathbb{Z}$. One way to compute this number is to consider the line bundle $\mathcal{L} = \mathcal{O}_X(D)$ associated to D. Then

$$D \cdot C = \deg \mathcal{L}|_C.$$

If D is ample then this intersection number is always positive. This implies that the class of every curve is non-trivial in homology.

Suppose the reducible fibres of E_1, E_2 and E_3 over their images are $A_1 + A_2$, $B_1 + B_2$ and $C_1 + C_3$. Suppose that the general fibres are A, B and C. We suppose that A_1 is attached to B and B_1 and C_1 to C. We have

$$\begin{aligned} [A] &= [A_1] + [A_2] \\ &= [B] + [A_2] \\ &= [B_1] + [B_2] + [A_2] \\ &= [C] + [B_2] + [A_2] \\ &= [C_1] + [C_2] + [B_2] + [A_2] \\ &= [A] + [C_2] + [B_2] + [A_2], \end{aligned}$$

in $H_2(X, \mathbb{Z})$, so that

$$[A_2] + [B_2] + [C_2] = 0 \in H_2(X, \mathbb{Z}).$$

Suppose that D were an ample divisor on X. Then

$$D \cdot ([A_2] + [B_2] + [C_2]) > D \cdot [A_2] + D \cdot [B_2] + D \cdot [C_2] > 0,$$

a contradiction.

There are a number of things to say about this way of looking at things, which lead in different directions. The first is that there is no particular reason to start with a triangle of curves. We could start with two conics intersecting transversally (so that they lie in different planes). We could even start with a nodal cubic, and just do something different over the two nodes. Neither of these examples are toric, of course. It is clear that in the first two examples, the morphism

$$\pi \colon X \longrightarrow \mathbb{P}^3,$$

is locally projective. It cannot be a projective morphism, since $\mathbb{P}^3$ is projective and the composition of projective is projective. It also follows that π is not the blow up of a coherent sheaf of ideals on $\mathbb{P}^3$. The third example is not even a variety. It is a complex manifold (and in fact it is something called an algebraic space). In particular the notion of the blow up in algebraic geometry is more delicate than it might first appear.

The second thing is to consider the difference between the order of blow ups of the two axes. Suppose we denote the composition of blowing up the x-axis and then the y-axis by $\pi_1 \colon X_1 \longrightarrow \mathbb{A}^3$ and the composition the other way by $\pi_2 \colon X_1 \longrightarrow \mathbb{A}^3$. Now X_1 and X_2 agree outside the origin. Let $\phi \colon X_1 \dashrightarrow X_2$ be the resulting birational map. If $A_1 + A_2$ is the fibre of π_1 over the origin and $B_1 + B_2$ is the fibre of π_2 over the origin, then ϕ is in fact an isomorphism outside A_2 and B_2. So ϕ is a birational map which is an isomorphism in codimension one, in fact an isomorphism outside a curve, isomorphic to $\mathbb{P}^1$. ϕ is an example of a *flop*. In terms of fans, we have four vectors v_1, v_2, v_3 and v_4, such that

$$v_1 + v_3 = v_2 + v4,$$

and any three vectors span the lattice. If σ is the cone spanned by these four vectors, then $Q = U_\sigma$ is the cone over a quadric. There are two ways to subdivide σ into two cones. Insert the edge connecting v_1 to v_3 or the edge corresponding to $v_2 + v_4$. The corresponding morphisms extract a copy of $\mathbb{P}^1$ and the resulting birational map between the two toric varieties is a (simple) flop. One can also insert the vector $w = v_1 + v_3$, to get a toric variety Y. The corresponding exceptional divisor is $\mathbb{P}^1 \times \mathbb{P}^1$. The toric varieties fit into a picture

$$\begin{tikzcd}
 & Y \arrow[dl] \arrow[dr] & \\
X_1 \arrow[rr, dashed] \arrow[dr] & & X_2 \arrow[dl] \\
 & Q. &
\end{tikzcd}$$

The two maps $Y \longrightarrow X_i$ correspond to the two projections of $\mathbb{P}^1 \times \mathbb{P}^1$ down to $\mathbb{P}^1$. By (10.7) $\pi_i \colon X_i \longrightarrow Q$ correspond to blowing up a coherent ideal sheaf. In fact it corresponds to blowing a Weil divisor (in fact this is a given, as π_i does not extract any divisors), the plane determined by either ruling.

Finally, it is interesting to wonder more about the original examples of varieties which are not projective. Note that in the case when we blow up either a triangle or a conic if we make one flop then we get a projective variety. Put differently, if we start with a projective variety then it is possible to get a non-projective variety by flopping a curve. When does flopping a curve mean that the variety is no longer projective? A variety is projective if it contains an ample divisor. Ample divisors intersect all curve positively. Note that any sum of ample divisors is ample.

Definition 10.8. *Let X be a proper variety. The* ***ample cone*** *is the cone in $H^2(X, \mathbb{R})$ spanned by the classes of the ample divisors.*

The ***Kleiman-Mori cone*** $\overline{\mathrm{NE}}(X)$ *is the cone in $H_2(X, \mathbb{R})$ is the closure of the cone spanned by the classes of curves.*

The significance of all of this is the following:

Theorem 10.9 (Kleiman's Criteria)**.** *Let X be a proper variety (or even algebraic space).*

A divisor D is ample if and only if the linear functional

$$\psi \colon H_2(X, \mathbb{R}) \longrightarrow \mathbb{R},$$

given by $\phi(\alpha) = [D] \cdot \alpha$ is strictly positive on $\overline{\mathrm{NE}}(X) - \{0\}$.

Using Kleiman's criteria, it is not hard to show that if $\phi \colon X \dashrightarrow Y$ is a flop of the curve C and X is projective then Y is projective if and only if the class of $[C]$ generates a one dimensional face of $\overline{\mathrm{NE}}(X)$.

11. Sheaf Cohomology

Definition 11.1. *Let X be a topological space. For every $i \geq 0$ there are functors H^i from the category of sheaves of abelian groups on X to the category of abelian groups such that*

(1) $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F})$.
(2) *Given a short exact sequence,*
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0,$$
there are boundary maps
$$H^i(X, \mathcal{H}) \longrightarrow H^{i+1}(X, \mathcal{F}).$$
which can be strung together to get a long exact sequence of cohomology.

In short, sheaf cohomology was invented to fix the lack of exactness, and in fact this property essentially fixes the definition.

Example 11.2. *If X is a simplicial complex (or a CW-complex) then $H^i(X, \mathbb{Z})$ agrees with the usual definition. The same goes for any other coefficient ring (considered as a local free sheaf).*

Like ordinary cohomology, sheaf cohomology inherits a cup product,
$$H^i(X, \mathcal{F}) \otimes H^j(X, \mathcal{G}) \longrightarrow H^{i+j}(X, \mathcal{F} \otimes \mathcal{G}),$$
where $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules. In particular if X is a projective scheme over A then
$$H^i(X, \mathcal{F}),$$
is an A-module, where $\mathcal{F}$ is an $\mathcal{O}_X$-module, since $A = H^0(X, \mathcal{O}_X)$. In particular if A is a field, then
$$H^i(X, \mathcal{F}),$$
are vector spaces.
$$h^i(X, \mathcal{F}),$$
denotes their dimension.

We would like to have a definition of these groups which allows us to compute. Let X be a topological space and let $\mathcal{U} = \{U_i\}$ be an open cover, which is locally finite. The group of k-cochains is
$$C^k(\mathcal{U}, \mathcal{F}) = \bigoplus_I \Gamma(U_I, \mathcal{F}),$$
where I runs over all $(k+1)$-tuples of indices and
$$U_I = \bigcap_{i \in I} U_i,$$

denotes intersection. k-cochains are skew-commutative, so that if we switch two indices we get a sign change.

Define a coboundary map

$$\delta^k \colon C^k(\mathcal{U}, \mathcal{F}) \longrightarrow C^{k+1}(\mathcal{U}, \mathcal{F}).$$

Given $\sigma = (\sigma_I)$, we have to construct $\tau = \delta(\sigma) \in C^{k+1}(\mathcal{U}, \mathcal{F})$. We just need to determine the components τ_J of τ. Now $J = \{i_0, i_2, \dots, i_k\}$. If we drop an index, then we get a k-tuple. We define

$$\tau_J = \left. \left(\sum_{i=0}^{k} (-1)^i \sigma_{J - \{i_i\}} \right) \right|_{U_J}.$$

The key point is that $\delta^2 = 0$. So we can take cohomology

$$H^i(\mathcal{U}, \mathcal{F}) = Z^i(\mathcal{U}, \mathcal{F}) / B^i(\mathcal{U}, \mathcal{F}).$$

Here Z^i denotes the group of i-cocycles, those elements killed by δ^i and B^i denotes the group of coboundaries, those cochains which are in the image of δ^{i-1}. Note that $\delta^i(B^i) = \delta^i \delta^{i-1}(C^{i-1}) = 0$, so that $B^i \subset Z^i$.

The problem is that this is not enough. Perhaps our open cover is not fine enough to capture all the interesting cohomology. A **refinement** of the open cover $\mathcal{U}$ is an open cover $\mathcal{V}$, together with a map h between the indexing sets, such that if V_j is an open subset of the refinement, then for the index $i = h(j)$ such that $V_j \subset U_i$. It is straightforward to check that there are maps,

$$H^i(\mathcal{U}, \mathcal{F}) \longrightarrow H^i(\mathcal{V}, \mathcal{F}),$$

on cohomology. Taking the (direct) limit, we get the Čech cohomology groups,

$$\check{H}^i(X, \mathcal{F}).$$

For example, consider the case $i = 0$. Given a cover, a cochain is just a collection of sections, (σ_i), $\sigma_i \in \Gamma(U_i, \mathcal{F})$. This cochain is a cocycle if $(\sigma_i - \sigma_j)|_{U_{ij}} = 0$ for every i and j. By the sheaf axiom, this means that there is a global section $\sigma \in \Gamma(X, \mathcal{F})$, so that in fact

$$H^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}).$$

It is also sometimes possible to untwist the definition of H^1. A 1-cocycle is precisely the data of a collection

$$(\sigma_{ij}) \in \Gamma(\mathcal{U}, \mathcal{F}),$$

such that

$$\sigma_{ij} - \sigma_{ik} + \sigma_{jk} = 0.$$

In general of course, one does not want to compute these things using limits. The question is how fine does the cover have to be to compute the cohomology? As a first guess one might require that

$$H^i(U_j, \mathcal{F}) = 0,$$

for all j, and $i > 0$. In other words there is no cohomology on each open subset. But this is not enough. One needs instead the slightly stronger condition that

$$H^i(U_I, \mathcal{F}) = 0.$$

Theorem 11.3 (Leray). *If X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups and $\mathcal{U}$ is an open cover such that*

$$H^i(U_I, \mathcal{F}) = 0,$$

for all $i > 0$ and indices I, then in fact the natural map

$$H^i(\mathcal{U}, \mathcal{F}) \simeq \check{H}^i(X, \mathcal{F}),$$

is an isomorphism.

It is in fact not too hard to prove:

Theorem 11.4 (Serre). *Let X be a notherian scheme. TFAE*

(1) *X is affine,*
(2) *$H^i(X, \mathcal{F}) = 0$ for all $i > 0$ and all quasi-coherent sheaves,*
(3) *$H^1(X, \mathcal{I}) = 0$ for all coherent ideals $\mathcal{I}$.*

Finally, we need to construct the coboundary maps. Suppose that we are given a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

We want to define

$$H^i(X, \mathcal{H}) \longrightarrow H^{i+1}(X, \mathcal{F}).$$

Cheating a little, we may assume that we have a commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & C^i(\mathcal{U}, \mathcal{F}) \arrow[r] \arrow[d] & C^i(\mathcal{U}, \mathcal{G}) \arrow[r] \arrow[d] & C^i(\mathcal{U}, \mathcal{H}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{F}) \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{G}) \arrow[r] & C^{i+1}(\mathcal{U}, \mathcal{H}) \arrow[r] & 0.
\end{tikzcd}$$

Suppose we start with an element $t \in H^i(X, \mathcal{H})$. Then t is the image of $t' \in H^i(\mathcal{U}, \mathcal{H})$, for some open cover $\mathcal{U}$. In turn t' is represented by $\tau \in Z^i(\mathcal{U}, \mathcal{H})$. Now we may suppose our cover is sufficiently fine, so that $\tau_I \in \Gamma(U_I, \mathcal{H})$ is the image of $\sigma_I \in \Gamma(U_I, \mathcal{G})$ (and this fixes the cheat). Applying the boundary map, we get $\delta(\sigma) \in C^{i+1}(\mathcal{U}, \mathcal{G})$. Now the image of $\delta(\sigma)$ in $C^{i+1}(\mathcal{U},\mathcal{H})$ is the same as $\delta(\tau)$, which is zero, as τ is a cocycle. But then by exactness of the bottom rows, we get $\rho \in C^{i+1}(\mathcal{U},\mathcal{F})$. It is straightforward to check that ρ is a cocycle, so that we get an element $r' \in H^{i+1}(\mathcal{U},\mathcal{F})$, whence an element r of $H^{i+1}(X,\mathcal{F})$, and that r does not depend on the choice of σ.

Thus sheaf cohomology does exist (at least when X is paracompact, which is not a problem for schemes). Let us calculate the cohomology of projective space.

Theorem 11.5. *Let A be a Noetherian ring. Let $X = \mathbb{P}^r_A$.*

(1) *The natural map $S \longrightarrow \Gamma_*(X,\mathcal{O}_X)$ is an isomorphism.*

(2)
$$H^i(X,\mathcal{O}_X(n)) = 0 \qquad \text{for all} \qquad 0 < i < r \text{ and } n.$$

(3)
$$H^r(X,\mathcal{O}_X(-r-1)) \simeq A.$$

(4) *The natural map*
$$H^0(X,\mathcal{O}_X(n)) \times H^r(X,\mathcal{O}_X(-n-r-1)) \longrightarrow H^r(X,\mathcal{O}_X(-r-1)) \simeq A,$$
is a perfect pairing of finitely generated free A-modules.

Proof. Let
$$\mathcal{F} = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n).$$
Then $\mathcal{F}$ is a quasi-coherent sheaf. Let $\mathcal{U}$ be the standard open affine cover. As every intersection is affine, it follows that we may compute using this cover. Now
$$\Gamma(U_I,\mathcal{F}) = S_{x_I},$$
where
$$x_I = \prod_{i\in I} x_i.$$
Thus Čech cohomology is the cohomology of the complex
$$\prod_{i=0}^r S_{x_i} \longrightarrow \prod_{i<j}^r S_{x_ix_j} \longrightarrow \dots \longrightarrow S_{x_0x_1,\dots x_r}.$$
The kernel of the first map is just $H^0(X,\mathcal{F})$, which we already know is S. Now let us turn to $H^r(X,\mathcal{F})$. It is the cokernel of the map
$$\prod_i S_{x_0x_1\dots\hat{x}_i\dots x_r} \longrightarrow S_{x_0x_1\dots x_r}.$$
The last term is the free A-module with generators all monomials in the Laurent ring (that is we allow both positive and negative powers).

The image is the set of monomials where x_i has non-negative exponent. Thus the cokernel is naturally identified with the free A-module generated by arbitrary products of reciprocals x_i^{-1},

$$\{\, x_0^{l_0} x_1^{l_1} \dots x_r^{l_r} \mid l_i < 0 \,\}.$$

The grading is then given by

$$l = \sum_{i=0}^{r} l_i.$$

In particular

$$H^r(X, \mathcal{O}_X(-r-1)),$$

is the free A-module with generator $x_0^{-1} x_1^{-1} \dots x_r^{-1}$. Hence (3).

To define a pairing, we declare

$$x_0^{l_0} x_1^{l_1} \dots x_r^{l_r},$$

to be the dual of

$$x_0^{m_0} x_1^{m_1} \dots x_r^{m_r} = x_0^{-1-l_0} x_1^{-1-l_1} \dots x_r^{-1-l_r}.$$

As $m_i \geq 0$ if and only if $l_i < 0$ it is straightforward to check that this gives a perfect pairing. Hence (4).

It remains to prove (2). If we localise the complex above with respect to x_r, we get a complex which computes $\mathcal{F}|_{U_r}$, which is zero in positive degree, as U_r is affine. Thus

$$H^i(X, \mathcal{F})_{x_r} = 0,$$

for $i > 0$ so that every element of $H^i(X, \mathcal{F})$ is annihilated by some power of x_r.

To finish the proof, we will show that multiplication by x_r induces an inclusion of cohomology. We proceed by induction on the dimension. Suppose that $r > 1$ and let $Y \simeq \mathbb{P}^{r-1}_A$ be the hyperplane $x_r = 0$. Then

$$\mathcal{I}_Y = \mathcal{O}_X(-Y) = \mathcal{O}_X(-1).$$

Thus there are short exact sequences

$$0 \longrightarrow \mathcal{O}_X(n-1) \longrightarrow \mathcal{O}_X(n) \longrightarrow \mathcal{O}_Y(n) \longrightarrow 0.$$

Now $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < r-1$ and the natural restriction map

$$H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(Y, \mathcal{O}_Y(n)),$$

is surjective (every polynomial of degree n on Y is the restriction of a polynomial of degree n on X). Thus

$$H^i(X, \mathcal{O}_X(n-1)) \simeq H^i(X, \mathcal{O}_X(n)),$$

for $0 < i < r-1$, and even if $i = r-1$, then we get an injective map. But this map is the one induced by multiplication by x_r. $\square$

Theorem 11.6 (Serre vanishing)**.** *Let X be a projective variety over a Noetherian ring and let $\mathcal{O}_X(1)$ be a very ample line bundle on X. Let $\mathcal{F}$ be a coherent sheaf.*

(1) *$H^i(X, \mathcal{F})$ are finitely generated A-modules.*
(2) *There is an integer n_0 such that $H^i(X, \mathcal{F}(n)) = 0$ for all $n \geq n_0$ and $i > 0$.*

Proof. By assumption there is an immersion $i\colon X \longrightarrow \mathbb{P}^r_A$ such that $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^r_A}(1)$. As X is projective, it is proper and so i is a closed immersion. If $\mathcal{G} = i_*\mathcal{F}$ then

$$H^i(\mathbb{P}^r_A, \mathcal{G}) \simeq H^i(X, \mathcal{F}).$$

Replacing X by $\mathbb{P}^r_A$ and $\mathcal{F}$ by $\mathcal{G}$ we may assume that $X = \mathbb{P}^r_A$.

If $\mathcal{F} = \mathcal{O}_X(q)$ then the result is given by (11.5). Thus the result also holds is $\mathcal{F}$ is a direct sum of invertible sheaves. The general case proceeds by descending induction on i. Now

$$H^i(X, \mathcal{F}) = 0,$$

if $i > r$ (clear, if we use Čech cohomology). On the other hand, $\mathcal{F}$ is a quotient of a direct sum $\mathcal{E}$ of invertible sheaves. Thus there is an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{R}$ is coherent. Twisting by $\mathcal{O}_X(n)$ we get

$$0 \longrightarrow \mathcal{R}(n) \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{F}(n) \longrightarrow 0.$$

Taking the long exact sequence of cohomology, we get isomorphisms

$$H^i(X, \mathcal{F}(n)) \simeq H^{i+1}(X, \mathcal{R}(n)),$$

for n large enough, and we are done by descending induction on i. $\square$

Theorem 11.7. *Let A be a Noetherian ring and let X be a proper scheme over A. Let $\mathcal{L}$ be an invertible sheaf on X. TFAE*

(1) *$\mathcal{L}$ is ample.*
(2) *For every coherent sheaf $\mathcal{F}$ on X there is an integer n_0 such that*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for $n > n_0$.

Proof. (1) implies (2) is proved using the division algorithm, as in the proof of (9.7).

Now suppose that (2) holds. Let $\mathcal{F}$ be a coherent sheaf. Let $p \in X$ be a closed point. Consider the short exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{O}_p \longrightarrow 0,$$

where $\mathcal{I}_p$ is the ideal sheaf of p. If we tensor this exact sequence with $\mathcal{L}^n$ we get an exact sequence

$$0 \longrightarrow \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \longrightarrow \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p \longrightarrow 0.$$

By hypotheses we can find n_0 such that

$$H^1(X, \mathcal{I}_p\mathcal{F} \otimes \mathcal{L}^n) = 0,$$

for all $n \geq n_0$. It follows that the natural map

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^n) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L}^n \otimes \mathcal{O}_p),$$

is surjective, for all $n \geq n_0$. It follows by Nakayama's lemma applied to the local ring $\mathcal{O}_{X,p}$ that that the stalk of $\mathcal{F} \otimes \mathcal{L}^n$ is generated by global sections. As $\mathcal{F}$ is a coherent sheaf, for each integer $n \neq n_0$ there is an open subset U, depending on n, such that sections of $H^0(X, \mathcal{F} \otimes \mathcal{L})$ generate the sheaf at every point of U.

If we take $\mathcal{L} = \mathcal{O}_X$ it follows that there is an integer n_1 such that $\mathcal{L}^{n_1}$ is generated by global sections over an open neighbourhood V of p. For each $0 \leq r \leq n_1 - 1$ we may find U_r such that $\mathcal{F} \otimes \mathcal{L}^{n_0+r}$ is generated by global sections over U_r. Now let

$$U_p = V \cap U_0 \cap U_1 \cap \dots \cap U_{n_1-1}.$$

Then

$$\mathcal{F} \otimes \mathcal{L}^n = (\mathcal{F} \otimes \mathcal{L}^{n_0+r}) \otimes (\mathcal{L}^{n_1})^m,$$

is generated by global sections over the whole of U_p for all $n \neq n_0$.

Now use compactness of X to conclude that we can cover X by finitely many U_p. □

Theorem 11.8 (Serre duality)**.** *Let X be a smooth projective variety of dimension n over an algebraically closed field. Then there is an invertible sheaf ω_X such that*

(1) $h^n(X, \omega_X) = 1$.

(2) *Given any other invertible sheaf $\mathcal{L}$ there is a perfect pairing*

$$H^i(X, \mathcal{L}) \times H^{n-i}(X, \omega_X \otimes \mathcal{L}^*) \longrightarrow H^n(X, \omega_X).$$

Example 11.9. *Let $X = \mathbb{P}^r_k$. Then $\omega_X = \mathcal{O}_X(-r-1)$ is a dualising sheaf.*

In fact, on any smooth projective variety, the dualising sheaf is constructed as the determinant of the cotangent bundle, which is a locally free sheaf. To construct the cotangent bundle, let $i\colon X \longrightarrow X \times X$ be the diagonal embedding. Let $\mathcal{I}$ be the ideal sheaf of the diagonal and let

$$\Omega^1_X = i^* \frac{\mathcal{I}}{\mathcal{I}^2}.$$

Ω^1_X is the dual of the tangent bundle. Ω^1_X is a locally free sheaf of rank n, known as the sheaf of Kähler differentials. The determinant sheaf is then the dualising sheaf,

$$\omega_X = \wedge^n \Omega^1_X.$$

This expresses a remarkable coincidence between the dualising sheaf, which is something defined in terms of sheaf cohomology and the determinant of the sheaf of Kähler differentials, which is something which comes from calculus on the variety.

Theorem 11.10. *Let $X = X(F)$ be a toric variety over $\mathbb{C}$ and let D be a T-Cartier divisor. Given $u \in M$ let*

$$Z(u) = \{\, v \in |F| \,|\, \langle u, v \rangle \geq \psi_D(v) \,\}.$$

Then

$$H^p(X, \mathcal{O}_X(D)) = \bigoplus_{u \in M} H^p(X, \mathcal{O}_X(D))_u \qquad \text{where} \qquad H^p(X, \mathcal{O}_X(D))_u = H^p_{Z(u)}(|F|).$$

Some explanation is in order. Note that the cohomology groups of X are naturally graded by M. (11.10) identifies the graded pieces.

$$H^p_{Z(u)}(|F|) = H^p(|F|, |F| - Z(u), \mathbb{C}).$$

denotes local cohomology. This comes with a long exact sequence for the pair. If X is an affine toric variety then both $|F|$ and $Z(u)$ are convex and the local cohomology vanishes. More generally, if D is ample, then then both $|F|$ and $Z(u)$ are convex and the local cohomology vanishes. This gives a slightly stronger result than Serre vanishing in the case of an arbitrary variety.

It is interesting to calculate the dualising sheaf in the case of a smooth toric variety. First of all note that the dualising sheaf is a line bundle, so that $\omega_X = \mathcal{O}_X(K_X)$, for some divisor K_X, which is called the **canonical divisor**. Note that the canonical divisor is only defined up to linear equivalence.

To calculate the canonical divisor, we need to write down a rational (or meromorphic in the case of $\mathbb{C}$) differential form. Note that if

$z_1, z_2, \dots, z_n$ are coordinates on the torus then

$$\frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_n}{z_n},$$

is invariant under the action of the torus, so that the assoiated divisor is supported on the invariant divisor. With a little bit of work one can show that this rational form has a simple pole along every invaraint divisor, that is

$$K_X + D \sim 0,$$

where D is a sum of the invariant divisors. For example,

$$-K_{\mathbb{P}^n} = H_0 + H_1 + \cdots + H_n \sim (n+1)H,$$

as expected.

Even if X is not smooth, it is possible to define the canonical divisor. Suppose that X is normal, so that the singular locus has codimension at least two. Let U be the smooth locus and let K_U be the canonical divisor of U. Let K_X be the divisor obtained by taking the closure of the components of K_U. Note that K_X is only defined as a Weil divisor in this case.

12. Singularities

It is the aim of this section to develop some of the theory and practice of the classification of singularities in algebraic geometry. If one want to classify singularities, then this is clearly a local problem. Unfortunately the Zariski topology is very weak, and the property of being local in the Zariski does not satisfactorily capture the correct notion of classification. In general the correct approach is to work with the formal completion. Since this is somewhat technical, we work instead over $\mathbb{C}$, and work locally analytically.

The most basic invariant of a singular point is the dimension of the Zariski tangent space.

Definition 12.1. *Let (X, p) be a germ of a singularity. The* ***embedding dimension*** *is the dimension of the Zariski tangent space of X at p.*

As the name might suggest, we have the following characterisation of the embedding dimension.

Lemma 12.2. *Let (X, p) be a germ of a singularity. The* ***embedding dimension*** *is equal to the smallest dimension of any smooth germ (M, q) such that X embeds in M.*

Proof. Let k be the embedding dimension of X, and suppose that $X \subset M$, where M is smooth. As $T_pX \subset T_pM$, and the dimension of M is equal to the dimension of T_pM, it is clear that the dimension of M is at least k.

Now consider embedding X into a smooth germ N and then projecting down to a smaller subspace M. Clearly we can always choose the projection to be an embedding of the Zariski tangent space to X at p, provided the dimension of M is at least k. Since the property that df is an isomorphism of Zariski tangent spaces is a local condition, it follows that possibly passing to a smaller open subset, we may assume that projection down to M induces an isomorphism of Zariski tangent spaces. But then the projection map is an isomorphism. $\square$

Definition 12.3. *We will say that X has a* ***hypersurface singularity*** *if the embedding dimension is one more than the dimension of X; we will say that a curve singularity is* ***planar*** *if it is a hypersurface singularity.*

Let $(X, p) \subset (M, p)$ be a hypersurface singularity. Pick coordinates $x_1, x_2, \dots, x_n$ on M and suppose that f defines X. Let m be the maximal ideal of M at p. The ***multiplicity of** X **at*** *p is equal to the smallest integer μ such that $f \in m^\mu$.*

Given X, let Y be the singularity given by $x^2 + f$, where x is a new variable, so that Y is a hypersurface singularity of dimension one more than X. Any singularity obtained by successively replacing f by $x^2 + f$ will be called a ***suspension of*** *X.*

*We say that X has type A_n**-singularities** if X is defined by the suspension of x^{n+1}. We say that X has type D_n**-singularities**, for $n \geq 4$, if X is defined by the suspension of $x^2y + y^{n-1}$. We say that X has a type E_6**-singularity**, if X is defined by the suspension of x^3+y^4, a type E_7**-singularity**, if X is defined by the suspension of $x^3 + xy^3$, and a type E_8**-singularity**, if X is defined by the suspension of x^3+y^5.*

Note that the multiplicity of X is independent of the choice of coordinates and that a hypersurface is smooth iff the multiplicity is one. Note that the multiplicity is upper semi-continuous in families.

There are a couple of basic results about power series that we will use time and again. First some basic notation. We say that a monomial m appears in f and write $m \in f$ if the coefficient of m in f is non-zero.

Lemma 12.4. *Let $f \in \mathbb{C}\{x_1, x_2, \dots, x_n\}$ be the germ of an analytic function.*

1. *If f has non-zero constant term then f is invertible and we may take nth roots.*
2. *If we write $f = ax_n^k + \dots$, where dots indicate terms divisible by x_n^k of higher degree and $a \neq 0$, then we may change coordinates so that $f = x_n^k$.*
3. *If $f = ux_n^k + \dots$, where $\dots$ indicate terms other than x_n^k and u is not in the maximal ideal, then we may change coordinates so that $f = x_n^k + f_{n-2}x_n^{k-2} + \dots + f_0$, where f_i are analytic functions in the first $n-1$ variables.*

Proof. (1) is well-known. Consider (2). By assumption we may write $f = ax_n^k + x_n^k g$, where g is an analytic function lying in the maximal ideal. In this case $f = x_n^k(a + g) = x_n^k u$, where by (1) u is a unit. In this case, also by (1), there is an analytic function v such that $v^k = u$. Replacing x_n by vx_n, f now has the correct form. This is (2).

Finally consider (3). Clearly we may expand f as

$$f = \sum_i f_i x_n^i,$$

where f_i are power series in the first $n-1$ variables. By assumption f_k is a unit. As before, we may then assume that $f_k = 1$. By (2) we may assume that $f_i = 0$ for $i > k$. Completing the nth power we may assume that $f_{k-1} = 0$. Now f has the required form. $\square$

Definition-Lemma 12.5. *Let X be a hypersurface singularity of multiplicity μ. Then we may choose coordinates $x_1, x_2, \dots, x_n$ such that X is given by*

$$x_n^{\mu} + f_{\mu-2}x_n^{\mu-2} + \dots + f_0,$$

*where f_i are analytic functions of the first $n-1$ variables. Any such polynomial is called a **Weierstrass polynomial**.*

Proof. By assumption f_μ is non-zero. Possibly changing coordinates, we may assume that $x_n^\mu \in f$. The result is now an easy consequence of (12.4). □

Lemma 12.6. *A planar curve singularity has multiplicity two iff it is of type A_n.*

Proof. After putting f into Weierstrass form, the result becomes easy. □

It is interesting to see what happens for small values of n. If $n = 1$, so that $f = y^2+x^2$, then we have a **node**. This corresponds to two smooth curves with distinct tangent directions. If $n = 2$, then $f = y^2+x^3$, then we have a **cusp**. In the case $n = 3$, we have $y^2 + x^4$, this represents two smooth curves which are tangent. We call this a **tacnode**. The case $n = 4$ is called a **ramphoid cusp**, $n = 5$ an **oscnode**, and $n = 6$ a **hyper-ramphoid cusp** and so on.

Definition 12.7. *Let C be a planar singularity of order μ. We say that C is **ordinary** if when we write $f = f_\mu + \dots$, where dots indicate higher order terms, then f_μ factors into μ distinct linear factors.*

It is not hard to show that that if C is ordinary, we may always choose coordinates so that $f = f_\mu$.

Definition 12.8. *Let X be a singular variety, a subset of $\mathbb{A}^n$. The **tangent cone** of X at a point p is the intersection of the strict transform of X with the exceptional divisor.*

If X is a hypersurface singularity, then the tangent cone is given by $f_\mu = 0$ a subset of $\mathbb{P}^{n-1} \simeq E$.

Example 12.9. *Consider ordinary planar curve singularities of multiplicity four. Then each linear factor defines an element of $\mathbb{P}^1$. But four unordered points in $\mathbb{P}^1$ have moduli (the j-invariant). Thus there is a one dimensional family of non-isomorphic planar curve singularities of multiplicity four. Indeed, since one can always choose the first three points to be* 0, 1 *and* ∞, *we can write*

$$f = xy(x-y)(x-\lambda y).$$

Definition 12.10. *Let (X,p) be the germ of a singularity. A **deformation** of X is a triple (π,σ,i), where $\pi\colon \mathcal{X} \longrightarrow B$ is a morphism, σ is a **section** of π (that is, $\pi\circ\sigma$ is the identity) such that for every $t\in B$ the pair $(X_t,\sigma(t))$ is a germ of a singularity and i is an **isomorphism** of the pair (X,p) and $(X_0,\sigma(0))$, the central fibre of π.*

In practice, it is customary to drop σ and i and refer to a deformation using only π. Note that since the multiplicity is upper semi-continuous in families, it follows that the multiplicity can only go down under deformation.

In other words, a deformation is to the germ of a singularity, as a family is to a variety. As such one might hope that there exists universal deformations, as there exists universal families. Equivalently, one might hope to write down the obvious functor and hope that there is a space which represents this functor. Unfortunately this is not so; the problem is that the central fibre might have more automorphisms, than the typical fibre (and this why we are careful to specify the isomorphism of the central fibre with the space to be deformed). Instead, the best we can hope for is

Definition 12.11. *Let (X,p) be a germ of a singularity. We say that a deformation π of X is **versal** if for every other deformation ψ there is a morphism $B'\longrightarrow B$ such that ψ is pulled back from π in the obvious way.*

Note that we do not require uniqueness of the versal family, and in fact we cannot, since if there is an automorphism of the central fibre that does not lift to the whole deformation space, for example if it does not lift to every fibre, then we get a different deformation, simply by composing with this automorphism (that is, we change the isomorphism i).

Fortunately, versal deformation spaces are easy to write down.

Definition 12.12. *Let X be a hypersurface singularity, defined by the equation $f=0$. Let*

$$T^1_f = \frac{\mathbb{C}[x_1,x_2,\dots,x_n]}{\langle f, \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots \frac{\partial f}{\partial x_n}\rangle}.$$

Theorem 12.13. *Let X be an isolated hypersurface singularity. Pick holomorphic functions $g_1,g_2,\dots,g_k$ such that their images in T^1_f form a basis. Then the deformation given by*

$$f_t = f + \sum_i t_i g_i,$$

where t_i are coordinates on the germ $(\mathbb{C}^k,0)$ is a versal deformation.

Another way to state (12.13), is that T^1_f is the Zariski tangent space to the versal deformation space. We will also need the following basic fact.

Lemma 12.14. *Let $\pi\colon \mathcal{X} \longrightarrow B$ be a versal deformation space, and let B' be a general subvariety of B. Then the restriction of π to B' defines a versal deformation space of the general point of B'.*

It is interesting to see what happens in a series of examples. Suppose we start with planar singularities. The simplest is an A_1-singularity. In this case

$$f = x^2 + y^2 \qquad \text{so that} \qquad \frac{\partial f}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 2y.$$

Thus

$$T^1_f = \frac{\mathbb{C}[x,y]}{\langle x, y\rangle}$$

and for g_1 we take 1. Thus the versal deformation space of a node is given as

$$f = x^2 + y^2 + t.$$

In other words, the only thing that we can do with a node is smooth it. Now consider what happens in the case of an A_n-singularity. In this case the derivatives are $(n+1)x^n$ and $2y$, so that we make take $g_i = x^i$, $i = 0 \dots n-1$. Thus the versal deformation space has dimension n, and it is given by

$$y^2 + x^n + t_0 + t_1 x + t_2 x^2 + \dots + t_{n-1} x^{n-1}.$$

For example consider the case of a cusp. In this case the versal deformation space has dimension two, and it is given by

$$y^2 + x^3 + ax + b.$$

where a and b are coordinates on the base. The point is that now we have two completely different one dimensional families. Either we can smooth the cusp, or we can partially smooth it to a node. In fact the locus of nodes forms a curve in the base.

Note that $y^2 = x^3 + ax + b$ is singular iff the polynomial $x^3 + ax + b$ has a double root. But then the singular locus is given by the discriminant, that is $4b^3 + 27a^2$, so that this locus is not smooth.

Similarly, it is not hard to see that the locus of A_n-singularities contains loci corresponding to the A_k-singularities, for $k \leq n$. In fact this locus will have codimension $n - k$.

In fact the converse is true, that is, one can only deform an A_n-singularity to an A_k singularity, for $k \leq n$. Compare this with the case of an ordinary four-fold point. Suppose that we start with $x^4 + y^4$.

Then the derivatives are $4x^3$ and $4y^3$ and so we can take for g_i, 1, x, y, x^2, xy, y^2, x^2y, xy^2, x^2y^2. In this case, there is a one dimensional locus corresponding to all other (nearby) ordinary four-fold singularities, given by $x^4+y^4+tx^2y^2$. In other words, there are infinitely many non-isomorphic germs in the versal deformation space.

It is interesting to look at some of these ideas from the point of view of blowing up and resolution of singularities.

Definition 12.15. *Let X be a variety and let $D=\sum D_i$ be a divisor, the sum of distinct prime divisors. We say that the pair (X,D) has normal crossings if X is smooth and locally about every point, the pair (X,D) is equivalent to $\mathbb{C}^n$ union some of the coordinate hyperplanes.*

*A **resolution of singularities** for X is a birational map $\pi\colon Y\longrightarrow X$ with the following properties.*

(1) *π is an isomorphism over the smooth locus of X,*
(2) *Y is smooth,*
(3) *the exceptional locus is a divisor with normal crossings.*

To date there is only one known way to resolve singularities (at this level of generality) and that is to embed X into a smooth variety M and then carefully choose an appropriate sequence of blow ups, at each stage blowing up M and replacing X by its strict transform. In this case we want the exceptional locus of $\psi\colon N\longrightarrow M$ to intersect the strict transform Y of X as transversally as possible. For example if X has a hypersurface singularity, then we want $Y+E$, where E is the exceptional locus, to have normal crossings.

We have already seen some examples of this. Perhaps the easiest example is the case of a nodal curve. In this case C sits inside a smooth surface M, and we simply blow up the singular point of C. At this point C is smooth and meets the exceptional locus smoothly in two points, so that the pair $C+E$ does have normal crossings.

Now suppose that we take a curve with a cusp. Pick local coordinates so that we have y^2+x^3. Blowing up once, we have already seen that C becomes smooth. However C is tangent to the exceptional locus. If we blow up, then the strict transform of C intersects the points where the two exceptional divisors intersect. Thus it is necessary to blow up once more to achieve normal crossings.

It is interesting to see what happens for an ordinary singularity. In this case we have seen that we may choose coordinates so that f is homogeneous. Thus f factors into μ distinct linear factors. Now each of those factors corresponds to a point of the exceptional locus and in fact when we blow up then C is smooth and meets the exceptional divisor at μ points. At this point $C+E$ has normal crossings.

For example, if the multiplicity is four, then C meets E in four points, and we get our j-invariant directly.

Theorem 12.16. *Let C be a planar curve singularity.*

Then the versal deformation space of C contains only finitely many isomorphism types iff C is one of A_n, D_n, E_6, E_7 or E_8.

Proof. Let f be a defining equation for C. Let us show that if there are only finitely many isomorphism types in the versal deformation space, then C must be one of the ADE-singularities. Suppose that the multiplicity of f is at least four. Then we may deform f to an ordinary multiplicity four singularity. But then there are infinitely many non-isomorphic singularities in the versal deformation space. On the other hand, if the multiplicity is two, then by (12.6) C must have type A_n.

Thus we may suppose that f has multiplicity three. Consider f_3. This factors into three linear factors. There are three cases; the three factors are distinct; there are two distinct factors, there is one.

Suppose that there are at least two distinct factors. Then there is a factor which occurs only once. We may assume that this factor is y. Since the multiplicity is three, in fact y must divide f, so that we may write

$$f = h \cdot y,$$

where h only depends on x and y. Now h has multiplicity two and its rank two part is not divisible by y. It follows that there is a change of variable, so that $h = x^2 + y^n$, where $n \geq 2$, which change of variable does not change y. But then $f = x^2y + y^{n+1}$ and we have a singularity of type D (more precisely, a D_{n+3}-singularity).

This final case is when $f_3 = y^3$, so that $f = y_3 + g$ and the multiplicity of g is at least four. Putting f into Weierstrass form, once again, we may assume that $f = y^3 + yg + h$, where g and h only involve x. Thus f can be put in the form $y^3 + ayx^k + x^l + \dots$, where the dots indicate higher powers of x, a is either zero or one and $k < l$. If $l = 4$, it follows that $k = 3$ so that completing the square we may assume that $a = 0$. In this case, it is not hard to show that we can choose coordinates so that $f = y^3 + x^4$ and we have an E_6-singularity. If $x^3y \in f$ and $l > 4$ then with some manipulation we can put f into the form $y^3 + x^3y$, so that we have an E_7-singularity; similarly if $y^5 \in f$ but we have no lower terms, then we have an E_8-singularity. Otherwise we may assume that $l > 5$ and that $k > 3$. In this case, we may as well assume that $f = y^3 + \lambda yx^4 = y(y^2 + \lambda x^4)$, which represents three smooth curves which are tangent. Suppose we blow up once; we get four curves passing through one point, the strict transform of the three tangent curves and the exceptional divisor. If we blow up the point, we get four curves intersecting the new exceptional divisor and the four points of intersection with the new exceptional gives one dimension of moduli (the j-invariant, which varies as we vary λ).

Now let us consider the converse problem, to show that there are only finitely many isomorphism types in the versal deformation space. Clearly it suffices to prove that we can only deform an ADE-singularity, to an ADE-singularity. This is clear for A_n-singularities, since under deformation the multiplicity can only go down.

Now consider the case of a D_n-singularity. We only need to consider deformations that preserve the multiplicity. In this case, the deformations of f_3 can only increase the number of distinct linear factors, and we cannot lose a term of the form y^k. Thus the deformation of an D_n-singularity is either a D_k-singularity, for some $k \leq n$ or an A_n-singularity.

Finally consider the three exceptional cases. Suppose we start with x^3+y^4. Then the only possible deformation which fixes the multiplicity, deforms to a singularity of type D_n, $n \leq 5$. Now suppose we start with $y^3 + x^3y$. Again we can only pick up a term of the form x^4 or increase the number of linear factors. Similarly for an E_8-singularity. $\square$

Here is a way to restate (12.16):

Proposition 12.17. *The ADE-singularities are the only singularities, which have multiplicity two and three, and such that after blowing up, the multiplicity of the total transform has multiplicity two or three.*

Proof. It is not hard to check that this is all we have used in the proof of (12.16) to characterise ADE-singularities. $\square$

The are four other obvious ways of creating examples of singularities other than writing down equations. The first is simply to take the cone over a closed subset of $\mathbb{P}^n$. Note that the cone is a degenerate example of the join of two varieties, where one of the two varieties to be joined is a point. Note also that if I is the ideal of $X \subset \mathbb{P}^n$, then I is also the ideal of the cone Y over X, where Y is the closure of the inverse image of X inside K^{n+1}. In particular, the classification of singularities is at least as hard as the classification of varieties. On the other hand, note that the resolution problem for such singularities is in fact easy. If X is smooth, then simply blowing up the vertex, we get a birational map $\pi\colon W \longrightarrow Y$, whose exceptional locus E is a copy of X, where W is smooth. In fact W is a $\mathbb{P}^1$-bundle over X, and E is simply a section of this bundle.

The next is to start with a configuration of divisors and contract them. Unfortunately it is quite hard to characterise which configurations are contractible. The third method is to take a quotient:

Definition 12.18. *Let G be an algebraic group acting on a variety X. We say that Y is a* ***categorical quotient of*** *X* ***by*** *G if there is a morphism $\pi\colon X \longrightarrow Y$ such that $\pi(g\cdot x) = \pi(x)$ for every $g \in G$, which is universal amongst all such morphisms:*

If $\phi\colon X \longrightarrow Z$ is a morphism such that $\psi(g \cdot x) = \psi(x)$ then there is a unique morphism $\psi\colon Y \longrightarrow Z$ which makes the diagram commute,

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ Y \arrow[r, "\psi"] & Z. \end{tikzcd}$$

It is common to denote the categorical quotient by X/G (if it exists at all). Fortunately there is one quite general existence theorem:

Theorem 12.19. *Let $X = \operatorname{Spec} A$ be an affine variety and let G be a finite group acting on X.*

Then the categorical quotient is the affine variety $Y = \operatorname{Spec} A^G$.

Proof. The key fact is that the ring of invariants

$$A^G = \{\, a \in A \mid g \cdot a = a \,\},$$

is a finitely generated k-algebra. □

Note that $Y = X/G$ will in general be a singular variety. It is however $\mathbb{Q}$-factorial, that is, every Weil divisor is $\mathbb{Q}$-Cartier, that is, given by any Weil divisor D, some multiple is Cartier (indeed, $r = |G|$ will do).

The final method is to use toric geometry. We start with the canonical example.

Let σ be the cone spanned by e_2 and $2e_1 - e_2$. The dual cone σ is spanned by f_1 and $f_1 + 2f_2$. Generators for the monoid are f_1, $f_1 + f_2$ and $f_1 + 2f_2$, so that

$$X = U_\sigma = \operatorname{Spec} \mathbb{C}[x, xy, xy^2] = \operatorname{Spec} \mathbb{C}[u, v, w]/\langle v^2 = uw\rangle.$$

Thus we have an A_1-singularity.

It is interesting to see how to resolve this singularity. Suppose we insert the vector e_1; this corresponds to a blow up with exceptional divisor isomorphic to $\mathbb{P}^1$. We get two cones σ_1 and σ_2, one spanned by e_1 and e_2 and the other spanned by e_1 and $2e_1 - e_2$. It follows that the blow up is smooth. Note that X is the cone over a conic; it follows once again that X can be resolved in one step.

Let's make this example a little more complicated. Let's start with the cone spanned by e_2, $re_1 - e_2$. The dual cone is the cone spanned by f_1 and $f_1 + rf_2$. Generators for the monoid are f_1, $f_1 + f_2$, $\dots$, $f_1 + rf_2$. We get

$$U_\sigma = \operatorname{Spec} \mathbb{C}[x, xy, \dots, xy^r] = \operatorname{Spec} \mathbb{C}[u^r, u^{r-1}v, \dots, v^r],$$

where $u^r = x$ and $v = y/x$, which is the cone over a rational normal curve of degree r. Note that the embedding dimension is $r + 1$. Note that this is again resolved in one step by inserting the vector e_1.

At the other extreme, consider the cone spanned by e_2 and $re_1 - (r - 1)e_2$. The dual cone is spanned by f_1 and $(r - 1)f_1 + rf_2$. Generators for the monoid are f_1, $(r - 1)f_1 + rf_2$ and $f_1 + f_2$. We get

$$U_\sigma = \operatorname{Spec} \mathbb{C}[x, xy, x^{r-1}y^r] = \operatorname{Spec} \mathbb{C}[u, v, w]/\langle v^r = uw\rangle,$$

which is an A_{r-1}-singularity. If we insert the vector e_1 then we the resulting blow up has two affine pieces. One is smooth, corresponding to the cone spanned by e_1 and e_2 and the other is the cone given by e_1 and $re_1 - (r - 1)e_2$. Switching the sign of e_2 we get e_1, $re_1 + (r - 1)e_2$. Switching e_1 and e_2 we get e_2 and $(r - 1)e_1 - re_2$. Replacing e_1 by $e_1 - 2e_2$ we get e_2 and $(r - 1)e_1 - (r - 2)e_2$ which as we have already seen is an A_{r-2}-singularity. Thus an A_r-singularity takes r-steps to resolve. On the resolution we get a chain of r-copies of $\mathbb{P}^1$.

More generally, we could consider the cone spanned by e_2 and $re_1 - ae_2$, where $0 < a < r$, is coprime to r. However the best way to proceed, is to look at all of this a different way.

We start with an example. The cyclic group $G = \mathbb{Z}_r$ acts on $\mathbb{C}^2$ via

$$(u, v) \longrightarrow (\omega u, \omega v),$$

where ω is a primitive rth root of unity. In this case the ring of invariants is precisely

$$\mathbb{C}[u, v]^G = \mathbb{C}[u^r, u^{r-1}v, \dots, v^r].$$

To see this using the toric structure, let $N' \subset N$ be the sublattice spanned by $e'_2 = e_2$ and $e'_1 = re_1 - e_2$. Then the cone σ' spanned by the same vectors e_2 and $re_1 - e_2$ now corresponds to a smooth toric variety. The dual lattice M' is an overlattice of M.

Thinking this way, we should make a basis for N' the standard vectors e'_1 and e'_2. The overlattice N is spanned by N' and the vector e_1 in the old coordinates. As

$$e_1 = 1/r(re_1 - e_2) + 1/re_2,$$

in the old coordinates, in the new coordinates we have that N' is spanned by e'_1, e'_2 and $1/r(e'_1 + e'_2)$. If we insert this vector, we get a basis for the lattice. If the dual lattice M' is the overlattice spanned by f_1' and f_2' then M is the sublattice spanned by all $af_1' + bf_2'$ such that $a+b$ is divisible by r.

In the other example, where we started with e_2 and $re_1 - (r-1)e_2$, then

$$e_1 = 1/r(re_1 - (r-1)e_2) + (r-1)/re_2.$$

So N is the lattice spanned by e_1, e_2 and $1/re_1 + (r-1)/re_2$. This suggests we should look at the action

$$(x, y) \longrightarrow (\omega x, \omega^{r-1} y) = (\omega x, \omega^{-1} y),$$

Indeed, the ring of invariants is $u = x^r$, $w = y^r$ and $v = xy$ and $v^r = uw$, as expected.

More generally still, for the action

$$(x, y) \longrightarrow (\omega x, \omega^a y),$$

we should look at the lattice N spaned by the standard lattice and the vector $1/r(1, a)$. Inserting this vector, gives two cones, one spanned by e_1, $1/r(1, a)$ and the other spanned by $1/r(1, a)$ and e_2. The second one is smooth. For the first, let us make the two vectors $1/r(1, a)$ and e_1 the standard generators for the lattice. As

$$(0, 1) = r/a(1/r, a/r) - 1/a(0, 1),$$

we then have the overlattice generated by $(-1/a, r/a)$. Now

$$r/a = k - b/a,$$

for some unique $0 \leq b < a$. So we get a singularity of type $1/a(1, b)$. Note that resolving the singularity corresponds to computing a continued fraction. The significance of k is the self-intersection of exceptional divisor (on the minimal resolution).

So the resolution graph of any cyclic surface singularity is a chain of $\mathbb{P}^1$'s. Singularities of type A_r correspond to a chain of r such curves, where each curve has self-intersection -2. In fact it is not hard to prove:

Theorem 12.20. *Let $S = \mathbb{C}^2/G$ be a two dimensional quotient singularity. Then $G \subset \mathrm{GL}(2, \mathbb{C})$ and there are three possibilities:*

(1) *G is cyclic and the dual graph of the (minimal) resolution corresponds to the Dynkin diagram A_n. The action is $(x, y) \longrightarrow (\omega x, \omega^a y)$, where ω and ω^a is both primtive roots of unity. S is isomorphic to a toric surface.*
(2) *G is a dihedral group and the dual graph corresponds to the Dynkin diagram for D_n, $n \geq 4$.*

(3) *G is one of three exceptional groups and the dual graph is the Dynkin diagram for E_6, E_7 or E_8.*

If in addition $G \subset \mathrm{SL}(2,\mathbb{C})$ then S has an ADE-singularity and the self-intersections of the exceptional curves are all -2.

More generally suppose that $\sigma \subset N \simeq \mathbb{Z}^n$ is a simplicial cone. As before let $N' \subset N$ be the sublattice spanned by the primitive generators of σ. Let $M \subset M'$ be the corresponding overlattice. Then there is a natural pairing

$$N/N' \times M'/M \longrightarrow \mathbb{Q}/\mathbb{Z}$$

This makes M the invariant sublattice of M', under the action of the finite abelian group $G = N/N'$ and under this action it is not hard to see that

$$A_\sigma = (A_{\sigma'})^G.$$

Note that G is a product of at most $n-1$ cyclic factors.

Let me end by talking a little about the problem of resolution of singularities. At it most basic we are given a finitely generated field extension K/k and we would like to find a smooth projective variety X over k with function field K.

Theorem 12.21. *Let X be a smooth projective variety and let $\mathcal{O}_X(1)$ be a very ample line bundle. Suppose that $X \subset \mathbb{P}^n$ has degree d.*

Then

$$h^0(X, \mathcal{O}_X(m)) = \frac{dm^n}{n!} + ...,$$

is a polynomial of degree n, for m large enough, with the given leading term.

Proof. Let Y be a hyperplane section. The trick is to compute $\chi(X, \mathcal{O}_X(m))$ by looking at the exact sequence

$$0 \longrightarrow \mathcal{O}_X(m-1) \longrightarrow \mathcal{O}_X(m) \longrightarrow \mathcal{O}_Y(m) \longrightarrow 0.$$

The Euler characteristic is additive so that

$$\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m-1)) = \chi(Y, \mathcal{O}_Y(m)).$$

By an easy induction, it follows that $\chi(X, \mathcal{O}_X(m))$ is a polynomial of degree n, with the given leading term. Now apply Serre vanishing. □

Definition 12.22. *Let $X \subset M$ be a subvariety of a smooth variety. The* ***multiplicity of*** *X* ***at*** *$p \in M$ is the smallest μ such that $\mathcal{I}_p \subset \mathfrak{m}^\mu$ where $\mathfrak{m}$ is the maximal ideal of M at p in $\mathcal{O}_{M,p}$ and $\mathcal{I}$ is ideal sheaf of X in M.*

The method of Albanese is to start with $X \subset \mathbb{P}^n$. Now re-embed X by the very ample line bundle $\mathcal{O}_X(m)$. The degree of the image is dm^n inside $\mathbb{P}^r$, where r is roughly $dm^n/n!$. Now suppose that there is a point p of multiplicity μ. If we project from p then we drop r by 1 and the degree by μ. So if we take m sufficiently large and always project from a point of highest multiplicity then we can also reduce to the case when the multiplicity is at most $n!$. Unfortunately it seems impossible to improve this bound.

Another intriguing method was proposed by Nash:

Definition 12.23. *Let $X \subset \mathbb{P}^N$ be a quasi-projective variety of dimension n. The **Gauss map** is the rational map*

$$X \longrightarrow \mathbb{G}(n, N) \qquad \text{given by} \qquad x \longrightarrow T_x X,$$

which sends a point to its (projective) tangent space.

Conjecture 12.24. *We can always resolve any variety by successively taking the Nash blow up and normalising.*

Despite the very appealing nature of this conjecture (consider for example the case of curves) we only know (12.24) in very special cases (the result for toric varieties is about six months old).

If X is a toric variety there is a pretty simple method to resolve singularities. First subdivide the cone until X is simplicial. It is not too hard to argue that one can resolve any simplicial toric variety (one keeps track of a simple invariant).

In general the only known method to find a strong resolution of singularities (only touch the smooth locus) goes back to Hironaka. The idea is embed X into a smooth variety M and choose a sequence of smooth blow ups in M. The problem reduces to two (closely related) parts. Determine the locus to blow up at every step and find some invariant which goes down if we blow this locus up. Forty years after Hironaka's original proof, we know now the only invariant we need to keep track of is the multiplicity.

Unfortunately it is also clear that we need to be quite careful how to choose the locus to blow up. For example consider

$$z^2 - x^3 y^3.$$

The singular locus consists of the x and y axis. If we blow up either axis it is clear that we are making progress (generically along the y-axis we have $z^2 - x^3$, which is resolved in three steps by blowing up the origin). But we are not allowed to blow up an axis. The problem is that this might only be the local analytic picture. We might globally have the singular locus be a nodal cubic (for example). So our resolution process must respect all local isomorphisms. There is an obvious x-y symmetry and so we cannot blow up only one axis. The only possible locus we could blow up which is in the singular locus is the origin. On the blow up we have coordinates $(x, y, z) \times [A : B : C]$, and equations expressing the rule $[x : y : z] = [A : B : C]$. On the coordinate patch $A \neq 0$ we have $y = bx$, $z = cx$ so that

$$z^2 - x^3y^3 = c^2x^2 - b^3x^6 = x^2(c^2 - b^3x^4).$$

Changing variables we have $z^2 - x^3y^4$ which is surely worse than before. The key thing is that the singular locus is given by $c = b = 0$ and $c = x = 0$. The first singular locus we created ourselves and so we know that we are allowed to blow up $c = b = 0$ and now we can desingularise.

So the blow up process must keep track of the sequence of blow ups.