Appendix to Chapter I
ALGEBRAIC LINKS

In this appendix we describe how a splice diagram for an algebraic link may be derived from Puiseux expansions. We also describe a simple method for deriving the power-series equation satisfied by a Puiseux expansion.

Algebraic links

Let $f(x,y)$ be a complex polynomial vanishing at 0, and let

$$C = \{(x,y) \in \mathbf{C}^2 | f(x,y) = 0\}$$

be the corresponding algebraic plane curve. For all sufficiently small $\varepsilon > 0$, the 3-sphere

$$S^2_\varepsilon = \{(x,y) \in \mathbf{C}^2 | \ |(x,y)| = \varepsilon\}$$

meets C transversely in a link, which has a natural orientation coming from that of C. An oriented link (S^3,K) obtained in this way is said to be an *algebraic link*.

To describe (S^3,K) more fully we may (changing variables if necessary) solve $f(x,y) = 0$ for y in terms of x, obtaining a set of solutions which are fractional power series, called *Puiseux series*, in x. Each fractional power series solution gives rise to a *branch* of the curve, and thus to one component of the link (two solutions which differ by a change of variable of the form $x \mapsto \zeta x$, with ζ a root of unity, may describe the same branch). It is not hard to show that all but finitely many terms of the power series can be removed without changing the topology of the link.

The resulting minimal Puiseux series are usually written in the form

$$y = c_1 x^{m_1/n_1} + c_2 x^{m_2/n_1 n_2} + \cdots$$

with $0 \neq c_i \in C$, $m_1/n_1 < m_2/n_1 n_2 < \cdots$,

and each pair (m_i, n_i) relatively prime, and then (m_i, n_i) are called the *Puiseux Pairs* for the corresponding branch. For us, however, it will be more convenient to write the solutions in the multiplicative form:

$$*) \qquad y = x^{q_1/p_1}(a_1 + x^{q_2/p_1 p_2}(a_2 + \cdots (a_{s-1} + x^{q_s/p_1 \cdots p_s}(a_s + \cdots)\cdots)))$$

with $p_i, q_i > 0$ and (p_i, q_i) relatively prime for all i. Expanding the product as a power series, one sees at once that the (p_i, q_i) are determined from the (m_i, n_i) by the formulas $p_i = n_i$, $q_1 = m_1$, $q_i = m_i - m_{i-1} n_i$ for $i > 1$. The pairs (p_i, q_i) might well be called the "Newton pairs" of the expansion, since they are the numbers which are produced by the direct application of Newtons method for computing the expansion ([Wa]).

Making a suitable change of variables we may assume that, in each of the expansions corresponding to C, we have $|y| << |x|$ for $|x|$ small. We may then replace the intersection of C with S^3_ε by the intersection of C with the solid torus

$$R = \{(x,y) \in \mathbf{C}^2 \mid |x| = \varepsilon,\ |y| \leq \varepsilon\},$$

which is naturally embedded as an unknotted solid torus in the (topological) sphere

$$R \cup \{(x,y) \in \mathbf{C}^2 \mid |x| \leq \varepsilon,\ |y| = \varepsilon\}.$$

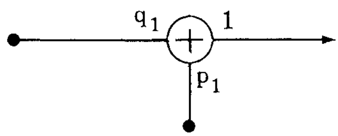
Isotopy arguments [K-N] show that the link $C \cap R$, embedded in S^3 as above, is the same as the link $C \cap S^3_\varepsilon$ in S^3_ε (the change to this point of view seems to be due to Kähler).

To understand this link, consider first the case of a knot, given, say, by the expansion *), in R. A first approximation to it is the p_1, q_1 torus knot, (S^3, K_1), given by

$$y = a_1 x^{q_1/p_1}$$

in $R \subset S^3$.

To see that this really is the (p_1, q_1) torus knot, set $x = \varepsilon t^{\phi}$, where t runs once around the complex unit circle $S^1 \subset C$. Then y is a constant times t^{q_1} so (x,y) runs p_1 times around in the longitudinal direction in R (the x-axis) while running q_1 times around the meridianal direction of R (the y-axis). We will represent (S^3, K_1) by the graph



(such graphs are introduced systematically in section 8).

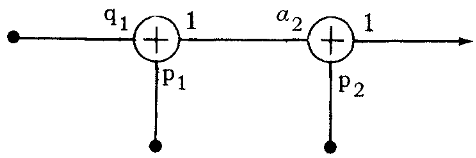
As a second approximation to the knot K, consider the knot K_2 given by

$$y = x^{q_1/p_1}(a_1 + a_2 x^{q_2/p_1 p_2}).$$

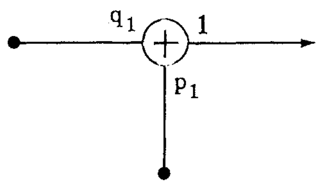
Changing our parametrization to $x = \varepsilon t^{p_1 p_2}$, we see that since ε is very small K_2 will lie in a small tubular neighborhood of K_1, and will in fact be a cable on K_1. Clearly, it will follow K_1 around p_2 times in a longitudinal direction, so for some integer α_2 it will be a (p_2, α_2) cable on K_1. If we let $L_2 = (S^3, K_0 \cup K_0(p_2, \alpha_2))$ be the link consisting of an unknotted circle K_0 and then (p_2, α_2)-cable on it, then by Proposition 1.1 we see that

$$(S^3, K_2) = (S^3, K_1) \underset{K_1 \qquad K_0}{\overline{\hspace{4em}}} L_2 .$$

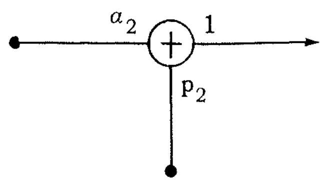
We will represent K_2 by the diagram



which should be regarded as a "composite" of the diagrams

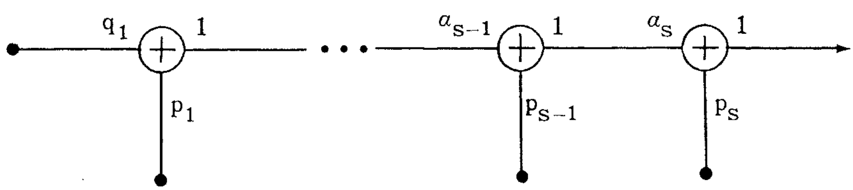


and



for (S^3, K_1) and L_2, respectively.

Repeating these constructions inductively, we see that the knot K represented by $*)$ is, for suitable integers $\alpha_1 = q_1$, $\alpha_2, \cdots, \alpha_s$, represented as the (p_s, α_s) cable on the (p_{s-1}, α_{s-1}) cable on the $\ldots (p_1, \alpha_1)$-cable on the unknot, and we represent it by the diagram



It remains to determine the α_i.

PROPOSITION 1A.1. *The* α_i *above are given from the Newton pairs* (p_i, q_i) *by the formulas*

$$\alpha_1 = q_1$$

and, for $i \geq 1$,

$$\alpha_{i+1} = q_{i+1} + p_i p_{i+1} \alpha_i .$$

Note that the *topological pair* (p_i, α_i) is relatively prime, since (p_i, q_i) is, and that the "algebraicity condition" $p_i, q_i > 0$ becomes $p_i > 0$, $\alpha_{i+1} > p_i p_{i+1} \alpha_i$ in terms of the topological pairs. (See section 9A for an extension of this.)

Proof. The fact that $\alpha_1 = q$ has already been noted. To prove the second formula, consider the knots K_{i-1}, K_i and K_{i+1}, where K_j is the knot in $R \subset S^3$ corresponding to the parametrization

$$y = x^{q_1/p_1}(\cdots(a_{j-1} + a_j x^{q_j/p_1 \cdots p_j})\cdots)$$

in R. If $\varepsilon = |x|$ is sufficiently small, then we may choose tubular neighborhoods $N(K_{i-1})$ and $N(K_i)$ such that $N(K_i)$ is contained in the interior of $N(K_{i-1}) - K_{i-1}$ and K_{i+1} is contained in the interior of $N(K_i)$. We will write M_j, L_j for the topologically standard meridian and longitude of K_j in $\partial N(K_j)$ for $j = i-1, i$.

It suffices to show that, in the homology of $N(K_i) - K_i$, we have $K_{i+1} \sim p_{i+1} L_i + (q_{i+1} + p_i p_{i+1} \alpha_i) M_i$. Let $L \subset N(K_{i-1}) - K_i$ be the knot obtained from K_i by moving each point of K_i a certain small distance in the direction "directly away" from K_{i-1}. We could describe L analytically by a parametrization of the form

$$y = x^{q_1/p_1}(\cdots(a_{i-1} + x^{q_i/p_1 \cdots p_i}(a_i + \delta))\cdots),$$

for suitably small real δ, and it follows from this description that, in the homology of $N(K_i) - K_i$,

$$K_{i+1} \sim p_{i+1} L + q_{i+1} M_i,$$

so it suffices to show that $L \sim L_i + p_i \alpha_i M_i$.

Since L is obviously homologous to K_i in $N(K_i)$, it suffices, by the definition of L_i, to show that $L - p_i \alpha_i M_i$ is null-homologous in $S^3 - K_i$. We will do this by showing that, in the homology of $N(K_{i-1}) - (K_{i-1} \cup K_i)$,

$$L - p_i \alpha_i M_i \sim p_i L_{i-1} ,$$

or, equivalently, $L \sim p_i L_{i-1} + p_i \alpha_i M_i$.

Now in the homology of $N(K_{i-1})-(K_{i-1} \cup K_i)$ we clearly have $p_i M_i \sim M_{i-1}$, so the desired formula becomes

$$L \sim p_i L_{i-1} + \alpha_i M_{i-1} .$$

On the other hand, from the construction of L, we see that we could continue to "push" away from K_{i-1} until reaching $\partial N(K_{i-1})$ without encountering K_i, so L is at least homologous to some linear combination of L_{i-1} and M_{i-1}. Since, in the homology of $N(K_{i-1})-K_{i-1}$ we have $L \sim K_i \sim p_i L_{i-1} + \alpha_i M_{i-1}$, this concludes the proof.

In describing algebraic links with several components we may restrict ourselves to the case of 2 components, since the general case offers no new difficulties.

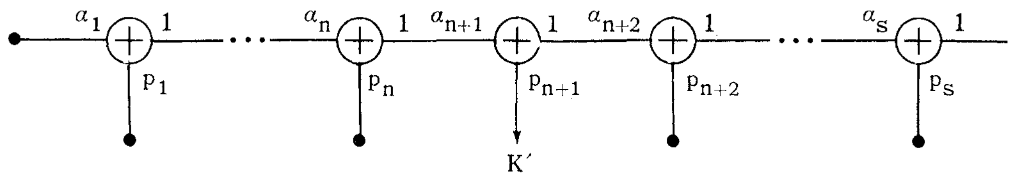
Suppose, then, that $(S^3, K \cup K')$ is an algebraic link with two components, corresponding to the two distinct developments

$$K: \; y = x^{q_1/p_1}(a_1 + \cdots (a_{s-1} + a_s x^{q_s/p_1 \cdots p_s}) \cdots)$$

$$K': \; y = x^{q_1'/p_1'}(a_1' + \cdots (a_{r-1}' + a_r' x^{q_r'/p_1' \cdots p_r'}) \cdots) .$$

Suppose, moreover, that n is the number of common terms; that is, $(p_1,q_1) = (p_1',q_1'), \cdots, (p_n,q_n) = (p_n',q_n')$ and $a_1 = a_1', \cdots, a_n = a_n'$ but $r = n$ or $s = n$ or $(p_{n+1}, q_{n+1}) \neq (p_{n+1}', q_{n+1}')$ or $a_{n+1} \neq a_{n+1}'$. There are several cases to consider:

First, suppose $r = n$ (the case $s = n$ is analogous). Adapting the notation for the successive approximations to K used above (so that $K = K_s$) we have $K' = K_n$. We represent this link by the diagram:



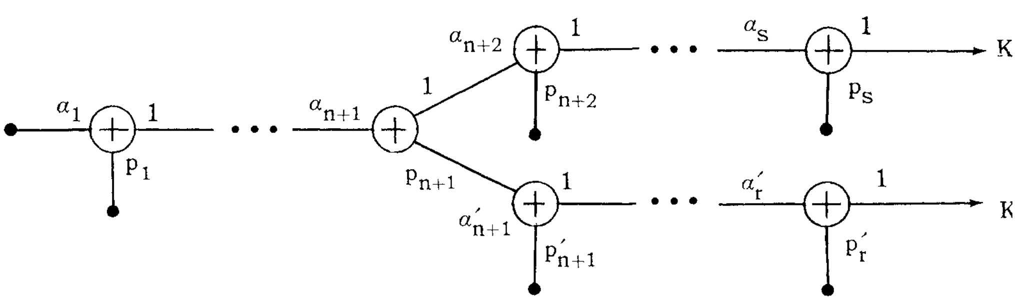
We may now suppose that r and s are both $> n$. Let K_n, K_{n+1} be the n^{th} and $(n+1)^{st}$ approximants to K as above, and let K'_{n+1} be the $(n+1)^{st}$ approximant to K', in the analogous way. Both K_{n+1} and K'_{n+1} are cables on K_n.

Suppose that $q_{n+1}/p_{n+1} < q'_{n+1}/p'_{n+1}$. Then, since $|x|$ is very small, we will have,

$$|a_{n+1} x^{q_{n+1}/p_1 \cdots p_{n+1}}| > |a'_{n+1} x^{q'_{n+1}/p'_1 \cdots p'_{n+1}}| ,$$

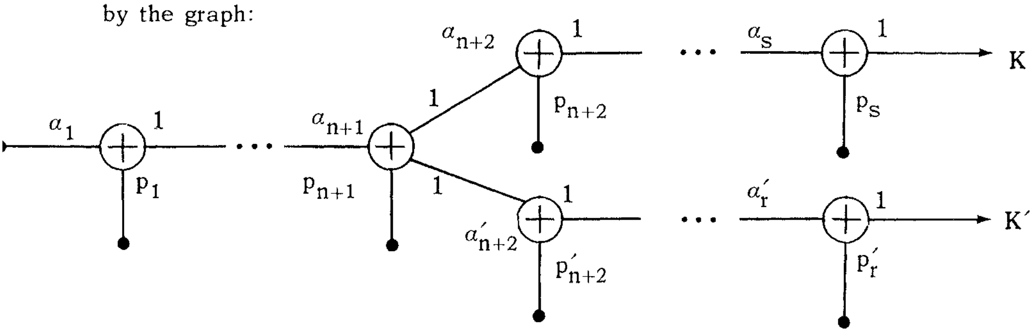
so that K'_{n+1} is a cable on the boundary of a *smaller* tubular neighborhood of K_n then is K_{n+1}.

Since the next cabling operations in the construction of K and K' take place in small tubular neighborhoods of K_{n+1} and K'_{n+1}, they do not interfere, so that with (p_i, α_i) derived from (p_i, q_i) and (p'_i, α'_i) derived from (p'_i, q'_i) as above, we may represent the link $(S^3, K \cup K')$ by the graph



Finally, suppose $q_{n+1}/p_{n+1} = q'_{n+1}/p'_{n+1}$, so that $(p_{n+1}, q_{n+1}) = (p'_{n+1}, q'_{n+1})$ but $a_{n+1} \neq a'_{n+1}$. In this case, K_{n+1} and K'_{n+1} are both

(p_{n+1}, α_{n+1})-cables on K_n, and after a topologically harmless change of a_{n+1} (say) we could assume that K'_{n+1} is formed on a smaller torus than K_n, and proceed as above. However, our notation allows a more symmetrical graph, as well. We may represent $(S^3, K \cup K')$ in this case by the graph:



This concludes our description of algebraic links via graphs.

A final remark on the "non-reduced" case is in order. The branches of C also correspond to distinct irreducible factors of f as a (convergent or formal) power series. If a factor of f occurs with multiplicity > 1, so that f is not "reduced", then it is natural to consider the corresponding branch of C to have multiplicity > 1 (this will be its natural scheme structure) and to consider the corresponding component of the algebraic link $C \subset R \subset S^3$ to have multiplicity > 1 (again, this will happen naturally if we treat $C \subset R$ as a real-analytic space instead of a set). The notion of a "multiple component of a link" was given corresponding topological significance above in section 3. We introduced there the terminology "multilink" for a link with an integer multiplicity assigned to each component. We also described how, by taking linking numbers, such an assignment of multiplicities can also be interpreted as a linear functional on the first homology of the link complement Σ_0, hence as a cohomology class $\underline{m} \in H^1(\Sigma_0; Z)$.

The equation of a plane branch

We present a simple method for computing an irreducible power series equation $f(x,y) = 0$ satisfied by a given Puiseux expansion

$$*)\quad \begin{cases} x = t^n \\ y = a_1 t^{m_1} + a_2 t^{m_2} + \cdots \\ a_i \neq 0,\ m_1 \geq n,\ \mathrm{GCD}(n, m_1, m_2, \cdots) = 1\ . \end{cases}$$

Of course the equation for a link with several branches is obtained by multiplying the equations of the individual branches.

The method is standard in the theory of integral extensions, but seems not widely noticed by those dealing with singularities. Everything here can be adapted to ground fields of arbitrary characteristic, but we leave this to the reader.

We write Id_n for an $n \times n$ identity matrix.

PROPOSITION. *Let* $x, y \in \mathbf{C}[[t]]$ *be such that* $\mathbf{C}[[t]]$ *is the integral closure of* $\mathbf{C}[[x,y]]$ *(or equivalently,* t *is a rational function of* x *and* y *). If* $u_0, \cdots, u_{n-1} \in \mathbf{C}[[t]]$ *reduces to a basis of* $\mathbf{C}[[t]]/(x)$*, then there exist unique elements* $v_{ij} \in \mathbf{C}[[x]]$ *such that*

$$yu_i = \sum_{j=0}^{n-1} v_{ij} u_j \qquad i = 0, \cdots, n-1\ .$$

If we set

$$f(x,u) = \det(u \cdot \mathrm{Id}_n - (v_{ij}))\ ,$$

then $f(x,y) = 0$ *and* $f(x,u)$ *is irreducible over the field of Laurent series* $\mathbf{C}((x))$*. In particular,* $f(x,u) = 0$ *defines the same branch of a plane curve as* $x(t), y(t)$.

REMARK. If x and y are convergent power series and the u_i are chosen similarly (they can always be taken to be $1, t, \cdots, t^{n-1}$ if one wishes), then the v_{ij} and f will be convergent.

In the situation of (*), and choosing the u_i as $1, t, \cdots, t^{n-1}$, if y is a polynomial then the v_{ij} and f will be polynomials.

Proof. By Nakayama's lemma, $\mathbb{C}[[t]]/(x^p)$ is a free $\mathbb{C}[[x]]/(x^p)$-module, with basis the classes of the u_i, for every p. Since $\mathbb{C}[[x]]$ is complete it follows that $\mathbb{C}[[t]]$ is itself a free $\mathbb{C}[[x]]$ module with basis the u_i. (In the convergent case the corresponding statement holds because $\mathbb{C}\{x\}$ is Henselian.) This implies the first statement of the proposition.

The element $y \in \mathbb{C}[[t]]$ acts $\mathbb{C}[[x]]$-linearly, by multiplication, on $\mathbb{C}[[t]]$ and $f(x,u)$ is its characteristic polynomial, so $f(x,y) = 0$. To prove irreducibility, it suffices to note that the quotient field of $\mathbb{C}[[x]][y]$ is $\mathbb{C}((t))$, and $[\mathbb{C}((t)) : \mathbb{C}((x))] = n$, the degree of f in y.

If x and y have the form given by (*), we can be much more explicit. Namely if we choose $u_j = t^i$ $(j = 0, \cdots, n-1)$ we obtain:

COROLLARY. *Suppose* x *and* y *are as in* (*), *and rewrite* y *as*

$$y(t) = \sum_{\ell \geq 0} b_\ell t^\ell .$$

For $s = 0, \cdots, n-1$ *set*

$$v_s = -\sum_{\ell \equiv s(n)} b_\ell x^{(\ell - s)/n} ,$$

$$f(x,u) = \det \begin{pmatrix} u+v_0 & v_1 & v_2 & \cdots & v_{n-2} & v_{n-1} \\ xv_{n-1} & u+v_0 & v_1 & v_2 & \cdots & v_{n-2} \\ xv_{n-2} & xv_{n-1} & \ddots & \ddots & \ddots & \vdots \\ \vdots & xv_{n-2} & \ddots & \ddots & \ddots & v_2 \\ xv_2 & \vdots & \ddots & \ddots & \ddots & v_1 \\ xv_1 & xv_2 & \cdots & xv_{n-2} & xv_{n-1} & u+v_0 \end{pmatrix}$$

then f *is irreducible over* $\mathbb{C}((x))$ *and* $f(x,y) \equiv 0$.

EXAMPLE. Let

$$x = t^4$$

$$y = t^6 + at^7 .$$

This is the simplest Puiseux series with more than one characteristic pair. We get

$$f(x,u) = \det \begin{pmatrix} u & 0 & -x & -ax \\ -ax^2 & u & 0 & x \\ -x^2 & -ax^2 & u & 0 \\ 0 & -x^2 & -ax^2 & u \end{pmatrix}$$

$$= u^4 - 2x^3u^2 - 4a^2x^5u + x^6 - a^4x^7 .$$

This is the example (with $a = 1$) given in part 1 of the introduction.