Dedicated to the memory of our friend and colleague
Denis Uglov (January 12, 1968–October 4, 1999)

Canonical Bases of Higher-Level q-Deformed Fock Spaces and Kazhdan–Lusztig Polynomials

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Abstract. The aim of this paper is to generalize some aspects of the recent work of Leclerc–Thibon and Varagnolo–Vasserot on the canonical bases of the level 1 q-deformed Fock spaces of Hayashi. Namely, we define canonical bases for the higher-level q-deformed Fock spaces of Jimbo–Misra–Miwa–Okado and establish a relation between these bases and (parabolic) Kazhdan–Lusztig polynomials for the affine Weyl group of type $A^{(1)}_{r-1}$. As an application, we derive an inversion formula for a subfamily of these polynomials.

1 Introduction

For any symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$, Kashiwara introduces in [11] the notion of a lower global crystal basis of an integrable module M of the universal quantum enveloping algebra $U_q(\mathfrak{g})$. Furthermore, he proves the existence and uniqueness of such a basis when M is irreducible.

The lower global crystal bases of irreducible modules were recently recognized to be closely related with the representation theory of Hecke algebras. It was observed in [16] that the vacuum irreducible module of $\widehat{\mathfrak{sl}}_n$ at level 1 can be identified with the direct sum $\oplus_m K_m$, where K_m is the complexified Grothendieck group of finitely generated projective modules of the Hecke algebra $H_m(v)$ at v, a complex nth root of unity. Furthermore, it was conjectured that, under this identification, the specialization of the lower global crystal basis at $q = 1$ corresponds to the basis of K formed by the indecomposable summands of the Hecke algebra. A proof of this conjecture and of a more general result providing a similar interpretation for the lower global crystal basis for any irreducible integrable module of $\widehat{\mathfrak{sl}}_n$ was given in [2].

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If an integrable module of a quantum enveloping algebra is not irreducible, it can have more than one lower global crystal basis. An example is the q-deformed Fock space of Hayashi [9, 20], which is a reducible integrable module of $U_q(\widehat{\mathfrak{sl}}_n)$ at level 1 spanned by the set of all partitions. One may then ask whether among the many lower global crystal bases of the q-deformed Fock space, one can single out a canonical one with particularly favorable properties.

Leclerc and Thibon gave a definition of such a canonical basis in [17]. The essential point of their definition was to introduce a bar-involution of the q-deformed Fock space by using the semiinfinite wedge construction of the latter given by Stern [S, 13]. With the involution in hand, the canonical basis is defined as the unique bar-invariant basis with a certain congruence property with respect to the $\mathbf{Q}[q]$-lattice spanned by partitions.

It was conjectured in [17] that the specialization at $q = 1$ of the transition matrix between the canonical basis of the q-deformed Fock space and the basis formed by partitions coincides with the decomposition matrix of the Weyl modules of the v-Schur algebra at v a complex nth root of unity. This conjecture was proved in [25], where it was shown that the entries of the transition matrix are given by certain parabolic Kazhdan–Lusztig polynomials for affine Weyl groups of type $A_{r-1}^{(1)}$. An excellent review of these developments can be found in [18].

The higher-level q-deformed Fock spaces, generalizing those of Hayashi, were introduced in [10] to compute the crystal graph of an arbitrary irreducible integrable module of $U_q(\widehat{\mathfrak{sl}}_n)$. For each sequence of l integers called the charge, there is a q-deformed Fock space which is an integrable module of $U_q(\widehat{\mathfrak{sl}}_n)$ at level l having as a basis the set of all l-tuples of partitions.

The main aim of this article is to give a construction of canonical bases for these q-deformed Fock spaces, generalizing the construction of Leclerc and Thibon for level 1.

As in that case, the first step of this construction is to give a semiinfinite wedge realization of each q-deformed Fock space. This was already done in [23], and we follow that paper except in some minor details. The wedge realization allows us to define a natural bar-involution on a q-deformed Fock space. Then the definition of the canonical basis proceeds exactly as in the level 1 case. In particular, as in [25], we find that the entries of the transition matrix between the canonical basis and the basis formed by the l-tuples of partitions are parabolic Kazhdan–Lusztig polynomials, the type of parabolic subgroup being determined by the charge of the q-deformed Fock space.

For a q-deformed Fock space of charge $(s_1, \dots, s_l)$, the $U_q(\widehat{\mathfrak{sl}}_n)$-submodule M generated by the l-tuple of empty partitions is isomorphic to the irreducible module with the highest-weight $\sum_{b=1}^{l} \Lambda_{s_b \bmod n}$. The canonical basis of the Fock space is, as in the case of level 1, a lower global crystal basis and contains the lower global crystal basis of M as a subset.

The definition of the canonical bases given in this article is constructive. It provides one with a simple algorithm for computation of the transition matrices between these bases and the bases formed by l-tuples of partitions. Due to the main result of [2], the submatrix of this transition matrix that corresponds to the expansion of the lower global crystal basis of M is known to be a q-analogue of the decomposition matrix of a certain Ariki–Koike algebra at the nth root of unity. We therefore get an algorithm for computing this decomposition matrix.

Now let us give an outline of the present article. In Section 2, we give the definitions of the q-deformed Fock spaces, of their crystal bases, and of the lower global crystal bases of their irreducible submodules. This part is entirely expository and mostly follows the work of [5]. In Section 3, after giving the necessary background on affine Weyl groups and affine Hecke algebras, we introduce wedge products and their canonical bases. We also establish the connection between these canonical bases and parabolic Kazhdan–Lusztig polynomials. Most of the results of this part are straightforward generalizations of results of [13] and of a small subset of results in [25]. In the first part of Section 4, we give the realizations of the q-deformed Fock spaces as subspaces of the semiinfinite wedge products. Here we follow [23], with some deviations. In the second part of this section, we introduce the bar-involution and define canonical bases. The content of this part is very close to that of [17]. In Section 5, we describe a symmetry of the bar-involution and derive by using this symmetry an inversion formula for certain parabolic Kazhdan–Lusztig polynomials. When the level l equals 1, this formula has already been established in [18].

Since most of the results in the present article are to be found, in the special case of level 1, in [18], we tried to organize the exposition so that it parallels the relevant parts of that work.

The preliminary version of this article appeared as the preprint [24].

2 The q-deformed Fock spaces

2.1 Definitions. Let $\mathfrak{h} = (\oplus_{i=0}^{n-1}\mathbf{Q}h_i) \oplus \mathbf{Q}\partial$ be the Cartan subalgebra of $\widehat{\mathfrak{sl}}_n$ and let $\mathfrak{h}^* = (\oplus_{i=0}^{n-1}\mathbf{Q}\Lambda_i) \oplus \mathbf{Q}\delta$ be its dual. Here $\Lambda_0, \ldots, \Lambda_{n-1}$ and δ are, respectively, the fundamental weights and the null root defined in terms of the pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$ by

$$\langle \Lambda_i, h_j \rangle = \delta_{ij}, \qquad \langle \Lambda_i, \partial \rangle = \langle \delta, h_i \rangle = 0, \qquad \langle \delta, \partial \rangle = 1.$$

The space $\mathfrak{h}^*$ is equipped with a bilinear symmetric form defined by

$$(\Lambda_i|\Lambda_j) = \min(i, j) - \frac{ij}{n}, \qquad (\Lambda_i|\delta) = 1, \qquad (\delta|\delta) = 0.$$

For $\Lambda \in \mathfrak{h}^*$, we shall write $|\Lambda|^2$ to mean $(\Lambda|\Lambda)$. It will be convenient to extend the index set of the fundamental weights to all integers by setting $\Lambda_i = \Lambda_{i \bmod n}$ for $i \in \mathbf{Z}$. Then the simple roots are defined for all integers i as $\alpha_i = 2\Lambda_i - \Lambda_{i+1} - \Lambda_{i-1} + \delta_{i \equiv 0 \bmod n}\delta$, where for a statement S we set $\delta_S = 1$ if S is true and $\delta_S = 0$ otherwise.

Let $U_q(\widehat{\mathfrak{sl}}_n)$ be the q-deformed universal enveloping algebra of $\widehat{\mathfrak{sl}}_n$. This is an algebra over $\mathbf{K} = \mathbf{Q}(q)$ with generators e_i, f_i, t_i, and $(t_i)^{-1}$ $(i = 0, \ldots, n-1)$ and ∂. The relations between e_i, f_i, t_i, and $(t_i)^{-1}$ are standard (see, e.g., [14]). We define the relations between the degree generator ∂ and the rest of the generators by

$$[\partial, e_i] = \delta_{i,0} e_i, \qquad [\partial, f_i] = -\delta_{i,0} f_i, \qquad [\partial, t_i] = 0.$$

For $l \in \mathbf{Z}$, a module M of $U_q(\widehat{\mathfrak{sl}}_n)$ is said to have *level* l if the canonical central element $t_0 t_1 \cdots t_{n-1}$ of $U_q(\widehat{\mathfrak{sl}}_n)$ acts on M as the multiplication by q^l. Let $U_q'(\widehat{\mathfrak{sl}}_n)$ be the subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by e_i, f_i, t_i, and $(t_i)^{-1}$.

For a nonnegative integer k, let Π_k be the set of partitions of k, i.e., the set of all nondecreasing sequences of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ summing to k. Let $\Pi = \sqcup_{k \geq 0} \Pi_k$ be the set of all partitions. For $l \in \mathbf{N}$, an element $\boldsymbol{\lambda}_l = (\lambda^{(1)}, \ldots, \lambda^{(l)})$ of Π^l is called an *l-multipartition*. It will be convenient to identify a multipartition $\boldsymbol{\lambda}_l$ with its diagram defined as the set

$$\{(i, j, b) \in \mathbf{N}^3 \mid 1 \leq b \leq l,\ 1 \leq j \leq \lambda_i^{(b)}\}.$$

An element of the diagram of a multipartition $\boldsymbol{\lambda}_l$ is called a *node* of $\boldsymbol{\lambda}_l$, and the total number of nodes of $\boldsymbol{\lambda}_l$ is denoted by $|\boldsymbol{\lambda}_l|$.

Let $s_l = (s_1, \ldots, s_l)$ be a sequence of l integers. With any such sequence one associates the *q-deformed Fock space* $\mathbf{F}_q[s_l]$ defined as

$$\mathbf{F}_q[s_l] = \bigoplus_{\boldsymbol{\lambda}_l \in \Pi^l} \mathbf{K}\, |\boldsymbol{\lambda}_l, s_l\rangle.$$

In other words, $\mathbf{F}_q[s_l]$ is a $\mathbf{K}$-linear space with a distinguished basis $|\boldsymbol{\lambda}_l, s_l\rangle$ labeled by the set of all l-multipartitions. The number l is called *the level* of $\mathbf{F}_q[s_l]$, and the sequence s_l is called the *charge* of $\mathbf{F}_q[s_l]$.

It was shown in [10] that $\mathbf{F}_q[s_l]$ can be endowed with a structure of an integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module. We shall describe this structure following the exposition given in [5]. To do this we introduce some notation. For a node $\gamma = (i, j, b)$ of a multipartition $\boldsymbol{\lambda}_l$, one defines its *n-residue* as $\mathrm{res}_n(\gamma) = (s_b + j - i) \bmod n$. For i between 0 and $n-1$, we say that $\gamma \in \boldsymbol{\lambda}_l$ is an *i-node* of $\boldsymbol{\lambda}_l$ if $\mathrm{res}_n(\gamma) = i$. Given two nodes $\gamma = (i, j, b)$ and $\gamma' = (i', j', b')$ of a multipartition $\boldsymbol{\lambda}_l$, we write $\gamma < \gamma'$ if either $(s_b + j - i) < (s_{b'} + j' - i')$ or $(s_b + j - i) = (s_{b'} + j' - i')$ and $b < b'$. If $\boldsymbol{\mu}_l$ and $\boldsymbol{\lambda}_l$ are two multipartitions such that $\boldsymbol{\mu}_l \supset \boldsymbol{\lambda}_l$, and $\gamma = \boldsymbol{\mu}_l \setminus \boldsymbol{\lambda}_l$ is an i-node of $\boldsymbol{\mu}_l$, we say that γ is a removable i-node of $\boldsymbol{\mu}_l$ and is an addable i-node of $\boldsymbol{\lambda}_l$. In this case, we define

$$\begin{aligned} N_i^{>}(\boldsymbol{\lambda}_l,\ \boldsymbol{\mu}_l | s_l, n) &= \sharp\{\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda}_l \text{ such that } \gamma' > \gamma\} \\ &\quad - \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda}_l \text{ such that } \gamma' > \gamma\}, \\ N_i^{<}(\boldsymbol{\lambda}_l,\ \boldsymbol{\mu}_l | s_l, n) &= \sharp\{\text{addable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda}_l \text{ such that } \gamma' < \gamma\} \\ &\quad - \sharp\{\text{removable } i\text{-nodes } \gamma' \text{ of } \boldsymbol{\lambda}_l \text{ such that } \gamma' < \gamma\}. \end{aligned}$$

Also, for a multipartition λ_l and i between 0 and $n-1$, we define

$$N_i(\lambda_l|s_l,n) = \sharp\{\text{addable } i\text{-nodes of } \lambda_l\} - \sharp\{\text{removable } i\text{-nodes of } \lambda_l\},$$
$$M_i(\lambda_l|s_l,n) = \sharp\{i\text{-nodes of } \lambda_l\},$$

and for $s_l = (s_1,\dots,s_l) \in \mathbf{Z}^l$, we set

$$\Delta(s_l|n) = \frac{1}{2}\sum_{b=1}^{l} |\Lambda_{s_b}|^2 + \frac{1}{2}\sum_{b=1}^{l}\left(\frac{s_b^2}{n} - s_b\right).$$

Now we can state the following.

Theorem 2.1 ([10, 5]). *The following formulas define on $\mathbf{F}_q[s_l]$ a structure of an integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module.*

$$f_i|\lambda_l, s_l\rangle = \sum_{\mathrm{res}_n(\mu_l/\lambda_l)=i} q^{N_i^{>}(\lambda_l,\mu_l|s_l,n)}\,|\mu_l, s_l\rangle,$$
$$e_i|\mu_l, s_l\rangle = \sum_{\mathrm{res}_n(\mu_l/\lambda_l)=i} q^{-N_i^{<}(\lambda_l,\mu_l|s_l,n)}\,|\lambda_l, s_l\rangle,$$
$$t_i|\lambda_l, s_l\rangle = q^{N_i(\lambda_l|s_l,n)}|\lambda_l, s_l\rangle,$$
$$\partial|\lambda_l, s_l\rangle = -(\Delta(s_l|n) + M_0(\lambda_l|s_l,n))|\lambda_l, s_l\rangle.$$

Remark 2.2. Our labeling of the basis vectors differs from that of [5] by the transformation reversing the order of components in $\lambda_l = (\lambda^{(1)},\dots,\lambda^{(l)})$ and $s_l = (s_1,\dots,s_l)$. Also, Theorem 2.1 as well as Theorem 2.4 below are stated in [5] only for s_l such that $n > s_1 \geq s_2 \geq \cdots \geq s_l \geq 0$. Generalizations for all $s_l \in \mathbf{Z}^l$ are straightforward.

Note that the vector $|\varnothing_l, s_l\rangle$, where $\varnothing_l$ denotes the l-tuple of empty partitions, is a highest-weight vector of $\mathbf{F}_q[s_l]$. Since $\mathbf{F}_q[s_l]$ is an integrable module, it follows that

$$\mathbf{F}_q[s_l] = U_q(\widehat{\mathfrak{sl}}_n)\,|\varnothing_l, s_l\rangle$$

is an irreducible submodule of $\mathbf{F}_q[s_l]$. Computing the weight of $|\varnothing_l, s_l\rangle$ in accordance with Theorem 2.1, we see that $\mathbf{F}_q[s_l]$ is isomorphic to the irreducible $U_q(\widehat{\mathfrak{sl}}_n)$-module $V_q(\Lambda)$ with highest-weight $\Lambda = -\Delta(s_l|n)\delta + \Lambda_{s_1} + \cdots + \Lambda_{s_l}$.

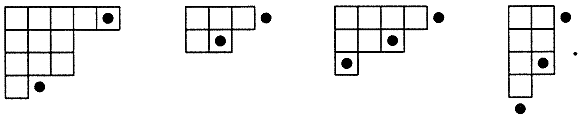
2.2 Crystal bases. The q-deformed Fock spaces were introduced in [10] in order to compute the crystal graphs of irreducible integrable modules of $U_q(\widehat{\mathfrak{sl}}_n)$. We have seen that any such module is embedded into a q-deformed Fock space as the component generated by the highest-weight vector labeled by the empty multipartition. From crystal base theory, it follows that the crystal graph of an irreducible module is embedded into the crystal graph of the corresponding Fock space. The last crystal graph was described in [10]. To recall how the arrows of this graph are obtained, we introduce, following [5], the notion of a *good node* of a multipartition λ_l.

First, observe that for each i between 0 and $n-1$ the relation $\gamma < \gamma'$ defines a total order on the set of all i-addable and i-removable nodes.

Example 2.3. Let $n = 3$, $l = 4$ and $s_l = (5, 0, 2, 1)$. Then, marking the 0-addable and the 0-removable nodes of the multipartition

$$\lambda_l = ((5, 3^2, 1), (3, 2), (4, 3, 1), (2^3, 1))$$

on the diagram of λ_l by $\bullet$, we get



Thus these nodes are ordered as

$$A_{-3,4} < R_{0,2} < R_{0,3} < R_{0,4} < A_{3,1} < A_{3,2} < R_{3,3} < A_{3,4} < A_{6,3} < R_{9,1},$$

where $A_{d,b}$ ($R_{d,b}$) denotes an addable (removable) node (i, j, b) with $s_b + j - i = d$.

Next, for a multipartition λ_l, write the sequence of its addable and removable i-nodes ordered as explained above. Then remove from this sequence recursively all pairs RA until no such pairs remain. The resulting sequence then has the form $A \dots AR \dots R$. The rightmost R-node in this sequence is called the *good removable i-node* of λ_l, and the leftmost A-node in this sequence is called the *good addable i-node* of λ_l. Clearly, there can be at most one of each. For instance, for the multipartition considered in Example 2.3, the nodes $A_{-3,4}$ and $R_{9,1}$ are good 0-nodes.

Let $A \subset \mathbf{Q}(q)$ be the ring of rational functions without pole at $q = 0$. Let $\mathcal{L}[s_l] = \oplus_{\lambda_l \in \Pi^l} A|\lambda_l, s_l\rangle$ and let $\mathcal{B}[s_l]$ be the $\mathbf{Q}$-basis of $\mathcal{L}[s_l]/q\mathcal{L}[s_l]$ given by $\mathcal{B}[s_l] = \{|\lambda_l, s_l\rangle \bmod q\mathcal{L}[s_l] \mid \lambda_l \in \Pi^l\}$.

Theorem 2.4 ([10, 5]). *The pair $(\mathcal{L}[s_l], \mathcal{B}[s_l])$ is a lower crystal basis of $\mathbf{F}_q[s_l]$ at $q = 0$. Moreover, the crystal graph $\mathcal{B}[s_l]$ contains the arrow*

$$|\lambda_l, s_l\rangle \bmod q\mathcal{L}[s_l] \xrightarrow{i} |\mu_l, s_l\rangle \bmod q\mathcal{L}[s_l]$$

if and only if μ_l is obtained from λ_l by adding a good i-node.

Let $\Pi^l(s_l)$ be the subset of Π^l such that $\mathcal{B}[s_l]^\circ = \{|\lambda_l, s_l\rangle \bmod q\mathcal{L}[s_l] \mid \lambda_l \in \Pi^l(s_l)\}$ is the set of vertices in the connected component of $|\varnothing_l, s_l\rangle \bmod q\mathcal{L}[s_l]$ in the crystal graph $\mathcal{B}[s_l]$ of $\mathbf{F}_q[s_l]$. Then [11, Theorem 3] implies that $\mathcal{B}[s_l]^\circ$ is isomorphic to the crystal graph of the irreducible submodule $\mathbf{F}_q[s_l]$ of $\mathbf{F}_q[s_l]$.

Let us now briefly review the notion of the global crystal base of an irreducible module $\mathbf{F}_q[s_l]$. First, recall the involution $x \mapsto \overline{x}$ of $U'_q(\widehat{\mathfrak{sl}}_n)$ defined as the unique algebra automorphism satisfying

$$\overline{q} = q^{-1}, \qquad \overline{t_i} = (t_i)^{-1}, \qquad \overline{e_i} = e_i, \qquad \overline{f_i} = f_i.$$

Now each vector v of $\mathbf{F}_q[s_l]$ can be written as $v = x|\varnothing_l, s_l\rangle$ for some $x \in U'_q(\widehat{\mathfrak{sl}}_n)$. Then we set $\overline{v} = \overline{x}|\varnothing_l, s_l\rangle$. Finally, denote by $U^-_{\mathbf{Q}}$ the $\mathbf{Q}[q, q^{-1}]$-subalgebra of $U'_q(\widehat{\mathfrak{sl}}_n)$ generated by the q-divided differences $f_i^k/[k]!$ and let $\mathbf{F}_q[s_l]_{\mathbf{Q}} = U^-_{\mathbf{Q}}|\varnothing_l, s_l\rangle$. (Here, $[k]!$ denotes the q-factorial, that is, $[k] = (q^k - q^{-k})/(q - q^{-1})$ and $[k]! = [k][k-1]\cdots[1]$.)

Theorem 2.5 ([11]). *There exists a unique* $\mathbf{Q}[q, q^{-1}]$*-basis* $\{\mathcal{G}(\boldsymbol{\lambda}_l, s_l) \mid \boldsymbol{\lambda}_l \in \Pi^l(s_l)\}$ *of* $\mathbf{F}_q[s_l]_{\mathbf{Q}}$ *such that*

$$\text{(i)} \qquad \overline{\mathcal{G}(\boldsymbol{\lambda}_l, s_l)} = \mathcal{G}(\boldsymbol{\lambda}_l, s_l),$$

$$\text{(ii)} \qquad \mathcal{G}(\boldsymbol{\lambda}_l, s_l) \equiv |\boldsymbol{\lambda}_l, s_l\rangle \bmod q\mathcal{L}[s_l].$$

The basis $\{\mathcal{G}(\boldsymbol{\lambda}_l, s_l) \mid \boldsymbol{\lambda}_l \in \Pi^l(s_l)\}$ is called the *lower global crystal basis* of $\mathbf{F}_q[s_l]$.

3 Canonical bases of wedge products

3.1 Affine Weyl group. Let $\mathfrak{t}^* = \oplus_{i=1}^r \mathbf{C}\varepsilon_i$ be the dual space of the Cartan subalgebra of $\mathfrak{gl}_r$. Let $\hat{\mathfrak{t}}^* = \mathfrak{t}^* \oplus \mathbf{C}\Lambda_0 \oplus \mathbf{C}\delta$ be the dual space of the Cartan subalgebra of $\widehat{\mathfrak{gl}}_r$. The space $\hat{\mathfrak{t}}^*$ is equipped with the bilinear symmetric form defined by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, $(\varepsilon_i, \Lambda_0) = (\varepsilon_i, \delta) = (\delta, \delta) = (\Lambda_0, \Lambda_0) = 0$, $(\Lambda_0, \delta) = 1$. The systems of roots R, positive roots R^+, and simple roots Π of type A_{r-1} are the subsets of $\mathfrak{t}^*$ defined by

$$\begin{aligned} R &= \{\alpha_{ij} = \varepsilon_i - \varepsilon_j \mid i \neq j\}, \\ R^+ &= \{\alpha_{ij} \mid i < j\}, \\ \Pi &= \{\alpha_1, \dots, \alpha_{r-1}\} \quad (\alpha_i := \alpha_{ii+1}). \end{aligned}$$

The systems of roots $\widehat{R}$, positive roots $\widehat{R}^+$, and simple roots $\widehat{\Pi}$ of type $A_{r-1}^{(1)}$ are the subsets of $\hat{\mathfrak{t}}^*$ defined by

$$\begin{aligned} \widehat{R} &= \{\alpha + k\delta \mid \alpha \in R,\ k \in \mathbf{Z}\}, \\ \widehat{R}^+ &= \{\alpha + k\delta \mid \alpha \in R^+,\ k \geq 0\} \sqcup \{-\alpha + k\delta \mid \alpha \in R^+,\ k > 0\}, \\ \widehat{\Pi} &= \{\alpha_0 := \delta - (\varepsilon_1 - \varepsilon_r)\} \sqcup \Pi. \end{aligned}$$

The Weyl group W of $\mathfrak{gl}_r$ is isomorphic to the symmetric group $\mathfrak{S}_r$ and has a realization as the group generated by the reflections $s_\alpha(\xi) = \xi - (\alpha, \xi)\,\alpha$ $(\alpha \in R)$ of $\mathfrak{t}^*$. Let $Q = \oplus_{i=1}^{r-1} \mathbf{Z}\alpha_i$ and $P = \oplus_{i=1}^r \mathbf{Z}\varepsilon_i$ be, respectively, the root and the weight lattices of $\mathfrak{gl}_r$. They both are preserved by W.

The affine Weyl group is defined as the semidirect product

$$\widehat{W} = W \ltimes P$$

with relations $wt_\eta = t_{w(\eta)}w$, where w and t_η are elements of $\widehat{W}$ that correspond to $w \in W$, $\eta \in P$. The group $\widehat{W}$ contains the Weyl group $\widetilde{W} = W \ltimes Q$ of type $A_{r-1}^{(1)}$ as a subgroup. The group $\widehat{W}$ acts on $\hat{\mathfrak{t}}^*$ by

$$\begin{aligned} s_\alpha(\zeta) &= \zeta - (\alpha, \zeta)\,\alpha && (\zeta \in \hat{\mathfrak{t}}^*,\ \alpha \in R), \\ t_\eta(\zeta) &= \zeta + (\delta, \zeta)\,\eta - \left((\eta, \zeta) + \frac{1}{2}(\eta, \eta)(\delta, \zeta)\right)\delta && (\zeta \in \hat{\mathfrak{t}}^*,\ \eta \in P). \end{aligned}$$

This action preserves the root system $\widehat{R}$, and the bilinear form on $\hat{\mathfrak{t}}^*$ is invariant with respect to this action.

For an affine root $\hat{\alpha} = \alpha + k\delta$ ($\alpha \in R, k \in \mathbf{Z}$), define the corresponding affine reflection as $s_{\hat{\alpha}} = t_{-k\alpha}\, s_\alpha$, and set $s_i = s_{\alpha_i}$ ($i = 0, 1, \ldots, r-1$), $\pi = t_{\varepsilon_1} s_1 \cdots s_{r-1}$. The group $\widehat{W}$ is generated by π, π^{-1}, $s_0, s_1, \ldots, s_{r-1}$ and is defined by the relations

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ s_i s_j &= s_j s_i \quad (i - j \neq \pm 1), \\ s_i^2 &= 1, \qquad \pi s_i = s_{i+1} \pi, \end{aligned}$$

where the subscripts are understood to be modulo r. In this presentation, $\widetilde{W}$ is the Coxeter subgroup generated by $s_0, s_1, \ldots, s_{r-1}$ and $\widehat{W} \cong \Omega \ltimes \widetilde{W}$, where $\Omega \cong \mathbf{Z}$ is the subgroup of $\widehat{W}$ generated by π, π^{-1}.

For $w \in \widehat{W}$, let $S(w) = \widehat{R}^+ \cap w^{-1}(\widehat{R}^-)$, where $\widehat{R}^- = \widehat{R} \setminus \widehat{R}^+$ is the set of negative roots. The length $l(w)$ of w is defined as the number $\sharp S(w)$ of elements in $S(w)$. The length of w is zero if and only if $w \in \Omega$. A partial order on $\widehat{W}$ is defined by $\pi^k w \preceq \pi^{k'} w'$ ($w, w' \in \widetilde{W}$) if $k = k'$ and $w \preceq w'$ in the Bruhat order of $\widetilde{W}$.

The next lemma follows immediately from the definition of $S(w)$.

Lemma 3.1.
(i) *For* $w \in \widehat{W}$, $S(w^{-1}) = -w(S(w))$.
(ii) *For* $u, v \in \widetilde{W}$, $S(u) = S(v)$ *implies* $u = v$.
(iii) *For* $w \in W$, $\lambda \in P$,

$$l(w\, t_\lambda) = \sum_{\alpha \in R^+,\ w(\alpha) \in R^+} |(\lambda, \alpha)| + \sum_{\alpha \in R^+,\ w(\alpha) \in R^-} |1 + (\lambda, \alpha)|.$$

A corollary to (iii) above is the equality $l(t_\lambda) = l(t_\mu)$ for $\lambda, \mu \in P$ such that $\lambda = w(\mu)$ for some $w \in W$.

The following lemma is contained in [1] as Definition and Proposition 2.2.2.

Lemma 3.2. *For* $\lambda \in P$, *let* w *be the shortest element of* W *such that* $w(\lambda) \in P^+$. *Then*

$$S(w) = \{\alpha \in R^+ \mid (\lambda, \alpha) < 0\}.$$

Proposition 3.3. *For every* $x \in \widehat{W}$, *there is a unique factorization of the form* $x = u\, t_\lambda v$, *where* $u, v \in W$, $\lambda \in P^+$, *and* $S(v) = \{\alpha \in R^+ \mid (\lambda, v(\alpha)) < 0\}$. *Moreover,* $l(x) = l(u) + l(t_\lambda) - l(v)$.

Proof. Every $x \in \widehat{W}$ can be factorized as $x = w\, t_\mu$, where $w \in W$, $\mu \in P$. Let $v \in W$ be the shortest element such that $v(\mu) \in P^+$. By Lemma 3.2, $S(v) = \{\alpha \in R^+ \mid (\mu, \alpha) < 0\}$. The desired factorization is afforded by $x = u\, t_\lambda v$, with $u = wv^{-1}$, $\lambda = v(\mu)$.

Assume that $x = u_1 t_{\lambda_1} v_1 = u_2 t_{\lambda_2} v_2$, where u_i, v_i, and λ_i satisfy the conditions listed in the statement of the proposition. Set $\mu_i = v_i^{-1}(\lambda_i)$ so that $x = u_i v_i t_{\mu_i}$. The presentation of x in the form $w\,t_\mu$, $(w \in W,\ \mu \in P)$ is unique; hence $\mu_1 = \mu_2$, $u_1 v_1 = u_2 v_2$. The equality $\mu_1 = \mu_2$ implies $S(v_1) = S(v_2)$, whence $v_1 = v_2$, and, therefore, $u_1 = u_2$. The factorization is unique.

It remains to show the relation $l(x) = l(u) + l(t_\lambda) - l(v)$. The length formula of Lemma 3.1 together with $S(v) = \{\alpha \in R^+ \mid (\mu, \alpha) < 0\}$ give

$$\begin{aligned} l(x) = l(w\,t_\mu) \ =\ & \sum_{\alpha \in R^+ \setminus S(v),\ w(\alpha) \in R^-} 1 - \sum_{\alpha \in S(v),\ w(\alpha) \in R^-} 1 \\ & + \sum_{\alpha \in R^+ \setminus S(v)} (\mu, \alpha) - \sum_{\alpha \in S(v)} (\mu, \alpha). \end{aligned}$$

On the other hand,

$$\begin{aligned} l(v) &= \sum_{\alpha \in S(v)} 1 = \sum_{\alpha \in S(v),\ w(\alpha) \in R^+} 1 + \sum_{\alpha \in S(v),\ w(\alpha) \in R^-} 1, \\ l(u) &= l(wv^{-1}) = \sum_{\alpha \in R^+ \setminus S(v),\ w(\alpha) \in R^-} 1 + \sum_{\alpha \in S(v),\ w(\alpha) \in R^+} 1, \\ l(t_\lambda) &= l(t_\mu) = \sum_{\alpha \in R^+ \setminus S(v)} (\mu, \alpha) - \sum_{\alpha \in S(v)} (\mu, \alpha). \end{aligned}$$

The relation $l(x) = l(u) + l(t_\lambda) - l(v)$ follows. □

3.1.1 *A right action of $\widehat{W}$ on P.* Let n be a positive integer, and define a right action of $\widehat{W}$ on P by

$$\begin{aligned} \zeta \cdot s_i &= s_i(\zeta) \qquad (\zeta \in P,\ 1 \le i < r), \\ \zeta \cdot t_\mu &= \zeta + n\mu \quad (\zeta \in P,\ \mu \in P). \end{aligned} \tag{1}$$

In coordinates $(\zeta_1, \dots, \zeta_r)$ of $\zeta = \sum_{i=1}^r \zeta_i \varepsilon_i$, this action looks as follows:

$$\begin{aligned} (\zeta_1, \dots, \zeta_r) \cdot s_i &= (\dots, \zeta_{i+1}, \zeta_i, \dots) \quad (1 \le i < r), \\ (\zeta_1, \dots, \zeta_r) \cdot t_{\varepsilon_i} &= (\dots, \zeta_i + n, \dots) \quad (1 \le i \le r). \end{aligned}$$

Hence

$$\begin{aligned} (\zeta_1, \dots, \zeta_r) \cdot \pi &= (\zeta_2, \dots, \zeta_r, \zeta_1 + n), \\ (\zeta_1, \dots, \zeta_r) \cdot s_0 &= (\zeta_r - n, \zeta_2, \dots, \zeta_{r-1}, \zeta_1 + n). \end{aligned}$$

Define $A^n \subset P$ by

$$A^n = \{a = (a_1, \dots, a_r) \in P \mid 1 \le a_1 \le a_2 \le \cdots \le a_r \le n\}.$$

Then A^n is a fundamental domain of the action given by (1). For $a \in A^n$, denote by W_a the stabilizer of a. The inequality $a_r - a_1 < n$ implies that $W_a \subset W$. Let ${}^a\widehat{W}$ (resp., ${}^aW_a$) be the set of minimal length representatives in the cosets $W_a \setminus \widehat{W}$ (resp., $W_a \setminus W$).

Lemma 3.4. *Let $x = u\, t_\lambda v$ be the factorized presentation of $x \in \widehat{W}$ given by Proposition 3.3. Then $x \in {}^a\widehat{W}$ if and only if $u \in {}^aW_a$.*

Proof. For any $w \in W_a$ the factorized presentation of wx is $wx = (wu)\, t_\lambda v$. It now follows from the length relation of Proposition 3.3 that $l(wx) \geq l(x) \Longleftrightarrow l(wu) \geq l(u)$, i.e., x is the shortest element of its coset if and only if u is the shortest element of *its* coset. □

Lemma 3.5 ([1]). *For $a \in A^n$, let $R_a^+ = \{\varepsilon_i - \varepsilon_j \in R^+ \mid a_i = a_j\}$. Then*

$$ {}^aW_a = \{u \in W \mid S(u^{-1}) \subset R^+ \setminus R_a^+\}. $$

For $w \in W$, let a map $w : \{1, \dots, r\} \to \{1, \dots, r\}$ be defined by $\varepsilon_{w^{-1}(i)} = w(\varepsilon_i)$. Note that for $u, v \in W$, $u(v(i)) = vu(i)$.

Lemma 3.6. *Let $u \in {}^aW_a$, and let $c = (c_1, \dots, c_r) = a \cdot u$. Then*

$$ S(u) = \{\varepsilon_i - \varepsilon_j \in R^+ \mid c_i > c_j\}. $$

Proof. Observe that $c_i = a_{u^{-1}(i)}$ for all $i = 1, \dots, r$. Also, $\varepsilon_i - \varepsilon_j \in S(u)$ if and only if $i < j, u^{-1}(i) > u^{-1}(j)$. Since the sequence a is nondecreasing, $i < j, a_{u^{-1}(i)} > a_{u^{-1}(j)}$ implies $i < j, u^{-1}(i) > u^{-1}(j)$, i.e., $\varepsilon_i - \varepsilon_j \in S(u)$. Conversely, $\varepsilon_i - \varepsilon_j \in S(u)$ implies, by Lemmas 3.1(i) and 3.5, that $u(\varepsilon_j - \varepsilon_i) = \varepsilon_{u^{-1}(j)} - \varepsilon_{u^{-1}(i)} \in R^+ \setminus R_a^+$. This gives $a_{u^{-1}(i)} > a_{u^{-1}(j)}$, and the lemma follows. □

Proposition 3.7. *For $a \in A^n$ and $x \in {}^a\widehat{W}$, let $a \cdot x = (c_1 + n\mu_1, \dots, c_r + n\mu_r)$, where $c_i \in \{1, \dots, n\}$ and $\mu_i \in \mathbf{Z}$. Then*

$$ \begin{aligned} l(x) &= \sharp\{i < j \mid c_i > c_j,\ \mu_i \geq \mu_j\} + \sharp\{i < j \mid c_i < c_j,\ \mu_i < \mu_j\} \\ &\quad + \sum_{i<j,\, \mu_i > \mu_j} (\mu_i - \mu_j) + \sum_{i<j,\, \mu_i < \mu_j} (\mu_j - \mu_i - 1). \end{aligned} $$

Proof. Let $x = u\, t_\lambda v$ be the factorized presentation of Proposition 3.3. The expression for $l(x)$ follows from $l(x) = l(u) + l(t_\lambda) - l(v)$, Lemmas 3.4 and 3.6, and the length formula of Lemma 3.1(iii) □

Proposition 3.8. *For $a \in A^n$, let $x \in {}^a\widehat{W}$, and let $\zeta = (\zeta_1, \dots, \zeta_r) = a \cdot x$. Set $\zeta_0 = \zeta_r - n$. Then for each $i = 0, 1, \dots, r-1$, one has the following complete set of alternatives:*

(i) $\zeta_i = \zeta_{i+1} \Longleftrightarrow xs_i \notin {}^a\widehat{W}$,

(ii) $\zeta_i > \zeta_{i+1} \Longleftrightarrow xs_i \in {}^a\widehat{W}, \quad l(xs_i) = l(x) - 1$,

(iii) $\zeta_i < \zeta_{i+1} \Longleftrightarrow xs_i \in {}^a\widehat{W}, \quad l(xs_i) = l(x) + 1$.

Moreover, in case (i), *$xs_i = s_j x$, where $s_j \in W_a$.*

PROOF. First, we show (i). Let $\zeta_i = \zeta_{i+1}$, then $\zeta \cdot s_i = \zeta$. Assuming that $xs_i \in {}^a\widehat{W}$, the length formula of Proposition 3.7 is applicable and immediately gives $l(xs_i) = l(x)$, which is impossible. Hence $\zeta_i = \zeta_{i+1} \Longrightarrow xs_i \notin {}^a\widehat{W}$. Now let $xs_i \notin {}^a\widehat{W}$. By [4, Lemma 2.1(iii)], in this case $xs_i = s_j x$ for some $s_j \in W_a$, which implies $\zeta \cdot s_i = \zeta$; hence $\zeta_i = \zeta_{i+1}$. Thus $\zeta_i = \zeta_{i+1} \Longleftarrow xs_i \notin {}^a\widehat{W}$.

Let $\zeta_i > \zeta_{i+1}$ (resp., $\zeta_i < \zeta_{i+1}$). Then $xs_i \in {}^a\widehat{W}$ and one may use the length formula of Proposition 3.7 to show $l(xs_i) = l(x) - 1$ (resp., $l(xs_i) = l(x) + 1$). Since (i) is already established, this proves (ii) and (iii).

Finally, [4, Lemma 2.1(iii)] states that in case (i), $xs_i = s_j x$, where $s_j \in W_a$. □

3.1.2 *A left action of $\widehat{W}$ on P.* Let l be a positive integer, and define a left action of $\widehat{W}$ on P by

$$\begin{aligned} s_i \cdot \eta &= s_i(\eta) \qquad (\eta \in P,\ 1 \le i < r), \\ t_\mu \cdot \eta &= \eta + l\mu \qquad (\eta \in P,\ \mu \in P). \end{aligned} \tag{2}$$

In coordinates $(\eta_1, \dots, \eta_r)$ of $\eta = \sum_{i=1}^r \eta_i \varepsilon_i$, this action looks as follows:

$$\begin{aligned} s_i \cdot (\eta_1, \dots, \eta_r) &= (\dots, \eta_{i+1}, \eta_i, \dots) \quad (1 \le i < r), \\ t_{\varepsilon_i} \cdot (\eta_1, \dots, \eta_r) &= (\dots, \eta_i + l, \dots) \qquad (1 \le i \le r). \end{aligned}$$

Hence

$$\begin{aligned} \pi \cdot (\eta_1, \dots, \eta_r) &= (\eta_r + l, \eta_1, \dots, \eta_{r-1}), \\ s_0 \cdot (\eta_1, \dots, \eta_r) &= (\eta_r + l, \eta_2, \dots, \eta_{r-1}, \eta_1 - l). \end{aligned}$$

Define $B^l \subset P$ by

$$B^l = \{b = (b_1, \dots, b_r) \in P \mid l \ge b_1 \ge b_2 \ge \cdots \ge b_r \ge 1\}.$$

Then B^l is a fundamental domain of the action given by (2). For $b \in B^l$, denote by W_b the stabilizer of b. The inequality $b_1 - b_r < l$ implies that $W_b \subseteq W$. Let $\widehat{W}^b$ (resp., W^b) be the set of minimal length representatives in the cosets $\widehat{W}/W_b$ (resp., W/W_b).

Lemma 3.9 ([1]). *For $b \in B^l$, let $R_b^+ = \{\varepsilon_i - \varepsilon_j \in R^+ \mid b_i = b_j\}$. Then*

$$W^b = \{v \in W \mid S(v) \subset R^+ \setminus R_b^+\}.$$

Lemma 3.10. *Let $v \in W^b$ and let $d = (d_1, \dots, d_r) = v \cdot b$. Then*

$$S(v^{-1}) = \{\varepsilon_i - \varepsilon_j \in R^+ \mid d_i < d_j\}.$$

Proposition 3.11. *For $b \in B^l$ and $x \in \widehat{W}^b$, let $x \cdot b = (d_1 + l\mu_1, \dots, d_r + l\mu_r)$, where $d_i \in \{1, \dots, l\}$ and $\mu_i \in \mathbf{Z}$. Then*

$$\begin{aligned} l(x) = {} & \sharp\{i < j \mid d_i < d_j,\ \mu_i \le \mu_j\} + \sharp\{i < j \mid d_i > d_j,\ \mu_i > \mu_j\} \\ & + \sum_{i<j,\ \mu_i > \mu_j} (\mu_i - \mu_j - 1) + \sum_{i<j,\ \mu_i < \mu_j} (\mu_j - \mu_i). \end{aligned}$$

Proposition 3.12. *For $b \in B^l$, let $x \in \widehat{W}^b$, and let $\eta = (\eta_1, \dots, \eta_r) = x \cdot b$. Set $\eta_0 = \eta_r + l$. Then for each $i = 0, 1, \dots, r-1$, one has the following complete set of alternatives:*

(i) $\eta_i = \eta_{i+1} \Longleftrightarrow s_i x \notin \widehat{W}^b$,

(ii) $\eta_i > \eta_{i+1} \Longleftrightarrow s_i x \in \widehat{W}^b,\quad l(s_i x) = l(x) + 1$,

(iii) $\eta_i < \eta_{i+1} \Longleftrightarrow s_i x \in \widehat{W}^b,\quad l(s_i x) = l(x) - 1$.

Moreover, in case (i), $s_i x = x s_j$, *where* $s_j \in W_b$.

We omit proofs of Lemma 3.10 and Propositions 3.11 and 3.12 because they are almost identical to the proofs of Lemma 3.6 and Propositions 3.7 and 3.8.

3.2 Affine Hecke algebra. The Hecke algebra $\widehat{H}$ of the Weyl group $\widehat{W}$ is the algebra over $\mathbf{K} = \mathbf{Q}(q)$ with basis T_x $(x \in \widehat{W})$ and relations

$$T_x T_y = T_{xy} \quad \text{whenever } l(xy) = l(x) + l(y),$$
$$(T_{s_i} - q^{-1})(T_{s_i} + q) = 0 \qquad \text{for all } i = 0, 1, \dots, r-1.$$

The subalgebra H of $\widehat{H}$ generated by T_w $(w \in W)$ is isomorphic to the Hecke algebra of the finite Weyl group W. A system of generators of $\widehat{H}$ is afforded by elements T_π, $T_{\pi^{-1}}$ and $T_0, T_1, \dots, T_{r-1}$, where, for simplicity, we put $T_i := T_{s_i}$.

Another system of generators is obtained as follows. For $\lambda \in P$, write $\lambda = \mu - \nu$, where $\mu, \nu \in P^+$, and define

$$X^\lambda = T_{t_\nu} T_{t_\mu}^{-1}.$$

Note that Lemma 3.1(iii) implies that $l(t_\mu t_\nu) = l(t_\mu) + l(t_\nu)$ for $\mu, \nu \in P^+$. From this it follows that X^λ does not depend on the choice of $\mu, \nu \in P^+$, and

$$X^\lambda X^\mu = X^{\lambda+\mu} \quad \text{for } \lambda, \mu \in P.$$

A proof of the following result is contained, e.g., in [15].

Lemma 3.13.

(i) *The elements $X^\lambda, \lambda \in P$, $T_1, \dots, T_{r-1}$ generate $\widehat{H}$.*

(ii) *For $\lambda \in P$, $i = 1, \dots, r-1$,*

$$X^\lambda T_i = T_i X^{s_i(\lambda)} + (q - q^{-1}) \frac{X^{s_i(\lambda)} - X^\lambda}{1 - X^{\alpha_i}},$$
$$T_i X^\lambda = X^{s_i(\lambda)} T_i + (q - q^{-1}) \frac{X^{s_i(\lambda)} - X^\lambda}{1 - X^{\alpha_i}}.$$

Following [21], let us now briefly recall the notions of the canonical bases and the Kazhdan–Lusztig polynomials of $\widehat{W}$.

First, we recall that there is a canonical involution $h \mapsto \overline{h}$ of $\widehat{H}$ defined as the unique algebra automorphism such that $\overline{T_x} = (T_{x^{-1}})^{-1}$ and $\overline{q} = q^{-1}$. A proof of the following lemma is straightforward.

Lemma 3.14. *For $u, v \in W$ and $\lambda \in P$,*

$$\overline{T_u X^\lambda (T_{v^{-1}})^{-1}} = T_{u\omega} X^{\omega(\lambda)} (T_{(\omega v)^{-1}})^{-1},$$

where ω is the longest element of W.

Let L^+ (resp., L^-) be the lattice spanned over $\mathbf{Z}[q]$ (resp., $\mathbf{Z}[q^{-1}]$) by T_x $(x \in \widehat{W})$. The canonical bases C'_x, C_x $(x \in \widehat{W})$ are the unique bases of $\widehat{H}$ with the properties

$$\overline{C'_x} = C'_x, \qquad \overline{C_x} = C_x,$$
$$C'_x \equiv T_x \bmod qL^+, \qquad C_x \equiv T_x \bmod q^{-1}L^-.$$

Let

$$C'_x = \sum_y \mathcal{P}^+_{y,x} T_y, \qquad C_x = \sum_y \mathcal{P}^-_{y,x} T_y.$$

The coefficients $\mathcal{P}^\pm_{y,x}$ are called the Kazhdan–Lusztig polynomials of $\widehat{W}$. They are nonzero only if $y \preceq x$, that is, only if $x = \pi^k \tilde{x}$, $y = \pi^k \tilde{y}$ for some $k \in \mathbf{Z}$, $\tilde{x}, \tilde{y} \in \widetilde{W}$ such that $\tilde{y} \preceq \tilde{x}$. In this case,

$$\mathcal{P}^+_{y,x} = q^{l(x)-l(y)} P_{\tilde{y},\tilde{x}}, \qquad \mathcal{P}^-_{y,x} = (-q)^{l(y)-l(x)} \overline{P_{\tilde{y},\tilde{x}}},$$

where $P_{\tilde{y},\tilde{x}} \in \mathbf{Z}_{\geq 0}[q^{-2}]$ are the Kazhdan–Lusztig polynomials of the Coxeter group $\widetilde{W}$.

3.2.1 *A right representation of $\widehat{H}$.* For $a \in A^n$, let H_a be the parabolic subalgebra of $\widehat{H}$ generated by T_w $(w \in W_a)$. Let $\mathbf{K1}^+_a$ be a one-dimensional right representation of H_a defined by

$$\mathbf{1}^+_a \cdot T_i = q^{-1}\, \mathbf{1}^+_a \quad (s_i \in W_a).$$

The induced right representation

$$\mathbf{K1}^+_a \otimes_{H_a} \widehat{H}$$

of $\widehat{H}$ has as its basis $\mathbf{1}^+_a \otimes_{H_a} T_x$ $(x \in {}^a\widehat{W})$. For $x \in {}^a\widehat{W}$, define

$$(\zeta| := \mathbf{1}^+_a \otimes_{H_a} T_x, \quad \text{where } \zeta = a \cdot x,\ \zeta \in P.$$

Then $(\zeta|$ $(\zeta \in a \cdot \widehat{W})$ is a basis of $\mathbf{K1}^+_a \otimes_{H_a} \widehat{H}$ and $(\zeta|$ $(\zeta \in P)$ is a basis of $\oplus_{a\in A^n} \mathbf{K1}^+_a \otimes_{H_a} \widehat{H}$. Proposition 3.8 allows us to describe the action of the affine Hecke algebra in the basis $(\zeta|$ explicitly. We have

$$(\zeta| \cdot T_i = \begin{cases} (\zeta \cdot s_i| & \text{if } \zeta_i < \zeta_{i+1}, \\ q^{-1}(\zeta| & \text{if } \zeta_i = \zeta_{i+1} \quad (0 \leq i < r), \\ (\zeta \cdot s_i| - (q - q^{-1})(\zeta| & \text{if } \zeta_i > \zeta_{i+1}, \end{cases} \tag{3}$$

$$(\zeta| \cdot T_\pi = (\zeta \cdot \pi|.$$

Define a canonical involution $v \mapsto \overline{v}$ of $\mathbf{K1}_a^+ \otimes_{H_a} \widehat{H}$ by

$$\overline{\mathbf{1}_a^+ \otimes_{H_a} h} = \mathbf{1}_a^+ \otimes_{H_a} \overline{h} \quad (h \in \widehat{H})$$

and two lattices by

$$L_a^+ := \bigoplus_{\zeta \in a \cdot \widehat{W}} \mathbf{Z}[q]\,(\zeta\,|, \qquad L_a^- := \bigoplus_{\zeta \in a \cdot \widehat{W}} \mathbf{Z}[q^{-1}]\,(\zeta\,|.$$

Theorem 3.15 ([4]). *There are unique bases $C_\zeta^\pm$ ($\zeta \in a \cdot \widehat{W}$) of $\mathbf{K1}_a^+ \otimes_{H_a} \widehat{H}$, such that*

$$\text{(i)} \qquad \overline{C_\zeta^\pm} = C_\zeta^\pm,$$
$$\text{(ii)} \qquad C_\zeta^\pm \equiv (\zeta\,| \bmod q^{\pm 1} L_a^\pm.$$

Moreover, if

$$C_\zeta^\pm = \sum_\eta P_{\eta,\zeta}^\pm\,(\eta\,|,$$

then

$$P_{\eta,\zeta}^+ = \mathcal{P}_{\omega_a y, \omega_a x}^+, \qquad P_{\eta,\zeta}^- = \sum_{u \in W_a} q^{-l(u)}\,\mathcal{P}_{uy,x}^-,$$

where x and y are unique elements of ${}^a\widehat{W}$ such that $a \cdot x = \zeta$, $a \cdot y = \eta$, and ω_a is the longest element of W_a.

In the proof of Theorem 3.26 below, we shall use the following relation [18, formula (31)]:

$$P_{\eta \cdot s_i, \zeta}^- = -q^{-1} P_{\eta,\zeta}^- \quad \text{if } \zeta_i > \zeta_{i+1},\ \eta_i > \eta_{i+1}. \tag{4}$$

Here $i = 0, 1, \ldots, r-1$.

3.2.2 *A left representation of $\widehat{H}$.* For $b \in B^l$, let H_b be the parabolic subalgebra of $\widehat{H}$ generated by T_w ($w \in W_b$). Let $\mathbf{K1}_b^-$ be a one-dimensional left representation of H_b defined by

$$T_i \cdot \mathbf{1}_b^- = -q\,\mathbf{1}_b^- \quad (s_i \in W_b).$$

The induced left representation

$$\widehat{H} \otimes_{H_b} \mathbf{K1}_b^-$$

of $\widehat{H}$ has as its basis $(T_{x^{-1}})^{-1} \otimes_{H_b} \mathbf{1}_b^-$ ($x \in \widehat{W}^b$). For $x \in \widehat{W}^b$, define

$$|\eta) := (T_{x^{-1}})^{-1} \otimes_{H_b} \mathbf{1}_b^-, \quad \text{where } \eta = x \cdot b,\ \eta \in P.$$

Then $|\eta)$ ($\eta \in \widehat{W} \cdot b$) is a basis of $\widehat{H} \otimes_{H_b} \mathbf{K1}_b^-$ and $|\eta)$ ($\eta \in P$) is a basis of $\oplus_{b \in B^l} \widehat{H} \otimes_{H_b} \mathbf{K1}_b^-$. Proposition 3.12 allows us to describe the action of the affine Hecke algebra in the basis $|\eta)$ explicitly:

$$T_i \cdot |\eta) = \begin{cases} |s_i \cdot \eta) & \text{if } \eta_i < \eta_{i+1}, \\ -q\,|\eta) & \text{if } \eta_i = \eta_{i+1} \\ |s_i \cdot \eta) - (q - q^{-1})\,|\eta) & \text{if } \eta_i > \eta_{i+1}, \end{cases} \qquad (0 \le i < r), \tag{5}$$
$$T_\pi \cdot |\eta) = |\pi \cdot \eta).$$

3.3 Wedge product. For $a \in A^n$ and $b \in B^l$, define a vector space $\Lambda^r(a, b)$ by

$$\Lambda^r(a, b) := \mathbf{K1}_a^+ \otimes_{H_a} \widehat{H} \otimes_{H_b} \mathbf{K1}_b^-.$$

Note that the maps

$$\mathbf{K1}_a^+ \otimes_{H_a} \widehat{H} \otimes_H H \otimes_{H_b} \mathbf{K1}_b^- \to \Lambda^r(a, b) \;:\; \mathbf{1}_a^+ \otimes \hat{h} \otimes h \otimes \mathbf{1}_b^- \mapsto \mathbf{1}_a^+ \otimes \hat{h}h \otimes \mathbf{1}_b^-, \tag{6}$$

$$\mathbf{K1}_a^+ \otimes_{H_a} H \otimes_H \widehat{H} \otimes_{H_b} \mathbf{K1}_b^- \to \Lambda^r(a, b) \;:\; \mathbf{1}_a^+ \otimes h \otimes \hat{h} \otimes \mathbf{1}_b^- \mapsto \mathbf{1}_a^+ \otimes h\hat{h} \otimes \mathbf{1}_b^- \tag{7}$$

are isomorphisms of vector spaces. Let

$$\Lambda^r(a) := \bigoplus_{b \in B^l} \Lambda^r(a, b) \quad (a \in A^n),$$
$$\Lambda^r(b) := \bigoplus_{a \in A^n} \Lambda^r(a, b) \quad (b \in B^l),$$
$$\Lambda^r := \bigoplus_{b \in B^l} \Lambda^r(b).$$

Let $v_1, \ldots, v_n$ (resp., $\dot{v}_1, \ldots, \dot{v}_l$) be a basis of $\mathbf{K}^n$ (resp., $\mathbf{K}^l$). With a sequence

$$v_{c_1} X^{\mu_1} \dot{v}_{d_1}, \ldots, v_{c_r} X^{\mu_r} \dot{v}_{d_r}, \quad \text{where } v_{c_i} X^{\mu_i} \dot{v}_{d_i} \in (\mathbf{K}^n \otimes \mathbf{K}^l)[X, X^{-1}],$$

we associate unique $a \in A^n$, $b \in B^l$, and unique $u \in {}^aW_a$, $v \in W^b$ such that

$$c = (c_1, \ldots, c_r) = a \cdot u, \quad d = (d_1, \ldots, d_r) = v \cdot b$$

and define the following vector of $\Lambda^r(a, b)$ (here ω_b is the longest element of W_b):

$$(v_{c_1} X^{\mu_1} \dot{v}_{d_1}) \wedge \cdots \wedge (v_{c_r} X^{\mu_r} \dot{v}_{d_r}) := (-q^{-1})^{l(\omega_b)}\, \mathbf{1}_a^+ \otimes_{H_a} T_u X^\mu (T_{v^{-1}})^{-1} \otimes_{H_b} \mathbf{1}_b^-. \tag{8}$$

Note that, using isomorphism (6) to identify the vector spaces, we have

$$(v_{c_1} X^{\mu_1} \dot{v}_{d_1}) \wedge \cdots \wedge (v_{c_r} X^{\mu_r} \dot{v}_{d_r}) = (-q^{-1})^{l(\omega_b)}\, (c| \cdot X^\mu \otimes_H |d), \tag{9}$$

and, using isomorphism (7), we have

$$(v_{c_1}X^{\mu_1}\dot{v}_{d_1}) \wedge \cdots \wedge (v_{c_r}X^{\mu_r}\dot{v}_{d_r}) = (-q^{-1})^{l(\omega_b)}\,(c| \otimes_H X^{\mu} \cdot |d). \tag{10}$$

We shall call a vector of the form (8) a *wedge* and the vector space Λ^r the *wedge product*. In what follows, it will often be convenient to use a slightly different indexation of wedges: in the notation of (8), set $u_{k_i} := v_{c_i}X^{\mu_i}\dot{v}_{d_i}$, where $k_i := c_i + n(d_i - 1) - nl\mu_i$. Since the integers c_i (resp., d_i) range from 1 to n (resp., from 1 to l), a wedge (hence c, d, μ, a, b, u, v) is completely determined by the sequence $\boldsymbol{k} = (k_1, k_2, \dots, k_r)$. To emphasize this, we write

$$c = c(\boldsymbol{k}),\quad d = d(\boldsymbol{k}),\quad \mu = \mu(\boldsymbol{k}),\quad a = a(\boldsymbol{k}),\quad b = b(\boldsymbol{k}),\quad u = u(\boldsymbol{k}),\quad v = v(\boldsymbol{k}). \tag{11}$$

Denote the left-hand side of (8) by

$$u_{\boldsymbol{k}} = u_{k_1} \wedge \cdots \wedge u_{k_r}.$$

Then $u_{\boldsymbol{k}}$ ($\boldsymbol{k} \in \mathbf{P} := \mathbf{Z}^r$) is a spanning set of Λ^r. However, the vectors of this set are not linearly independent. Indeed, using, e.g., (9), it follows that there are relations among these vectors that come from

$$(c| \cdot X^{\mu}T_i \otimes_H |d) = (c| \cdot X^{\mu} \otimes_H T_i \cdot |d) \quad (i = 1, 2, \dots, r-1). \tag{12}$$

The exchange formula for X^{μ} and T_i of Lemma 3.13(ii) and the formulas for the action of T_i on $(c|$ and $|d)$ given, respectively, by (3) and (5) allow us to compute the relations among the wedges explicitly. Note that the relations for general r follow from those for $r = 2$.

Let us call a wedge $u_{\boldsymbol{k}}$ *ordered* if $\boldsymbol{k} \in \mathbf{P}^{++} := \{\boldsymbol{k} \in \mathbf{P} \mid k_1 > k_2 > \cdots > k_r\}$.

Proposition 3.16. (i) *Let $r = 2$. For integers k_1 and k_2 such that $k_1 \leq k_2$, let $c_i \in \{1, \dots, n\}$, $d_i \in \{1, \dots, l\}$, and $\mu_i \in \mathbf{Z}$ be the unique numbers satisfying $k_i = c_i + n(d_i - 1) - nl\mu_i$. Let γ (resp., δ) be the residue of $c_2 - c_1$ (resp., $n(d_2 - d_1)$) modulo nl. Then*

$$u_{k_1} \wedge u_{k_2} = -\, u_{k_2} \wedge u_{k_1} \qquad \textit{if } \gamma = 0,\ \delta = 0, \tag{R1}$$

$$\begin{aligned} u_{k_1} \wedge u_{k_2} = &-q^{-1} u_{k_2} \wedge u_{k_1} \\ &+ (q^{-2} - 1)\sum_{m \geq 0} q^{-2m} u_{k_2-\gamma-nlm} \wedge u_{k_1+\gamma+nlm} \\ &- (q^{-2} - 1)\sum_{m \geq 1} q^{-2m+1} u_{k_2-nlm} \wedge u_{k_1+nlm} \qquad \textit{if } \gamma > 0,\ \delta = 0, \end{aligned} \tag{R2}$$

$$\begin{aligned} u_{k_1} \wedge u_{k_2} = &\, q u_{k_2} \wedge u_{k_1} \\ &+ (q^2 - 1)\sum_{m \geq 0} q^{2m} u_{k_2-\delta-nlm} \wedge u_{k_1+\delta+nlm} \end{aligned} \tag{R3}$$

$$+ (q^2 - 1) \sum_{m \geq 1} q^{2m-1} u_{k_2 - nlm} \wedge u_{k_1 + nlm} \qquad \textit{if } \gamma = 0,\ \delta > 0,$$

$$\begin{aligned} u_{k_1} \wedge u_{k_2} = u_{k_2} \wedge u_{k_1} & \\ + (q - q^{-1}) \sum_{m \geq 0} \frac{(q^{2m+1} + q^{-2m-1})}{(q + q^{-1})} & u_{k_2 - \delta - nlm} \wedge u_{k_1 + \delta + nlm} \\ + (q - q^{-1}) \sum_{m \geq 0} \frac{(q^{2m+1} + q^{-2m-1})}{(q + q^{-1})} & u_{k_2 - \gamma - nlm} \wedge u_{k_1 + \gamma + nlm} \\ + (q - q^{-1}) \sum_{m \geq 1} \frac{(q^{2m} - q^{-2m})}{(q + q^{-1})} & u_{k_2 + nl - \gamma - \delta - nlm} \wedge u_{k_1 - nl + \gamma + \delta + nlm} \\ + (q - q^{-1}) \sum_{m \geq 1} \frac{(q^{2m} - q^{-2m})}{(q + q^{-1})} & u_{k_2 - nlm} \wedge u_{k_1 + nlm} \\ & \textit{if } \gamma > 0,\ \delta > 0, \end{aligned} \tag{R4}$$

where summations continue as long as wedges appearing under the sums remain ordered.

(ii) *Let* $r > 2$. *Then the relations of* (i) *hold in every pair of adjacent factors of* $u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_r}$.

It follows from this proposition that ordered wedges span Λ^r. Relations (R1)–(R4) can then be thought of as ordering rules that allow us to straighten an arbitrary wedge as a linear combination of ordered wedges.

Remark 3.17. (i) In order to compute the ordering rules of the wedges one can use—instead of isomorphism (6) and relations (12)—isomorphism (7) and relations $(c| \cdot T_i \otimes_H X^\mu \cdot |d) = (c| \otimes_H T_i X^\mu \cdot |d)$. The result is easily seen to be the same.

(ii) The ordering rules given in Proposition 3.16 differ from those used in [23, 24]. This difference is due to a different definition of wedges adopted here. In the present notation, the wedge of [24] is $(-1)^{l(v)} u_{\mathbf{k}}$, where v is the same as in (8).

The next lemma follows easily from Proposition 3.16.

Lemma 3.18. *Let* $k \geq m$. *Then*

(i) $$u_m \wedge u_k \wedge u_{k-1} \wedge \cdots \wedge u_m = 0,$$

(ii) $$u_k \wedge u_{k-1} \wedge \cdots \wedge u_m \wedge u_k = 0.$$

Now our aim is to show that ordered wedges form a basis of the wedge product. To this end, for $b \in B^l$ and $\zeta \in P$, we define $[\zeta]_b \in \Lambda^r(b)$ by

$$[\zeta]_b := (-q^{-1})^{l(\omega_b)} (\zeta | \otimes_{H_b} \mathbf{1}_b^- .$$

Then (3) implies that

$$[\zeta]_b = \begin{cases} 0 & \text{if } \zeta_i = \zeta_{i+1},\ b_i = b_{i+1}, \\ -q^{-1}[\zeta \cdot s_i]_b & \text{if } \zeta_i < \zeta_{i+1},\ b_i = b_{i+1}. \end{cases} \tag{13}$$

From this it follows that $[\zeta]_b$ $(\zeta \in P_b^{++})$, where $P_b^{++} := \{\zeta \in P \mid (\alpha_i, \zeta) > 0$ if $b_i = b_{i+1}\}$, span $\Lambda^r(b)$. For b such that $P_b^{++} = P^{++}$, a proof of the following lemma is given in [18]; a proof for general b is completely similar.

Lemma 3.19. *For each $b \in B^l$, the set $\{[\zeta]_b \mid \zeta \in P_b^{++}\}$ is a basis of $\Lambda^r(b)$.*

For $\boldsymbol{k} \in \mathbf{P}$, we define $\zeta(\boldsymbol{k}) \in P$ by

$$\zeta(\boldsymbol{k}) := v(\boldsymbol{k})^{-1} \cdot (c(\boldsymbol{k}) - n\mu(\boldsymbol{k})),$$

where $v(\boldsymbol{k}), c(\boldsymbol{k})$ and $\mu(\boldsymbol{k})$ are defined in (11).

Proposition 3.20. *Suppose $\boldsymbol{k} \in \mathbf{P}^{++}$. Then $\zeta(\boldsymbol{k}) \in P_{b(\boldsymbol{k})}^{++}$ and $u_{\boldsymbol{k}} = [\zeta(\boldsymbol{k})]_{b(\boldsymbol{k})}$. Conversely, for $b \in B^l$ and $\zeta \in P_b^{++}$, there is $\boldsymbol{k} \in \mathbf{P}^{++}$ such that $b = b(\boldsymbol{k})$, $\zeta = \zeta(\boldsymbol{k})$.*

Proof. First, let us show that $\boldsymbol{k} \in \mathbf{P}^{++}$ implies that $u_{\boldsymbol{k}} = [\zeta(\boldsymbol{k})]_{b(\boldsymbol{k})}$. We set $\zeta := \zeta(\boldsymbol{k})$ and use the notation of (8) and (11). Let $\lambda := -\mu$. Observe that $\boldsymbol{k} \in P^{++}$ implies that

$$\lambda \in P^+, \tag{14}$$
$$i < j,\ d_i < d_j \Longrightarrow \lambda_i > \lambda_j, \tag{15}$$
$$i < j,\ d_i = d_j \Longrightarrow c_i + n\lambda_i > c_j + n\lambda_j. \tag{16}$$

From the dominance of λ, it follows that $X^\mu = T_{t_\lambda}$. Using $v \in W^b$, Lemma 3.10, and (15), we get $\alpha \in S(v^{-1}) \Longrightarrow (\alpha, \lambda) > 0$. Since $S(v^{-1}) = -v(S(v))$, this gives $\alpha \in S(v) \Longrightarrow (v(\alpha), \lambda) < 0$, i.e., $S(v) \subset \{\alpha \in R^+ \mid (\alpha, \nu) < 0\}$, where we set $\nu := v^{-1}(\lambda)$. Let $w \in W$ be the shortest element such that $w(\nu) \in P^+$; clearly, $w(\nu) = \lambda$. By Lemma 3.2, $S(w) = \{\alpha \in R^+ \mid (\alpha, \nu)\}$; hence $S(v) \subset S(w)$. On the other hand, the length of w does not exceed the length of v. This is possible only if $S(v) = S(w)$; hence $v = w$. Thus $S(v) = \{\alpha \in R^+ \mid (v(\alpha), \lambda) < 0\}$. Now, Proposition 3.3 implies $l(u\,t_\lambda v) = l(u) + l(t_\lambda) - l(v)$. This, $l(u) + l(t_\lambda) = l(u\,t_\lambda)$, and $X^\mu = T_{t_\lambda}$ give

$$T_u\, X^\mu (T_{v^{-1}})^{-1} = T_{ut_\lambda v}.$$

Since $u \in {}^aW_a$, by Lemma 3.4, we have $u\,t_\lambda v \in {}^a\widehat{W}$; hence

$$u_{\boldsymbol{k}} = [a \cdot u\,t_\lambda v]_b.$$

It remains to observe that

$$a \cdot u\,t_\lambda v = (c + n\lambda) \cdot v = v^{-1} \cdot (c + n\lambda) = \zeta.$$

Next, we show that $\boldsymbol{k} \in \mathbf{P}^{++}$ implies that $\zeta \in P_b^{++}$. It follows from $v \in W^b$ and Lemma 3.9 that for $i < j$,

$$b_i = b_j \Rightarrow v^{-1}(i) < v^{-1}(j); \quad \text{equivalently,} \quad d_{v^{-1}(i)} = d_{v^{-1}(j)} \Rightarrow v^{-1}(i) < v^{-1}(j).$$

Therefore, using $c_{v^{-1}(i)} + n\lambda_{v^{-1}(i)} = \zeta_i$ and (16), we get $\zeta \in P_b^{++}$.

Finally, $u_{\boldsymbol{k}}$ with $\boldsymbol{k} \in \mathbf{P}^{++}$ span Λ^r. Hence for $b \in B^l$ and $\zeta \in P_b^{++}$, we have

$$[\zeta]_b = \sum_{\boldsymbol{l} \in \mathbf{P}^{++}} e_{\boldsymbol{l}} u_{\boldsymbol{l}} \quad (e_{\boldsymbol{l}} \in \mathbf{K}).$$

However, $\{[\zeta]_b | b \in B^l, \zeta \in P_b^{++}\}$ is a basis of Λ^r. Therefore, writing $u_{\boldsymbol{l}}$ as $[\zeta(\boldsymbol{l})]_{b(\boldsymbol{l})}$, we get $[\zeta]_b = u_{\boldsymbol{k}}$, where $\boldsymbol{k} \in \mathbf{P}^{++}$ is such that $b = b(\boldsymbol{k})$ and $\zeta = \zeta(\boldsymbol{k})$. □

This proposition and Lemma 3.19 immediately imply the following.

Proposition 3.21. *$\{u_{\boldsymbol{k}} \mid \boldsymbol{k} \in \mathbf{P}^{++}\}$ is a basis of Λ^r.*

3.4 Canonical bases of the wedge product. Define an involution $x \mapsto \overline{x}$ of Λ^r by

$$\overline{\mathbf{1}_a^+ \otimes_{H_a} h \otimes_{H_b} \mathbf{1}_b^-} = \mathbf{1}_a^+ \otimes_{H_a} \overline{h} \otimes_{H_b} \mathbf{1}_b^-, \quad \overline{q} = q^{-1} \quad (h \in \widehat{H}).$$

Lemma 3.22. *Let $u \in {}^aW_a$, $v \in W^b$, and $\mu \in P$. Then $\omega_a u \omega \in {}^aW_a$, $\omega v \omega_b \in W^b$, and*

$$\overline{T_u X^\mu (T_{v^{-1}})^{-1}} = T_{\omega_a} T_{\omega_a u \omega} X^{\omega(\mu)} (T_{(\omega v \omega_b)^{-1}})^{-1} (T_{\omega_b})^{-1}.$$

Proof. Since for $w \in W$, we have $l(w\omega) = l(\omega) - l(w)$; $u\omega$ is the longest element of the coset $W_a u\omega$. Hence $\omega_a u \omega$ is the shortest element of this coset, i.e., $\omega_a u \omega \in {}^aW_a$. Therefore, $l(u\omega) = l(\omega) + l(\omega_a u \omega)$ and

$$T_{u\omega} = T_{\omega_a} T_{\omega_a u \omega}.$$

In a completely similar fashion, we get $\omega v \omega_b \in W^b$ and

$$(T_{(\omega v)^{-1}})^{-1} = (T_{(\omega v \omega_b)^{-1}})^{-1} (T_{\omega_b})^{-1}.$$

Lemma 3.14 implies the remaining statement. □

Proposition 3.23. *For $u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_r} \in \Lambda^r(a, b)$, we have*

$$\overline{u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_r}} = (-q)^{l(\omega_b)} q^{-l(\omega_a)} u_{k_r} \wedge \cdots \wedge u_{k_2} \wedge u_{k_1}.$$

Proof. Using (9) in the right-hand side of (8), we have, by Lemma 3.22,

$$\overline{\mathbf{1}_a^+ \otimes_{H_a} T_u X^\mu (T_{v^{-1}})^{-1} \otimes_{H_b} \mathbf{1}_b^-} = q^{-l(\omega_a)} (-q)^{-l(\omega_b)} (a \cdot u\omega| \cdot X^{\omega(\mu)} \otimes_H |\omega v \cdot b).$$

The result follows. □

Remark 3.24. It is easy to see that in the notation of (8), we have

$$l(\omega_a) = \sharp\{i < j \mid c_i = c_j\}, \qquad l(\omega_b) = \sharp\{i < j \mid d_i = d_j\}.$$

For $\boldsymbol{k} \in \mathbf{P}^{++}$, set

$$\overline{u_{\boldsymbol{k}}} = \sum_{\boldsymbol{l} \in \mathbf{P}^{++}} R_{\boldsymbol{k},\boldsymbol{l}}(q)\, u_{\boldsymbol{l}}.$$

By Proposition 3.16, the entries of matrix $\|R_{\boldsymbol{k},\boldsymbol{l}}(q)\|$ are Laurent polynomials in q with integral coefficients, and by Remark 3.24, we have $R_{\boldsymbol{k},\boldsymbol{k}}(q) = 1$.

We define a partial order on $\mathbf{P}^{++}$ by

$$\boldsymbol{k} \geq \boldsymbol{l} \quad \text{iff} \quad \left| \begin{array}{l} \sum_{i=1}^{j} k_i \geq \sum_{i=1}^{j} l_i \quad \text{for all } j = 1, \ldots, r, \\ \sum_{i=1}^{r} k_i = \sum_{i=1}^{r} l_i, \\ k_r \leq l_r. \end{array} \right.$$

The ordering rules of Proposition 3.16 imply that the matrix $\|R_{\boldsymbol{k},\boldsymbol{l}}(q)\|$ is lower triangular with respect to this order. That is, $R_{\boldsymbol{k},\boldsymbol{l}}(q)$ is not zero only if $\boldsymbol{k} \geq \boldsymbol{l}$.

Define two lattices of Λ^r by

$$\mathcal{L}^+ := \bigoplus_{\boldsymbol{k} \in \mathbf{P}^{++}} \mathbf{Q}[q]\, u_{\boldsymbol{k}}, \qquad \mathcal{L}^- := \bigoplus_{\boldsymbol{k} \in \mathbf{P}^{++}} \mathbf{Q}[q^{-1}]\, u_{\boldsymbol{k}}.$$

The unitriangularity of $\|R_{\boldsymbol{k},\boldsymbol{l}}(q)\|$ implies, by the standard argument going back to Kazhdan and Lusztig, the following.

Theorem 3.25. *There are unique bases $\{G_{\boldsymbol{k}}^+ \mid \boldsymbol{k} \in \mathbf{P}^{++}\}$, $\{G_{\boldsymbol{k}}^- \mid \boldsymbol{k} \in \mathbf{P}^{++}\}$ of Λ^r such that*

(i) $\qquad \overline{G_{\boldsymbol{k}}^+} = G_{\boldsymbol{k}}^+, \qquad \overline{G_{\boldsymbol{k}}^-} = G_{\boldsymbol{k}}^-,$

(ii) $\qquad G_{\boldsymbol{k}}^+ \equiv u_{\boldsymbol{k}} \bmod q\mathcal{L}^+, \qquad G_{\boldsymbol{k}}^- \equiv u_{\boldsymbol{k}} \bmod q^{-1}\mathcal{L}^-.$

Set

$$G_{\boldsymbol{k}}^+ = \sum_{\boldsymbol{l} \in \mathbf{P}^{++}} \Delta_{\boldsymbol{k},\boldsymbol{l}}^+(q)\, u_{\boldsymbol{l}}, \qquad G_{\boldsymbol{k}}^- = \sum_{\boldsymbol{l} \in \mathbf{P}^{++}} \Delta_{\boldsymbol{k},\boldsymbol{l}}^-(q)\, u_{\boldsymbol{l}}.$$

It is clear that $\Delta_{\boldsymbol{k},\boldsymbol{l}}^+(q)$ or $\Delta_{\boldsymbol{k},\boldsymbol{l}}^-(q)$ is nonzero only if $u_{\boldsymbol{k}}$ and $u_{\boldsymbol{l}}$ belong to the same subspace $\Lambda^r(a,b)$. That is, in the notation of (11), only if $a(\boldsymbol{k}) = a(\boldsymbol{l})$ and $b(\boldsymbol{k}) = b(\boldsymbol{l})$.

Theorem 3.26. *For $\boldsymbol{k}, \boldsymbol{l} \in \mathbf{P}^{++}$ such that $a(\boldsymbol{k}) = a(\boldsymbol{l})$, $b(\boldsymbol{k}) = b(\boldsymbol{l})$, set $\xi = \zeta(\boldsymbol{k})$, $\eta = \zeta(\boldsymbol{l})$ and $a = a(\boldsymbol{k})$, $b = b(\boldsymbol{k})$. Then*

(i) $\qquad \Delta_{\boldsymbol{k},\boldsymbol{l}}^-(q) = P_{\eta,\xi}^-,$

(ii) $\qquad \Delta_{\boldsymbol{k},\boldsymbol{l}}^+(q) = \sum_{v \in W_b} (-q)^{l(v)} P_{\eta\cdot\omega_b v,\, \xi\cdot\omega_b}^+,$

where $P_{\eta,\xi}^-$ and $P_{\eta,\xi}^+$ are the parabolic Kazhdan–Lusztig polynomials associated with $\mathbf{1}_a^+ \otimes_{H_a} \widehat{H}$.

PROOF. Set $D_\xi = C^-_\xi \otimes \mathbf{1}^-_b$. Then $\overline{D_\xi} = D_\xi$. Using (4) and (13), we obtain

$$D_\xi = (-q)^{l(\omega_b)} \sum_{\eta \in P} P^-_{\eta,\xi}\,[\eta]_b = z_b \sum_{\eta \in P^{++}_b} P^-_{\eta,\xi}\,[\eta]_b,$$

where $z_b = (-q)^{l(\omega_b)} \sum_{v \in W_b} q^{-2l(v)}$. Since $\overline{z_b} = z_b$, we have $D_\xi = z_b\, G^-_k$ and (i) follows.

Next, let $E_\xi = C^-_{\xi\cdot\omega_b} \otimes \mathbf{1}^-_b$. Then $\overline{E_\xi} = E_\xi$, and

$$E_\xi = \sum_{\eta \in P^{++}_b} \sum_{v \in W_b} (-q)^{l(\omega_b)-l(v)} P^+_{\eta\cdot v,\xi\cdot\omega_b}\,[\eta]_b = \sum_{\eta \in P^{++}_b} \sum_{v \in W_b} (-q)^{l(v)} P^+_{\eta\cdot\omega_b v,\xi\cdot\omega_b}\,[\eta]_b.$$

Hence $E_\xi = G^+_k$, which implies (ii). □

Remark 3.27. For $\xi \in a \cdot \widehat{W}$, let $x(\xi)$ be the unique element of ${}^a\widehat{W}$ such that $\xi = a \cdot x(\xi)$. It follows from Proposition 3.8 that for $\xi \in P^{++}_b$ we have $x(\xi \cdot v) = x(\xi)v$ for all $v \in W_b$. Hence, one can rewrite the formulas of Theorem 3.26 in terms of (ordinary) Kazhdan–Lusztig polynomials for $\widehat{H}$ as

(i) $$\Delta^-_{k,l}(q) = \sum_{u \in W_a} q^{-l(u)} \mathcal{P}^-_{u\,x(\eta),\,x(\xi)},$$

(ii) $$\Delta^+_{k,l}(q) = \sum_{v \in W_b} (-q)^{l(v)} \mathcal{P}^+_{\omega_a\,x(\eta)\,\omega_b\,v,\;\omega_a\,x(\xi)\,\omega_b}.$$

3.5 Actions of quantum affine algebras on the wedge product. It is easy to verify that the formulas

$$\begin{aligned}
e_i(v_c X^m) &= \delta_{i+1 \equiv c \bmod n}\; v_{c-1} X^{m+\delta_{i0}},\\
f_i(v_c X^m) &= \delta_{i \equiv c \bmod n}\; v_{c+1} X^{m-\delta_{i0}},\\
t_i(v_c X^m) &= q^{\delta_{i\equiv c \bmod n} - \delta_{i+1\equiv c \bmod n}}\; v_c X^m,\\
\partial(v_c X^m) &= m v_c X^m
\end{aligned}$$

define a level zero action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\mathbf{K}^n[X, X^{-1}]$. Here it is understood that $v_0 = v_n$, $v_{n+1} = v_1$. Also, for a statement S we set $\delta_S = 1$ (resp., 0) if S is true (resp., false). Using the coproduct

$$\Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i, \qquad \Delta(t_i) = t_i \otimes t_i,$$
$$\Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i, \qquad \Delta(\partial) = \partial \otimes 1 + 1 \otimes \partial,$$

we extend this action on the tensor product $(\mathbf{K}^n[X, X^{-1}])^{\otimes r}$. Next, using the isomorphism

$$(\mathbf{K}^n[X, X^{-1}])^{\otimes r} \xrightarrow{\sim} \bigoplus_{a \in A^n} \mathbf{K}\mathbf{1}^+_a \otimes_{H_a} \widehat{H}, \quad v_{c_1}X^{\mu_1} \otimes \cdots \otimes v_{c_r}X^{\mu_r} \mapsto (c| \cdot X^\mu,$$

where $c = (c_1, \ldots, c_r)$ and $\mu = (\mu_1, \ldots, \mu_r)$, to identify the involved vector spaces, we obtain an action of $U_q(\widehat{\mathfrak{sl}}_n)$ on the $\widehat{H}$–module $M_R = \oplus_{a \in A^n} \mathbf{K}\mathbf{1}_a^+ \otimes_{H_a} \widehat{H}$. The following proposition is due to [8]. It is easily verified by reducing to the case $r = 2$ (cf. [18, proof of Proposition 7.1]).

Proposition 3.28. *The actions of $U_q(\widehat{\mathfrak{sl}}_n)$ and $H \subset \widehat{H}$ on M_R commute.*

By isomorphism (6), for any $b \in B^l$, the commutativity of $U_q(\widehat{\mathfrak{sl}}_n)$ and H allows us to restrict the action of $U_q(\widehat{\mathfrak{sl}}_n)$ on $\Lambda^r(b)$, which then gives an action of $U_q(\widehat{\mathfrak{sl}}_n)$ on Λ^r. In terms of the wedge vectors $u_k = u_{k_1} \wedge \cdots \wedge u_{k_r}$ ($k = (k_1, \ldots, k_r) \in \mathbf{P}$), this action is written as

$$e_i(u_k) = \sum_{j=1}^{r} u_{k_1} \wedge \cdots \wedge u_{k_{j-1}} \wedge e_i(u_{k_j}) \wedge t_i^{-1}(u_{k_{j+1}}) \wedge \cdots \wedge t_i^{-1}(u_{k_r}), \tag{17}$$

$$f_i(u_k) = \sum_{j=1}^{r} t_i(u_{k_1}) \wedge \cdots \wedge t_i(u_{k_{j-1}}) \wedge f_i(u_{k_j}) \wedge u_{k_{j+1}} \wedge \cdots \wedge u_{k_r}, \tag{18}$$

$$t_i(u_k) = t_i(u_{k_1}) \wedge \cdots \wedge t_i(u_{k_r}), \tag{19}$$

$$\partial(u_k) = \sum_{j=1}^{r} u_{k_1} \wedge \cdots \wedge u_{k_{j-1}} \wedge \partial(u_{k_j}) \wedge u_{k_{j+1}} \wedge \cdots \wedge u_{k_r}. \tag{20}$$

Here we set $e_i(u_{k_j}) = e_i(v_{c_j} X^{\mu_j} \dot{v}_{d_j}) := e_i(v_{c_j} X^{\mu_j}) \dot{v}_{d_j}$ and similarly for the rest of the generators.

In a completely similar fashion, we define on the wedge product an action of the quantum affine algebra $U_p(\widehat{\mathfrak{sl}}_l)$, where $p := -q^{-1}$. In order to distinguish between this action and the action of $U_q(\widehat{\mathfrak{sl}}_n)$, we put dots over the generators of $U_p(\widehat{\mathfrak{sl}}_l)$. Define a level-zero action of $U_p(\widehat{\mathfrak{sl}}_l)$ on $\mathbf{K}^l[X, X^{-1}]$ by

$$\dot{e}_i(X^m \dot{v}_d) = \delta_{i+1 \equiv d \bmod l}\, X^{m+\delta_{i0}} \dot{v}_{d-1}, \qquad \dot{t}_i(X^m \dot{v}_d) = p^{\delta_{i \equiv d \bmod l} - \delta_{i+1 \equiv d \bmod l}}\, X^m \dot{v}_d,$$
$$\dot{f}_i(X^m \dot{v}_d) = \delta_{i \equiv d \bmod l}\, X^{m-\delta_{i0}} \dot{v}_{d+1}, \qquad \dot{\partial}(X^m \dot{v}_d) = m X^m \dot{v}_d.$$

Here it is understood that $\dot{v}_0 = \dot{v}_l$, $\dot{v}_{l+1} = \dot{v}_1$. Using the coproduct

$$\Delta(\dot{e}_i) = \dot{e}_i \otimes \dot{t}_i^{-1} + 1 \otimes \dot{e}_i, \qquad \Delta(\dot{t}_i) = \dot{t}_i \otimes \dot{t}_i,$$
$$\Delta(\dot{f}_i) = \dot{f}_i \otimes 1 + \dot{t}_i \otimes \dot{f}_i, \qquad \Delta(\dot{\partial}) = \dot{\partial} \otimes 1 + 1 \otimes \dot{\partial},$$

we extend this action on the tensor product $(\mathbf{K}^l[X, X^{-1}])^{\otimes r}$. Next, using the isomorphism

$$(\mathbf{K}^l[X, X^{-1}])^{\otimes r} \xrightarrow{\sim} \bigoplus_{b \in B^l} \widehat{H} \otimes_{H_b} \mathbf{K}\mathbf{1}_b^-,$$
$$X^{\mu_1} \dot{v}_{d_1} \otimes \cdots \otimes X^{\mu_r} \dot{v}_{d_r} \mapsto X^{\mu} \cdot |d\rangle\, (-q^{-1})^{l(\omega_b)},$$

where $d = (d_1, \ldots, d_r)$, $\mu = (\mu_1, \ldots, \mu_r)$, and b is the unique point of B^l in the orbit $W \cdot d$, to identify the involved vector spaces, we obtain an action of $U_p(\widehat{\mathfrak{sl}}_l)$

on the $\widehat{H}$-module $M_L = \oplus_{b \in B'} \widehat{H} \otimes_{H_b} \mathbf{K}\mathbf{1}_b^-$. The following result is an analogue of Proposition 3.28 and is verified similarly.

Proposition 3.29. *The actions of $U_p(\widehat{\mathfrak{sl}}_l)$ and $H \subset \widehat{H}$ on M_L commute.*

By the isomorphism (7), for any $a \in A^n$, the commutativity of $U_p(\widehat{\mathfrak{sl}}_l)$ and H allows us to restrict the action of $U_p(\widehat{\mathfrak{sl}}_l)$ on $\Lambda^r(a)$, which gives then an action of $U_p(\widehat{\mathfrak{sl}}_l)$ on Λ^r. In terms of the wedge vectors $u_{\boldsymbol{k}} = u_{k_1} \wedge \cdots \wedge u_{k_r}$ ($\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbf{P}$), this action is written as

$$\dot{e}_i(u_{\boldsymbol{k}}) = \textstyle\sum_{j=1}^r u_{k_1} \wedge \cdots \wedge u_{k_{j-1}} \wedge \dot{e}_i(u_{k_j}) \wedge \dot{t}_i^{-1}(u_{k_{j+1}}) \wedge \cdots \wedge \dot{t}_i^{-1}(u_{k_r}), \quad (21)$$

$$\dot{f}_i(u_{\boldsymbol{k}}) = \textstyle\sum_{j=1}^r \dot{t}_i(u_{k_1}) \wedge \cdots \wedge \dot{t}_i(u_{k_{j-1}}) \wedge \dot{f}_i(u_{k_j}) \wedge u_{k_{j+1}} \wedge \cdots \wedge u_{k_r}, \quad (22)$$

$$\dot{t}_i(u_{\boldsymbol{k}}) = \dot{t}_i(u_{k_1}) \wedge \cdots \wedge \dot{t}_i(u_{k_r}), \quad (23)$$

$$\dot{\partial}(u_{\boldsymbol{k}}) = \textstyle\sum_{j=1}^r u_{k_1} \wedge \cdots \wedge u_{k_{j-1}} \wedge \dot{\partial}(u_{k_j}) \wedge u_{k_{j+1}} \wedge \cdots \wedge u_{k_r}. \quad (24)$$

Here we put $\dot{e}_i(u_{k_j}) = \dot{e}_i(v_{c_j} X^{\mu_j} \dot{v}_{d_j}) := v_{c_j} \dot{e}_i(X^{\mu_j} \dot{v}_{d_j})$ and similarly for the rest of the generators.

A well-known result of Bernstein (see, e.g., [15]) says that the centre $Z(\widehat{H})$ of $\widehat{H}$ is generated by symmetric Laurent polynomials in $X_i := X^{\varepsilon_i}$. It follows by either using (6) or (7) that $Z(\widehat{H})$ acts on the wedge product Λ^r. This action may be computed in terms of the wedge vectors $u_{\boldsymbol{k}} = u_{k_1} \wedge \cdots \wedge u_{k_r}$ ($\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbf{P}$) by using either (9) or (10). In particular, for $B_m := \sum_{i=1}^r X_i^m$ ($m \in \mathbf{Z}^*$), we get

$$B_m(u_{\boldsymbol{k}}) = \sum_{j=1}^r u_{k_1} \wedge \cdots \wedge u_{k_{j-1}} \wedge u_{k_j - nlm} \wedge u_{k_{j+1}} \wedge \cdots \wedge u_{k_r}. \quad (25)$$

Proposition 3.30. *The actions of $U_q'(\widehat{\mathfrak{sl}}_n)$, $U_p'(\widehat{\mathfrak{sl}}_l)$, and $Z(\widehat{H})$ on Λ^r are pairwise mutually commutative.*

Proof. The commutativity of $U_q'(\widehat{\mathfrak{sl}}_n)$ (resp., $U_p'(\widehat{\mathfrak{sl}}_l)$) with $Z(\widehat{H})$ is immediate by Proposition 3.28 (resp., Proposition 3.29). The commutativity of $U_q'(\widehat{\mathfrak{sl}}_n)$ with $U_p'(\widehat{\mathfrak{sl}}_l)$ follows from (17)–(19) and (21)–(23). □

The actions of $U_q(\widehat{\mathfrak{sl}}_n)$, $U_p(\widehat{\mathfrak{sl}}_l)$ and $Z(\widehat{H})$ on Λ^r are compatible with the bar involution in the following sense.

Proposition 3.31. *For $v \in \Lambda^r$*

$$\overline{e_i(v)} = e_i(\overline{v}), \quad \overline{f_i(v)} = f_i(\overline{v}), \quad \overline{t_i(v)} = t_i^{-1}(\overline{v}), \quad \overline{\partial(v)} = \partial(\overline{v}),$$
$$\overline{\dot{e}_j(v)} = \dot{e}_j(\overline{v}), \quad \overline{\dot{f}_j(v)} = \dot{f}_j(\overline{v}), \quad \overline{\dot{t}_j(v)} = \dot{t}_j^{-1}(\overline{v}), \quad \overline{\dot{\partial}(v)} = \dot{\partial}(\overline{v}),$$
$$\overline{B_m(v)} = B_m(\overline{v})$$

for $i = 0, 1, \ldots, n-1$, $j = 0, 1, \ldots, l-1$, and $m \in \mathbf{Z}^*$.

PROOF. It is enough to show this for $v = u_{\boldsymbol{k}}$ ($\boldsymbol{k} = (k_1, \ldots, k_r) \in \mathbf{P}$). As in Section 3.3, we set $k_i = c_i + n(d_i - 1) - nl\mu_i$ with $c_i \in \{1, \ldots, n\}$, $d_i \in \{1, \ldots, l\}$, and $\mu_i \in \mathbf{Z}$. By Proposition 3.23 and Remark 3.24, we have

$$\overline{u_{\boldsymbol{k}}} = q^{-\kappa(c)}(-q)^{\kappa(d)} u_{k_r} \wedge \cdots \wedge u_{k_1} = p^{-\kappa(d)}(-p)^{\kappa(c)} u_{k_r} \wedge \cdots \wedge u_{k_1},$$

where $\kappa(c)$ (resp., $\kappa(d)$) is the number of pairs (i, j) such that $c_i = c_j$ (resp., $d_i = d_j$). Using (22), we have

$$\overline{\dot{f}_0(u_{\boldsymbol{k}})} = (-p)^{\kappa(c)} \sum_{j,\, d_j = l} p^{-\kappa(d'_{(j)}) + \sum_{i<j} \delta_{d_i=1} - \delta_{d_i=l}} \\ u_{k_r} \wedge \cdots \wedge (v_{c_j} X^{\mu_j - 1} \dot{v}_1) \wedge \cdots \wedge u_{k_1}$$

and

$$\dot{f}_0(\overline{u_{\boldsymbol{k}}}) = (-p)^{\kappa(c)} \sum_{j,\, d_j = l} p^{-\kappa(d) + \sum_{i>j} \delta_{d_i=l} - \delta_{d_i=1}} \\ u_{k_r} \wedge \cdots \wedge (v_{c_j} X^{\mu_j - 1} \dot{v}_1) \wedge \cdots \wedge u_{k_1},$$

where

$$\kappa(d'_{(j)}) = \kappa(d + \varepsilon_j - l\varepsilon_j) = \kappa(d) + \sum_{i \neq j} \delta_{d_i=1} - \delta_{d_i=l}.$$

This gives $\overline{\dot{f}_0(u_{\boldsymbol{k}})} = \dot{f}_0(\overline{u_{\boldsymbol{k}}})$. The relations for the rest of the generators of $U_p(\widehat{\mathfrak{sl}}_l)$ and for the generators of $U_q(\widehat{\mathfrak{sl}}_n)$ are verified similarly. Finally, $\overline{B_m(v)} = B_m(\overline{v})$ follows from Lemma 3.14. □

4 Canonical bases of the q-deformed Fock spaces

4.1 Semiinfinite wedge product. For each integer s, we define a vector space $\Lambda^{s+\frac{\infty}{2}}$ as the inductive limit $\lim_{\rightarrow} \Lambda^r$, where the maps $\Lambda^r \to \Lambda^t$ ($t > r$) are given by

$$v \mapsto v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}.$$

The vector space $\Lambda^{s+\frac{\infty}{2}}$ will be called the semiinfinite wedge product of charge s. For $v \in \Lambda^r$, we shall use the semiinfinite expression

$$v \wedge u_{s-r} \wedge u_{s-r-1} \wedge u_{s-r-2} \wedge \cdots$$

to denote the image of v with respect to the canonical map from Λ^r to $\lim_{\rightarrow} \Lambda^r$. Let $\mathbf{P}(s)$ be the set of semiinfinite sequences $\boldsymbol{k} = (k_1, k_2, \ldots) \in \mathbf{Z}^\infty$ such that $k_i = s - i + 1$ for all $i \gg 1$. By definition, $\Lambda^{s+\frac{\infty}{2}}$ is spanned by $u_{\boldsymbol{k}} := u_{k_1} \wedge u_{k_2} \wedge \cdots$ with $\boldsymbol{k} \in \mathbf{P}(s)$. We shall call a semiinfinite wedge $u_{\boldsymbol{k}}$ *ordered* if $\boldsymbol{k} \in \mathbf{P}^{++}(s) := \{(k_1, k_2, \ldots) \in \mathbf{P}(s) \mid k_1 > k_2 > \cdots\}$.

Proposition 4.1 ([23]). *Ordered wedges form a basis of* $\Lambda^{s+\frac{\infty}{2}}$.

In what follows, we shall use, besides the indexation by $\mathbf{P}^{++}(s)$, three other indexations of the basis formed by ordered wedges. The first of these is the obvious indexation by the set Π of all partitions. Namely, with $\boldsymbol{k} = (k_1, k_2, \dots) \in \mathbf{P}^{++}(s)$, we associate a partition $\lambda = (\lambda_1, \lambda_2, \dots)$ by taking $\lambda_i = k_i - s + i - 1$ for all $i \in \mathbf{N}$, and we set $|\lambda, s\rangle := u_{\boldsymbol{k}}$.

Another indexation is by the set of pairs $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$, where $\boldsymbol{\lambda}_l = (\lambda^{(1)}, \dots, \lambda^{(l)})$ is an l-multipartition and $\boldsymbol{s}_l = (s_1, \dots, s_l)$ is a sequence of integers summing up to s. For any $d \in \{1, \dots, l\}$ the partition $\lambda^{(d)}$ and the integer s_d are defined as follows. For each $i \in \mathbf{N}^*$ write $k_i = c_i + n(d_i - 1) - nlm_i$, where $c_i \in \{1, \dots, n\}$, $d_i \in \{1, \dots, l\}$, and $m_i \in \mathbf{Z}$. Then let $k_1^{(d)}$ be equal $c_i - nm_i$, where i is the smallest such that $d_i = d$, let $k_2^{(d)}$ be equal to $c_j - nm_j$, where j is the second smallest such that $d_j = d$, and so on. In this way, we obtain a strictly decreasing semiinfinite sequence $(k_1^{(d)}, k_2^{(d)}, \dots)$ such that $k_i^{(d)} = s_d - i + 1$ $(i \gg 1)$ for some unique integer s_d. Now we define the partition $\lambda^{(d)} = (\lambda_1^{(d)}, \lambda_2^{(d)}, \dots)$ by $\lambda_i^{(d)} = k_i^{(d)} - s_d + i - 1$. It is easy to check that the s_d obtained in this way satisfies $s_1 + \cdots + s_l = s$.

In a completely similar fashion, we associate with each $\boldsymbol{k} \in \mathbf{P}^{++}(s)$ a pair $(\boldsymbol{\lambda}_n, \boldsymbol{s}_n)$, where $\boldsymbol{\lambda}_n$ is an n-multipartition and $\boldsymbol{s}_n$ is a sequence of n integers summing up to s. The $(\boldsymbol{\lambda}_n, \boldsymbol{s}_n)$ is obtained by the same procedure as the $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$, reversing everywhere the roles of n and l and the roles of c_i and d_i.

For any $s \in \mathbf{Z}$, let $\mathbf{Z}^l(s)$ be the set of l-tuples of integers summing to s. Define the map $\tau_l^s : \Pi \to \Pi^l \times \mathbf{Z}^l(s)$ (resp., the map $\tau_n^s : \Pi \to \Pi^n \times \mathbf{Z}^n(s)$) by

$$\tau_l^s \ : \ \lambda \mapsto (\boldsymbol{\lambda}_l, \boldsymbol{s}_l), \quad \text{(resp., by)} \quad \tau_n^s \ : \ \lambda \mapsto (\boldsymbol{\lambda}_n, \boldsymbol{s}_n). \tag{26}$$

It is not difficult to see that for each s, the maps τ_l^s and τ_n^s are bijections. Hence, setting $|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle := |\lambda, s\rangle$ (resp., $|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle := |\lambda, s\rangle$) if $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l) = \tau_l^s(\lambda)$ (resp., if $(\boldsymbol{\lambda}_n, \boldsymbol{s}_n) = \tau_n^s(\lambda)$), we obtain that $B(s) := \{|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l, \ \boldsymbol{s}_l \in \mathbf{Z}^l(s)\} = \{|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle \mid \boldsymbol{\lambda}_n \in \Pi^n, \ \boldsymbol{s}_n \in \mathbf{Z}^n(s)\} = \{|\lambda, s\rangle \mid \lambda \in \Pi\}$ is a basis of $\Lambda^{s+\frac{\infty}{2}}$.

Remark 4.2. (i) The pair $(\boldsymbol{\lambda}_n = (\lambda^{(1)}, \dots, \lambda^{(n)}), \boldsymbol{s}_n = (s_1, \dots, s_n))$ can be read off the diagram of the partition λ in the following manner. For $r \in \{0, 1, \dots, n-1\}$, let R_r (resp., C_r) be the set of all rows (resp., columns) of λ that have a node of residue r as their rightmost (resp., bottom) node. Then for each $c \in \{1, \dots, n\}$, the diagram of $\lambda^{(c)}$ is embedded into the diagram of λ by

$$\lambda^{(c)} = \lambda \cap C_c \cap R_{c-1},$$

where we set $C_n = C_0$. Hence $\boldsymbol{\lambda}_n$ is the n-quotient of λ. On the other hand, the sequence $\boldsymbol{s}_n$ is obtained as follows. Let $N_r(\lambda|s, n)$ be the number of addable nodes of residue r minus the number of removable nodes of residue r. Then for each $c \in \{1, \dots, n-1\}$, we have $s_c - s_{c+1} = N_c(\lambda|s, n)$. These equalities, together with $\sum_{a=1}^n s_a = s$, determine $\boldsymbol{s}_n$ completely. It follows that $\boldsymbol{s}_n$ is a particular parameterization of the n-core (cf. [19]) of λ.

(ii) In a similar manner one can describe the pair $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ corresponding to λ and s. In this case, we first associate with λ and s another partition, which we denote by

$\sigma^s(\lambda)$. Let $\boldsymbol{k} = (k_1, k_2, \ldots)$ be the element of $\mathbf{P}^{++}(s)$ defined by (λ, s). For each $i \in \mathbf{N}$, we set $k_i = e_i - nlm_i$, where $e_i \in \{1, \ldots, nl\}$, $m_i \in \mathbf{Z}$. Next, for every $e \in \{1, \ldots, nl\}$, we define $(k_1^{(e)}, k_2^{(e)}, \ldots)$ to be the semiinfinite strictly decreasing sequence $(1 - m_i \mid e_i = e)$. This sequence stabilizes, for $i \gg 1$, to $t_e - i + 1$, where t_e is a uniquely defined integer. Then we define a partition $\mu^{(e)} = (\mu_1^{(e)}, \mu_2^{(e)}, \ldots)$ by $\mu_i^{(e)} = k_i^{(e)} - t_e + i - 1$. In this way, we get an nl-multipartition $(\mu^{(1)}, \ldots, \mu^{(nl)})$ and a sequence of nl integers $(t_1, \ldots, t_{nl})$ summing to s. Of course, these are simply the nl-quotient and the nl-core of λ. Let σ be the permutation of $\{1, \ldots, nl\}$ defined by

$$\sigma : d + l(c-1) \mapsto c + n(d-1) \quad (c \in \{1, \ldots, n\},\ d \in \{1, \ldots, l\}).$$

Now $\sigma^s(\lambda)$ is defined to be the unique partition with the nl-quotient $(\mu^{(\sigma(1))}, \ldots, \mu^{(\sigma(nl))})$ and the nl-core $(t_{\sigma(1)}, \ldots, t_{\sigma(nl)})$. Finally, $\boldsymbol{\lambda}_l$ and s_l can be read off the diagram of $\sigma^s(\lambda)$ in exactly the same way as $(\boldsymbol{\lambda}_n, s_n)$ are read off the diagram of λ, i.e., as explained in (i) above, but reversing everywhere the roles of n and l.

Example 4.3. Table 1 illustrates, for partitions λ of size 7, the three indexations defined above. Here $n = 3$, $l = 2$, and $s = 0$.

λ	$\sigma^s(\lambda)$	$\boldsymbol{\lambda}_l$	s_l	$\boldsymbol{\lambda}_n$	s_n
(7)	(7)	$((3), \varnothing)$	$(1, -1)$	$((2), \varnothing, \varnothing)$	$(1, 0, -1)$
$(6, 1)$	$(6, 1^2)$	$(\varnothing, (3, 1))$	$(0, 0)$	$(\varnothing, \varnothing, (1))$	$(0, -1, 1)$
$(5, 2)$	$(4, 2, 1)$	$(\varnothing, (3))$	$(1, -1)$	$(\varnothing, (2), \varnothing)$	$(1, 0, -1)$
$(5, 1^2)$	$(4, 1^4)$	$(\varnothing, (2, 1^2))$	$(0, 0)$	$(\varnothing, (1), \varnothing)$	$(-1, 1, 0)$
$(4, 3)$	$(3^2, 1)$	$((1), (2))$	$(1, -1)$	$((1), (1), \varnothing)$	$(1, 0, -1)$
$(4, 2, 1)$	$(2^2, 1^3)$	$(\varnothing, (2, 1))$	$(1, -1)$	$((1^2), \varnothing, \varnothing)$	$(1, 0, -1)$
$(4, 1^3)$	$(2, 1)$	$(\varnothing, \varnothing)$	$(-1, 1)$	$((1), \varnothing, (1))$	$(1, 0, -1)$
$(3^2, 1)$	$(5, 4, 1^2)$	$((1^2), (1))$	$(2, -2)$	$(\varnothing, \varnothing, (1))$	$(-1, 1, 0)$
$(3, 2^2)$	$(5, 2^3, 1)$	$((1), (2))$	$(2, -2)$	$((1), \varnothing, \varnothing)$	$(0, -1, 1)$
$(3, 2, 1^2)$	$(5, 2)$	$((2, 1), \varnothing)$	$(1, -1)$	$(\varnothing, \varnothing, (2))$	$(1, 0, -1)$
$(3, 1^4)$	$(5, 1^3)$	$((3, 1), \varnothing)$	$(0, 0)$	$(\varnothing, (1), \varnothing)$	$(0, -1, 1)$
$(2^3, 1)$	$(3, 2^2)$	$((1^2), (1))$	$(1, -1)$	$(\varnothing, (1), (1))$	$(1, 0, -1)$
$(2^2, 1^3)$	$(3, 2, 1^2)$	$((1^3), \varnothing)$	$(1, -1)$	$(\varnothing, (1^2), \varnothing)$	$(1, 0, -1)$
$(2, 1^5)$	$(3, 1^5)$	$((2, 1^2), \varnothing)$	$(0, 0)$	$((1), \varnothing, \varnothing)$	$(-1, 1, 0)$
(1^7)	(1^7)	$(\varnothing, (1^3))$	$(1, -1)$	$(\varnothing, \varnothing, (1^2))$	$(1, 0, -1)$

Table 1.

4.2 Actions of quantum affine algebras on $\Lambda^{s+\frac{\infty}{2}}$. Taking the action (17)–(20) of $U_q(\widehat{\mathfrak{sl}}_n)$ and the action (21)–(24) of $U_p(\widehat{\mathfrak{sl}}_l)$ as the input, we shall define actions of $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$ on the semiinfinite wedge product $\Lambda^{s+\frac{\infty}{2}}$. First, we assign to each vector u of $\Lambda^{*+\frac{\infty}{2}} = \oplus_{s\in\mathbf{Z}}\Lambda^{s+\frac{\infty}{2}}$ a weight $\mathrm{wt}(u)$ of $\widehat{\mathfrak{sl}}_n$ and a weight $\dot{\mathrm{wt}}(v)$ of $\widehat{\mathfrak{sl}}_l$. We shall write $\mathrm{Wt}(u)$ for the sum of $\mathrm{wt}(u)$ and $\dot{\mathrm{wt}}(u)$. For $s \in \mathbf{Z}$, set

$|s\rangle := u_s \wedge u_{s-1} \wedge \cdots$. Let $\mathrm{wt}(|0\rangle) := l\Lambda_0$ and $\dot{\mathrm{wt}}(|0\rangle) := n\dot{\Lambda}_0$. Here and in what follows we put dots over the fundamental weights, fundamental roots, and the null root of $\widehat{\mathfrak{sl}}_l$ in order to distinguish them from those of $\widehat{\mathfrak{sl}}_n$. For each nonzero integer s, let

$$\mathrm{Wt}(|s\rangle) := \mathrm{Wt}(|0\rangle) + \begin{cases} -\mathrm{Wt}(u_0 \wedge u_{-1} \wedge \cdots \wedge u_{s+1}) & \text{if } s < 0, \\ \mathrm{Wt}(u_s \wedge u_{s-1} \wedge \cdots \wedge u_1) & \text{if } s > 0. \end{cases}$$

Here $\mathrm{Wt}(u_0 \wedge u_{-1} \wedge \cdots \wedge u_{s+1})$ and $\mathrm{Wt}(u_s \wedge u_{s-1} \wedge \cdots \wedge u_1)$ are defined by (19)–(20) and (23)–(24). Then for $r \le t$ and $v \in \Lambda^r$, the expression

$$\mathrm{Wt}(v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}) + \mathrm{Wt}(|s-t\rangle)$$

is independent of the choice of t. Hence the assignment

$$\mathrm{Wt}(v \wedge |s-r\rangle) := \mathrm{Wt}(v) + \mathrm{Wt}(|s-r\rangle)$$

gives a well-defined weight for the vector $v \wedge |s-r\rangle$ of $\Lambda^{s+\frac{\infty}{2}}$. Thus we have obtained a weight decomposition of $\Lambda^{s+\frac{\infty}{2}}$. It is straightforward to verify that in terms of the basis $\{|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l,\ \mathbf{s}_l \in \mathbf{Z}^l(s)\} = \{|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle \mid \boldsymbol{\lambda}_n \in \Pi^n,\ \mathbf{s}_n \in \mathbf{Z}^n(s)\}$ this decomposition looks as follows (here we use the notation of Section 2.1):

$$\mathrm{wt}(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle) = -\Delta(\mathbf{s}_l, n)\,\delta + \Lambda_{s_1} + \cdots + \Lambda_{s_l} - \sum_{i=0}^{n-1} M_i(\boldsymbol{\lambda}_l|\mathbf{s}_l, n)\alpha_i, \tag{27}$$

$$\begin{aligned} \dot{\mathrm{wt}}(|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle) = {} & -(\Delta(\mathbf{s}_l, n) + M_0(\boldsymbol{\lambda}_l|\mathbf{s}_l, n))\,\dot{\delta} \\ & + (n - s_1 + s_l)\dot{\Lambda}_0 + (s_1 - s_2)\dot{\Lambda}_1 + \cdots + (s_{l-1} - s_l)\dot{\Lambda}_{l-1}, \end{aligned} \tag{28}$$

$$\dot{\mathrm{wt}}(|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle) = -\Delta(\mathbf{s}_n, l)\,\dot{\delta} + \dot{\Lambda}_{s_1} + \cdots + \dot{\Lambda}_{s_n} - \sum_{i=0}^{l-1} M_i(\boldsymbol{\lambda}_n|\mathbf{s}_n, l)\dot{\alpha}_i, \tag{29}$$

$$\begin{aligned} \mathrm{wt}(|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle) = {} & -(\Delta(\mathbf{s}_n, l) + M_0(\boldsymbol{\lambda}_n|\mathbf{s}_n, l))\,\delta \\ & + (l - s_1 + s_n)\Lambda_0 + (s_1 - s_2)\Lambda_1 + \cdots + (s_{n-1} - s_n)\Lambda_{n-1}. \end{aligned} \tag{30}$$

Now we define actions of the Cartan parts of $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$ by

$$t_i\, u = q^{\langle \mathrm{wt}(u), h_i\rangle} u, \qquad \partial\, u = \langle \mathrm{wt}(u), \partial\rangle u \quad (i = 0, 1, \ldots, n-1), \tag{31}$$

$$\dot{t}_j\, u = p^{\langle \dot{\mathrm{wt}}(u), \dot{h}_j\rangle} u, \qquad \dot{\partial}\, u = \langle \dot{\mathrm{wt}}(u), \dot{\partial}\rangle u \quad (j = 0, 1, \ldots, l-1). \tag{32}$$

Next, we define on $\Lambda^{s+\frac{\infty}{2}}$ actions of the raising generators of $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$. Let $v \in \Lambda^r$ for some $r \in \mathbf{N}$. Lemma 3.18(i), (17), and (21) imply that the expressions

$$e_i(v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}) \wedge (t_i)^{-1}|s-t\rangle \quad (i = 0, \ldots, n-1),$$

$$\dot{e}_j(v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}) \wedge (\dot{t}_j)^{-1}|s-t\rangle \quad (j = 0, \ldots, l-1)$$

are independent of t for $t \geq r$. Hence the assignments

$$e_i(v \wedge |s-r\rangle) := e_i(v) \wedge (t_i)^{-1}|s-r\rangle \qquad (i = 0, \ldots, n-1),$$
$$\dot{e}_j(v \wedge |s-r\rangle) := \dot{e}_j(v) \wedge (\dot{t}_j)^{-1}|s-r\rangle \qquad (j = 0, \ldots, l-1)$$

determine well-defined endomorphisms of $\Lambda^{s+\frac{\infty}{2}}$.

Finally, we define actions of the lowering generators of $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_p(\widehat{\mathfrak{sl}}_l)$ on $\Lambda^{s+\frac{\infty}{2}}$. Now using Lemma 3.18(ii), (18), and (22), we can check that the expressions

$$f_i(v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}) \wedge u_{s-t} \wedge u_{s-t+1} \wedge \cdots \wedge u_{s-m+1},$$
$$\dot{f}_j(v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}) \wedge u_{s-t} \wedge u_{s-t+1} \wedge \cdots \wedge u_{s-m+1}$$
$$(i = 0, \ldots, n-1, \ j = 0, \ldots, l-1)$$

are independent of $t \leq m$ provided $t > r$ and t is sufficiently large. Hence we obtain well-defined endomorphisms of $\Lambda^{s+\frac{\infty}{2}}$ by setting

$$f_i(v \wedge |s-r\rangle) := f_i(v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}) \wedge |s-t\rangle,$$
$$\dot{f}_j(v \wedge |s-r\rangle) := \dot{f}_j(v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}) \wedge |s-t\rangle,$$

where t is arbitrary such that $t \gg r$ $(i = 0, \ldots, n-1, \ j = 0, \ldots, l-1)$.

The actions of f_i and e_i defined above are easily described in terms of the basis $B(s) = \{|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l, \ \mathbf{s}_l \in \mathbf{Z}^l(s)\}$. Using the notation of Section 2.1, we have

$$f_i|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = \sum_{\mathrm{res}_n(\boldsymbol{\mu}_l/\boldsymbol{\lambda}_l)=i} q^{N_i^>(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l|\mathbf{s}_l, n)} \, |\boldsymbol{\mu}_l, \mathbf{s}_l\rangle, \tag{33}$$

$$e_i|\boldsymbol{\mu}_l, \mathbf{s}_l\rangle = \sum_{\mathrm{res}_n(\boldsymbol{\mu}_l/\boldsymbol{\lambda}_l)=i} q^{-N_i^<(\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l|\mathbf{s}_l, n)} \, |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle. \tag{34}$$

These actions and the actions of the generators t_i and ∂ defined by (27) are identical to those defined on the combinatorial Fock space $\mathbf{F}_q[\mathbf{s}_l]$. It follows that e_i, f_i, t_i, and ∂ satisfy the defining relations of $U_q(\widehat{\mathfrak{sl}}_n)$. Moreover, we see that for each $\mathbf{s}_l = (s_1, \ldots, s_l) \in \mathbf{Z}^l$, the Fock space $\mathbf{F}_q[\mathbf{s}_l]$ is realized inside $\Lambda^{s+\frac{\infty}{2}}$ with $s = s_1 + \cdots + s_l$ as the subspace spanned by $\{|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l\}$. Note that by (28), the Fock space $\mathbf{F}_q[\mathbf{s}_l]$ is the set of all vectors $u \in \Lambda^{s+\frac{\infty}{2}}$ such that $\dot{\mathrm{wt}}(u)$ is congruent to $(n - s_1 + s_l)\dot{\Lambda}_0 + (s_1 - s_2)\dot{\Lambda}_1 + \cdots + (s_{l-1} - s_l)\dot{\Lambda}_{l-1}$ modulo $\mathbf{Z}\dot{\delta}$.

Similarly, one can describe the actions of $\dot{f}_j$ and $\dot{e}_j$ in terms of the basis $B(s)$. Now the formulas acquire a simple form if we use the indexation of $B(s)$ by $(\boldsymbol{\lambda}_n, \mathbf{s}_n)$ $(\boldsymbol{\lambda}_n \in \Pi^n, \mathbf{s}_n \in \mathbf{Z}^n(s))$. We have

$$\dot{f}_j|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle = \sum_{\mathrm{res}_l(\boldsymbol{\mu}_n/\boldsymbol{\lambda}_n)=j} p^{N_j^>(\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n|\mathbf{s}_n, l)} \, |\boldsymbol{\mu}_n, \mathbf{s}_n\rangle, \tag{35}$$

$$\dot{e}_j|\boldsymbol{\mu}_n, \mathbf{s}_n\rangle = \sum_{\mathrm{res}_l(\boldsymbol{\mu}_n/\boldsymbol{\lambda}_n)=j} p^{-N_j^<(\boldsymbol{\lambda}_n, \boldsymbol{\mu}_n|\mathbf{s}_n, l)} \, |\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle. \tag{36}$$

These actions and the actions of the generators $\dot{t}_j$ and $\dot{\partial}$ defined by (29) are identical to those defined on the combinatorial Fock space $\mathbf{F}_q[\mathbf{s}_n]$. It follows that $\dot{e}_j$, $\dot{f}_j$, $\dot{t}_j$, and $\dot{\partial}$ satisfy the defining relations of $U_p(\widehat{\mathfrak{sl}}_l)$. Moreover, for each $\mathbf{s}_n = (s_1, \dots, s_n) \in \mathbf{Z}^n$, the Fock space $\mathbf{F}_q[\mathbf{s}_n]$ is realized inside $\Lambda^{s+\frac{\infty}{2}}$ with $s = s_1 + \cdots + s_n$ as the subspace spanned by $\{|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle \mid \boldsymbol{\lambda}_n \in \Pi^n\}$. Note that by (30), the Fock space $\mathbf{F}_q[\mathbf{s}_n]$ is the set of all vectors $u \in \Lambda^{s+\frac{\infty}{2}}$ such that $\mathrm{wt}(u)$ is congruent to $(l - s_1 + s_n)\Lambda_0 + (s_1 - s_2)\Lambda_1 + \cdots + (s_{n-1} - s_n)\Lambda_{n-1}$ modulo $\mathbf{Z}\delta$.

4.3 Action of the Heisenberg algebra on $\Lambda^{s+\frac{\infty}{2}}$. Let $v \in \Lambda^r$ for some $r \in \mathbf{N}$. Lemma 3.18(i) and (25) imply that for $m > 0$ the expression

$$B_m(v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}) \wedge |s-t\rangle$$

is independent of t for $t \geq r$. Hence the assignment

$$B_m(v \wedge |s-r\rangle) := B_m(v) \wedge |s-r\rangle \quad (m > 0)$$

determines a well-defined endomorphism of $\Lambda^{s+\frac{\infty}{2}}$. Now using Lemma 3.18(ii), one can check that for $m < 0$ the expression

$$B_m(v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}) \wedge u_{s-t} \wedge u_{s-t+1} \wedge \cdots \wedge u_{s-k+1}$$

is independent of $t \leq k$ provided that $t > r$ and t is sufficiently large. Hence we obtain a well-defined endomorphism of $\Lambda^{s+\frac{\infty}{2}}$ by setting

$$B_m(v \wedge |s-r\rangle) := B_m(v \wedge u_{s-r} \wedge u_{s-r-1} \wedge \cdots \wedge u_{s-t+1}) \wedge |s-t\rangle \qquad (m < 0),$$

where t is arbitrary such that $t \gg r$. It is clear that B_m $(m \in \mathbf{N})$ has weight $m\delta + m\dot{\delta}$ with respect to the weight decomposition of $\Lambda^{s+\frac{\infty}{2}}$ defined in the previous section. This implies that the subspaces $\mathbf{F}_q[s_l]$ and $\mathbf{F}_q[\mathbf{s}_n]$ are preserved by B_m. The next proposition shows that B_m generates a Heisenberg algebra $\mathcal{H}$.

Proposition 4.4. *There are nonzero $\gamma_m \in \mathbf{K}$ (independent of s) such that*

$$[B_m, B_{m'}] = \delta_{m+m',0}\gamma_m.$$

Proof. Since for each r the actions of B_m on Λ^r commute, for any $v \in \Lambda^r$ we have

$$[B_m, B_{m'}](v \wedge |s-r\rangle) = v \wedge [B_m, B_{m'}]|s-r\rangle. \tag{37}$$

It is therefore enough to show that the statement of the proposition holds when $[B_m, B_{m'}]$ is applied to $|s\rangle$ with arbitrary $s \in \mathbf{Z}$. Let us first assume that $m + m' > 0$. Then it is easy to see from (27)–(30) that $\mathrm{Wt}(|s\rangle) + (m + m')(\delta + \dot{\delta})$ is not a weight of $\Lambda^{s+\frac{\infty}{2}}$. Hence $[B_m, B_{m'}]|s\rangle = 0$ in this case. Next, let $m + m' < 0$. Write $[B_m, B_{m'}]|s\rangle$ as a linear combination of ordered wedges

$$[B_m, B_{m'}]|s\rangle = \sum_{\nu} c_\nu u_{k_1^\nu} \wedge u_{k_2^\nu} \wedge \cdots,$$

where c_ν are nonzero coefficients. Since $\mathrm{Wt}(|s\rangle) + (m+m')(\delta+\dot{\delta})$ is distinct from the weight of $|s\rangle$, we have $k_1^\nu > s$ for all ν. For any $t > 0$, (37) gives

$$[B_m, B_{m'}]|s\rangle = u_s \wedge u_{s-1} \wedge \cdots \wedge u_{s-nlt+1} \wedge [B_m, B_{m'}]|s - nlt\rangle.$$

From the structure of the ordering rules of Proposition 3.16, it follows that

$$[B_m, B_{m'}]|s - nlt\rangle = \sum_\nu c_\nu u_{k_1^\nu - nlt} \wedge u_{k_2^\nu - nlt} \wedge \cdots .$$

Choosing t sufficiently large so that $s \geq k_1^\nu - nlt$ holds for all ν and taking into account the inequality $k_1^\nu - nlt > s - nlt$, we obtain by Lemma 3.18(ii) that $u_s \wedge u_{s-1} \wedge \cdots \wedge u_{s-nlt+1} \wedge u_{k_1^\nu} \wedge u_{k_2^\nu} \wedge \cdots$ vanishes for all ν. Hence $[B_m, B_{m'}]|s\rangle$ is zero.

Finally, let $m + m' = 0$. Then the weight of $[B_m, B_{m'}]|s\rangle$ equals $\mathrm{Wt}(|s\rangle)$. It is easy to see that the vector $|s\rangle$ coincides with $|\varnothing_l, s_l\rangle$ for a certain $s_l \in \mathbf{Z}^l(s)$. However, from (27)–(30), it is clear that the weight subspace of $\mathbf{F}_q[s_l]$ with the weight $\mathrm{Wt}(|\varnothing_l, s_l\rangle)$ is one dimensional. Since B_m preserves $\mathbf{F}_q[s_l]$, it follows that $[B_m, B_{m'}]|s\rangle = \gamma_m|s\rangle$ for some $\gamma_m \in \mathbf{K}$. Using $[B_m, B_{m'}]|s\rangle = u_s \wedge [B_m, B_{m'}]|s-1\rangle$ shows that γ_m does not depend on s. Specializing to $q = 1$, we obtain $\gamma_m|_{q=1} = mnl$. Hence $\gamma_m \neq 0$. $\square$

Proposition 4.5. *For $m > 0$, we have*

$$\gamma_m = m\frac{1-q^{-2mn}}{1-q^{-2m}}\frac{1-q^{2ml}}{1-q^{2m}}.$$

For $l = 1$, a proof of this proposition is given in [13]. We give a proof for all $l \in \mathbf{N}$ in Section 5.1.1.

Following the same reasoning that was used above to show that B_m commutes with $B_{m'}$ unless $m + m' = 0$, we obtain the following.

Proposition 4.6. *The actions of $U_q'(\widehat{\mathfrak{sl}}_n)$, $U_p'(\widehat{\mathfrak{sl}}_l)$, and $\mathcal{H}$ on $\Lambda^{s+\frac{\infty}{2}}$ are pairwise mutually commutative.*

Taking into account the weight decomposition of $\Lambda^{s+\frac{\infty}{2}}$ given by (27)–(30) and the fact that weight of B_m equals $m(\delta + \dot{\delta})$, we see that vectors $|\varnothing_l, s_l\rangle$ ($s_l \in \mathbf{Z}^l(s)$) are singular vectors of $\mathcal{H}$, i.e., are annihilated by B_m with positive m. Clearly, these vectors are also singular vectors for $U_q(\widehat{\mathfrak{sl}}_n)$. Likewise, the vectors $|\varnothing_n, s_n\rangle$ ($s_n \in \mathbf{Z}^n(s)$) are singular for both $\mathcal{H}$ and $U_p(\widehat{\mathfrak{sl}}_l)$. Define the sets $A_l^n(s)$ and $A_n^l(s)$ by

$$A_l^n(s) := \{s_l = (s_1, \ldots, s_l) \in \mathbf{Z}^l(s) \mid s_1 \geq s_2 \geq \cdots \geq s_l,\ s_1 - s_l \leq n\},$$
$$A_n^l(s) := \{s_n = (s_1, \ldots, s_n) \in \mathbf{Z}^n(s) \mid s_1 \geq s_2 \geq \cdots \geq s_n,\ s_1 - s_n \leq l\}.$$

The definitions of $|\boldsymbol{\lambda}_l, s_l\rangle$ and $|\boldsymbol{\lambda}_n, s_n\rangle$ given in Section 4.1 imply that

$$\{|\varnothing_l, s_l\rangle \mid s_l \in A_l^n(s)\} = \{|\varnothing_n, s_n\rangle \mid s_n \in A_n^l(s)\}. \tag{38}$$

Hence $|\varnothing_l, s_l\rangle$ ($s_l \in A_l^n(s)$) or, equivalently, $|\varnothing_n, s_n\rangle$ ($s_n \in A_n^l(s)$) are the only vectors of the basis $B(s)$ that are simultaneously singular for $U_q(\widehat{\mathfrak{sl}}_n)$, $U_p(\widehat{\mathfrak{sl}}_l)$ and $\mathcal{H}$. Equality (38) shows that we have a bijection $A_l^n(s) \to A_n^l(s)$ such that

$$s_l \mapsto s_n \quad \text{if and only if} \quad |\varnothing_l, s_l\rangle = |\varnothing_n, s_n\rangle.$$

This bijection is completely determined by comparing the weights of $|\varnothing_l, s_l\rangle$ and $|\varnothing_n, s_n\rangle$ according to (27)–(30). Namely, $(t_1, \ldots, t_n) \in A_n^l(s)$ is the image of $(s_1, \ldots, s_l) \in A_l^n(s)$ if and only if

$$\Lambda_{s_1} + \cdots + \Lambda_{s_l} = (l - t_1 + t_n)\Lambda_0 + (t_1 - t_2)\Lambda_1 + \cdots + (t_{n-1} - t_n)\Lambda_{n-1}$$

or, equivalently, if and only if

$$\dot{\Lambda}_{t_1} + \cdots + \dot{\Lambda}_{t_n} = (n - s_1 + s_l)\dot{\Lambda}_0 + (s_1 - s_2)\dot{\Lambda}_1 + \cdots + (s_{l-1} - s_l)\dot{\Lambda}_{l-1}.$$

Example 4.7. Let $n = 5$, $l = 2$, and $s = 11$. Then $A_2^5(11)$ contains three elements: $(6, 5)$, $(7, 4)$, and $(8, 3)$. On the other hand, $A_5^2(11)$ is formed by the elements

$$(3, 2, 2, 2, 2), (3, 3, 2, 2, 1), (3, 3, 3, 1, 1).$$

The bijective correspondence between $A_2^5(11)$ and $A_5^2(11)$ is given by

$$\begin{aligned}
&|\varnothing_2, (6,5)\rangle = |\varnothing_5, (3,2,2,2,2)\rangle, && -(\delta + \dot{\delta}) + \Lambda_0 + \Lambda_1 + 4\dot{\Lambda}_0 + \dot{\Lambda}_1,\\
&|\varnothing_2, (7,4)\rangle = |\varnothing_5, (3,3,2,2,1)\rangle, && -2(\delta + \dot{\delta}) + \Lambda_2 + \Lambda_4 + 2\dot{\Lambda}_0 + 3\dot{\Lambda}_1,\\
&|\varnothing_2, (8,3)\rangle = |\varnothing_5, (3,3,3,1,1)\rangle, && -3(\delta + \dot{\delta}) + 2\Lambda_3 + 5\dot{\Lambda}_1.
\end{aligned}$$

Here we have listed the weights of the corresponding vectors in the right column.

The next theorem shows that $\{|\varnothing_l, s_l\rangle \mid s_l \in A_l^n(s)\} = \{|\varnothing_n, s_n\rangle \mid s_n \in A_n^l(s)\}$ is the complete set of singular vectors in $\Lambda^{s+\frac{\infty}{2}}$. A proof follows immediately from [6, Theorem 1.6] (see also [7, Theorem 3.2]).

Theorem 4.8.

$$\Lambda^{s+\frac{\infty}{2}} = \bigoplus_{s_l \in A_l^n(s)} U_q'(\widehat{\mathfrak{sl}}_n) \cdot \mathcal{H} \cdot U_p'(\widehat{\mathfrak{sl}}_l)\, |\varnothing_l, s_l\rangle;$$

equivalently,

$$\Lambda^{s+\frac{\infty}{2}} = \bigoplus_{s_n \in A_n^l(s)} U_q'(\widehat{\mathfrak{sl}}_n) \cdot \mathcal{H} \cdot U_p'(\widehat{\mathfrak{sl}}_l)\, |\varnothing_n, s_n\rangle.$$

Corollary 4.9. (i) *For $t_l = (t_1, \ldots, t_l) \in A_l^n(s)$, let $s_l = (s_1, \ldots, s_l)$ be any element of $\mathbf{Z}^l(s)$ such that $\Lambda_{s_1} + \cdots + \Lambda_{s_l} = \Lambda_{t_1} + \cdots + \Lambda_{t_l}$. Then $|\varnothing_l, t_l\rangle \in U_p'(\widehat{\mathfrak{sl}}_l)|\varnothing_l, s_l\rangle$.*

(ii) *For $t_n = (t_1, \ldots, t_n) \in A_n^l(s)$, let $s_n = (s_1, \ldots, s_n)$ be any element of $\mathbf{Z}^n(s)$ such that $\dot{\Lambda}_{s_1} + \cdots + \dot{\Lambda}_{s_n} = \dot{\Lambda}_{t_1} + \cdots + \dot{\Lambda}_{t_n}$.*

Then $|\varnothing_n, t_n\rangle \in U_q'(\widehat{\mathfrak{sl}}_n)|\varnothing_n, s_n\rangle$.

PROOF. Since the proofs of (i) and (ii) are almost identical, we show only (i). Since $|\emptyset_l, s_l\rangle$ is a singular vector for $U_q(\mathfrak{sl}_n) \cdot \mathcal{H}$, we have, by Theorem 4.8,

$$|\emptyset_l, s_l\rangle \in \bigoplus_{r_l \in A_l^n(s)} U_p'(\widehat{\mathfrak{sl}}_l)\,|\emptyset_l, r_l\rangle.$$

Observe that for two distinct elements r_l and t_l of $A_l^n(s)$, we have $\Lambda_{r_1} + \cdots + \Lambda_{r_l} \neq \Lambda_{t_1} + \cdots + \Lambda_{t_l}$. Therefore, comparing $\widehat{\mathfrak{sl}}_n$-weights, we have

$$|\emptyset_l, s_l\rangle \in U_p'(\widehat{\mathfrak{sl}}_l)\,|\emptyset_l, t_l\rangle.$$

Since $U_p'(\widehat{\mathfrak{sl}}_l)\,|\emptyset_l, t_l\rangle$ is an irreducible representation of $U_p'(\widehat{\mathfrak{sl}}_l)$, the claim follows. □

4.4 Canonical bases of the q-deformed Fock spaces. Fix an arbitrary integer s and define a gradation of the semiinfinite wedge product $\Lambda^{s+\frac{\infty}{2}}$by setting $\deg|\lambda, s\rangle = |\lambda|$.

Lemma 4.10. *Let $k = (k_1, k_2, \ldots) \in \mathbf{P}^{++}(s)$. Then for any $t, r \in \mathbf{N}$ such that $t > r \geq \deg u_k$, we have*

$$\overline{u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_r} \wedge u_{k_{r+1}} \wedge \cdots \wedge u_{k_t}} = \overline{u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_r}} \wedge u_{k_{r+1}} \wedge \cdots \wedge u_{k_t}.$$

For $l = 1$, a proof of this lemma is given in [18, proof of Lemma 7.7]. For arbitrary l, a proof is virtually identical and will be omitted.

From this lemma, it follows that for $k = (k_1, k_2, \ldots) \in \mathbf{P}^{++}(s)$, the assignment

$$\overline{u_k} := \overline{u_{k_1} \wedge u_{k_2} \wedge \cdots \wedge u_{k_r}} \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \cdots \quad (r \geq \deg u_k) \tag{39}$$

determines a well-defined semilinear involution $u \mapsto \overline{u}$ of $\Lambda^{s+\frac{\infty}{2}}$. It is easily seen from the weight decomposition of $\Lambda^{s+\frac{\infty}{2}}$ defined in Section 4.2 that $\mathrm{Wt}(\overline{u}) = \mathrm{Wt}(u)$ for any weight vector u of $\Lambda^{s+\frac{\infty}{2}}$. Hence for $s_l \in \mathbf{Z}^l(s)$ (resp., $s_n \in \mathbf{Z}^n(s)$), the Fock space $\mathbf{F}_q[s_l]$ (resp., $\mathbf{F}_q[s_n]$) is invariant with respect to the bar-involution. Therefore, in particular, we have

$$\overline{|\lambda_l, s_l\rangle} = \sum_{\mu_l \in \Pi^l} R_{\lambda_l,\mu_l}(s_l|q)\,|\mu_l, s_l\rangle,$$

where $R_{\lambda_l,\mu_l}(s_l|q)$ is a Laurent polynomial in q with integral coefficients. From (27) and the fact that the involution preserves the weight subspaces of $\Lambda^{s+\frac{\infty}{2}}$, it follows that $R_{\lambda_l,\mu_l}(s_l|q)$ is nonzero only if $|\lambda_l| = |\mu_l|$.

For $\lambda_l, \mu_l \in \Pi^l$ and $s_l \in \mathbf{Z}^l(s)$, let $k = (k_1, k_2, \ldots)$ and $l = (l_1, l_2, \ldots)$ be the unique elements of $\mathbf{P}^{++}(s)$ such that $|\lambda_l, s_l\rangle = u_k$ and $|\mu_l, s_l\rangle = u_l$. Then (39) implies that

$$R_{\lambda_l,\mu_l}(s_l|q) = R_{(k)_r,(l)_r}(q), \tag{40}$$

where $(k)_r := (k_1, k_2, \dots, k_r)$, $(l)_r := (l_1, l_2, \dots, l_r)$, and r is an arbitrary integer satisfying $r \geq \deg u_k, \deg u_l$. Here the coefficient $R_{(k)_r,(l)_r}(q)$ is defined in Section 3.4. The unitriangularity of the matrix $\|R_{k,l}(q)\|$ $(k, l \in \mathbf{P}^{++})$ described in that section immediately leads to the following.

Proposition 4.11. *For $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l$ and $s_l \in \mathbf{Z}^l(s)$, the coefficient $R_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q)$ is zero unless the partition $\lambda = (\tau_l^s)^{-1}(\boldsymbol{\lambda}_l, s_l)$ is greater than or equal to the partition $\mu = (\tau_l^s)^{-1}(\boldsymbol{\mu}_l, s_l)$ with respect to the dominance order on partitions. Moreover, $R_{\boldsymbol{\lambda}_l,\boldsymbol{\lambda}_l}(s_l|q) = 1$.*

The unitriangularity of the involution matrix $\|R_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q)\|$ allows us to define canonical bases $\{\mathcal{G}^+(\boldsymbol{\lambda}_l, s_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ and $\{\mathcal{G}^-(\boldsymbol{\lambda}_l, s_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ of the Fock space $\mathbf{F}_q[s_l]$ for arbitrary $s_l \in \mathbf{Z}^l(s)$. These bases are characterized by

$$\text{(i)}\quad \overline{\mathcal{G}^+(\boldsymbol{\lambda}_l, s_l)} = \mathcal{G}^+(\boldsymbol{\lambda}_l, s_l), \qquad \overline{\mathcal{G}^-(\boldsymbol{\lambda}_l, s_l)} = \mathcal{G}^-(\boldsymbol{\lambda}_l, s_l),$$
$$\text{(ii)}\quad \mathcal{G}^+(\boldsymbol{\lambda}_l, s_l) \equiv |\boldsymbol{\lambda}_l, s_l\rangle \bmod q\mathcal{L}^+(s), \qquad \mathcal{G}^-(\boldsymbol{\lambda}_l, s_l) \equiv |\boldsymbol{\lambda}_l, s_l\rangle \bmod q^{-1}\mathcal{L}^-(s),$$

where $\mathcal{L}^+(s)$ (resp., $\mathcal{L}^-(s)$) is the $\mathbf{Q}[q]$-lattice (resp., $\mathbf{Q}[q^{-1}]$-lattice) of $\Lambda^{s+\frac{\infty}{2}}$ generated by the basis

$$\begin{aligned} B(s) &= \{|\lambda, s\rangle \mid \lambda \in \Pi\} \\ &= \{|\boldsymbol{\lambda}_l, s_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l,\ s_l \in \mathbf{Z}^l(s)\} \\ &= \{|\boldsymbol{\lambda}_n, s_n\rangle \mid \boldsymbol{\lambda}_n \in \Pi^n,\ s_n \in \mathbf{Z}^n(s)\}. \end{aligned}$$

Set

$$\begin{aligned} \mathcal{G}^+(\boldsymbol{\lambda}_l, s_l) &= \sum_{\boldsymbol{\mu}_l \in \Pi^l} \Delta^+_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q)\, |\boldsymbol{\mu}_l, s_l\rangle, \\ \mathcal{G}^-(\boldsymbol{\lambda}_l, s_l) &= \sum_{\boldsymbol{\mu}_l \in \Pi^l} \Delta^-_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q)\, |\boldsymbol{\mu}_l, s_l\rangle. \end{aligned}$$

Then, keeping notation as in (40), we have

$$\Delta^+_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q) = \Delta^+_{(k)_r,(l)_r}(q), \qquad \Delta^-_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q) = \Delta^-_{(k)_r,(l)_r}(q),$$

where the matrices $\|\Delta^\pm_{k,l}(q)\|$ $(k, l \in \mathbf{P}^{++})$ are defined in Section 3.4. Hence Theorem 3.26 shows that $\Delta^\pm_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q)$ are parabolic Kazhdan–Lusztig polynomials. Note that $R_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q) \neq 0$ only if $|\boldsymbol{\lambda}_l| = |\boldsymbol{\mu}_l|$ implies that $\Delta^\pm_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q) \neq 0$ only if $|\boldsymbol{\lambda}_l| = |\boldsymbol{\mu}_l|$. For each nonnegative integer k, let us set

$$\|\Delta^\pm_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q)\|_k = \|\Delta^\pm_{\boldsymbol{\lambda}_l,\boldsymbol{\mu}_l}(s_l|q)\| \quad (|\boldsymbol{\lambda}_l| = |\boldsymbol{\mu}_l| = k).$$

A proof of the next proposition in the special case $l = 1$ is given in [18]. A proof of the general case is similar and will be omitted.

Proposition 4.12. *For each $u \in \Lambda^{s+\frac{\infty}{2}}$, one has*

$$\overline{e_i u} = e_i \overline{u}, \qquad \overline{f_i u} = f_i \overline{u} \qquad (i \in \{0, 1, \ldots, n-1\}),$$
$$\overline{\dot{e}_j u} = \dot{e}_j \overline{u}, \qquad \overline{\dot{f}_j u} = \dot{f}_j \overline{u} \qquad (j \in \{0, 1, \ldots, l-1\}),$$
$$\overline{B_{-m} u} = B_{-m} \overline{u}, \qquad \overline{B_m u} = q^{2m(n-l)} B_m \overline{u} \quad (m > 0).$$

Let us now describe how the canonical bases relate to the global crystal bases of Kashiwara. As in Section 2.2, let $\mathcal{L}[s_l] = \oplus_{\lambda_l \in \Pi^l} A\,|\lambda_l, s_l\rangle$ be the lower crystal lattice of $\mathbf{F}_q[s_l]$ at $q = 0$. Proposition 4.12 then implies that $\overline{\mathcal{L}[s_l]} = \oplus_{\lambda_l \in \Pi^l} \overline{A}\ \overline{|\lambda_l, s_l\rangle}$, where $\overline{A} \subset \mathbf{Q}(q)$ is the subring of rational functions regular at $q = \infty$, is a lower crystal lattice of $\mathbf{F}_q[s_l]$ at $q = \infty$ (cf. [11, 12]). Let $U_q(\widehat{\mathfrak{sl}}_n)_{\mathbf{Q}}$ be the $\mathbf{Q}[q, q^{-1}]$-subalgebra of $U_q(\widehat{\mathfrak{sl}}_n)$ generated by the q-divided differences $e_i^{(m)} and f_i^{(m)}$ and

$$\prod_{k=1}^{m} \frac{q^{1-k} t_i - (t_i)^{-1} q^{k-1}}{q^k - q^{-k}}$$

with $m \in \mathbf{N}$. One can show [3, Lemma 2.7] that

$$\mathbf{F}_q[s_l]_{\mathbf{Q}} = \oplus_{\lambda_l \in \Pi^l} \mathbf{Q}[q, q^{-1}]\,|\lambda_l, s_l\rangle$$

is invariant with respect to the action of $U_q(\widehat{\mathfrak{sl}}_n)_{\mathbf{Q}}$ on $\mathbf{F}_q[s_l]$.

The existence and uniqueness of the basis $\{\mathcal{G}^+(\lambda_l, s_l) \mid \lambda_l \in \Pi^l\}$ can, by using the unitriangularity of the bar-involution, be reformulated as the existence of an isomorphism

$$\mathbf{F}_q[s_l]_{\mathbf{Q}} \cap \mathcal{L}[s_l] \cap \overline{\mathcal{L}[s_l]} \xrightarrow{\sim} \mathcal{L}[s_l]/q\mathcal{L}[s_l]$$

such that the preimage of $|\lambda_l, s_l\rangle$ mod $q\mathcal{L}[s_l]$ is $\mathcal{G}^+(\lambda_l, s_l)$. Therefore, in the terminology of [11, 12],

$$\{\mathcal{G}^+(\lambda_l, s_l) \mid \lambda_l \in \Pi^l\}$$

is a lower global crystal basis of the integrable $U_q(\widehat{\mathfrak{sl}}_n)$-module $\mathbf{F}_q[s_l]$.

Now let us use the indexation of the basis $B(s)$ by pairs (λ_n, s_n) with $\lambda_n \in \Pi^n$ and $s_n \in \mathbf{Z}^n(s)$. Certainly, we may label the canonical bases by these pairs as well so that

$$\{\mathcal{G}^\pm(\lambda_n, s_n) \mid \lambda_n \in \Pi^n,\ s_n \in \mathbf{Z}^n(s)\} = \{\mathcal{G}^\pm(\lambda_l, s_l) \mid \lambda_l \in \Pi^l,\ s_l \in \mathbf{Z}^l(s)\}$$

and $\mathcal{G}^\pm(\lambda_n, s_n)$ are congruent to $|\lambda_n, s_n\rangle$ modulo $q^{\pm 1}\mathcal{L}^\pm(s)$. Comparing (35)–(36) with (33)–(34) and taking into account Theorem 2.4, we see that $\mathcal{L}[s_n] = \oplus_{\lambda_n \in \Pi^n} \overline{A}\ |\lambda_n, s_n\rangle$ is a lower crystal lattice of the $U_p(\widehat{\mathfrak{sl}}_l)$-module $\mathbf{F}_q[s_n]$ at $p := -q^{-1} = 0$. Then by Proposition 4.12 again, $\overline{\mathcal{L}[s_n]} = \oplus_{\lambda_n \in \Pi^n} A\ \overline{|\lambda_n, s_n\rangle}$ is a lower crystal lattice of $\mathbf{F}_q[s_n]$ at $p = \infty$, and the existence and uniqueness of the basis $\{\mathcal{G}^-(\lambda_n, s_n) \mid \lambda_n \in \Pi^n\}$ imply that there is an isomorphism

$$\mathbf{F}_q[s_n]_{\mathbf{Q}} \cap \mathcal{L}[s_n] \cap \overline{\mathcal{L}[s_n]} \xrightarrow{\sim} \mathcal{L}[s_n]/p\mathcal{L}[s_n]$$

taking $\mathcal{G}^-(\lambda_n, s_n)$ to $|\lambda_n, s_n\rangle$ mod $p\mathcal{L}[s_n]$. Therefore,

$$\{\mathcal{G}^-(\lambda_n, s_n) \mid \lambda_n \in \Pi^n\}$$

is a lower global crystal basis of the integrable $U_p(\widehat{\mathfrak{sl}}_l)$-module $\mathbf{F}_q[s_n]$.

Let us now comment on how the canonical basis $\{\mathcal{G}^+(\lambda_l, s_l) \mid \lambda_l \in \Pi^l\}$ is related to the lower global crystal basis $\{\mathcal{G}(\lambda_l, s_l) \mid \lambda_l \in \Pi^l(s_l)\}$ of the irreducible $U_q(\widehat{\mathfrak{sl}}_n)$-submodule $\mathbf{F}_q[s_l]$ generated by $|\varnothing_l, s_l\rangle$ (cf. Theorem 2.5). Using [3, Lemma 2.7], one can show that the rational form $\mathbf{F}_q[s_l]_{\mathbf{Q}}$ of $\mathbf{F}_q[s_l]$ belongs to $\mathbf{F}_q[s_l]_{\mathbf{Q}}$. From the definition of $\mathcal{G}(\lambda_l, s_l)$, it now follows that $\mathcal{G}(\lambda_l, s_l)$ belongs to $\mathcal{L}^+(s)$ and hence has the same congruence property with respect to $\mathcal{L}^+(s)$ as does $\mathcal{G}^+(\lambda_l, s_l)$. On the other hand, by Proposition 4.12, the restriction of the bar-involution on $\mathbf{F}_q[s_l]$ coincides with the involution of $\mathbf{F}_q[s_l]$ defined in Section 2.2. By the uniqueness of $\mathcal{G}^+(\lambda_l, s_l)$, it now follows that $\mathcal{G}^+(\lambda_l, s_l) = \mathcal{G}(\lambda_l, s_l)$ for all $\lambda_l \in \Pi^l(s_l)$. Hence

$$\{\mathcal{G}^+(\lambda_l, s_l) \mid \lambda_l \in \Pi^l(s_l)\}$$

is the lower global crystal basis of the irreducible $U_q(\widehat{\mathfrak{sl}}_n)$-submodule $\mathbf{F}_q[s_l]$.

For $s_n \in \mathbf{Z}^n$, let $\mathbf{F}_q[s_n]$ be the irreducible $U_p(\widehat{\mathfrak{sl}}_l)$-submodule of $\mathbf{F}_q[s_n]$ generated by the highest-weight vector $|\varnothing_n, s_n\rangle$. By the same argument as above, we conclude that

$$\{\mathcal{G}^-(\lambda_n, s_n) \mid \lambda_n \in \Pi^n(s_n)\}$$

is the lower global crystal basis of $\mathbf{F}_q[s_n]$.

Note that the involution matrix $R_{\lambda_l,\mu_l}(s_l|q)$, because of its unitriangularity, can be computed by using the ordering rules of Proposition 3.16. Therefore, we have an algorithm for computation of the transition matrices $\|\Delta^{\pm}_{\lambda_l,\mu_l}(s_l|q)\|$. By the deep result of [2], the coefficients $\Delta^{+}_{\lambda_l,\mu_l}(s_l|1)$ for $\lambda_l \in \Pi^l(s_l)$, $\mu_l \in \Pi^l$ are identified with the decomposition numbers of Specht modules for an Ariki–Koike algebra and hence are nonnegative integers. Tables of the transition matrices suggest that for all $\lambda_l, \mu_l \in \Pi^l$ the entries $\Delta^{+}_{\lambda_l,\mu_l}(s_l|q)$ are in $\mathbf{Z}_{\geq 0}[q]$ (and those of $\|\Delta^{-}_{\lambda_l,\mu_l}(s_l|q)\|$ are in $\mathbf{Z}_{\geq 0}[p]$).

5 An inversion formula for Kazhdan–Lusztig polynomials

The aim of this section is to prove Theorem 5.15, which gives an inversion formula relating the matrices $\|\Delta^{+}_{\lambda_l,\mu_l}(s_l|q)\|$ with $\|\Delta^{-}_{\lambda_l,\mu_l}(s_l|q)\|$. In the $l = 1$ case, this formula has already been proved by Leclerc and Thibon in [18].

5.1 Some properties of the Heisenberg algebra action on $\Lambda^{s+\frac{\infty}{2}}$.

Definition 5.1. Let $m \in \mathbf{Z}_{\geq 0}$. We shall say that a pair $(\lambda_r = (\lambda^{(1)}, \dots, \lambda^{(r)}), s_r = (s_1, \dots, s_r)) \in \Pi^r \times \mathbf{Z}^r$ is *m-dominant* if for all $a = 1, 2, \dots, r-1$ we have the inequalities

$$s_a - s_{a+1} \geq m + |\lambda_r|,$$

where $|\boldsymbol{\lambda}_r| = |\lambda^{(1)}| + \cdots + |\lambda^{(r)}|$.

Also, we shall say that a basis vector $|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ (resp., $|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle$) is m-dominant if the pair $(\boldsymbol{\lambda}_l, \mathbf{s}_l)$ (resp., $(\boldsymbol{\lambda}_n, \mathbf{s}_n)$) is. To explain the reason for introducing this definition, we need to present some notation. Let $n \in \mathbf{N}$, $l = 1$. Let x be a linear operator on $\Lambda^{s+\frac{\infty}{2}}$ acting on the elements of the basis $B(s)$ by

$$x\,|\lambda, s\rangle = \sum_{\mu \in \Pi} x_{\lambda,\mu}(s)\,|\mu, s\rangle,$$

where $x_{\lambda,\mu}(s)$ are coefficients in $\mathbf{K}$. Now let $n \in \mathbf{N}$, $l \in \mathbf{N}$. For each $b = 1, 2, \ldots, l$, we define an endomorphism $x^{(b)}[n, 1]$ of $\Lambda^{s+\frac{\infty}{2}}$ by

$$x^{(b)}[n, 1]\,|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = \sum_{\mu \in \Pi} x_{\lambda^{(b)},\mu}(s_b)\,|(\lambda^{(1)}, \ldots, \lambda^{(b-1)}, \mu, \lambda^{(b+1)}, \ldots, \lambda^{(l)}), \mathbf{s}_l\rangle.$$

Similarly, let $n = 1$ and $l \in \mathbf{N}$. For an endomorphism y of $\Lambda^{s+\frac{\infty}{2}}$, we introduce the corresponding matrix elements $y_{\lambda,\mu}(s)$ on the basis $B(s)$ by

$$y\,|\lambda, s\rangle = \sum_{\mu \in \Pi} y_{\lambda,\mu}(s)\,|\mu, s\rangle.$$

Again, for arbitrary $n \in \mathbf{N}$ and $l \in \mathbf{N}$, we define for each $a = 1, 2, \ldots, n$ an endomorphism $y^{(a)}[1, l]$ of $\Lambda^{s+\frac{\infty}{2}}$ by

$$y^{(a)}[1, l]\,|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle = \sum_{\mu \in \Pi} y_{\lambda^{(a)},\mu}(s_a)\,|(\lambda^{(1)}, \ldots, \lambda^{(a-1)}, \mu, \lambda^{(a+1)}, \ldots, \lambda^{(n)}), \mathbf{s}_n\rangle.$$

Example 5.2. For $n = 2$, $l = 1$, and $s \in 2\mathbf{Z}$, one finds using the ordering rules of Proposition 3.16 that

$$\begin{aligned} B_{-2}|\varnothing, s\rangle &= |(4), s\rangle - q^{-1}|(3,1), s\rangle + (q^{-2} - 1)|(2^2), s\rangle \\ &\quad + q^{-1}|(2, 1^2), s\rangle - q^{-2}|(1^4), s\rangle. \end{aligned}$$

Hence for $n = 2$ and $l = 2$, taking $\mathbf{s}_l = (s_1, s_2)$ such that $s_1, s_2 \in 2\mathbf{Z}$, we have

$$\begin{aligned} B^{(1)}_{-2}[2, 1]\,|\boldsymbol{\varnothing}_l, \mathbf{s}_l\rangle &= |((4), \varnothing), \mathbf{s}_l\rangle - q^{-1}|((3,1), \varnothing), \mathbf{s}_l\rangle \\ &\quad + (q^{-2} - 1)|((2^2), \varnothing), \mathbf{s}_l\rangle \\ &\quad + q^{-1}|((2, 1^2), \varnothing), \mathbf{s}_l\rangle - q^{-2}|((1^4), \varnothing), \mathbf{s}_l\rangle, \\ B^{(2)}_{-2}[2, 1]\,|\boldsymbol{\varnothing}_l, \mathbf{s}_l\rangle &= |(\varnothing, (4)), \mathbf{s}_l\rangle - q^{-1}|(\varnothing, (3,1)), \mathbf{s}_l\rangle \\ &\quad + (q^{-2} - 1)|(\varnothing, (2^2)), \mathbf{s}_l\rangle \\ &\quad + q^{-1}|(\varnothing, (2, 1^2)), \mathbf{s}_l\rangle - q^{-2}|(\varnothing, (1^4)), \mathbf{s}_l\rangle. \end{aligned}$$

Proposition 5.3.

(i) *Let* $(\boldsymbol{\lambda}_l, \mathbf{s}_l) \in \Pi^l \times \mathbf{Z}^l$ *be* nm*-dominant for some* $m \in \mathbf{N}$. *Then*

$$B_{-m} \, |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = \sum_{b=1}^{l} q^{(b-1)m} B_{-m}^{(b)}[n, 1] \, |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle.$$

(ii) *Let* $(\boldsymbol{\lambda}_l, \mathbf{s}_l) \in \Pi^l \times \mathbf{Z}^l$ *be* 0*-dominant. Then for any* $m \in \mathbf{N}$,

$$B_{m} \, |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle = \sum_{b=1}^{l} q^{(b-1)m} B_{m}^{(b)}[n, 1] \, |\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle.$$

(i′) *Let* $(\boldsymbol{\lambda}_n, \mathbf{s}_n) \in \Pi^n \times \mathbf{Z}^n$ *be* lm*-dominant for some* $m \in \mathbf{N}$. *Then*

$$B_{-m} \, |\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle = \sum_{a=1}^{n} p^{(a-1)m} B_{-m}^{(a)}[1, l] \, |\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle.$$

(ii′) *Let* $(\boldsymbol{\lambda}_n, \mathbf{s}_n) \in \Pi^n \times \mathbf{Z}^n$ *be* 0*-dominant. Then for any* $m \in \mathbf{N}$,

$$B_{m} \, |\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle = \sum_{a=1}^{n} p^{(a-1)m} B_{m}^{(a)}[1, l] \, |\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle.$$

A proof of this proposition is given in Section 5.4.

Example 5.4. To illustrate Proposition 5.3, take $n = 2$, $l = 2$, and $\mathbf{s}_l = (2, -2)$. A straightforward computation using Proposition 3.16 gives

$$\begin{aligned} B_{-2} \, |\varnothing_l, \mathbf{s}_l\rangle = {} & |((4), \varnothing), \mathbf{s}_l\rangle - q^{-1}|((3,1), \varnothing), \mathbf{s}_l\rangle + (q^{-2} - 1)|((2^2), \varnothing), \mathbf{s}_l\rangle \\ & + q^{-1}|((2, 1^2), \varnothing), \mathbf{s}_l\rangle - q^{-2}|((1^4), \varnothing), \mathbf{s}_l\rangle \\ & + q^2|(\varnothing, (4)), \mathbf{s}_l\rangle - q|(\varnothing, (3,1)), \mathbf{s}_l\rangle + (1 - q^2)|(\varnothing, (2^2)), \mathbf{s}_l\rangle \\ & + q|(\varnothing, (2, 1^2)), \mathbf{s}_l\rangle - |(\varnothing, (1^4)), \mathbf{s}_l\rangle. \end{aligned}$$

The pair $(\varnothing_l, \mathbf{s}_l)$ is 4-dominant. Taking into account the formulas of Example 5.2, we see that the relation

$$B_{-2} \, |\varnothing_l, \mathbf{s}_l\rangle = B_{-2}^{(1)}[2, 1] \, |\varnothing_l, \mathbf{s}_l\rangle + q^2 B_{-2}^{(2)}[2, 1] \, |\varnothing_l, \mathbf{s}_l\rangle$$

is indeed satisfied.

Remark 5.5. Simple decompositions for the actions of the bosons described in Proposition 5.3 fail to hold in general when bosons are applied to vectors that are not dominant. For example, let $n = l = 2$ and $\mathbf{s}_l = (0, 0)$. Then the pair $(\varnothing_l, \mathbf{s}_l)$ is not m-dominant for any $m \in \mathbf{N}$. In this case, an explicit computation yields

$$\begin{aligned} B_{-2} \, |\varnothing_l, \mathbf{s}_l\rangle = {} & q|((4), \varnothing), \mathbf{s}_l\rangle - |((3,1), \varnothing), \mathbf{s}_l\rangle + (1 - q^2)|((2^2), \varnothing), \mathbf{s}_l\rangle \\ & + q|((2, 1^2), \varnothing), \mathbf{s}_l\rangle - |((1^4), \varnothing), \mathbf{s}_l\rangle \\ & + |(\varnothing, (4)), \mathbf{s}_l\rangle - q^{-1}|(\varnothing, (3,1)), \mathbf{s}_l\rangle + (q^{-2} - 1)|(\varnothing, (2^2)), \mathbf{s}_l\rangle \\ & + |(\varnothing, (2, 1^2)), \mathbf{s}_l\rangle - q^{-1}|(\varnothing, (1^4)), \mathbf{s}_l\rangle \\ & + (q^2 - 1)|((2), (2)), \mathbf{s}_l\rangle + (q^{-1} - q)|((1), (2, 1)), \mathbf{s}_l\rangle \\ & + (q^{-1} - q)|((2, 1), (1)), \mathbf{s}_l\rangle + (1 - q^{-2})|((1^2), (1^2)), \mathbf{s}_l\rangle. \end{aligned}$$

For $m \in \mathbf{Z}$, let e_m and h_m be, respectively, the elementary symmetric function and the complete symmetric function (cf. [19]). In terms of the power-sum basis of the ring of symmetric functions, one has

$$e_m = e_m(p_1, p_2, \dots) = \sum_{\nu \in \Pi, |\nu| = m} a_{m,\nu}\, p_\nu,$$
$$h_m = h_m(p_1, p_2, \dots) = \sum_{\nu \in \Pi, |\nu| = m} b_{m,\nu}\, p_\nu,$$

where, as usual, we set $p_\nu = p_{\nu_1} p_{\nu_2} \cdots$ for a partition $\nu = (\nu_1, \nu_2, \dots)$. It will be understood that $e_0 = h_0 = 1$ and $e_m = h_m = 0$ for $m < 0$. Now, for all integers m, we define

$$E_m := e_m(B_1, B_2, \dots), \qquad \widetilde{E}_m := e_m(B_{-1}, B_{-2}, \dots),$$
$$H_m := h_m(B_1, B_2, \dots), \qquad \widetilde{H}_m := h_m(B_{-1}, B_{-2}, \dots).$$

Corollary 5.6.

(i) *Let* $(\boldsymbol{\lambda}_l, s_l) \in \Pi^l \times \mathbf{Z}^l$ *be nm-dominant for some* $m \in \mathbf{N}$. *Then*

$$\widetilde{E}_m\, |\boldsymbol{\lambda}_l, s_l\rangle = \sum_{m_1 + \cdots + m_l = m} \prod_{b=1}^{l} q^{(b-1)m_b}\, \widetilde{E}^{(b)}_{m_b}[n,1]\, |\boldsymbol{\lambda}_l, s_l\rangle,$$
$$\widetilde{H}_m\, |\boldsymbol{\lambda}_l, s_l\rangle = \sum_{m_1 + \cdots + m_l = m} \prod_{b=1}^{l} q^{(b-1)m_b}\, \widetilde{H}^{(b)}_{m_b}[n,1]\, |\boldsymbol{\lambda}_l, s_l\rangle.$$

(ii) *Let* $(\boldsymbol{\lambda}_l, s_l) \in \Pi^l \times \mathbf{Z}^l$ *be* 0*-dominant. Then for any* $m \in \mathbf{N}$,

$$E_m\, |\boldsymbol{\lambda}_l, s_l\rangle = \sum_{m_1 + \cdots + m_l = m} \prod_{b=1}^{l} q^{(b-1)m_b}\, E^{(b)}_{m_b}[n,1]\, |\boldsymbol{\lambda}_l, s_l\rangle,$$
$$H_m\, |\boldsymbol{\lambda}_l, s_l\rangle = \sum_{m_1 + \cdots + m_l = m} \prod_{b=1}^{l} q^{(b-1)m_b}\, H^{(b)}_{m_b}[n,1]\, |\boldsymbol{\lambda}_l, s_l\rangle.$$

(i′) *Let* $(\boldsymbol{\lambda}_n, s_n) \in \Pi^n \times \mathbf{Z}^n$ *be lm-dominant for some* $m \in \mathbf{N}$. *Then*

$$\widetilde{E}_m\, |\boldsymbol{\lambda}_n, s_n\rangle = \sum_{m_1 + \cdots + m_n = m} \prod_{a=1}^{n} p^{(a-1)m_a}\, \widetilde{E}^{(a)}_{m_a}[1,l]\, |\boldsymbol{\lambda}_n, s_n\rangle,$$
$$\widetilde{H}_m\, |\boldsymbol{\lambda}_n, s_n\rangle = \sum_{m_1 + \cdots + m_n = m} \prod_{a=1}^{n} p^{(a-1)m_a}\, \widetilde{H}^{(a)}_{m_a}[1,l]\, |\boldsymbol{\lambda}_n, s_n\rangle.$$

(ii′) *Let* $(\boldsymbol{\lambda}_n, \mathbf{s}_n) \in \Pi^n \times \mathbf{Z}^n$ *be* 0*-dominant. Then for any* $m \in \mathbf{N}$,

$$E_m\,|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle = \sum_{m_1+\cdots+m_n=m} \prod_{a=1}^{n} p^{(a-1)m_a} E_{m_a}^{(a)}[1, l]\,|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle,$$

$$H_m\,|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle = \sum_{m_1+\cdots+m_n=m} \prod_{a=1}^{n} p^{(a-1)m_a} H_{m_a}^{(a)}[1, l]\,|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle.$$

Proof. It follows from (27)–(30) that for each $m \in \mathbf{Z}^*$, a vector $B_m|\boldsymbol{\lambda}_l, \mathbf{s}_l\rangle$ (resp., a vector $B_m|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle$) is a linear combination of $|\boldsymbol{\mu}_l, \mathbf{s}_l\rangle$ (resp., $|\boldsymbol{\mu}_n, \mathbf{s}_n\rangle$) with $|\boldsymbol{\mu}_l| = |\boldsymbol{\lambda}_l| - nm$ (resp., with $|\boldsymbol{\mu}_n| = |\boldsymbol{\lambda}_n| - lm$). Also, if Δ is the comultiplication on the ring of symmetric functions defined by $\Delta p_m = p_m \otimes 1 + 1 \otimes p_m$, then $\Delta e_m = \sum_{r+s=m} e_r \otimes e_s$, $\Delta h_m = \sum_{r+s=m} h_r \otimes h_s$ (cf. [19]). These facts and Proposition 5.3 imply the claims. □

5.1.1 *Proof of Proposition* 4.5. To emphasize the dependence on n and l, let us set, in the notation of Proposition 4.5, $\gamma_m[n, l] := \gamma_m$. First, let $n = l = 1$. In this case, the ordering rules of Proposition 3.16 reduce to $u_{k_1} \wedge u_{k_2} = -u_{k_2} \wedge u_{k_1}$ for all $k_1, k_2 \in \mathbf{Z}$. This makes it easy to verify that $\gamma_m[1, 1] = m$. Next, let $n > 1$, $l = 1$. It is clear that there is $\mathbf{s}_n \in \mathbf{Z}^n$ such that the pair $(\varnothing_n, \mathbf{s}_n)$ is m-dominant. Hence applying Proposition 5.3(i′), we obtain

$$\gamma_m[n, 1]|\varnothing_n, \mathbf{s}_n\rangle = B_m B_{-m}|\varnothing_n, \mathbf{s}_n\rangle = B_m \sum_{a=1}^{n} p^{(a-1)m} B_{-m}^{(a)}[1, 1]|\varnothing_n, \mathbf{s}_n\rangle.$$

However, $B_{-m}|\varnothing_n, \mathbf{s}_n\rangle$ is a linear combination of $|\boldsymbol{\lambda}_n, \mathbf{s}_n\rangle$ with $|\boldsymbol{\lambda}_n| = m$ and hence a linear combination of 0-dominant vectors. Therefore, one may apply Proposition 5.3(ii′) and get

$$B_m \sum_{a=1}^{n} p^{(a-1)m} B_{-m}^{(a)}[1, 1]|\varnothing_n, \mathbf{s}_n\rangle = \sum_{a=1}^{n} p^{2(a-1)m} B_{m}^{(a)}[1, 1] B_{-m}^{(a)}[1, 1]|\varnothing_n, \mathbf{s}_n\rangle.$$

This implies

$$\gamma_m[n, 1]|\varnothing_n, \mathbf{s}_n\rangle = \left(\sum_{a=1}^{n} p^{2(a-1)m}\gamma_m[1, 1]\right)|\varnothing_n, \mathbf{s}_n\rangle.$$

Thus

$$\gamma_m[n, 1] = m\frac{1 - p^{2mn}}{1 - p^{2m}}.$$

Finally, let $n \geq 1$ and $l > 1$. Obviously, there is $\mathbf{s}_l \in \mathbf{Z}^l$ such that the pair $(\varnothing_l, \mathbf{s}_l)$ is nm-dominant. Therefore, by Proposition 5.3(i), we have

$$\gamma_m[n, l]|\varnothing_l, \mathbf{s}_l\rangle = B_m B_{-m}|\varnothing_l, \mathbf{s}_l\rangle = B_m \sum_{b=1}^{n} q^{(b-1)m} B_{-m}^{(b)}[n, 1]|\varnothing_l, \mathbf{s}_l\rangle.$$

Again, it is clear that $B_{-m}|\varnothing_l, s_l\rangle$ is a linear combination of 0-dominant vectors. Hence using Proposition 5.3(ii), we obtain

$$\gamma_m[n,l]|\varnothing_l, s_l\rangle = \left(\sum_{b=1}^{l} q^{2(b-1)m}\gamma_m[n,1]\right)|\varnothing_l, s_l\rangle.$$

It follows that

$$\gamma_m[n,l] = m\frac{1-p^{2mn}}{1-p^{2m}}\frac{1-q^{2ml}}{1-q^{2m}}.$$

Recalling that $p := -q^{-1}$, we get the desired result. □

5.2 A scalar product of $\Lambda^{s+\frac{\infty}{2}}$. For each $s \in \mathbf{Z}$, we define on the semiinfinite wedge product $\Lambda^{s+\frac{\infty}{2}}$ a $\mathbf{K}$-bilinear scalar product by $\langle b, b'\rangle = \delta_{b,b'}$, where b and b' are any two elements of the basis

$$\begin{aligned} B(s) &= \{|\lambda, s\rangle \mid \lambda \in \Pi\} \\ &= \{|\boldsymbol{\lambda}_l, s_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l,\ s_l \in \mathbf{Z}^l(s)\} \\ &= \{|\boldsymbol{\lambda}_n, s_n\rangle \mid \boldsymbol{\lambda}_n \in \Pi^n,\ s_n \in \mathbf{Z}^n(s)\}. \end{aligned}$$

It is clear that this scalar product is symmetric, and that for two weight vectors u and v, $\langle u, v\rangle$ is nonzero only if $\mathrm{Wt}(u) = \mathrm{Wt}(v)$.

Proposition 5.7. *For $u, v \in \Lambda^{s+\frac{\infty}{2}}$, one has*

$$\begin{aligned} &\langle e_i u, v\rangle = \langle u, q^{-1}(t_i)^{-1} f_i v\rangle, \quad \langle f_i u, v\rangle = \langle u, q^{-1} t_i e_i v\rangle \quad (i = 0, 1, \ldots, n-1), \\ &\langle \dot{e}_j u, v\rangle = \langle u, p^{-1}(\dot{t}_j)^{-1} \dot{f}_j v\rangle, \quad \langle \dot{f}_j u, v\rangle = \langle u, p^{-1} \dot{t}_j \dot{e}_j v\rangle \quad (j = 0, 1, \ldots, l-1). \end{aligned}$$

Proof. This follows immediately from (33)–(36). □

Proposition 5.8. *For $m \in \mathbf{N}$ and $u, v \in\in \Lambda^{s+\frac{\infty}{2}}$, one has*

$$\langle B_{-m} u, v\rangle = \langle u, B_m v\rangle.$$

To prove this proposition, we use the following lemma.

Lemma 5.9. (i) *Assume that the statement of Proposition* 5.8 *is valid for some $n \in \mathbf{N}$ and $l = 1$. Then it is also valid for the same n and all $l \in \mathbf{N}_{>1}$.*

(ii) *Assume that the statement of Proposition* 5.8 *is valid for some $l \in \mathbf{N}$ and $n = 1$. Then it is also valid for the same l and all $n \in \mathbf{N}_{>1}$.*

Proof. Since the proofs of (i) and (ii) are similar, we only give a proof of (i). Let us keep notation as in Proposition 5.8. Using Theorem 4.8, we assume without loss of generality that

$$u = \sum_k x_k y_k |\varnothing_l, t_l\rangle,$$

where x_k is an element of $U_q(\widehat{\mathfrak{sl}}_n)^- \cdot \mathcal{H}^-$, y_k is an element of $U_p(\widehat{\mathfrak{sl}}_l)^-$ and $\boldsymbol{t}_l = (t_1, \ldots, t_l)$ is an element of $A_l^n(s)$. By Corollary 4.9, for any $\boldsymbol{s}_l = (s_1, \ldots, s_l) \in \mathbf{Z}^l(s)$ such that $\Lambda_{s_1} + \cdots + \Lambda_{s_l} = \Lambda_{t_1} + \cdots + \Lambda_{t_l}$, we have $|\varnothing_l, \boldsymbol{t}_l\rangle = Y(\boldsymbol{s}_l)|\varnothing_l, \boldsymbol{s}_l\rangle$ for some $Y(\boldsymbol{s}_l) \in U'_p(\widehat{\mathfrak{sl}}_l)$. Hence

$$\langle B_m u, v\rangle = \sum_k \langle B_{-m} x_k |\varnothing_l, \boldsymbol{s}_l\rangle, Y_k^*(\boldsymbol{s}_l) v\rangle,$$

where $Y_k^*(\boldsymbol{s}_l)$ is the adjoint of $Y_k(\boldsymbol{s}_l) := y_k Y(\boldsymbol{s}_l)$. Note that by Proposition 5.7, $Y_k^*(\boldsymbol{s}_l) \in U'_p(\widehat{\mathfrak{sl}}_l)$. We may and do assume that for each k the element $x_k \in U_q(\widehat{\mathfrak{sl}}_n)^- \cdot \mathcal{H}^-$ has a definite weight $\mathrm{wt}(x_k) = -(r_{k,0}\alpha_0 + \cdots + r_{k,n-1}\alpha_{n-1})$, where $r_{k,i}$ are some nonnegative integers. For all k, we have $x_k|\varnothing_l, \boldsymbol{s}_l\rangle \in \mathbf{F}_q[\boldsymbol{s}_l]$, and using (27), we see that $x_k|\varnothing_l, \boldsymbol{s}_l\rangle$ is a linear combination of $|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle$ with $|\boldsymbol{\lambda}_l| = r_k := r_{k,0} + \cdots + r_{k,n-1}$. Now let $r := \max_k\{r_k\}$ and choose $\boldsymbol{s}_l$ so that

$$s_b - s_{b+1} \geq r + nm \qquad (b = 1, \ldots, l-1). \tag{41}$$

Then for each k, the vector $x_k|\varnothing_l, \boldsymbol{s}_l\rangle$ is a linear combination of nm-dominant vectors. Hence we may apply Proposition 5.3(i) and get

$$\langle B_m u, v\rangle = \sum_k \sum_{b=1}^{l} q^{(b-1)m} \langle B_{-m}^{(b)}[n,1] x_k|\varnothing_l, \boldsymbol{s}_l\rangle, Y_k^*(\boldsymbol{s}_l) v\rangle.$$

Now using the assumption in statement (i) of the lemma, we obtain

$$\langle B_m u, v\rangle = \sum_k \sum_{b=1}^{l} q^{(b-1)m} \langle x_k|\varnothing_l, \boldsymbol{s}_l\rangle, B_m^{(b)}[n,1] Y_k^*(\boldsymbol{s}_l) v\rangle.$$

For each k, the scalar product $\langle B_{-m} x_k|\varnothing_l, \boldsymbol{s}_l\rangle, Y_k^*(\boldsymbol{s}_l) v\rangle$ is nonzero only if $Y_k^*(\boldsymbol{s}_l)v \in \mathbf{F}_q[\boldsymbol{s}_l]$ and

$$\mathrm{wt}(Y_k^*(\boldsymbol{s}_l)v) = \mathrm{wt}(B_{-m} x_k|\varnothing_l, \boldsymbol{s}_l\rangle) = \mathrm{wt}|\varnothing_l, \boldsymbol{s}_l\rangle - \sum_{i=0}^{n-1} (r_{k,i} + m)\alpha_i.$$

It follows that $Y_k^*(\boldsymbol{s}_l)v$ is a linear combination of $|\boldsymbol{\mu}_l, \boldsymbol{s}_l\rangle$ with $|\boldsymbol{\mu}_l| = r_k + nm$, whence by (41) it is a linear combination of 0-dominant vectors. Therefore, we may apply Proposition 5.3(ii) and obtain

$$\langle B_{-m} u, v\rangle = \sum_k \langle x_k|\varnothing_n, \boldsymbol{s}_n\rangle, B_m Y_k(\boldsymbol{s}_n)^* v\rangle = \langle u, B_m v\rangle. \qquad \square$$

Proof of Proposition 5.8. In view of Lemma 5.9, it is sufficient to show that the statement of the proposition is valid for $n = l = 1$. However, in this case, the relation $\langle B_{-m} u, v\rangle = \langle u, B_m v\rangle$ is just a restatement of the fact that the endomorphism $m\partial/\partial p_m$ of the ring of symmetric functions is adjoint to the multiplication by p_m with respect to the scalar product orthonormalizing the basis of Schur functions. $\square$

Proposition 5.7 implies that for $m \in \mathbf{Z}_{\geq 0}$ and $u, v \in \Lambda^{s+\frac{\infty}{2}}$, one has

$$\langle \widetilde{E}_m u, v\rangle = \langle u, E_m v\rangle, \qquad \langle \widetilde{H}_m u, v\rangle = \langle u, H_m v\rangle. \tag{42}$$

5.3 A symmetry of the bar-involution. Define a semilinear involution $u \mapsto u'$ of $\Lambda^{*+\frac{\infty}{2}} = \oplus_{s\in\mathbf{Z}}\Lambda^{s+\frac{\infty}{2}}$ by

$$|\lambda, s\rangle' = |\lambda', -s\rangle, \qquad q' = q^{-1}.$$

Here λ' stands for the conjugate partition of λ. The description of the indexations of $|\lambda, s\rangle$ given in Remark 4.2 implies that for an l-multipartition $\boldsymbol{\lambda}_l = (\lambda^{(1)}, \dots, \lambda^{(l)})$ and $\boldsymbol{s}_l = (s_1, \dots, s_l) \in \mathbf{Z}^l$, we have

$$|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle' = |\boldsymbol{\lambda}_l', \boldsymbol{s}_l'\rangle,$$

where $\boldsymbol{\lambda}_l' = (\lambda^{(l)'}, \dots, \lambda^{(1)'})$ and $\boldsymbol{s}_l' = (-s_l, \dots, -s_1)$. Likewise, for an n-multipartition $\boldsymbol{\lambda}_n$ and $\boldsymbol{s}_n \in \mathbf{Z}^n$, we have

$$|\boldsymbol{\lambda}_n, \boldsymbol{s}_n\rangle' = |\boldsymbol{\lambda}_n', \boldsymbol{s}_n'\rangle.$$

Proposition 5.10. *For* $u \in \Lambda^{s+\frac{\infty}{2}}$ $(s \in \mathbf{Z})$, *we have*

$$\begin{aligned} &(e_i u)' = q^{-1}t_{-i}e_{-i}u', && (f_i u)' = q^{-1}(t_{-i})^{-1}f_{-i}u' && (i = 0, 1, \dots, n-1), \\ &(\dot{e}_j u)' = p^{-1}\dot{t}_{-j}\dot{e}_{-j}u', && (\dot{f}_j u)' = p^{-1}(\dot{t}_{-j})^{-1}\dot{f}_{-j}u' && (j = 0, 1, \dots, l-1). \end{aligned}$$

Proof. This follows from (33)–(36). □

Proposition 5.11. *For* $u \in \Lambda^{s+\frac{\infty}{2}}$ $(s \in \mathbf{Z})$ *and* $m \in \mathbf{Z}_{\geq 0}$, *we have*

(i) $$(\widetilde{E}_m u)' = (-q)^{m(n-1)}(-p)^{m(l-1)}\widetilde{H}_m u',$$

(ii) $$(E_m u)' = (-q)^{m(n-1)}(-p)^{m(l-1)}H_m u'.$$

To prove this proposition, we use the following lemma.

Lemma 5.12. (i) *Assume that statement* (i) *of Proposition* 5.11 *is valid for some* $n \in \mathbf{N}$ *and* $l = 1$. *Then it is also valid for the same* n *and all* $l \in \mathbf{N}_{>1}$.

(ii) *Assume that statement* (i) *of Proposition* 5.11 *is valid for some* $l \in \mathbf{N}$ *and* $n = 1$. *Then it is also valid for the same* l *and all* $n \in \mathbf{N}_{>1}$.

Proof. Since the proofs of (i) and (ii) are virtually identical, we give only a proof of (i). Let us keep notation as in Proposition 5.11. Taking into account Theorem 4.8, we may and do assume without loss of generality that

$$u = \sum_k x_k y_k |\varnothing_l, \boldsymbol{t}_l\rangle,$$

where $\boldsymbol{t}_l = (t_1, \dots, t_l)$ is an element of $A_l^n(s)$, x_k is an element of $U_q(\widehat{\mathfrak{sl}}_n)^- \cdot \mathcal{H}^-$, and y_k is an element of $U_p(\widehat{\mathfrak{sl}}_l)^-$. Choose any sequence $\boldsymbol{s}_l = (s_1, \dots, s_l) \in \mathbf{Z}^l(s)$ such that the relation $\Lambda_{s_1} + \dots + \Lambda_{s_l} = \Lambda_{t_1} + \dots + \Lambda_{t_l}$ is satisfied. Corollary 4.9(i)

It is clear that we may assume all x_k to be weight vectors of $U_q(\widehat{\mathfrak{sl}}_n)^- \cdot \mathcal{H}^-$. Then $\mathrm{wt}(x_k) = -(r_{k,0}\alpha_0 + \cdots + r_{k,n-1}\alpha_{n-1})$ for some nonnegative integers $r_{k,i}$. Moreover, $x_k \in U_q(\widehat{\mathfrak{sl}}_n)\cdot\mathcal{H}$ implies that $x_k|\varnothing_l, s_l\rangle$ belongs to $\mathbf{F}_q[s_l]$. Hence $x_k|\varnothing_l, s_l\rangle$ is a linear combination of vectors $|\boldsymbol{\lambda}_l, s_l\rangle$ ($\boldsymbol{\lambda}_l \in \Pi^l$), and from formula (27) for the weight of $|\boldsymbol{\lambda}_l, s_l\rangle$, we see that for all these vectors $|\boldsymbol{\lambda}_l| = r_k := r_{k,0} + \cdots + r_{k,n-1}$. Now let $r := \max_k\{r_k\}$ and choose the sequence s_l so that the inequalities

$$s_b - s_{b+1} \geq r + nm \tag{43}$$

are satisfied for all $b = 1, \dots, l-1$. Then for each k, the vector $x_k|\varnothing_l, s_l\rangle$ is a linear combination of nm-dominant $|\boldsymbol{\lambda}_l, s_l\rangle$, whereupon Corollary 5.6(i) gives

$$\widetilde{E}_m u = \sum_k Y_k(s_l) \sum_{m_1+\cdots+m_l=m} \prod_{b=1}^{l} q^{(b-1)m_b} \widetilde{E}^{(b)}_{m_b}[n,1]\, x_k|\varnothing_l, s_l\rangle.$$

Now we use the assumption in statement (i) of the lemma and obtain

$$\begin{aligned}(\widetilde{E}_m u)' &= q^{-m(l-1)}(-q)^{m(n-1)} \\ &\quad \times \sum_k Y_k(s_l)' \sum_{m_1+\cdots+m_l=m} \prod_{b=1}^{l} q^{(b-1)m_b} \widetilde{H}^{(b)}_{m_b}[n,1]\, (x_k|\varnothing_l, s_l\rangle)',\end{aligned}$$

where $Y_k(s_l)'$ is the element of $U'_p(\widehat{\mathfrak{sl}}_l)$ defined by $Y_k(s_l)'v' = (Y_k(s_l)v)'$ (cf. Proposition 5.10). Next, observe that if a vector $|\boldsymbol{\lambda}_l, s_l\rangle$ is nm-dominant, then so is $|\boldsymbol{\lambda}_l, s_l\rangle'$. Hence from (43), it follows that for each k the vector $(x_k|\varnothing_l, s_l\rangle)'$ is a linear combination of nm-dominant $|\boldsymbol{\lambda}_l, s'_l\rangle$. Therefore, we may again apply Corollary 5.6(i) and obtain

$$\begin{aligned}(\widetilde{E}_m u)' &= q^{-m(l-1)}(-q)^{m(n-1)} \sum_k Y_k(s_l)' \widetilde{H}_m (x_k|\varnothing_l, s_l\rangle)' \\ &= (-p)^{m(l-1)}(-q)^{m(n-1)} \widetilde{H}_m u'.\end{aligned}$$

Thus (i) is proved. □

Proof of Proposition 5.11. By Lemma 5.12, the statement (i) of the proposition will be proved once it is shown to hold for $n = l = 1$. However, in this case, (i) is just a restatement of the relation $\omega(e_m) = h_m$ for the standard involution of the ring of symmetric functions defined in terms of the Schur functions by $\omega(s_\lambda) = s_{\lambda'}$.

It remains to observe that, the scalar product being nondegenerate, relation (ii) follows from (i), relations (42), and the easily checked formula $\langle u', v\rangle = \overline{\langle u, v'\rangle}$. □

Proposition 5.13. *For $u, v \in \Lambda^{s+\frac{\infty}{2}}$ ($s \in \mathbf{Z}$), one has*

$$\langle \overline{u}, v\rangle = \langle u', \overline{v'}\rangle.$$

PROOF. Using the decomposition of $\Lambda^{s+\frac{\infty}{2}}$ described in Theorem 4.8 we define a gradation of $\Lambda^{s+\frac{\infty}{2}}$ in the following way. We set the degrees of all the singular vectors $|\varnothing_l, s_l\rangle$ ($s_l \in A_l^n(s)$) to be zero, and we require that with respect to our gradation the operators f_i, $\dot{f}_j$, and B_{-m} ($m \in \mathrm{N}$) be homogeneous of degrees 1, 1, and m, respectively. Then the operators e_i, $\dot{e}_j$ are homogeneous of degree -1 and the operators $\widetilde{E}_m$, $\widetilde{H}_m$, E_m, and H_m are homogeneous of respective degrees m, m, $-m$, and $-m$.

Now we show the claim by induction. In degree zero, we have

$$\langle \overline{|\varnothing_l, s_l\rangle}, |\varnothing_l, t_l\rangle\rangle = \langle |\varnothing_l, s_l\rangle, |\varnothing_l, t_l\rangle\rangle = \delta_{s_l, t_l},$$
$$\langle |\varnothing_l, s_l'\rangle, \overline{|\varnothing_l, t_l'\rangle}\rangle = \langle |\varnothing_l, s_l'\rangle, |\varnothing_l, t_l'\rangle\rangle = \delta_{s_l, t_l}$$

for all $s_l, t_l \in A_l^n(s)$. Hence the claim holds in this case.

Assume that the proposition is proved for all u, v with degrees $< k$. By Theorem 4.8, to prove the proposition for all u, v, it is enough to show that

$$\langle \overline{(f_i u)}, v\rangle = \langle (f_i u)', \overline{v'}\rangle, \tag{44}$$
$$\langle \overline{(\dot{f}_j u)}, v\rangle = \langle (\dot{f}_j u)', \overline{v'}\rangle, \tag{45}$$
$$\langle \overline{(\widetilde{E}_m w)}, v\rangle = \langle (\widetilde{E}_m w)', \overline{v'}\rangle \tag{46}$$

for u, v, and w with degrees $k-1$, k, and $k-m$, respectively.

Let us show (44). We have

$$\langle \overline{(f_i u)}, v\rangle = \langle f_i \overline{u}, v\rangle = \langle \overline{u}, q^{-1} t_i e_i v\rangle = \langle u', \overline{(q^{-1} t_i e_i v)'}\rangle.$$

Here the first equality follows from Proposition 4.12, the second follows from Proposition 5.7, and the third follows from the induction assumption. Further,

$$\langle u', \overline{(q^{-1} t_i e_i v)'}\rangle = \langle u', \overline{e_{-i} v'}\rangle = \langle u', e_{-i}\overline{v'}\rangle = \langle q^{-1}(t_{-i})^{-1} f_{-i} u', \overline{v'}\rangle = \langle (f_i u)', \overline{v'}\rangle.$$

Here we used Propositions 5.10, 4.12, and 5.7. Thus (44) is established. A proof of (45) is similar.

Finally, let us show (46). We have

$$\langle \overline{(\widetilde{E}_m w)}, v\rangle = \langle \widetilde{E}_m \overline{w}, v\rangle = \langle \overline{w}, E_m v\rangle = \langle w', \overline{(E_m v)'}\rangle.$$

Here the first equality follows from Proposition 4.12, the second follows from (42), and the third follows from the induction assumption. Continuing, we have

$$\begin{aligned}\langle w', \overline{(E_m v)'}\rangle &= \langle w', \overline{(-q)^{m(n-1)}(-p)^{m(l-1)} H_m v'}\rangle \\ &= \langle w', (-q)^{m(n-1)}(-p)^{m(l-1)} H_m \overline{v'}\rangle \\ &= \langle (-q)^{m(n-1)}(-p)^{m(l-1)} \widetilde{H}_m w', \overline{v'}\rangle = \langle (\widetilde{E}_m w)', \overline{v'}\rangle.\end{aligned}$$

Here we used Propositions 5.11 and 4.12 and (42). Thus (46) is proved. □

For $s_l = (s_1, \ldots, s_l) \in \mathbf{Z}^l$, let $\{\mathcal{G}^*(\boldsymbol{\lambda}_l, s_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ be the basis of $\mathbf{F}_q[s_l]$ dual to $\{\mathcal{G}^+(\boldsymbol{\lambda}_l, s_l) \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ with respect to the scalar product $\langle u, v\rangle$. Write

$$\mathcal{G}^*(\boldsymbol{\lambda}_l, s_l) = \sum_{\boldsymbol{\mu}_l \in \Pi^l} \Delta^*_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(s_l|q)\,|\boldsymbol{\mu}_l, s_l\rangle.$$

Since the basis $\{|\boldsymbol{\lambda}_l, s_l\rangle \mid \boldsymbol{\lambda}_l \in \Pi^l\}$ of $\mathbf{F}_q[s_l]$ is orthonormal relative to the scalar product, the matrix $\|\Delta^*_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(s_l|q)\|$ is the transposed inverse of the matrix $\|\Delta^+_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(s_l|q)\|$.

Proposition 5.14. *For $s_l \in \mathbf{Z}^l$ and $\boldsymbol{\lambda}_l \in \Pi^l$, one has*

$$\mathcal{G}^*(\boldsymbol{\lambda}_l, s_l)' = \mathcal{G}^-(\boldsymbol{\lambda}_l', s_l').$$

Proof. Since the matrix $\|\Delta^+_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(s_l|q)\|$ is unitriangular with off-diagonal entries in $q\mathbf{Z}[q]$, the same is true for its transposed inverse $\|\Delta^*_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}(s_l|q)\|$. It follows that

$$\mathcal{G}^*(\boldsymbol{\lambda}_l, s_l)' \equiv |\boldsymbol{\lambda}_l', s_l'\rangle \bmod q^{-1}\mathcal{L}^-(s),$$

where $s = s_1 + \cdots + s_l$. Hence $\mathcal{G}^*(\boldsymbol{\lambda}_l, s_l)'$ has the required congruence property relative to the basis $\{|\boldsymbol{\lambda}_l, s_l'\rangle\}$.

It remains to show that $\mathcal{G}^*(\boldsymbol{\lambda}_l, s_l)'$ is invariant with respect to the bar-involution. Since $\mathcal{G}^*(\boldsymbol{\lambda}_l, s_l)$ is dual to $\mathcal{G}^+(\boldsymbol{\lambda}_l, s_l)$, this is equivalent to

$$\langle \overline{\mathcal{G}^*(\boldsymbol{\lambda}_l, s_l)'}, \mathcal{G}^+(\boldsymbol{\mu}_l, s_l)'\rangle = \delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}.$$

Using Proposition 5.13, we obtain

$$\begin{aligned}\langle \overline{\mathcal{G}^*(\boldsymbol{\lambda}_l, s_l)'}, \mathcal{G}^+(\boldsymbol{\mu}_l, s_l)'\rangle &= \langle \mathcal{G}^*(\boldsymbol{\lambda}_l, s_l), \overline{\mathcal{G}^+(\boldsymbol{\mu}_l, s_l)}\rangle \\ &= \langle \mathcal{G}^*(\boldsymbol{\lambda}_l, s_l), \mathcal{G}^+(\boldsymbol{\mu}_l, s_l)\rangle = \delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}.\end{aligned}$$ □

This proposition immediately implies the following.

Theorem 5.15 (Inversion formula). *For $s_l \in \mathbf{Z}^l$, $\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l \in \Pi^l$, one has*

$$\sum_{\boldsymbol{\nu}_l \in \Pi^l} \Delta^-_{\boldsymbol{\lambda}_l', \boldsymbol{\nu}_l'}(s_l'|q^{-1})\, \Delta^+_{\boldsymbol{\mu}_l, \boldsymbol{\nu}_l}(s_l|q) = \delta_{\boldsymbol{\lambda}_l, \boldsymbol{\mu}_l}.$$

5.4 Proof of Proposition 5.3. Let us prove part (i) of Proposition 5.3. First, we introduce some notation. Let s be an integer. For any pair $(\boldsymbol{\lambda}_l, s_l) \in \Pi^l \times \mathbf{Z}^l(s)$, where

$$\boldsymbol{\lambda}_l = (\lambda^{(1)}, \ldots, \lambda^{(l)}), \qquad s_l = (s_1, \ldots, s_l),$$

let $\boldsymbol{k} = (k_1, k_2, \ldots)$ be the unique sequence from $\mathbf{P}^{++}(s)$ (cf. Section 4.1) such that

$$|\boldsymbol{\lambda}_l, s_l\rangle = u_{\boldsymbol{k}}.$$

As in Section 4.1, we write $k_i = a_i + n(b_i - 1) - nlm_i$, where $a_i \in \{1, \ldots, n\}$, $b_i \in \{1, \ldots, l\}$, and $m_i \in \mathbf{Z}$.

For any natural number r, set $(k)_r = (k_1, k_2, \dots, k_r)$. Then $(k)_r \in \mathbf{P}^{++}$ and $u_k = u_{(k)_r} \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \cdots$, where $u_{(k)_r} = u_{k_1} \wedge \cdots \wedge u_{k_r}$ is an element of Λ^r.

We define $(k)_r^+ = (k_1^+, \dots, k_r^+)$ to be the unique permutation of the sequence $(k)_r$ characterized by the following two conditions:

$$\begin{aligned} b_i^+ &\le b_j^+ \quad \text{for all } i < j, \\ k_i^+ &> k_j^+ \quad \text{for all } i < j \text{ such that } b_i^+ = b_j^+. \end{aligned}$$

Here we set $k_i^+ = a_i^+ + n(b_i^+ - 1) - nlm_i^+$, where $a_i^+ \in \{1, \dots, n\}$, $b_i^+ \in \{1, \dots, l\}$, and $m_i^+ \in \mathbf{Z}$.

Example 5.16. Let $n = 2$, $l = 3$, and $s = -2$. Take $\boldsymbol{\lambda}_l = ((1^2), (1), (2))$ and $s_l = (5, 0, -7)$. Let $r = 25$. In this case,

$$\begin{aligned} (k)_r = (&14, 13, 7, 3, 2, 1, -3, -4, -5, -8, -9, -10, -11, -13, -14, \\ &-15, -16, -17, -20, -21, -22, -23, -24, -25, -26), \end{aligned}$$

and

$$\begin{aligned} (k)_r^+ = (&14, 13, 7, 2, 1, -4, -5, -10, -11, -16, -17, -22, -23, 3, \\ &-3, -8, -9, -14, -15, -20, -21, -26, -13, -24, -25). \end{aligned}$$

Recall that in Section 4.1 we associated with k the semiinfinite sequences

$$k^{(b)} = (k_1^{(b)}, k_2^{(b)}, \dots) \quad (b = 1, 2, \dots, l)$$

such that $k_i^{(b)} = s_b + i - 1 + \lambda_i^{(b)}$ for all $i \in \mathbf{N}$. The wedge $u_{(k)_r^+} = u_{k_1^+} \wedge \cdots \wedge u_{k_r^+}$ may be expressed in terms of these sequences in the following way. For $a \in \{1, \dots, n\}$, $b \in \{1, \dots, l\}$, and $m \in \mathbf{Z}$, set $u^{(b)}_{a-nm} := u_{a+n(b-1)-nlm}$. Then

$$\begin{aligned} u_{(k)_r^+} = u^{(1)}_{k_1^{(1)}} \wedge u^{(1)}_{k_2^{(1)}} \wedge \cdots \wedge u^{(1)}_{k_{r_1}^{(1)}} \wedge u^{(2)}_{k_1^{(2)}} \wedge u^{(2)}_{k_2^{(2)}} \wedge \cdots \wedge u^{(2)}_{k_{r_2}^{(2)}} \wedge \cdots \\ \cdots \\ \cdots \wedge u^{(l)}_{k_1^{(l)}} \wedge u^{(l)}_{k_2^{(l)}} \wedge \cdots \wedge u^{(l)}_{k_{r_l}^{(l)}}, \end{aligned} \tag{47}$$

where for each $b \in \{1, \dots, l\}$ we set $r_b := \sharp\{1 \le i \le r \mid b_i = b\}$.

Note that in general the wedge $u_{(k)_r^+}$ is not ordered, and using the ordering rules of Proposition 3.16 to straighten $u_{(k)_r^+}$ as a linear combination of ordered wedges, one typically obtains a linear combination with many terms. The first step towards the proof of the proposition is to show that if the pair $(\boldsymbol{\lambda}_l, s_l)$ we started with is 0-dominant, then for any $r \in \mathbf{N}$, the straightening of $u_{(k)_r^+}$ produces only one term, which, up to a power of q, coincides with $u_{(k)_r}$.

Lemma 5.17. *Let $b_1, b_2 \in \{1, \dots, l\}$ and $a_1, a_2 \in \{1, \dots, n\}$ satisfy the inequalities $b_1 < b_2$, $a_1 \ge a_2$. Let $m \in \mathbf{Z}$. Then for any $t \in \mathbf{Z}_{\ge 0}$, one has the following relation:*

$$\begin{aligned} &u^{(b_1)}_{a_1-nm} \wedge u^{(b_1)}_{a_1-nm-1} \wedge u^{(b_1)}_{a_1-nm-2} \wedge \cdots \wedge u^{(b_1)}_{a_1-nm-t} \wedge u^{(b_2)}_{a_2-nm} \\ &= q^{\sum_{k=0}^{t} \delta(a_1-k) \equiv a_2 \bmod n} u^{(b_2)}_{a_2-nm} \wedge u^{(b_1)}_{a_1-nm} \wedge u^{(b_1)}_{a_1-nm-1} \wedge u^{(b_1)}_{a_1-nm-2} \wedge \cdots \wedge u^{(b_1)}_{a_1-nm-t}. \end{aligned}$$

PROOF. This is shown by induction on t using relations (R3) and (R4) of Proposition 3.16 and Lemma 3.18. □

Keeping $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$, $\boldsymbol{k}$, r, $(\boldsymbol{k})_r$, and $(\boldsymbol{k})_r^+$ as above, let us define

$$c_r(\boldsymbol{k}) := \sharp\{1 \le i < j \le r \mid b_i > b_j,\ a_i = a_j\},$$

and

$$|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle_r^+ := u_{(\boldsymbol{k})_r^+} \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \cdots .$$

Lemma 5.18. *Suppose that $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ is 0-dominant. Then*

$$|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle_r^+ = q^{c_r(\boldsymbol{k})} |\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle.$$

PROOF. Since $|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle = u_{\boldsymbol{k}} = u_{(\boldsymbol{k})_r} \wedge u_{k_{r+1}} \wedge u_{k_{r+2}} \wedge \cdots$, we must prove that

$$u_{(\boldsymbol{k})_r^+} = q^{c_r(\boldsymbol{k})} u_{(\boldsymbol{k})_r}.$$

First of all, let us examine what the 0-dominance of $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ implies for the semiinfinite sequence $\boldsymbol{k}$. For each $b \in \{1, \dots, l\}$, let p_b be the minimal number such that $k_i^{(b)} = s_b - i + 1$ for all $i \ge p_b$. Then $p_b = l(\lambda^{(b)}) + 1$, where $l(\lambda)$ denotes the length of a partition λ, and we have $k_{p_b}^{(b)} = s_b - l(\lambda^{(b)})$. On the other hand, $k_1^{(b)} = s_b + \lambda_1^{(b)}$. Hence using the assumption that $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ is 0-dominant (cf. Definition 5.1), we find that for all $b = 1, 2, \dots, l-1$,

$$k_{p_b}^{(b)} - k_1^{(b+1)} = s_b - s_{b+1} - l(\lambda^{(b)}) - \lambda_1^{(b+1)} \ge s_b - s_{b+1} - |\boldsymbol{\lambda}_l| \ge 0.$$

The fact that we have the inequalities $k_{p_b}^{(b)} \ge k_1^{(b+1)}$ for all $b = 1, 2, \dots, l-1$, implies that to straighten $u_{(\boldsymbol{k})_r^+}$ on the basis of ordered wedges, we need only to repeatedly apply Lemma 5.17. The result follows. □

Example 5.19. Let us illustrate the proof of Lemma 5.18 for $|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle_r^+$, where $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ and r are the same as in Example 5.16. Note that the pair $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ is 0-dominant.

In this case, $u_{(\boldsymbol{k})_r}$ is given by the following expression:

$$\begin{aligned} &u_{2+2\cdot2}^{(1)} \wedge u_{1+2\cdot2}^{(1)} \wedge u_{1+2}^{(1)} \wedge u_{1}^{(2)} \wedge u_{2}^{(1)} \wedge u_{1}^{(1)} \wedge u_{1-2\cdot1}^{(2)} \wedge u_{2-2\cdot1}^{(1)} \wedge u_{1-2\cdot1}^{(1)} \wedge u_{2-2\cdot2}^{(2)} \\ &\quad \wedge u_{1-2\cdot2}^{(2)} \wedge u_{2-2\cdot2}^{(1)} \wedge u_{1-2\cdot2}^{(1)} \wedge u_{1-2\cdot3}^{(3)} \wedge u_{2-2\cdot3}^{(2)} \wedge u_{1-2\cdot3}^{(2)} \wedge u_{2-2\cdot3}^{(1)} \wedge u_{1-2\cdot3}^{(1)} \\ &\quad \wedge u_{2-2\cdot4}^{(2)} \wedge u_{1-2\cdot4}^{(2)} \wedge u_{2-2\cdot4}^{(1)} \wedge u_{1-2\cdot4}^{(1)} \wedge u_{2-2\cdot5}^{(3)} \wedge u_{1-2\cdot5}^{(3)} \wedge u_{2-2\cdot5}^{(2)}. \end{aligned}$$

Now let us rearrange (taking care of powers of q) the factors in this wedge by repeatedly applying Lemma 5.17. The rearrangement involves the following seven steps:

$$
\begin{aligned}
u_{(k)_r} = {} & u^{(1)}_{2+2\cdot 2} \wedge u^{(1)}_{1+2\cdot 2} \wedge u^{(1)}_{1+2} \wedge u^{(2)}_{1} \wedge u^{(1)}_{2} \wedge u^{(1)}_{1} \wedge u^{(2)}_{1-2\cdot 1} \wedge u^{(1)}_{2-2\cdot 1} \wedge u^{(1)}_{1-2\cdot 1} \\
& \wedge u^{(2)}_{2-2\cdot 2} \wedge u^{(2)}_{1-2\cdot 2} \wedge u^{(1)}_{2-2\cdot 2} \wedge u^{(1)}_{1-2\cdot 2} \wedge u^{(3)}_{1-2\cdot 3} \wedge u^{(2)}_{2-2\cdot 3} \wedge u^{(2)}_{1-2\cdot 3} \wedge u^{(1)}_{2-2\cdot 3} \\
& \wedge u^{(1)}_{1-2\cdot 3} \wedge u^{(2)}_{2-2\cdot 4} \wedge u^{(2)}_{1-2\cdot 4} \wedge u^{(1)}_{2-2\cdot 4} \wedge u^{(1)}_{1-2\cdot 4} \wedge \underline{u^{(3)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 5} \wedge u^{(2)}_{2-2\cdot 5}} \\
= {} & q^{-1} u^{(1)}_{2+2\cdot 2} \wedge u^{(1)}_{1+2\cdot 2} \wedge u^{(1)}_{1+2} \wedge u^{(2)}_{1} \wedge u^{(1)}_{2} \wedge u^{(1)}_{1} \wedge u^{(2)}_{1-2\cdot 1} \wedge u^{(1)}_{2-2\cdot 1} \wedge u^{(1)}_{1-2\cdot 1} \\
& \wedge u^{(2)}_{2-2\cdot 2} \wedge u^{(2)}_{1-2\cdot 2} \wedge u^{(1)}_{2-2\cdot 2} \wedge u^{(1)}_{1-2\cdot 2} \wedge u^{(3)}_{1-2\cdot 3} \wedge u^{(2)}_{2-2\cdot 3} \wedge u^{(2)}_{1-2\cdot 3} \wedge u^{(1)}_{2-2\cdot 3} \\
& \wedge u^{(1)}_{1-2\cdot 3} \wedge \underline{u^{(2)}_{2-2\cdot 4} \wedge u^{(2)}_{1-2\cdot 4} \wedge u^{(1)}_{2-2\cdot 4} \wedge u^{(1)}_{1-2\cdot 4}} \wedge u^{(2)}_{2-2\cdot 5} \wedge u^{(3)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 5} \\
= {} & q^{-3} u^{(1)}_{2+2\cdot 2} \wedge u^{(1)}_{1+2\cdot 2} \wedge u^{(1)}_{1+2} \wedge u^{(2)}_{1} \wedge u^{(1)}_{2} \wedge u^{(1)}_{1} \wedge u^{(2)}_{1-2\cdot 1} \wedge u^{(1)}_{2-2\cdot 1} \wedge u^{(1)}_{1-2\cdot 1} \\
& \wedge u^{(2)}_{2-2\cdot 2} \wedge u^{(2)}_{1-2\cdot 2} \wedge u^{(1)}_{2-2\cdot 2} \wedge u^{(1)}_{1-2\cdot 2} \wedge u^{(3)}_{1-2\cdot 3} \wedge \underline{u^{(2)}_{2-2\cdot 3} \wedge u^{(2)}_{1-2\cdot 3} \wedge u^{(1)}_{2-2\cdot 3}} \\
& \underline{\wedge u^{(1)}_{1-2\cdot 3} \wedge u^{(1)}_{2-2\cdot 4} \wedge u^{(1)}_{1-2\cdot 4}} \wedge u^{(2)}_{2-2\cdot 4} \wedge u^{(2)}_{1-2\cdot 4} \wedge u^{(2)}_{2-2\cdot 5} \wedge u^{(3)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 5} \\
= {} & q^{-7} u^{(1)}_{2+2\cdot 2} \wedge u^{(1)}_{1+2\cdot 2} \wedge u^{(1)}_{1+2} \wedge u^{(2)}_{1} \wedge u^{(1)}_{2} \wedge u^{(1)}_{1} \wedge u^{(2)}_{1-2\cdot 1} \wedge u^{(1)}_{2-2\cdot 1} \wedge u^{(1)}_{1-2\cdot 1} \\
& \wedge u^{(2)}_{2-2\cdot 2} \wedge u^{(2)}_{1-2\cdot 2} \wedge u^{(1)}_{2-2\cdot 2} \wedge u^{(1)}_{1-2\cdot 2} \wedge \underline{u^{(3)}_{1-2\cdot 3} \wedge u^{(1)}_{2-2\cdot 3} \wedge u^{(1)}_{1-2\cdot 3} \wedge u^{(1)}_{2-2\cdot 4}} \\
& \underline{\wedge u^{(1)}_{1-2\cdot 4} \wedge u^{(2)}_{2-2\cdot 3} \wedge u^{(2)}_{1-2\cdot 3} \wedge u^{(2)}_{2-2\cdot 4} \wedge u^{(2)}_{1-2\cdot 4} \wedge u^{(2)}_{2-2\cdot 5}} \wedge u^{(3)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 5} \\
= {} & q^{-11} u^{(1)}_{2+2\cdot 2} \wedge u^{(1)}_{1+2\cdot 2} \wedge u^{(1)}_{1+2} \wedge u^{(2)}_{1} \wedge u^{(1)}_{2} \wedge u^{(1)}_{1} \wedge u^{(2)}_{1-2\cdot 1} \wedge u^{(1)}_{2-2\cdot 1} \wedge u^{(1)}_{1-2\cdot 1} \\
& \wedge \underline{u^{(2)}_{2-2\cdot 2} \wedge u^{(2)}_{1-2\cdot 2} \wedge u^{(1)}_{2-2\cdot 2} \wedge u^{(1)}_{1-2\cdot 2} \wedge u^{(1)}_{2-2\cdot 3} \wedge u^{(1)}_{1-2\cdot 3} \wedge u^{(1)}_{2-2\cdot 4} \wedge u^{(1)}_{1-2\cdot 4}} \\
& \wedge u^{(2)}_{2-2\cdot 3} \wedge u^{(2)}_{1-2\cdot 3} \wedge u^{(2)}_{2-2\cdot 4} \wedge u^{(2)}_{1-2\cdot 4} \wedge u^{(2)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 3} \wedge u^{(3)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 5} \\
= {} & q^{-17} u^{(1)}_{2+2\cdot 2} \wedge u^{(1)}_{1+2\cdot 2} \wedge u^{(1)}_{1+2} \wedge u^{(2)}_{1} \wedge u^{(1)}_{2} \wedge u^{(1)}_{1} \wedge \underline{u^{(2)}_{1-2\cdot 1} \wedge u^{(1)}_{2-2\cdot 1} \wedge u^{(1)}_{1-2\cdot 1}} \\
& \underline{\wedge u^{(1)}_{2-2\cdot 2} \wedge u^{(1)}_{1-2\cdot 2} \wedge u^{(1)}_{2-2\cdot 3} \wedge u^{(1)}_{1-2\cdot 3} \wedge u^{(1)}_{2-2\cdot 4} \wedge u^{(1)}_{1-2\cdot 4}} \wedge u^{(2)}_{2-2\cdot 2} \wedge u^{(2)}_{1-2\cdot 2} \\
& \wedge u^{(2)}_{2-2\cdot 3} \wedge u^{(2)}_{1-2\cdot 3} \wedge u^{(2)}_{2-2\cdot 4} \wedge u^{(2)}_{1-2\cdot 4} \wedge u^{(2)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 3} \wedge u^{(3)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 5} \\
= {} & q^{-21} u^{(1)}_{2+2\cdot 2} \wedge u^{(1)}_{1+2\cdot 2} \wedge u^{(1)}_{1+2} \wedge \underline{u^{(2)}_{1} \wedge u^{(1)}_{2} \wedge u^{(1)}_{1} \wedge u^{(1)}_{2-2\cdot 1} \wedge u^{(1)}_{1-2\cdot 1} \wedge u^{(1)}_{2-2\cdot 2}} \\
& \underline{\wedge u^{(1)}_{1-2\cdot 2} \wedge u^{(1)}_{2-2\cdot 3} \wedge u^{(1)}_{1-2\cdot 3} \wedge u^{(1)}_{2-2\cdot 4} \wedge u^{(1)}_{1-2\cdot 4}} \wedge u^{(2)}_{1-2\cdot 1} \wedge u^{(2)}_{2-2\cdot 2} \wedge u^{(2)}_{1-2\cdot 2} \\
& \wedge u^{(2)}_{2-2\cdot 3} \wedge u^{(2)}_{1-2\cdot 3} \wedge u^{(2)}_{2-2\cdot 4} \wedge u^{(2)}_{1-2\cdot 4} \wedge u^{(2)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 3} \wedge u^{(3)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 5} \\
= {} & q^{-26} u^{(1)}_{2+2\cdot 2} \wedge u^{(1)}_{1+2\cdot 2} \wedge u^{(1)}_{1+2} \wedge u^{(1)}_{2} \wedge u^{(1)}_{1} \wedge u^{(1)}_{2-2\cdot 1} \wedge u^{(1)}_{1-2\cdot 1} \wedge u^{(1)}_{2-2\cdot 2} \wedge u^{(1)}_{1-2\cdot 2} \\
& \wedge u^{(1)}_{2-2\cdot 3} \wedge u^{(1)}_{1-2\cdot 3} \wedge u^{(1)}_{2-2\cdot 4} \wedge u^{(1)}_{1-2\cdot 4} \wedge u^{(2)}_{1} \wedge u^{(2)}_{1-2\cdot 1} \wedge u^{(2)}_{2-2\cdot 2} \wedge u^{(2)}_{1-2\cdot 2} \\
& \wedge u^{(2)}_{2-2\cdot 3} \wedge u^{(2)}_{1-2\cdot 3} \wedge u^{(2)}_{2-2\cdot 4} \wedge u^{(2)}_{1-2\cdot 4} \wedge u^{(2)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 3} \wedge u^{(3)}_{2-2\cdot 5} \wedge u^{(3)}_{1-2\cdot 5} \\
= {} & q^{-26} u_{(k)_r^+}.
\end{aligned}
$$

Here at each step we underline the part to which we apply Lemma 5.17. Note that we use this lemma twice at steps 1, 2, 3, 4, 5 and we use it once at steps 6 and 7.

Let $l = 1$ temporarily and for partitions λ and μ, define the matrix elements $(B_{-m})_{\lambda,\mu}(s)$ by

$$B_{-m}|\lambda, s\rangle = \sum_{\mu\in\Pi}(B_{-m})_{\lambda,\mu}(s)\,|\mu, s\rangle.$$

Now we proceed with the proof of the proposition. Assume that the pair $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ is nm-dominant for some $m \in \mathbf{N}$. Then

$$B_{-m}|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle = B_{-m}u_{\boldsymbol{k}} = (B_{-m}u_{(\boldsymbol{k})_r}) \wedge |s-r\rangle = q^{-c_r(\boldsymbol{k})}(B_{-m}u_{(\boldsymbol{k})_r^+}) \wedge |s-r\rangle,$$

where the second equality is obtained by taking r sufficiently large and the third equality follows from Lemma 5.18. Using (47), we have

$$B_{-m}u_{(\boldsymbol{k})_r^+} = \sum_{b=1}^{l}\sum_{i=1}^{r_b} u^{(1)}_{(\boldsymbol{k}^{(1)})_{r_1}} \wedge u^{(2)}_{(\boldsymbol{k}^{(2)})_{r_2}} \wedge\cdots\wedge u^{(b)}_{(\boldsymbol{k}^{(b)})_{r_b}-\epsilon_i nm} \wedge\cdots\wedge u^{(l)}_{(\boldsymbol{k}^{(l)})_{r_l}},$$

where we set $(\boldsymbol{k}^{(b)})_{r_b} = (k_1^{(b)}, \ldots, k_{r_b}^{(b)})$ and $\epsilon_i = (0, \ldots, 0, 1, 0\ldots, 0)$ with 1 on the ith position. Let us now straighten the expression

$$\sum_{i=1}^{r_b} u^{(b)}_{(\boldsymbol{k}^{(b)})_{r_b}-\epsilon_i nm}$$

on the basis of ordered wedges. It is clear that to do so we only need to use relations (R1) and (R2) of Proposition 3.16. However, these two relations are the same as in the $l = 1$ case. Hence assuming (as we may by choosing large enough r) that $r_1, r_2, \ldots, r_l$ are sufficiently large, we get

$$\begin{aligned}&B_{-m}|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle\\ &\quad= q^{-c_r(\boldsymbol{k})}\sum_{b=1}^{l}\sum_{\mu\in\Pi}(B_{-m})_{\lambda^{(b)},\mu}(s_b)\,|(\lambda^{(1)}, \ldots, \lambda^{(b-1)}, \mu, \lambda^{(b+1)}, \ldots, \lambda^{(l)}), \boldsymbol{s}_l\rangle_r^+.\end{aligned}$$

Observe now that in the above sum we have for all b and μ,

$$|\mu| = |\lambda^{(b)}| + nm.$$

Hence the nm-dominance of $(\boldsymbol{\lambda}_l, \boldsymbol{s}_l)$ implies the 0-dominance of each pair

$$((\lambda^{(1)}, \ldots, \lambda^{(b-1)}, \mu, \lambda^{(b+1)}, \ldots, \lambda^{(l)}), \boldsymbol{s}_l). \tag{48}$$

It follows that we may apply Lemma 5.18 in order to straighten each wedge under the sum. Doing so, we get

$$\begin{aligned}B_{-m}|\boldsymbol{\lambda}_l, \boldsymbol{s}_l\rangle &= \sum_{b=1}^{l}\sum_{\mu\in\Pi} q^{c_r(\boldsymbol{l})-c_r(\boldsymbol{k})}\\ &\quad\times (B_{-m})_{\lambda^{(b)},\mu}(s_b)\,|(\lambda^{(1)}, \ldots, \lambda^{(b-1)}, \mu, \lambda^{(b+1)}, \ldots, \lambda^{(l)}), \boldsymbol{s}_l\rangle,\end{aligned}$$

where $\boldsymbol{l}$ is the unique element of $\mathbf{P}^{++}(s)$ such that

$$|(\lambda^{(1)},\ldots,\lambda^{(b-1)},\mu,\lambda^{(b+1)},\ldots,\lambda^{(l)}),s_l\rangle = u_l.$$

Finally, using the 0-dominance of (48) and the 0-dominance of $(\boldsymbol{\lambda}_l, s_l)$, it is not difficult to see that $c_r(\boldsymbol{l}) - c_r(\boldsymbol{k}) = (b-1)m$ for all large enough r. Proposition 5.3(i) follows.

A proof of Proposition 5.3 (i$'$)is obtained by following the same steps as above but interchanging everywhere the roles of n and l and the roles of p and q. The proofs of (ii) and (ii$'$) are similar to the proofs of (i) and (i$'$) and will be omitted. □

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