Algebra 3 Complements

I write several things which I hoped to explain in the class but could not explain. Today, I write about zeta functions and about my friend Kurokawa.

1 Was there only one zeta function when the universe started?

1.1. I have a friend Nobushige Kurokawa who studies zeta functions. He is a little crazy.

There are four kinds of zeta functions.

1. Zeta functions $\zeta_A(s)$ for commutative rings A which are finitely generated over some finite field $\mathbb{F}_q$. (These zeta functions are rational functions of q^{-s}.)

2. Zeta functions $\zeta_A(s)$ for integral domains A which are finitely generated over $\mathbb{Z}$ and whose quotient fields are of characteristic 0. Example: Riemann's zeta function $\zeta(s) = \zeta_{\mathbb{Z}}(s)$.

3. Zeta functions of modular forms.

4. Selberg zeta functions. This 4 is not explained in the class. A small explanation is given in 1.6 below.

Riemann's hypothesis has been proved for 1 and 4, but not yet proved for 2 and 3.

For 1, 3 and 4, it has been proved that the zeta function has meromorphic continuation to the whole complex plane $\mathbb{C}$, but this is not yet proved for 2.

Langlands correspondence is a conjecture which says $2 = 3$. Hence if we have Langlands correspondence as a theorem (not only a conjecture), we can prove that the zeta functions in 2 have meromorphic analytic continuations to $\mathbb{C}$.

Thus Langlands corresponding is a unification theory of different kinds of zeta.

This suggests that if we have some unification theory of all zeta functions 1-4, then we can prove Riemann's hypothesis.

1.2. Kurokawa wishes to prove Riemann's hypothesis. He wished too much to stay sane.

1.3. Kurokawa wishes to have the unification theory of four kinds of zeta functions. He compares this with the problem of the unification theory in physics. There are four kinds of forces in physics:

1. Electromagnetic forces.

2. Strong nuclear forces.

3. Weak nuclear forces.

4. Gravitational forces.

Kurosawa believes that these four forces correspond to the above four zeta functions.

Kurokawa also thinks that there are four kinds of lives; animals, plants, (I forgot the other two).

Kurokawa believes that to try to unify four kinds of zeta functions, a good method is to think about the analogy with the four kinds of forces and with the four kinds of lives.

Hence in his class on zeta functions, a big drawing of a cell appears on the blackboard and Kurokawa considers mitochondrion and flagellum in the cell, and then he discusses whether we find mitochondrion or flagellum in the zeta functions in each 1–4. He thinks the studies of mitochondrion and flagellum are the important keys for the study of zeta functions. I saw in his class, the poor students were copying the drawing of the cell on the blackboard faithfully in their notebooks, not understanding that their teacher is just a crazy person.

1.4. People in physics think that the above four kinds of forces were just one force when the universe started. People in biology think that lives were not separated to animals, plants, ... when the lives started. Hence Kurokawa believes that the above four kinds of zeta functions were just one zeta function when the universe started.

1.5. An organizer of a conference once asked Kurokawa to give a talk on the history of zeta function. The organizer wished that Kurokawa would talk about the history of the study of zeta functions. He was shocked by the talk of Kurokawa who explained that when the universe started there was only one zeta function, and talked about mitochondrion and flagellum drawing a cell on the blackboard. The life on the earth was simple when it started. Kurosawa told that the first zeta function which appeared in the universe was also simple and like $1/s$ (as a function in s). This simple zeta function was swimming in the ancient ocean by moving its flagellum.

1.6. Here I explain Selberg zeta function (but my understanding is not sufficient and I may make a mistake). For a hyperbolic manifold X (I do not explain this), the Selberg zeta function $\zeta_X(s)$ is defined by

$$\zeta_X(s) := \prod_{\gamma} \frac{1}{1 - N(\gamma)^{-s}}$$

where γ ranges over all geodesics on X and $N(\gamma) = e^{l(\gamma)}$ with $l(\gamma)$ the length of γ.