Herman Dyudin, 30.09.2003

Second Theory Test

Task 1

Problem: Characterize equation $y' - 30y = 9y^2$, $where\ y$: $[0, 0.5] \to$ **R**; write (and explain) polynomial (of the 3rd degree) approximation for the IVP: $y(0) = 2003$; define (and explain) the order of the polynomial approximation in some neighborhood of 0.

Solution:

Firstly, it is nonlinear Bernoulli differential equation with degree=2 that matches pattern from lecture notes (topic I): $y' + g(x) \times y = f(x) \times y^k$, $k \in$ **R .** In our case k = 2**.**

To write polynomial approximation for the IVP, we will use series approach (as in lecture slides of topic IV)

Solution in the form of Maclaurin series:

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(0) \times x^n}{n!}$$

For polynomial (of the 3rd degree as it given in statements) approximation, we should find $y(0)$, $y'(0)$, $y''(0)$ and $y'''(0)$. $y(0) = 2003$ – our initial value from statements.

$y' - 30y = 9y^2$, then $y' = 30y + 9y^2 = y \times (30 + 9y)$. (move $30y$ to the right)

$y'(0) = y(0) \times (30 + 9 \times y(0)) = 2003 \times (30 + 9 \times 2003) = 2003 \times 18057$

Then we differentiate both sides of $y' = 30y + 9y^2$, we get the following:

$y'' = 30 \times y' + 18 \times y \times y'$. So:

$$y''(0) = 30 \times y'(0) + 18 \times y(0) \times y'(0) =$$
$$= 2003 \times 18057 \times (30 + 18 \times 2003) = 2003 \times 18057 \times 36084$$

Again differentiate both sides: $y'' = 30 \times y' + 18 \times y \times y'$, then:

$y''' = 30 \times y'' + 18 \times (y \times y'' + y' \times y') = y'' \times (30 + 18 \times y) + 18 \times y' \times y'$. For point x=0:

$$y'''(0) = y''(0) \times (30 + 18 \times y(0)) + 18 \times y'(0) \times y'(0) =$$
$$= 2003 \times 18057 \times 36084 \times (30 + 18 \times 2003) + 18 \times 2003^2 \times 18057^2 =$$
$$= 2003 \times 18057 \times (36084^2 + 18 \times 2003 \times 18057) = 2003 \times 18057 \times 1953082134$$

Hence, by Maclaurin series formula that was mentioned above:

$$y(x) = 2003 + 2003 \times 18057 \times x + \frac{2003 \times 18057 \times 36084}{2} \times x^2 + \frac{2003 \times 18057 \times 1953082134}{6} \times x^3 + \ldots$$

So, 3rd degree approximation:

$$y(x) = 2003 + 2003 \times 18057 \times x + \frac{2003\times18057\times36084}{2} \times x^2 + \frac{2003\times18057\times1953082134}{6} \times x^3$$

But*:* In my equation there is a discontinuity point. Using analytical solution of IVP we get: $y(x) = \frac{20030e^{30x}}{6019-6009e^{30x}}$, which has discontinuity point at $x = \frac{1}{30} \times ln(\frac{6019}{6009})$, that lies in $[0, 0.5]$. So, approximation is valid only up to this point.

To define the order of the polynomial approximation in some neighborhood of 0, Taylor's theorem with Peano's remainder should be used:

$$f(x + a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + ... + \frac{x^n}{n!}f^{(n)}(a) + o(x^n)$$

where $o(x^n)$ represents a function g(x) with g(x)/x^n → 0 as x → 0

Taking a as 0 and $f(x + a)$ as y(x), y(x) can be written as:

$$y(x) = 2003 + 2003 \times 18057 \times x + \frac{2003\times18057\times36084}{2} \times x^2 + \frac{2003\times18057\times1953082134}{6} \times x^3 + o(x^3)$$

So, the order of the polynomial approximation of y(x) in some neighborhood of 0 is 3.

Answer:

$$y(x) = 2003 + 2003 \times 18057 \times x + \frac{2003\times18057\times36084}{2} \times x^2 + \frac{2003\times18057\times1953082134}{6} \times x^3$$

for x in [0, 0.5] that are up to point of discontinuity, y ∈ R. For bigger values of x, approximation is not valid. The order of the polynomial approximation of y(x) in some neighborhood of 0 is 3.

Task 2

Problem: $30 \times y'' + 9 \times y = 0,$ where y: [0, +∞) → **R .** Characterize it, define what is stability of the equation at equilibrium point (0,0) and prove stability

Firstly, it is a second-order linear homogeneous ordinary differential equation.

Proof:

Let $x_1(t)$ be equal to y(x) and $x_2(t) = y'(x)$

$x'_1 = y' = x_2$ and $x'_2 = y''$. From the initial equation, $y'' = -\frac{3}{10}y$, $x'_2 = y'' = -\frac{3}{10}y = -\frac{3}{10}x_1$.

So we have: $x'_1 = x_2$, $x'_2 = -\frac{3}{10}x_1$.

Consider the system:

$x = f(\chi)$, $\chi_{t=t_0} = \chi_0$, where $\chi \in R^2$, $\chi = (x_1, x_2)$, f(χ) = $(x_2, -\frac{3}{10}x_1)$

<u>Using Tutorial 7 notes, topic V</u>: Assume that solution exists for all $t > t_0$, then define the stability of the equation at the equilibrium point $x_e = (0,0)$:

"The solution is called stable by Lyapunov

if $\forall \varepsilon > 0$ $\exists \delta > 0$, such that $\forall t > t_0$

$\|\chi(t) - x_e\| < \varepsilon$ ***(In our problem: $\|\chi(t) - (0,0)\| = \|\chi(t)\| < \varepsilon$),*** **(*)**

if $\|\chi(t_0) - x_e\| < \delta$ ***(In our problem:*** **$\|\chi(t_0)\| < \delta$). "**

Now, consider the system $\dot{x}$ = f(χ), f(0) = 0,

where $\chi = (x_1, x_2)$, f(χ) = $(x_2, -\frac{3}{10}x_1)$

We will use Second Lyapunov's stability theorem from Tutorial 7 slides to prove that the equation at equilibrium point (0,0) is stable:

"If there exist the Lyapunov function for the system

$\dot{x}$ = f(χ), f(0) = 0, **(**)**

then the zero equilibrium is stable."

Definition of the Lyapunov function from the Tutorial 7 slides:

"A continuously differentiable function L(x) is called Lyapunov function for the system

$\dot{x}$ = f(x), f (0) = 0,

at the equilibrium x = 0 if:

(***)

- **L $\in$ (C^1), L(x): $R^n \to$ R;**
- **L(x) = 0, x = 0;**
- **L(x) > 0, x $\neq$ 0;**
- **$\exists\ \epsilon > 0$ such that $\acute{L}$(x(t)) $\leq$ 0, as $\forall \|x\| < \epsilon$."**

Let us define and consider new function: L(χ) = $x_1^2 + \frac{10}{3}x_2^2$, where χ = (x_1, x_2), L: $R^2 \rightarrow R$.

- **L is continuously differentiable, L(χ): $R^2 \rightarrow R$,**
- L(0, 0) = $x_1^2 + \frac{10}{3}x_2^2 = 0 + \frac{10}{3} \times 0 = 0$,
- **Since** $x_1^2 \geq 0$, $\frac{10}{3}x_2^2 \geq 0$, **if** $\chi \neq (0, 0)$, **then L(χ) > 0,**
- $L(\chi)' = 2x_1 \times x'_1 + \frac{20}{3}x_2 \times x'_2 = 2x_1 \times x_2 + \frac{20}{3}x_2 \times (-\frac{3}{10}x_1) =$

$$= 2x_1 \times x_2 - 2x_2 \times x_1 = 0.$$

So, $\acute{L}(\chi) \leq 0$ **for all** χ, **then** $\exists\ \epsilon$ **> 0 :** $\acute{L}$**(χ) ≤ 0, as ∀||χ|| <** ϵ

Using this properties, L(x) actually is a Lyapunov function for system **(*)**, because of the definition from Tutorial that was mentioned above **(**)**.

Answer: So, by the theorem from Tutorial **(***)**, **because there exists a Lyapunov function for the system, then the initially given equation is stable at equilibrium point (0,0)**.