Introduction

Welcome!

This resource is an open-source textbook for second-year classical mechanics and flipped classroom instruction that is currently under development for Queen's University by Prof Sarah Sadavoy with support from Queen's students Cora Sleegers and Lance Schonberg. It covers the main topics of dynamics courses. These topics include:

1. Newtonian Mechanics
2. Simple Harmonic Motion
3. Non-inertial and Rotating Reference Frames
4. Linear and Angular Momentum
5. Torque and Rotation
6. Work and Energy Conservation
7. Central Forces
8. Motion in Space
9. Orbits and Kepler's Laws
10. The Euler-Lagrange Method

How to Best Use this Book

This book follows the design and format from *Introductory Physics: Building Models to Describe Our World*, which is an open-source textbook for first-year physics developed by Prof Ryan Martin and students Emma Neary, Joshua Rinaldo, and Olivia Woodman among others.

To help broaden the physics instruction, this book contains a number of different elements to help with the learning process, including check point questions, worked examples, student commentary, and real-world applications of the physics. There are also videos and extra problems using python available via our online repository. Each textbook feature is represented by a different tag, as summarized below:

Learning Objectives

- Summary of the learning objectives at the start of each chapter

Comments and Discussion

Additional information to help your reflect or research a topic.

Quick Questions

1. Short checkpoint questions to test your understanding

Student's Thoughts

Commentary from a student's perspective on challenging concepts.

Sample Problem

Example problems with full solutions

Key Takeaways

Summary of the main topics, methods, and objectives from the chapter.

Important Equations

List of the most important equations from the chapter.

Real World Applications

Examples of how physics concepts are integrated into experimentation and every-day life, as well as viewpoints from across the world.

Practice Problem

Problems at the end of each chapter to test your understanding of the concepts.

Physics Mindset and Strategies

Physics teaches **problem solving**. The main goals of this textbook are to develop your **physics toolkit** and to teach **critical thinking** so that you can break down big problems into smaller, easier to implement pieces. There are often multiple ways to solve a physics problem, and by building your foundation, you will be able to select which of those ways is most efficient or most ideal. When solving problems, think about the different methods you could use to solve physics problems and under which situations you would favour one method over another.

Thinking like a physicist: while there are lots of ways to approach physics problems. As you practice solving problems, try to recognize and develop the strategies that make learning best for you. Everyone is different, but here are some common tips and strategies to consider:

I) Make sure you understand the system set up: Before attempting any problem, be sure you know what the system looks like given the description and all the moving parts. If it helps to visualize it, draw a diagram (it doesn't need to be perfect). Taking a few moments to think about the system as a whole can save you a lot of time and trouble when solving the problem later on.

II) Plan how you will approach the problem: There are a number of ways you can solve most physics questions (e.g., Newton's laws, conservation laws, Euler-Lagrange method). Before you decide on an approach, consider what information is given to you, what the questions is asking, and what assumptions you can or cannot make. The more you practice, the better you will intuit how to approach most questions.

III) Stick with variables/symbols as long as possible: Symbols are less messy than numbers and you don't have to worry about units. It is also easier to check your answer at the end if you only have to plug numbers into your calculator once.

IV) Look for tricks to simplify the problem: Often, physicists can use math tricks or approximations to simplify the calculations or set up of a physics problem. Examples include math substitutions, scaling relations, or Taylor series simplifications. The textbook will highlight these approaches and give insights on when they are appropriate. Use these to your advantage.

V) Check your answer: After working through a complex problem, consider what you got at the end. Do your units make sense? Does the order of magnitude of a numerical value make sense? For example, if you get the motion of a simple pendulum to be faster than the speed of light, that should be an automatic flag that something went wrong. Think about your answer critically before moving on.

Dimensional Analysis Tips and Tricks

Dimensional analysis can be an effective way to check your answer. Unit conversions can be tricky and in some cases, you may be given specialized units over standard units (e.g., distances in astronomy are often in parsec or light years). Dimensional analysis lets you verify that the units broadly match your expectations. For example, the dimensional analysis of kinetic energy, $K = \frac{1}{2}mv^2$ is:

$$\begin{aligned}[\mathrm{J}] &= [\mathrm{kg}][\mathrm{m\ s^{-1}}]^2 \\ [\mathrm{J}] &= [\mathrm{kg\ m^2\ s^{-2}}] \\ [\mathrm{J}] &= [\mathrm{J}]\end{aligned}$$

For more information, see lists of Standard International (SI) units and some simple conversions from Wolfram, National Physical Laboratory, and Wikipedia.

VI) Physics is best learned with others: Whether you are hearing the thoughts and perspective of a classmate or vocalizing your own understanding, **you learn more and retain that knowledge better** by sharing with others. Share ideas with your classmates. Work on problems with each other. And have fun!

Contents

1 Calculus and Vectors

Learning Objectives

- Review of vector notation and basic calculus
- Review of coordinate systems
- Application to simple physical systems

This chapter reviews basic calculus and vector notation and coordinate systems. Please also refer to Appendix A for helpful equations and identities.

1.1 Coordinates and Motion in Vector Notation

1.1.1 Linear Motion

For linear motion, we can use the Cartesian coordinate system. The position of an object is described by a vector $\vec{r}$ in x, y, and z, the linear velocity of the object is the time derivative of position, and the linear acceleration is the time derivative of velocity.

$$\boxed{\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \langle x, y, z\rangle} \tag{1.1}$$

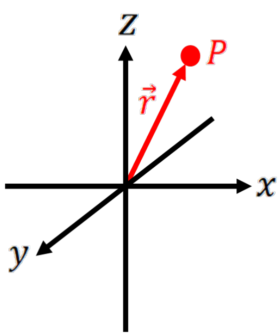


Figure 1.1: Position of a vector in Cartesian coordinates where $\hat{\imath}$ is the unit vector for x, $\hat{\jmath}$ is the unit vector for y, and $\hat{k}$ is the unit vector for z.

The equations of linear motion in 1-D:

$$v_x = \frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} \qquad a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \ddot{x}$$

$$v_y = \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y} \qquad a_y = \frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \ddot{y}$$

$$v_z = \frac{\mathrm{d}z}{\mathrm{d}t} = \dot{z} \qquad a_z = \frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{\mathrm{d}^2z}{\mathrm{d}t^2} = \ddot{z}$$

Helpful Nomenclature

A dot over a variable can be used as shorthand for the *time* derivative of that variable. Two dots would be the second time derivative, and so forth.

Note that dots over variables are only a shorthand for the time derivative. If you have a $\frac{\mathrm{d}}{\mathrm{d}x}$ derivative, then do not use a dot.

Equations of linear motion in 3-D:

$$\begin{aligned}
\vec{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k} = \langle x, y, z\rangle \\
\vec{v} &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \dot{\vec{r}} = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\jmath} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{k} = \left\langle \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle \\
\vec{a} &= \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = \ddot{\vec{r}} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\hat{\imath} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\hat{\jmath} + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\hat{k} = \left\langle \frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \frac{\mathrm{d}^2y}{\mathrm{d}t^2}, \frac{\mathrm{d}^2z}{\mathrm{d}t^2} \right\rangle
\end{aligned}$$

The above equations highlight that there are several ways to write a parameter in vector notation. For example, $\vec{r}$ can be expressed as $x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ or $\langle x, y, z\rangle$. You should try to be consistent within a problem so that there is less chance for confusion or error in your solutions.

Definitions

The time derivative of position gives you *instantaneous* velocity and the time derivative of velocity gives you the *instantaneous* acceleration. These are instantaneous because they correspond to the velocity or acceleration in that exact instant or moment in time. By contrast, the *average* velocity ($\bar{v} = \frac{\Delta\vec{r}}{\Delta t}$) and *average* acceleration ($\bar{a} = \frac{\Delta\vec{v}}{\Delta t}$) are measured over a longer duration of time, Δt. The average quantity is denoted by a bar (−) over the variable.

Note that in the limit as $\Delta t \to 0$, $\frac{\Delta\vec{r}}{\Delta t} \to \vec{v}$ and $\frac{\Delta v}{\Delta t} \to \vec{a}$. So for a very short duration of time, $\Delta t \to \mathrm{d}t$, the average velocity and average acceleration are equivalent to the instantaneous quantities.

Quick Questions

1. If your velocity is constant between t_1 and t_2, how does your instantaneous velocity at $t = t_2$ compare to the average velocity between t_1 and t_2?
2. If your acceleration is constant between t_1 and t_2, is the instantaneous velocity at $t = t_2$ the same as the average velocity between t_1 and t_2?

1.1.2 Rotational Motion

Rotational motion is when you have a body spinning about a rotation axis.

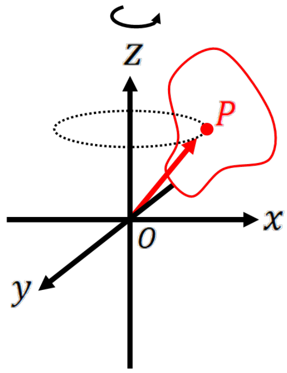


Figure 1.2: For a rigid body rotating on a fixed axis, a point P on the body will travel in a circle with radius r about the rotation axis.

For rotational motion, it is useful to describe the motion in terms of angles: angular position (θ), angular velocity (ω), and angular acceleration (α). Note that a radius r is also necessary to describe the motion, and we will assume this is constant for now. For this coordinate system to work, you need a reference axis (reference point).

Consider the figure below. In time t_1 to t_2 the object has rotated from the first position at θ_1 to the second position at θ_2. The distance from the origin to both points (radius) is constant. Thus, the angular position that the object moves is $\Delta\theta = \theta_2 - \theta_1$ in time $\Delta t = t_2 - t_1$. The distance traveled is the arc, s, as traced out by the angle $\Delta\theta$.

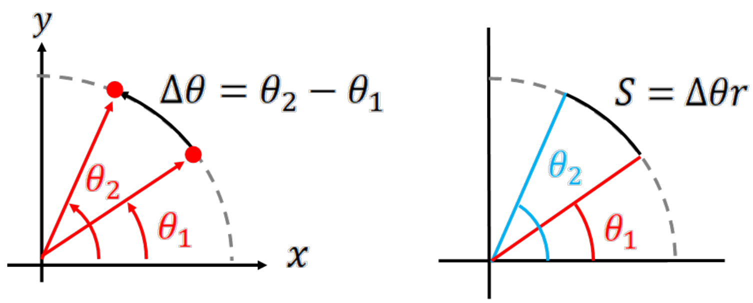


Figure 1.3: Left panel shows the change in angular position ($\Delta\theta$) that the object moves. The right panel defines the arc, S.

Definitions

The above example defines θ as increasing counter-clockwise. It is important that you define your axes at the start and that you keep consistent with that defined reference axis.

For angular motion, we generally measure angles in radians not degrees. One complete circle is when $\theta = 2\pi$ or $s = 2\pi r$ (the perimeter of a circle). Note that a complete rotation does not start back at $\theta = 0$.

2π rad $= 360° = 1$ revolution

1 rad $= 57.296° = 0.159$ revolutions

The average angular velocity of the object is then given by $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$. As we shrink Δt to a very small time interval ($\Delta t \to \mathrm{d}t$), then we get the instantaneous angular velocity or the angular velocity, ω. The angular velocity is given by the time derivative of θ and the angular acceleration is given by the time derivative of the angular velocity.

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta}$$

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \ddot{\theta}$$

The units of ω is rad s^{-1} and the units of α are rad s^{-2}, although we often drop the radians and give s^{-1} and s^{-2}, respectively. If you see s^{-1} or s^{-2} for ω and α, the radians are implied.

For a rigid body, all points in the object move with the same angular velocity and angular acceleration because every point is moving together (the object doesn't deform during rotation).

Quick Questions

1. What is the angle θ (in radians) for two complete rotations?
2. A wheel with radius of 1 m rotates at 2.5 revolutions per second. What is the angular displacement (in radians) of the wheel after 1 minute?

1.2 Introduction to Plane Polar Coordinates

In cases of circular motion, it is often easier to solve a problem by changing your coordinate system from Cartesian plane (x, y) to polar coordinates (r, θ). The two coordinate systems are connected, where $x = r\cos\theta$ and $y = r\sin\theta$, where r is the radius length and θ is the polar angle (see Figure 1.4). Solving for r and θ, we get:

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Of course, r and θ are vector quantities, where $\hat{r}$ points away from the origin of the system and $\hat{\theta}$ is orthogonal to $\hat{r}$ in the counter-clockwise direction (usually). Note that the ˆ symbol indicates a unit vector (direction only). Figure 1.4 shows these vector directions.

The position vector in plane polar coordinates can be written as $\vec{r} = r\hat{r}$ and the angle vector can be written as $\vec{\theta} = \theta\hat{\theta}$.

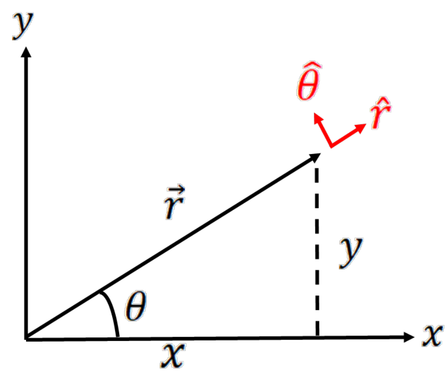


Figure 1.4: Visual definitions of $\hat{r}$ and $\hat{\theta}$ in plane polar coordinates. The unit vector for radius extends away from the origin and the unit vector for angle points counter clockwise. Note that $\hat{\theta}$ is always tangent to the radius by definition.

Real World Applications

Aircraft and naval navigation are both based on cylindrical coordinate systems, using a distance (radius), direction (angle), and altitude or depth (z). These coordinate systems are often slightly modified to use North, either magnetic or true, as the zero angle. The use of polar coordinates are helpful in putting context to the positions of objects and obstacles relative to the moving vehicle.

Now we want to find an equation for $\vec{v}$ and $\vec{a}$ in polar coordinates instead of Cartesian coordinates. This is a variation of how we defined velocity and acceleration previously (because previously we used Cartesian coordinates).

Let's look at velocity first.

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \frac{\mathrm{d}(r\hat{r})}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t}$$

where $\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} \neq 0$. Consider the line moving in Figure 1.4. The $\hat{r}$ unit vector will point in a different direction as the radius vector moves around the circle.

Figure 1.5 defines the $\hat{r}$ and $\hat{\theta}$ unit vectors in terms of Cartesian axes. Both $\hat{r}$ and $\hat{\theta}$ have components in x and y.

$$\begin{aligned}\hat{r} &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}\\ \hat{\theta} &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}\end{aligned}$$

Note that the Cartesian unit vectors ($\hat{\imath}$, $\hat{\jmath}$) are fixed, whereas the polar-axes ($\hat{r}$, $\hat{\theta}$) are moving relative to them because the radial vectors is moving.

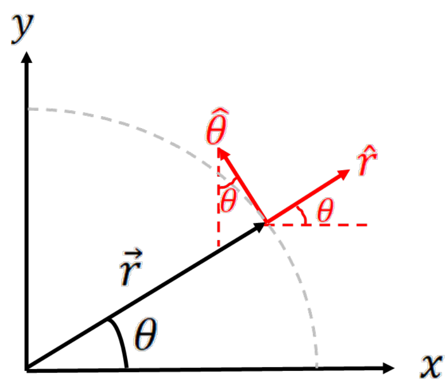


Figure 1.5: Sketch showing how $\hat{r}$ and $\hat{\theta}$ can be described in terms of $\hat{\imath}$ and $\hat{\jmath}$. Note that the Cartesian system ($\hat{\imath}$ and $\hat{\jmath}$) do not change with time, but the plane polar system ($\hat{r}$ and $\hat{\theta}$) do change with time.

If we take the derivative of $\hat{r}$ with respect to time, we get:

$$\begin{aligned}
\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\
&= -\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\imath} + \cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\jmath} \quad \Longrightarrow \hat{\imath} \text{ and } \hat{\jmath} \text{ don't change with time} \\
&= \frac{\mathrm{d}\theta}{\mathrm{d}t}\underbrace{(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath})}_{\hat{\theta}} \\
&= \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta} \quad \Longrightarrow \omega\hat{\theta}
\end{aligned}$$

The above equation applies a full time derivative to $\hat{r}$, which means that you must not only take the time derivative of $\cos\theta$ and $\sin\theta$, but also time derivative of θ. See the textbook repository for a video showing the difference between full and partial derivatives.

Now that we have $\frac{\mathrm{d}\hat{r}}{\mathrm{d}t}$, we can go back to our velocity equation from before. For polar coordinates we get:

$$\vec{v} = \frac{\mathrm{d}r}{\mathrm{d}t}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t}$$

$$\boxed{\vec{v} = \underbrace{\frac{\mathrm{d}r}{\mathrm{d}t}}_{\vec{v}_r}\hat{r} + \underbrace{r\frac{\mathrm{d}\theta}{\mathrm{d}t}}_{\vec{v}_\theta}\hat{\theta}} \tag{1.2}$$

where the first term is the radial velocity component $\vec{v}_r$ and the second term is the tangential velocity component $\vec{v}_\theta$. The radial velocity component indicates how the point is moving in and out along the direction of the radius vector, whereas the tangential velocity component of the motion describes how the point is moving along a circle (motion that is tangent to the radius vector). For circular motion, the radius is constant, such that $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$ and you get $\vec{v}_{circ} = r\omega\hat{\theta}$.

By definition the speed (or $|\vec{v}|$) is given by $\vec{v}\cdot\vec{v} = \sqrt{v_r^2 + v_\theta^2} = \sqrt{v_r^2 + (r\omega)^2}$, using the vector dot product (see also, Chapter 1.5.1).

We can also take the time derivative of $\hat{\theta}$, using the definition of $\hat{\theta}$ in Cartesian coordinates.

$$\begin{aligned}
\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) \\
&= -\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\imath} - \sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\jmath} \\
&= -\frac{\mathrm{d}\theta}{\mathrm{d}t}\underbrace{(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})}_{\hat{r}} \\
&= -\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{r} \quad \Longrightarrow -\omega\hat{r}
\end{aligned}$$

For acceleration, we want the time derivative of velocity. Following a similar procedure,

$$\begin{aligned}
\vec{a} &= \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\hat{r} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta}\right) \\
&= \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\hat{r} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\hat{\theta} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} \quad \Longrightarrow \text{sub in } \frac{\mathrm{d}\hat{r}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta}, \frac{\mathrm{d}\hat{\theta}}{\mathrm{d}t} = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{r} \\
&= \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\hat{r} + \frac{\mathrm{d}r}{\mathrm{d}t}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta}\right) + \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\hat{\theta} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\left(-\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{r}\right) \\
&= \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\hat{r} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta} + r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\hat{\theta} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\hat{r} \quad \Longrightarrow \text{simplify}
\end{aligned}$$

$$\boxed{\vec{a} = \underbrace{\left[\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]\hat{r}}_{a_r} + \underbrace{\left(r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{\theta}}_{a_\theta}} \quad \Longrightarrow \text{collect } \hat{r} \text{ and } \hat{\theta} \text{ terms} \tag{1.3}$$

where the first term is the acceleration in the radial direction (a_r) and the second term is the acceleration in the tangential direction (a_θ). That is:

$$\begin{aligned}
\vec{a}_r &= \left[\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]\hat{r} \\
\vec{a}_\theta &= \left(r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{\theta}
\end{aligned}$$

These acceleration terms are key in rotating reference frames (Chapter 5).

Quick Questions

1. Consider an object moving such that θ is constant with time. What is the acceleration term in this case? Does this make sense?
2. Consider a simple pendulum in plane polar coordinates. How does the radial acceleration ($\vec{a}_r$) change as a function of time? Comment.

1.3 Equations of Motion

In the previous section, we defined the positions, velocities, and accelerations for linear and circular motion. To describe the motion, however, you need to solve these equations.

$$\begin{aligned}
\vec{v} &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \quad \Longrightarrow \quad \vec{r} = \int \vec{v}\ \mathrm{d}t \\
\vec{a} &= \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \quad \Longrightarrow \quad \vec{v} = \int \vec{a}\ \mathrm{d}t \\
\omega &= \frac{\mathrm{d}\theta}{\mathrm{d}t} \quad \Longrightarrow \quad \theta = \int \omega\ \mathrm{d}t \\
\alpha &= \frac{\mathrm{d}\omega}{\mathrm{d}t} \quad \Longrightarrow \quad \omega = \int \alpha\ \mathrm{d}t
\end{aligned}$$

The solutions to these integrals depends on how the system moves with time. For example, consider the case when α and $\vec{a}$ are constant with time. Let's look at the case of linear motion with constant acceleration in 1-D so that we can drop the vector notation.

$$\begin{aligned}
v_x &= \int a_x \mathrm{d}t \\
v_x &= a_x t + C_1 \qquad \Longrightarrow a \text{ is constant (definition), } C_1 \text{ is constant of integration}
\end{aligned}$$

$$\begin{aligned}
x &= \int v_x \mathrm{d}t \\
x &= \int (a_x t + C_1)\ \mathrm{d}t \qquad \Longrightarrow \text{sub in equation for } v \\
x &= \frac{1}{2} a_x t^2 + C_1 t + C_2 \qquad \Longrightarrow C_2 \text{ is constant of integration}
\end{aligned}$$

The constant C_1 is the initial velocity $v_{x,0}$ (at $t = 0$) and C_2 is the initial position x_0 (at $t = 0$). Subbing in those definitions for the constants of integration, we get:

$$v_x = a_x t + v_{x,0}$$

$$x = \frac{1}{2} a_x t^2 + v_{x,0} t + x_0$$

Hopefully these equations look familiar. Of course, since position and velocity are vectors quantities, you need to solve for the motion along the different coordinate axes (e.g., x, y, z) separately. For example, the acceleration may be zero along one axis and non-zero along another axis (e.g., such is the case with gravity).

Note that if you take the time derivative of $x = \frac{1}{2} a_x t^2 + v_{x,0} t + x_0$, you recover the equation for $v_x = a_x t + v_{x,0}$ as you should. In general, it is a good idea to check the consistency of your equations.

Definitions

The above equations of motion for v_x and x (and the equivalent for y and z) are only applicable if the acceleration is constant. If your acceleration is changing as a function of time, then the above equations will not apply and you need to solve the equations of motion (see Chapter 2).

Real World Applications

We typically use Standard Internation (SI) units to describe position, velocity, acceleration, and time. But historically and around the world, there have been may different ways of looking at those measurements. One interesting example are water clocks from the Babylonian Empire, where time had the same units as mass. These clocks used the weight of water passing through the clock as a measure of time. Since the Babylonian Empire wasn't directly on the equator, the amount of water used to break up the day had to be adjusted throughout the year.

1.4 Linear and Rotational Motion

We can also connect circular motion to linear motion. Consider two points associated with rotational motion with a circular radius of r_0 as shown below.

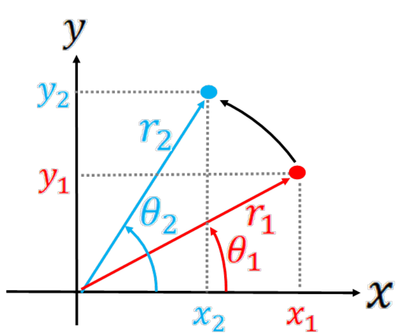


Figure 1.6: The system is rotating from position θ_1 to θ_2. The vectors $\vec{r}_1$ and $\vec{r}_2$ represent those two positions. Note that $|\vec{r}_1| = |\vec{r}_2|$ in this simple case.

We can write the radius vectors in terms of their x and y values. For example, $x_1 = r_1 \cos\theta_1$ and $y_1 = r_1 \sin\theta_1$. The same can be applied to the second position. In vector form, we get:

$$\begin{aligned} \vec{r} &= x(t)\hat{\imath} + y(t)\hat{\jmath} \\ \vec{r} &= r_0 \cos\theta\hat{\imath} + r_0 \sin\theta\hat{\jmath} \end{aligned}$$

For circular motion, θ changes with time. Let's consider the simplest case where $\dot{\theta} = \omega =$ constant, such that we can solve for $\theta(t)$ as $\theta = \int \omega \mathrm{d}t = \omega t + \theta_0$, where θ_0 is a constant of integration and represents the initial angle. It is often convenient to define the initial angle as $\theta_0 = 0$ so $\theta(t) = \omega t$. Therefore, we get:

$$\boxed{\vec{r} = \underbrace{r_0 \cos(\omega t)\hat{\imath}}_{x(t)} + \underbrace{r_0 \sin(\omega t)\hat{\jmath}}_{y(t)}} \tag{1.4}$$

We can then look at the velocity and acceleration of the system by just taking the time derivatives of $\vec{r}$.

$$\begin{aligned}
\vec{v} &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \\
\vec{v} &= [-r_0\omega \sin(\omega t)]\,\hat{\imath} + [r_0\omega\cos(\omega t)]\,\hat{\jmath} \\
\\
\vec{a} &= \frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \\
\vec{a} &= \left[-r_0\omega^2\cos(\omega t)\right]\hat{\imath} + \left[-r_0\omega^2\sin(\omega t)\right]\hat{\jmath} \\
\vec{a} &= -\omega^2\vec{r}
\end{aligned}$$

Quick Questions

1. Show that the speed $|\vec{v}|$ in the above example is equal to ωr_0.
2. For constant ω, compare the linear acceleration $\vec{a}$ to the angular acceleration α?

Sample Problem 1-1

You throw a ball straight up into the air with a constant velocity of v_0 and at an initial height of r_0. **(1) What is the maximum height that the ball reaches? (2) How long does it take for the ball to hit the ground?**

Solution

This is a 1-D motion problem under constant acceleration ($a = -g\ \hat{y}$). First consider how the ball will move. For vertical motion upward with a downward acceleration, the ball will initially rise. But due to the pull downward by gravity, the ball will slow down, and then momentarily come to a stop ($v = 0$ at the crest of motion) before it falls back down again, accelerating as it falls. Figure 1.7 shows a cartoon of this motion.

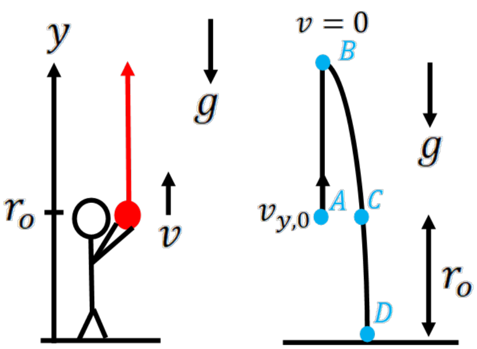


Figure 1.7: Cartoon of 1-D vertical motion.

For vertical motion upward with an acceleration downward, the ball rises initially (A $\rightarrow$ B) and slows down as it moves upward until it has $v = 0$ (point B at the peak), before it falls back down again (B $\rightarrow$ D).

Note that we are using the y axis only at this time because all the motion is in the vertical.

1. **What is the maximum height that the ball reaches?** This is the height at point B in Figure 1.7. This problem is a linear motion question with constant acceleration. We just solved that equation in Section 1.3, so we will need to use the equation, $y = \frac{1}{2}at^2 + v_{y,0}t + r_0$ to solve for y when the ball is at its heights point. We aren't given that time when this happens, but we can solve for it, because when the ball has reached its maximum height, $v_y = 0$ (requirement of the physics). So the first step is to get the time when the ball has reached its maximum height.

$$\begin{aligned} v_y &= at + v_{y,0} && \Longrightarrow && \text{see Section 1.3} \\ 0 &= at + v_{y,0} && \Longrightarrow && \text{set } v_y = 0 \text{ at the maximum height} \\ t &= -\frac{v_{y,0}}{a} \\ t &= \frac{v_{y,0}}{g} && \Longrightarrow && \text{because } a = -g \end{aligned}$$

For simplicity, we can drop the vector notation because everything is happening in 1-D. Here positive corresponds to $+\hat{y}$ and negative corresponds to $-\hat{y}$.

So now we have the time when the ball reaches the maximum height. We can put this time into our distance equation to solve for the maximum height.

$$\begin{aligned} y &= \frac{1}{2}at^2 + v_{y,0}t + r_0 \\ y &= \frac{1}{2}(-g)\left(\frac{v_{y,0}}{g}\right)^2 + v_{y,0}\left(\frac{v_{y,0}}{g}\right) + r_0 && \Longrightarrow && t = \frac{v_{y,0}}{g},\ a = -g \\ y &= -\frac{1}{2}\left(\frac{v_{y,0}^2}{g}\right) + \left(\frac{v_{y,0}^2}{g}\right) + r_0 && \Longrightarrow && \text{simplify} \\ y &= \frac{1}{2}\left(\frac{v_{y,0}^2}{g}\right) + r_0 \end{aligned}$$

Now we have our equation for the maximum height given our initial velocity $v_{y,0}$ and initial height r_0. This is the generic solution for all initial values of $v_{y,0}$ and r_0. If you are given these quantities, you can plug them in to solve the problem.

Quick Questions

(a) Check the dimensional analysis for t and y in the above equations.

(b) What is the maximum height of a ball when $v_{y,0} = 10$ m s^{-1} and $r_0 = 1.8$ m (assume $g = 9.8$ m s^{-2})?

(c) Consider the case where the acceleration is positive, not negative. What does $t = \frac{-v_{y,0}}{a}$ mean for a positive acceleration?

2. **How long does it take for the ball to reach the ground?** So this is at the end of the motion (point D in Figure 1.7). We don't know the speed at which the ball reaches the ground or the time, but we do know the ball hits the ground when $y = 0$. So we want to solve for the time when $y = 0$. Using the height equation, we get $y = -\frac{1}{2}gt^2 + v_{y,0}t + r_0 = 0$, which is a quadratic equation. The solution for a quadratic equation of the form $0 = Ax^2 + Bx + C$ is:

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

In this case, t is our variable, $A = -\frac{1}{2}g$, $B = v_{y,0}$, and $C = r_0$. Plugging those numbers in gives:

$$t = \frac{v_{y,0} \pm \sqrt{v_{y,0}^2 + 2gr_0}}{g}$$

There are two solutions. Since $2gr_0$ is positive, the term $\sqrt{v_{y,0}^2 + 2gr_0} > v_{y,0}$ for all values of $v_{y,0}$ and r_0. So there will be one value of $t > 0$ and one value of $t < 0$. The latter case ($t < 0$) is unphysical given the set up of this problem, however. While it mathematically solves the problem, we know that the ball's motion started from a height r_0 at $t = 0$. Effectively, the $t < 0$ case corresponds to the time when the ball would need to be thrown from $y = 0$ such that it has a speed of $v_{y,0}$ at $t = 0$ and height r_0. But that wasn't our question, so we are instead interested in the $t > 0$ case.

So our solution to this problem is:

$$t = \frac{v_{y,0} + \sqrt{v_{y,0}^2 + 2gr_0}}{g} \quad \Longrightarrow \quad \text{Drop the } - \text{ case because it is unphysical}$$

Quick Questions

(a) Check the dimensional analysis for t from the quadratic equation.

(b) What is the velocity of the ball when it hits the ground?

(c) What is the velocity of the ball at point C in Figure 1.7?

Sample Problem 1-2

A wheel rotating at an angular speed of ω_0 is allowed to decelerate. After τ seconds the new angular speed is ω_τ. If the angular acceleration is constant, **how long does it take the wheel to come to rest and how many revolutions does the wheel make before coming to a rest?**

Solution

Let's first consider the motion. The wheel is fixed in place and spinning along an axis. The rate at which it is spinning is slowing down with time, but we do not know the angular acceleration (only that it is negative). We also do not know how long it takes to come to rest. But we are given the initial angular speed (ω_0), and the speed ω_τ at a specific time τ, where $\omega_0 > \omega_\tau$. We also know the final angular speed $\omega_t = 0$ at time t.

To solve this problem, we need to look at equations for angular motion. With constant angular acceleration, these have the same form as the equations for rotational motion that we went through earlier in Section 1.2.

$$\omega = \alpha t + \omega_0, \qquad \theta = \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0$$

1. **How long does it take the wheel to come to a rest?** We will use the equation for angular speed to solve this problem. (You may notice a degree of similarity with the last problem. This was intentional to show the how similar problems can have slight differences in answers and methodology.)

   Here, we don't know α or t. If we set $\omega = 0$ at time t, we have $0 = \alpha t + \omega_0$, where the only known quantity is ω_0. But we can solve for α because we are told that the acceleration is constant. That means that the instantaneous acceleration at any time is equal to the average acceleration between any fixed time. Between $t = 0$ and $t = \tau$, the angular velocity decreased from ω_0 to ω_τ such that the average acceleration is:

$$
\begin{aligned}
\alpha &= \frac{\Delta\omega}{\Delta t} \\
\alpha &= \frac{\omega_\tau - \omega_0}{\tau} \quad \Longrightarrow \quad \text{for } \omega_0 \text{ at } t = 0 \text{ to } \omega_\tau \text{ at } t = \tau
\end{aligned}
$$

Now that we have α, we can solve for the time at which the wheel has reached rest.

$$
\begin{aligned}
t &= -\frac{\omega_0}{\alpha} \quad \Longrightarrow \quad 0 = \alpha t + \omega_0 \text{ as given above} \\
t &= -\frac{\omega_0 \tau}{\omega_\tau - \omega_0} \Longrightarrow \quad \text{sub in the equation for } \alpha
\end{aligned}
$$

Note that for $t > 0$, you must have $\omega_\tau < \omega_0$ (true by definition).

2. **How many revolutions does the wheel make before coming to a rest?** This is a question of how large an angle, θ, the wheel rotates through. In Section 1.1.2, we defined revolutions and angular displacement. Recall the quick question in Section 1.1.2 about how many radians are in 2 revolutions and 3 revolutions.

   To get the total angular displacement ($\Delta\theta$) from $t = 0$ until the wheel comes to rest at t, we can use the above equation for θ, because we have α, t, and ω_0.

$$
\begin{aligned}
\Delta\theta &= \frac{1}{2}\alpha t^2 + \omega_0 t + \theta_0 \quad \Longrightarrow \quad \text{set } \theta_0 = 0\text{, sub in equations for } \alpha \text{ and } t \\
\Delta\theta &= \frac{1}{2}\left(\frac{\omega_\tau - \omega_0}{\tau}\right)\left(-\frac{\omega_0 \tau}{\omega_\tau - \omega_0}\right)^2 + \omega_0\left(-\frac{\omega_0 \tau}{\omega_\tau - \omega_0}\right) \\
\Delta\theta &= \frac{1}{2}\left(\frac{\omega_0^2 \tau}{\omega_\tau - \omega_0}\right) + \left(-\frac{\omega_0^2 \tau}{\omega_\tau - \omega_0}\right) \quad \Longrightarrow \quad \text{simplify} \\
\Delta\theta &= -\frac{1}{2}\left(\frac{\omega_0^2 \tau}{\omega_\tau - \omega_0}\right)
\end{aligned}
$$

   Note that we are interested in the angular displacement. As such, the initial angle θ_0 does not matter. We are counting revolutions from $t = 0$ where θ_0 is our reference angle and set to $\theta_0 = 0$.

Quick Questions

(a) How many revolutions do you get if $\omega_0 = 240$ revolutions per minute, $\omega_\tau = 180$ revolutions per minute, and $\tau = 10$ s?

(b) How many revolutions do you get if $\omega_0 = 19$ radians per second, $\omega_\tau = 12$ radians per second, and $\tau = 1$ minute?

1.5 Vector Calculus

1.5.1 Vector Dot Product

For any two vectors, $\vec{a} = x_a\hat{\imath} + y_a\hat{\jmath} + z_a\hat{k}$ and $\vec{b} = x_b\hat{\imath} + y_b\hat{\jmath} + z_b\hat{k}$, the vector dot product (also called the vector scalar product) is given by:

$$\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b + z_a z_b$$

Since the dot product is just scalar multiplication of terms, vector order does not matter (e.g., $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$).

In principle, the dot product represents the projection of one vector onto the other. For example, when you want to calculate the x-component of a vector, you take the projection of that vector on the x-axis. This is equivalent to $\vec{a} \cdot \hat{\imath}$, where only the x-component is retained.

The dot product can also be expressed as:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

where $|\vec{a}|$ is the magnitude of $\vec{a}$, $|\vec{b}|$ is the magnitude of $\vec{b}$ and θ is the angle between the two vectors when the vectors are tail-to-tail. So the angle between any two vectors can be calculated from:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

A vector magnitude is given by:

$$b = |\vec{b}| = \sqrt{b_x^2 + b_y^2 + b_z^2}$$

which is essentially the dot product of a vector with itself ($\theta = 0$).

$$b^2 = |\vec{b}|^2 = \vec{b} \cdot \vec{b}$$

1.5.2 Vector Cross Product

For any two vectors, $\vec{a}$ and $\vec{b}$, the vector cross product is given by:

$$\vec{c} = \vec{a} \times \vec{b}$$

Unlike the dot product, the vector cross product results in a vector, which has both magnitude and direction, and the vector $\vec{c}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. In other words, the vector cross product $\vec{c}$ is normal (perpendicular) to a plane that is defined by $\vec{a}$ and $\vec{b}$.

The magnitude of $\vec{c}$ can be given as:

$$|\vec{c}| = |\vec{a}||\vec{b}| \sin\theta$$

where θ is the angle between the two vectors when the vectors are tail-to-tail. But this is only the magnitude. To get the direction of the cross product, you can use one of two methods: (1) the right hand rule (RHR) or (2) the matrix determinant method to solve the vector cross product.

Figure 1.8 shows how to solve for the cross product direction with the RHR.

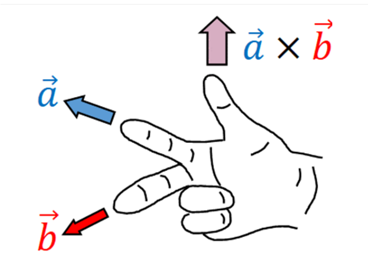


Figure 1.8: Vector orientation from the right hand rule. For $\vec{a} \times \vec{b}$, align your index finger with the direction of $\vec{a}$ and your middle finger with the direction of $\vec{b}$. Your thumb then points in the direction given by $\vec{a} \times \vec{b}$.

The matrix determinant method gives you the full vector solution for the cross product:

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \\ &= (a_y b_z - a_z b_y)\hat{\imath} + (a_z b_x - a_x b_z)\hat{\jmath} + (a_x b_y - a_y b_x)\hat{k} \end{aligned}$$

For the vector cross product, order matters. Here are a few helpful identities:

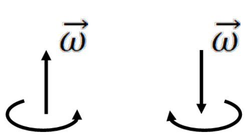
$$\begin{aligned} \vec{a} \times \vec{b} &= -\vec{b} \times \vec{a} \\ \vec{c} \times (\vec{a} + \vec{b}) &= (\vec{c} \times \vec{a}) + (\vec{c} \times \vec{b}) \\ n(\vec{a} \times \vec{b}) &= (n\vec{a}) \times \vec{b} \;=\; \vec{a} \times (n\vec{b}) \;=\; (\vec{a} \times \vec{b})n \end{aligned}$$

Definitions

In general, angular velocity (ω) and angular acceleration (α) are vectors, although we often drop the vector symbol. The true definition of these terms are:

$$\vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2} \qquad\qquad \vec{\alpha} = \frac{\vec{r} \times \vec{a}}{r^2}$$

where the direction is given by the right-hand rule. For rotational motion:



$\Longrightarrow$ curl your fingers in direction of rotation

$\Longrightarrow$ thumb points in the direction of $\vec{\omega}$

1.6 Approximations

In physics, you can often make approximations to simplify the math based on the conditions of your system. For example, if you have a complicated force acting on a system, but you are only interested in short distances or short times, you can often simplify the equation for that force making the calculations easier. Deviations between the true value (considering the full complicated force equation) and the approximation (with the simplified force equation) would be considered small, such that you get a good idea of how the system will move without needing to do the complicated math. (Of course, with some high-precision physics, you cannot make this approximation.)

A very common approach is to use the Taylor series expansion. The idea here is that any function can be broken up into a series of polynomials following:

$$
\begin{aligned}
f(x) = f(x_0) + \frac{\mathrm{d}f(x_0)}{\mathrm{d}x}(x-x_0) + \frac{1}{2}\frac{\mathrm{d}^2 f(x_0)}{\mathrm{d}x^2}(x-x_0)^2 + \\
\frac{1}{3!}\frac{\mathrm{d}^3 f(x_0)}{\mathrm{d}x^3}(x-x_0)^3 + \cdots + \frac{1}{n!}\frac{\mathrm{d}^n f(x_0)}{\mathrm{d}x^n}(x-x_0)^n
\end{aligned} \tag{1.5}
$$

where $f(x)$ is the function and x_0 is a reference value for the function and the ! symbol is the factorial symbol. That is $f(x)$ describes the entire function for all values x, whereas $f(x_0)$ is the value of the function at the specific value of $x = x_0$. See Appendix B.2 for more details and other approximation techniques.

For Equation 1.5, consider values of $x \approx x_0$. That is, you are only looking at cases of your variable, x when it is close to your reference value. In this case, $x - x_0$ is small. Thus, higher order terms like $(x-x_0)^2$ and $(x-x_0)^3$ are very small and can be dropped. Suddenly, your function has become very simple.

Let's look at an example. Consider the Taylor series expansion for e^x for small values of x. In this case, we can set $x_0 = 0$ because we are looking at small values of x. The expansion is:

$$f(x) \approx f(0) + f'(0)(x) + \frac{1}{2}f''(0)(x)^2 + \frac{1}{3!}f'''(0)(x)^3 + \cdots +$$

where $f'(0)$ means take the derivative of $f(x)$ with respect to x and evaluate that for $x = 0$. For $f = e^x$, we have:

$$
\begin{aligned}
f(0) &= e^0 = 1 \\
f'(0) &= e^0 = 1 \quad \Longrightarrow \quad \frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x \\
f''(0) &= e^0 = 1 \quad \Longrightarrow \quad \frac{\mathrm{d}^2 e^x}{\mathrm{d}x^2} = e^x
\end{aligned}
$$

and so forth. Taking these terms, we can approximate the solution to e^x at $x \approx 0$ as:

$$e^x \approx 1 + x$$

where we drop the higher order terms because if x is small ($|x| \ll 1$), then the higher order terms which have x^2 and x^3 become negligible. A function of $1 + x$ is much simpler to work with than a function of e^x. This highlights the power of a Taylor series expansion.

Figure 1.9 demonstrates this approximation. The figure compares a function of e^x with a function of $1+x$. For small values of x (such as $-0.3 < x < 0.3$), the two functions are very similar. For larger values of x, however, the approximation breaks down. Note, however that you can include additional higher order terms when necessary. That is, $e^x \approx 1+x+\frac{1}{2}x^2$ would give a better approximation than $e^x \approx 1 + x$.

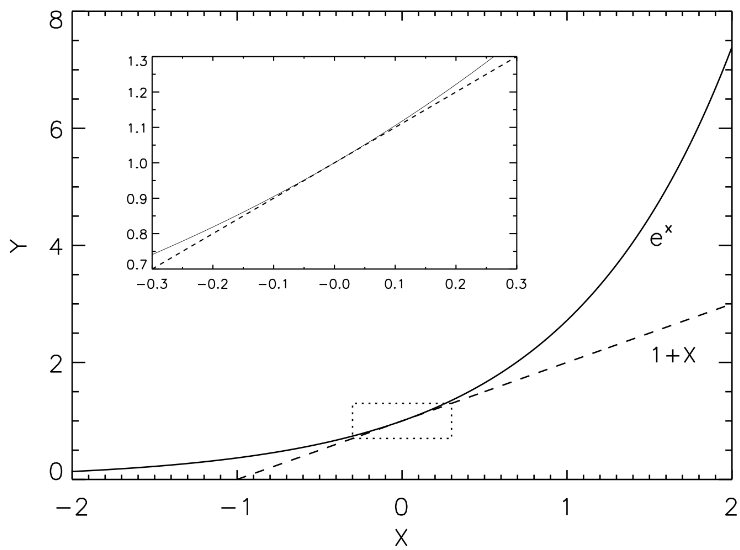


Figure 1.9: The left figure compares $y = e^x$ with the Taylor approximation of $y = 1+x$ for different values of x. The inset shows a zoom-in of the region between $x = -0.3$ and $x = 0.3$.

Real World Applications

A common application of Taylor Series approximations is the small angle case, where for small angles $\sin\theta \approx \theta$ and $\tan\theta \approx \theta$ (for angles in radians). In astronomy, nearby stars make small shifts in position relative to more distant background stars due to Earth's orbit around the Sun. The size of the shift, called a stellar parallax, is measured as the angle on the sky from the apparent shift in position. From trigonometry, the parallax angle is given by $\tan\theta_p = \frac{Earth\ orbit}{star\ distance}$. Since stars are very far away, stellar parallaxes are very small angles ($\ll 1$ rad), so we can simplify the parallax equation as $\theta_p = \frac{Earth\ orbit}{star\ distance}$. Knowing the Earth's orbit size, we can therefore find the distances to stars by measuring their parallax angles with telescopes. For example, the *Gaia* space telescope has measured stellar parallaxes to over 1 billion stars providing unprecedented maps of our Galaxy.

See Appendix A.3 for a list of common Taylor series approximations and Appendix B.2 for more details on this method and other approximations. Taylor series approximations may seem confusing at first, but they can work for you when applied properly. When we use the Taylor approximation throughout this text, think about why we are using it and how the approximation simplifies the calculations.

Sample Problem 1-3

Simplify the function $f(x) = \sqrt{3+e^x}$ assuming x is very small ($x \to 0$). Take the first two terms of the expansion only.

Solution

We just solved $e^x \approx 1 + x$. So for small values of x, we can simplify the e^x term. Thus,

$$\sqrt{3+e^x} \approx \sqrt{3+1+x} \approx \sqrt{4+x}$$

But we can go further. We can also simplify the square root function. The Taylor series equation will be:

$$f(x) \approx f(0) + f'(0)(x)$$

taking the first two terms only. So we need to evaluate the function and the derivatives for $x = 0$.

$$\begin{aligned} f(0) &= \sqrt{4+0} = 2 \\ f'(0) &= \frac{1}{2}\frac{1}{\sqrt{4+0}} = \frac{1}{4} \end{aligned}$$

Thus, our expansion becomes:

$$\begin{aligned} f(x) &\approx f(0) + f'(0)(x) \\ \sqrt{3+e^x} &\approx 2 + \frac{1}{4}x \end{aligned}$$

which is a much simpler function than the original one. You can visualize and calculate problems with $f(x) = 2 + \frac{1}{4}x$, but it is much harder to picture and use $f(x) = \sqrt{3+e^x}$

Lance's Thoughts

You might run across a very complicated equation you need to approximate, something that combines trigonometric and exponential or logarithmic functions, or a complex polynomial. The key is to think of it in the same way as doing a multiple integral: work from the inside out, one function at a time.

Challenge Question

1. Find the first three terms of the Taylor series for $y = e^{\cos(2x)}$ assuming x is close to 0. Plot the function and your approximation using a programming language. See the online repository for examples using python.

Number of Terms

How many terms of the Taylor series expansion should you take? Generally, it will depend on the problem and the degree of accuracy you need. The best way to determine the number of terms is to stop when you have a good representation for how a function changes. For example, if you want to know how a function is changing for small values of x but the first two terms of the Taylor series expansion give you zero or a constant, then you will want to go to higher order terms.

Consider a Taylor series expansion of $\cos x$ for small values of x. From the first two terms, you get:

$$\begin{aligned} f(x) &\approx f(0) + f'(0)(x) \\ \cos x &\approx 1 - \cancelto{0}{\sin(0)}(x) \qquad \Longrightarrow \qquad f'(x) = \frac{\mathrm{d}(\cos x)}{\mathrm{d}x} = -\sin x \\ \cos x &\approx 1 \end{aligned}$$

which is a constant. The first two terms alone are not helpful if you want to know how $\cos x$ varies with x for small values of x. To get around this, add an additional term:

$$\begin{aligned} f(x) &\approx f(0) + f'(0)(x) + \frac{1}{2} f''(0)(x)^2 \\ \cos x &\approx 1 - \cancelto{0}{\sin(0)}(x) - \frac{1}{2}\cos(0)(x)^2 \qquad \Longrightarrow \qquad f''(x) = \frac{\mathrm{d}^2(\cos x)}{\mathrm{d}x^2} = -\cos x \\ \cos x &\approx 1 - \frac{1}{2}x^2 \end{aligned}$$

Now our expansion gives us a simple function for how $\cos x$ varies with x for small values.

1.7 Real-World Application: LIGO

One of the simplest, fundamental concepts in physics is the case of constant motion where $d = vt$. And this basic equation is at the core of one of the most ground-breaking discoveries in the 21st century, gravitational waves.

First predicted by Einstein in 1916, gravitational waves are a natural outcome of General Relativity and can be described as “ripples” in spacetime. They are incredibly small in magnitude, where *strong* gravitational waves have magnitudes on the order of 10^{-18} m, which is around one thousandth the diameter of a proton. With the level of sensitivity needed, it was roughly a century between prediction and detection.

The first gravitational waves were detected on September 14, 2015 by the Laser Interferometer Gravitational-wave Observatory (LIGO) experiment. The experiment itself uses interferometry where identical laser beams reflect off mirrors and then converge on a detector producing an interference pattern (see Figure 1.10). When a gravitational wave passes through the Earth, it temporarily warps space and changes the distance between the mirrors which subsequently changes the arrival time of the reflected beams at the detector. LIGO can detect a change in distance between its mirrors on the order of 10^{-19} m.

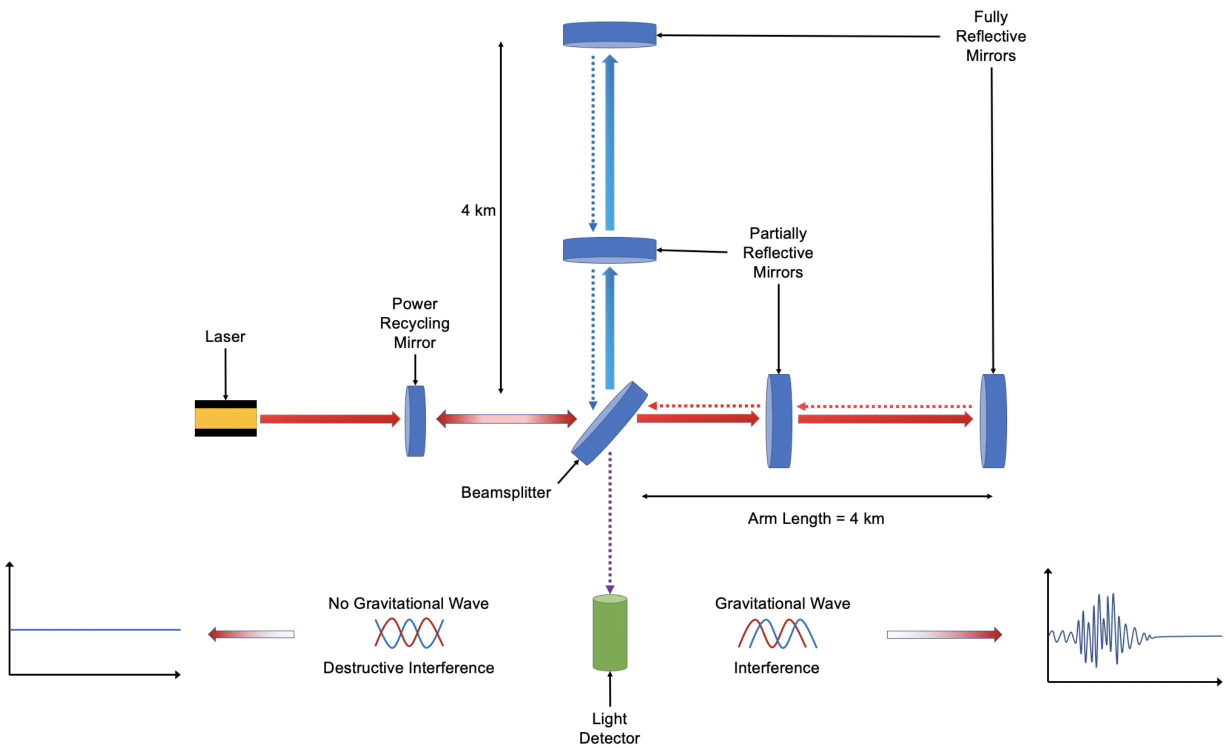


Figure 1.10: Cartoon showing the basic concept behind the LIGO experiment. Laser light is split into two orthogonal beams and reflects off distant mirrors that are 4 km away. The reflected light combine at a detector. The distance between the mirrors is so precise that the reflected waves should destructively interfere at the detector. A gravitational wave alters the mirror separations causing the combined wave to produce an interference pattern instead.

The slight change in distance from a passing gravitational wave alters the interference pattern measured at the detector. Figure 1.11 shows the gravitational wave signal from the first detection, which were generated by a pair of merging intermediate-mass black holes located 1.3 billion light years away. The interference pattern is often described as a "chirp", because it rises to higher frequencies toward the end. Research into gravitation waves include LIGO in the USA, VIRGO in Italy, and GEO600 in Germany, with a third site, KAGRA, under construction in Japan. Multiple experiments all over the world are necessary to pinpoint the direction of the gravitational wave events because each site will measure a difference in signal and arrival time.

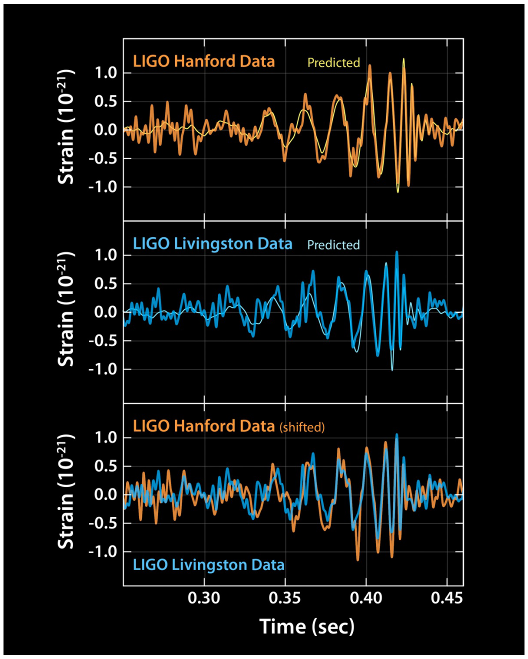


Figure 1.11: First gravitational wave signal from LIGO. The strain (y axis) indicates the fractional change in distance between the mirrors (positive means further, negative means closer) for two different experiments located in Washington and Louisiana. The lower panel overlays both experiments (with a shift in the Hanford data because the gravitational waves reached each detector at slightly different times). The thin "predicted" lines show the best-fit merging black hole model, where black holes of 36 $M_\odot$ and 29 $M_\odot$ merged to form a black hole of 62 $M_\odot$. The missing mass (~ 3 $M_\odot$) was converted into the energy that created the gravitational waves. Credit: Caltech/MIT/LIGO Lab.

LIGO is an international collaboration including more than 1200 scientists from over 100 institutions located in 18 different countries. The ground-breaking discovery has significant implications for general relativity, black holes, and our universe. But recall that the basic principle at the heart of this experiment is a change in arrival time from a change in distance.

For more information:
The LIGO Scientific Collaboration website has a lot of information about the original detection and process.
This video translated the merging event into a sound bite that showcases the "chirp" from the merger.
Sky & Telescope also has a nice article (with lots of links) describing the first detection.

1.8 Summary

Key Takeaways

This chapter is about setting up the coordinate systems and solving for the equation of motion knowing the acceleration. This section focuses on Cartesian coordinates, where motion is described in terms of x, y, z:

$$\begin{aligned}\vec{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k}\\ \vec{v} &= \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}\\ \vec{a} &= \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}\end{aligned}$$

And plane-polar coordinates, where the position vector is given by $\langle r, \theta\rangle$, and the velocity is described by:

$$\begin{aligned}\vec{v} &= \frac{\mathrm{d}r}{\mathrm{d}t}\hat{r} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta}\\ \vec{a} &= \left[\frac{\mathrm{d}^2r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]\hat{r} + \left(r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)\hat{\theta}\end{aligned}$$

where there is a radial and transverse (θ) component to velocity and acceleration.

For purely circular rotation, we the coordinates are easy to relate:

$$\vec{r} = r_0\cos(\omega t)\hat{\imath} + r_0\sin(\omega t)\hat{\jmath}$$

And the radial vector is constant such that:

$$\begin{aligned}\vec{v} &= r\omega\hat{\theta}\\ \vec{a} &= -r\omega^2\hat{r}\end{aligned}$$

Note that we assume that the rotation rate is not changing.

This Chapter also described the vector dot ($\vec{a}\cdot\vec{b}$) and vector cross product ($\vec{a}\times\vec{b}$), which will be used more explicitly in later chapters. The vector dot product is essentially a projection of $\vec{a}$ onto $\vec{b}$ and yields a scalar answer. The vector cross product gives the vector that is normal to the surface described by $\vec{a}$ and $\vec{b}$.

Finally, the Chapter introduced Taylor series expansion as a method to simplify complex functions. We will use this method to more efficiently solve physics problems.

Important Equations

Cartesian Coordinates:

$$\begin{aligned}\vec{r} &= x\hat{\imath} + y\hat{\jmath} + z\hat{k}\\ \vec{v} &= \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}\\ \vec{a} &= \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}\end{aligned}$$

Circular rotation:

$$\vec{r} = \underbrace{r_0 \cos(\omega t)\hat{\imath}}_{x(t)} + \underbrace{r_0 \sin(\omega t)\hat{\jmath}}_{y(t)}$$

(in Cartesian coordinates)

$$\begin{aligned}\vec{v} &= \omega r\hat{\theta}\\ \vec{a} &= -\omega^2\vec{r}\end{aligned}$$

(in plane-polar coordinates)

Plane-Polar Coordinates:

$$\vec{v} = \underbrace{\frac{\mathrm{d}r}{\mathrm{d}t}}_{\vec{v}_r}\hat{r} + r\underbrace{\frac{\mathrm{d}\theta}{\mathrm{d}t}}_{\vec{v}_\theta}\hat{\theta}$$

$$\vec{a} = \underbrace{\left[\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - r\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]}_{a_r}\hat{r} + \underbrace{\left(r\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + 2\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)}_{a_\theta}\hat{\theta}$$

Vector Dot Product:

$$\begin{aligned}\vec{a}\cdot\vec{b} &= x_a x_b + y_a y_b + z_a z_b\\ \vec{a}\cdot\vec{b} &= |\vec{a}||\vec{b}|\cos\theta\end{aligned}$$

Vector Cross Product:

$$\begin{aligned}\vec{a}\times\vec{b} &= (a_y b_z - a_z b_y)\hat{\imath} + (a_z b_x - a_x b_z)\hat{\jmath} + (a_x b_y - a_y b_x)\hat{k}\\ |\vec{a}\times\vec{b}| &= |\vec{a}||\vec{b}|\sin\theta\end{aligned}$$

Taylor series approximation (for small x):

$$f(x) \approx f(0) + f'(0)(x) + \frac{1}{2}f''(0)(x)^2 + \frac{1}{3!}f'''(0)(x)^3 + \cdots +$$

1.9 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 1-1

The equation of acceleration for a system is $a = Ce^{-t/\tau}$. If the system starts with $v = v_0$, what is the equation for velocity, $v(t)$?

Practice Problem 1-2

Two vectors are $\vec{a} = c\hat{\imath} + 4c\hat{\jmath}$ and $\vec{b} = 3\hat{\imath} + 5\hat{\jmath}$, where c is a constant. What is the vector dot product $\vec{a} \cdot \vec{b}$ and cross product $\vec{a} \times \vec{b}$?

Practice Problem 1-3

Two vectors are $\vec{a} = 2k\hat{\imath} + 2\hat{\jmath}$ and $\vec{b} = 3\hat{\imath} + 3k\hat{\jmath}$, where k is a constant. What is the angle between these vectors?

Practice Problem 1-4

Find $\vec{a} \cdot \vec{b}$ if:

a) $\vec{a} = -4\hat{\imath} + 4\hat{\jmath} + 4\hat{k}$, $\vec{b} = 4\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$
b) $\vec{a} = 7\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$, $\vec{b} = 2\hat{\imath} + 4\hat{\jmath} - 8\hat{k}$
c) $\vec{a} = 3\hat{\imath} + 3\hat{\jmath} + 2c\hat{k}$, $\vec{b} = 3c\hat{\imath} - 3\hat{\jmath} - c\hat{k}$
d) What value(s) of the constant c in part c) would make $\vec{a}$ and $\vec{b}$ perpendicular?

Practice Problem 1-5

For the vectors $\vec{a} = 9\hat{\imath} - 3\hat{\jmath} + 2\hat{k}$, $\vec{b} = -4\hat{\imath} - 5\hat{\jmath} + 2\hat{k}$, and $\vec{c} = 3s\hat{\imath} + 3\hat{\jmath} + 9s\hat{k}$, solve:

a) $\vec{a} \times \vec{b}$
b) $\vec{b} \times \vec{c}$
c) $\vec{c} \times \vec{b}$
d) $\vec{c} \times \vec{a}$

Practice Problem 1-6

Two vectors are $\vec{a} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = \hat{\imath} + 2\hat{\jmath} + \hat{k}$. Solve the following triple cross products:

a) $\vec{a} \times (\vec{a} \times \vec{b})$
b) $\vec{b} \times (\vec{b} \times \vec{a})$
c) $(\vec{b} \times \vec{a}) \times \vec{b}$

Practice Problem 1-7

For each of the following, find the plane-polar (r,θ) coordinates or equation.

a) $P(x, y) = (3, 4)$
b) $y = x$
c) $y = x^2 + x$

Practice Problem 1-8

A particle moves in a cloud chamber such that its position can be described by, $r = e^{2t}$ and $\theta = t^2$. Find its velocity and acceleration. Assume all quantities are unitless.

Practice Problem 1-9

A toy car on a racing track is moving in a circle of constant radius, R. The speed of the car is increasing as $v = bt$, where b is a positive constant. What is the angle between the total velocity and total acceleration vectors at time $t = \sqrt{\frac{R}{b}}$?

Practice Problem 1-10

A cat goes for a walk in a spiral path. In polar coordinates, the cat's radius and angle coordinates are given by $r(t) = be^{kt}$ and $\theta(t) = ct$, where b, k, and c are all positive constants.

a) What is the cat's velocity, in polar coordinates?
b) What is the cat's acceleration, in polar coordinates?
c) Show that the angle between the velocity and acceleration is a constant.

Practice Problem 1-11

For small x, solve the first *three* terms of the Taylor series for

a) $f(x) = \tan x$

b) $f(x) = \dfrac{1}{1-x}$

c) $f(x) = \ln(1-x)$

Practice Problem 1-12

For small x, solve the first *two* terms of the Taylor series for

a) $f(x) = (1-x^2)^3$

b) $f(x) = e^{\sin 5x}$

c) $\ln(x^2 + 3x + 2)$

Practice Problem 1-13

Plot the functions and Taylor series approximations (first three non-zero terms) for the following functions (sample python script).

a) $f(x) = e^{-x^2}$ for $x \approx 0$

b) $f(x) = 2e^{x^2}$ for $x \approx 0$

c) $\ln(1+x)$ for $x \approx 0$

d) e^x for $x \approx 2$

2 Newtonian Motion

Learning Objectives

- Review Newton's Laws and their meaning
- Apply free-body diagrams to force problems in equilibrium and in motion
- Solve equations of motion with simple forces

In this chapter, we will look at the universal laws of motion with emphasis on review of Newton's Law's (mainly the second law) and free-body diagrams.

2.1 Universality of the Laws of Motion:

Classical mechanics describes how objects move. While this chapter is called *Newtonian Motion* after Sir Isaac Newton, it is important to recognize that Newton was not the first person to develop theories about the motion of objects. Physics is universal. Historically, physicists from all over the world also sought laws of motion hundreds to thousands of years before Newton.

Early Laws of Motion

One of the earliest individuals to connect forces to changes in motion is the Persian scholar Abu ʻAlī ibn Sīnā (980-1037), known as Avicenna in Europe. Note the following translation from ibn Sīnā's work and its similarity to Newton's first law (the law of inertia) given in the next section:

"...[N]obody begins to move or comes to rest of itself" (Hecht 2015, p. 1)

For more information:

Hecht, E. (2015), *Origins of Newton's First Law*, The Physics Teacher, 53, 2, (pp. 80-83)

Standford historical pages

American Institute of Physics historical pages

2.2 Newton's Three Laws of Motion:

Newton's three laws of motion are a mathematical description connecting motion to forces. These laws apply to all objects of any size (above the atomic level), any shape, and any internal structure (solid or even liquid). The three laws are:

1. **The law of inertia**: A body moves with constant velocity unless acted on by a force.
2. **The equation of motion**: The change of momentum of a body equals the net force acting on it.
3. **The law of action and reaction**: For every force acting on a body, there is an equal and opposite reactive force.

The **first law** corresponds to the conservation of momentum. The idea here, is that an object with no forces acting on it will be at rest or moving at a constant velocity. It is important to note that the first law requires that you define an appropriate inertial frame (a frame of reference). Your inertial frame can be static (at rest) or in motion (with constant velocity, no acceleration). If your reference frame is accelerating, we call that a non-inertial frame and the physics is a bit different. We will discuss non-inertial frames in Chapters 4 and 5.

The **second law** corresponds to the rate of change of momentum, $\vec{p}$. The net force acting on a system is:

$$\boxed{\sum \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}} \tag{2.1}$$

where the momentum is $\vec{p} = m\vec{v}$. If your system has *constant mass*, then the second law can be written as:

$$\boxed{\sum \vec{F} = m\vec{a} = m\ddot{\vec{r}}} \tag{2.2}$$

See Chapter 1 for a review of vector notation.

The second law connects kinematics (changes in momentum or the acceleration of an object) to a dynamical force. In the case of a constant mass, Equation (2.1) has the more familiar form of Equation (2.2). However, there can be physics problems where the mass of the system is also allowed to change (e.g., if you are in a rocket that is using fuel). In these cases, you *cannot* use the more familiar $\vec{F} = m\vec{a}$ equation. We will discuss variable mass problems in Chapter 6.

Finally, the **third law** says that forces come in pairs. For two objects that are exerting forces on each other, those forces will be equal in magnitude and opposite in sign. That is,

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}} \tag{2.3}$$

For example, when you stand on the ground, you push downward on the surface due to gravity. But you don't fall into the surface, because the ground pushes back up on you in an equal and opposite force, typically called the normal force.

Limitations of Newton's Laws

For most "everyday life physics", Newtonian mechanics applies just fine. Nevertheless, there are cases where Newtonian mechanics break down because the assumption of absolute space or an absolute time is not quite correct. For example, in special relativity, objects moving close to the speed of light do not behave the same way as objects that are moving at much slower speeds. In general relativity, space and time are warped by mass, an effect that cannot be fully explained by Newtonian mechanics. A great example of this is the shape of Mercury's orbit (it is not elliptical but rosette shaped).

For more information:

Mercury's unusual orbit

Introduction to General Relativity

Introduction to special relativity

2.3 Static Systems

Static systems are systems that are not in motion. In this case, the sum of all forces ($\sum \vec{F}$) and the sum of all momenta ($\sum \vec{p}$) are both zero. Let's look at a problem for a static system.

Sample Problem 2-1

Figure 2.1 shows two masses connected to each other by an ideal pulley. If this system is at rest, **what is the magnitude of the force of friction?**

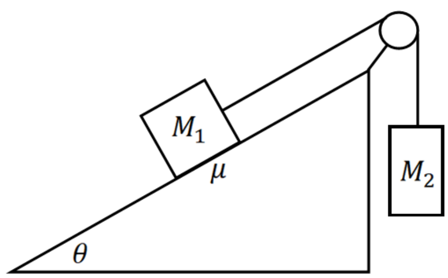


Figure 2.1: Two masses are attached to an ideal string that runs along ideal pulley at the edge of an incline of angle θ. One mass sits on the incline and the other hangs off the edge of the incline. The incline has a coefficient of friction, μ. The system is at rest.

Solution

First, we find all the forces acting on the system. Since there are two masses, they will each have gravity M_1g and M_2g. M_1 is sitting on the incline, so it is pushing down on the incline due to gravity and the incline is pushing up on M_1 via the normal force N. Since M_2 is hanging, it does not have a normal force. Both masses are attached by a string so that means there is a tension T for both. Since the string and pulley are *ideal*, they have no mass or friction. But there is a coefficient of friction acting on M_1 from the incline and the friction force is $f = \mu N$, where N is the normal force.

With the forces, let's draw the free-body diagram. For simplicity, we'll split up M_1 and M_2 into two different panels. The left panel in Figure 2.2 shows the free-body diagram for M_1 on the incline and the right panel is the free-body diagram for M_2.

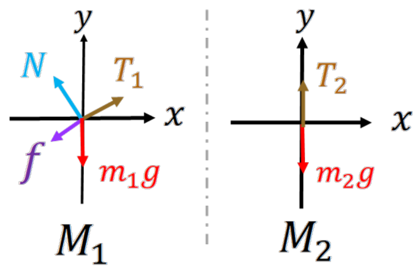


Figure 2.2: Free-body diagram showing gravity (mg), tension (T), friction (f), and the normal force (N) associated with Figure 2.1. Left panel is for M_1, right panel is for M_2.

Figure 2.2 shows the directions for all the forces, where gravity points down, the normal force is perpendicular to the incline, tension is along the string, and friction is parallel to the surface. Note that we need to assume a direction for friction. The actual direction of friction will depend on the relative masses for M_1 and M_2 (we do not know if M_1 wants to slide up or down the incline at this time). So we will make a guess for the direction of friction for now. If we guessed wrong, we will just get a negative force.

Quick Question

1. If you remove M_2, will M_1 necessarily slide down the ramp?

To find the net force on the system, we sum all forces for both masses. Since the string is ideal, it does not stretch or deform, which means that the tension on both sides of the pulley must be the same and $T_1 = T_2$.

For M_2, there are only two forces, T_2 and $M_2 g$. Since both masses are at rest, the sum of all forces on M_2 must equal zero (Second Law). So we get $T_2 = M_2 g$.

Combining $T_1 = T_2$ and $T_2 = M_2 g$, we can revisit the free-body diagram of M_1. Figure 2.3 is an updated free-body diagram of M_1 only.

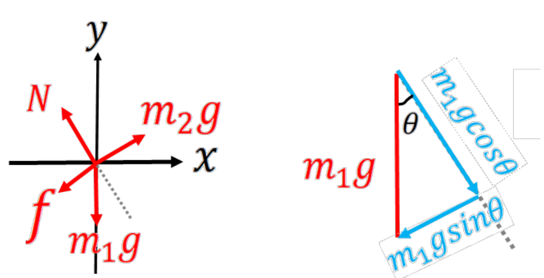


Figure 2.3: Left: *Free-body diagram of M_1 with $T_1 = T_2 = M_2 g$.* Right: *Vector diagram for the gravitational force on M_1.*

With Figure 2.3, we can solve for the normal force N and friction f. We break up the gravitational force into the component that is parallel to the incline and perpendicular to the incline. We can get these from trigonometry (see the right panel of Figure 2.3):

$$F_{g,//} = -M_1 g \sin\theta$$
$$F_{g,\perp} = -M_1 g \cos\theta$$

Note that we have defined up the incline and outward from the incline as positive.

For the system to be at rest, the sum of all forces must be zero. Since the components parallel to the incline $(f, F_{g,//}, T_1)$ are orthogonal to the components perpendicular to the incline $(N, F_{g,\perp})$, they must each add to zero. That is, the sum of all parallel components must be zero and the sum of all perpendicular components must be zero. The two groups of forces act independently. Adding the individual forces in each group:

$$0 = \sum F_\perp \qquad\qquad 0 = \sum F_{//}$$
$$0 = N - M_1 g \cos\theta \qquad\qquad 0 = M_2 g - M_1 g \sin\theta - f$$
$$N = M_1 g \cos\theta \qquad\qquad f = M_2 g - M_1 g \sin\theta$$

Thus, we have our magnitude of f. The true direction of f will depending on θ, M_1, and M_2. If $M_2 g > M_1 g \sin\theta$, then f is positive and our assumed direction for f is correct. If $M_2 g < M_1 g \sin\theta$, then f will be negative, indicating that our assumed direction for f was wrong. This case means that M_1 is so heavy that it will pull on M_2 (e.g., M_1 wants to move down the incline so friction is acting up the incline) .

Quick Questions

1. If M_2 and θ are held constant, what is the maximum mass for M_1 before our assumed direction of friction from Figure 2.2 is wrong?
2. Recall that friction is defined by $f = \mu N$, where μ is the coefficient of friction. If $M_1 = M_2$ and the angle is 30°, what is the value of μ?

Lance's Thoughts

There's no rule about which axes to use when solving a problem or which direction is positive. Often, the standard $x-y$ (or $x-y-z$) axes will make your math harder than it needs to be. Look at the problem and decide what makes the most sense to you and align your coordinate system to maximum advantage.

Things to think about: consider how the system is moving and align the coordinates to best fit that motion, including which directions are most convenient to make positive. If the motion is circular or along a curve, polar coordinates might be a better choice than Cartesian. Whatever choices you make, stick with them through your solution.

Figure 2.1 shows an ideal pulley system. When you have an ideal pulley system, it means that the pulley and rope extending over the pulley each have no mass and there is no friction between them. It also means that the rope will not deform (e.g., stretch) due to tension. So you can assume that the tension is the same everywhere in the rope.

An Atwood machine is an ideal pulley system with masses attached by an ideal rope that hang from an ideal pulley. Figure 2.4 shows an example single Atwood machine (left image) and a double Atwood machine (right image). For the single Atwood machine, a single rope holds two masses, M_1 and M_2, over an ideal pulley. The double Atwood machine has two ideal ropes: one connecting M_1 and M_2 and a second connecting M to the lower pulley.

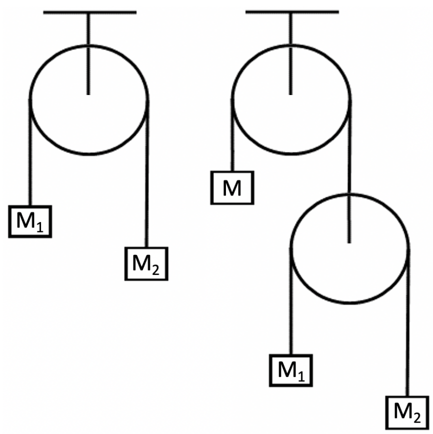


Figure 2.4: A single Atwood (left) and double Atwood (right) machine. The ropes and pulleys in each machine are ideal. The single Atwood machine has one rope (connecting M_1 and M_2), whereas the double Atwood machine has lower rope connecting M_1 and M_2 and an upper rope connecting M and the lower pulley.

Since ideal pulleys have no mass, they will have no net force acting on them (Newton's second law). That condition makes them useful when equating forces to solve problems. Consider the free-body diagrams for the mass and pulley systems above. Whichever forces act on the pulleys will have to balance to zero.

Lance's Thoughts

The key to free body diagrams (FBDs) for Atwood machines is to remember that every mass and every pulley needs one. If you're working with ideal pulleys, the FBDs for those will give you the ratios of the tensions in your system and having those will make your life a lot easier. Remember that for ideal ropes that the tension is equal everywhere on the same rope and the sum of forces on the pulley is zero. Things get a little more complicated if the pulleys have mass, but drawing the FBD will still help you.

Questions on Atwood Machines

1. Draw the free-body diagrams for the masses and pulley in the single Atwood machine in Figure 2.4
2. Find the equations of acceleration for both masses in terms of M_1, M_2, and g. How will the system move if $M_1 = M_2$ or $M_1 \gg M_2$?
3. Draw the free-body diagrams for the double Atwood machine in Figure 2.4.

2.4 Systems with Constant Acceleration

The simplest case for Newton's laws is a system with constant acceleration and constant mass such that the net force is also constant ($\sum \vec{F} = m\vec{a} = \text{constant}$).

Sample Problem 2-2

A person of height r_0 throws a ball at an angle of θ upwards from the ground with an initial speed of v_0. **What is the maximum height that the ball reaches? How far does the ball travel horizontally when it hits the ground?**

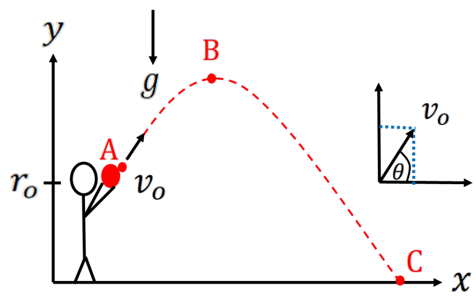


Figure 2.5: Sketch of the ball's trajectory. The ball starts at point A, rises up to a maximum at point B, and then hits the ground at point C a distance x_C from the original starting point.

Solution

We will assume that the ball stays close to the ground such that we can approximate gravity as a constant force (we revisit gravity as a non-constant force in Chapter 10).

First, consider how the ball will move. Since the ball is given both a vertical and horizontal initial motion, this is a 2D problem. But the only source of acceleration is from gravity, which is constant and points downwards (gravity does not affect the horizontal motion).

Therefore, the ball will make an arc, with its vertical motion changing (due to the acceleration with gravity) and its horizontal motion held constant (we will ignore any air resistance). The motion along x and y are independent, so we can solve this problem by looking at each component separately.

1. **What is the maximum height?** This is point B in Figure 2.5. The maximum height depends only on the motion along the y-axis, so we can ignore motion on the x-axis. For the y-axis motion, we have an acceleration of $a = -g$, defining up as positive. Since $a = \frac{\mathrm{d}v}{\mathrm{d}t}$, we can integrate to get the equation for velocity as a function of time. See also, Chapter 1 for more details.

$$
\begin{aligned}
v_y &= \int a\mathrm{d}t &\implies& \quad v_y \text{ is given by the integral of } a \text{, which is a constant}\\
v_y &= -gt + C &\implies& \quad a = -g,\ C \text{ is the initial velocity along the } y\text{-axis}\\
v_y &= -gt + v_0 \sin\theta &\implies& \quad v_0 \sin\theta \text{ is the initial velocity (see Figure 2.5)}
\end{aligned}
$$

For height, $y = \int v_y \mathrm{d}t$. Integrating the velocity equation gives,

$$
y = r_0 + v_0 \sin\theta t - \frac{1}{2}gt^2 \quad \implies \quad r_0 \text{ is the initial height of the ball}
$$

To find the maximum height, y_B, we need the time when the ball reaches the peak of motion, t_B. At the peak, the vertical component of the velocity will be instantaneously zero.

$$
\begin{aligned}
0 &= -gt_B + v_0 \sin\theta &\implies& \quad \text{at the peak of motion, } v_y = 0\\
t_B &= \frac{v_0 \sin\theta}{g}
\end{aligned}
$$

$$
\begin{aligned}
y_B &= -\frac{1}{2}gt_B^2 + (v_0 \sin\theta)t_B + r_0\\
y_B &= -\frac{1}{2}g\left(\frac{v_0 \sin\theta}{g}\right)^2 + v_0 \sin\theta\left(\frac{v_0 \sin\theta}{g}\right) + r_0 &\implies& \quad \text{set } t_B = v_0 \sin\theta/g\\
y_B &= -\frac{1}{2}\left(\frac{v_0^2 \sin^2\theta}{g}\right) + \left(\frac{v_0^2 \sin^2\theta}{g}\right) + r_0 &\implies& \quad \text{simplify}\\
y_B &= \frac{1}{2}\left(\frac{v_0^2 \sin^2\theta}{g}\right) + r_0
\end{aligned}
$$

Note that this equation has the same form as the 1-D case (see Example 1-1), but with a $\sin\theta$ term. If $\theta = 90°$, then the ball is being thrown straight up and we recover the 1-D case exactly, as we should. So the 2-D equation is a more generic form of how the ball moves, whereas the 1-D situation is a specific case.

2. **How far does the ball travel horizontally when it hits the ground?** Unlike the vertical motion, the horizontal motion does not have an acceleration. So the horizontal component of the motion remains constant throughout the ball's travels. The horizontal component of the motion is given by $v_x = v_0 \cos\theta$. Assuming that the ball starts at $x = 0$, we want to calculate the position it has traveled after time t_C. That distance is simply given by $x_C = v_x t_C$, because the ball starts at $x = 0$ (definition) and $a_x = 0$. That means we need to know how long the ball was in the air to know how far it traveled horizontally.

To get the length of time that the ball was in the air, we need to solve for the time when $y = 0$ because the ball has hit the ground (see Example 1-1). The height at point C is given by,

$$y_C = -\frac{1}{2}gt_C^2 + (v_0 \sin\theta)t_C + r_0 = 0$$

which is a quadratic equation with a solution of:

$$t_C = \frac{v_0 \sin\theta \pm \sqrt{v_0^2 \sin^2\theta + 2gr_0}}{g}$$

There are two solutions, one that gives a positive time and one that gives a negative time. Only the positive case is correct given the motion of the ball as defined by the problem (see Example 1-1 for more information on why we reject the negative case). Thus, the time necessary to hit the ground is:

$$t_C = \frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gr_0}}{g}$$

And the horizontal distance traveled by the ball in that time is:

$$x_C = v_x t_C = v_0 \cos\theta \left[\frac{v_0 \sin\theta + \sqrt{v_0^2 \sin^2\theta + 2gr_0}}{g}\right]$$

Note, if $\theta = 90°$ (ball is thrown straight up), then we get $x_C = 0$ as expected. For $\theta = 90°$, there is no horizontal motion, and we recover the 1-D case where the ball travels only with vertical motion.

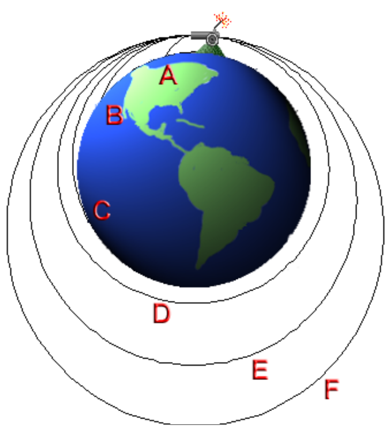
Quick Questions

1. Give the maximum height and horizontal distance in the $\theta = 0°$ case. Describe the trajectory of the motion.
2. What is the direction and magnitude of the velocity vector at point B and point C in Figure 2.5?
3. How would the maximum height and the horizontal distance change if you were to throw the ball from the surface of the Moon (with 1/6th the force of gravity) compared to Earth?

Real World Applications

We tend to treat gravity as a constant acceleration near the surface of the Earth, but in practice drag forces from the air can still cause objects to fall at different rates. During the Apollo 15 mission to the Moon in 1971, astronauts dropped a hammer and a feather and showed that they fell at the same time. This experiment demonstrated that gravity can indeed be considered a constant in the complete absence of atmospheric drag. See the Apollo 15 Hammer-Feather Drop video.

From Projectiles to Orbits



Refer to the trajectories shown in the adjacent figure. With a low initial velocity, a projectile will arc and hit the ground a short distance away (case A). As you increase the velocity, the projectile arcs more and hits further away such that the curvature of the Earth becomes a factor (cases B and C). At high enough speeds, the Earth curves under the projectile at the same rate that its trajectory curves. Basically, gravity changes the direction of motion as the Earth's surface curves away. When this happens, the projectile is in a circular orbit (case D) or elliptical orbit (cases E and F).

How fast do you need to go? The Earth's surface curves down ~ 5 m every ~ 8 km. At an acceleration of 9.8 m s^{-2}, an object will drop 5 m in ~ 1 s. So the projectile must travel about 8 km in 1 s to maintain a constant height over the Earth. A speed of 8 km/s is about 29,000 km/h, which is also about the speed of low-Earth orbit satellites and the International Space Station. So low-Earth orbit satellites are falling back to Earth at the same rate as the Earth curves.

Try at Home

There are some helpful web applications that can help you visualize 2-D projectile motion and test your calculations with different input parameters. Give them a try and test your calculations for different circumstances.

Projectile Motion simulator from the University of Colorado Boulder

Projectile motion simulator from the University of Virginia

2.5 Systems with Varying Acceleration

Now consider cases where the acceleration is not constant. As a result, the force will also vary with time, $F = F(t)$. We will consider how the these forces affect the motion of a system.

2.5.1 Exponential Force

Consider a force that is changing exponentially with time. You can get exponential forces in some cases of drag and damping (e.g., in the critical case). Let us assume there is one force and it has a form of $F = m\alpha e^{-\beta t}$, where α and β are positive constants, and m is the mass of the system. Note that this is our net force such that $F = ma = m\alpha e^{-\beta t}$, so $a = \alpha e^{-\beta t}$. For the equation to be dimensionally consistent with a, the units of α are [m s^{-2}] and the units of β are [s^{-1}]. **Find the equations for $x(t)$ and $v(t)$ assuming that the system has $v = v_0$ and $x = 0$ at $t = 0$.**

To solve this problem, we use $\frac{\mathrm{d}v}{\mathrm{d}t} = a$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = v$. Starting with a:

$$
\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}t} &= a \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= \alpha e^{-\beta t} \quad \Longrightarrow \quad \text{sub in our equation for } a \\
\mathrm{d}v &= \alpha e^{-\beta t}\mathrm{d}t \\
\int \mathrm{d}v &= \int \alpha e^{-\beta t}\mathrm{d}t \\
v &= -\frac{\alpha}{\beta}e^{-\beta t} + C \quad \Longrightarrow \quad \text{where } C \text{ is a constant of integration}
\end{aligned}
$$

We can solve for C using the initial conditions that $v = v_0$ at $t = 0$.

$$
\begin{aligned}
C &= v + \frac{\alpha}{\beta}e^{-\beta t} \\
C &= v_0 + \frac{\alpha}{\beta} \quad \Longrightarrow \quad \text{set } v = v_0 \text{ at } t = 0
\end{aligned}
$$

Subbing C into our velocity equation:

$$
\begin{aligned}
v &= -\frac{\alpha}{\beta}e^{-\beta t} + v_0 + \frac{\alpha}{\beta} \\
v &= v_0 + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)
\end{aligned}
$$

Definite VS the Indefinite Integral

The example above uses an *indefinite* integral, which is where you do not include limits on the integral. With this approach, you end up with a constant of integration and you need to apply the initial conditions to solve for it. The *definite* integral is where you put limits on the integral and solve the problem without the need for a constant. For the above problem, the definite integral would be:

$$\int_{v_0}^{V} \mathrm{d}v = \int_0^{\tau} \alpha e^{-\beta t}\mathrm{d}t$$

where V and τ are dummy variables to represent the velocity at a later time (we use dummy variables to avoid overlap with v and t in the integrand). Note that the definite integral contains the initial conditions (in the lower bounds), so solving this equation will give you the full equation for velocity without needing to solve for a constant of integration. We will show examples of *both* cases in this textbook. See the online repository for a video that directly compares these cases.

Let's look at some limits. **First, what happens as $t \to 0$?** This is not the same as $t = 0$. Basically, we want t to be very small, but not quite zero yet. When $t \to 0$, the exponential can be simplified by its Taylor series (see Chapter 1 and Appendix B). Using the approximation that $e^x \approx 1 + x$ for small values of x, we get,

$$v = v_0 + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) \approx v_0 + \frac{\alpha}{\beta}[1 - (1 - \beta t)] \approx v_0 + \alpha t$$

This makes sense, because at early times (small t), the force is $F(t) = m\alpha e^{-\beta t} \approx m\alpha$, which means that the force and the acceleration are close to being constant. If you have a constant acceleration, your velocity is just a linear function with time.

What happens as $t \to \infty$ (so t is very big)? As t becomes very large, the exponential term goes to zero. With this condition, we have

$$v = v_0 + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) \approx v_0 + \frac{\alpha}{\beta} = \text{constant}$$

So at very large times, the velocity approaches a constant. This makes sense, because as t becomes very large, the force and acceleration both approach zero, $F(t) = m\alpha e^{-\beta t} \approx 0$. If you have no acceleration, then you have a constant velocity.

Finally, let's solve for the position, x using the *definite* integral:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= v \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= v_0 + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) && \Longrightarrow \quad \text{sub in our equation for } v \\
\mathrm{d}x &= v_0 \mathrm{d}t + \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)\mathrm{d}t \\
\int_0^X \mathrm{d}x &= \int_0^\tau \left(v_0 + \frac{\alpha}{\beta} - \frac{\alpha}{\beta}e^{-\beta t}\right)\mathrm{d}t && \Longrightarrow \quad \text{apply limits of } x = 0 \text{ at } t = 0 \\
\left(x\Big|_0^X\right. &= \left(v_0 t + \frac{\alpha}{\beta}t + \frac{\alpha}{\beta^2}e^{-\beta t}\right|_0^\tau \\
X - 0 &= v_0\tau + \frac{\alpha}{\beta}\tau + \frac{\alpha}{\beta^2}e^{-\beta\tau} - \frac{1}{\beta^2} \\
x &= v_0 t + \frac{\alpha}{\beta}t + \frac{\alpha}{\beta^2}e^{-\beta t} - \frac{1}{\beta^2} && \Longrightarrow \quad \text{replace dummy variables } X \text{ and } \tau
\end{aligned}$$

In this above example, we use X and τ to represent the position at some unknown time. They are just representative variables for position and time to avoid confusion and can be swapped out with the generic x and t at the end.

Let's look at the limiting case of x as $t \to 0$. We will again use the Taylor series expansion for the exponential function. This time, however, we need three terms rather than just two terms. When t is very small, $x(t)$ becomes:

$$\begin{aligned} x = v_0 t + \frac{\alpha}{\beta}\left(t + \frac{1}{\beta}e^{-\beta t} - \frac{1}{\beta}\right) &\approx v_0 t + \frac{\alpha}{\beta}\left(t + \frac{1}{\beta}\left[1 - \beta t + \frac{1}{2}\beta^2 t^2\right] - \frac{1}{\beta}\right) \\ &= v_0 t + \frac{\alpha}{\beta}\left(t + \frac{1}{\beta} - t + \frac{1}{2}\beta t^2 - \frac{1}{\beta}\right) \\ &= v_0 t + \frac{\alpha}{\beta}\left(\frac{1}{2}\beta t^2\right) \\ &= v_0 t + \frac{1}{2}\alpha t^2 \end{aligned}$$

So when t is very small, our equation for position goes as $x \approx v_0 t + \frac{1}{2}\alpha t^2$, which is the equation you would get for constant acceleration. This also makes sense, because at very early times, the acceleration is roughly constant.

Quick Questions

1. In the previous case (solving for x at early times), we had to take the first three terms in the Taylor series approximation of the exponential. What happens if we only take the first or first two terms? Why did we need three terms?
2. Why don't we consider the position as $t \to \infty$ (becomes very large)? What would happen to an object in this case?

2.5.2 Force is Proportional to Velocity

Consider a force that is proportional to the velocity of the system. Examples of such forces are the magnetic force (magnitude is proportional to velocity, although in a vector cross product), viscous friction of a body in a fluid, and drag forces.

Consider a force, $F(v)$ acting on a particle with the magnitude of $F(v) = -m\alpha v$, where m is the mass of the particle, α is a positive constant, and v is the velocity of the particle. Assume that the system moves only in 1-D (along x) and that $v = v_0$ and $x = 0$ at $t = 0$. **Find the equations for $x(t)$ and $v(t)$ for this particle**.

Let's start with $v(t)$. Starting from the Second Law, we have $F = ma = -m\alpha v$. Thus, we get that $a = -\alpha v$ or

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\alpha v$$

What we have now is a differential equation. The time derivative of v depends on v itself. This differential equation has a simple solution, fortunately. We must re-arrange the equation by moving the v to the left side of the equation and the dt to the right side of the equation. We can now easily integrate both sides to solve this problem.

$$\begin{aligned}
\frac{\mathrm{d}v}{v} &= -\alpha \mathrm{d}t && \Longrightarrow \quad \textcolor{blue}{\text{using prime variables because we're solving for } v \text{ at } t} \\
\int_{v_0}^{V} \frac{\mathrm{d}v}{v} &= -\alpha \int_0^{\tau} \mathrm{d}t && \Longrightarrow \quad \textcolor{blue}{v = v_0 \text{ at } t = 0,\ V \text{ and } \tau \text{ are dummy variables}} \\
\left(\ln v \Big|_{v_0}^{V}\right. &= -\alpha(\tau - 0) \\
\ln V - \ln v_0 &= -\alpha\tau \\
\ln\left(\frac{V}{v_0}\right) &= -\alpha\tau && \Longrightarrow \textcolor{blue}{\text{recall that } \ln a - \ln b = \ln(\tfrac{a}{b})} \\
\frac{V}{v_0} &= e^{-\alpha\tau} && \Longrightarrow \textcolor{blue}{\text{remove the natural logarithm}} \\
v &= v_0 e^{-\alpha t} && \Longrightarrow \textcolor{blue}{\text{replace dummy variables } V \text{ and } \tau \text{ with } v \text{ and } t}
\end{aligned}$$

Quick Questions

1. What is the velocity as $t \to 0$ and $t \to \infty$? Do these values make sense?
2. Find the equation for the acceleration of the particle and the units for any constants? What is the the acceleration as $t \to 0$ and $t \to \infty$?

What about x? Well, using our equation for v and the condition of $x = 0$ at $t = 0$.

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= v \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= v_0 e^{-\alpha t'} \\
\mathrm{d}x &= v_0 e^{-\alpha t} \mathrm{d}t \\
\int \mathrm{d}x &= \int v_0 e^{-\alpha t} \mathrm{d}t \\
x &= -\frac{v_0}{\alpha} e^{-\alpha t} + C && \Longrightarrow \textcolor{blue}{\text{recall that } \textstyle\int e^{-x}\mathrm{d}x = -e^{-x}} \\
x &= -\frac{v_0}{\alpha} e^{-\alpha\tau} + \frac{v_0}{\alpha} && \Longrightarrow \textcolor{blue}{\text{from initial conditions } (x = 0 \text{ at } t = 0),\ C = \frac{v_0}{\alpha}} \\
x &= \frac{v_0}{\alpha}\left(1 - e^{-\alpha t}\right)
\end{aligned}$$

Quick Questions

1. What is the position as $t \to 0$ and $t \to \infty$? Do these values make sense?
2. Plot position, velocity, and acceleration for the particle assuming $\alpha = 0.5\ \mathrm{s}^{-1}$ and $v_0 = 2\mathrm{m\ s}^{-1}$. You can use any programming language (e.g., python, MATLAB) or you can try and plot it by hand. Check your limits against your plots.

There are other ways that force can be proportional to velocity. Let's look at an example of a viscous force.

Sample Problem 2-3

A metal block of mass m slides on a horizontal surface that has a layer of heavy oil so that the block experiences a viscous force that varies as the 3/2 power of the speed, that is, $F(v) = -bmv^{3/2}$, here b is a positive constant. Let the initial speed of the block be v_0 at $x = 0$. **What is the equation of maximum distance that the block will travel before it comes to rest in terms of m, v_0, and b?**

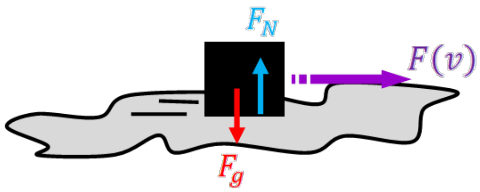


Figure 2.6: Block of mass m slides forward in an oil slick, experiencing $F(v)$, the viscous force F_v. The gravitational force is F_g and the normal force is F_N.

Solution

Looking at the above diagram, and knowing intuitively that the metal block does not leave the horizontal surface, the vertical y-component forces can be ignored as $F_g = F_N$ (the gravitational force equals the normal force). So the only forces that matter are in the horizontal direction.

The block starts with a velocity v_0 and the viscous force acts to slow it down (the block has negative acceleration). When the block reaches its maximum displacement, then $v = 0$ (e.g., if the block is still moving forward, then it has not reached its maximum distance yet).

If we apply Newton's second law (Equation 2.1) to the viscous force given, then we obtain a differential equation of motion:

$$\begin{aligned} ma &= F(v) \\ m\frac{\mathrm{d}v}{\mathrm{d}t} &= -bmv^{3/2} \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= -bv^{3/2} \end{aligned}$$

Since we are trying to solve for the maximum distance (when $v = 0$), we could solve the problem by getting $v(t)$ and then $x(t)$ using all the initial conditions, find the time when the velocity goes to zero, and then get x at that time. But a faster way to solve this problem is to get $v(x)$. We can do this using the *chain rule*, where

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}.$$

A way to think of the chain rule for derivatives is to think in terms of finite changes in a function like Δx. That is,

$$
\begin{aligned}
a &= \frac{\Delta v}{\Delta t} \\
a &= \frac{\Delta v}{\Delta t}\frac{\Delta x}{\Delta x} \quad \Longrightarrow \quad \text{note that } \frac{\Delta x}{\Delta x} = 1 \\
a &= \frac{\Delta v}{\Delta x}\frac{\Delta x}{\Delta t} \quad \Longrightarrow \quad \text{rearrange the terms} \\
a &= \frac{\Delta v}{\Delta x} v \quad \Longrightarrow \quad \frac{\Delta x}{\Delta t} = v \\
\frac{\Delta v}{\Delta t} &= v\frac{\Delta v}{\Delta x} \\
\frac{\mathrm{d}v}{\mathrm{d}t} &= v\frac{\mathrm{d}v}{\mathrm{d}x} \quad \Longrightarrow \quad \text{in the limit where } \Delta t \to 0
\end{aligned}
$$

The chain rule simplifies the math needed to solve the problem. Tricks like this are helpful to more efficiently tackle physics problems. It may not be intuitive to you yet, but the more you practice using this trick, the more you will be able to know when to apply it.

Using the chain rule, we can get the equation of motion in terms of $v(x)$:

$$
\begin{aligned}
v\frac{\mathrm{d}v}{\mathrm{d}x} &= -bv^{3/2} \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= -bv^{1/2} \quad \Longrightarrow \quad \text{simplify} \\
v^{-1/2}\mathrm{d}v &= -b\mathrm{d}x \quad \Longrightarrow \quad \text{rearrange}
\end{aligned}
$$

Now we can integrate both sides. In this case, we consider v at different positions x. At $t = 0$ we are told that $x = 0$ and $v = v_0$ in the problem. At our maximum distance, x_{max}, the block stops moving, so $v = 0$. So we can solve this differential equation using these limits:

$$
\begin{aligned}
\int_{v_0}^{0} v^{-1/2}\mathrm{d}v &= \int_0^{x_{max}} -b\mathrm{d}x \\
\left(2v^{1/2}\right|_{v_0}^{0} &= -b\left(x\right|_0^{x_{max}} \\
2(0 - v_0^{1/2}) &= -b(x_{max} - 0) \\
-2v_0^{1/2} &= -b(x_{max}) \\
x_{max} &= \frac{2}{b}\sqrt{v_0}
\end{aligned}
$$

You will get the same answer if you solve for $v(t)$, $x(t)$, and the time t_{max} when $v = 0$ to then find the position $x(t_{max})$. Try it out and compare the time and number of steps.

2.6 Real-World Application

Drag is often considered a problem in design, but it has many constructive uses as well. One of the most obvious ways to see a drag force in action is by considering a parachute. In the case of a skydiver, the parachute opens behind/above them and creates a much larger surface area perpendicular into the motion, increasing the drag force to counter most of the acceleration due to Earth's gravity, and lowering the diver's terminal velocity enough to allow the parachutist to reach the ground with only a mild impact.

Parachutes are used for other purposes as well, like slowing a race car down quickly after it hits the finish line in a short-track race, increased resistance for a runner trying to build strength, and landing a space capsule for retrieval, and planetary exploration.

For more information:
For demonstrations of the drag force in action to slow down short track race cars, check out this video, courtesy of the National Hot Rod Association.

This video shows the descent of Perseverance using a parachute to slow from 450 m/s to only about 30 m/s before it deployed to the surface of Mars.

2.7 Summary

Key Takeaways

This section is about Newton's three laws and their application to solve physics problems. The key laws covered in this section are the second law,

$$\sum \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

and the third law,

$$\vec{F}_{12} = -\vec{F}_{21}$$

When a system has constant mass, the second law can be written as,

$$\sum \vec{F} = m\vec{a} = m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$$

Newton's laws can be used to solve for the equations of motion, by integrating the acceleration to get velocity and then integrating the velocity to get position. It is important to consider how the acceleration of the system (or net force) varies with time prior to applying the integration.

To help you solve the equations of motion:

1. draw a free-body diagram
2. ensure that you know how the system is moving
3. consider the initial conditions

before attempting any problems.

Important Equations

Newtons 2nd Law:

$$\sum_i \vec{F}_i = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

$$\sum \vec{F} = m\vec{a}$$

(note: this only for constant mass)

Newtons 3rd Law:

$$\vec{F}_{12} = -\vec{F}_{21}$$

2.8 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 2-1

Two masses, m_1 and m_2 are attached by a massless string that passes over an *ideal* pulley (m_p) as shown in the figure. The pulley is then lifted upwards by a constant acceleration a_F due to an external force. What are the correct equations for Newton's Second Law for the two masses and the pulley?

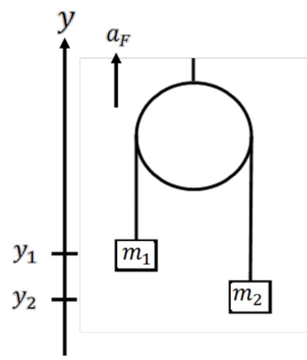


Figure 2.7: Masses m_1 and m_2 begin at positions y_1 and y_2, and accelerate upwards at a_F.

Practice Problem 2-2

You are standing a distance d from a building and your friend is on the roof (height h). You throw an object at an angle of θ from head level (y_0) so that it reaches your friend. What is the minimum initial speed for the object to just make it on the roof?

Practice Problem 2-3

For the following problems, plot the position, velocity, and acceleration for the first three seconds of motion. Example codes for python are provided in the online repository.

a) A ball with an initial velocity of 10 m s^{-1} and an acceleration of 9.8 m s^{-2}, both in the $+x$ direction. Assume the initial position is $x_0 = 0$.
b) A projectile launched upward with an initial velocity of 10 m s^{-1} at an angle of 45° relative to the ground. Assume the only force acting on the projectile is gravity and that the projectile starts from the ground at $x_0 = 0$ and $y_0 = 0$.
c) A particle accelerates such that $a = 4e^{-2t}$, where all quantities are unitless. Assume initial conditions of $v_0 = 5$ and $x_0 = 3$.

Practice Problem 2-4

A particle of mass m is moving in one dimension under the influence of a force, $F = F_0 - \alpha t^2$, where α and F_0 are positive constants. At time $t = 0$, the initial velocity is v_0. Find the equation of $v(t)$.

Practice Problem 2-5

A particle of mass m experiences a force of $F = -kmvx$, where k is a positive constant, v is the velocity, and x is the position. The system starts with $x = 0$ and $v = v_0$.

a) Find $v(x)$.
b) Find $x(t)$. For this problem, you may want to use the following to help solve the integral:

$$\int \frac{\mathrm{d}x}{a^2 - x^2} = \frac{1}{2a} \ln\left(\frac{a+x}{a-x}\right)$$

Practice Problem 2-6

An alien parachutist of mass m decides to go skydiving on their Earth-like home planet. The alien jumps out of a plane at a height h above the surface and feels a drag force $F_{drag} = -bmv$, where b is a positive constant. Assume h is small enough that gravity can be approximated as a constant, g.

a) What is the equation for the net force acting on the alien?
b) What is the terminal velocity? Hint: Terminal velocity is when the velocity reaches a constant, so you do not need to solve the equations of motion.
c) Find an equation for $y(t)$, the height of the alien after they jump from the plane.

Practice Problem 2-7

See figure below. A heavy block of mass M needs to be pulled across a surface with a constant velocity. The coefficient of friction between the block and the surface is μ.

a) Draw a free-body diagram of the block.
b) At what angle should you pull the block to minimize the force required to move it across the surface?
c) What is the minimum force required to move the block across the surface?

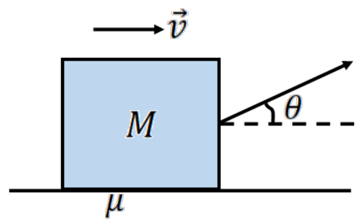


Figure 2.8: Mass M being pulled at angle theta, but moving along the horizontal.

Practice Problem 2-8

See figure below. A spherical ball of mass m and radius R is dropped into a vat of liquid as shown in the figure. As the ball sinks to bottom of the vat, it experiences a viscous force of $\vec{F}_v = -\alpha\vec{v}$, where α is a constant, and a buoyancy force of magnitude $F_b = \rho V g$, where ρ is the density of the liquid (constant), V is the volume of the ball (constant), and g is the acceleration due to gravity.

a) Draw a free-body diagram for the ball.
b) Use your answer for part a) to find $\sum F$ and the differential equation of motion for the system.
c) Solve the differential equation of motion and get $v(t)$, assuming that $v = 0$ at $t = 0$.
d) Consider the limit where $t \to \infty$. What is the velocity at such long times? Does this make sense?

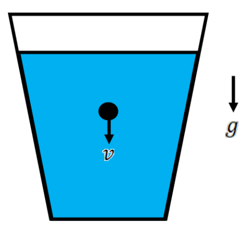
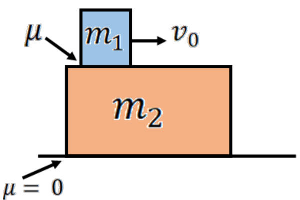


Figure 2.9: The ball of mass m sinks to the bottom of the vat at velocity v.

Practice Problem 2-9

Two blocks are sitting on top of each other on a frictionless surface. The top block has a mass m_1 and the bottom block has a mass m_2. There is a coefficient of friction μ between the two blocks. At $t = 0$, m_1 is moving with a speed of v_0 relative to m_2 and m_2 is at rest relative to the frictionless surface. After a certain time, $t = t_r$, m_1 will beat rest with respect to m_2 (e.g., the two boxes are traveling at the same velocity).

Draw the free-body diagram for both masses.Find time $t = t_r$ when the two masses are traveling at the same velocity. Find the velocity of m_1 and m_2 at $t = t_r$.



a)

b)

Figure 2.10: System configuration for Problem 2-6.

Practice Problem 2-10

A double Atwood machine has three masses. Assume $M_1 = 2M$ and $M_2 = 3M$.

a) Draw the free-body diagram for the two pulleys and the three masses.
b) Describe how the system should move.
c) Find the acceleration of M relative to the accelerations of masses M_2 and M_3.
d) Find the acceleration of M_2.

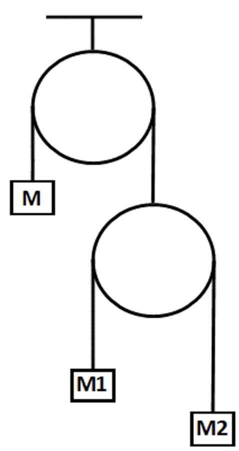


Figure 2.11: The double Atwood machine for Problem 2-7.

3 Simple Harmonic Motion

Learning Objectives

- Introduce and define simple harmonic motion
- Solve the equations of motion for simple mass-spring systems and pendulums

In this chapter, we will apply Newton's Laws to cases of simple harmonic motion.

3.1 Force is Proportional to Position

Let's consider a force that is proportional to the position of the system. Examples of such forces are found in springs, pendulums, and torsion oscillators. Figure 3.1 shows a case of a spring and mass, where the force acting on the mass from the spring is $F = -kx$, where k is a positive constant. (Note this equation is also called Hooke's Law.)

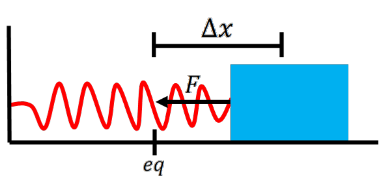


Figure 3.1: Example of the spring force acting on a mass.

A force in the form of $F = -kx$ is also called a *restoring force*, because the force seeks to return a system to a state of equilibrium ($x = 0$). For example, in Figure 3.1, the spring is stretched from where it wants to be, x_0. The spring force will try to return the mass back to its equilibrium state.

Restoring Forces

The key element to a restoring force is that the force is the negative sign in the $F = -kx$ equation. Because of that negative sign, the force vector will always point in the opposite direction as the displacement. If $x = 0$ is our equilibrium position, then for $x < 0$ the restoring force will move the system toward a positive x, and for $x > 0$ the restoring force will move the system towards negative x (in either case, the force tries to get the system back to equilibrium). If $x = 0$, then there is no force.

If the net force acting on the mass is the spring force, then we can use Newton's second law, $\sum F = ma = -kx$ to get,

$$\boxed{a = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{k}{m}x} \tag{3.1}$$

Equation 3.1 is a second-order differential function, where the second time derivative of displacement is proportional to the displacement (k and m are constants). Thus, we need a function that when differentiated twice gives you the negative of that original function multiplied by a constant. This type of problem has a well known solution. Two familiar functions that meet these conditions are the cos and sin functions.

The solution of a $\cos t$ or $\sin t$ function should make sense. Picture a mass hanging from a spring. If you move the mass upward and let go, the mass will initially move downwards until it reaches a maximum drop at which point it will be pulled back upwards until it reaches its original position then it will move back downwards. Essentially, the mass will move down and up in a periodic manner. Figure 3.2 shows the up-down displacement of this mass as a function of time; note that the displacement looks like a cos (or sin) function.

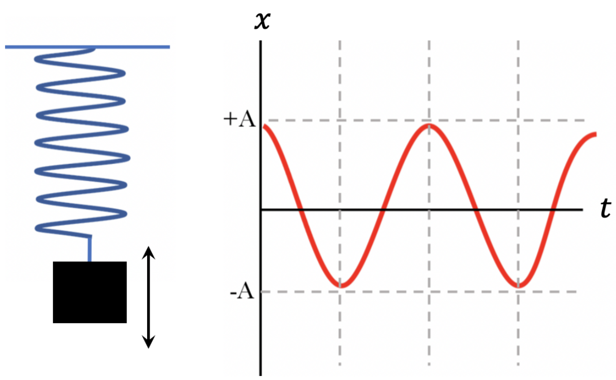


Figure 3.2: Motion of a simple harmonic oscillator. The left panel shows that the mass will move up and down in periodic motion. The right panel shows a sketch of its displacement over time. The mass starts with a maximum displacement (e.g., maximum compression of the spring) and moves to the other end (e.g., maximum extension of the spring) and back again. This back-and-forth motion continues.

Therefore, the solution to the second-order differential equation (Eq 3.1) is met with:

$$\boxed{x = A\cos(\omega_0 t + \phi)} \tag{3.2}$$

where A, ω_0, and ϕ are all constants.

- A is the amplitude of the motion, the maximum displacement from from equilibrium.
- ω_0 is the angular frequency. This is not the same as angular velocity (recall that we used $\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}$). Instead, ω_0 is a fundamental property of the system itself. See details below for more details.
- ϕ is the phase constant (sets where you are in the motion at $t = 0$).

Equation 3.2 is a generic solution to the second-order differential equation that works for any simple harmonic oscillator (not just a mass and spring). Note also that instead of cos, we can use $B\sin(\omega_0 t + \phi_2)$, where B, ω_0, and ϕ_2 are all constants. Indeed, the cos and sin forms of the equation are interchangeable if you just alter the value of the phase constant. In practice, the most general solution for simple harmonic motion would be a superposition of cos and sin functions. For this textbook, however, we will assume that the motion can be described via a single periodic function and we will use the cos function by default.

Now that we have $x(t)$, we just need to differentiate once to get the velocity.

$$\begin{aligned} v &= \dot{x} \\ v &= \frac{\mathrm{d}}{\mathrm{d}t}[A\cos(\omega_0 t + \phi)] \end{aligned}$$

$$\boxed{v \;=\; -\omega_0 A \sin(\omega_0 t + \phi)} \tag{3.3}$$

And we can differentiate again to get the acceleration.

$$\begin{aligned}
a &= \dot{v} \\
a &= \frac{\mathrm{d}}{\mathrm{d}t}[-\omega_0 A \sin(\omega_0 t + \phi_1)] \\
a &= -\omega_0^2 \underbrace{[A\cos(\omega_0 t + \phi)]}_{x(t)}
\end{aligned}$$

$$\boxed{a \;=\; -\omega_0^2 x} \tag{3.4}$$

Thus, we find that $a = -\omega_0^2 x$, where ω_0 is the angular frequency constant. Going back to our original definition of the force in Equation (3.1), we had $a = -\frac{k}{m}x$ for the force $F = -kx$. Thus, the generic differential equation of motion solves Hooke's Law if:

$$\omega_0 \;=\; \sqrt{\frac{k}{m}}.$$

Note that for other restoring forces, the solution for ω_0 will be different.

3.2 Simple Harmonic Motion: Springs

3.2.1 Horizontal Springs

A spring is a coil of wire. When stretched or compressed, the spring will try to return to its equilibrium position via a restoring force of $F = -kx$ that acts against the spring's displacement from equilibrium. The constant, k, is the spring constant and it is a measure of the spring's stiffness.

We just solved the differential equation of motion for a simple spring-mass system in the previous section. So we know that the solution to this motion is

$$x = A\cos(\omega_0 t + \phi)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$. To get the values for A and ϕ, you need to be given information about the motion at a particular time. These are constants (similar to constants of integration) and require initial conditions to be solved.

The angular frequency, ω_0, is a fundamental property of the system itself (depends on the mass and spring constant) and it also relates to the period of motion. A cos function repeats every 2π radians, so a full period T occurs when $\omega_0 T = 2\pi$ or:

$$\boxed{T = \frac{2\pi}{\omega_0}} = 2\pi\sqrt{\frac{m}{k}} \tag{3.5}$$

So the physical properties of the system itself (mass, spring constant) determine the period of motion. That is, the system itself sets the period of motion, not the force that is applied.

3.2.2 Vertical Springs

Consider the case of a vertical spring. If the spring is vertical, we have an additional force to consider: gravity. Figure 3.3 shows a spring hanging from the ceiling. Because there is a force pulling down on the spring, the spring has a different equilibrium point from the case when there is no mass hanging off it.

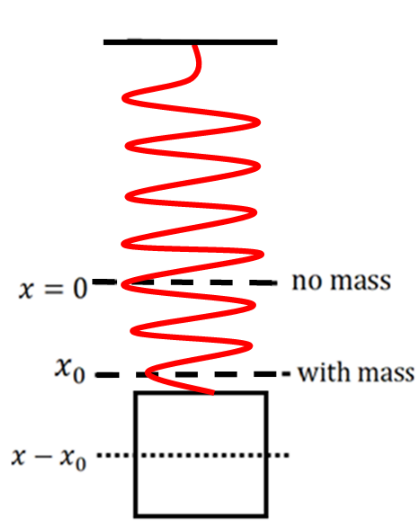


Figure 3.3: Example of a vertical mass-spring system. Without the mass, the spring will have an equilibrium point at $x = 0$. With the mass, gravity pulls down the spring until it reaches a new equilibrium point at $x = x_0$, where $x_0 < 0$. If the mass is displaced from this new equilibrium point it will undergo simple harmonic motion.

In this case, gravity stretches the spring downward, but the spring also pulls upward to counteract gravity. At some point, the spring force will balance gravity, and the system is in a new equilibrium. To find the new equilibrium position x_0 and the equation of motion, we go back to Newton's second law.

$$\begin{aligned} \sum F &= ma \\ F_g + F_s &= ma \quad \Longrightarrow \sum F \text{ is gravity } (F_g) \text{ and the spring force } (F_s) \\ ma &= -kx - mg \quad \Longrightarrow F_g = -mg \text{ and } F_s = -kx \text{ by definition } (+\hat{x} \text{ is up}) \end{aligned}$$

If the system is static, it is in equilibrium. Here, $ma = 0$ and $-kx - mg = 0$, which means that the new equilibrium position is $x_0 = -\frac{mg}{k}$. Note that x_0 is negative because we defined $x = 0$ to be at the original equilibrium point when there is no mass on the spring and we defined x as positive pointing up.

For any other position, x, the system will feel a net force and $a \neq 0$:

$$\begin{aligned} ma &= -kx - mg \\ m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -kx + kx_0 \quad \Longrightarrow \text{from } x_0 = -\tfrac{mg}{k}, \text{ we get } -mg = kx_0 \\ 0 &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{k}{m}x - \frac{k}{m}x_0 \end{aligned}$$

So we have an additional (constant) term in our differential equation of motion. Nevertheless, we can still solve this second order differential equation. The trick here is that,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(x - x_0) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$$

if x_0 is a constant. The reason is that the derivative of a constant is always zero. So the constant does not factor into the differential at all. That means that the solution to this differential equation of motion is just what we had before, but with an offset. For example, substitute $X = x - x_0$. Doing this gives us:

$$\begin{aligned}
\frac{\mathrm{d}^2X}{\mathrm{d}t^2} &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2} && \Longrightarrow \text{where } x_0 \text{ is a constant} \\
\frac{\mathrm{d}^2X}{\mathrm{d}t^2} &= -\frac{k}{m}X && \Longrightarrow \tfrac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\tfrac{k}{m}(x - x_0) = -\tfrac{k}{m}X \\
X &= A\cos(\omega_0 t + \phi)
\end{aligned}$$

which is the solution for a generic simple harmonic oscillator. But since $X = x - x_0$, the equation for vertical displacement x is then $x - x_0 = A\cos(\omega_0 t + \phi)$, where x_0 is our new equilibrium position. The equation of motion is,

$$x = A\cos(\omega_0 t + \phi) + x_0$$

Quick questions

1. Consider a vertical spring-mass system with an equilibrium position at $x_0 = -\frac{mg}{k}$. What is the magnitude of the spring force at $x = x_0$ and $x = 0$?
2. How does increasing (or decreasing) the spring constant affect the motion and the equilibrium point of a vertical spring-mass system.
3. If you moved the spring-mass system from Earth to the Moon (lower gravity) or Jupiter (higher gravity), how would the motion and equilibrium position change?

Test Your Understanding

Test your answers to the above questions with this simulator:
https://phet.colorado.edu/sims/html/masses-and-springs/latest/masses-and-springs_en.html

3.3 Brief Aside on the Differential Equation of Motion

The Differential Equation of Motion is a convenient tool to solve cases of simple harmonic motion. If you can put your physics into this format,

$$0 = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + Cx$$

where C is constant with time, then you can get the angular frequency, ω_0 (and by default, the period T and frequency f) directly from the equation alone. In this form, where the differential has no coefficient, $\omega_0^2 = C$. Other constants do not matter (e.g., consider the vertical spring) to solving ω_0. Thus, you can read off the value of ω_0 directly from the equation.

Quick Questions

1. After equating all forces, you obtain a differential equation of motion of the form:

$$A\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{4R_1k}{R_2}y + M(R_1^2 + R_2^2)g = 0$$

where all parameters are constants except y. What is the angular frequency (ω_0) of this system? Note, you do not need to solve the differential equation of motion.

A) $\omega_0 = \frac{4R_1k}{R_2}$

B) $\omega_0 = \sqrt{\frac{4R_1k}{AR_2}}$

C) $\omega_0 = \sqrt{\frac{4R_1k}{R_2} + M(R_1^2 + R_2^2)g}$

D) $\omega_0 = \frac{4R_1k}{AR_2} + \frac{M(R_1^2+R_2^2)g}{A}$

3.4 Simple Harmonic Motion: Pendulum

3.4.1 Simple Pendulum

Now let's consider a simple pendulum. A simple pendulum is a mass that hangs at the end of a string and is allowed to swing (see Figure 3.4).

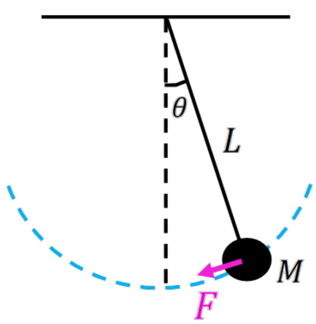


Figure 3.4: Example of a simple pendulum. The mass m is in equilibrium when it is vertically downward and displaced from equilibrium when shifted an angle θ from the vertical axis. A restoring force (F) moves the pendulum back to equilibrium.

A pendulum is a simple harmonic oscillator as well, because it has an equilibrium position (straight down) and a restoring force that is proportional to the displacement will seek to return the pendulum to that position.

Keep in Mind

We're going to discuss the simple pendulum in two ways. Here, we use $F = ma$ to describe the motion of a simple pendulum. Later in Chapter 7, we will revisit the simple pendulum using torques to show you how the two approaches differ. One of the key elements of this textbook is determine which methodology is ideal to use for a given physics problem. So when going through both, think about the pros (and cons) of each method.

To solve for the force, let's look at the free-body diagram of this system (Figure 3.5).

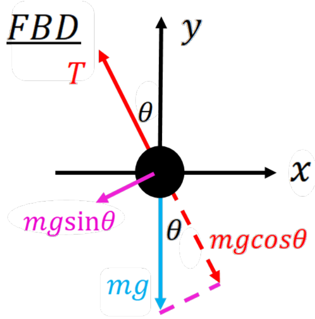


Figure 3.5: Free-body diagram of the simple pendulum from Figure 3.4. The labeled forces are tension (T) in red, gravity (mg)in blue, and the restoring force ($mg \sin\theta$) in magenta. Shown in dotted-red is the component of gravity that balances tension ($mg \cos\theta$).

The restoring force is caused by a component of gravity that is perpendicular to the tension in the string. Because the mass-string system has an angular displacement (θ) from the equilibrium line, there is a component of gravity along the string and a component of gravity perpendicular to the string. It is the perpendicular component that is our restoring force (see magenta arrow in Figure 3.5). From trigonometry, the component parallel to the string can be written as $mg \cos\theta$ and the component perpendicular to the string is $mg \sin\theta$. The $mg \cos\theta$ component is equal (and opposite) to the tension in the string. The $mg \sin\theta$ component is our restoring force and it will be driving our motion. So we have,

$$F = -mg \sin\theta$$

where the negative sign is present because this is a restoring force (it will act in the opposite direction to our angular displacement). Since T and $mg \cos\theta$ cancel (equal and opposite forces because the string is not deforming), our net force is equal to this restoring force.

You'll notice that this force equation does not depend on x, but instead depends on the angular displacement. If we want to use $F = ma$, we need to get the displacement in units of x because $a = \ddot{x}$. Using the small angle approximation (see Appendix B), we can write $\sin\theta = \dfrac{x}{L}$:

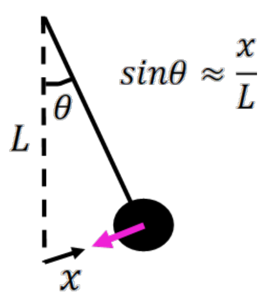


Figure 3.6: Small angle approximation diagram for small values of θ, as this is approximately a right angle triangle where the y-axis (L), and displacement x meet.

The true path of the pendulum is an arc, so this assumption requires that θ isn't too big so that there is very little difference between an arc and a straight line. See Appendix B for a review on applying small angle approximations.

So with the small angle assumption, we get

$$F = -\frac{mg}{L}x$$

where m, g, and L are all constants.

This equation has the exact same form as what we used for the spring $F = -kx$, only with different constants. We can solve the equation of motion.

$$
\begin{aligned}
\sum F &= ma \\
-\frac{mg}{L}x &= m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \\
-\frac{g}{L}x &= \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \\
0 &= \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{g}{L}x
\end{aligned}
$$

Once more we have a Differential Equation of Motion that we can solve just by looking at it. This equation has the same structure as the spring and mass system. The general solution to this problem is $x(t) = A\cos(\omega_0 t + \phi_1) + B\sin(\omega_0 t + \phi_2)$, but in this case, we have a different value for the angular frequency.

$$
\begin{aligned}
\omega_0^2 &= \frac{g}{L} \quad \Longrightarrow \text{Recall that } \omega_0^2 \text{ equals the coefficient in front of } x \\
\omega_0 &= \sqrt{\frac{g}{L}}
\end{aligned}
$$

And the angular frequency relates to the period of motion by,

$$
T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{L}{g}} \quad \Longrightarrow \quad \text{The period is independent of the mass of the pendulum}
$$

Quick Questions

1. You have two identical pendulum clocks, but one is on Earth and one is on the Moon. How would the clock on the Moon keep time relative to the one on Earth?
2. How could you adjust the pendulum clock on the Moon for it to keep the same time as the one on Earth?

The physics of pendulums

The period (or the time necessary for the pendulum to swing back to its starting point) depends entirely on the length of the pendulum and the gravitational field where it is located. With the right series of lengths, you can get some interesting harmonics.
https://www.youtube.com/watch?v=yVkdfJ9PkRQ

3.4.2 Physical Pendulum

Technically, any object can be made into a pendulum if displaced from its equilibrium position and allowed to swing freely from a pivot point. We call these cases a physical pendulum. Figure 3.7 shows an example of a physical pendulum.

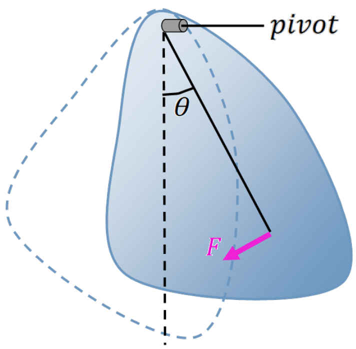


Figure 3.7: A physical pendulum. The irregular object has an equilibrium position as shown by the black dashed outline. When rotated out of this equilibrium position, a restoring force F will seek to move it back toward equilibrium.

The solution for a physical pendulum via $F = ma$ is non-trivial, because you need to consider the acceleration of every individual particle ($F_i = m_i a_i$) in the system and the linear acceleration a_i will differ throughout the system. Instead, we will revisit the physical pendulum when we discuss *angular acceleration* and torques in Chapter 7.

3.5 Sample Problems

Sample Problem 3-1

A block of mass m is attached to two springs on a frictionless surface as shown in the figure below. If the block is displaced from equilibrium and set into simple harmonic motion, **find the period of oscillations for the block**.

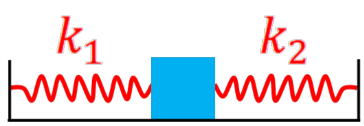


Figure 3.8: Diagram of the system, with spring one having the constant k_1 and spring 2 having the constant k_2.

Solution

Let's first look at the free-body diagram of the system. There is the gravitational force and the normal force, which will be equal and opposite (no vertical motion). There are also restoring forces from each of the springs. If the block is displaced a distance x from the equilibrium position, then the forces will be $F_1 = -k_1 x$ and $F_2 = -k_2 x$ (both springs are displaced by the same amount). The spring forces are in the same direction, because both act to move the block to equilibrium.

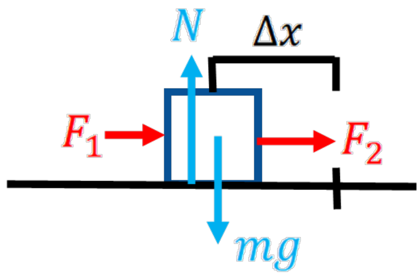


Figure 3.9: Free-body diagram of the block and two spring system where F_1 comes from spring 1 and F_2 comes from spring 2, and the mass has been displaced Δx to the left.

Using Newton's second law, the sum of all (horizontal) forces is;

$$
\begin{aligned}
\sum F &= F_1 + F_2 = ma \quad \Longrightarrow \quad F_g = N, \text{ so we ignore the vertical forces} \\
m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= F_1 + F_2 \\
m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= -k_1 x - k_2 x \quad \Longrightarrow \quad \text{use the equations for } F_1 \text{ and } F_2 \\
\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} &= -\frac{k_1 + k_2}{m} x \\
0 &= \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{k_1 + k_2}{m} x
\end{aligned}
$$

Again, we have a Differential Equation of Motion, and in this form, we can read off ω_0^2 from the coefficient in front of the x term.

$$
\begin{aligned}
\omega_0^2 &= \frac{k_1 + k_2}{m} \\
\omega_0 &= \sqrt{\frac{k_1 + k_2}{m}}
\end{aligned}
$$

But the question asked for the period of oscillations. For the period, we get

$$
\begin{aligned}
T &= 2\pi \frac{1}{\omega_0} \\
T &= 2\pi \sqrt{\frac{m}{k_1 + k_2}}
\end{aligned}
$$

This is the same solution as the simple (one spring) case, but $k \to k_1 + k_2$ because there are two springs working together. As a consequence of having these two springs, the period decreased compared to if there was one spring alone.

Sample Problem 3-2

A simple pendulum of mass m and length L is also attached to spring with spring constant k as shown in the figure below. The equilibrium point for both the pendulum and spring is given by the vertical with the pivot point. If the pendulum is displaced from this equilibrium by an angle θ (like a pendulum), **find the period of oscillations**.

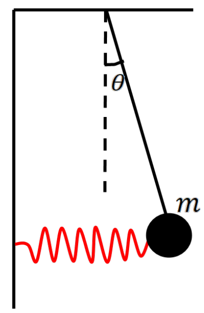


Figure 3.10: A mass m forms a simple pendulum with a massless rope of length L. The mass is also attached to a horizontal spring with spring constant k. The equilibrium point for both the pendulum and spring is shown by the dashed vertical line (right below the pivot point).

Solution

This problem contains two simple harmonic oscillators. Let's look at a free-body diagram for the mass. At the position of the mass, the forces are gravity on the mass, tension in the rope, and the spring force. Breaking up the gravitational force into its components along the axis of the rope and perpendicular to that axis, we get the restoring force from the pendulum as $F_p = F_g \sin\theta \approx \frac{mg}{L}x$ for small angles, θ. Note that for small angles, both the restoring forces act in the same direction.

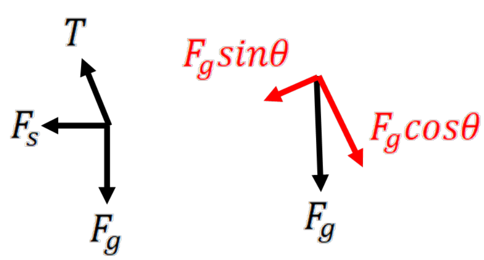


Figure 3.11: On the left is the free-body diagram of the spring-mass system, and on the right is the component break-down of the force of gravity, (F_g) in red.

Putting these forces together, we can solve the equation of motion from the sum of all forces:

$$\begin{aligned}
\sum F &= F_s + F_p = ma \quad \Longrightarrow \quad \text{T and $F_g \cos\theta$ cancel, so we can ignore them} \\
ma &= -kx - \frac{mg}{L}x \\
m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -\left(k + \frac{mg}{L}\right)x \\
\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -\left(\frac{kL+mg}{mL}\right)x \\
0 &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\frac{kL+mg}{mL}\right)x
\end{aligned}$$

Again, we have a Differential Equation of Motion, and in this form, we can read off ω_0^2 from the coefficient in front of the x term.

$$\begin{aligned}
\omega_0^2 &= \frac{kL+mg}{mL} \\
\omega_0 &= \sqrt{\frac{kL+mg}{mL}} \\
\omega_0 &= \sqrt{\frac{k}{m} + \frac{g}{L}}
\end{aligned}$$

The angular frequency has a term for the spring and the pendulum, which increases the value of ω_0 compared to the value from either the spring or pendulum alone. A larger ω_0 will decrease the period (they are inversely proportional). This means that by adding simple harmonic oscillators, the motion goes faster (shorter period).

For this ω_0, the period is:

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{mL}{kL+mg}}$$

3.6 Aside on Damping and Driven Motion

The harmonic motion described above is all perfectly conserved (e.g., there is no loss of energy from friction). In practice, most oscillators undergoing harmonic motion are damped or driven. Examples of damped (energy lost) oscillations include the suspension in a vehicle (this is on purpose to limit the oscillations from bumps on the road) and tuned mass dampers in tall buildings to limit motion at high floors from earthquakes or strong winds. Examples of driven (energy gained) oscillations include pushing a child on a swing (when timed right, the child goes higher and higher) or resonances in bridges. Damped and driven motion will not be covered here.

But for fun, here are some videos that show damping motion in action. An excellent example of a tuned mass damper is the Taipei 101 building in Taiwan. Unlike most skyscrapers, the

tuned mass damper in Taipei 101 is available to be seen.
Here is a nice video showing the Taipei 101 building tuned mass damper in action
This video does a nice job illustrating why these dampers work.

And to also showcase driven motion, here is a video from 1940 which shows the collapse of the Tacoma Narrows bridge in the USA during a strong wind after less than four months in operation.
Here is the Millennium pedestrian bridge in the UK. It did not collapse, but note how the oscillations are driven; as the bridge sways, more and more people become unbalanced at the same time and then take steps in sequence driving stronger oscillations.

3.7 Real-World Application

Not all oscillations are simple harmonic motion. Nevertheless, other types of periodic behaviour can be expressed with similar base mathematics even if the physics behind them is very different than a simple restoring force. These more complex cases consequently produce more complex oscillatory motions, extending the concepts of *simple* harmonic motion into more varied phenomena.

Seismology is the study of seismic (sound) waves that move around and through the Earth. Studying these waves can provide us information about the structure of our planet's interior that we couldn't otherwise constrain. The strongest seismic waves are generated by movements of tectonic plates but waves may also be caused by volcanoes, landslides, explosions, and other energetic events on and under the Earth's surface.

Seismographs are used to record the motion of the ground due to seismic waves. Those waves travel through layers with different compositions and densities, and so are refracted and reflected. Using multiple instruments, the amount of time it takes seismic waves to travel through the Earth can be calculated and the type of material the waves are travelling through can be deduced, giving a picture of the Earth's interior.

Since seismic waves can cause widespread damage, many agencies around the world have developed early warning systems to detect earthquakes as quickly as possible. The nationwide Earthquake Early Warning (EEW) system operated in Japan is the most advanced detection system in use, with a network of more than 4,000 seismometers.

For more information:

For some introductory science on seismic waves, you can visit the Science Learning Hub - Pokapū Akoranga Pūtaiao.

This web site contains information on earthquake warning systems in use around the world.

This interactive map uses real time data from Japan's Earthquake Early Warning system.

3.8 Summary

Key Takeaways

This chapter focuses on simple harmonic motion and solving problems of simple harmonic motion using Newton's laws. Simple harmonic motion is a periodic motion (system moves back and forth) that arises when a force has the form of

$$\vec{F} = -k\vec{r}$$

where k is a constant. These types of forces are also called restoring forces, because the force itself seeks to return the system back to an equilibrium position (where the displacement is zero).

Solving Newton's laws for simple harmonic motion yields a second-order differential equation of motion in the form of:

$$0 = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + Cx$$

where C is a positive constant. This differential equation has a very well known solution of a cos (or sin) function, such as

$$x = A\cos(\omega_0 t + \phi)$$

A key property of simple harmonic motion is the angular frequency, ω_0, which can be read directly off the differential equation of motion,

$$\omega_0^2 = C$$

The angular frequency is a fundamental property of the system. It depends only on the constants of the system (e.g., mass, rope length, spring constant) and it also sets the period for the periodic motion.

$$T = \frac{2\pi}{\omega_0}$$

Thus, if you can get a simple harmonic motion problem into its differential equation of motion, you can automatically solve for the period of motion and its angular frequency.

Important Equations

General 2nd-Order Differential:

$$\begin{aligned} 0 &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + Cx \\ \text{where:} & \\ C &= \omega_0^2 \quad \text{for positive constant C} \\ \text{and} & \\ T &= \frac{2\pi}{\omega_0} \end{aligned}$$

2nd-Order Differential for Spring-Mass:

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -\frac{k}{m}x \\ \omega_0 &= \sqrt{\frac{k}{m}} \end{aligned}$$

2nd-Order Differential for Simple Pendulum:

$$\begin{aligned} \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -\frac{g}{L}x \\ \omega_0 &= \sqrt{\frac{g}{L}} \end{aligned}$$

Simple Solution to 2nd-Order Differential:

$$\begin{aligned} x &= A\cos\left(\omega_0 t + \phi_1\right) \\ v &= -\omega_0 A\sin\left(\omega_0 t + \phi_1\right) \\ a &= -\omega_0^2 x \end{aligned}$$

General Solution to 2nd-Order Differential:

$$\begin{aligned} x &= A\cos\left(\omega_0 t + \phi_1\right) + B\sin\left(\omega_0 t + \phi_2\right) \\ v &= -\omega_0 A\sin\left(\omega_0 t + \phi_1\right) + \omega_0 B\cos\left(\omega_0 t + \phi_2\right) \\ a &= -\omega_0^2 x \end{aligned}$$

3.9 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 3-1

If you are given a clock made from a simple mass and spring system, but the clock is running slow such that each oscillation takes twice as long as it should, what change can you make to the mass to correct the clock?

Practice Problem 3-2

A standard pendulum clock on Earth has a period of 2s. If NASA wants to engineer a series of simple pendulum clocks that can keep proper time on all of the planets in the Solar System, what arm lengths would be necessary for each planet?

Practice Problem 3-3

An antique pendulum clock uses a uniform rod of length L and operates with an angular frequency of $\omega_0 = \sqrt{\frac{3g}{L}}$.

a) What is the differential equation of motion for this clock?

b) Plot the period as function of rod length for lengths between 10 cm and 1 m.

c) What length would give you a period of 1s? Check your answer against your plots from part b).

Practice Problem 3-4

Two springs with different spring constants are attached to a mass (m) as shown. What is the angular frequency of this simple harmonic motion produced by this system?

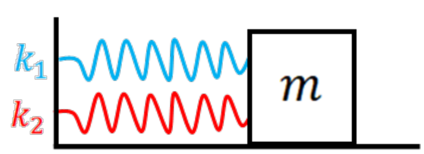


Figure 3.12: Mass m sits on a frictionless surface attached to springs 1 and 2 in parallel.

Practice Problem 3-5

An elevator is falling at nearly the free-fall acceleration. If the elevator also contains a simple pendulum clock, what happens to the periodic motion of the pendulum clock when,

a) The elevator was stationary?

b) While it is in free-fall? Note: free-fall means that the elevator is traveling at near the gravitational acceleration.

Practice Problem 3-6

Consider a pendulum-spring system where the angular frequency is:

$$\omega_0 = \sqrt{\frac{k}{m} + \frac{g}{L}}$$

a) Write out the equations for position, velocity, and acceleration assuming small displacements, A in x.

b) Plot these equations for different values for A, k, m, and L and observe how the graphs change.

Practice Problem 3-7

See the figure below. A mass, m, is sitting on an incline and attached to a spring. Assume the surface is frictionless. The mass is displaced from equilibrium by a small distance, x.

a) Draw a free-body diagram for the mass.

b) What is the equilibrium point of the mass?

c) What is the differential equation of motion for the mass?

d) What is the period of oscillations for this motion? How does the period depend on θ? Comment.

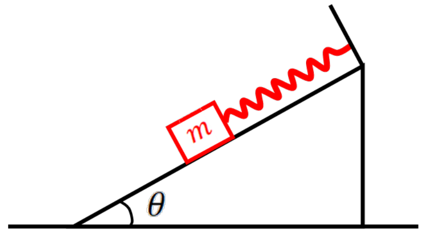


Figure 3.13: Mass m sits on a frictionless incline of angle θ, attached to the top of the incline.

Practice Problem 3-8

A mass m sits on a frictionless surface with one spring attached to each side. The springs have spring constants of k_1 and $3k_1$, respectively. If the mass is displaced from equilibrium with an amplitude A, it will oscillate.

a) Draw a free-body diagram for the mass.

b) Find the differential equation of motion for the mass.

c) Find the angular frequency and period of the mass as it oscillates.

d) Find the equations for displacement, velocity, and acceleration in terms of the variables given.

e) Plot the functions from part d). Sample python codes are available in the online repository.

Practice Problem 3-9

Two ideal springs with constants k_1 and k_2 are connected to each other and hang vertically from a ceiling. A mass is attached to the lower spring and the system is set in simple harmonic motion. Note that the two springs are connected in series with this arrangement.

a) Draw a free-body diagram for the mass.

b) We want to replace the two springs with a single spring but retain the same simple harmonic motion. Show that the effective spring constant for this new spring is:

$$k_{eff} = \frac{k_1 k_2}{k_1 + k_2}$$

Hint: For ideal (massless) springs that are connected, they will have the same force, but different displacements.

Practice Problem 3-10

See the figure below. Three springs are attached to a mass, m in series as shown in the figure below. Assume the surface is frictionless. The mass is displaced from equilibrium by a small distance, x.

a) Draw a free-body diagram for the mass.

b) What is the differential equation of motion for the mass?

c) Show that this system has a period of oscillations of:

$$T = 2\pi\sqrt{\frac{m(k_1 + k_2)}{k_1 k_2 + k_3(k_1 + k_2)}}$$

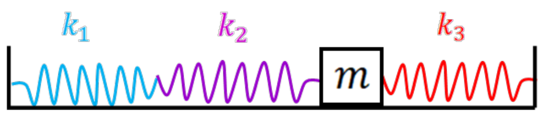


Figure 3.14: Mass m sits on a frictionless surface attached to springs 1 and 2 to its left and spring 3 to its right, displaced a distance x, from its equilibrium.

4 Introduction to Non-Inertial and Rotating Frames

Learning Objectives

- Review of relative motion and moving coordinates
- Introduction to non-inertial frames of reference and definition of fictitious forces
- Introduction to rotating frames
- Types of acceleration

In this chapter, we will review inertial frames and reference and introduce non-inertial and rotating frames. A frame of reference represents your observer. Frames can be stationary, accelerating, or rotating. The physics in each of cases will need to be treated differently.

4.1 Review of Reference Frames

Figure 4.1 shows two reference frames with a point P common to both. The black S frame is stationary and the red S' frame is moving. (Imagine two observers looking at point P with one observer standing still and the other is moving.) The vector from the S to the point is $\vec{r}_{PS}$ and the vector from S' to the point is $\vec{r}_{PS'}$. The vector from S to S' is $\vec{r}_{S'S}$. Using vector addition, you can show that $\vec{r}_{PS} = \vec{r}_{S'S} + \vec{r}_{PS'}$.

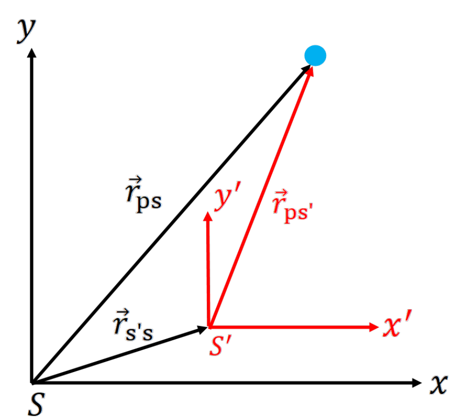


Figure 4.1: Sketch of two frames of reference. The black coordinate axes correspond to the inertial frame, S. The red coordinate axes correspond to a moving frame, S'. A point P is shown with vectors from the origin of S and S'. The point is located at $\vec{r}_{PS}$ in S and the point is located at $\vec{r}_{PS'}$ in S'. The vector from S to S' is $\vec{r}_{S'S}$

Taking the time derivative and second time derivative of $\vec{r}_{PS} = \vec{r}_{S'S} + \vec{r}_{PS'}$, you get velocity and acceleration.

$$\begin{aligned}
\vec{r}_{PS} &= \vec{r}_{S'S} + \vec{r}_{PS'} \\
\vec{v}_{PS} &= \vec{v}_{S'S} + \vec{v}_{PS'} && \Longrightarrow \text{taking the first time derivative of all terms} \\
\vec{a}_{PS} &= \vec{a}_{S'S} + \vec{a}_{PS'} && \Longrightarrow \text{taking the second time derivative of all terms}
\end{aligned}$$

These equations show the relative velocity and relative acceleration of P between the two frames. If S' is an inertial frame, then S' is moving with a constant velocity. For a constant velocity, $\vec{a}_{SS'} = 0$ and we get $\vec{a}_{PS} = \vec{a}_{PS'}$, the acceleration is the same in both frames. Note that the same result happens if S' is stationary.

For two different *inertial* frames, an observer in each frame would measure the same acceleration. There could be a difference in velocity (e.g., relative motion), but there is no difference in acceleration. As a result, there is no difference in the net forces ($\sum \vec{F} = m\vec{a}$).

4.2 Introduction to Non-Inertial Reference Frames

In a non-inertial frame, the frame of reference is accelerating or rotating. Going back to our example from Section 4.1, now $\vec{a}_{SS'} \neq 0$ and the acceleration for point P measured in both frames will be different because,

$$\vec{a}_{PS} = \vec{a}_{S'S} + \vec{a}_{PS'}$$

Let's consider motion from the perspective of an observer in an inertial frame (S) and an observer in a non-inertial frame (S'). Imagine that the two observers are sitting at the origins of each frame. They would register the motion of point P relative to their frame only. So the observer in S would say that the acceleration of P is $\vec{a}_{PS}$ and the observer in S' would measure $\vec{a}_{PS'}$, where $\vec{a}_{PS} \neq \vec{a}_{PS'}$.

If both observers were to apply Newton's second law, they would get:

Observer in S: $\sum \vec{F}_S = m\vec{a}_{PS}$

Observer in S': $\sum \vec{F}_{S'} = m\vec{a}_{PS'}$

But $m\vec{a}_{PS} \neq m\vec{a}_{PS'}$, so that means $\sum \vec{F}_S \neq \sum \vec{F}_{S'}$. The two observers will measure different solutions from Newton's laws.

But there can be only one true net force. Physics cannot change just because the reference frame has changed. It may seem like Newton's laws have failed, but in practice, we need to apply a "correction" for the accelerating frame. This correction can be written as:

$$\begin{aligned}
\sum \vec{F}_{S'} &= m\vec{a}_{PS'} \\
\sum \vec{F}_{S'} &= m(\vec{a}_{PS} - \vec{a}_{S'S}) \qquad \Longrightarrow \vec{a}_{PS} = \vec{a}_{S'S} + \vec{a}_{PS'} \\
\sum \vec{F}_{S'} &= m\vec{a}_{PS} - m\vec{a}_{S'S} \\
\sum \vec{F}_{S'} &= \sum \vec{F}_S - m\vec{a}_{S'S} \\
\sum \vec{F}_{S'} &= \sum \vec{F}_S + F_{fic}
\end{aligned}$$

where we have introduced a "new force", $\vec{F}_{fic}$. We call this "new force" a *fictitious force* or an inertial force. For the second law to match in both the inertial and non-inertial (accelerating) frame, we add in these fictitious forces to the inertial frame, where

$$\boxed{\vec{F}_{fic} = -m\vec{a}_{S'S}} \tag{4.1}$$

Note that the fictitious forces do not represent actual forces. Fictitious forces do not arise from the interaction between the two frames S and S' or from an interaction between the moving object and another object. Instead, they arise from the non-inertial frame having an acceleration. It is a "force" that an observer in a non-inertial frame would feel acting on them only because they are in an accelerating frame.

Cora's Thoughts

Overall fictitious forces are forces that *appear* to act on an object to explain its motion. A good way to think of fictitious forces is in the context of driving a car. If you are driving a car down a straight road with cruise control on (traveling at a constant linear velocity), then you are in an inertial frame and you do not feel any forces from the motion of the car. However, when you hit a bend in the road you accelerate as you turn making the car a non-inertial frame. When the car turns left, it accelerates to the left, and you feel a "force" that pushes you to the right. That "force" is the fictitious force. It is the force felt in the opposite the direction of the acceleration, that comes from being an observer in a non-inertial frame, as you only know that you are in an accelerating frame due to feeling this fictitious force.

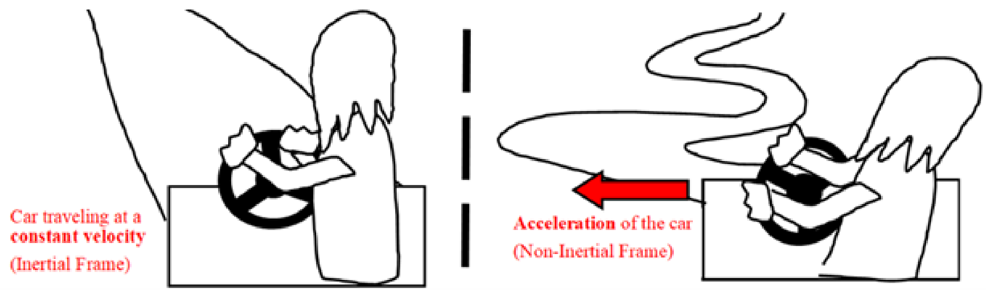


Figure 4.2: On the left is the inertial frame and on the right is the non-inertial frame.

Mathematically, the acceleration of the non-inertial frame causes the object to have an extra term in the force equation as measured from the perspective of someone in a true inertial frame. That extra term has the form of a force (mass times acceleration). If we need to add the fictitious force(s) to the inertial forces, then we can apply Newton's second law to the non-inerial frame and get the same answer:

$$\begin{aligned}
\sum \vec{F}_{S'} &= \sum \vec{F}_S + \vec{F}_{fic} \\
m\vec{a}_{PS'} &= m\vec{a}_{PS} - m\vec{a}_{S'S} \\
m\vec{a}_{PS'} &= m(\vec{a}_{S'S} + \vec{a}_{PS'}) - m\vec{a}_{S'S} \quad \Longrightarrow \quad \text{definition of } \vec{a}_{PS} \\
m\vec{a}_{PS\prime} &= m\vec{a}_{PS\prime} \quad \Longrightarrow \quad \text{left side} = \text{right side}
\end{aligned}$$

So now we have matching physics in both reference frames. That is, the two observers would come to the same answer if we include a new "force". Ultimately, an observer in a non-inertial frame must correct their net force (compared to an inertial frame) using a fictitious force.

$$\boxed{\underbrace{\sum \vec{F}_{S'}}_{\substack{\text{non-inertial}\\\text{frame}}} = \underbrace{\sum \vec{F}_S}_{\substack{\text{inertial}\\\text{frame}}} + \underbrace{\vec{F}_{fic}}_{\text{correction}}} \tag{4.2}$$

Equivalence Principle of Mechanics

The equivalence principle of mechanics describes how fictitious forces apply to non-inertial frames. Consider a moving particle. The motion of this particle as seen by an observer in the non-inertial frame can be described by the applied forces on the particle (e.g., gravity, tension, etc) and an additional fictitious force in the direction of $-a$. This fictitious force acts like a force, and can be thought of as a modification of one of the inertial forces (like gravity). That is, we can think of the fictitious force as an effective gravity term, since gravity is just given by m and an acceleration.

4.3 Example Problems with Linear Acceleration

Lets put non-inertial frames into practice with a couple of examples where the acceleration is linear (no rotation).

Sample Problem 4-1

The mass is hanging from the ceiling of an elevator by a rope, and elevator is moving upwards with an acceleration of a_e. **Compare the tension in the rope as measured by an observer in (a) an inertial frame and (b) the elevator's moving frame.** See Figure 4.3.

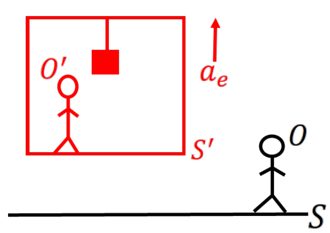


Figure 4.3: In the red we have the elevator as the non-inertial frame accelerating upwards at a_e, and in the black we have the ground as the inertial frame.

Solution

a) Inertial Frame: The observer O on the ground is in an inertial (stationary) frame. This observer sees the mass moving upward with an acceleration of a_e.

We have just two forces, tension and gravity, acting on the mass. And the mass has a net acceleration upwards of a_e. So from Newton's second law we have:

$$\begin{aligned}
\sum \vec{F} &= m\vec{a_e} \\
T - mg &= ma_e \\
T &= m(a_e + g)
\end{aligned}$$

where a_e is the acceleration of the mass because of the elevator.

b) Elevator Frame: Now consider the observer O' in the elevator with the mass. From the perspective of this observer, the mass is stationary because both the observer and the mass are moving upwards (there is no relative motion between O' and m). So $\sum \vec{F}_{S'} = 0$. But we cannot say that $\sum \vec{F}_{S'}$ is given by tension and gravity alone, because $'$ is in a non-inertial frame. This observer must take into account the acceleration of their own frame and include a fictitious force acting on mass.

For the moving reference frame, we need to correct the second law using the fictitious force. Taking up as positive, we have:

$$\begin{aligned}
\sum \vec{F}_{S'} &= \sum \vec{F}_S + \vec{F}_{fic} \\
\sum \vec{F}_{S'} &= \sum \vec{F}_S - ma_e \quad \Longrightarrow \quad \vec{F}_{fic} = -m\vec{a}_{S'S} = -ma_e \\
0 &= T - mg - ma_e \quad \Longrightarrow \quad m \text{ at rest in } S'\text{, so } \sum \vec{F}_{S'} = 0 \\
T &= m(a_e + g)
\end{aligned}$$

which is the same answer as before from the inertial frame as expected. We need our physics to match in both reference frames (or we have a problem with physics!).

Quick Questions

1. Draw a free-body diagram of the mass in the inertial and non-inertial frames. What is the vector direction of the fictitious force?
2. Now the elevator is accelerating downward. Solve for the tension in the rope for both frames. What is the direction of the fictitious force?

Sample Problem 4-2

A truck is carrying a box of mass m. When the truck decelerates at a rate of $a_0 = 0.6g$, the box in the rear of the truck begins to slide forward relative to an observer sitting in the truck. If the coefficient of friction between the box and the truck is $\mu = 0.4$, **what is the acceleration of the box relative to the (a) the ground and (b) the truck?**

Solution

Case 1: We will first solve this problem from the perspective of an observing standing on the ground (so from the inertial frame).

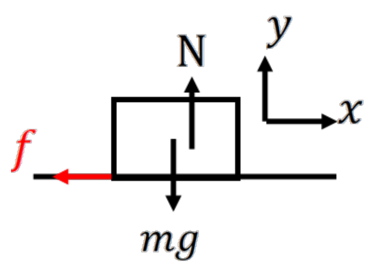


Figure 4.4: Free-body diagram of the mass m in the inertial frame. The forces acting on the mass are the force of friction (f) in red, the normal force (N) and gravity (mg).

Inertial Frame: Figure 4.4, shows the free-body diagram of the box relative to an observer in an inertial frame (e.g., an observer on the ground). If the box is moving forward ($+x$ direction), then friction is acting in the opposite direction ($-x$ direction). We also have gravity and the normal force.

All motion is horizontal. The box does not move up or down, so $\sum \vec{F}_y = 0$. There are only two vertical forces, gravity and the normal force. Therefore, we can say that $N = mg$.

a) To the observer on the ground (in the inertial frame), the net force acting on the box is just from friction, $\sum \vec{F}_g = f$. From Newton's second law, we have:

$$\begin{aligned}\sum \vec{F}_g &= \vec{f}\\ ma_b &= -\mu mg\\ a_b &= -0.4g\end{aligned}$$

So the acceleration of the box relative to the ground is $-0.4g$ (taking forward to be positive).

b) To get the acceleration of the box relative to the truck, $a_{b'}$, we use coordinate transformation. Note that the acceleration of the truck is $\vec{a}_0 = -0.6g\hat{\imath}$ because the truck is decelerating and we set the forward direction as the positive x direction.

$$\begin{aligned}\vec{a}_{b'} &= \vec{a}_b - \vec{a}_0 \\ a_{b'} &= -0.4g - (-0.6g) \\ a_{b'} &= 0.2g\end{aligned}$$

So the acceleration of the box relative to the truck is $0.2g$. Note that this is positive. That makes sense as the box is sliding forward relative to the the observer sitting (stationary) in the truck.

Case 2: We can also solve this problem using the non-inertial frame of the truck. That is, we can solve the accelerations from the perspective of a person sitting in the truck.

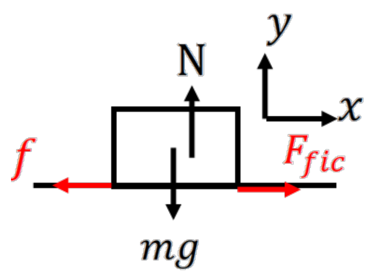


Figure 4.5: Free-body diagram of the mass m in the non-inertial frame. The forces acting on the mass are the force of friction (f) and the fictitious force (f_{fic}) in red, the normal force (N) and gravity (mg).

Truck Frame: Figure 4.5 shows the free-body diagram of the box relative to an observer in the truck (non-inertial frame). We have the same forces as the inertial frame (friction, gravity, normal), but there is also the fictitious force.

Again, all motion is horizontal. But in the non-inertial frame of the truck, there are two horizontal forces. First, is friction given by $f = -\mu N = -\mu mg$ (negative because it acts in the $-x$ direction). Second, is the fictitious force because the truck is a non-inertial frame. Recall that fictitious forces act in the opposite direction of the acceleration of the frame relative to an inertial frame. Since the truck is decelerating ($-x$ direction) relative to the inertial frame ($a_0 = -0.6g$), the fictitious force acts in the $+x$ direction.

For the non-inertial frame, we will first find $a_{b'}$, the acceleration of the box relative to an observer on the truck.

$$\begin{aligned}\sum \vec{F}_{S'} &= -\mu mg + \vec{F}_{fic} \\ a_{b'} &= -0.4g + 0.6g \quad \Longrightarrow \quad \vec{F}_{fic} = -ma_0 \text{ and } a_0 = -0.6g \\ a_{b'} &= 0.2g\end{aligned}$$

Which is exactly what we had before from Case 1 when solving the problem from the inertial frame as expected.

Quick Questions

1. Between solving the problem in the inertial frame versus the non-inertial frame, which method did you like better?
2. What if the deceleration of the truck is $a_0 = 0.2g$? Will the box slide?

4.4 Rotating Frames

For a review of rotational motion see Chapter 1.1.2 and for an example practice problem with rotation in an inertial frame, see Example 1-2.

4.4.1 Rotating Systems

In this section, we will introduce rotating non-inertial frames. We often call the Earth's surface an inertial frame in physics, but this assumption neglects the rotation of the Earth about its axis (and the rotation of the Earth around the Sun, the rotation of the Sun around the centre of our galaxy, and the motion of our galaxy within our Local Group of galaxies...). For simple problems, we can often assume the Earth's surface is an inertial frame. But there are physics problems that require that you take into account Earth's own rotation.

For inertial frames, an object that is rotating with a constant angular velocity of $\vec{\omega}$ around a fixed axis has the following equations of motion.

$$\begin{aligned} \vec{v} &= \vec{\omega} \times \vec{r} \\ \vec{a} &= \vec{\omega} \times (\vec{\omega} \times \vec{r}) \end{aligned}$$

Consider an object rotating with ω in the $\hat{k}$ direction and that this axis of rotation is fixed (see Figure 4.6). A point P in this system has the vector position $\vec{r}$. This vector position can also be described by $\vec{r} = \rho\hat{\rho} + z\hat{k}$, where $\vec{\rho}$ is the projection of $\vec{r}$ onto the plane perpendicular to $\vec{\omega}$ (for $\vec{\omega}$ along $\hat{k}$, this plane is the $x - y$ plane).

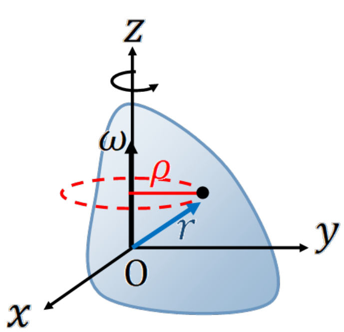


Figure 4.6: An irregular object rotating in the counter-clockwise direction around the z-axis of an xyz-axis coordinate system.

From this definition of $\vec{r}$ and $\vec{\rho}$, we can show that the velocity is:

$$\vec{v} = \vec{\omega} \times \vec{r} = \omega r \sin\theta\hat{\theta} = \omega\rho\hat{\theta}$$

where $\hat{\theta}$ is an azimuthal angle between $\vec{\omega}$ and $\vec{r}$.

Similarly, we can show that the magnitude of acceleration is:

$$\vec{a} \quad = \quad -\omega^2\vec{\rho}$$

since $\vec{a} = \vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega} \times (\omega\rho\hat{\theta})$, and $\vec{\omega} \perp \hat{\theta}$. The negative sign arises from the cross product and indicates that the acceleration is directed toward the rotation axis. See Chapter 1.5.2 for a review of vector cross products and the right-hand rule.

4.4.2 Coordinate System of a Rotating Frame: Velocity

Figure 4.7 shows the coordinates for an inertial frame, S, in black and a non-inertial rotating frame, S', in red. The inertial frame is represented by a fixed coordinate system of x, y, z and the rotating frame is represented by the coordinate system x', y', z'. The corresponding unit vectors are $\hat{\imath}, \hat{\jmath}, \hat{k}$ for the x, y, z system and $\hat{\imath}', \hat{\jmath}', \hat{k}'$ for the x', y', z'.

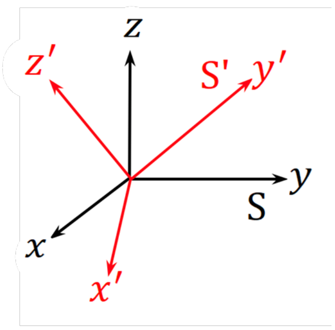


Figure 4.7: Coordinates for an inertial frame (S) in black and a rotating frame (S′) in red. Both coordinate axes share the same origin. The only difference is that S is fixed and S′ is rotating about the origin. Note that as S′ rotates, the positions of the $\hat{\imath}', \hat{\jmath}', \hat{k}'$ unit vectors change.

A point in these coordinate systems would have a vector position of:

Inertial Frame (S): $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$

Rotating Frame (S'): $\vec{r}\,' = x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}'$

Both frames have the same origin, which means that $\vec{r} = \vec{r}\,'$. To therefore get the relative velocity and relative acceleration, we need to take the time derivative of both vectors.

$$\begin{aligned}
\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} &= \frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t} \\
\frac{\mathrm{d}}{\mathrm{d}t}(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) &= \frac{\mathrm{d}}{\mathrm{d}t}(x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}') \\
\frac{\mathrm{d}x}{\mathrm{d}t}\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\jmath} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{k} &= \frac{\mathrm{d}x'}{\mathrm{d}t}\hat{\imath}' + x'\frac{\mathrm{d}\hat{\imath}'}{\mathrm{d}t} + \frac{\mathrm{d}y'}{\mathrm{d}t}\hat{\jmath}' + y'\frac{\mathrm{d}\hat{\jmath}'}{\mathrm{d}t} + \frac{\mathrm{d}z'}{\mathrm{d}t}\hat{k}' + z'\frac{\mathrm{d}\hat{k}'}{\mathrm{d}t}
\end{aligned}$$

$\Longrightarrow$ Note that $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$ are all constant with time (no derivative)

$$\begin{aligned}
\underbrace{\frac{\mathrm{d}x}{\mathrm{d}t}\hat{\imath} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\jmath} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{k}}_{\vec{v}} &= \underbrace{\frac{\mathrm{d}x'}{\mathrm{d}t}\hat{\imath}' + \frac{\mathrm{d}y'}{\mathrm{d}t}\hat{\jmath}' + \frac{\mathrm{d}z'}{\mathrm{d}t}\hat{k}'}_{\vec{v}\,'} + x'\frac{\mathrm{d}\hat{\imath}'}{\mathrm{d}t} + y'\frac{\mathrm{d}\hat{\jmath}'}{\mathrm{d}t} + z'\frac{\mathrm{d}\hat{k}'}{\mathrm{d}t} \\
\vec{v} &= \vec{v}\,' + x'\frac{\mathrm{d}\hat{\imath}'}{\mathrm{d}t} + y'\frac{\mathrm{d}\hat{\jmath}'}{\mathrm{d}t} + z'\frac{\mathrm{d}\hat{k}'}{\mathrm{d}t}
\end{aligned}$$

The above equation says that the velocity of the point, P, between the inertial (non-rotating) frame and the non-inertial (rotating) frame are related by an extra term corresponding to the rotation of the coordinate system itself.

We need to solve for $\frac{\mathrm{d}\hat{\imath}'}{\mathrm{d}t}$, $\frac{\mathrm{d}\hat{\jmath}'}{\mathrm{d}t}$, and $\frac{\mathrm{d}\hat{k}'}{\mathrm{d}t}$ to fully complete the coordinate transformation. The unit vectors in S' are rotating at a rate of $\vec{\omega}$, which is the angular velocity:

$$\vec{\omega} = \omega\hat{n}$$

where $\hat{n}$ is unit vector in the direction of $\vec{\omega}$ (the normal to the plane of rotation). Recall that using the right-hand rule, if your fingers curl in the direction of rotation, extending your thumb gives the direction of the angular velocity vector.

Figure 4.8 shows the rotation of the $\hat{\imath}'$ coordinate axis.

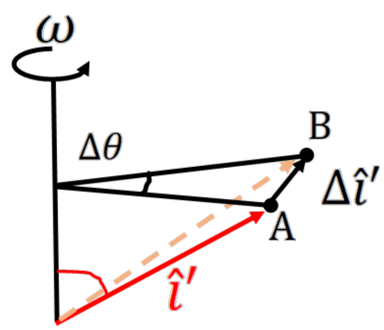


Figure 4.8: The $\hat{\imath}'$ coordinate is offset by an angle ϕ from the rotation axis (angle in red). In time Δt, $\hat{\imath}'$ moves from position A to position B due to rotation. The change in the vector position is shown by the angle $\Delta\hat{\imath}'$ and angle $\Delta\theta$.

From Figure 4.8, the $\hat{\imath}'$ axis moves a distance $\Delta\hat{\imath}'$ between points A and B in a time Δt. That displacement in time Δt is:

$$\begin{aligned}
\Delta\hat{\imath}' &= (\hat{\imath}'\sin\phi)\Delta\theta \\
\frac{\Delta\hat{\imath}'}{\Delta t} &= (\hat{\imath}'\sin\phi)\frac{\Delta\theta}{\Delta t} \quad \Longrightarrow \quad \text{divide by } \Delta t
\end{aligned}$$

Assuming that Δt is sufficiently short, we can set $\Delta t \to \mathrm{d}t$, $\Delta\hat{\imath}' \to \mathrm{d}\hat{\imath}'$, and $\Delta\theta \to \mathrm{d}\theta$:

$$\begin{aligned}
\frac{\mathrm{d}\hat{\imath}'}{\mathrm{d}t} &= (\hat{\imath}'\sin\phi)\frac{\mathrm{d}\theta}{\mathrm{d}t} \\
\frac{\mathrm{d}\hat{\imath}'}{\mathrm{d}t} &= (\hat{\imath}'\sin\phi)\omega \quad \Longrightarrow \quad \omega = \frac{\mathrm{d}\theta}{\mathrm{d}t}
\end{aligned}$$

This form of this equation should look familiar. It looks like a vector cross product. Recall that $\vec{a}\times\vec{b} = ab\sin\theta$, where θ is the angle between the vectors. So we can say that

$$\frac{\mathrm{d}\hat{\imath}'}{\mathrm{d}t} = \vec{\omega}\times\hat{\imath}'$$

You can apply similar arguments to get

$$\begin{aligned}
\frac{\mathrm{d}\hat{\jmath}'}{\mathrm{d}t} &= \vec{\omega}\times\hat{\jmath}' \\
\frac{\mathrm{d}\hat{k}'}{\mathrm{d}t} &= \vec{\omega}\times\hat{k}'
\end{aligned}$$

Quick Questions

1. Apply the right-hand rule to show that $\frac{\mathrm{d}\hat{\imath}'}{\mathrm{d}t} = \vec{\omega} \times \hat{\imath}'$ and not $\hat{\imath}' \times \vec{\omega}$.
2. A system is rotating with an angular speed of ω along the $\hat{\imath}'$ direction. What is $\frac{\mathrm{d}\hat{\imath}'}{\mathrm{d}t}$? Does this answer make sense?

Combining these definitions of the motion for the S' coordinates, we get:

$$\begin{aligned} x'\frac{\mathrm{d}\hat{\imath}'}{\mathrm{d}t} + y'\frac{\mathrm{d}\hat{\jmath}'}{\mathrm{d}t} + z'\frac{\mathrm{d}\hat{k}'}{\mathrm{d}t} &= x'(\vec{\omega} \times \hat{\imath}') + y'(\vec{\omega} \times \hat{\jmath}') + z'(\vec{\omega} \times \hat{k}') \\ &= \vec{\omega} \times (x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}') \\ &= \vec{\omega} \times \vec{r}\,' \end{aligned}$$

Thus, our coordinate transformation is:

$$\boxed{\vec{v} = \vec{v}\,' + \vec{\omega} \times \vec{r}\,'} \tag{4.3}$$

where $\vec{v}$ is the velocity relative to the inertial frame, $\vec{v}\,'$ is the velocity relative to the rotating frame, and $\vec{\omega} \times \vec{r}\,'$ is the coordinate transformation of the rotating frame.

4.4.3 Coordinate System of a Rotating Frame: Acceleration

Before we solve for the acceleration, we are going to modify our velocity equation slightly so we don't need to take the second time derivative of any of the position vectors. In Equation, we have $\vec{v} = \vec{v}\,' + \vec{\omega} \times \vec{r}\,'$. Since velocity is the time derivative of position, we can say,

$$\begin{aligned} \vec{v} &= \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_I \\ \vec{v}\,' &= \left(\frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t}\right)_R \end{aligned}$$

where the two differentials correspond to the time derivative of the position vector in the inertial frame (subscript "I") and the time derivative of the position vector in the rotating frame (subscript "R"). That is, we do not take the time derivative of the unit vectors in either case because we are applying the time derivative in each of their frames (from the perspective of an observer in those frames, the unit vectors are fixed).

Coming back to our velocity Equation (4.3), we have:

$$\begin{aligned} \vec{v} &= \vec{v}\,' + \vec{\omega} \times \vec{r}\,' \\ \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_I &= \left(\frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t}\right)_R + \vec{\omega} \times \vec{r}\,' \\ \left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_I &= \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_R + \vec{\omega}\times\right]\vec{r}\,' \end{aligned}$$

Recall however that $\vec{r} = \vec{r}\,'$ since both frames have the same origin and same end point, P. As a result, we can say that the above equation can be written as:

$$\left(\frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\right)_I = \underbrace{\left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_R + \vec{\omega}\times\right]}_{operator}\vec{r} \tag{4.4}$$

where the term in front of $\vec{r}$ acts like a coordinate transformation operator on vector $\vec{r}$ to go from the rotating frame to the inertial frame. But you can technically apply an operator to any vector, it doesn't have to be position. So if we apply this vector operator to $\vec{v}$ instead of $\vec{r}$, we get acceleration in the inertial frame.

$$\left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)_I = \left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)_R + \vec{\omega}\times\vec{v}$$

However, taking the time derivative of $\vec{v}$ in the rotating frame will require a coordinate transformation of the unit vectors. Fortunately, we can re-write $\vec{v}$ as $\vec{v}\,' + \vec{\omega}\times\vec{r}\,'$, which is relative to the rotating frame (so we don't need to worry about the moving unit vectors).

$$\begin{aligned}
\left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)_I &= \left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)_R + \vec{\omega}\times\vec{v} \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_R (\vec{v}\,' + \vec{\omega}\times\vec{r}\,') + \vec{\omega}\times(\vec{v}\,' + \vec{\omega}\times\vec{r}\,') \\
&= \left(\frac{\mathrm{d}\vec{v}\,'}{\mathrm{d}t}\right)_R + \left(\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}\right)_R \times\vec{r}\,' + \vec{\omega}\times\left(\frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t}\right)_R + \vec{\omega}\times\vec{v}\,' + \vec{\omega}\times(\vec{\omega}\times\vec{r}\,')
\end{aligned}$$

This looks like a mess, but we can simplify it a bit. First, by definition, the accelerations in each reference frame are:

$$\begin{aligned}
\vec{a} &= \left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)_I \\
\vec{a}\,' &= \left(\frac{\mathrm{d}\vec{v}\,'}{\mathrm{d}t}\right)_R
\end{aligned}$$

where $\vec{a}$ is the acceleration of the point in the inertial frame and $\vec{a}\,'$ is the acceleration of the point in the rotating frame.

In addition, recall that the velocity of the object from the perspective of the rotating frame:

$$\vec{v}\,' = \left(\frac{\mathrm{d}\vec{r}\,'}{\mathrm{d}t}\right)_R$$

Combining these definitions, our acceleration is:

$$\vec{a} = \vec{a}\,' + \left(\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}\right)_R \times\vec{r}\,' + 2\vec{\omega}\times\vec{v}\,' + \vec{\omega}\times(\vec{\omega}\times\vec{r}\,')$$

The only term that is still unclear is the time derivative of the angular velocity. If you apply the coordinate transformation operator from Equation 4.4 to $\vec{\omega}$ (recall that operators can be applied on any vector), you will get that

$$\left(\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}\right)_R = \left(\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}\right)_I$$

So we will define the time derivative of $\vec{\omega}$ as $\vec{\alpha}$, which is the angular acceleration.

Thus, we obtain the equation for the coordinate transformation of:

$$\boxed{\vec{a} = \vec{a}\,' + \vec{\alpha} \times \vec{r}\,' + 2\vec{\omega} \times \vec{v}\,' + \vec{\omega} \times (\vec{\omega} \times \vec{r}\,')} \tag{4.5}$$

Quick Question

1. Apply the coordinate operator to show that $\left(\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}\right)_R = \left(\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t}\right)_I$

Here $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$ and $\vec{a}\,' = \ddot{x}'\hat{\imath}' + \ddot{y}'\hat{\jmath}' + \ddot{z}'\hat{k}'$ in Cartesian coordinates. That is, these are the accelerations as seen from the inertial frame and rotating frame, respectively. The remaining terms are additional forms of acceleration. These are the "fictitious forces" for a rotating non-inertial frame of reference.

4.5 Types of Acceleration and Fictitious Forces

Equation 4.5 equates the acceleration between an inertial frame and a non-inertial rotating frame where the two have the same origin. In practice, the origin of the rotating frame can move relative to the origin of the inertial frame. This motion would add a linear acceleration term that is independent of the rotation, so we can just add an additional acceleration term, $\vec{A}$ to represent the acceleration of the origin of the rotating frame as viewed by an observer in the inertial frame.

Thus, our final equation for the acceleration (relative to the non-inertial frame) is:

$$\boxed{\underbrace{\vec{a}\,'}_{1} = \underbrace{\vec{a}}_{2} - \underbrace{\vec{\alpha} \times \vec{r}}_{3} - \underbrace{2\vec{\omega} \times \vec{v}\,'}_{4} - \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{5} - \underbrace{\vec{A}}_{6}} \tag{4.6}$$

1. <u>Linear acceleration in the rotating frame</u> (what an observer in the rotating frame would measure as the acceleration). This would be equivalent to the net acceleration from the perspective of the rotating frame.

2. <u>Linear acceleration in the inertial frame</u> (what an observer in the inertial frame would measure as the acceleration). This would be equivalent to the net acceleration from the perspective of the inertial frame.

3. Azimuthal acceleration. This is a fictitious force acceleration that arises due to a change in the angular velocity of rotation (either magnitude or direction). Some call this the transverse acceleration.

4. Coriolis acceleration. This is a fictitious force acceleration if you have an object moving in a rotating frame.

5. Centrifugal acceleration. This is a fictitious force acceleration if you have an object offset from the origin of a rotating frame.

6. Translational acceleration. This is a fictitious force acceleration that represents how the origin of the rotating frame moves relative to the origin of the inertial frame.

To solve for the force in the rotating frame, multiple all the accelerations by the mass, m:

$$\begin{aligned} m\vec{a}\,' &= m\vec{a} - m\vec{\alpha}\times\vec{r} - 2m\vec{\omega}\times\vec{v}\,' - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\vec{A} \\ \boxed{m\vec{a}\,' = \sum\vec{F}_I + \vec{F}_{az} + \vec{F}_{Cor} + \vec{F}_{cent} + \vec{F}_{trans}} \end{aligned} \tag{4.7}$$

where $\sum\vec{F}_I$ is the net force in the inertial frame and the remaining terms are all the fictitious forces. Note that for a given problem, not all fictitious forces may be present. You will need to consider the physics in the problem to identify which ones are applicable.

4.6 Simple Example of a Rotating Reference Frame

This chapter looks at simple cases of rotating reference frames. We will get to more complicated cases in Chapter 5.

Sample Problem 4-3

Consider a particle of mass m in circular motion with a radius R around a star of mass M due to gravity at a constant angular speed of ω. **Describe the motion of the particle in the inertial and the rotating frame.**

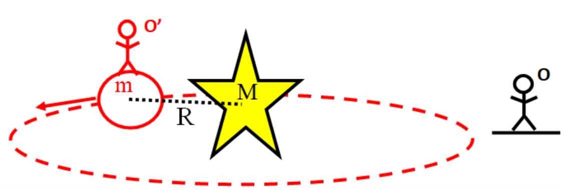


Figure 4.9: In red is the rotating frame where the particle is moving in the counter-clockwise direction with the observer O'. In black is the star and observer in the inertial frame.

Solution

Inertial Frame: In the inertial frame, an observer, O is looking at this particle orbit the star. There is a single force, gravity, acting on the particle.

$$\sum \vec{F}_I = -\frac{GMm}{R^2}\hat{r} = m\vec{a}$$

The equation of motion for circular rotation is r = constant, so that $\dot{r} = \ddot{r} = 0$. The motion comes entirely from a change in angle. For circular motion, we can use the centripetal acceleration, $\vec{a} = -\omega^2 R\hat{r}$ to describe the circular motion. Recall that this equation comes directly from plane-polar coordinates (see Chapter 1.2).

$$-\frac{GMm}{R^2}\hat{r} = -m\omega^2 R\hat{r}$$

<u>Rotating Frame:</u> In the rotating frame, an observer sitting on the particle doesn't think the particle is moving. So this observer would measure no rotation and no acceleration relative to their position. So this observer would measure:

$$m\vec{a}\,' = 0$$

But this observer has not considered that they are on a rotating reference frame. As a result, they need to consider the fictitious forces that come with that frame.Fortunately, many of the terms are equal to zero.

$$\cancelto{0}{m\vec{a}\,'} = \sum \vec{F}_I - \cancelto{0}{m\vec{\alpha} \times \vec{r}} - \cancelto{0}{2m\vec{\omega} \times \vec{v}\,'} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - \cancelto{0}{m\vec{A}}$$

There is no angular acceleration ($\alpha = 0$), the person is not moving within the rotating frame ($\vec{v}\,' = 0$), and the origins are not changing ($\vec{A} = 0$). The only fictitious force left is the centrifugal force. Therefore,

$$0 = \sum \vec{F}_I - \underbrace{m\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{F_{cent}}$$

For this circular rotation, the angular velocity and radial vectors are perpendicular to each other. Thus, F_{cent} has a magnitude of $m\omega^2 R$. You'll notice that this force has the same magnitude as the centripetal acceleration in the inertial frame.

What about the direction of the centrifugal force? In the rotating frame, the centrifugal force points outward from the origin. This makes sense, since fictitious forces act in the opposite direction to the acceleration relative to the inertial frame (equivalence principle). The centripetal acceleration always points inward. The figure below shows a breakdown of the $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ cross product terms.

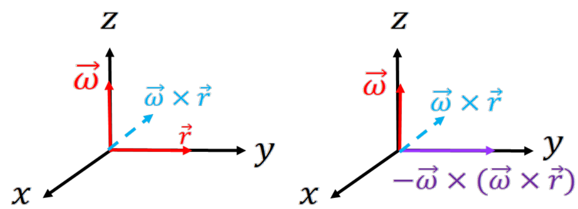


Figure 4.10: For the direction of the centrifugal force, use the right-hand rule.

Using our value for the centrifugal force, we get:

$$\begin{aligned} 0 &= \sum \vec{F}_I + \vec{F}_{cent} \\ 0 &= -\frac{GMm}{R^2}\hat{r} + m\omega^2 R\hat{r} \\ -\frac{GMm}{R^2}\hat{r} &= -m\omega^2 R\hat{r} \end{aligned}$$

which is exactly what we had for the inertial frame.

4.7 Summary

Key Takeaways

This chapter introduces the concepts of inertial and non-inertial frames. Inertial frames are reference frames that are either stationary or move with a constant velocity, whereas non-inertial frames are either accelerating or rotating.

To solve physics problems in non-inertial and rotating frames, we introduced the concept of *fictitious forces*. These are not real forces, in the sense that they arise from any specific interactions. They are forces that appear to act on an object to explain its properties. The fictitious force always acts opposite the direction of the acceleration (equivalence principle).

$$\vec{F}_{fic} = -m\vec{a}_{S'S}$$

In this chapter, we derive the coordinate transformations between inertial and non-inertial frames. The general case is,

$$\vec{a}\,' \;=\; \vec{a} - \vec{\alpha} \times \vec{r}\,' - 2\vec{\omega} \times \vec{v}\,' - \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') - \vec{A}$$

If there is no rotation, then the coordinate transformation is:

$$\vec{a}\,' \;=\; \vec{a} - \vec{A}$$

Due to the extra acceleration terms, Newton's second law in a non-inertial frame needs to include extra "force" terms from the fictitious forces:

$$\sum \vec{F}_{S'} = \sum \vec{F}_S + F_{fic}$$

Using the naming convention for the different fictitious forces, Newton's second law in a non-inertial frame is:

$$m\vec{a}\,' \;=\; \sum \vec{F}_I + \vec{F}_{az} + \vec{F}_{Cor} + \vec{F}_{cent} + \vec{F}_{trans}$$

Depending on your physics problem, it can be easier to solve a question in the non-inertial frame than in the inertial frame. Recognizing which frame to use is part of the challenge. When working on the practice problems, think about which frame of reference is easier to work with.

Important Equations

Fictitious Forces:

$$\sum \vec{F}_{S'} = \sum \vec{F}_S + F_{fic}$$

$$\vec{F}_{fic} = -m\vec{a}_{S'S}$$

Coordinate Transformation for velocity:

$$\vec{v} = \vec{v}\,' + \vec{\omega} \times \vec{r}\,'$$

Acceleration in a Rotating Frame:

$$\vec{a}\,' \quad = \quad \vec{a} - \vec{\alpha} \times \vec{r}\,' - 2\vec{\omega} \times \vec{v}\,' - \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') - \vec{A}$$

Newton's Second Law in a Rotating Frame:

$$m\vec{a}\,' \quad = \quad \sum \vec{F}_I + \vec{F}_{az} + \vec{F}_{Cor} + \vec{F}_{cent} + \vec{F}_{trans}$$

4.8 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 4-1

A mass M hangs from the ceiling of a train that is accelerating in the $+x$-direction.

a) Draw the free-body diagram for this mass in the frame of a person standing outside the train.

b) Draw the free-body diagram for this mass in the frame of a person standing inside the train.

Practice Problem 4-2

A 70 kg person stands on a bathroom scale in a moving elevator.

a) If the elevator has a *downward* acceleration of $a = \frac{g}{4}$, what is the force of the person on the scale?

b) If the elevator has a *upward* acceleration of $a = \frac{g}{4}$, what is the force of the person on the scale?

Practice Problem 4-3

A small object of mass m is attached to an ideal rope and the top of the rope is held fixed on a moving train. What is the angle of deflection if the train is accelerating forward with a constant acceleration of $a = 0.5g$?

Practice Problem 4-4

A physics student stuck inside a cargo container of a train feels the train begin to accelerate. The student ties their shoe to the ceiling of the container and estimates the angle of deflection of the shoe from vertical to be θ.

a) What is the acceleration of the train?

b) What is the "effective gravity" the student feels?

Practice Problem 4-5

A funicular train is accelerating up an incline with an angle θ above the horizontal.

a) Would the effective gravity felt by the passengers be higher or lower than g?

b) If $a = 0.1g$ and $\theta = 30°$, what is the magnitude of g_{eff}

Practice Problem 4-6

Small object of mass m is suspended from the ceiling of a train by an ideal rope of length L. If the mass oscillates like a simple pendulum, how does the period of oscillations change if the train goes from rest to an acceleration of $a = \frac{1}{3}g$?

Practice Problem 4-7

A wheel of radius R rolls on the ground without slipping in the $+x$-direction with a constant speed at its center of mass of v_0. What is the magnitude of the centrifugal acceleration and the Coriolis acceleration of a point on the rim of the wheel?

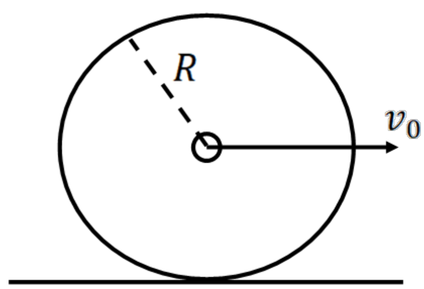


Figure 4.11: The wheel of radius R rolls in the $+x$-direction moving with a linear velocity v_0.

Practice Problem 4-8

A fun house at a local amusement park has a circular room with a rotating floor that has a constant angular speed of $\omega_0\hat{k}$ (up direction). A physics student enters the room. Which fictitious forces does the student feel if they:

a) sit in the very center of the room?

b) sit at a radius r from the center?

c) move with a constant velocity from a radius r_1 to r_2?

Practice Problem 4-9

Consider two astronauts in space far from any source of gravity. The spaceship is accelerating upwards (relative to an inertial observer watching the spaceship) at an acceleration of $a = 9.8$ m s^{-2}. Inside, the two astronauts are throwing a ball back and forth. The ball has a mass m and the two astronauts are 10 m away from each other.

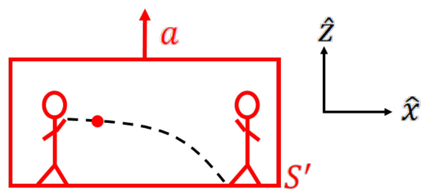


Figure 4.12: Astronaut 1 throws the ball to Astronaut 2 while in the non-inertial frame.

a) In the frame of the two astronauts (the non-inertial frame), what is the magnitude and direction of the fictitious force acting on the ball?

b) In the frame of the two astronauts (the non-inertial frame), what is the effective gravity acting on the ball? Draw a free-body diagram.

c) Astronaut 1 is throwing the ball to Astronaut 2. In the frame of the two astronauts (the non-inertial frame), what is the minimum speed needed for the ball to travel the 10 m between the two astronauts and how long does it take to reach Astronaut 2? You can assume that Astronaut 1 has a height of 2 m and tosses the ball to Astronaut 2 at an angle of 0 deg relative to the horizontal.

d) Now consider an observer that is outside of the spaceship and not moving (the inertial frame). What forces are acting on the ball after Astronaut 1 throws it?

e) Solve for the minimum speed and time from the perspective of an observer in the inertial frame. Do you get the same answers as c)?

f) Describe the trajectory of the ball in the frame of the inertial observer and the frame of the two astronauts?

5 Application of Non-Inertial and Rotating Frames

Learning Objectives

- Describe the fictitious forces of a rotating frame
- Solve physics problems with the Earth as a non-inertial frame
- Solve the Foucault Pendulum problem

In this chapter, we will use non-inertial and rotating frames of reference to solve physics problems. Please see Chapter 4 for an introduction to non-inertial and rotating frames.

5.1 Rotating versus Accelerating Frames

In Chapter 4, we derived the equations for acceleration and velocity in a non-inertial frame when there was only linear acceleration (Chapter 4.2) and when there was rotation (Chapter 4.4). Note that these two equations are connected.

The equation for acceleration in a rotating frame is:

$$\boxed{\vec{a}\,' = \vec{a} - \vec{\alpha} \times \vec{r}\,' - 2\vec{\omega} \times \vec{v}\,' - \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') - \vec{A}} \tag{5.1}$$

where the parameters with primes are in the rotating reference frame and the parameters without primes are measured in an inertial frame. See Chapter 4.5 for what each of these terms mean. If your system is *not* rotating, then $\vec{\alpha} = \vec{\omega} = 0$ and we recover the same equation for a linear non-inertial frame from Chapter 4.2,

$$\boxed{\vec{a}\,' = \vec{a} - \vec{A}} \quad \Longrightarrow \quad \text{for no rotation} \tag{5.2}$$

where $\vec{A}$ is the acceleration of the non-inertial frame relative to the inertial frame.

Similarly, the equation for velocity in a rotating frame is:

$$\boxed{\vec{v}\,' = \vec{v} - \vec{\omega} \times \vec{r}\,' - \vec{u}} \tag{5.3}$$

where we have added an extra term, $\vec{u}$, to represent the velocity of the origin in the non-inertial frame relative to the inertial frame.

Finally, we defined the fictitious forces as,

$$\boxed{m\vec{a}\,' = m\vec{a} + \underbrace{\vec{F}_{az}}_{-m\vec{\alpha}\times\vec{r}\,'} + \underbrace{\vec{F}_{Cor}}_{-2m\vec{\omega}\times\vec{v}\,'} + \underbrace{\vec{F}_{cent}}_{-m\vec{\omega}\times(\vec{\omega}\times\vec{r}\,')} + \underbrace{\vec{F}_{trans}}_{-m\vec{A}}} \tag{5.4}$$

where the four labeled terms are the four fictitious force: the azimuthal force, the Coriolis force, the centrifugal force, and the translational force. Note how each of these forces are defined with negative signs because they act opposite the direction of acceleration.

In the next two sections, we will look at examples of the centrifugal force and the Coriolis force. The azimuthal force will be left for practice. See Chapter 4 for examples of the translation force.

5.2 Centrifugal Fictitious Force

The centrifugal force is a consequence of a rotating frame and has the form of

$$\boxed{\vec{F}_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}\,')} \tag{5.5}$$

Note that the acceleration from the centrifugal force is $\vec{\omega} \times (\vec{\omega} \times \vec{r}\,')$ and this acceleration has the same form as the centripetal acceleration associated with circular motion (see Chapter 1). If you have circular motion, $\vec{\omega} \perp \vec{r}$ and $|\vec{F}_{circ}| = m\omega^2 r = \frac{mv^2}{r}$.

Centrifugal vs Centripetal Acceleration

In rotational motion, we have two similar sounding accelerations, the centrifugal and centripetal acceleration. These should be treated differently, and neither should really be considered a real force. The centrifugal force is a fictitious force that arises from being in a rotating reference frame. The centripetal acceleration is a description of the acceleration coming from a different force (e.g., gravity, friction, tension). As a result, we do not include a "centripetal force" on any free-body diagrams (although the centrifugal force would be required in a non-inertial frame free-body diagram).

While similar in magnitude, the direction of the centrifugal force is not the same as the direction of the centripetal acceleration. The centrifugal force points radially outward for rotating frames, whereas the centripetal acceleration points radially inward. This should make intuitive sense as fictitious forces act in the opposite direction to the acceleration in the inertial frame (negative sign in the Equation (5.5)).

To prove that the centrifugal force is radially outward, let's go through an example vector cross product for uniform circular motion with its axis of rotation pointing up ($\hat{k}'$). Even though we have a radial dependence with our cross product, we will use Cartesian coordinates for the rotating frame ($\hat{\imath}', \hat{\jmath}', \hat{k}'$). The reason is, in our rotating frame, the radial vector will move with the non-inertial coordinate system. That is, from the perspective of a non-inertial observer rotating with the coordinate system, the radial vector does not change. So we can define our radial vector as being along the x-axis ($\vec{r}\,' = r\hat{\imath}'$) for example, and as the system rotates, our radial vector will remain along the $\hat{\imath}'$ direction (both the position and the coordinates are rotating in this inertial frame).

Using $\vec{\omega} = \omega\hat{k}'$ and $\vec{r}\,' = r\hat{\imath}'$, the centrifugal acceleration is $\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') = \omega\hat{k}' \times (\omega\hat{k}' \times r\hat{\imath}')$. To solve this problem, we need to do two cross products. First, we will do the cross product in brackets. (See Chapter 1.5.2 for review on computing the cross product).

$$\vec{\omega} \times \vec{r}\,' = \begin{vmatrix} \hat{\imath}' & \hat{\jmath}' & \hat{k}' \\ 0 & 0 & \omega \\ r & 0 & 0 \end{vmatrix} = \hat{\imath}'(0-0) + \hat{\jmath}'(r\omega - 0) + \hat{k}'(0-0) = r\omega\hat{\jmath}'$$

Then we will take our solution to that first cross product and apply that to the second cross product.

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') = \begin{vmatrix} \hat{\imath}' & \hat{\jmath}' & \hat{k}' \\ 0 & 0 & \omega \\ 0 & r\omega & 0 \end{vmatrix} = \hat{\imath}'(0 - r\omega^2) + \hat{\jmath}'(0-0) + \hat{k}'(0-0) = -r\omega^2\hat{\imath}' \quad (5.6)$$

So the direction of the resulting vector from $\vec{\omega} \times (\vec{\omega} \times \vec{r}\,')$ is along the radial line, pointing inward toward the origin (negative value). But the centrifugal fictitious force is equal to $\vec{F}_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') = mr\omega^2\hat{\imath}'$, which means that $\vec{F}_{cent}$ is pointing radially outward (away from the origin).

Centrifugal Force in Practice

The outward acceleration of the centrifugal force is why laundry sticks to the sides of a top-loading washing machine during the high spin cycle, why you stick to the walls of an amusement park ride that spins very quickly, and why centrifuges are called centrifuges. These are all cases where the objects in question (clothes, people, lab materials) are in a non-inertial frame that is spinning. The rotating frame pushes things outward.

Sample Problem 5-1

A cat with mass m sits on a turntable at a radial position of R. The turntable is spinning at a constant angular velocity of $\vec{\omega} = \omega\hat{k}$ (see Figure 5.1). If the coefficient of static friction between the cat and the table is μ, **what is the maximum rotation rate before the cat starts to slip?**

Solution

<u>Inertial Frame:</u> In the inertial frame, the cat's acceleration is

$$\vec{a} = -\omega^2 R\hat{r}$$

which is just the centripetal acceleration. The force behind this acceleration is actually friction with the table. The friction force points inward toward the center, because the rotation makes the cat want to move outward (centrifugal force).

$$\vec{f} = -\mu N\hat{r} = -\mu mg\hat{r}$$

where N is the normal force. Since there is no vertical motion, $N = mg$. The cat will start to lose balance when the acceleration from rotation equals the (static) friction force. Any additional rotation, and the cat will start to move. From Newton's second law, we get:

$$\begin{aligned}
\sum \vec{F} &= -m\vec{a} \\
f &= -m\omega^2 R\hat{r} \\
-\mu mg\hat{r} &= -m\omega^2 R\hat{r} \\
\omega &= \sqrt{\frac{\mu g}{R}}
\end{aligned}$$

So in the inertial frame, we can describe the cat's motion and the condition for slipping fairly easily. **What about the non-inertial frame?**

Cat's Frame: The cat is our observer in the rotating frame, which means that the cat will experience fictitious forces. Figure 5.1 shows the free-body diagram from the cat's perspective with vectors showing the gravitational force and the centrifugal force. Note that no other fictitious forces act on the cat, since it isn't moving and the table is rotating at a constant rate.

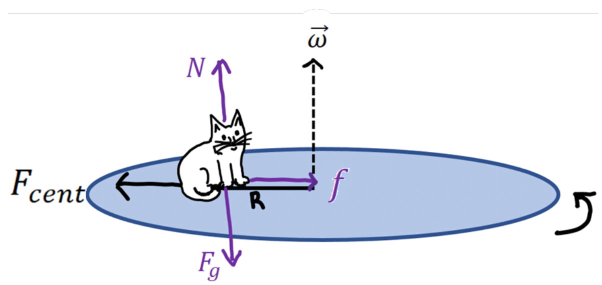


Figure 5.1: Free-body diagram for the cat on a spinning turntable. The labeled forces are the gravitational force (F_g), the normal force (N), friction (f), and the centrifugal force (F_{cent}). The three forces in purple are the forces that we would identify in an inertial frame. The centrifugal force in black is only in the cat's frame.

Compared to the inertial frame, the cat would identify one additional force. The centrifugal force. So the sum of all forces would be:

$$m\vec{a}\,' = \vec{F}_I + \vec{F}_{cent}$$

where $\vec{F}_I$ is the total force in the inertial frame (the real forces). From the cat's perspective, however, it isn't moving. The cat is just sitting and the world is moving around it. So to the cat, $\vec{a}\,' = 0$.

Taking $\vec{a}\,' = 0$, we get:

$$\begin{aligned}
0 &= \vec{F}_I + \vec{F}_{cent} \\
0 &= f + F_{cent} \quad \Longrightarrow \quad \text{only inertial force is friction} \\
0 &= -\mu mg + mR\omega^2 \quad \Longrightarrow \quad f \text{ and } F_{cent} \text{ act in opposite directions} \\
\omega &= \sqrt{\frac{\mu g}{R}}
\end{aligned}$$

which is the same solution as the inertial frame (as expected). The difference is that we have identified the fictitious centrifugal force for the non-inertial frame.

Quick Questions

1. Fictitious force act like a modification of gravity (e.g., effective gravity) in the non-inertial frame. Depending on the problem, the effective gravity vector can be at an angle relative to the vertical. Assuming there is only the centrifugal force acting on the cat, what is the equation for the angle for the effective gravity in terms of ω, R, and g? Hint, add the F_g and F_{cent} vectors.
2. What is the rotation rate if the angle from the vertical is $2°$ and $R = 2$ m?
3. If the coefficient of static friction with the table is $\mu = 0.2$, what is the maximum angular speed (ω_{max}) before the cat starts to slip?
4. What is the angle, relative to the vertical, of the effective gravity acting on the cat when $\omega = \omega_{max}$?

Lance's Thoughts

For a rotating frame of reference, you start with five potential fictitious forces. Most of the time, you'll be able to eliminate at least some of them based on whether the object is moving or accelerating inside the rotating frame and whether the non-inertial frame is rotating or moving at a constant rate. Look for terms that will equal zero.

In the sample problem above, we only had centrifugal force to worry about. Since the cat, seen from inside the rotating frame, isn't moving at all, we can eliminate the linear and Coriolis accelerations. Since the rotation is at a constant rate and the axis of rotation isn't moving, we can eliminate the azimuthal and translational accelerations. With four out of five fictitious forces gone, this becomes a much easier problem to solve.

5.3 Coriolis Fictitious Force

The Coriolis force is a consequence of an object moving in a rotating frame. The force equation is:

$$\boxed{\vec{F}_{Cor} = -2m\vec{\omega} \times \vec{v}\,'} \tag{5.7}$$

Like the centrifugal force, the Coriolis force depends on a vector cross product. So we need to look at the direction.

Let's assume that we have a rotating system with $\vec{\omega} = \omega\hat{k}'$ and we have a particle of mass m in this system moving radially outward with a constant velocity in the rotating frame with $\vec{v}\,' = v'\hat{\imath}'$. As in the previous section, we will use Cartesian coordinates for simplicity, because the mass is rotating with the system such that our radial vector will remain along the $\hat{\imath}'$ direction.

Using $\vec{\omega} = \omega\hat{k}'$ and $\vec{v}\,' = v'\hat{\imath}'$, we need to solve $\vec{\omega} \times \vec{v}\,' = \omega\hat{k}' \times (v'\hat{\imath}')$ for the Coriolis force.

$$\vec{\omega} \times \vec{v}\,' = \begin{vmatrix} \hat{\imath}' & \hat{\jmath}' & \hat{k}' \\ 0 & 0 & \omega \\ v' & 0 & 0 \end{vmatrix} = \hat{\imath}'(0-0) + \hat{\jmath}'(v'\omega - 0) + \hat{k}'(0-0) = v'\omega\hat{\jmath}' \tag{5.8}$$

But the Coriolis fictitious force is equal to $\vec{F}_{Cor} = -2m\vec{\omega} \times \vec{v}\,' = -2mv'\omega\hat{\jmath}'$. So the Coriolis force is in the negative $\hat{\jmath}'$ direction. You may notice this force if you've ever tried walking on a rotating surface (e.g., a merry-go-round). You feel off balance.

Figure 5.2 shows the fictitious forces for a particle that is moving outward with a constant velocity on a rotating reference frame. There are two fictitious forces in this case, the Coriolis force (due to motion in a rotating frame) and the centrifugal force (due to the rotating frame itself).

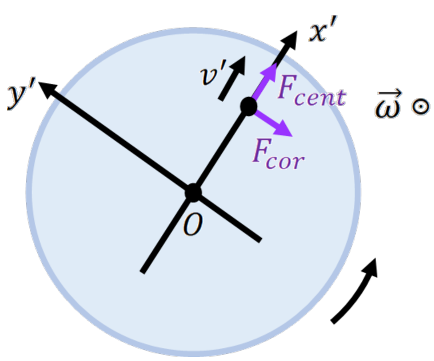


Figure 5.2: Overhead view of a rotating frame with a particle of mass m moving at a speed v' in the rotating frame. From the particle's perspective, there are two fictitious forces, the centrifugal force (F_{cent}) and the Coriolis force (F_{Cor}). Both forces only exist in the non-inertial (rotating) frame.

Quick Questions

1. Can you have a case where the centrifugal force is zero and the Coriolis force is non-zero? If yes, under which circumstances?

Sample Problem 5-2

Let's return to the question of the cat sitting on a turntable from Sample Problem 5-1. Now, the cat starts to move radially outward at a constant speed of $v'\hat{\imath}'$. **At what radius will the cat start to slip?**

Solution

Inertial Frame: In the inertial frame, the net force acting on the cat is still friction with the table. But due to the combination of a rotation frame and the cat is moving, friction is no longer radial (e.g., we cannot say that the cat is undergoing simple circular motion). It is *much easier* to solve this problem in the cat's reference frame.

The Cat's Frame: From the cat's perspective, there is no acceleration because it is moving at a constant velocity. So we can simplify the equation of motion with $\vec{a}\,' = 0$. But there are two fictitious forces acting on the cat. So the force equation becomes:

$$\begin{aligned} m\vec{a}\,' &= m\vec{a} + \vec{F}_{fic} \\ 0 &= \vec{F}_I + \vec{F}_{cent} + \vec{F}_{Cor} \end{aligned}$$

where $\vec{F}_I$ is the net force in the inertial frame.

In this problem, we have $\vec{\omega} = \omega\hat{k} = \omega\hat{k}'$ and both the radial position and the velocity vectors are along the $\hat{\imath}'$ in the cat's frame (these do not change from $\hat{\imath}'$, because the cat is rotating with the reference frame). That means we can use our previous solutions for the centrifugal and Coriolis forces (see Equations 5.6 and 5.8).

The Coriolis force is constant because the cat's velocity and the table's angular velocity are both constant. The Coriolis force is:

$$\vec{F}_{Cor} = -2m\vec{\omega} \times \vec{v}\,' = -2m\omega\hat{k}' \times (v'\hat{\imath}') = -2mv'\omega\hat{\jmath}'$$

The centrifugal force is not constant, however, because the cat's position is changing. The cat's position can be described by $\vec{r}\,' = x'\hat{\imath}'$, where $\dot{x}' = v'$.

$$\vec{F}_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}\,') = -m\omega\hat{k}' \times (\omega\hat{k}' \times x'\hat{\imath}') = m\omega^2 x'\hat{\imath}'$$

Combining these equations into the force equation from the cat's perspective, we have:

$$\begin{aligned} 0 &= \vec{F}_I + \vec{F}_{cent} + \vec{F}_{Cor} \\ 0 &= \vec{F}_I + m\omega^2 x'\hat{\imath}' - 2mv'\omega\hat{\jmath}' \\ \vec{F}_I &= 2mv'\omega\hat{\jmath}' - m\omega^2 x'\hat{\imath}' \end{aligned}$$

We now have an equation for the net inertial force, which is only friction in this case (gravity and the normal force will cancel because the cat doesn't leave the surface of the table). Figure 5.3 shows the direction of $\vec{F_I}$ in pink. Note that $\vec{F_I}$ is not radial because the two fictitious forces make the cat want to move in two perpendicular directions. Solving for $\vec{F_I}$ is very challenging in the inertial frame.

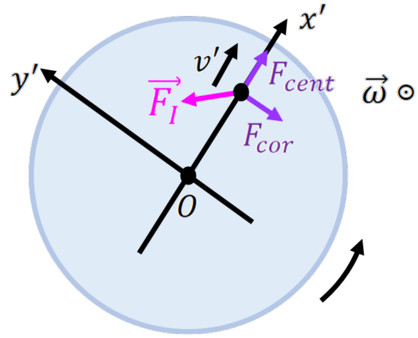


Figure 5.3: Same as Figure 5.2, but with the net force from the inertial frame included in pink. Note that $\vec{F_I} = -\vec{F}_{cent} - \vec{F}_{Cor}$ by definition (see above for the equation). So the net force in the inertial frame is acting on an angle.

We can use the net inertial force to get the net acceleration in the inertial frame as $\vec{a} = 2v'\omega\hat{\jmath}' - \omega^2 x'\hat{\imath}'$. Note that this acceleration is not constant and it is not radial.

The condition for slipping is when the net force acting on a system has a magnitude that is equal to the static friction force, $|\vec{F_I}| \leq f = \mu mg$.

$$\begin{aligned}
\mu mg &= |\vec{F_I}| \\
\mu mg &= |2mv'\omega\hat{\jmath}' - m\omega^2 x'\hat{\imath}'| \\
\mu mg &= \sqrt{(2mv'\omega)^2 + (m\omega^2 x')^2} \\
\mu^2 g^2 &= 4(v')^2\omega^2 + \omega^4 (x')^2 \\
(x')^2 &= \frac{\mu^2 g^2 - 4(v')^2\omega^2}{\omega^4} \\
x' &= \frac{\sqrt{\mu^2 g^2 - 4(v')^2\omega^2}}{\omega^2}
\end{aligned}$$

So this is the maximum radial distance that the cat can reach before the combination of fictitious forces exceed the condition for slipping.

Quick Questions

1. Consider the case where the cat walks inward toward the origin instead of outward. What is the *Coriolis* force in this case?
2. Sketch Figure 5.3 in the case where the cat walks toward the origin.

5.4 Earth as a Non-Inertial Frame

The Earth is rotating, which means the Earth is a non-inertial reference frame. To think about the fictitious forces acting on the Earth, it helps to think in 3-D.

Consider Figure 5.4, which shows the position of a person on Earth's surface. Ignoring Earth's orbit around the Sun, we can set the inertial reference frame to the center of the planet (rotation is zero there) and we can set the non-inertial reference frame to the position of the person at the surface. Note: this example is a case where the inertial and rotating frames do not have the same origin. But the distance between them, R is fixed.

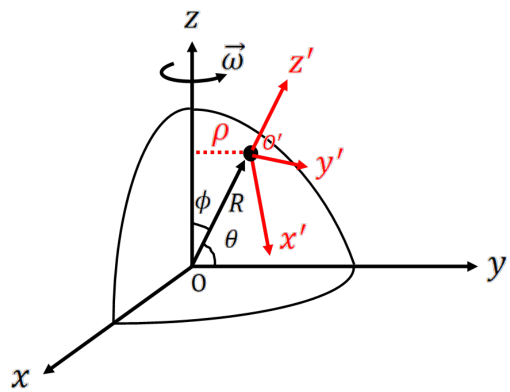


Figure 5.4: A coordinate system on Earth. The point shown is fixed to the surface of the Earth. The red coordinates show the non-inertial reference frame for an observer at this location (O'). The black coordinates show the inertial reference frame at the center of the Earth, a distance R from the point. Also shown are the latitude θ, polar angle ϕ, and distance to the rotation axis ρ.

Our observer is located at the point shown in Figure 5.4. This person is at a latitude of θ, where the equator is located at the x, y-plane of the inertial frame. We can also describe the person's position using the polar angle ϕ (also called the colatitude), where $\phi = 90 - \theta$. In the inertial frame, these angles do not change (e.g., the latitude of a fixed point on Earth does not change), but the x, y axes rotate about the z axis due to Earth's spin.

The observer on Earth's surface has a different coordinate system that is fixed from their perspective (they don't see the rotation). For example, here on Earth we define up and down, North and South, East and West, and those directions are fixed from our reference, even though we are on a moving surface. North is always north. Up is always up. The Earth's motion does not change your perspective on those directions.

In this textbook, we will define up as $+\hat{k}'$, East as $+\hat{\imath}'$, and North as $+\hat{\jmath}'$ for an observer in the rotating reference frame. Subsequently, down, West, and South will be the negative unit vector directions. Note that from the reference of the observer, the axis of rotation for the Earth is not along any of the unit vector axes. Figure 5.5 shows the $\hat{k}'$ and $\hat{\jmath}'$ components of Earth's angular velocity vector for an observer at a latitude of θ. Note that the component of ω in the non-inertial frame depends on latitude.
From Figure 5.5, Earth's angular velocity vector can be described as

$$\boxed{\vec{\omega} = \omega \cos\theta \hat{\jmath}' + \omega \sin\theta \hat{k}'} \tag{5.9}$$

Thus, non-inertial motion on Earth's surface will vary with latitude.

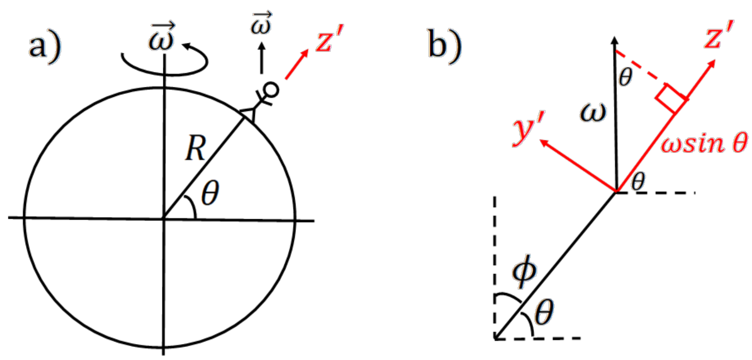


Figure 5.5: (a) *A person at a latitude of θ. The moving coordinate system has $\hat{k}'$ normal to the surface whereas $\vec{\omega}$ is in the $\hat{k}$ direction of the inertial frame (see also, Figure 5.4).* (b) *Zoom in of the reference frame at a latitude of θ showing the components of $\vec{\omega}$ in the $\hat{k}'$ and $\hat{\jmath}'$ coordinates.*

For a person on Earth's surface moving with a constant velocity, there will be fictitious forces acting on them because they are in a non-inertial frame. Let's look at what is at play.

$$\vec{a}\,' = \vec{a} - \vec{\alpha} \times \vec{r}\,' - 2\vec{\omega} \times \vec{v}\,' - \vec{\omega} \times (\vec{\omega} \times \vec{r}\,') - \vec{A}$$

We can simplify this equation. We will assume that (1) the Earth's rotation is constant[1], so $\alpha = \dot{\omega} = 0$ and (2) the origin of the non-inertial frame has no translation acceleration relative to the origin of the inertial frame (R is constant, $A = \ddot{R} = 0$). Moreover, if the person is moving with a constant velocity on Earth's surface, $\vec{a}\,' = 0$. Taking these simplifications, the remaining fictitious forces are the centrifugal and the Coriolis forces.

$$0 \;=\; \vec{a} - 2\vec{\omega} \times \vec{v}\,' - \vec{\omega} \times (\vec{\omega} \times \vec{r}\,')$$

Let's start with the centrifugal force, $F_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}\,')$.

Figure 5.6 shows the breakdown of the vector directions from the two cross products. Since $\vec{\omega}$ is not along the $\hat{k}'$ axis and it is not perpendicular to $\vec{R}$, getting the direction is not intuitive. You can use the right hand rule (see Chapter 1.5.2), to estimate the direction, where $\vec{\omega} \times \vec{R}$ points mostly into the page, and $\vec{\omega} \times (\vec{\omega} \times \vec{R})$ points mostly toward the Earth's axis of rotation. Thus, we should expect the centrifugal force to mostly point away from the axis of rotation.

The total magnitude of the centrifugal force should therefore be given by

$$|\vec{F}_{cen}| \;=\; m\omega^2 R \sin\phi = m\omega^2 R \cos\theta$$

where ϕ is the angle between the axis of rotation and the radius vectors (see Figure 5.6). Note that $\sin\phi = \cos\theta$, and that the $\sin\phi$ term only comes from the first cross product, $(\vec{\omega} \times \vec{r}\,')$.

As for the direction, the centrifugal force should be along an axis that is perpendicular to the rotation axis (e.g., $\hat{\rho}$). We can verify this by using the vector cross product, where the

[1]The Earth is actually slowing down in rotation due to torques with the Moon, but the change is very small and can be considered negligible.

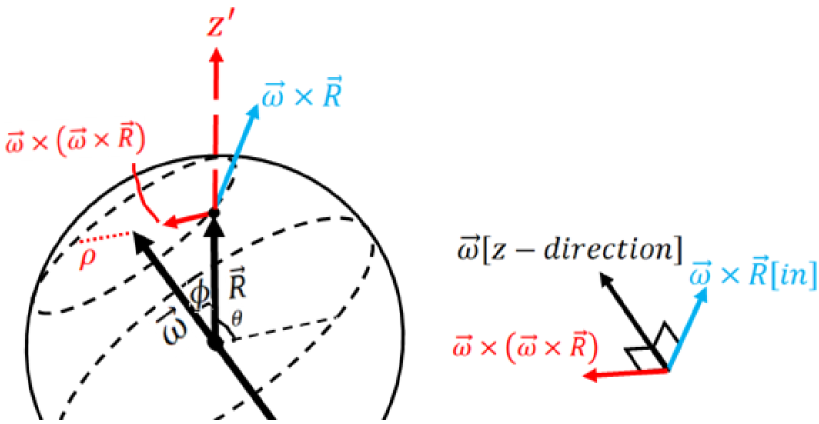


Figure 5.6: The vector cross product solution for the centrifugal force at a position that is at a latitude of θ from the equator. The point P is undergoing circular motion with a radius of ρ, with $\rho = R\sin\phi = R\cos\theta$.

observer is located at $\vec{R}\hat{k}'$ and the Equation for Earth's angular motion in the non-inertial frame from Equation 5.9.

$$\vec{\omega} \times \vec{r}' = \begin{vmatrix} \hat{\imath}' & \hat{\jmath}' & \hat{k}' \\ 0 & \omega\cos\theta & \omega\sin\theta \\ 0 & 0 & R \end{vmatrix} = \hat{\imath}'(R\omega\cos\theta) + \hat{\jmath}'(0-0) + \hat{k}'(0-0) = R\omega\cos\theta\hat{\imath}'$$

$$\begin{aligned} \vec{\omega} \times (\vec{\omega} \times \vec{r}') &= \begin{vmatrix} \hat{\imath}' & \hat{\jmath}' & \hat{k}' \\ 0 & \omega\cos\theta & \omega\sin\theta \\ R\omega\cos\theta & 0 & 0 \end{vmatrix} \\ &= \hat{\imath}'(0-0) + \hat{\jmath}'(\omega^2 R\cos\theta\sin\theta) + \hat{k}'(-\omega^2 R\cos^2\theta) \\ &= \omega^2 R\cos\theta\sin\theta\hat{\jmath}' - \omega^2 R\cos^2\theta\hat{k}' \end{aligned}$$

Thus, the centrifugal force will be:

$$\vec{F}_{cen} = -m\omega^2 R\cos\theta\sin\theta\hat{\jmath}' + m\omega^2 R\cos^2\theta\hat{k}'$$

which is pointing in a direction that is South and up. Looking at Figure 5.6, that direction point away from the rotation axis.

Quick Question

1. What is the direction of the centrifugal force if you are located at the Equator? Does this answer make sense? Consider Figure 5.6 from the perspective of someone standing on the equator.

By contrast, gravity from the Earth is directed toward the center of the Earth, which will be along the $-\hat{k}'$ direction, by definition. That means we have two vectors with different directions. The effective gravity will be the sum of these two vectors.

Taking the magnitude of $\vec{F}_{cen}$, we have:

$$\begin{aligned} |\vec{F}_{cen}| &= \sqrt{(-m\omega^2 R\cos\theta\sin\theta)^2 + (m\omega^2 R\cos^2\theta)^2} \\ &= m\omega^2 R\cos\theta\sqrt{\underbrace{\sin^2\theta + \cos^2\theta^2}_{=1}} \\ &= m\omega^2 R\cos\theta \end{aligned}$$

which is what we expected using the right-hand rule and the simple vector cross product.

The total magnitude of the centrifugal force is quite small. The centrifugal force is *largest* at the equator ($\theta = 0$). Taking $R = 6370$ km for the Earth's radius and $\omega = 7.27 \times 10^{-5}$ s^{-1} for the rotation rate, the centrifugal acceleration is 0.034 m s^{-2}, which is $< 1\%$ of the magnitude of acceleration from Earth's gravitational field at the surface.

So for an observer on the surface of the Earth, we can generally ignore the centrifugal force from Earth's rotation. It has a negligible effect. Thus, our vector equation for a person on Earth's surface becomes:

$$\vec{a}' = \vec{a} - 2\vec{\omega} \times \vec{v}'$$

where the remaining motion is just from the inertial forces and the Coriolis force.

5.5 Foucault's Pendulum

Foucault's pendulum is a classic example of the Coriolis force in action. Consider a simple pendulum (mass hanging from an ideal string) that is also frictionless at its pivot point. Only two forces act on this pendulum, tension and gravity (see Chapter 3 and Figure 5.7).

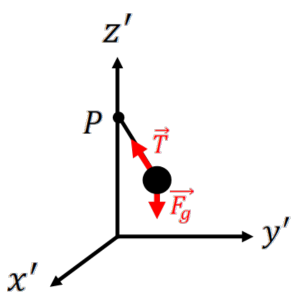


Figure 5.7: A Foucault pendulum of mass m and length ℓ. The pivot point at P does not move and has no friction. Two forces act in the inertial frame, tension ($\vec{T}$) and gravity ($\vec{F}_g$). Gravity points down, tension is directed to the pivot. Note that the pendulum is moving near Earth's surface, so we will want to use only the non-inertial coordinate system.

When set in motion, the pendulum will have a non-zero Coriolis force. Ignoring the azimuthal and centrifugal forces (negligible), we can simplify the non-inertial frame acceleration as (see previous section):

$$\vec{a}' = \vec{a} - 2\vec{\omega} \times \vec{v}'$$

where a is the acceleration due to the net (real) forces acting on the pendulum in the inertial frame and $-2\vec{\omega} \times \vec{v}'$ is from the Coriolis force.

1. Finding the inertial forces: The inertial forces are gravity and tension. Gravity acts down (-$\hat{k}'$ direction) in the non-inertial frame. So we need to convert tension to our non-inertial reference frame by finding its components along $\hat{\imath}', \hat{\jmath}', \hat{k}'$. Figure 5.8 shows the break down of the tension, $\vec{T}$ in red, into the non-inertial coordinate system.

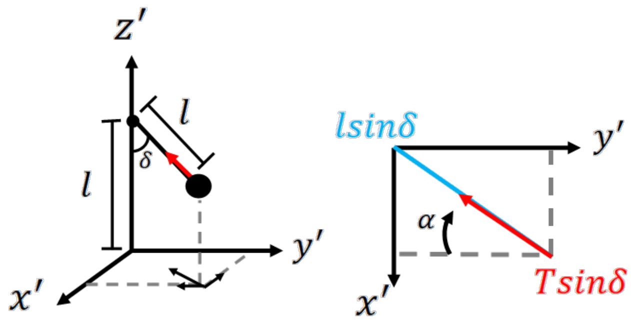


Figure 5.8: Vector diagram for tension in a pendulum relative to $\hat{\imath}', \hat{\jmath}', \hat{k}'$. Left: *A 3-D view of the tension. The tension is shown by the red arrow. The pendulum makes an angle δ with respect to $\hat{k}'$. The black arrows at the bottom show the component of tension in the x', y' plane and along the x' and y' axes.* Right: *A bird's eye view of the x', y' plane with the component of tension and the pendulum rope length in this plane.*

The component of tension in the $x' - y'$ plane is $T \sin\delta$, where δ is the angle between the pendulum and the vertical. This vector points inward toward the origin (because it is a restoring force). Figure 5.8 also shows the component of the pendulum rope length in the $x' - y'$ plane in blue, given by $\ell \sin\delta$ for a rope of length ℓ.

Using Figure 5.8 with a bit of algebra, you can get,

$$\vec{T} = -\frac{Tx'}{\ell}\hat{\imath}' - \frac{Ty'}{\ell}\hat{\jmath}' + \frac{T(\ell - z')}{\ell}\hat{k}'$$

Note the negative signs for the x' and y' components. This should make sense as these would be a restoring force and restoring forces are always negative.

For the z' component, we can use the vector dot product because we know the angle between tension and $\hat{k}'$ is δ. So $T_{z'} = T\cos\delta$. We can define $\cos\delta$ from the length of the rope because it is fixed. The pendulum height is given by $z' = \ell - \ell\cos\delta$, so we can solve for $\cos\delta = \frac{\ell - z'}{\ell}$. For small angles, $z' \approx 0$ so $T_{z'} \approx T$.

For the x' and y' components, we use the projection of T into the $x' - y'$ plane. The right panel of Figure 5.8 shows this projection. The x'-component is given by $T_{x'} = T\sin\delta\sin\alpha$, where $\sin\alpha = \frac{x'}{\ell\sin\delta}$, so $T_{x'}$ simplifies to $T_{x'} = \frac{Tx'}{\ell}$. Similar arguments can be made for $T_{y'}$.

Quick Question

1. Go through the algebra and verify that you get $\vec{T} = -\frac{Tx'}{\ell}\hat{\imath}' - \frac{Ty'}{\ell}\hat{\jmath}' + \frac{T(\ell - z')}{\ell}\hat{k}'$ for a pendulum that is displaced by an angle δ from the vertical.

2. Finding the non-inertial forces: For our pendulum, the only non-inertial force we need to consider is the Coriolis force (the centrifugal force is negligible). In Section 5.3 we found the Coriolis force for a velocity in 1-D. The Foucault pendulum, however, moves in $\hat{\imath}', \hat{\jmath}', \hat{k}'$. If we assume small angles, then $z' \ll \ell$ and any motion in the vertical direction ($\hat{k}'$) will be negligible and we can approximate the velocity by $\vec{v}\,' = \vec{v}\,' = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}'$.

Using $\vec{\omega}$ from Equation 5.9 and $\vec{v}\,' = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}'$, we can solve for the Coriolis force.

$$\vec{\omega} \times \vec{v}\,' = \begin{vmatrix} \hat{\imath}' & \hat{\jmath}' & \hat{k}' \\ 0 & \omega\cos\theta & \omega\sin\theta \\ \dot{x}' & \dot{y}' & 0 \end{vmatrix} = (-\dot{y}'\omega\sin\theta)\hat{\imath}' + (\dot{x}'\omega\sin\theta)\hat{\jmath}' + (-\dot{x}'\omega\cos\theta)\hat{k}'$$

The solution to the Coriolis force is then:

$$\begin{aligned} \vec{F}_{Cor,x'} &= 2m\dot{y}'\omega\sin\theta \\ \vec{F}_{Cor,y'} &= -2m\dot{x}'\omega\sin\theta \end{aligned}$$

for the x' and y' axes, respectively. Again, we're going to ignore the $\hat{k}'$ component and focus on the deflection in $\hat{\imath}'$ and $\hat{\jmath}'$.

3. Finding the acceleration We have descriptions for gravity, tension, and the Coriolis force in our non-inertial reference frame. We can now solve for the acceleration. For simplicity, we will do this for the x' and y' components separately. Since we are assuming negligible motion in z', we can ignore all forces (inertial or fictitious) in the z' direction. The remaining forces in x' and y' are the inertial tension force and the fictitious Coriolis force.

$$\begin{aligned} m\ddot{x}' &= -\frac{Tx'}{l} + 2m\omega\sin\theta\dot{y}' \\ m\ddot{y}' &= -\frac{Ty'}{l} - 2m\omega\sin\theta\dot{x}' \end{aligned}$$

where θ is the latitude of the observer. if we assume that the angle of displacement is small, then $T \approx mg$. So we can simplify the above as:

$$\begin{aligned} \ddot{x}' &= -\frac{g}{l}x' + (2\omega\sin\theta)\dot{y}' \\ \ddot{y}' &= -\frac{g}{l}y' - (2\omega\sin\theta)\dot{x}' \end{aligned}$$

The above equations are differential equations of motion. Note that for a given observer on Earth, ω and θ are constant. The first term should look familiar. This is the solution for a simple pendulum that is displaced by a small angle from equilibrium. If $\omega = 0$, then we recover the differential equation of motion for an ordinary pendulum in an inertial frame.

The second terms comes from the Coriolis force and describe a deflection in the pendulum's swing. This deflection always acts perpendicular to the velocity vector in the plane of motion. So instead of just oscillating back and forth in a straight line, the pendulum will slowly turn (precess) as it oscillates back and forth. The magnitude of the Coriolis force is small, but it changes the direction of the pendulum just enough that it will trace out a circle over time.

Foucault Pendulums

Stirling Hall at Queen's University has a Foucault pendulum! Visit the pendulum at different times of day and note which direction it is swinging in and how that direction changes from morning to afternoon relative to the hall. If you want to Wait and watch a full rotation, note that it takes many hours (see quick questions below).

Alternatively, here is a great video that shows the full rotation of a Foucault pendulum sped up.

For northern latitudes ($\theta > 0$), the pendulum will rotate clockwise due to the Coriolis force, and for southern latitudes ($\theta < 0$), the pendulum will rotate counter-clockwise due to the Coriolis force. At the equator, $\theta = 0$, and there is no deflection in the x' and y' plane. So a Foucault pendulum at the equator is just an ordinary pendulum that oscillates back and forth.

The time it takes the pendulum to complete one full circle depends on the latitude. The period for one full circle due to the Coriolis force is given by:

$$t_F = \frac{2\pi}{\omega \sin\theta} = \frac{24h}{\sin\theta}$$

So the Foucault pendulum offers a direct way to measure your latitude. At the North and South pole, this period is exactly 24 hours (the length of 1 day). You can say that the Earth is rotating below the pendulum as it oscillates in place due to the Coriolis force. As you approach the equator, the period gets longer. The experiment of Foucault's pendulum was monumental for showing Earth's rotation and that the Earth is a non-inertial frame.

Quick Questions

1. How long would it take the Striling Hall Foucault pendulum to complete one full rotation (Kingston has $\theta = 45\,\text{deg}$)?
2. The length of a day on Venus is almost the same as its year. If the Earth had a spin that was almost the same length as its year, what would that mean for the motion of a Foucault pendulum?

5.6 Real-World Application

Although forces like the centrifugal force and Coriolis force are fictitious, we can see the effects of rotating references frames on objects and ourselves. A centrifuge is a device that rotates an object around a fixed axis very quickly. In laboratories, these high rotation speeds are used to separate out different substances into layers by their densities allowing pristine samples to be collected. The effective force can be hundreds or thousands of times that of a standard Earth gravity.

Rotating rides at amusement parks operate at lower speeds that centrifuges, but those on the rides feel similar effects. When on one of these rides, you would feel your body move outward, often against the wall. Space agencies also use systems like centrifuges for high-gravity simulation during astronaut training. Astronauts leaving or returning to Earth feel changes in effective gravity that can affect the blood flow to their heads and make them pass out. With training, the astronauts can simulate those conditions and learn to function.

Courtesy of the ESA astronaut Andreas Mogensen, this video shows the view from outside and inside a training centrifuge in operation.

Fisher Scientific provides a primer on centrifuge theory.

5.7 Summary

Key Takeaways

This chapter applies the basic concepts of non-inertial frames from Chapter 4 to more complex problems. In particular, this chapter expands on the Coriolis and centrifugal fictitious forces in rotating frames. The general equation for the fictitious forces are,

$$m\vec{a}\,' = m\vec{a} + \underbrace{\vec{F}_{az}}_{-m\vec{\alpha}\times\vec{r}\,'} + \underbrace{\vec{F}_{Cor}}_{-2m\vec{\omega}\times\vec{v}\,'} + \underbrace{\vec{F}_{cent}}_{-m\vec{\omega}\times(\vec{\omega}\times\vec{r}\,')} + \underbrace{\vec{F}_{trans}}_{-m\vec{A}}$$

This chapter also introduces the Earth to be a non-inertial frame. To first order, the physics problems from Chapter 2 and 3 assume that the Earth is an inertial frame of reference. This approximation is generally fine, as the fictitious forces do not greatly affect these types of physics problems. Try the practice problems below to see the magnitude of some of these forces.

But for some types of high-precision physics, such as weather patterns, satellite orbits, and Foucault pendulums, you need to take into account the fictitious forces that arise from a non-inertial Earth-bound reference frame. This chapter goes through a few examples, highlighting how to breakdown and simplify such problems.

Important Equations

For No Rotation:

$$\begin{aligned} \vec{a}\,' &= \vec{a} - \vec{A} \\ \vec{v}\,' &= \vec{v} - \vec{u} \end{aligned}$$

Rotating Frame:

$$\begin{aligned} \vec{a}\,' &= \vec{a} - \vec{\alpha}\times\vec{r}\,' - 2\vec{\omega}\times\vec{v}\,' - \vec{\omega}\times(\vec{\omega}\times\vec{r}\,') - \vec{A} \\ \vec{v}\,' &= \vec{v} - \vec{\omega}\times\vec{r}\,' - \vec{u} \end{aligned}$$

Rotation Frame Fictitious Forces:

$$m\vec{a}\,' = m\vec{a} + \underbrace{\vec{F}_{az}}_{-m\vec{\alpha}\times\vec{r}\,'} + \underbrace{\vec{F}_{Cor}}_{-2m\vec{\omega}\times\vec{v}\,'} + \underbrace{\vec{F}_{cent}}_{-m\vec{\omega}\times(\vec{\omega}\times\vec{r}\,')} + \underbrace{\vec{F}_{trans}}_{-m\vec{A}}$$

Earth's rotation axis for an observer on the surface:

$$\vec{\omega} = \omega\cos\theta\hat{\jmath}' + \omega\sin\theta\hat{k}'$$

Centrifugal Force:

$$\vec{F}_{cent} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r}\,')$$

Coriolis Force:

$$\vec{F}_{Cor} = -2m\vec{\omega}\times\vec{v}\,'$$

5.8 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 5-1

For the following questions, which fictitious force(s) are non-zero?

a) A cannon fires a cannonball from the surface of the Earth.

b) A ladybug is sitting at the edge of a stationary on a merry-go-round that is spinning at a constant speed.

c) A ladybug is sitting at the edge of a decelerating merry-go-round.

d) A lady bug is walking towards the edge of a merry-go-round that is spinning at a constant speed.

e) A lady bug is running towards the edge of a merry-go-round that is decelerating its spin speed and has gone off the rails and is moving away from its starting point with a constant acceleration.

Practice Problem 5-2

Astronauts use a training centrifuge to simulate the high gravity conditions they can experience during launch and re-entry. NASA uses a centrifuge that is 8.84 m in diameter for astronaut training.

a) Find the expression for the centrifugal acceleration.

b) The centrifuge can simulate accelerations that are equivalent to $20g$ (20 times Earth's gravity). How fast must it be rotating to reach that acceleration? Give your units in revolutions per second.

c) Plot angular velocity versus centrifugal acceleration from $1g$ to $20g$. Adjust your plot for different size centrifuges and see how the values change.

Practice Problem 5-3

A cat sits on a rotating table that is rotating in the counter-clockwise direction with an angular velocity that is decelerating. Draw a free-body diagram showing the correct directions of the fictitious forces acting on the cat.

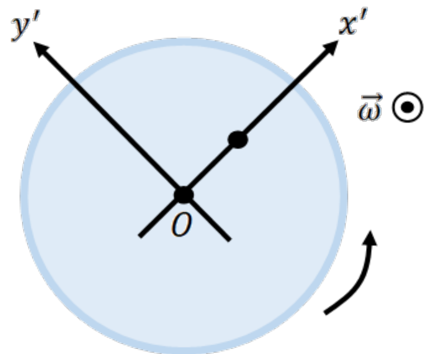


Figure 5.9: A cat sits on the turn table as it turns in the counter-clockwise direction.

Practice Problem 5-4

Typical vinyl records spin $33\frac{1}{3}$ times per minute. What is the magnitude and direction of the Coriolis Force experienced by a ladybug (m = 0.02 g) that is crawling radially outward with a velocity of 1 cm s^{-1} at a distance of 10 cm from the axis of rotation?

Practice Problem 5-5

An amusement park ride spins its riders in a large circle and then tilts the circle into a vertical position. Before it begins to tilt, it has to accelerate to an angular velocity that will allow its riders to be safe at the top of the tilted circle. Assume the ride has a radius of 12 m.

a) During the acceleration phase of the ride, what fictitious forces will the riders experience?

b) The ride reaches a maximum centrifugal acceleration of $1g$ before it tilts. Find the angular acceleration if it takes 90 seconds to reach its maximum speed and it gets there with a constant angular acceleration.

c) Find the magnitude of the azimuthal force and centrifugal force halfway through the acceleration phase (e.g., at 45 seconds).

Practice Problem 5-6

A race car driver at the Indy 500 races (latitude is 40 deg N) is traveling south in their reference frame.

a) What is the direction of the Coriolis force acting on the driver?

b) If the race car driver hits a speed of 200 km/h going due north at the Indy 500 races, what is the magnitude of the Coriolis force acting on this driver relative to the force of gravity (e.g., F_{Cor}/F_g)? You can assume that the angular velocity for the Earth is $7.27 \times 10^{-5}\text{s}^{-1}$).

Practice Problem 5-7

A group of astronauts crash land on a planet with an unknown angular velocity of rotation. The astronauts have a Foucault pendulum, and they measure the period of precession of 38 hours at their unknown latitude of θ. The astronauts then walk north and estimate their new position as being $10°$ above the original latitude. They then measure a new period of precession for the Foucault pendulum of 31 hours. What was the original latitude of the crash landing? (Hint, you may find the following trig identity helpful: $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$)

Practice Problem 5-8

On a physics field trip, you drop a pebble from rest in the elevator shaft of the CN Tower (latitude of 43.5 deg, height of 500 m). As the pebble falls, it is deflected along the x'-axis. If $+\hat{\imath}'$ is East and $-\hat{\imath}'$ is West, what is the magnitude and direction of the deflection in x'? (Hint, you will need to solve for the deflection as a function of $\dot{z}'$ to answer this question. You can assume that the deflection is small such that only the velocity from $\dot{z}'$ matters for the Coriolis force. Use an angular velocity of 7.27×10^{-5} s^{-1} for the Earth.)

Practice Problem 5-9

A bead of mass m sits at the end of a smooth frictionless rod of length L that is rotating about one end at a rate of ω as shown in the figure below. The bead is given a little push which so that it starts moving.

a) What is the magnitude and direction of the centrifugal and Coriolis forces?

b) What is the magnitude and direction of the inertial force?

c) Draw a free-body diagram for the bead in the rotating frame. What inertial force(s) are acting on the bead?

d) Use the differential equation of motion to find the equation for $x(t)$ if the initial velocity given to the bead is ωL. (Hint, you may find the following equation helpful: $\ddot{x} - Cx = 0 \rightarrow x(t) = Ae^{ct} + Be^{-ct}$, where C is a constant)

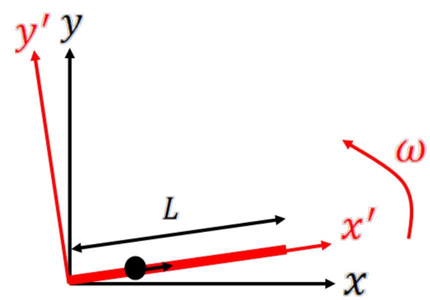


Figure 5.10: The bead is pushed along the positive x'-axis.

6 Momentum and Variable Mass

Learning Objectives

- Review linear momentum and momentum conservation
- Define momentum with external forces, impulse, and collision
- Define center of mass for N-body systems
- Investigate problems with variable mass

In this chapter, we will review linear momentum and the conservation of momentum. We will also discuss impulse, collisions, and variable mass problems.

6.1 Linear Momentum

Momentum is a dynamic property of a system, equal to mass times velocity.

$$\boxed{\vec{p} = m\vec{v}} \tag{6.1}$$

In Chapter 2, we related momentum to Newton's second law. Namely, the net force on a system is equal to the change in momentum for that system.

$$\boxed{\sum \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}} \tag{6.2}$$

If we assume that mass is constant we can substitute in Equation (6.1):

$$\begin{aligned}
\sum \vec{F} &= \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} \\
\sum \vec{F} &= m\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} \qquad \Longrightarrow \qquad \text{assuming mass is constant} \\
\sum \vec{F} &= m\vec{a}
\end{aligned}$$

We will look at the case where the mass changes with time in Section 6.5.

If the net external force is equal to zero ($\sum \vec{F} = 0$), then the total momentum of a system is constant, $\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = 0$ and $\vec{p}$ is a constant. This result is the conservation of linear momentum.

6.2 Conservation of Linear Momentum

Consider an isolated system of n particles that have distinct masses and velocities. The total momentum of the system is given by:

$$\boxed{\vec{p}_{tot} = \sum \vec{p}_i = \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_n} \tag{6.3}$$

For simplicity, let's take a case with 3 particles. Such that the total momentum is:

$$\vec{p}_{tot} = \sum \vec{p}_i = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$$

and the time derivative of the total momentum is:

$$\begin{aligned}
\frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \\
\frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} &= \frac{\mathrm{d}\vec{p}_1}{\mathrm{d}t} + \frac{\mathrm{d}\vec{p}_2}{\mathrm{d}t} + \frac{\mathrm{d}\vec{p}_3}{\mathrm{d}t} \\
\frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} &= \sum \vec{F}_1 + \sum \vec{F}_2 + \sum \vec{F}_3 \quad \Longrightarrow \quad \text{set } \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F}
\end{aligned}$$

where $\sum \vec{F}_1$ is the net force on particle 1, $\sum \vec{F}_2$ is the net force on particle 2, and $\sum \vec{F}_3$ is the net force on particle 3.

The net force on particle 1 should be the force from particle 2 ($\vec{F}_{21}$) and the force from particle 3 ($\vec{F}_{31}$). There are no other forces on particle 1 because the system is <u>isolated</u> (e.g., the system has no outside influences). The same argument can be made for particles 2 and 3. So the time derivative of our net momentum becomes:

$$\frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} = \underbrace{\vec{F}_{21} + \vec{F}_{31}}_{\sum \vec{F}_1} + \underbrace{\vec{F}_{12} + \vec{F}_{32}}_{\sum \vec{F}_2} + \underbrace{\vec{F}_{13} + \vec{F}_{23}}_{\sum \vec{F}_3}$$

But because of Newton's third law (every action has an equal and opposite reaction), the force of particle 2 on particle 1 ($\vec{F}_{21}$) must be equal and opposite to the force of particle 1 on particle 2 ($\vec{F}_{12}$). You can think of two masses in space pulling on each other due to gravity. Or two isolated charges attracting or repelling each other. As a result, $\vec{F}_{21} = -\vec{F}_{12}$, $\vec{F}_{31} = -\vec{F}_{13}$, and $\vec{F}_{32} = -\vec{F}_{23}$. So we finally obtain:

$$\begin{aligned}
\frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} &= 0 \\
\vec{p}_{tot} &= \text{constant}
\end{aligned}$$

The above example is for three particles, but we can easily generalize the solution to N particles as long as the system is isolated (no external forces). For a system of n-particles,

$$\begin{aligned}
\vec{p}_{tot} &= \sum_{i=1}^{N} \vec{p}_i \\
\frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} \frac{\mathrm{d}\vec{p}_i}{\mathrm{d}t} \\
\frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} &= \sum_{i=1}^{N} (\Sigma \vec{F}_i) \\
\frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} &= \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \vec{F}_{ij} = 0
\end{aligned}$$

where the unique pairs of forces are represented by double sums. To break down what the nested sums mean, first let's consider a single particle, represented by i. We can write

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \sum_{\substack{j=1 \\ j\neq i}} F_{ij}$$

which is basically saying that the time derivative of the momentum for the ith particle is just the sum of all the forces from the other particles. The condition of $j \neq i$ is needed because each particle acts on the other particles in the system, but not on themselves ($\vec{F}_{11}$, $\vec{F}_{22}$, and $\vec{F}_{33}$ are not allowed).

To then get the the total momentum of an isolated system, we need to sum over individual particles, $\vec{p}_{tot} = \sum \vec{p}_i$. This gives us our nested sums,

$$\boxed{\frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} = \sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} \vec{F}_{ij} = 0} \tag{6.4}$$

where you start with the outer summation and set a value for i before cycling through the inner summation and setting all possible values for j. For an example, if $N = 3$, the nested sums would give $\vec{F}_{12}, \vec{F}_{13}, \vec{F}_{21}, \vec{F}_{23}, \vec{F}_{31}$, and $\vec{F}_{32}$ in that order.

6.3 Momentum with an External Force

The conservation of linear momentum draws directly from Newton's third law. In an isolated system, all forces balance (equal and opposite reactions) such that the total momentum of the system is constant. But if there is an external force, then the total linear momentum is no longer constant. For a simple particle in a system,

$$\boxed{\frac{\mathrm{d}\vec{p}_i}{\mathrm{d}t} = \sum \vec{F}_i = \vec{F}_{i,int} + \vec{F}_{i,ext}} \tag{6.5}$$

where $\vec{F}_{i,int}$ is the force on the particle from the system itself (internal force) and $\vec{F}_{i,ext}$ is the external force on the particle. If you then look at all particles in the system:

$$\begin{aligned} \frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} &= \sum \frac{\mathrm{d}}{\mathrm{d}t}\vec{p}_i = \sum \vec{F}_{i,int} + \sum \vec{F}_{i,ext} \\ \frac{\mathrm{d}\vec{p}_{tot}}{\mathrm{d}t} &= \sum \vec{F}_{i,ext} \quad \Longrightarrow \quad \sum \vec{F}_{i,int} = 0 \text{ (see Chapter 6.2)} \end{aligned}$$

So in a system with an external force, the net change in momentum of that system is given by the net external force acting on the system.

6.3.1 Impulse

The impulse of a force is defined as:

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}\mathrm{d}t$$

where F is the net force acting on the system. We can re-define the net force in terms of momentum, however.

$$\boxed{\vec{I} = \int_{t_1}^{t_2} \vec{F}\mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\mathrm{d}t = \int_{t_1}^{t_2} \mathrm{d}\vec{p} = \vec{p}_2 - \vec{p}_1 = \Delta\vec{p}} \tag{6.6}$$

The impulse of a force represents the change in linear momentum in the system. Note that mass does not need to be constant in the definition of an impulse.

An impulse is an instantaneous event, where the interaction time is short ($\Delta t \approx 0$). We can assume that the system does not change during the impulse, but it will change drastically as a result of the impulse (it just hasn't had time for that to happen).

In general practice, an impulse is a fast, powerful force that quickly changes the momentum of a system rather than a slow process that slowly changes the momentum of a system. The distinction between fast and slow, however, is not well defined. What is necessary is that your force can be approximated by an average value over the small time duration. If you have a relatively constant force, then you can take a longer time duration, but if your force changes quickly, you need a shorter time interval.

Quick Question

1. A hockey player hits a puck. If the puck was initially going at a speed of $v_i\hat{\imath}$ and ends up going $v_f\hat{\jmath}$, what was the direction of the impulse vector?

Figure 6.1 shows two identical impulses with the same area $\int \vec{F}\mathrm{d}t$, which means they have the same impulse magnitude. Note the differences between them. The average impulse is $\vec{I}_{avg} = \vec{F}_{avg}\Delta t$. So if you increase the interaction time, Δt, you decrease the average force necessary for the same impulse. In the figure, the blue curve has a shorter Δt and requires a stronger force than the red curve to produce the same change in momentum. You can minimize the force by increasing the time of interaction.

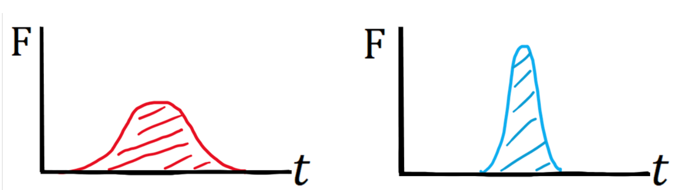


Figure 6.1: Sketch of two forces with the same impulse (area under the curve).

Impulse in Everyday Life

Increasing the impact time to decrease the force occurs many times in everyday life. For example, when jumping, most people bend their knees as they land. Bending your knees extends the time of impact and lessens the force on your legs.

Cora's Thoughts

Generally we consider impulses to be short bursts. An impulse is powerful force that quickly changes the momentum of a system, hence why $\Delta t \to \mathrm{d}t$.

An airbag is an excellent example of impulse in everyday life. When an accident occurs and the vehicle stops, the drivers momentum carries them forward towards the steering wheel, where they will experience a quick change in momentum (impulse). The airbag extends the time of that impact lessening the force on the driver, often saving lives. In Figure 6.1, the red graph may be what the force with the airbag would look like, and the blue would be what the force without the airbag would look like.

6.3.2 Collisions

Collisions are a way that the momentum of a system can change. There are two types of collisions that we'll consider, *inelastic* and *elastic* collisions.

Case 1: Inelastic Collision

In a totally inelastic collision, the colliding systems merge (stick together). Figure 6.2 illustrates the basic case of an inelastic collision in 1D, where two masses (m and M) collide and stick together. After the collision, the two masses move as one system ($m + M$).

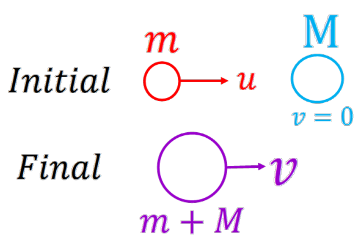


Figure 6.2: Example of an inelastic collision. A mass m moving at a speed u approaches another mass M that is at rest. The two objects collide and merge into one system $m + M$ that moves at a new speed, v.

Since there are no external forces (m and M are internal to the system), momentum must be conserved between the initial state and final state of the system. Thus, $\vec{p}_i = \vec{p}_f$.

$$\begin{aligned} \vec{p}_i &= \vec{p}_f \\ mu\hat{\imath} + 0 &= (m+M)v\hat{\imath} \\ v &= \frac{m}{m+M}u \end{aligned}$$

If $M \gg m$, then v is small relative to u. If $M \ll m$, then $v \approx u$.

Sample Problem 6-1

Two particles of equal mass m and identical speeds u collide and stick. The first particle initially moves at an angle of θ_1 above the horizontal and the second particle initially moves at an angle of θ_2 below the horizontal. If both θ_1 and θ_2 are $< 90°$, **what is the final velocity of the system after collision?**

Solution

We want to find the velocity (speed and direction) of the system after an inelastic collision. The first thing we need to do is draw the initial set up of the problem. Figure 6.3 shows the motion of particle 1, m_1, and particle 2, m_2, as described by the problem. The two masses collide at the origin and then stick together.

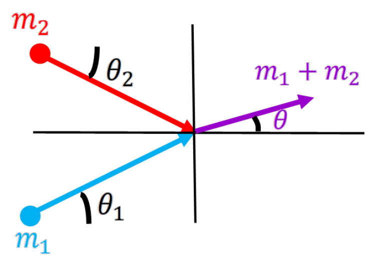


Figure 6.3: The motion of two identical particles, m_1 and m_2, on a coordinate grid. The velocity of the post-collision system is shown by the purple vector.

When the masses collide, they continue their journey as $m_1 + m_2 = 2m$ (purple vector). We do not know the exact direction of motion for the post-collision system, however. The $m_1 + m_2$ system may move at an angle θ above or below the horizontal. Figure 6.3 shows the above case. If we chose wrong, we will get a negative angle.

Since this is an isolated system, the total momentum must be conserved and it must be conserved in both x and y, where

$$\begin{aligned} p_{x,tot} &= \text{constant} \\ p_{y,tot} &= \text{constant} \end{aligned}$$

We will solve the x and y components of the motion separately.

For the x component

$$\begin{aligned} p_{x,f} &= p_{x,i} \quad \Longrightarrow \quad \text{conservation of momentum} \\ (m_1 + m_2)v_x &= m_1 v_{1,x} + m_2 v_{2,x} \\ 2mv_x &= m(u\cos\theta_1) + m(u\cos\theta_2) \\ v_x &= \frac{u}{2}(\cos\theta_1 + \cos\theta_2) \end{aligned}$$

For the y component

$$\begin{aligned} p_{y,f} &= p_{y,i} \quad \Longrightarrow \quad \text{conservation of momentum} \\ (m_1 + m_2)v_y &= m_1 v_{1,y} + m_2 v_{2,y} \\ 2mv_y &= m(u\sin\theta_1) - m(u\sin\theta_2) \quad \Longrightarrow \quad v_{2,y} \text{ is in the negative direction} \\ v_y &= \frac{u}{2}(\sin\theta_1 - \sin\theta_2) \end{aligned}$$

With the x and y components of the motion, the final speed is:

$$\begin{aligned}
|\vec{v}| &= \sqrt{v_x^2 + v_y^2} \\
&= \sqrt{\frac{u^2}{4}(\cos\theta_1 + \cos\theta_2)^2 + \frac{u^2}{4}(\sin\theta_1 - \sin\theta_2)^2} \\
&= \frac{u}{2}\sqrt{(\cos\theta_1 + \cos\theta_2)^2 + (\sin\theta_1 - \sin\theta_2)^2} \\
&= \frac{u}{2}\sqrt{2 + 2\cos(\theta_1 + \theta_2)} \quad \Longrightarrow \quad \text{use trig identities to simplify}
\end{aligned}$$

To get the angle, we can again use trigonometry. The v_x component is the x-component of the vector and the v_y component is the y-component of the vector.

$$\begin{aligned}
\tan\theta &= \frac{v_y}{v_x} \\
\tan\theta &= \frac{\frac{u}{2}(\sin\theta_1 - \sin\theta_2)}{\frac{u}{2}(\cos\theta_1 + \cos\theta_2)} \\
\theta &= \tan^{-1}\left[\frac{\sin\theta_1 - \sin\theta_2}{\cos\theta_1 + \cos\theta_2}\right]
\end{aligned}$$

Quick Questions

1. If $\theta_1 = 45°$, $\theta_2 = 30°$ and $u = 10$ m s^{-1}, what is the velocity of the system after the collision? Give speed and angle.
2. In Figure 6.3 we guessed that $\theta > 0°$ (above the horizontal). For what values of θ_1 and θ_2 will you have $\theta > 0°$?

Case 2: Elastic Collision

For a completely elastic collision, the particles rebound off each other and there is no change in their mass pre-collision and post-collision. In this case, there is no loss of energy (kinetic energy is completely conserved) and once again, momentum is conserved.

Examples of Elastic Collisions

A good example of elastic collisions is a game of pool / billiards, where you use one billiard ball to hit other ones into pockets. Or even a game of domino's, where one domino piece hits the next and so forth. In practice, these are not fully elastic collisions as there will be some loss of energy, but there is little energy loss.

For some nice examples of elastic-like collisions, here are videos showing Newton's cradle and Dominos with cats.

Sample Problem 6-2

Consider two balls moving toward each other on the x-axis as shown in Figure 6.4. The first ball has a mass of m_1 and is moving at $v_{1,i}$, and the second ball has a mass of m_2 and is moving at a speed of $v_{2,i}$. After collision, m_1 is moving at $v_{1,f}$ and m_2 is moving at $v_{2,f}$. **Find equations for the final speeds in terms of the initial parameters?**

Solution

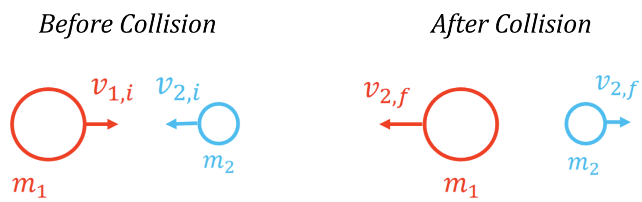


Figure 6.4: Two masses, m_1 and m_2 collide in an elastic collisions before and after. Their speeds change after the collision.

This may sound like an easy problem, but there are some tricks to it. We want to find $v_{1,f}$ and $v_{2,f}$ in terms of m_1, m_2, $v_{1,i}$ and $v_{2,i}$. For a completely elastic collision, the total momentum is conserved, so the initial momentum equals the final momentum.

$$\begin{aligned} \vec{p}_i &= \vec{p}_f \\ m_1\vec{v}_{1,i} + m_2\vec{v}_{2,i} &= m_1\vec{v}_{1,f} + m_2\vec{v}_{2,f} \end{aligned} \tag{6.7}$$

But to fully solve this problem, we cannot just use the conservation of momentum, because we have two unknown quantities ($v_{1,f}$ and $v_{2,f}$). To have a unique solution with two unknown quantities, you need to have at least two unique equations. For our second equation, we will use the conservation of kinetic energy (we will return to kinetic energy in Chapter 8), where kinetic energy can be expressed as $K = \frac{1}{2}mv^2$.

$$\begin{aligned} K_i &= K_f \implies \text{total kinetic energy is conserved} \\ \frac{1}{2}m_1v_{1,i}^2 + \frac{1}{2}m_2v_{2,i}^2 &= \frac{1}{2}m_1v_{1,f}^2 + \frac{1}{2}m_2v_{2,f}^2 \implies \text{can drop } \tfrac{1}{2} \\ m_1v_{1,i}^2 + m_2v_{2,i}^2 &= m_1v_{1,f}^2 + m_2v_{2,f}^2 \end{aligned} \tag{6.8}$$

Now we have two equations and two unknowns and can solve the problem. To make the math easier, we will re-arrange Equation 6.7 to bring all m_1 terms to one side and all m_2 terms to the other side.

$$m_1(v_{1,i} - v_{1,f}) = m_2(v_{2,f} - v_{2,i}) \tag{6.9}$$

Similarly, we can re-arrange Equation 6.8 as

$$\begin{aligned} m_1(v_{1,i}^2 - v_{1,f}^2) &= m_2(v_{2,f}^2 - v_{2,i}^2) \\ m_1(v_{1,i} - v_{1,f})(v_{1,i} + v_{1,f}) &= m_2(v_{2,f} - v_{2,i})(v_{2,f} + v_{2,i}) \end{aligned} \tag{6.10}$$

where we use the difference of squares, $(a^2 - b^2) = (a - b)(a + b)$ to simplify the equation.

With Equations 6.9 and 6.10 you now have two equations with which to solve two unknowns. Dividing 6.10 by 6.9 gives:

$$\begin{aligned} \frac{m_1(v_{1,i} - v_{1,f})(v_{1,i} + v_{1,f})}{m_1(v_{1,i} - v_{1,f})} &= \frac{m_2(v_{2,f} - v_{2,i})(v_{2,f} + v_{2,i})}{m_2(v_{2,f} - v_{2,i})} \\ v_{1,i} + v_{1,f} &= v_{2,f} + v_{2,i} \quad \Longrightarrow \quad \text{simplify} \end{aligned}$$

So all that math went into showing that the sum of the two velocities for m_1 equals the sum of the two velocities for m_2. Note that this conclusion is not immediately obvious from either the conservation of momentum or the conservation of kinetic energy equations themselves. But for us to have a unique solution to this problem, the sum of the individual object velocities must be equal.

We can re-arrange the above equation to give $v_{1,f} = v_{2,f} + v_{2,i} - v_{1,i}$ and then plug this into Equation 6.7 to solve for $v_{2,f}$.

$$\begin{aligned} m_1 v_{1,i} + m_2 v_{2,i} &= m_1(v_{2,f} + v_{2,i} - v_{1,i}) + m_2 v_{2,f} \\ m_1 v_{1,i} + m_2 v_{2,i} &= m_1 v_{2,f} + m_1 v_{2,i} - m_1 v_{1,i} + m_2 v_{2,f} \\ 2m_1 v_{1,i} + m_2 v_{2,i} - m_1 v_{2,i} &= m_1 v_{2,f} + m_2 v_{2,f} \\ 2m_1 v_{1,i} + (m_2 - m_1) v_{2,i} &= v_{2,f}(m_1 + m_2) \\ v_{2,f} &= \frac{2m_1 v_{1,i} + (m_2 - m_1) v_{2,i}}{m_1 + m_2} \end{aligned}$$

And now we have the velocity in terms of the initial velocities and masses.

You can follow the same procedure to get the equation for $v_{1,f}$.

$$v_{1,f} = \frac{2m_2 v_{2,i} + (m_1 - m_2) v_{1,i}}{m_1 + m_2}$$

Quick Questions

1. Find $v_{1,f}$ and $v_{2,f}$ if $m_1 = 5m_2$, $v_{1,i} = 2$ m s^{-1}, and $v_{2,i} = -4$ m s^{-1}?
2. Under which circumstances would m_1 travel in the $-\hat{\imath}$ direction after collision? Under which circumstances would m_2 travel in the $-\hat{\imath}$ direction after collision?

Test your Understanding

Here is a fun web application that you can use to test your understanding of both elastic and inelastic collisions. Try to solve the set up problems before running the application and see how well you do.

6.4 Centre of Mass

Another important concept in motion and momentum is the centre of mass. Whether you have a system of independent particles (e.g., a cluster of stars) or an irregularly shaped rigid body (e.g., a car), every system has a special point called the *centre of mass*. The centre of mass is not a mass, but a position. It's the centroid position and it is defined as:

$$\vec{R}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + \cdots m_n} = \frac{\sum m_i\vec{r}_i}{\sum m_i}$$

where $\vec{r}_i$ is the position of the ith particle relative to the origin and m_i is the mass of that particle. Since $\sum m_i = M$ is the total mass of the system, the centre of mass is:

$$\boxed{\vec{R}_{cm} = \frac{\sum m_i\vec{r}_i}{M}} \tag{6.11}$$

Note that you can think of the centre of mass as a mass-weighted average position. The formal definition of an average quantity is:

$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i}$$

where x_i is the quantity and w_i is a weight. Note that the $\sum$ used above has the limits of $\sum\limits_{i=1}^{N}$, where N is the total number of particles.

Definition of Average

You are probably familiar with the definition of an average as $\bar{x} = \frac{1}{N}\sum x_i$, where N is the total number of elements of x. This equation for average comes from the assumption that all quantities of x have equal weight. If all quantities have *equal weight*, then you can factor out the weight from the summation terms and they cancel, leaving you with $\bar{x} = \frac{1}{N}\sum x_i$.

The centre of mass is where you can perfectly balance a system and it doesn't need to be at the centre of the object. For example, if you try to hold a hammer at its centre, it will feel unbalanced. That's because a hammer has an uneven distribution of mass. The head of the hammer is much heavier than the handle, so the centre of mass for the hammer will be closer to the head than the the middle of the handle because most of the mass is located near the head (R_{cm} will be weighted more heavily toward the head than the handle).

Try at Home

This https://phet.colorado.edu/en/simulation/balancing-actweb applet will let you play around with balancing masses and finding the centre of mass. Examine different structures and test your understanding.

In Cartesian coordinates, we can also describe the centre of mass in terms of the x, y, and z axes. The position of a particle in the system is given by $\vec{r}_i = x_i\hat{\imath} + y_i\hat{\jmath} + z_i\hat{k}$. The centre of mass for the system is then determined by $\vec{R}_{cm} = x_{cm}\hat{\imath} + y_{cm}\hat{\jmath} + z_{cm}\hat{k}$, where

$$x_{cm} = \frac{\sum m_i x_i}{M}$$
$$y_{cm} = \frac{\sum m_i y_i}{M}$$
$$z_{cm} = \frac{\sum m_i z_i}{M}$$

If the particle mass is constant, then the total momentum of a system of particles can be written as:

$$\boxed{\vec{p} = \sum m_i \vec{v}_i = \sum m_i \frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum (m_i \vec{r}_i)} \tag{6.12}$$

Note that the term in the summation from the above equation is equivalent to $M\vec{R}_{cm}$ from Equation 6.11. Thus, we can put the total momentum in terms of the centre of mass.

$$\boxed{\vec{p} = \frac{\mathrm{d}}{\mathrm{d}t}(M\vec{R}_{cm}) = M\vec{v}_{cm}} \tag{6.13}$$

where $\vec{v}_{cm}$ is the velocity of the centre of mass. In other words, the total momentum of a system of particles is equivalent to the total mass of the system times the velocity of the centre of mass (how the centre of mass of the system is moving).

Equation 6.13 is a way to approximate a complicated system. In physics, we like to simplify problems as much as possible. Rather than trying to solve a complicated problem of a system of particles or an irregularly shaped body, you can instead use one giant particle with a mass given by the total mass of the system located at and moving with the centre of mass and moving. You are basically condensing the problem from a collection of particles down to a representative particle at a mass-weighted average position.

It can also be useful to consider a coordinate system relative to the centre of mass rather than a stationary observer. Figure 6.5 shows the difference between an initial reference frame from a stationary observer, S, and a moving frame, S', located at the centre of mass of an irregular object. For simplicity, the centre of mass is moving with a constant velocity, $\vec{u}$ (so S' is also an inertial frame). To an observer in S', the irregular object would appear to be stationary (both the observer and the object are moving together). This means that the total momentum in the CM frame is zero.

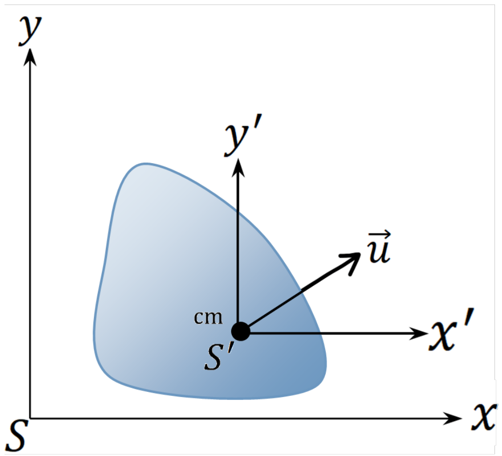


Figure 6.5: Comparison between a stationary observer coordinate system (S) and a centre-of-mass coordinate system (S′). An irregular object is moving in the stationary frame. The centre of mass (cm) of this object has a speed $\vec{u}$ relative to the stationary frame. The S′ frame is fixed relative to the centre of mass and moves with it (such that the object would be stationary in the centre-of-mass frame).

Consider the same particle in both reference frames. The particle has a velocity $\vec{v}_i$ in frame S and a velocity $\vec{v}_i'$ in frame S′. Since the two frames differ by a relative velocity $\vec{u}$, the velocity in S and S′ are connected by,

$$\vec{v}_i = \vec{v}_i' + \vec{u}$$

This equation implies that if the total momentum must be conserved in both frames, because the final and initial momentum shifted by a constant amount ($\vec{u}$). This case is true if there are no external forces (only internal forces) and no acceleration.

Sample Problem 6-3

Let's look at an inelastic collision question. Figure 6.6 shows an inelastic collision between two particles, m, and M. The particle m is moving toward M with a velocity of $\vec{u}$ and M is at rest. After collision, the combined mass is moving at a velocity $\vec{v}$. **Find the final velocity v of the $M + m$ system.**

Solution

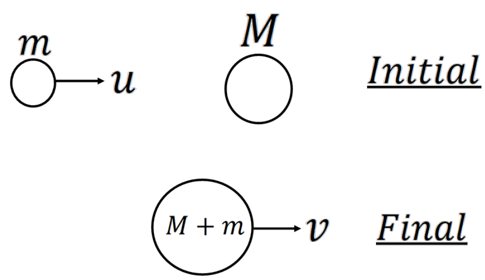


Figure 6.6: Two masses collide in an inelastic collision. The first mass m is initially moving with a velocity $\vec{u}$ and the second mass M is initially at rest. After the collision, the masses move together at velocity v.

You can solve this problem using a stationary frame where the two masses are moving. But here we're going to solve this problem using both a stationary and the moving centre-of-mass (CM) frame.

Stationary frame: To a stationary observer, the masses are moving before and after collision. From the conservation of momentum, we have:

$$\begin{aligned}\vec{p}_i &= \vec{p}_f \\ mu\hat{\imath} + 0 &= (M+m)v\hat{\imath} \\ v &= \frac{m}{M+m}u\end{aligned}$$

CM frame: In the CM frame, the observer is moving with a speed of $\vec{v}_{cm}$ corresponding to the centre of mass of the system. Note that the centre-of-mass velocity must be the same after the collision as before the collision because the centre-of-mass momentum is conserved (no external forces).

First, we can find $\vec{v}_{cm}$. For our two particles, their individual masses are constant and $M_{tot} = M + m$.

$$\begin{aligned}\vec{v}_{cm} &= \frac{1}{M+m}\left(\frac{\mathrm{d}}{\mathrm{d}t}\sum m_i\vec{r}_i\right) \\ \vec{v}_{cm} &= \frac{1}{M+m}\left(\sum m_i\frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t}\right) \\ \vec{v}_{cm} &= \frac{1}{M+m}\left(m\frac{\mathrm{d}\vec{r}_m}{\mathrm{d}t} + M\frac{\mathrm{d}\vec{r}_M}{\mathrm{d}t}\right) \\ \vec{v}_{cm} &= \frac{1}{M+m}(m\vec{u}+0) \quad \Longrightarrow \quad \text{in S, } \frac{\mathrm{d}\vec{r}_m}{\mathrm{d}t} = \vec{u} \text{ and } \frac{\mathrm{d}\vec{r}_M}{\mathrm{d}t} = 0 \\ \vec{v}_{cm} &= \frac{m}{M+m}\vec{u}\end{aligned}$$

Now we need to relate $\vec{v}_{cm}$ to the final velocity, $\vec{v}$. After the collision, M and m are stuck together and the system moves as one particle with a speed of v in the observer's frame. In the CM frame, the post-collision system has a speed of:

$$\vec{v}\,'_{(M+m)} = \vec{v} - \vec{v}_{cm}$$

But with only one particle, that particle represents the centre-of-mass position for the post-collision system. In the CM frame, the observer is moving with the centre of mass of the system, so there is no net velocity. That means $\vec{v}\,'_{(M+m)} = 0$.

Thus, we can now find the final velocity, v in the CM frame.

$$\begin{aligned} 0 &= \vec{v} - \vec{v}_{cm} \\ \vec{v} &= \vec{v}_{cm} \\ \vec{v} &= \frac{m}{M+m}\vec{u} \end{aligned}$$

which is exactly what we had before from the stationary frame.

Quick Questions

1. Find the velocities of m and M before the collision in the CM frame.
2. How does the centre-of-mass position change as the two particles approach each other?

Switching to the centre of mass frame can be convenient when you have complicated systems with an irregular rigid mass or a large system of masses.

6.5 Variable Mass

Up until now, we have applied Newton's second law as $\sum \vec{F} = ma$. This equation is applicable if the system mass is constant with time. But you can have problems in physics where the mass changes.

In general, Newton's second law follows;

$$\boxed{\sum \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = \left(\frac{\mathrm{d}m}{\mathrm{d}t}\right)\vec{v} + m\left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)}$$

Note that you recover $\sum \vec{F} = m\vec{a}$ if the mass is constant ($\dot{m} = 0$). But if the mass is changing, then you must include the $\dot{m}$ term as well when applying Newton's second law.

Sample Problem 6-4

A rope with a linear mass density of λ (in kg m^{-1}) and length L is coiled in a heap on the floor. You grab one end of the rope and pull it up at a constant speed of v. **What is the force as a function of height y that you must apply to raise rope?**

Solution

This is a variable mass problem, because you're not moving the whole rope at once. The amount of mass you are raising is increasing as more of the rope is lifted off the ground. You are basically giving momentum to "new" atoms in the rope as they leave the ground and additional force is needed to apply that change in momentum.

Since the rope is moving with a constant speed, we know that $\frac{dv}{dt} = 0$. But there is still a net force acting on the rope because the mass is changing.

$$\sum \vec{F} = \left(\frac{\mathrm{d}m}{\mathrm{d}t}\right)\vec{v} + m\cancelto{0}{\left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)} = \left(\frac{\mathrm{d}m}{\mathrm{d}t}\right)\vec{v}$$

There are only two forces acting on the system, the external force from you lifting the rope and gravity, both of which are being applied along the y-axis:

$$\begin{aligned}\Sigma\vec{F} &= \vec{F}_{ext} + \vec{F}_g \\ &= \vec{F}_{ext} - m(t)g\hat{\jmath} \\ &= \vec{F}_{ext} - (\lambda y)g\hat{\jmath}\end{aligned}$$

where λy represents the amount of mass above the ground. The total mass of the rope is given by $m_{tot} = \lambda L$.

To solve for the external force, we apply Newton's second law. We don't now how quickly the mass is changing (we don't have $\dot{m}$, but we can use the general form of Newton's second law:

$$\sum \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

and solve for the equation of the momentum.

Momentum is defined as $\vec{p} = m\vec{v} = m(t)\vec{v}$ in this case. The mass at any given time, t, depends on the amount of rope that is lifted off the ground. If we call this height y, then the mass is $m(t) = \lambda y$ and the velocity is $\vec{v} = \dot{y}\hat{\jmath}$.

$$\vec{p} = \lambda y\dot{y}\hat{\jmath}$$

Taking the full time derivative of the momentum, we have:

$$\begin{aligned}\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda y\dot{y}\right)\hat{\jmath} \\ &= \lambda(\dot{y})^2\hat{\jmath} + \lambda y\ddot{y}\hat{\jmath} \\ &= \lambda(\dot{y})^2\hat{\jmath} \quad \Longrightarrow \quad \dot{y} \text{ is constant}\end{aligned}$$

Thus, we can can express F_{ext} from Newton's second law:

$$\begin{aligned}\Sigma\vec{F} &= \lambda(\dot{y})^2\hat{\jmath} \\ \vec{F}_{ext} - (\lambda y)g\hat{\jmath} &= \lambda(\dot{y})^2\hat{\jmath} \\ \vec{F}_{ext} &= \left(\lambda(\dot{y})^2 + \lambda yg\right)\hat{\jmath}\end{aligned}$$

A classic variable mass problem is a rocket (or car or airplane) using fuel. These objects lose mass with time, which will affect their momentum.

Sample Problem 6-5

A rocket has an initial mass of M before launch. To move the rocket, its engines burn fuel at a constant rate and expel the gases from the back at a speed of v_{ex} relative to the speed of the rocket. Ignoring gravity and drag, **find the speed v_f when the rocket mass has decreased from its initial mass to a mass of M_f?**

Solution

First, describe the motion. Figure 6.7 shows a cartoon of how the rocket is moving at time $t + \mathrm{d}t$. The rocket expels exhaust from its back and the gas is moving at a speed of v_{ex} in the $-x$ direction relative to the motion of the rocket (and exhaust) system.

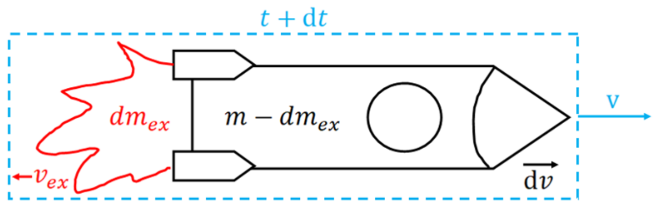


Figure 6.7: At time $t + \mathrm{d}t$, the rocket expels $\mathrm{d}m_{ex}$ of exhaust at a speed of v_{ex} relative to the speed of the rocket (v). As a consequence, the rocket is given an impulse and increases speed by $\mathrm{d}v$.

The rocket and exhaust form an isolated system (there are no other forces), so the total momentum of the system must be conserved. We will consider time t before there is any burning of fuel. So at t there is no contribution of momentum from the exhaust and all the momentum is from the rocket. We will assume that the rocket has a mass of m and velocity v. The total momentum at time t is therefore,

$$\vec{p}_{tot}(t) = mv\hat{\imath}$$

At time $t+\mathrm{d}t$, the rocket expels exhaust in the $-x$ direction. Under the conservation of momentum, the rocket must be given equal momentum in the $+x$ direction (this is the impulse given to the rocket that propels it forward). The rocket gains a bit of velocity $\mathrm{d}v$ and speeds up. But as the rocket is gaining velocity, it is also losing mass because it is using up fuel by expelling the exhaust. The mass of exhaust will equal the mass lost by the rocket (we need to obey the conservation of mass). The exhaust at time $t + \mathrm{d}t$ has a mass of $\mathrm{d}m_{ex}$ and a speed of $v_{ex}(-\hat{\imath})$ relative to the speed of the system as a whole (see Figure 6.7).

In the frame of a stationary observer at time $t + \mathrm{d}t$, the exhaust is moving a speed of $(v - v_{ex})\hat{\imath}$ and the rocket has a mass $m - \mathrm{d}m_{ex}$ and a velocity of $(v + \mathrm{d}v)\hat{\imath}$. The total momentum at time $t + \mathrm{d}t$ is therefore,

$$\vec{p}_{tot}(t + \mathrm{d}t) = \underbrace{(\mathrm{d}m_{ex})(v - v_{ex})\hat{\imath}}_{exhaust} + \underbrace{(m - \mathrm{d}m_{ex})(v + \mathrm{d}v)\hat{\imath}}_{rocket}$$

Since the total momentum of the rocket + exhaust system is conserved, the total momentum at time t must be equal to the total momentum at time $t + \mathrm{d}t$.

$$\begin{aligned}
\vec{p}_{tot}(t) &= \vec{p}_{tot}(t + \mathrm{d}t) \\
mv &= (\mathrm{d}m_{ex})(v - v_{ex}) + (m - \mathrm{d}m_{ex})(v + \mathrm{d}v) \quad \Longrightarrow \quad \text{all motion in 1D, drop } \hat{\imath} \\
mv &= v\mathrm{d}m_{ex} - v_{ex}\mathrm{d}m_{ex} + mv + m\mathrm{d}v - v\mathrm{d}m_{ex} - \mathrm{d}m_{ex}\mathrm{d}v \quad \Longrightarrow \quad \text{expand} \\
0 &= -v_{ex}\mathrm{d}m_{ex} + m\mathrm{d}v - \mathrm{d}m_{ex}\mathrm{d}v \quad \Longrightarrow \quad \text{simplify the terms} \\
0 &= -v_{ex}\mathrm{d}m_{ex} + m\mathrm{d}v \quad \Longrightarrow \quad \text{ignore the } \mathrm{d}m\mathrm{d}v \text{ term, it is very small} \\
\mathrm{d}v &= v_{ex}\frac{\mathrm{d}m_{ex}}{m}
\end{aligned}$$

This equation relates a differential velocity to a differential mass. But $\mathrm{d}m_{ex}$ corresponds to the increasing mass of the exhaust, whereas the m term in the above equation corresponds to the mass of the rocket. The increasing mass of the exhaust is equal to the decrease in mass in the rocket. In other words, $\mathrm{d}m_{ex} = -\mathrm{d}m$, where $\mathrm{d}m$ is the change in mass of the rocket.

$$\begin{aligned}
\mathrm{d}v &= -v_{ex}\frac{\mathrm{d}m}{m} \quad \Longrightarrow \quad \text{sub } \mathrm{d}m_{ex} = -\mathrm{d}m \\
\int_0^{v_f} \mathrm{d}v &= -v_{ex}\int_M^{M_f} \frac{\mathrm{d}m}{m} \quad \Longrightarrow \quad v_{ex} \text{ is a constant} \\
v_f - 0 &= -v_{ex}\Big(\ln m\Big|_M^{M_f} \quad \Longrightarrow \quad \text{initial is } v = 0,\ m = M \\
v_f &= -v_{ex}\left(\ln M_f - \ln M\right) \\
v_f &= v_{ex}\ln\left(\frac{M}{M_f}\right)
\end{aligned}$$

Thus, we found the equation for the velocity when the mass decreased from M to M_f. Note that v_f increases with time because M_f decreases with time.

6.6 Real-World Application

Recreational activities like air hockey, billiards (pool), and bumper cars are all built around the principle of collisions, whether elastic or inelastic. In general, these applications will always involve friction which will make the puck, balls, or car come to a halt given time. Each uses a different method to try to reduce that friction as far as possible: a layer of air to keep the puck off the table for air hockey, smooth paint on the balls and low-friction felt for billiards, and graphite sprinkled across a smooth metal floor for bumper cars.

On a larger scale, collisions and momentum conservation are also crucial for particle physics. The Large Hadron Collider (LHC) at CERN, routinely collides particle beams. The particle beams travel in opposite directions around a 27-km accelerator ring, guided and accelerated to very high energies (very close to the speed of light) using thousands of super-cooled superconducting magnets, before being made to collide. While operating at relativistic velocities and energies, the same basic physics of conservation of momentum and energy applies in these collisions.

The objective of studying these ultra-high-energy collisions is to understand more about matter and how the universe evolved. The LHC is able to simulate energy levels and temperatures similar to those that existed approximately 10^{-12} seconds after the Big Bang. In relativistic collisions between free particles, energy and momentum are always conserved. The LHC has detectors to measure the speed, mass, and charge of the post-collision particles, which enables them to identify new particles based on the fundamental requirement that momentum must be conserved.

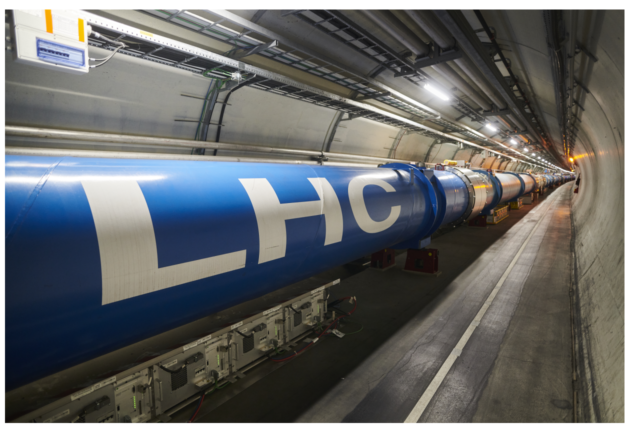


Figure 6.8: A very small section of the Large Hadron Collider tunnel. Image credit: CERN.

For more information:

See the CERN website on the LHC.

Let's Talk Science has a nice introduction to momentum and billiards, with links to videos.

6.7 Summary

Key Takeaways

This chapter describes linear momentum for systems of particles in more detail. Linear momentum is defined as $\vec{p} = m\vec{v}$. For a system with only internal forces, the total linear momentum is conserved.

$$\sum \vec{p}_i = \text{constant}$$

Newton's second law is also more formally defined based on the total linear momentum. If your system mass is allowed to change, then the full equation for Newton's second law is:

$$\sum \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \left(\frac{\mathrm{d}m}{\mathrm{d}t}\right)\vec{v} + m\left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)$$

In systems where the total linear momentum is conserved, the net force is zero. If the system has variable mass, then one must use the general form of Newton's second law to solve for the equations of motion.

This chapter also defines impulse and collisions, which is how momentum of a system can change. Impulse is when an external force acts on a system over a short time duration such that the system has not had time to move between the start of the event and the end.

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}\mathrm{d}t = \Delta\vec{p}$$

Collisions are when momentum is exchanged between objects in a system. There are two types, inelastic collisions (the objects stick together) and elastic collisions (the objects rebound away). If the total linear momentum from the collisions is conserved, then we can say:

$$\vec{p}_i = \vec{p}_f$$

An important property for systems of particles in the centre of mass. The centre of mass is a special radius vector that represents the mass-weighted average position vector.

$$\vec{R}_{cm} = \frac{\sum m_i \vec{r}_i}{M}$$

Large or complex systems of particles can be simplified to a single giant particle at the position of the centre of mass that is moving with the centre of mass. The centre-of-mass reference frame can also be helpful in simplifying problems, because the net linear momentum of the centre-of-mass frame is zero by definition.

Important Equations

Linear Momentum:

$$\vec{p} = m\vec{v}$$

Newton's Second Law:

$$\sum \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\vec{v})}{\mathrm{d}t} = \left(\frac{\mathrm{d}m}{\mathrm{d}t}\right)\vec{v} + m\left(\frac{\mathrm{d}\vec{v}}{\mathrm{d}t}\right)$$

Conservation of Momentum:

$$\vec{p}_{tot} = \sum \vec{p}_i = \text{constant}$$

Momentum with an External Force:

$$\frac{\mathrm{d}\vec{p}_i}{\mathrm{d}t} = \sum \vec{F}_i = \vec{F}_{i,int} + \vec{F}_{i,ext}$$

Impulse:

$$\vec{I} = \int_{t_1}^{t_2} \vec{F}\mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\mathrm{d}t = \int_{t_1}^{t_2} \mathrm{d}\vec{p} = \vec{p}_2 - \vec{p}_1 = \Delta\vec{p}$$

Centre of Mass:

$$\vec{R}_{cm} = \frac{\sum m_i\vec{r}_i}{M}$$

Momentum of a System of Particles:

$$\vec{p} = \sum m_i\vec{v}_i = \sum m_i\frac{\mathrm{d}\vec{r}_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\sum (m_i\vec{r}_i)$$

Momentum of Particles with Fixed Masses:

$$\vec{p} = M\frac{\mathrm{d}\vec{R}_{cm}}{\mathrm{d}t} = M\vec{v}_{cm}$$

6.8 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 6-1

Three identical particles are moving as follows:

Particle 1: $2\hat{\imath}$

Particle 2: $\hat{\jmath}$

Particle 3: $\hat{\imath} + \hat{\jmath} + \hat{k}$

Where all values are in SI units. What is the centre-of-mass velocity v_{cm} of this system?

Practice Problem 6-2

A particle with mass M_1 and velocity $v_1\hat{\imath}$ collides with a particle of mass M_2, initially at rest.

a) After the collision, M_1 is at rest. What is the velocity of M_2?

b) After collision, the two particle stick together and continue in the same direction. What is their velocity?

Practice Problem 6-3

Two particles of mass M_1 and M_2 collide. Before the collision, the first particle has a speed of $v_1 = 4\hat{\imath} - 3\hat{\jmath}$ and the second particle has a speed of $v_2 = 4\hat{\imath} + 3\hat{\jmath}$.

a) If both particles have the same mass, what is the speed of M_1 if M_2 comes to a stop after the collision?

b) Consider that $M_2 = 2M_1$ and the collision is instead perfectly inelastic (particles stick together). What is their new velocity?

Practice Problem 6-4

A student is late running from Depuis Hall to Stirling Hall. As they run past the speedometer on University Ave, they notice that they are running at 18 km/h. They then run into another student who is waiting for the bus. (Assume that the running student weighs 65 kg).

a) What is the impulse required for the student waiting for the bus to stop the running student and not fall over?

b) If this impulse is delivered to the student in 0.20 seconds, then what is the magnitude of the force acting between the stationary student and the running student?

Practice Problem 6-5

A bullet traveling at a velocity of $v\hat{\imath}$ is shot through a stationary block of wood head on. When it emerges from the other side, the bullet has lost half its speed. If the block has a mass of M and the bullet has a mass of m, what is the velocity of the block of wood after the bullet emerges? (Ignore any friction or loss of energy. Assume the block of wood loses no mass.)

Practice Problem 6-6

An artillery shell is launched upwards with a speed of v_0 at an angle of θ with respect to the ground. When it reaches its maximum point, the shell explodes into two fragments of equal mass. One of the fragments goes straight up at a speed of $v_0/2$. What is the speed of the other fragment?

Practice Problem 6-7

A Velcro block target of mass M hangs from the cross-bar of a hockey net with an ideal rope of length L. Iconic hockey player Wayne Gretzky's slap-shot fires a puck (covered in Velcro), of mass m_p straight into the block target and becomes stuck to the block. The block is initially at rest and the puck has an initial speed of v_0. The impact causes the block to oscillate with a maximum angle of θ_{max} from the vertical.

a) Find the speed of the block at the moment the puck becomes stuck to it.

b) Find the amplitude (x_{max}) of the simple harmonic motion of the block. You can assume that the block is like a point mass at the bottom of the rope.

c) Find the maximum angle, θ_{max} from the vertical. You can assume that θ_{max} is a small angle.

d) Consider the case where $m_p = 160$g, $M = 100$g, $L = 1.4$m, and $\theta_{max} = 14$ degrees. What is the initial speed of the puck? Does that value make sense?

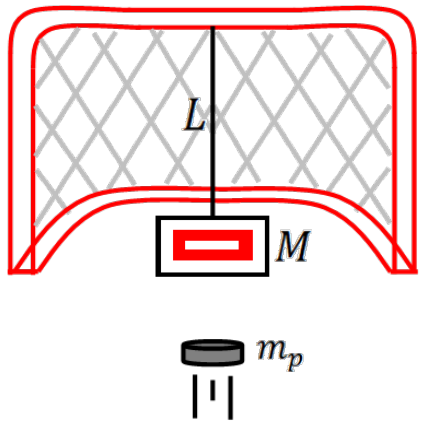


Figure 6.9: The puck of mas m_p is fired at the target.

Practice Problem 6-8

On a strange alien planet, Grog has invented a rudimentary car by attaching four wheels to a long piece of wood. Grog then fixes another piece of wood vertically to the front of it and operates the car by throwing 0.25-kg rocks perfectly horizontally at 20 m/s to bounce off the vertical piece. The rock rebounds with only one quarter of its original speed (Grog ducks) and lands behind the wooden car. If there is no loss of energy to friction, what is Grog's speed after throwing 10 rocks? Grog's mass is 70 kg and the wheeled vehicle has a mass of 30 kg. Neglect the mass of the rocks still on the car after each throw.

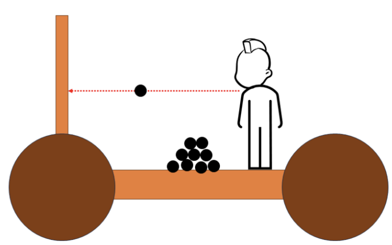


Figure 6.10: Grog's rudimentary car.

Note, this problem could be considered a variable mass question as the loss of the rocks ends up changing the mass of the car. If you want to challenge yourself, trying solving the problem including the mass loss.

Practice Problem 6-9

See the figure below. A mass m_1 is moving toward a second mass, m_2, with a speed of u. The second mass m_2 is stationary and connected to a spring (see figure). After m_1 collides with m_2 they stick and compress the spring.

a) Find the centre-of-mass velocity of the system after collision.

b) Use Newton's laws to find the differential equation of motion of velocity for the spring.

c) Solve the differential equation of motion to find the maximum compression of the spring. (Hint, you want to find a maximum displacement, x, corresponding to a specific change in velocity.)

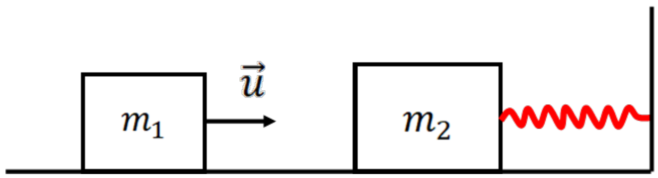


Figure 6.11: Set up with m_1 approaching m_2 at velocity $\vec{u}$.

Practice Problem 6-10

A car of mass M is moving at an initial speed of v_0 when it snags a heavy rope sitting in a pile on the road. The rope has a mass density of λ (in kg m^{-1}) and length L. Assume the rope starts off at rest and only a small piece attaches initially to the car and that the rope does not deform during the motion. Assume there is no friction acting on the rope.

a) What is the mass of the car after a length x of rope has uncoiled?

b) What is the speed of the car after a length x of rope has uncoiled? Assume that the driver of the car does not exert any change in force that would affect the speed.

c) What is the tension in the rope in the piece of rope that is right next to the pile on the ground

Practice Problem 6-11

A rocket ship of mass M_0 drifts in space with a speed of v_0. At time $t = 0$, it drifts into a dust cloud that is stationary. The cloud has a volume density of ρ (in kg m^{-3}) and dust from the cloud sticks to the rocket over its cross-sectional area, A.

a) Show that the change in mass for the rocket is given by $\dfrac{\mathrm{d}m}{\mathrm{d}t} = A\rho v$ at time t.

b) What is the mass of the rocket when it is moving at speed v?

c) Solve for $v(t)$. Hint, use Newton's second law to set up a differential equation in terms of v and t.

7 Angular Momentum and Torque

Learning Objectives

- Introduce angular momentum and rotational dynamics
- Introduce torques and identify Newton's second law for torques
- Solve equations of motion with simple torques
- Solve equations of motion with both rotational and translational motion

In this chapter, we will discuss rotational motion in the context of angular momentum and torques, and we will apply Newton's Law's to force problems that involve rotation.

7.1 Angular Momentum

In Chapter 6, we introduced linear momentum, $\vec{p} = m\vec{v}$. But objects can also move by rotation (spinning), and the momentum of a body as it undergoes rotation is called *angular momentum*. We define the angular momentum of a particle as:

$$\boxed{\vec{l}_i = \vec{r}_i \times \vec{p}_i} \tag{7.1}$$

where $\vec{r}_i$ is the position of the particle relative to the origin, and $\vec{p}_i$ is the momentum of that particle. The total angular momentum of a system of particles is the sum of all the particles angular momentum's:

$$\boxed{\vec{L} = \sum \vec{l}_i = \sum (\vec{r}_i \times \vec{p}_i)} \tag{7.2}$$

Note: The angular momentum is a vector quantity. The direction of the vector is given by the vector cross product of $\vec{r}$ and $\vec{p}$ (see Chapter 1.5.2 for a review of cross products).

Angular momentum is tied to circular motion. It describes any motion where there is rotation about an axis or arc-like movement. For example, an object moving in a straight line will have no angular momentum because $\vec{r}_i$ is parallel to $\vec{p}_i$ (cross product is zero). An object also has no angular momentum if it is stationary ($\vec{p}_i = 0$) or it is at the origin ($\vec{r}_i = 0$).

Like linear momentum ($\vec{p}$), the angular momentum in an isolated system is conserved.

7.2 Rotational Dynamics

7.2.1 Rotational Dynamics from Newton's Laws

Another form of Newton's laws comes from the conservation of angular momentum rather than the conservation of linear momentum. With Newton's second law, the change in linear momentum with time is equal to the net force acting on the system.

$$\sum \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$$

Now, consider the time derivative of angular momentum. For simplicity, let's look at a single particle of mass m_i located at a distance r_i from the origin:

$$\begin{aligned}
\frac{\mathrm{d}\vec{l}_i}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r}_i \times \vec{p}_i) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}[\vec{r}_i \times (m_i\dot{\vec{r}}_i)] \quad \Longrightarrow \quad \text{assume mass is constant} \\
&= m_i(\dot{\vec{r}}_i \times \dot{\vec{r}}_i + \vec{r}_i \times \ddot{\vec{r}}_i) \quad \Longrightarrow \quad \text{apply the time derivative to each term} \\
&= m_i(0 + \vec{r}_i \times \ddot{\vec{r}}_i) \quad \Longrightarrow \quad \text{the cross product of two identical vectors is zero} \\
&= \vec{r}_i \times (m_i\ddot{\vec{r}}_i) \quad \Longrightarrow \quad \text{mass is a constant, so you can put it anywhere} \\
&= \vec{r}_i \times \vec{F}_i \quad \Longrightarrow \quad \text{recall that } F = ma = m\ddot{r} \text{ for constant mass}
\end{aligned}$$

We get that the time derivative of the angular momentum equals to the cross product of $\vec{r}$ and $\vec{F}$. This cross product is also known as the torque, $\vec{\tau}$.

$$\boxed{\vec{\tau}_i = \vec{r}_i \times \vec{F}_i} \tag{7.3}$$

Now, if we have a collection of particles, then we need to sum up all their individual contributions. This yields:

$$\boxed{\sum \vec{\tau}_i = \sum \frac{\mathrm{d}\vec{l}_i}{\mathrm{d}t} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}} \tag{7.4}$$

where $\vec{L}$ is the total angular momentum of a system. Thus, we find that the net torque acting on a system is equal to the time derivative of the total angular momentum of that system. Equation 7.4 is Newton's second law for rotation.

If you have a rigid body instead of a system of independent particles, then all the mass elements in the body will rotate together with the same angular velocity, ω, and angular acceleration, α (e.g., the body does not deform). The magnitude of the total angular momentum, L, of a body is:

$$\boxed{L = I\omega} \tag{7.5}$$

where I is the moment of inertia of the system of particles (see Section 7.2.2 for details).

Combining our equation for the total angular momentum (Equation 7.5) with the equation for the net torque (Equation 7.4), we get:

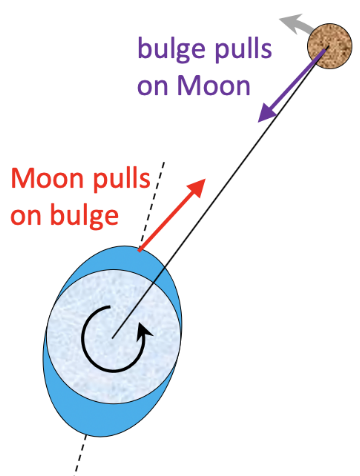
$$\begin{aligned} \sum \vec{\tau} &= \frac{\mathrm{d}\vec{L}}{\mathrm{d}t} && \implies \text{The vector notation is dropped to give magnitudes only} \\ &= I\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} && \implies \text{Note that } I \text{ is constant with time (system does not deform)} \\ &= I\vec{\alpha} \end{aligned}$$

So similarly to $F = ma$ for the net force, we have $\tau = I\alpha$ for the net torque.

Quick Question

1. Consider a planet in a circular orbit around a star with only the gravitational force acting on the planet. What is the equation for the net torque? Use $\sum \tau = I\alpha$ and $\vec{\tau} = \vec{r} \times \vec{F}$.

Torques in the Earth-Moon System



You've probably heard that the Moon causes tides on Earth. The tides don't occur right when the Moon is overhead, instead the tides are ahead of the Moon. As a consequence, the Earth's tides bulge at an angle to the Moon. This bulge pulls on the Moon and the Moon in return pulls on the bulge (equal and opposite reactions), see figure (not to scale). Because these tiny forces are at an angle relative to the Earth-Moon radial line, they will each cause a torque. The torque on the Moon pulls the Moon ahead in its orbit slightly (the Moon gains momentum), whereas the torque on Earth drags the Earth slightly back in its spin (the Earth loses momentum). This is a case of angular momentum conservation! The net effect is very small, but the Moon is slowly moving away from us (at a rate of $\sim$ 40 mm per year) and the Earth's day is slowing increasing (by $\sim$ 2 ms per century). For more information, see the Wikipedia webpage and Explaining Science's webpage on tidal acceleration and the day on Earth.

7.2.2 Moment of Inertia

The moment of inertia, I, is an important quantity in rotation. It represents how the mass of the system is distributed as a function of position and describes how efficiently the system can be rotated. It is defined as:

$$\boxed{I = \sum_i m_i r_i^2} \tag{7.6}$$

where m_i is the mass of a tiny piece of the system and r_i is the distance between that mass and the rotation axis (pivot point) of that system.

For example, Figure 7.1 shows an irregular shaped mass that is free to rotate back and forth about a pivot point toward its top. The position vector $\vec{r_i}$ is defined for each mass element in the object as measured from the pivot point. You must sum up all mass elements to measure the full moment of inertia for any object. Note that any mass elements at the position of the rotation axis have zero contribution to the moment of inertia because $\vec{r_i} = 0$.

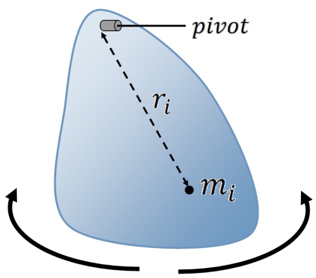


Figure 7.1: Definition of the moment of inertia. This object will rotate about the fixed pivot point. A tiny section of mass m_i is located a distance r_i from that pivot point. The total moment of inertia of this system is $I = \sum (m_i r_i^2)$ for the whole system.

Quick Questions

1. Try calculating the moment of inertia for a simple geometric shape like a uniform ring or disk for an axis through the centre (see Appendix A.4 for the solutions to many shapes).
2. Show that $L = I\omega$ from $\vec{L} = \vec{r} \times \vec{p}$ for circular motion. Hint: Recall that $\vec{r} \perp \vec{p}$ for circular motion.

Changing your moment of inertia

You can change the moment of inertia by changing the distribution of mass. For example, you could re-arrange your mass. A figure skater is an example of this. When they spin with their arms out, their moment of inertia is at its highest because they have arranged their mass (their arms) at larger radii, $I = \sum (m_i r_i)$. Conversely, when bring their arms in, their moment of inertia is smaller. Since their total angular momentum L must be conserved, when the skater's arms are out, their speed will be slower and when their arms are tucked in, their speed will be faster (L = constant = $I\omega$, so if I increases, ω decreases and vice versa). See also:
Video connecting rotation with the moment of inertia
Video on figure skating

See Appendix A.4 for a chart of basic shapes and their moments of inertia. Most of these equations are relative to a rotation axis through the centre of mass, whereas in practice, the rotation axis could be at a different location. If you change the location of the rotation axis, you can also change the mass distribution and the moment of inertia. We can calculate the new moment of inertia using the *parallel axis theorem*.

Equation 7.7 gives the parallel axis theorem. Consider an object that has a moment of inertia about its centre of mass of I_{cm}. If you were to pivot that object at a point P that is a distance d from the centre of mass, the moment of inertia about point P would be.

$$\boxed{I_p = I_{cm} + Md^2} \tag{7.7}$$

where I_p is the moment of inertial about P and M is the total mass of the object.

Lance's Thoughts

The power of the parallel axis theorem shines through when you have a strange or unusual object that you can break into parts with individually easy moments. Once you've got those, you essentially stack them using the parallel axis theorem and add them all up. Don't forget to calculate the centre of mass, too.

7.2.3 Direction of Torque and Angular Momentum

Torque and angular momentum are vectors, where their directions are defined by a vector cross product, which makes finding their directions more challenging. There are several ways to get the directions. First, you can use the RHR or matrix determinant to get the direction from the definition of each vector, e.g., $\vec{\tau} = \vec{r} \times \vec{F}$. See Chapter 1.5.2 for a review of the vector cross product.

Second, you can use the RHR for rotation to connect the rotation of a system to the direction of its torque or angular momentum vectors. To apply the RHR for rotation, curl your fingers in the direction or rotation and your thumb will point in the direction of the torque vector (see also Figure 7.2). Note you can also use the RHR for rotation to get the direction of rotation if you know the direction of torque.

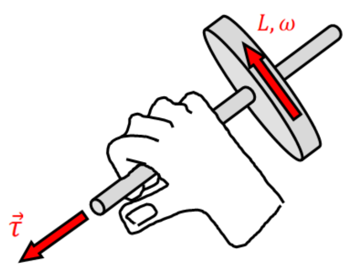


Figure 7.2: Right hand rule for connecting the direction of rotation with the direction of the torque vector.

If the torque vector points out of the page (e.g., toward you), then the system is rotating counter-clockwise. If the torque vector points into the page (away from you), then the system is rotating clockwise. When solving problems, you will want to define which of these two rotation directions (counter-clockwise versus clockwise) is positive.

Cora's Thoughts

Another way to think of torques is in the context of screws. When twisting a screw clockwise, it gets tighter and moves into the page which is the direction of that torque. When twisting a screw counterclockwise it loosens and moves out of the page, which it the direction of its torque. The direction of the movement of a screw it the same as the direction of its torque.

Figure 7.3: The motion of the screws can be remembered by the old axiom "righty-tighty and lefty-loosey.

7.3 The Pendulum Revisited

Let's return to the pendulum program from Chapter 3.4, but this time, we're going to solve it using torques and angular motion instead of forces and linear motion. Here is our sketch of the pendulum and the free-body diagram from before.

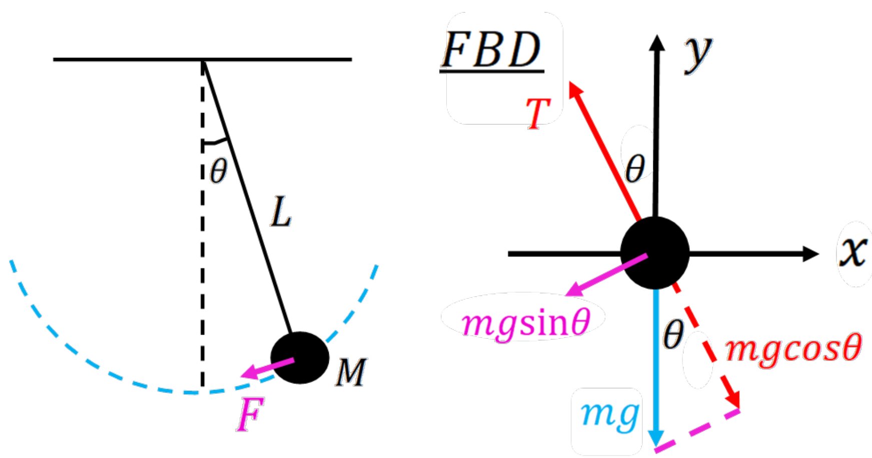


Figure 7.4: Example of a simple pendulum. Left: The mass is in equilibrium when it is vertically downward and displaced from equilibrium when shifted an angle θ from the vertical axis. A restoring force (F) moves the pendulum back to equilibrium. Right: The free-body diagram shows the labeled forces tension (T) in red, gravity (mg) in blue, and the restoring force ($mg\sin\theta$) in magenta. Shown in dotted-red is the component of gravity that balances tension ($mg\cos\theta$).

The restoring force acting on the pendulum is given by $F = -mg\sin\theta$. The torque acting on the pendulum is then $\vec{\tau} = \vec{r} \times \vec{F}$. So we need to find $\vec{r}$, $\vec{F}$, and the angle between them.

Quick Question

1. What is the direction of the torque vector for the pendulum shown in the above figure? Use the RHR to find it.
2. Describe the torque vector through one full period of the pendulum's motion. What is the torque at the maxima versus the equilibrium point?

For our simple pendulum, $r = L$ is the distance from the pivot point to where the force is applied, which is fixed. F is the restoring force, $F = -mg\sin\theta$. By definition, the restoring force is perpendicular to the radius vector (the restoring force is given by the component of gravity that is perpendicular to the radius vector along the string, see Figure 7.4). If $\vec{r} \perp \vec{F}$, then $\tau = rF$. Putting this information in, we have:

$$\begin{aligned} \tau &= |\vec{r} \times \vec{F}| \\ \tau &= rF \\ \tau &= -mgL\sin\theta \end{aligned}$$

From Newton's second law for rotation, the net torque is equal to:

$$\boxed{\sum \tau = I\alpha} \tag{7.8}$$

Since there is only one torque acting on the system (from the restoring force),

$$\begin{aligned} I\alpha &= -mgL\sin\theta \\ \alpha &= -\frac{mgL}{I}\sin\theta \\ \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} &= -\frac{mgL}{I}\sin\theta \end{aligned}$$

This form of the differential equation of motion is difficult to solve. But, we can make it solvable by assuming that the angle formed by the pendulum and the vertical axis is small ($\theta \ll 1$ in radian units). If θ is small, then $\sin\theta \approx \theta$ (See Appendix B) and

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{mgL}{I}\theta = 0$$

Now our equation is in the form of a simple differential equation of motion (see Chapter 3), and we know how to solve an equation in this format. The solution for $x(t)$ is a cos function with an angular frequency given by the coefficient in from of the θ term.

But wait, that isn't the exact same solution as what we had before in Chapter 3.4. Well, there is one more step we need to do. We need to define the moment of inertia, I.

For a point mass located a distance L from the pivot point, the moment of inertia is just $I = mL^2$. If you plug in $I = mL^2$ into the differential equation of motion, we get:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{L}\theta = 0$$

which is exactly the same as what we had before in Chapter 3 and it once again gives us an angular frequency of $\omega_0 = \sqrt{\frac{g}{L}}$).

Force vs Torque

The torque method gives $\ddot{\theta} + \frac{mgL}{I}\theta = 0$, which is a more general solution to describe the motion of a pendulum. This solution holds for a single pendulum of any shape. If you can write down the moment of inertia for that pendulum, you can solve its equation of motion. See Appendix A.4 for basic geometric shapes. Or you can find a table of solutions online.

Complex pendulum shapes are hard to solve with the linear force method. Think about the problem you are trying to solve, and consider using torques rather than forces!

Quick Questions

1. Find the angular frequency, ω_0, of a pendulum that consists of a rod hanging from the pivot point at one end. Assume the rod has mass M and length L.
2. You construct a pendulum by attaching a ring to a massless rod and setting it into periodic motion. Find the angular frequency, ω_0, of this pendulum if the rod has a length L, and the ring has a mass M and radius R.
3. You have a massless rod of length L to which you can attached either a solid sphere or a spherical shell. The solid sphere and spherical shells have masses and radii of (1) M, R, (2) $2M$, $\frac{1}{2}R$, or (3) $\frac{1}{2}M$ $2R$. Which object will give your pendulum the shortest period and which will give you the longest period of oscillation?

7.4 The Physical Pendulum

By definition, a physical pendulum is any rigid body that is free to swing about a pivot point. Figure 7.5 is an example of a physical pendulum.

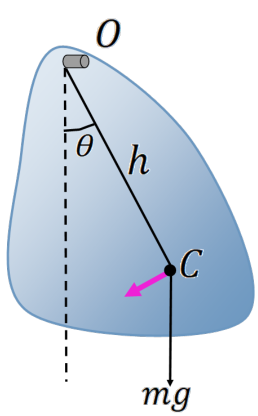


Figure 7.5: A physical pendulum. The body is suspended from the point O and allowed to rotate freely by an angle θ. The centre of mass of the system C is located a distance h from the pivot point. The total mass of the objects is m. The purple arrow shows the restoring force acting on this pendulum.

Although the object has an irregular shape, the problem can be simplified by expressing the motion for the centre of mass rather than for each individual mass element of the object.

You can think of this as compressing the mass of the entire object to a single point located at the centre of mass (point C) and then determining how the restoring force acts on that compressed object. The force acting on this physical pendulum is $F = -mg \sin\theta$ at the position C. This simplification is another strength of the centre of mass.

The torque acting at the centre of mass is given by $\vec{\tau} = \vec{r}_{cm} \times F$. The $\vec{r}_{cm}$ vector is the vector from the pivot point to the centre of mass. We know that $\vec{r}_{cm} \perp F$, which means that our torque has a magnitude of $\tau = r_{cm}F = -mgh\sin\theta \approx -mgh\theta$ for small angles.

If this is the only torque acting on our system, Equation (7.8) becomes:

$$\begin{aligned}
\sum \tau &= I\alpha \\
I\alpha &= -mgh\theta \\
0 &= I\alpha + mgh\theta \\
0 &= \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{mgh}{I}\theta
\end{aligned}$$

This is the exact same equation of motion as the simple pendulum, only that the simple pendulum had the length of the rope to the mass, L, and the physical pendulum has the distance between the pivot and the centre of mass h.

So for a physical pendulum of any shape swinging from a pivot point that is a distance h from its centre of mass, we find that the motion can be described with an angular frequency of $\omega_0 = \sqrt{\frac{mgh}{I}}$, where h is the distance to the centre of mass and I is the moment of inertia for the body. Note, that for an object to be a physical pendulum, the pivot point must be located away from the centre of mass (at the centre of mass, $h = 0$).

7.5 Example of a Physical Pendulum

In this example, we will determine the equation of motion for a physical pendulum corresponding to a single simple harmonic oscillator.

Sample Problem 7-1

A disk of mass m_d and radius R is attached to a rod of mass m_r and length L. **What is the period of oscillations if this object is set in motion about the other end of the rod?** See Figure 7.6.

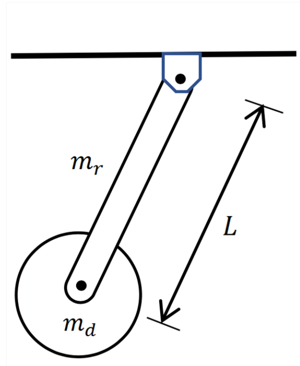


Figure 7.6: Diagram of the physical pendulum. This physical pendulum is constructed from a disk and rod. The disk is attached to the rod at one end and allowed to rotate freely at the other end of the rod. The disk has mass m_d and radius R. The rod as mass m_r and length L.

Solution

This system is not a simple pendulum (e.g., a point source at the end of a rope), because the rod has mass and disk has mass and dimension. So you need to consider this as a physical pendulum.

The solution for a physical pendulum is:

$$0 = \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{Mgh\theta}{I}$$

where h is the distance to the centre of mass, $M = m_r + m_d$ is the total mass of the system, and I is the moment of inertia for the system (see Chapter 7.4). The solution is a cos function with an angular frequency of $\omega_0 = \sqrt{\frac{Mgh}{I}}$. So the solution for the period of rotation is:

$$\begin{aligned} T &= \frac{2\pi}{\omega_0} \\ T &= 2\pi\sqrt{\frac{I}{Mgh}} \end{aligned}$$

Getting the equation for the period isn't the hard part. The trick for this problem is defining h and I.

Let's start with h, which is the distance from the pivot to the centre of mass of the pendulum. Since both the rod and the disk have mass, the centre of mass of the two combined is located at a mass-averaged position between the two. We will need to calculate the position of the centre of mass (see Chapter 6.4 for a definition of the centre of mass).

Fortunately, the centre of mass for each component of the pendulum is easy to calculate. The centre of mass for a uniform rod would be its midpoint and the centre of mass for a uniform disk would be its midpoint. For the rod, $r_{cm,r} = \frac{L}{2}$ (location of the midpoint of the rod from the pivot) whereas for the disk, $r_{cm,d} = L$ (location of the midpoint of the disk from the pivot). So we can treat both systems as effective point masses with all their mass at the respective centre-of-mass positions.

Thus, the centre of mass for this physical pendulum is:

$$h = \frac{m_r r_{cm,r} + m_d r_{cm,d}}{m_r + m_d}$$
$$h = \frac{m_r(\frac{1}{2}L) + m_d L}{M}$$

where $M = m_r + m_d$ is the total mass of the pendulum.

Thus, we have a position for our centre of mass. Note that if our rod mass is very small (e.g., $m_r \to 0$), then $M \to m_d$ and $r_{cm} \to L$, or the centre of the disk. This recovers the solution for a simple pendulum.

Now let's look at I. We have two objects, a rod and a disk. To get the moment of inertia for the combined rod+disk pendulum, we can simply add the I components from each object separately.

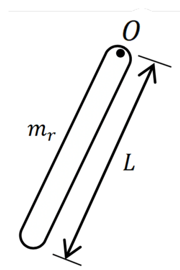


Figure 7.7: Sketch of the rod with the pivot at one end.

The moment of inertia for a rod with the axis of rotation at one end is (Appendix A.4):

$$I_{rod} = \frac{1}{3} m_r L^2$$

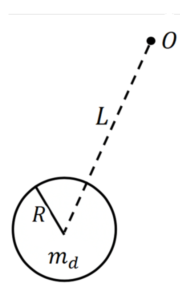


Figure 7.8: Sketch of the disk with the pivot a distance L from the centre of mass.

The moment of inertia for a disk with the axis of rotation through its centre of mass is (Appendix A.4):

$$I_{disk} = \frac{1}{2} m_d R^2$$

But the pivot is not located at the centre of mass. The pivot is located a distance L from the centre of mass. Therefore, we need to find I for the disk about the pivot point, O. Using the parallel axis theorem (Chapter 7.2.2), we have,

$$I_{disk,0} = \frac{1}{2} m_d R^2 + m_d L^2$$

Thus, the moment of inertia for the entire physical pendulum is

$$I = I_{rod} + I_{disk,0} = \frac{1}{3} m_r L^2 + \frac{1}{2} m_d R^2 + m_d L^2$$

Taking our equations for h and I, we can now solve for the period:

$$\begin{aligned} T &= 2\pi \sqrt{\frac{I}{Mgh}} \\ &= 2\pi \sqrt{\frac{\frac{1}{3} m_r L^2 + \frac{1}{2} m_d R^2 + m_d L^2}{(Mg)\frac{1}{M}(\frac{1}{2} m_r L + m_d L)}} \qquad \Longrightarrow \text{note } M \text{ cancels in denominator} \\ &= 2\pi \sqrt{\frac{(\frac{1}{3} m_r + m_d) L^2 + \frac{1}{2} m_d R^2}{gL(\frac{1}{2} m_r + m_d)}} \end{aligned}$$

Quick Question

1. Find the period if the disk was attached at the midpoint of the rod instead.

7.6 Example with Rolling Motion

In this case, we will consider a system rolling on a surface. In ideal cases, rolling motion occurs without slipping, which means that friction at the point of contact between the rolling object and the surface is sufficient to keep the system moving continuously by rolling motion. If the system is slipping, then you can get forward motion without rolling.

Sample Problem 7-2

A light cord is wrapped around the inner drum of a wheel of mass m and pulled with a constant force F to make the wheel roll. The wheel has a radius R and the inner drum has a radius of r. If the wheel rolls without slipping, **what is the force of friction at the point of contact between the wheel and horizontal surface?**

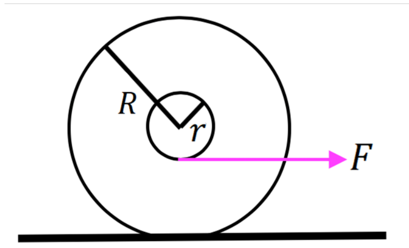


Figure 7.9: The applied force (F) is represented by the magenta arrow, (r) is the radius of the inner drum, and (R) is the radius of the wheel.

Solution

First, make sure you know how this system will move. Pulling the cord in the direction shown will cause the wheel to rotate. Will it rotate clockwise or counter-clockwise? Well, let's draw a free-body diagram to describe the forces at play in this motion and figure out how this system is going to move.

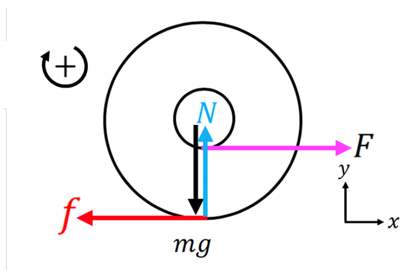


Figure 7.10: Free-body diagram of the system. The force F is applied to the wheel (magenta). Also acting on the wheel are gravity (black), the normal force (blue), and friction (red). The positive x and y axes are shown. We have also defined the clockwise rotation direction as positive.

Overall, the wheel is moving forward in the same direction as F but it is also rotating. So there are two types of motion we need to consider, translation for the forward movement and rotation for the spin of the wheel. This means we will need to consider both forms of Newton's second law.

$$\begin{aligned} \sum F &= ma_{cm} \quad \Longrightarrow \text{for translation} \\ \sum \tau &= I\alpha \quad \Longrightarrow \text{for rotation} \end{aligned}$$

Note that for the translation motion, we're interested only in how the centre of mass is moving. That's because the centre of mass has no rotation motion, only linear motion.

Let's set up our coordinate system. We define $+x$ toward the right, $+y$ up, and $+\omega$ in the clockwise direction. These choices are intentional. If the linear motion is in the $+\hat{\imath}$, then the rotation should be in the clockwise direction. While we set up the coordinate system to be most intuitive, as long as you are consistent with your defined coordinate system you will still get the correct answer.

Let's look at $\sum F = ma_{cm}$ to start. What forces do we need to worry about for the forward motion? Both gravity and the normal force act along the y-axis. These will be equal and opposite forces. The wheel does not rise above the ground nor does it sink below the ground. So we only care about F and f. These are opposite in direction (Figure 7.10). Based on our definition of the $+x$ axis, we have $\sum F = F - f$.

What about a_{cm}? Keep in mind that the acceleration corresponds to the bulk forward motion of the system. If the wheel was a square box that didn't rotate, then a_{cm} would be how fast you were able to drag the box. But the magnitude of a_{cm} depends on the rate of rotation because all the motion happens due to rotation (condition of rolling without slipping).

Figure 7.11 shows a schematic of our rolling wheel. The wheel is rolling forward a distance s represented by the red arc. As a result of moving forward, the centre of mass has changed position from x_1 to x_2, where $\Delta x = s$ (the translation motion is relative to the ground). That is, the system goes forward an equal distance given by the arc of the circle traveled.

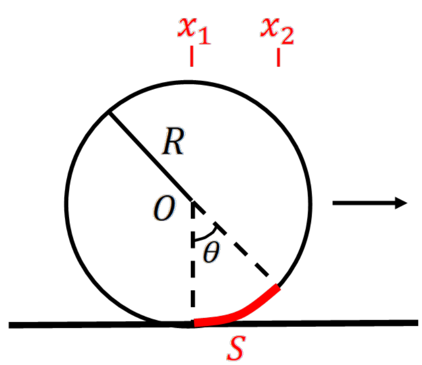


Figure 7.11: The rolling wheel of radius R. The centre of mass is given by the origin (O) and the system moves forward a distance s given by the red arc.

If the system has moved a distance s in time Δt. If you have *rolling without slipping*, then the centre of mass motion is given by $v_{cm} = \dfrac{\Delta x}{\Delta t} = \dfrac{s}{\Delta t}$. For very small times,

$\Delta t \rightarrow \mathrm{d}t$, v_{cm} and a_{cm} can be instead written as:

$$\begin{aligned} v_{cm} &= \frac{\mathrm{d}s}{\mathrm{d}t} \\ a_{cm} &= \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \end{aligned}$$

But the arc length, s can be written in terms of the angle θ and the radius R. That is, $s = R\theta$. Substituting $s = R\theta$ into our acceleration equation gives:

$$a_{cm} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 R\theta}{\mathrm{d}t^2} = R\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = R\alpha$$

The above equation gives the magnitude of a_{cm} in terms of the rotation.

Assuming positive clockwise rotation, we need to check whether our α is also clockwise. To get the rotation direction, note that rolling happens at the point of contact where the base of the wheel meets the ground. There is only one force acting at the contact point (friction) and friction will point against the direction of motion. Using the RHR for rotation, the torque produced by the friction force is into the page and the rotation of the wheel will be clockwise. So α is in the clockwise direction and positive based on our definition.

Thus, combining the values of $\sum F$ and a_{cm}, we get:

$$\begin{aligned} \sum F &= ma_{cm} \\ F - f &= mR\alpha \end{aligned}$$

Definitions

Just like with x and y, you need to define a positive and negative direction for rotation. And it is important that you stay consistent with that choice. For this example, we defined the positive axis as clockwise. By this definition, we get $a_{cm} = R\alpha$. But had we defined the positive rotation axis as counter-clockwise, then we would need to set $a_{cm} = -R\alpha$, because in this case, the rotation of the system would be counter to the defined axis. Either way is fine, just be consistent.

Here is a video demonstration that shows positive and negative rotation.

A key feature of this problem is the condition of rolling without slipping. This condition specifies that the rotation is entirely responsible for any forward motion such that the rotation rate can be equated to the centre of mass motion. Under this condition,

$$v_{cm} = \omega R$$
$$a_{cm} = \alpha R$$

where ω is the angular velocity and α is the angular acceleration. Note that the vector directions are not the same for these quantities. Only the magnitudes apply.

For a rigid object, v_{cm} applies equally in magnitude and direction to the whole object (it is moving forward and doesn't deform), whereas the motion from rotation depends on the radius and can be either forward or backwards. Consider the motion from translation and rotation at the contact point (where the wheel meets the ground). There are two velocities acting at that point, the translation velocity from v_{cm} and the rotation velocity, $R\omega$. These two velocity are equal in magnitude, but opposite in direction (at the contact point, the wheel is moving forward with v_{cm} but backwards with ωR from rotation). Therefore, the contact point is *instantaneously* at rest. If you had rolling will slipping, then the contact point would have excess motion from translation and not be at rest.

Now let's switch to $\sum \tau = I\alpha$. This equation describes how the wheel is going to rotate. Again, rotation and translation are two separate actions, although their magnitudes are connected due to the condition of rolling without slipping. To describe the rotation, we will want to look at how the wheel is being torqued. There are two torques acting on the wheel from F and f, so we want to find τ_F and τ_f.

Quick Questions

1. Why do we not consider torques produced by the gravitational force or normal force?

The external force is applied at the inner radius, r, whereas friction is acting at the outer radius R of the wheel (where it hits the ground). Both forces are perpendicular to their radius vectors, which makes the math much easier. Figure 7.12 shows a sketch of these vectors.

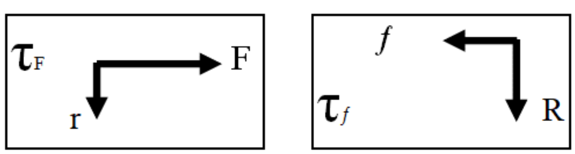


Figure 7.12: Sketch of how the two forces produce torques. The radius vectors are defined by the origin (centre of mass location) and the forces are shown with their directions.

Since the forces are perpendicular to the radii vectors, we can simplify the torques to:

$$\begin{aligned}\tau_F &= rF \\ \tau_f &= Rf\end{aligned}$$

But direction also matters. If you use the RHR, you will get that the external force produces a torque that is directed out of the page and the friction force produces a torque that is into the page. As a consequence, τ_F will produce rotation that is counter-clockwise and τ_f will produce rotation that is clockwise. Based on our definition of positive clockwise rotation, the total torque of our system is:

$$\sum \tau = Rf - rF$$

It may seem counter intuitive to have the external force as the negative term, but this is due to our choice to define the clockwise direction as positive. Had we defined the counter-clockwise direction as positive, then we would have the external force as the positive term (but we would need a negative factor relating a_{cm} and α; see prior comment).

Now we have both forms of Newton's law's:

$$\begin{aligned}\sum F &= ma_{cm} \quad \Longrightarrow \quad F - f = mR\alpha \\ \sum \tau &= I\alpha \quad \Longrightarrow \quad Rf - rF = I\alpha\end{aligned}$$

With two equations and two unknowns, f (which we want) and α, we can re-arrange these equations to solve for f.

$$\begin{aligned}\alpha &= \frac{Rf - rF}{I} \qquad (1) \\ \alpha &= \frac{F - f}{mR} \qquad (2)\end{aligned}$$

$$\frac{Rf - rF}{I} = \frac{F - f}{mR} \qquad \Longrightarrow (1) = (2)$$

$$\begin{aligned}f\left(\frac{R}{I} + \frac{1}{mR}\right) &= F\left(\frac{r}{I} + \frac{1}{mR}\right) \\ f\left(\frac{mR^2 + I}{mRI}\right) &= F\left(\frac{mRr + I}{mRI}\right) \\ f &= F\left(\frac{mRr + I}{mR^2 + I}\right)\end{aligned}$$

To fully solve this problem, however, we need to know the moment of inertia I. What is I for a wheel? We will assume that the wheel consists of a thick ring with an inner radius of r and an outer radius of R. In this case, the rotation axis is through the centre of the wheel, so we don't need to apply the parallel axis theorem (Chapter 7.2.2). The moment of inertia for a thick ring with an axis through its centre is $I_{CM} = \frac{1}{2}M(r_1^2 + r_2^2)$ (see Appendix A.4). For our values, this gives $I = \frac{1}{2}m(r^2 + R^2)$.

Adding our equation for I to the problem, we get:

$$
\begin{aligned}
f &= F\left(\frac{mRr + \frac{1}{2}m(r^2 + R^2)}{mR^2 + \frac{1}{2}m(r^2 + R^2)}\right) \\
f &= F\left(\frac{2Rr + r^2 + R^2}{3R^2 + r^2}\right)
\end{aligned}
$$

Quick Questions

1. Find the *maximum* amount of friction for the wheel if the mass is 45 kg and the coefficient of kinetic friction is $\mu_k = 0.15$.
2. How does the maximum amount of friction compare to the friction necessary to keep the wheel rolling without slipping? Assume $r = 7.5$ cm, $R = 12.5$ cm, and $F = 180$ N.
3. What force should you exert on the wheel (from b) to have it move without slipping?

7.7 Real-World Application

The conservation of angular momentum is a fundamental physics concept. It is sometimes referred to as Gyroscopic Motion, the tendency of a rotating object to maintain its orientation of motion. A common application you may be familiar with are fidget spinners. Fidget spinners are essentially miniature gyroscopes with a low-friction bearing to allow it to rotate longer. If you set the spinner in motion and then tilt it slowly to one side, you'll feel it resisting the tilt, pulling back toward its original position to conserve angular momentum.

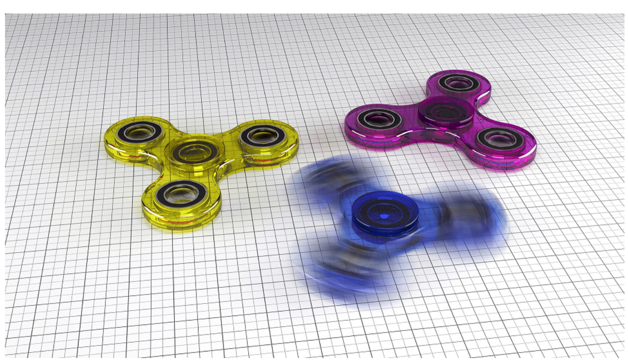


Figure 7.13: Fidget spinners. Image credit: Matthias Wewering from Pixabay

While the fidget spinner is an example of a simple low-weight mechanical gyroscope, there are other types, including fluid, laser, fibre-optic, and vibrational, all working on same basic principles of rotational motion. For example, with vibrational or MEMS (Micro Electro-Mechanical System) gyroscopes, the angular velocity in the sensor produces torques on vibration elements, providing measurable displacements that can then be amplified to produce an angular velocity signal. Three sensors arranged orthogonally in a single chip provide three dimensional components and track changes in orientation. This is the type of gyroscope is used in smart phones to provide image stabilization in a camera or auto-rotation, track step counts in fitness programs, and help give accurate location and positioning with accelerometers in GPS satellites.

For more information:

For lots of detail on the physics of fidget spinners, check out this article from the International Journal for Research in Applied Science and Engineering Technology by Vandana Kaushik.

For some detail on the different types of gyroscopes, see this article from SM Lease Design.

7.8 Summary

Key Takeaways

This chapter introduces angular momentum and torques. Angular momentum is defined as:

$$\vec{L} = \vec{r} \times \vec{p}$$

And torque is defined as:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

where both properties are vector cross products. Angular momentum and torque describe systems that are rotation. In many respects, angular momentum and torque are analogous to linear momentum and force. Newton's second law can be written for rotation, where the net torque on a system is equal to the time derivative of the total angular momentum.

$$\sum \vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$$
$$\sum \tau = I\alpha$$

A key feature of angular momentum and torques is the moment of inertia, which represents how easily a system is able to rotate.

$$I = \sum_i m_i r_i^2$$

In this chapter, we applied Newton's second law for rotation to simple harmonic motion and rolling problems. In particular, we discussed the condition of rolling without slipping, which is a special physical case.

$$v_{cm} = \omega R$$

This condition allows you to simplify rolling problems. Rolling without slipping means that any translation (centre of mass) motion occurs due to rolling, such that forward motion can be directly connected to the rotation. If slipping occurs, then you can get forward motion independent of rotation.

Important Equations

Angular Momentum of a Particle:

$$\vec{l}_i = \vec{r}_i \times \vec{p}_i$$

Total Angular Momentum:

$$\vec{L} = \sum \vec{l}_i = \sum (\vec{r}_i \times \vec{p}_i)$$

Torque:

$$\vec{\tau}_i = \vec{r}_i \times \vec{F}_i$$

Newton's second law for rotation:

$$\sum \vec{\tau} = \frac{\mathrm{d}\vec{L}}{\mathrm{d}t}$$
$$\sum \tau = I\alpha$$

Magnitude of Total Angular Momentum:

$$L = I\omega$$

Moment of Inertia:

$$I = \sum_i m_i r_i^2$$

Parallel Axis Theorem:

$$I_P = I_{cm} + Md^2$$

Rolling Without Slipping Condition:

$$v_{cm} = \omega R$$

7.9 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 7-1

Use the the Parallel Axis Theorem to find expressions for the moments of inertia for each of the figures below. Note the location of the axis of rotation in each case.

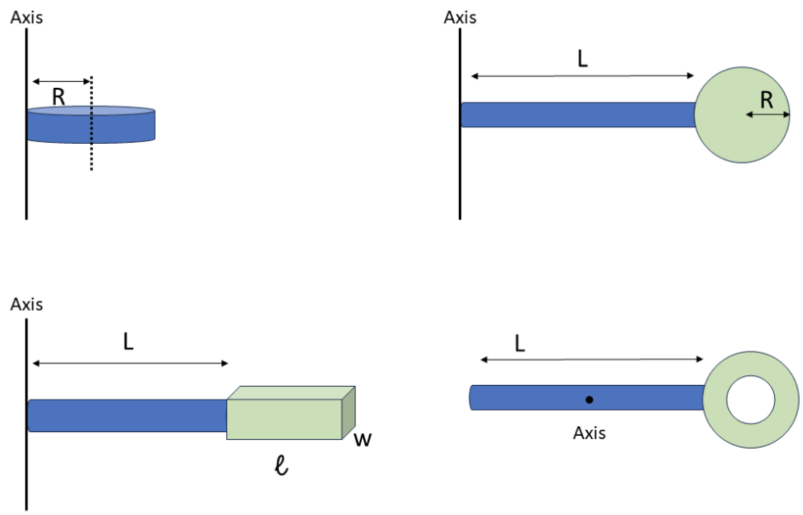


Figure 7.14: The four cases described in the question.

a) A thin disk of radius R and mass M rotating around an axis at its edge.

b) A rod of length L and mass M_R with a sphere of radius R and mass M_S attached to one end and the axis of rotation at the opposite end.

c) A rod of length L and mass M_R with a thin rectangular plate of mass M_P, length ℓ, and width w attached to one end and the axis of rotation at the opposite end.

d) A rod of length L and mass M_R with a hollow cylinder of mass M_C, inner radius R_1 and outer radius R_2 attached to one end. The axis of rotation is through the centre of the rod.

Practice Problem 7-2

A red giant star has a mass fifteen times that of our Sun (15 $M_\odot$) and a radius of one astronomical unit (1.5×10^8 km). It undergoes a sudden supernova, producing a neutron star with a radius of 20 km. Assuming only 1/10the of the star's mass ends up in the neutron star, what happens to its rotation rate (angular speed)? Assume both the red giant star and the neutron star can be approximated as perfect spheres.

Practice Problem 7-3

Find expressions for the torque (magnitude and direction) in each of the following figures.

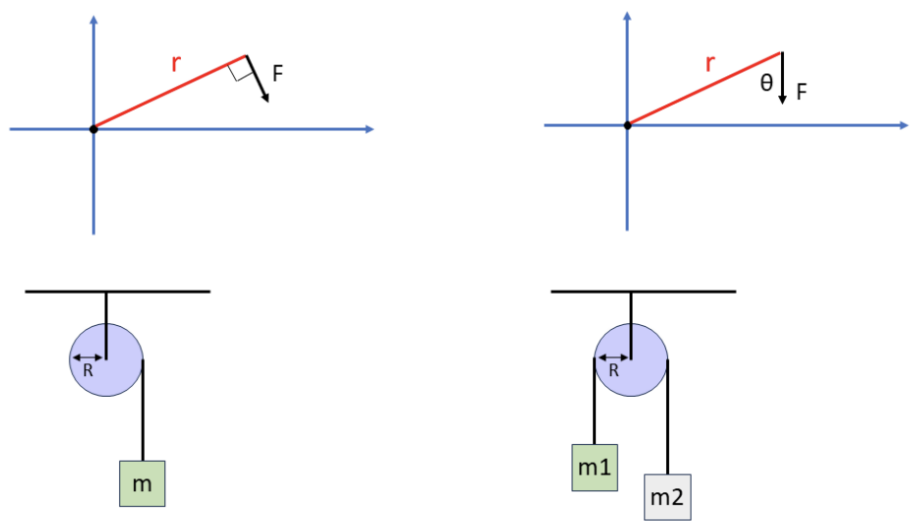


Figure 7.15: The four cases discussed.

a) A force F is applied orthogonaly to the end of a fulcrum of length r.

b) A force F is applied at an angle θ to the end of a fulcrum of length r.

c) A mass M hangs from a pulley of radius R. Find the torque on the pulley.

d) Two masses, M_1 and M_2 hang from opposite end of a rope over a pulley of radius R. Find the torque on the pulley.

Practice Problem 7-4

See figure below. A piece of sticky putty of mass m moves with speed v_0 and collides with a rod of length ℓ and mass M. The rod is pivoted at its centre and the putty hits the rod (and sticks to it) at the far end at an angle perpendicular to the axis of the rod.

a) Write an equation for the angular momentum before and after the collision. Assume that $M \gg m$ such that the centre of mass of the system remains at the centre of the rod.

b) What is the angular velocity ω of the resulting rotation.

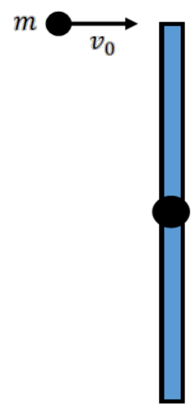


Figure 7.16: The rod and particle system.

Practice Problem 7-5

Two identical rods of mass M and length L are welded together forming a right angle. They are then allowed to rotate about a pivot point at their corner, producing a physical pendulum. What is the distance between the centre of mass of the physical pendulum and the pivot point?

Practice Problem 7-6

See figure below. You have a circular disk of mass M and radius R. The disk is hanging from a pivot point located a distance s from the centre of mass as shown.

a) What is the moment of inertia for the disk about the pivot point s?

b) What is the period of oscillations, assuming the disk is displaced a small angle from the vertical?

c) What value of s gives you the smallest possible period of oscillations?

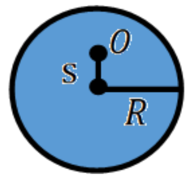


Figure 7.17: The physical pendulum hangs from its pivot point O.

Practice Problem 7-7

A metre stick is pivoted at the 20cm mark and allowed to freely oscillate. What is the angular frequency of those oscillations assuming that the metre stick is displaced from equilibrium by a small amount?

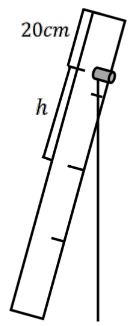


Figure 7.18: A meter stick as a physical pendulum.

Practice Problem 7-8

A Physical pendulum made of a rod of length L and a sphere of radius R, as shown in Figure 7-14 (top right). The rod and the sphere have the same mass, M. Consider the pivot point to be through the opposite end of the rod from the sphere.

a) Where is the center of mass?

b) What is the moment of inertia?

c) Find the period of oscillations for small angles.

Practice Problem 7-9

A circular disk of mass M and radius R rolls down an incline (angle for the incline in θ) without slipping. The moment of inertia for a disk is $\frac{1}{2}MR^2$.

a) Draw a free body diagram for the system.

b) Find the equations for $\sum F$ and $\sum \tau$.

c) What is the acceleration of the disk centre of mass?

d) If the coefficient of static friction is μ, what is the steepest angle θ before the disk starts to slip?

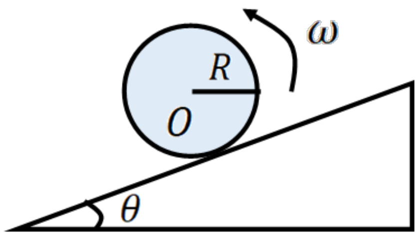


Figure 7.19: The circular disk of mass M rolls down an incline of angle θ.

8 Work and Energy

Learning Objectives

- Define work and kinetic energy
- Review the work-energy theorem
- Introduce conservative forces and gravity
- Introduce potential energy

In this chapter, we will switch to using energy to solve physical problems instead of Newton's laws and momentum. We will review the concepts of work, kinetic energy, and potential energy, and we will introduce conservative forces.

8.1 Introduction to Work and Energy

When a force is applied to an object such that it moves a displacement of $\Delta\vec{r}$, work is done by that force. Work is defined as:

$$\boxed{W = \int \vec{F} \cdot \mathrm{d}\vec{r}} \tag{8.1}$$

where $\vec{F}$ is the force and $\mathrm{d}\vec{r}$ represents a small displacement.

Units of Work

Work has units of energy. The SI units for energy is the Joule, abbreviated as [J].

$[1\ \mathrm{J}] = [1\ \mathrm{N\ m}] = [1\ \mathrm{kg\ m^2\ s^{-2}}]$.

If the force is *constant*, then you can simplify the above equations to:

$$W = \vec{F} \cdot \int \mathrm{d}\vec{r} = \vec{F} \cdot \Delta\vec{r} = F_x\Delta x + F_y\Delta y + F_z\Delta z$$

where $\Delta\vec{r}$ represents the displacement between two points, r_1 and r_2. Alternatively, the vector dot product can be solved following $\vec{F} \cdot \Delta\vec{r} = F\Delta r\cos\theta$, where θ is the angle between the two vectors.

If $\vec{F}$ and $\mathrm{d}\vec{r}$ are parallel ($\theta = 0$), then the force is acting in the same direction as the displacement and the force does maximum positive work. If the force is perpendicular to the displacement ($\theta = \frac{\pi}{2}$), then the force does zero work. That is, the force is not responsible for the displacement and contributed no work to that displacement. If the force is antiparallel to the displacement ($\theta = \pi$), then the force does maximum negative work (the force acts to counter the motion as much as it can).

Quick Question

1. Describe a situation where a force is at an angle of $0°, 90°$, and $180°$ from a displacement. Consider how that force affects the motion.

Work is a *scalar* quantity. It has magnitude but no direction. Since work is calculated by a vector dot product between force and displacement, only the component of the force that is along the displacement vector matters for the work calculation. Thus, solving problems with energy can make the math much easier if you select the right coordinate system (e.g., Cartesian versus polar coordinates) when defining your force and displacement.

Describing Work

Since work is measured from one position (r_1) to a second position (r_2), we will use $W(r_1 \to r_2)$ to illustrate that the work is being done from r_1 to r_2. This form is convenient to specify the direction, because you can also measure the work from r_2 to r_1. For conservative forces (Chapter 8.5),

$$\boxed{W(r_1 \to r_2) = -W(r_2 \to r_1)} \tag{8.2}$$

The difference between these two cases is the direction of the displacement vector for $\mathrm{d}\vec{r}$. In one case, the force will be against the displacement (negative work), and in the other case, the force will be in the same direction as the displacement (positive work).

8.2 Work-Energy Theorem

Consider a net 1-D force F that acts on a single particle in the x direction. From Newton's second law (in one dimension), we have:

$$F = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = m\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}v}{\mathrm{d}x} = mv\frac{\mathrm{d}v}{\mathrm{d}x}$$

using the chain rule (see Sample Problem 2-3 for more information). We can re-write the above as:

$$\begin{aligned}
F\mathrm{d}x &= mv\mathrm{d}v \\
F\mathrm{d}x &= \frac{1}{2}m\mathrm{d}(v^2) \quad \Longrightarrow \quad \text{note that } \mathrm{d}(v^2) = 2v\mathrm{d}v \\
F\mathrm{d}x &= \mathrm{d}\left(\frac{1}{2}mv^2\right) \quad \Longrightarrow \quad m \text{ is constant} \\
F\mathrm{d}x &= \mathrm{d}K
\end{aligned}$$

where K corresponds to the kinetic energy of the system.

The work done by the force F going from position x_1 to x_2 can be found by integrating both sides of the above equation:

$$\begin{aligned}\int_{x_1}^{x_2} F(x)\mathrm{d}x &= \int_{x_1}^{x_2} \mathrm{d}K \\ W_2 - W_1 &= K_2 - K_1\end{aligned}$$

$$\boxed{W(x_1 \to x_2) = \Delta K} \tag{8.3}$$

Equation 8.3 is the *work-kinetic energy theorem.*

Work-Kinetic Energy Theorem

The work done by all forces acting on a system corresponds to the change in kinetic energy of the system. That is, if positive work is done by the force (movement is with the force), then there is an increase in kinetic energy, and if there is negative work (movement is against the force), there is a decrease in kinetic energy.

Similarly, one can also define the work-kinetic energy theorem for rotation and torques. Recall from Chapter 7 that the net torque is:

$$\tau = I\alpha = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = I\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = I\omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta}$$

Now, torque is a vector quantity given by $\vec{\tau} = \vec{r} \times \vec{F}$. But work is a scalar quantity and the work done on a particle to move it from one position to another applies only to the component of the force in the direction of motion. If a force is perpendicular to the direction of motion, that force does no work. And if a force is parallel to the direction of motion, it does maximum work.

So if the path length in the direction of the force is $\mathrm{d}s = r\mathrm{d}\theta$, then we can say that the work done by the force is:

$$\begin{aligned}\mathrm{d}W &= F_s\mathrm{d}s \implies && F_s \text{ is the component parallel to the path } s \\ &= F_s r\mathrm{d}\theta \implies && \text{sub in } \mathrm{d}s = r\mathrm{d}\theta \\ &= \tau\mathrm{d}\theta \implies && \tau = rF_s \text{ because } \vec{F}_s \perp \vec{r} \text{ if } \vec{F}_s \,//\, \mathrm{d}\vec{s} \text{ (e.g., } \vec{r} \perp \mathrm{d}\vec{s})\end{aligned}$$

Combining this with our definition of torque from before, we get:

$$\begin{aligned}\mathrm{d}W &= \tau\mathrm{d}\theta \\ &= \left(I\omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\right)\mathrm{d}\theta \implies && \text{equation for } \tau \text{ from the second law} \\ &= I\omega\mathrm{d}\omega \\ &= \frac{1}{2}I\mathrm{d}(\omega^2) \implies && \text{note that } \mathrm{d}(\omega^2) = 2\omega\mathrm{d}\omega \\ &= \mathrm{d}\left(\frac{1}{2}I\omega^2\right) \implies && I \text{ is constant} \\ &= \mathrm{d}K\end{aligned}$$

Integrating both sides will give you the same answer as the linear case: $W(\theta_1 \to \theta_2) = \Delta K$. So the work-kinetic energy theorem applies for both translation and rotation movement. The kinetic energy equations are slightly different, however.

$$\boxed{K = \frac{1}{2}mv^2} \quad \Longrightarrow \quad \text{for translation} \tag{8.4}$$

$$\boxed{K = \frac{1}{2}I\omega^2} \quad \Longrightarrow \quad \text{for rotation} \tag{8.5}$$

8.3 Work in Different Frames

The work-kinetic energy theorem applies to all inertial frames (constant velocity) whether they are moving or stationary. So the change in work is the same within a stationary (S) frame or in a moving frame S' (e.g., the centre-of-mass frame such as in Chapter 6).

Figure 8.1 show a particle starting from rest and moving under a constant force F in a laboratory. This particle will have a constant acceleration a due to this force. The work done to move this particle from point x_1 to x_2 in the laboratory frame (S) is simply $W_S = F\Delta x = \Delta K$ because all the motion is in 1-D.

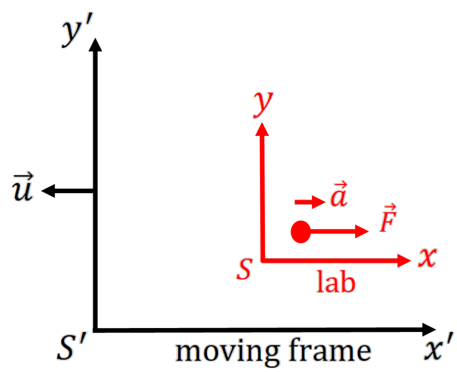


Figure 8.1: The motion of a particle in two different inertial frames. The lab frame S is stationary (red) and the second frame S' is moving at a constant velocity u relative to the lab frame. A particle moves under a constant force F with an acceleration a in the lab frame. The work done to move that particle from points x_1 to x_2 in the lab frame is equal to the work done to move the particle in the moving inertial frame.

Now, consider what an observer in a moving frame, S' would measure for the work done by that force. Here, S' is moving at a constant velocity and is also an inertial frame. Because S' is moving at a constant velocity, the observer in S' would find the same acceleration as the observer in S (e.g., $\frac{\mathrm{d}}{\mathrm{d}t}(v + v_0) = \frac{\mathrm{d}v}{\mathrm{d}t}$). As a result, both the acceleration and force are unchanged in the moving frame. So

$$\boxed{F_S = F_{S'} = ma} \tag{8.6}$$

with the same value of a for both frames.

If the force is constant (given in the question), then the work in the moving frame is $W_{S'} = \vec{F}\Delta\vec{x}_{S'}$, where $\Delta\vec{x}_{S'}$ is the displacement in the moving frame. Note that since the motion still takes place in 1-D we can drop vectors (if there are more dimensions, you just need to break up the movement by each coordinate).

For the displacement, $\Delta\vec{x}_{S'}$, we need to know how far the particle travels in time Δt. For a system with constant acceleration (see Chapters 1 and 2), the displacement $\Delta x = \frac{1}{2}at^2 + v_i t$, where v_i is the initial velocity of the system. While the particle is initially at rest in the stationary frame, from the perspective of the moving frame, the particle does not start stationary. The moving frame has a velocity of $-u\hat{\imath}$ relative to the lab frame. That means that the moving frame would see the particle as having an initial speed of $u\hat{\imath}$. So we can say that $v_i = u$ and the displacement is:

$$\Delta x = \frac{1}{2}at^2 + v_i t = \frac{1}{2}at^2 + ut$$

Taking our equations for F and Δx in the moving frame, the work done by the force is

$$\begin{aligned}
\Delta W &= F\Delta x \\
&= (ma)\left(\frac{1}{2}at^2 + ut\right) \quad \Longrightarrow \quad \text{sub } F = ma \text{ and our equation for } \Delta x \\
&= \frac{1}{2}m[(at)^2 + 2uat] \\
&= \frac{1}{2}m[(at)^2 + 2uat + u^2 - u^2] \quad \Longrightarrow \quad \text{add and subtract } u^2 \text{ (same as adding zero)} \\
&= \frac{1}{2}m[(at+u)^2 - u^2] \quad \Longrightarrow \quad \text{recall that } (a+b)^2 = a^2 + 2ab + b^2 \\
&= \frac{1}{2}m[v_f^2 - u^2] \quad \Longrightarrow \quad \text{for constant acceleration, } v_f = at + u \\
&= \frac{1}{2}m[v_f^2 - v_i^2] \quad \Longrightarrow \quad u \text{ is just the initial velocity, } v_i \\
&= K_f - K_i \\
&= \Delta K
\end{aligned}$$

where K_f is the final kinetic energy and K_i in the initial kinetic energy.

So while the values of W and K as measured in the two frames (stationary and moving) may be different, the requirement that $\Delta W = \Delta K$ holds in both frames. Again, this is only the case for *inertial* frames. In non-inertial frames (e.g., rotating or accelerating frames), the net force will include fictitious forces due to the non-inertial frame and the measured accelerations would be different (see Chapters 4 and 5).

8.4 Gravity

8.4.1 Simple Approximation

Near the Earth's surface, we often describe the gravitational force as $\vec{F}_g = m\vec{g}$, where $\vec{g}$ is a constant vector that points down vertically and has a constant magnitude.

Consider the case where the position of a heavy box of mass m changes y_1 to y_2 in vertical height and that both positions y_2 and y_1 are near the Earth's surface. Work is done by gravity as the box moves by this displacement. The vector describing the displacement of the box is $\Delta\vec{y} = (y_2 - y_1)\hat{\jmath}$, whereas the gravitational force is $\vec{F}_g = -mg\hat{\jmath}$, which is constant. Taking the equation for work with a constant force, the work is:

$$\begin{aligned}\Delta W &= \vec{F}_g \cdot \Delta\vec{y} \\ &= (-mg\hat{\jmath}) \cdot (y_2 - y_1)\hat{\jmath} \\ &= -mg(y_2 - y_1) \quad \Longrightarrow \quad \hat{\jmath}\cdot\hat{\jmath} = 1\end{aligned}$$

Remember, this is the work done by gravity. So if a box is lifted upward ($y_2 > y_1$), gravity acts against the displacement and $W < 0$. If the box is lowered ($y_2 < y_1$), gravity acts with (helps) the displacement and $W > 0$. And if $y_2 = y_1$ then $W = 0$ (no work is done by gravity).

8.4.2 General Equation

In general, the true form of the gravitational force is:

$$\boxed{\vec{F}_g = -\frac{GMm}{r^2}\hat{r}} \tag{8.7}$$

where G is the gravitational constant of 6.67×10^{-11} N m^2 kg^{-2}, M is the mass of the object producing the gravitational field, m is the mass of an object being accelerated in the gravitational field, and r is the distance between the centers of the two objects. Thus, the true form for the acceleration due to gravity is given by:

$$\boxed{\vec{g} = -\frac{GM}{r^2}\hat{r}} \tag{8.8}$$

Note that the acceleration due to gravity always points toward the centre of the source of the gravitational field (points radially inward).

Quick Question

1. Show that the acceleration due to gravity near the surface of the Earth is roughly 9.8 m s^{-2}. The radius of the Earth is $\approx$ 6370 km and the mass of the Earth is $M \approx 5.97 \times 10^{24}$ kg.

The force of gravity is a radial force with no azimuthal (angle) dependence. That means that any point that has the same radial distance ($|\vec{r}|$) will have the same magnitude of gravity $|\vec{g}|$. We often describe gravity as a gravitational field, a sphere of influence where any object with a mass m will be subjected to a gravitational force. The magnitude of the gravitational field changes as a function of radial distance. For example, Figure 8.2 shows a cartoon of the Earth with two different vectors, $\vec{r}_1$ and $\vec{r}_2$. Their magnitudes of gravitational acceleration are:

$$|\vec{g}_1| = \frac{GM}{r_1^2} \quad \text{and} \quad |\vec{g}_2| = \frac{GM}{r_2^2}$$

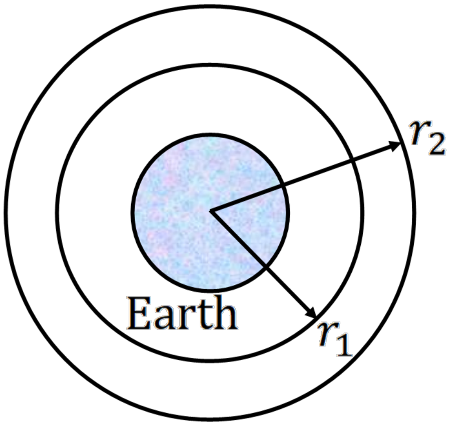


Figure 8.2: A cartoon of the Earth with two different vectors, $\vec{r}_1$ and $\vec{r}_2$. Each vector also has a circle with corresponding radii of r_1 and r_2. Every point on the r_1 circle will have the same gravitational field magnitude and similarly, every point on the r_2 circle will have the same gravitational field magnitude.

Since $r_2 > r_1$, $|\vec{g}_1| > |\vec{g}_2|$. But any point on a sphere with radius r_1 around the Earth will have the same magnitude of acceleration due to gravity (e.g., the circles in Figure 8.2), and likewise for all the points on a sphere with radius r_2. You can think of gravity as a sphere of influence, where any point that has the same distance from the centre of the Earth has the same magnitude of $|\vec{g}|$.

Note that the direction of gravity will be different depending on where you are, because the force always points inward toward the centre of the Earth (in its true form, gravity is not vertical, but radial). For very small distances and positions near the Earth's surface, you can still assume a vertical direction and constant magnitude for a frame at that surface.

What is the work done by gravity using this general equation?

Well, consider going from r_1 to r_2 as shown in Figure 8.2.

$$\begin{aligned}
W(r_1 \to r_2) &= \int_{r_1}^{r_2} \vec{F}\cdot \mathrm{d}\vec{r} &&\Longrightarrow \quad \text{force is not constant}\\
&= \int_{r_1}^{r_2} \left(-\frac{GMm}{r^2}\right)\hat{r}\cdot \mathrm{d}\vec{r} &&\Longrightarrow \quad \text{use the equation for the force of gravity}\\
&= -GMm\int_{r_1}^{r_2}\left(\frac{1}{r^2}\hat{r}\right)\cdot(\mathrm{d}r\hat{r}) &&\Longrightarrow \quad \mathrm{d}\vec{r} = \mathrm{d}r\hat{r}\\
&= -GMm\int_{r_1}^{r_2}\left(\frac{1}{r^2}\mathrm{d}r\right)\hat{r}\cdot\hat{r} &&\Longrightarrow \quad \hat{a}\cdot\hat{a} = 1 \text{ for any unit vector}\\
&= -GMm\int_{r_1}^{r_2}\frac{1}{r^2}\mathrm{d}r\\
&= -GMm\left(-\frac{1}{r}\right|_{r_1}^{r_2}\\
&= \frac{GMm}{r_2} - \frac{GMm}{r_1}
\end{aligned}$$

Note that if $r_2 > r_1$, $W < 0$ as we would expect (e.g., you are doing work against gravity to move an object further away). You may recognize $-\frac{GMm}{r}$ as the gravitational potential energy. We will discuss potential energies in more detail in Section 8.6.

8.4.3 Escape velocity

The gravitational force follows an inverse square ($\frac{1}{r^2}$) law. So at very large distances from the source of the gravitational field, the gravitational force goes to zero and the work necessary to move a particle also goes to zero (if there is no force, there is no work).

Consider an object that is on the surface of Earth and is launched so that it reaches a very large distance away (assume infinity). Due to Earth's gravitational field, the object will feel a force that opposes it's motion to leave. Gravity will be doing negative work and the kinetic energy of the object will decrease. **What speed is needed for this object to just reach infinity?** Assume Earth's atmosphere does not affect its motion.

To solve this problem, we will use the work-kinetic energy theorem (Equation (8.3)).

$$W(r_1 \to r_2) = \Delta K = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

We just solved for the work to move an object between two radii in a gravitational field. We start with Earth's surface ($r_1 = R_E$) and we end very far away ($r_2 = \infty$). That means that:

$$W(r_1 \to r_2) = \frac{GMm}{r_2} - \frac{GMm}{r_1} = 0 - \frac{GmM_E}{R_E}$$

where R_E is the radius of the Earth, M_E is the mass of the Earth, and m is the mass of our object being moved. For kinetic energy, $v_2 = 0$ because the object just reaches infinity. As such:

$$\begin{aligned}
W(r_1 \to r_2) &= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \\
-\frac{GmM_E}{R_E} &= 0 - \frac{1}{2}mv_1^2 \\
v_1^2 &= \frac{2GM_E}{R_E} \\
\boxed{v_{esc} = \sqrt{\frac{2GM_E}{R_E}}}
\end{aligned} \tag{8.9}$$

Equation 8.9 describes the *escape velocity* and for Earth, which is roughly 11 km s^{-1}. The escape velocity is the minimum speed for rockets and satellites to leave Earth's surface and travel great distances away.

Quick Question

1. Verify that $v_{esc} = 11$ km s^{-1} is the escape velocity for the Earth. Assume that $R_E \approx 6370$ km and $M_E \approx 5.97 \times 10^{24}$ kg.

Note that an object starting from rest at a great distance from the Earth will hit the Earth at a speed equivalent to the escape velocity. This converse situation is true because gravity is a *conservative force*, which we will discuss in the next section.

8.5 Conservative Forces

Conservative forces are a class of forces that depend only on position. These forces are also ones where *energy is conserved*. Common examples of conservative forces are gravity, the spring force, and the electric force. Conservative forces must meet the following criteria:

1. A conservative force does no total work on an object during a round trip.
2. The work done by a conservative force is independent of the path taken between two points.

The first requirement states:

$$\boxed{W(r_1 \rightarrow r_1) = \oint \vec{F} \cdot \mathrm{d}\vec{r} = 0} \tag{8.10}$$

where $\oint$ indicates an integral over a closed loop. For example, if you measure the work between $r_1 \rightarrow r_2$ and then $r_2 \rightarrow r_1$, the work function would be:

$$W(r_1 \rightarrow r_2 \rightarrow r_1) = \int_{r_1}^{r_2} \vec{F} \cdot \mathrm{d}\vec{r} + \int_{r_2}^{r_1} \vec{F} \cdot \mathrm{d}\vec{r}$$

For a conservative force, that equation would equal zero.

Cora's Thoughts

Let's show how it equals zero by connecting to what we saw in Chapter 8.1. We can say that:

$$W(r_1 \rightarrow r_2 \rightarrow r_1) = W(r_1 \rightarrow r_2) + W(r_2 \rightarrow r_1) = \int_{r_1}^{r_2} \vec{F} \cdot \mathrm{d}\vec{r} + \int_{r_2}^{r_1} \vec{F} \cdot \mathrm{d}\vec{r}$$

So if we substitute in Equation (8.2), which is true for conservative forces, we get:

$$\begin{aligned} W(r_1 \rightarrow r_2) + W(r_2 \rightarrow r_1) &= \int_{r_1}^{r_2} \vec{F} \cdot \mathrm{d}\vec{r} - \int_{r_1}^{r_2} \vec{F} \cdot \mathrm{d}\vec{r} \\ W(r_1 \rightarrow r_2 \rightarrow r_1) &= 0 \\ W &= 0 \end{aligned}$$

Showing that the force does no total work on a round trip.

Now consider the case of gravity,

$$\int_{r_1}^{r_2} \vec{F} \cdot \mathrm{d}\vec{r} = \frac{GMm}{r_2} - \frac{GMm}{r_1}$$
$$\int_{r_2}^{r_1} \vec{F} \cdot \mathrm{d}\vec{r} = \frac{GMm}{r_1} - \frac{GMm}{r_2}$$

If you add those two segments, you indeed get $W = 0$. This is true for any number of stops in the closed loop. As long as your starting and final positions are the same, you will get $W = 0$ for a conservative force.

Quick Question

1. Verify that the electric force and the spring force also satisfy criterion 1 and give you the condition that $W = 0$ for a closed loop.

The second requirement states that the total work done to move an object between two points only depends on the initial and final positions. How you get from point one to point two doesn't matter. Figure 8.3 shows two paths that connect points r_1 and r_2. The work done by a conservative force will be the same no matter the path chosen.

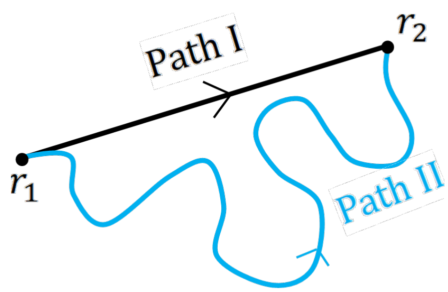


Figure 8.3: Comparison of two paths between points r_1 and r_2. For a conservative force, the work done to go from r_1 to r_2 is the same for both paths, even though the paths are very different.

And of course, if you go from r_1 to r_2 and then from r_2 back to r_1, the total work is zero (first criterion) no matter which path you take in either case (e.g., Path I from r_1 to r_2 and then Path II from r_2 back to r_1).

Note that forces like friction are *not* conservative forces. First, friction doesn't depend on position. Second, friction removes energy from a system. Third, the work done by friction in a closed loop is not zero. For example, consider a hockey puck moving in a circle of radius R on a rough horizontal surface that has friction. The friction force on the hockey puck is $\vec{f} = \mu_K mg\hat{\theta}$, and the puck moves so that it is always antiparallel (180°) to the displacement. So $W_f = -f\Delta d = -\mu_K mg(2\pi R)$ for a closed loop. Thus, $W_f \neq 0$.

8.5.1 Identifying Conservative Forces

A conservative force can be identified mathematically using the following condition:

$$\boxed{\vec{\nabla} \times \vec{F} = 0} \tag{8.11}$$

Note that $\vec{\nabla} \times \vec{F}$ is a vector cross product called the curl of the vector $\vec{F}$. The $\vec{\nabla}$ symbol is a vector differential operator sometimes called the del or nabla operator. In Cartesian coordinates, it has the form of:

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}$$

where ∂ indicates the partial derivative. For partial derivatives, you ignore any other variable. For example, $\frac{\partial}{\partial x}f(y) = 0$ because x is not present in the function (you treat y as a constant for a partial derivative with respect to x). For $\vec{\nabla}$ in other coordinate systems, see Appendix A.5.

The curl of $\vec{A}$ (in Cartesian coordinates) is then given by the following matrix.

$$\begin{aligned}
\vec{\nabla} \times \vec{A} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)\hat{\imath} + \left(\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right)\hat{\jmath} + \left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right)\hat{k}
\end{aligned}$$

A conservative force has $\vec{\nabla} \times \vec{F} = 0$ by definition. We will discuss why in Chapter 8.6.

8.5.2 Example Conservative Forces

Sample Problem 8-1

Show that the gravitational force is a conservative force.

Solution

The gravitational force is:

$$\vec{F}_g = -\frac{GMm}{r^2}\hat{r}$$

Since this is a radial force, use the spherical coordinates for the curl. You can use Cartesian, but there is a lot more math involved. See Appendix A.5 for the definition of curl in spherical coordinates.

$$\begin{aligned}
\vec{\nabla} \times \vec{F}_g &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ -\frac{GMm}{r^2} & 0 & 0 \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(0)\hat{r} + \frac{\partial}{\partial \phi}\left(-\frac{GMm}{r^2}\right)r\hat{\theta} + \frac{\partial}{\partial \theta}\left(-\frac{GMm}{r^2}\right)r\sin\theta\hat{\phi}\right] \\
&= 0
\end{aligned}$$

because there is no θ or ϕ dependence on the force. That is, $\frac{\partial}{\partial\theta}f(r) = \frac{\partial}{\partial\phi}f(r) = 0$ because you would treat r as a constant for a partial derivative with respect to θ or ϕ. Since we have $\vec{\nabla} \times \vec{F} = 0$, the gravitational force is conservative.

Quick Question

1. Verify that $\vec{\nabla} \times \vec{F} = 0$ also for the 1-D spring force, $F = -kx$.

Sample Problem 8-2

Is the force $\vec{F} = x^2yz\hat{\imath} - xyz^2\hat{k}$ conservative?

Solution

Here we have some unnamed force that depends on position. To find out if this force is conservative, we have to solve $\vec{\nabla} \times \vec{F}$. Note that the force is in Cartesian coordinates, so we will want to use the curl in Cartesian coordinates.

$$
\begin{aligned}
\vec{\nabla} \times \vec{F} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & 0 & -xyz^2 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}[-xyz^2] - 0\right)\hat{\imath} + \left(\frac{\partial}{\partial z}[x^2yz] - \frac{\partial}{\partial x}[-xyz^2]\right)\hat{\jmath} + \left(0 - \frac{\partial}{\partial y}[x^2yz]\right)\hat{k} \\
&= (-xz^2)\hat{\imath} + (x^2y + yz^2)\hat{\jmath} + (-x^2z)\hat{k}
\end{aligned}
$$

Since $\vec{\nabla} \times \vec{F} \neq 0$, this force is not a conservative force.

8.6 Potential Energy

The potential energy represents the energy of a system based on its position or configuration. It represents the capacity of the system to do work,

$$\Delta U = -\Delta W_{con}$$

where ΔW_{con} is the work done by a conservative force. More formally,

$$\boxed{U(r_1 \to r_2) = -\int_{r_1}^{r_2} \vec{F} \cdot \mathrm{d}\vec{r} = -W_{con}(r_1 \to r_2)} \tag{8.12}$$

Potential energy is associated with conservative forces only. Consider a heavy box. If you lift that box upward, the work done by gravity is negative (gravity opposes the motion), but you increase the potential energy of the box because work is being done against gravity ($W < 0$). If you lower that box or let go, then gravity is doing positive work on the box ($W > 0$) and it will have a decrease in potential energy.

Let's look at the gravitational potential energy between positions r_1 and r_2.

$$\begin{aligned}
U(r_1 \to r_2) &= -W_{con}(r_1 \to r_2) \\
&= -\int_{r_1}^{r_2} \vec{F} \cdot \mathrm{d}\vec{r} \\
&= -\int_{r_1}^{r_2} \left(-\frac{GMm}{r^2}\hat{r}\right) \cdot (\mathrm{d}r\hat{r}) \\
&= \int_{r_1}^{r_2} \left(\frac{GMm}{r^2}\right) \mathrm{d}r \\
&= \left(-\frac{GMm}{r}\right|_{r_1}^{r_2} \\
&= -\frac{GMm}{r_2} + \frac{GMm}{r_1}
\end{aligned}$$

In general, the gravitational potential energy is measured between two points. It is a *relative* energy. You will want to set a convenient reference point. For example, we can set $r_1 = \infty$, where there would be no contribution from the force, such that:

$$U = -\frac{GMm}{r} \quad \text{(compared to infinity)}$$

For gravity problems near Earth's surface, setting $U = 0$ at the surface is often convenient.

The potential energy for the spring force going from x_1 to x_2 is:

$$U(x_1 \to x_2) = -\int_{x_1}^{x_2} (-kx\hat{x}) \cdot (\mathrm{d}x\hat{x}) = \int_{x_1}^{x_2} kx\mathrm{d}x = k\left(\frac{1}{2}x^2\right|_{x_1}^{x_2} = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$$

Once again, we want to set a convenient initial value like $x_1 = 0$ (the equilibrium position), so that the potential energy of a spring is simply $U = \frac{1}{2}kx^2$ relative to that point.

Quick Questions

1. Show that that the potential energy of a pendulum can be written as $U = \frac{1}{2}mgh\theta^2$ for small angles of θ and setting $\theta_1 = 0$.
2. Show that the potential energy of the electric force $F_e = \frac{kQq}{r^2}\hat{r}$ is given by $U = \frac{kQq}{r}$, where k is the electric constant, Q is the electric charge of the central object forming the electric field, and q is the charge of the test particle in the field.

8.7 Conservation of Energy

The work-kinetic energy theorem states that $\Delta W = \Delta K$. That is, the total work on a system by a force is equal to the change in the kinetic energy. The total work is the sum of work by conservative forces (ΔW_{con}) and the work by non-conservative forces (ΔW_{nc}):

$$\begin{aligned}
\Delta W &= \Delta K \\
\Delta W_{con} + \Delta W_{nc} &= \Delta K \quad \Longrightarrow \quad \Delta W = \Delta W_{con} + \Delta W_{nc} \\
\Delta W_{nc} &= \Delta K - \Delta W_{con} \\
\Delta W_{nc} &= \Delta K + \Delta U \quad \Longrightarrow \quad \Delta W_{con} = -\Delta U
\end{aligned}$$

Setting $\Delta W_{nc} = \Delta E$, or the change in mechanical energy (non-conservative work), we get:

$$\boxed{\Delta E = \Delta(K + U)} \tag{8.13}$$

For small times $\mathrm{d}t$, you can say that $\mathrm{d}E = \mathrm{d}(K + U)$. Integrating these functions then gives $E = K + U$.

In the absence of non-conservative forces (e.g., if all forces acting on a system are conservative), then $\Delta W_{nc} = 0$ and $\Delta E = 0$ such that we get $\Delta(K + U) = 0$ or $K + U = \text{constant}$. This is the *conservation of energy*. When energy is conserved, $\Delta K = -\Delta U$ or the change in kinetic energy directly corresponds to a change in potential energy. If kinetic energy increases, then potential energy decreases and vice versa.

Real World Applications

Hydroelectric power generation works by converting gravitational potential energy to kinetic kinetic energy and ultimately electrical energy. The basic principle behind hydroelectric power is a large volume of water experiencing a drop in elevation. Water flows from the intake downward to a turbine, gaining kinetic energy which can be captured by the turbine. The larger the elevation change, the more power that can potentially be generated. Hydroelectric installations often involve huge dams and are some of the largest construction projects on Earth.

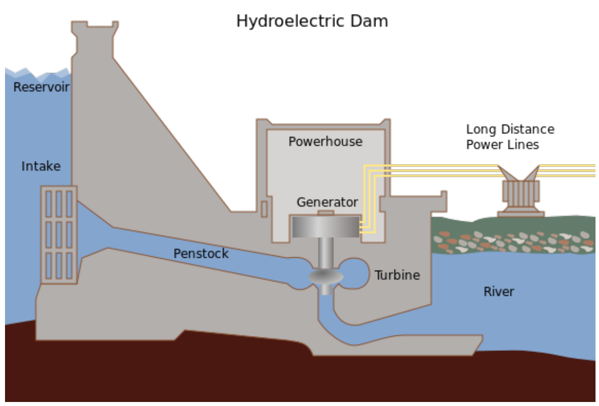


Figure 8.4: Simplified hydroelectric dam. Image credit: Tennessee Valley Authority.

8.8 Application of Potential Energy

A conservative force can also be written in terms of the potential energy,

$$\boxed{\vec{F} = -\vec{\nabla} U = -\left(\frac{\partial U}{\partial x}\hat{\imath} + \frac{\partial U}{\partial y}\hat{\jmath} + \frac{\partial U}{\partial z}\hat{k}\right)} \tag{8.14}$$

for Cartesian coordinates (see Appendix A.5 for other coordinate systems). The $\vec{\nabla}$ operator in this context means the gradient of U.

We can use this definition of a conservative force to show that $U(r_1 \to r_2) = -W(r_1 \to r_2)$.

$$\begin{aligned}
W(r_1 \to r_2) &= \int_{r_1}^{r_2} \vec{F} \cdot \mathrm{d}\vec{r} \\
&= \int_{r_1}^{r_2} (-\vec{\nabla} U) \cdot \mathrm{d}\vec{r} \implies \text{substitute in our scalar field} \\
&= -\int_{r_1}^{r_2} \mathrm{d}U_r \implies \text{only the component along the path is non-zero} \\
&= -\Big(U_r\Big|_{r_1}^{r_2} \implies \text{only the end points matter for conservative forces} \\
&= -U(r_1 \to r_2)
\end{aligned}$$

Exactly as we expect. Thus, the potential energy of a conservative force satisfies $\vec{F} = -\vec{\nabla} U$. Moreover, $\vec{\nabla} \times \vec{F} = 0$ for a conservative force. We can substitute in $\vec{F} = -\vec{\nabla} U$,

$$\vec{\nabla} \times (-\vec{\nabla} U) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial U}{\partial x} & \dfrac{\partial U}{\partial y} & \dfrac{\partial U}{\partial z} \end{vmatrix}$$

$$\begin{aligned}
&= -\left(\frac{\partial}{\partial y}\frac{\partial U}{\partial z} - \frac{\partial}{\partial z}\frac{\partial U}{\partial y}\right)\hat{\imath} - \left(\frac{\partial}{\partial z}\frac{\partial U}{\partial x} - \frac{\partial}{\partial x}\frac{\partial U}{\partial z}\right)\hat{\jmath} - \left(\frac{\partial}{\partial x}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\frac{\partial U}{\partial x}\right)\hat{k} \\
&= -\left(\frac{\partial^2 U}{\partial y\,\partial z} - \frac{\partial^2 U}{\partial z\,\partial y}\right)\hat{\imath} - \left(\frac{\partial^2 U}{\partial z\,\partial x} - \frac{\partial^2 U}{\partial x\,\partial z}\right)\hat{\jmath} - \left(\frac{\partial^2 U}{\partial x\,\partial y} - \frac{\partial^2 U}{\partial y\,\partial x}\right)\hat{k} \\
&= 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} = 0
\end{aligned}$$

Note for the above we are assuming that U is twice continuously differentiable. If U can be differentiated twice, then its partial derivatives are independent of the order and all terms cancel (e.g, $\frac{\partial U}{\partial y \partial z} = \frac{\partial U}{\partial z \partial y}$).

Sample Problem 8-3

A potential energy has the function of $U(r) = U_0 - \frac{1}{2} A\sigma^2 e^{-r^2/\sigma^2}$, where U_0, A, and σ are all constants. **What is the force for this potential?**

Solution

Since the potential energy has a radial component only, we only need the radial component of the gradient ($\vec{F} = -\vec{\nabla} U$) to find the force.

$$\vec{F} = -\vec{\nabla} U = -\frac{\partial U(r)}{\partial r}\hat{r} = -\left(\frac{1}{2}A\sigma^2\right)(-\frac{2r}{\sigma^2}e^{-r^2/\sigma^2})\hat{r} = -rAe^{-r^2/\sigma^2}\hat{r}$$

For a potential, U, there can be points where $-\vec{\nabla} U = \vec{F} = 0$. Mathematically, these points are located where the derivative of the potential is zero and correspond to points of local maxima or local minima. Figure 8.5) shows a sketch of a potential with a local maximum and local minimum. These locations are also known as *equilibrium points* or saddle points.

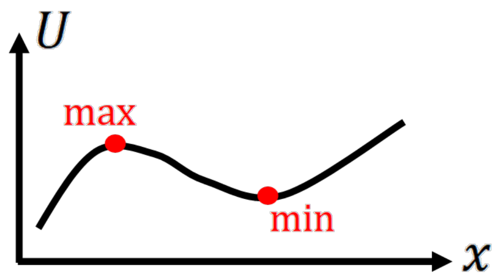


Figure 8.5: Example potential with a local maximum and local minimum.

A local minimum is a *stable* equilibrium point. At the minimum, if a particle is slightly perturbed, it wouldn't really go anywhere. The particle will feel a force that just brings it back to the minimum saddle point. Recall that the force is the negative derivative of the potential, so if you perturb the particle to a lower x value, the potential has a negative slope and the force will be positive back toward the saddle point. And if you perturb the particle to a higher x value, the potential has a positive slope and the force will be negative back toward the saddle point. So for a small shift in position, your particle more or less stays at the saddle point.

At the local maximum saddle point, however, a slight perturbation will have a huge effect on the particle's motion. If you slightly perturb the particle to a lower x value, the potential has a positive slope so the force will be negative toward even more negative x values. (Similar case if you perturb the particle to a higher x value). So a slight perturbation at the maximum of a potential will cause the system to be unstable and move away from that position.

Quick Questions

1. Do the math for this problem and show yourself that the minimum is an equilibrium saddle point where the force will always point back to the minimum and the maximum is an unstable saddle point where the force will always point away from the maximum.

Cora's Thoughts

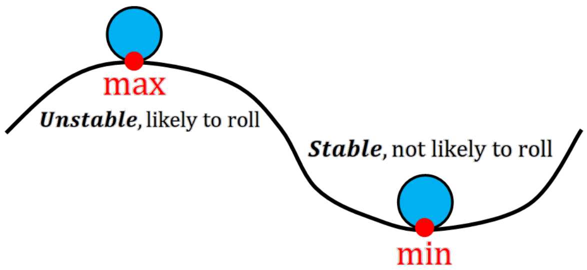


Figure 8.6: A ball rolling on the same potential graph from Figure 8.5.

When looking at potential graphs it can be useful to consider the functions as hills for a ball to roll on. At the exact max of the function, the ball will not roll, however, with a small perpetration, it will roll making it an unstable position. At the minimum, the ball will also be stationary, but with a small perpetration, it will not roll much. The valley that the ball is in keeps it in a is stable position.

Sample Problem 8-4

A particle is moving in one dimension under the influence of the potential $U(x) = 2x^3 - 3x^2 - 12x + 30$, where all values are in SI units. **Find the equilibrium point(s) of this potential. Which point(s) are stable?**

Solution

Figure 8.5 shows a sketch of this potential (a cubic function). So we should expect two equilibrium points, one maximum and one minimum. Since equilibrium points are where $\vec{F} = 0$, we can set the derivative of the potential to zero to find their location.

$$0 = \frac{\partial U(x)}{\partial x} = 6x^2 - 6x - 12 = x^2 - x - 2$$

which is a quadratic equation with the solutions $x = -1$ m and $x = 2$ m.

Although there are two equilibrium points, only one of these is a *stable* equilibrium solution.

Visually, we can see from the figure of the potential, that our minimum saddle point is at $x = 2$ m. But if we wanted to calculate the saddle point without plotting, the true equilibrium position is when the second derivative of the potential is positive. The second derivative of our potential is:

$$\frac{\partial^2 U(x)}{\partial x^2} = \frac{\partial}{\partial x}(6x^2 - 6x - 12) = 12x - 6$$

At $x = -1$ m, $U'' < 0$ and at $x = 2$ m, $U'' > 0$. So the stable point is at $x = 2$ m.

Saddle Points in Space

Lagrange points are saddle points that correspond to maxima or minima in gravitational potential between the planets and the Sun. These are special points where gravity from the Sun and planet and the centrifugal force balance. For example, Jupiter has a collection of "moons" called Trojan asteroids that located in two clusters within Jupiter's orbit at the Lagrange Points L4 and L5, which are potential minima. These asteroids are effectively trapped by a local potential miminum saddle point and they orbit the Sun (not Jupiter) in lock-step with Jupiter. For more information, the hyperphysics webpage for Trojan satellites and NASA's webpage on Lagrange points.

Sample Problem 8-5

A particle of mass m moves in a potential given by the following equation: $U = k(\frac{1}{3}x^2 + 4y^2)$. **What is the equation for acceleration for the particle in both x and y coordinates?**

Solution

To solve for the acceleration, we need to start by finding the force, since Newton's second law states that $\sum F = ma$, assuming the particle mass is constant.

The force is given by the gradient of the potential, Equation (8.14). Since our potential has Cartesian coordinates, we'll use Cartesian coordinates for the gradient.

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} = -\frac{2}{3}kx \\ F_y &= -\frac{\partial U}{\partial y} = -8ky \end{aligned}$$

Using Newton's Second Law:

$$\begin{aligned} F_x &= m\ddot{x} = -\frac{2}{3}kx \\ F_y &= m\ddot{y} = -8ky \end{aligned}$$

So we have:

$$\begin{aligned} \ddot{x} &= -\frac{2}{3m}kx \\ \ddot{y} &= -\frac{8}{m}ky \end{aligned}$$

Note that both of these accelerations have the form of simple harmonic motion (see Chapter 3). Thus, this potential represents a 2-D harmonic oscillator.

Quick Question

1. Solve the differential equation of motion for each axis for this 2-D harmonic oscillator.

Sample Problem 8-6

A particle of mass m is moving under the influence of a potential of $U = Ay^2e^{-x}$, where A is a constant. No other forces act on the particle. **Find the work required to move the particle under a straight path from points $(2,1)$ to $(2,3)$.**

Solution

To solve for the work, you can do one of two things. You can use $U = -\Delta W$ (true if there are no non-conservative forces) or you can solve for F and specify that $\Delta W = \int F \mathrm{d}r$.

Case 1: $\Delta W = -\Delta U$ (this is the simplest of the methods).

$W(1 \to 2) = -[U(2,3) – U(2,1)] = U(2,1) – U(2,3) = A(e^{-2}) – A(9e^{-2}) = -8Ae^{-2}$

Case 2: Solve for F using the gradient of U

$$F_x = -\frac{\mathrm{d}U}{\mathrm{d}x} = Ay^2e^{-x}, \qquad F_y = -\frac{\mathrm{d}U}{\mathrm{d}y} = -2Aye^{-x}$$

With the force, you can get work from:

$$\begin{aligned}
W(2,1 \to 2,3) &= \int F_x \mathrm{d}x + \int F_y \mathrm{d}y \\
&= \cancelto{0}{\int_2^2 (Ay^2e^{-x})\mathrm{d}x} + \int_1^3 (-2Aye^{-x})\mathrm{d}y \quad \Longrightarrow \quad \text{no change in } x \\
&= -2Ae^{-2}\int_1^3 y\mathrm{d}y \quad \Longrightarrow \quad \text{Take } e^{-x} \text{ out of the integral, } x = 2 \\
&= -2Ae^{-2}\left(\frac{1}{2}y^2\right|_1^3 \\
&= -8Ae^{-2}
\end{aligned}$$

As you can see we can get the same answer with both cases.

8.9 Summary

Key Takeaways

This chapter introduces kinetic energy, potential energy, and work. These are fundamental concepts in physics. Energy is a scalar quantity, which can make physics problems easier to solve.

We defined the concept of work as

$$W = \int \vec{F} \cdot \mathrm{d}\vec{r}$$

Work is always measured as a relative quantity and it can be positive or negative. Broadly, work is defined based on how a force affects the motion of a system. If the force helps the motion, then it does positive work. If the force opposes the motion, then it does negative work.

The change in work equals the change in kinetic energy through the work-kinetic energy theorem.

$$W(r_1 \to r_2) = \Delta K$$

where the kinetic energy can be either translation (linear motion) or rotation.

$$\begin{aligned} K &= \frac{1}{2}mv^2 \implies \text{for translation} \\ K &= \frac{1}{2}I\omega^2 \implies \text{for rotation} \end{aligned}$$

This chapter also introduced conservative forces, such as gravity, which are a class of forces where the work done is independent of the path taken, and energy is conserved.

$$\begin{aligned} W(r_1 \to r_1) = \oint \vec{F} \cdot \mathrm{d}\vec{r} &= 0 \\ \vec{\nabla} \times \vec{F} &= 0 \end{aligned}$$

Conservative forces are also associated with a potential energy, defined by:

$$\vec{F} = -\vec{\nabla} U$$

The potential energy is important to setting how a system, when released, will move because it directly connects to a force. The chapter briefly introduces saddle points, which are regions of local stability or instability within a potential field. Examples of conservative forces and potential energy are given for gravity.

For systems with only conservative forces,

$$\Delta E = \Delta(K + U)$$

which is the equation of energy conservation.

Important Equations

Work:

$$\begin{aligned} W &= \int \vec{F} \cdot \mathrm{d}\vec{r} \\ W(r_1 \to r_2) &= -W(r_2 \to r_1) \quad \Longrightarrow \quad \text{for conservative forces} \end{aligned}$$

Work Kinetic Energy Theorem:

$$W(r_1 \to r_2) = \Delta K$$

Kinetic Energy:

$$\begin{aligned} K &= \frac{1}{2}mv^2 \quad \Longrightarrow \quad \text{for translation} \\ K &= \frac{1}{2}I\omega^2 \quad \Longrightarrow \quad \text{for rotation} \end{aligned}$$

Gravitational Force:

$$\begin{aligned} \vec{F}_g &= -\frac{GMm}{r^2}\hat{r} \\ \vec{g} &= -\frac{GM}{r^2}\hat{r} \end{aligned}$$

Escape Velocity:

$$v = \sqrt{\frac{2GM_E}{R_E}}$$

Conservative Forces:

$$\begin{aligned} W(r_1 \to r_1) = \oint \vec{F} \cdot \mathrm{d}\vec{r} &= 0 \\ \vec{\nabla} \times \vec{F} &= 0 \\ \vec{F} &= -\vec{\nabla}U \end{aligned}$$

Potential Energy:

$$U(r_1 \to r_2) = -\int_{r_1}^{r_2} \vec{F} \cdot \mathrm{d}\vec{r} = -W_{con}(r_1 \to r_2)$$

Conservation of Energy:

$$\Delta E = \Delta(K + U)$$

8.10 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 8-1

Find the work done in each of the following cases:

a) Lifting a crate of mass m from the floor to a table of height h.

b) Pushing a 1000 kg car 100 m up a 10° incline at constant speed. Neglect friction.

c) A particle moving under a force $\vec{F} = (x^2 + 3x + 4)\hat{\imath}$ from $x = 1$ to $x = 3$.

Practice Problem 8-2

Calculate the escape velocity for the Sun at:

a) The surface of the Sun.

b) At the average orbital radius of the Earth (1 au). Compare this to the escape velocity of the Earth.

c) The average orbital radius of Neptune (30.1 au).

d) Plot the escape velocity as a function of distance from the Sun.

Practice Problem 8-3

For each of the following forces, determine if the force is conservative.

a) $\vec{F} = (xy)\,\hat{\imath} + (yz)\,\hat{\jmath} + (xz)\,\hat{k}$

b) $\vec{F} = (xz)\,\hat{\imath} + (y^3z^2)\,\hat{\jmath} + (x^2z)\,\hat{k}$

c) $\vec{F} = (yz)\,\hat{\imath} + (xz)\,\hat{\jmath} + (xy)\,\hat{k}$

d) $\vec{F} = r\sin(2\phi)\hat{r} + r\cos(2\phi)\hat{\phi} + 3\hat{z}$ (Hint: use cylindrical coordinates)

Practice Problem 8-4

A force has the form $\vec{F} = (ax + by^2)\hat{\imath} + (cxy)\hat{\jmath}$, where a, b, and c are all constants. Under what condition(s) is this force conservative?

Practice Problem 8-5

What value of c will make $\vec{F} = \left(\frac{z}{y}\right)\hat{\imath} + c\left(\frac{xz}{y^2}\right)\hat{\jmath} + \left(\frac{x}{y}\right)\hat{k}$ conservative force?

Practice Problem 8-6

Find the equation for the conservative force that produces the following potentials:

a) $U = 2x + 3y^2 + 4z^2$

b) $U = x^2y^2 + z^3$

Practice Problem 8-7

A particle can move only along the x-axis. It is in a potential defined as $U = Bx + \frac{A}{x}$, where B and A are constants. What is the equilibrium position of this particle?

Practice Problem 8-8

A particle of mass m moves under a potential of $U(x, y, z) = ax + by^2 + cz^3$ with no other forces acting on it. The parameters a, b, and c are all constants. If the particle is momentarily at rest at position $(1, 1, 1)$, what is its speed at the origin? (Hint: you do not need to solve any differential equations of motion for this problem.)

Practice Problem 8-9

Famous tennis player Serena Williams is playing a match when her opponent sends the 0.0577 kg tennis ball is 1.5 m above the ground and has a speed of 20.0 m/s. Williams hits the ball, doing 50 J of work on it. However, due to air resistance the energy of the ball halves by the time it hits the ground.

a) Determine the potential energy of the ball before Williams hits it.

b) Determine the kinetic energy of the ball before Williams spikes it.

c) Determine the total mechanical energy of the ball before Williams hits it.

d) Determine the total mechanical energy of the ball upon hitting the ground on the opponent's side of the net.

e) Determine the speed of the ball upon hitting the ground on the opponent's side of the net.

Practice Problem 8-10

Consider the forces $F_1 = x\hat{\imath} + y\hat{\jmath}$, and $F_2 = y\hat{\imath} - x\hat{\jmath}$.

a) Find the work done by both forces to move a particle from position $(0, 0)$ to $(0, 1)$ and then the work done to move the particle from position $(0, 1)$ to $(1, 1)$

b) Find the work done by both forces to move a particle from position $(0, 0)$ to $(1, 1)$ using a direct path (e.g., using the line $y = x$).

c) Note that the initial and final points in a) and b) are the same. How does the total work done from both forces compare between a) and b)? What does that mean for the forces?

d) Verify your answer for c) by taking the curl of both forces.

9 Application of Energy Conservation

Learning Objectives

- Apply energy conservation to problems involving rolling and translation
- Apply energy conservation to problems involving simple harmonic motion
- Identify the differential equation of motion from energy conservation

In this chapter, we will apply energy conservation to solve problems in physics. Throughout this chapter, compare the method of energy conservation to using Newton's laws.

9.1 Energy Conservation

In Chapter 8, we showed that conservative forces had the property of

$$\boxed{\Delta K + \Delta U = 0} \tag{9.1}$$

where K is the kinetic energy and U is the potential energy. This result indicates that $K + U = E = \text{constant}$.

$$\boxed{K + U = E = \text{constant}} \tag{9.2}$$

where E is the mechanical energy of the system. If E is a constant, then your mechanical energy equals the total (kinetic plus potential) energy of your system.

Because energy is scalar instead of a vector quantity, it is sometimes easier to solve a question using energy conservation than using Newton's Laws. The important points to consider are:

1. What are your sources of kinetic energy (translation versus rotation)?
2. What are your sources of potential energy?
3. How can you write both forms of energy in terms of the parameters needed and in terms of time?

If you can answer those questions, you can solve physics problems using energy conservation.

If energy is conserved, you can compare the energy before and after the motion. The total energy initially must equal the total energy at the end. Any gain or loss in kinetic energy corresponds to a gain or loss in potential energy. This is similar to how we applied the conservation of momentum and angular momentum to problems.

$$\begin{aligned} E_i &= E_f \\ K_i + U_i &= K_f + U_f \end{aligned}$$

For this method to be applicable, you need to have a clearly defined energy (potential and kinetic) for at least one point in the motion.

Alternatively, if energy is conserved, then E is a constant and

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = 0} \tag{9.3}$$

For this method to be applicable, you need to express the energy as a function of a time t.

9.2 Energy Conservation in 3-D

Consider a particle moving in 3-D,

$$\begin{aligned} K &= \frac{1}{2}m(\vec{v}\cdot\vec{v}) \quad \Longrightarrow \quad \text{Note that } \vec{v}\cdot\vec{v} = v^2 \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \end{aligned}$$

in the case of Cartesian coordinates and linear motion.

Now, the rate of change of the kinetic energy for this particle is given by $\frac{\mathrm{d}K}{\mathrm{d}t}$. This does not need to be constant with time (although E is assumed to be constant).

$$\begin{aligned} \frac{\mathrm{d}K}{\mathrm{d}t} &= \frac{1}{2}m\frac{\mathrm{d}}{\mathrm{d}t}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \\ \frac{\mathrm{d}K}{\mathrm{d}t} &= \frac{1}{2}m(2\dot{x}\ddot{x} + 2\dot{y}\ddot{y} + 2\dot{z}\ddot{z}) \\ \frac{\mathrm{d}K}{\mathrm{d}t} &= m(\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z}) \quad \Longrightarrow \quad \text{Note the bracket term is } \dot{\vec{r}}\cdot\ddot{\vec{r}} \\ \frac{\mathrm{d}K}{\mathrm{d}t} &= m\dot{\vec{r}}\cdot\ddot{\vec{r}} \\ \frac{\mathrm{d}K}{\mathrm{d}t} &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}\cdot\vec{F} \quad \Longrightarrow \quad \text{Recall that } \vec{F} = m\vec{a} = m\ddot{\vec{r}} \\ \mathrm{d}K &= \mathrm{d}\vec{r}\cdot\vec{F} \end{aligned}$$

Due to symmetry with the dot product, we can say that $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$. So we end up with:

$$\begin{aligned} \mathrm{d}K &= \vec{F}\cdot\mathrm{d}\vec{r} \\ \mathrm{d}K &= \mathrm{d}W_{net} \end{aligned}$$

which is just our work-kinetic energy theorem again.

What about the potential energy? Well, our potential energy must be dependent on position (a conservative force depends on position). Assume that $U \to U(\vec{r})$.

$$\begin{aligned} \frac{\mathrm{d}U}{\mathrm{d}t} &= \frac{\mathrm{d}U(\vec{r})}{\mathrm{d}t} \\ \frac{\mathrm{d}U}{\mathrm{d}t} &= \frac{\partial U}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial U}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial U}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \quad \Longrightarrow \quad \text{apply each coordinate} \end{aligned}$$

Recall that $\vec{\nabla} U = \dfrac{\partial U}{\partial x}\hat{\imath} + \dfrac{\partial U}{\partial y}\hat{\jmath} + \dfrac{\partial U}{\partial z}\hat{k}$. Therefore, we can re-write the above as:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \vec{\nabla} U \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$$

So if we assume that we only have conservative forces, then $E = K + U$ then the time derivative of the energy is:

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}K}{\mathrm{d}t} + \frac{\mathrm{d}U}{\mathrm{d}t} \\ &= \vec{F} \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} + \vec{\nabla} U \cdot \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \cdot (\vec{F} + \vec{\nabla} U)\end{aligned}$$

Since $\vec{F} = -\vec{\nabla} U$ for a conservative force, we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \cdot (\vec{F} + \vec{\nabla} U) = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \cdot (\vec{F} - \vec{F}) = 0$$

for any velocity for the particle. So if you have a system that is fully described by conservative forces in any dimension, then that system will have its total energy conserved.

Keep in mind that not all forces that are functions of position are conservative forces. For a system to have a conservative force, $\vec{\nabla} \times \vec{F} = 0$ (see Chapter 8) or the work done on the system within a close loop must be zero.

Sample Problem 9-1

Consider a force $\vec{F} = -ky\hat{\imath} + kx\hat{\jmath}$. **Show that the work done on a particle by this force in a closed loop does not equal to zero.**

Solution

While one could demonstrate that $\vec{F}$ is a conservative force by showing $\vec{\nabla} \times \vec{F} \neq 0$, the problem specifically asks to show the solution for a closed loop. To solve this problem, we need to evaluate the integral:

$$\oint \vec{F} \cdot \mathrm{d}\vec{r}$$

where $\oint$ indicates a closed loop. You can take any closed loop for a conservative force. So it is handy to take a direct (simple) closed path. Figure 9.1 shows a very simple closed loop from $(0,0) \to (a,0) \to (a,b) \to (0,b) \to (0,0)$. You can technically do any closed loop, but make the math easy for yourself and take something simple.

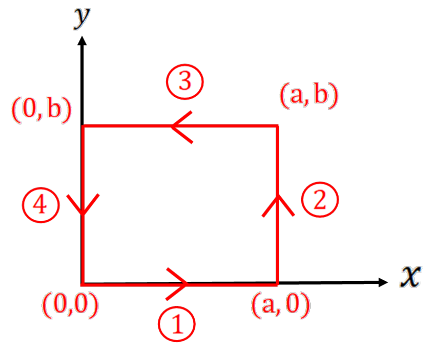


Figure 9.1: A simple closed loop. You can define any closed loop for conservative forces, but direct paths are the most mathematically simple to use. Here we have four paths that are labeled as (1), (2), (3), and (4).

So we can break up the work for this system using each of the different path legs. For the first path length (1), it is entirely along the x-axis, so we can solve the work from just F_x because no work is being done in y along that axis.

$$W_1 = W(0,0) \to W(a,0) = \int_0^a F_x \mathrm{d}x = -ky \int_0^a \mathrm{d}x = -kya = 0$$

because $y = 0$ for this section, (e.g., $W_1 = -kya = 0$ because $y = 0$).

We can make similar calculations for the other sections.

$$\begin{aligned}
W_2 &= W(a,0) \to W(a,b) = \int_0^b F_y \mathrm{d}y = kx \int_0^b \mathrm{d}y = kxb = kab \quad (x = a) \\
W_3 &= W(a,b) \to W(0,b) = \int_a^0 F_x \mathrm{d}x = -ky \int_a^0 \mathrm{d}x = kya = kab \quad (y = b) \\
W_4 &= W(0,b) \to W(0,0) = \int_b^0 F_y \mathrm{d}y = kx \int_b^0 \mathrm{d}y = -kxb = 0 \quad (x = 0)
\end{aligned}$$

So the total work done in this closed loop is $W = W_1 + W_2 + W_3 + W_4 = 0 + kab + kab + 0 = 2kab \neq 0$. So the above force is *not* conservative.

Quick Questions

1. Confirm that the force $\vec{F} = -ky\hat{\imath} + kx\hat{\jmath}$ is not conservative by taking $\vec{\nabla} \times \vec{F}$.
2. Is $\vec{F} = (xy)\hat{\imath} + (xz)\hat{\jmath} + (yz)\hat{k}$ conservative? Try both the closed loop and the curl methods. Which do you like better?

9.3 Example Problems: Gravity and Rotation

Sample Problem 9-2

A uniform spherical shell of mass M and radius R is able to rotate about a vertical axis without friction (Figure 9.2). A massless rope passes around the shell at its equator and over a pulley to a smaller mass m that is hanging over the edge of a table. **If there are no losses in energy, what is the velocity of the mass m after it has fallen a distance h from rest?** Assume the pulley is a disk with mass M_p and radius R_p.

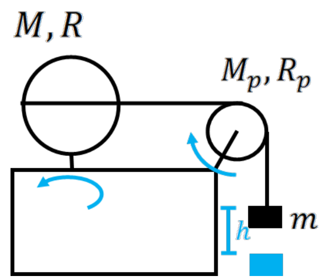


Figure 9.2: The set up for this problem shows the spherical shell of mass M and R that can rotate on an axis. There is a disk-pulley with mass M_p and radius R_P that can likewise rotate about an axis. There is a mass m hanging over the edge that can descend due to gravity. There is no friction (no energy loss) and that the rope has no mass and cannot stretch.

Solution

First, how will this system move? At time $t = 0$, everything is at rest. Then the mass m is released and drops a distance h, pulling on the rope and subsequently rotating the pulley and the spherical shell (see blue marks in Figure 9.2).

You can try to solve this problem using forces, torques, and Newton's laws, but we will use energy here. We are told that there are no losses in energy, so the only force driving the motion is gravity, which is a conservative force.

Energy conservation requires measuring ΔU and ΔK. So we need to consider all the sources of potential energy and all the sources of kinetic energy.

Potential energy: In this case, there is only one source of potential energy, the little mass m. It drops in height where the change in height is $\Delta y = -h$, where the negative indicates that the object decreased in height from its original reference position (it's position at $t = 0$). Recall that potential energy from gravity is equal to $U_g = mg\Delta y$, if you assume that the gravitational acceleration is constant (see Chapter 8). Thus, we can take any convenient reference position for Δy. The height at $t = 0$ is a convenient reference position, so we'll use that. Therefore, we have

$$\Delta U = -mgh$$

There is a loss of potential energy, which means there will be a gain in kinetic energy, as expected.

Kinetic energy: Initially, everything is at rest, so $K_i = 0$. But after the mass m moves, there are three sources of kinetic energy. There is the moving mass m, the rotating pulley, and the rotating spherical shell.

$$\begin{aligned} K &= K_m + K_p + K_s \\ K &= \frac{1}{2}mv^2 + \frac{1}{2}I_p\omega_p^2 + \frac{1}{2}I_s\omega_s^2 \end{aligned}$$

where $v = \dot{y}$ is the speed of the mass, I_p and ω_p are the moment of inertia and angular velocity of the pulley, and I_s and ω_s are the moment of inertia and angular velocity for the spherical shell.

The pulley is a disk, so $I_p = \frac{1}{2}M_pR_p^2$. The moment of inertia for a spherical shell is $I_s = \frac{2}{3}M_sR_s^2$ (see Appendix A.4).

Since the rope is massless and cannot be stretched (it is inextensible), the velocity at any point of the rope must be constant. If we say that the mass moves at a velocity v, then the velocity vector where the rope meets the pulley has a speed v and the velocity vector where the rope meets the shell has a speed v. So the pulley and shell have the same linear velocity v at the radii where the rope contacts them. That means the linear velocity is v at a radius of R_s for the shell at at a radius of R_p for the pulley.

Since we have only rolling motion, we can say:

$$\omega_p = \frac{v}{R_p}$$
$$\omega_s = \frac{v}{R_s}$$

Taking our equations for the angular speeds and the moments of inertia, we get:

$$\begin{aligned} K &= \frac{1}{2}mv^2 + \frac{1}{2}I_p\omega_p^2 + \frac{1}{2}I_s\omega_s^2 \\ K &= \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}M_pR_p^2\right)\left(\frac{v}{R_p}\right)^2 + \frac{1}{2}\left(\frac{2}{3}M_sR_s^2\right)\left(\frac{v}{R_s}\right)^2 \\ K &= \frac{1}{2}mv^2 + \frac{1}{4}M_pv^2 + \frac{1}{3}M_sv^2 \end{aligned}$$

So our change in kinetic energy is:

$$\begin{aligned} \Delta K &= K_f - K_i \\ \Delta K &= \frac{1}{2}mv^2 + \frac{1}{4}M_pv^2 + \frac{1}{3}M_sv^2 \end{aligned}$$

From the conservation of energy, we have $\Delta K = -\Delta U$. Sub in our equations for ΔK and ΔU.

$$\begin{aligned}
-\Delta U &= \Delta K \\
mgh &= v^2\left(\frac{1}{2}m + \frac{1}{4}M_p + \frac{1}{3}M_s\right) \\
v^2 &= \frac{mgh}{\frac{1}{2}m + \frac{1}{4}M_p + \frac{1}{3}M_s} \\
v &= \sqrt{\frac{2gh}{1 + \frac{1}{2}\frac{M_p}{m} + \frac{2}{3}\frac{M_s}{m}}}
\end{aligned}$$

So we have solved for the speed of the mass. To get the velocity, we need to specify a direction. In this case, we know that the mass is falling down, so the direction would be down.

Sample Problem 9-3

Let's try to solve the motion of a Atwood Machine where the pulley is a disk of mass M and radius R. See Figure 9.3. Assume that the rope connecting the masses is light (negligible mass) and inextensible (any stretch of the rope is negligible) and that the pulley has a frictionless ball bearing (so no energy losses). **What is the acceleration of the two masses?**

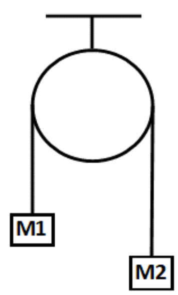


Figure 9.3: The Atwood machine. Assume that the rope is inextensible and that there is no friction on the bearing for the pulley. The pulley is a disk of mass M and radius R.

Solution

While this problem can also be solved using Newton's laws and forces, let's look at energy conservation.

$$K + U = E = \text{constant}$$

First step is to consider all sources of kinetic energy and all sources of potential energy.

For potential energy, we have the two masses within a gravitational field. For small distances, we can assume that $F_g = mg$ and that means that $U = mg\Delta y$, where Δy indicates the change in vertical. If we set $y = 0$ to be at the midpoint of the pulley (see Figure 9.4), then the potential energy of the masses are $U_1 = -mgy_1$ and $U_2 = -mgy_2$, where y_1 and y_2 are the positions of the masses relative to the pulley.

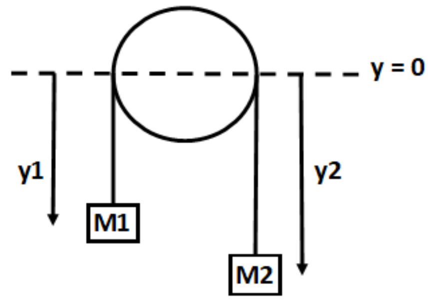


Figure 9.4: Position of masses in the Atwood machine. The midpoint of the pulley sets the $y = 0$ point, with the masses distances y_1 and y_2 being measured from the $y = 0$.

Note that there is no potential energy from the pulley because the pulley does not move vertically. So there is no work done by gravity in moving the pulley (by its centre of mass).

So our potential energy of the system is given by:

$$U = -mgy_1 - mgy_2$$

This is the potential energy for a given time, t. We don't know which mass will move up and which one will move down. All we know is that m_1 and m_2 are at specific positions y_1 and y_2 at t.

For the kinetic energy, there are three sources of kinetic energy in this system. We have the translation motion of m_1, the translation motion of m_2, and the rotational motion of the pulley. For the translation motion, we have $K_1 = \frac{1}{2}m_1(\dot{y}_1)^2$ and $K_2 = \frac{1}{2}m_2(\dot{y}_2)^2$. For the rotational motion, we have $K_p = \frac{1}{2}I\omega^2$.

$$\begin{aligned} K &= K_1 + K_2 + K_p \\ K &= \frac{1}{2}m_1(\dot{y}_1)^2 + \frac{1}{2}m_2(\dot{y}_2)^2 + \frac{1}{2}I\omega^2 \end{aligned}$$

Similar to the previous problem, we need to connect the rotational motion to the translation motion. The pulley rotates at an angular speed of ω. Since the rope is inextensible (does not stretch), we can assume that the two masses move at the same speed ($|\dot{y}_1| = |\dot{y}_2| = v$) and with the same linear speed as the contact point of the pulley, which is $v = R\omega$ (e.g., see Figure 9.5).

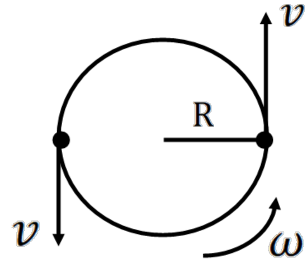


Figure 9.5: Rotation of the pulley assuming $m_1 > m_2$. The pulley rotates at the angular speed ω. The velocity of that angular speed at the two points shown will be $v = \omega R$ where v is the speed of the masses.

Re-writing our kinetic energy equation, we have:

$$
\begin{aligned}
K &= \frac{1}{2}m_1(\dot{y}_1)^2 + \frac{1}{2}m_2(\dot{y}_2)^2 + \frac{1}{2}I\omega^2 \quad \Longrightarrow \quad \text{where } |\dot{y}_1| = |\dot{y}_2| = \omega R \\
K &= \frac{1}{2}m_1(\dot{y}_1)^2 + \frac{1}{2}m_2(\dot{y}_1)^2 + \frac{1}{2}I\left(\frac{\dot{y}_1}{R}\right)^2 \quad \Longrightarrow \quad \text{use } \dot{y}_1 \text{ for simplicity} \\
K &= \frac{1}{2}m_1(\dot{y}_1)^2 + \frac{1}{2}m_2(\dot{y}_1)^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{\dot{y}_1}{R}\right)^2 \quad \Longrightarrow \quad I = \tfrac{1}{2}MR^2 \text{ for a disk} \\
K &= \frac{1}{2}m_1(\dot{y}_1)^2 + \frac{1}{2}m_2(\dot{y}_1)^2 + \frac{1}{4}M(\dot{y}_1)^2
\end{aligned}
$$

Similar to the potential energy, this kinetic energy is for time t when the masses are moving at a speed of v. If the system starts at rest, we would need to calculate the change in position of y_1 and y_2 to get the change in kinetic energy. But we were not told of an initial configuration. Instead, we have determined U and K at time t. So we will use Equation 9.3:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

Our total energy is $E = U + K$ which is a constant. So at time t, the sum of the potential energy and kinetic energy is:

$$
\begin{aligned}
E &= U + K \\
E &= -mgy_1 - mgy_2 + \frac{1}{2}m_1(\dot{y}_1)^2 + \frac{1}{2}m_2(\dot{y}_1)^2 + \frac{1}{4}M(\dot{y}_1)^2
\end{aligned}
$$

Since the total energy is constant for a system with only conservative forces, the time derivative of the total energy is zero (Equation 9.3):

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= 0 \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[-mgy_1 - mgy_2 + \frac{1}{2}m_1(\dot{y}_1)^2 + \frac{1}{2}m_2(\dot{y}_1)^2 + \frac{1}{4}M(\dot{y}_1)^2\right] \\
0 &= -m_1 g\dot{y}_1 - m_2 g\dot{y}_2 + m_1\dot{y}_1\ddot{y}_1 + m_2\dot{y}_1\ddot{y}_1 + \frac{1}{2}M\dot{y}_1\ddot{y}_1 \\
0 &= -m_1 g\dot{y}_1 + m_2 g\dot{y}_1 + m_1\dot{y}_1\ddot{y}_1 + m_2\dot{y}_1\ddot{y}_1 + \frac{1}{2}M\dot{y}_1\ddot{y}_1 \quad \Longrightarrow \quad \text{sub } \dot{y}_2 = -\dot{y}_1 \\
0 &= -m_1 g + m_2 g + m_1\ddot{y}_1 + m_2\ddot{y}_1 + \frac{1}{2}M\ddot{y}_1 \quad \Longrightarrow \quad \text{eliminate } \dot{y}_1 \\
0 &= g(m_2 - m_1) + \ddot{y}_1\left(m_1 + m_2 + \frac{1}{2}M\right) \\
\ddot{y}_1 &= \frac{g(m_1 - m_2)}{m_1 + m_2 + \frac{1}{2}M}
\end{aligned}$$

So our acceleration of m_1 is given by the above equation. And we can find the acceleration of m_2 from $\ddot{y}_2 = -\ddot{y}_1$.

Quick Question

1. Consider the difference between using Newton's second law versus energy conservation for the above Atwood question. Which method do you find better or easier to use? Why?

9.4 Application to Simple Harmonic Motion

In general, you can use Newton's laws or energy conservation to solve simple harmonic motion problems. But there are many cases where energy conservation can save you a lot of extra work. Consider using Newton's laws to calculate the following problem instead.

Sample Problem 9-4

Consider a mass m hanging from a disk pulley of mass M and radius R as shown in Figure 9.6. The pulley is supported by an inextensible and massless rope that is fixed to the ceiling at one end and attached to a spring of spring constant k on the other end. If the pulley rotates without slipping, **find the equilibrium position and the period of oscillations if the small mass m is pulled down a small distance.** Assume there is no loss of energy from friction.

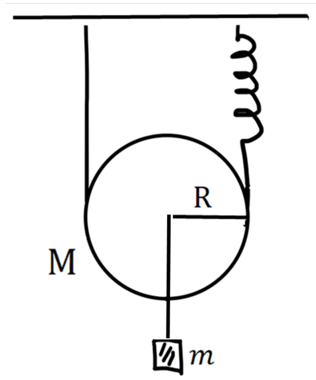


Figure 9.6: A "simple" harmonic oscillator formed by a pulley and spring. The pulley is a disk of radius R and mass M that is held up by an inextensible cord that is attached to the ceiling on one end and attached to a spring of spring constant k on the other end. A small mass m hangs from the centre of the disk.

Solution

Find the equilibrium position. When you are in equilibrium, there is no movement, so there is no rotation and no velocity. That means that all forces are zero. But before we can answer this question, how do the pulley, mass, and spring move relative to each other?

We can solve for this equilibrium point by setting the net force and net torque on the pulley equal to zero (that will be the equilibrium point). Figure 9.7 shows the free-body diagram for the pulley.

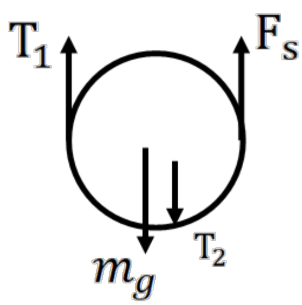


Figure 9.7: Free-body diagram of the pulley. There is a tension T_1 from the rope on the left, and a tension T_2 from the small mass m acting at the centre of mass. The pulley has its own gravity mg. And there is the spring force F_s acting on the right side of the pulley.

Since we're in equilibrium, the net torque must be zero. Therefore, $T_1 = F_s$, otherwise the pulley would rotate. For a spring, $F_s = -kx = T_1$. The other unknown force is T_2, but that is simply the tension caused by the hanging mass m and therefore $T_2 = mg$.

So for our sum of all forces, we have:

$$\begin{aligned}
\sum F &= T_1 + F_s - Mg - mg \\
0 &= -2kx - g(M+m) \quad \Longrightarrow \quad \text{for equilibrium, } \sum F = 0 \\
x_0 &= -\frac{g(M+m)}{2k}
\end{aligned}$$

What is the period of small oscillations? We want the differential equation of motion. If you can get the equation in the form of $\ddot{x} + Cx = 0$, then you can read off ω_0^2 and can get the period.

You can solve this problem using forces and torques, but we will use the conservation of energy here.

The potential energy is given by the gravitational potential energy of the two masses and the potential energy of the spring. Thus, our potential energy is:

$$U = -Mgx - mgx + \frac{1}{2}k\Delta x^2$$

where we have specified that the potential energy is zero for the masses at $x = 0$. A convenient reference point (e.g., setting $x = 0$ for the gravitational energy) is at x_0, since this is a known reference point. Note that the spring potential is *not* zero at $x = x_0$. So we need to consider Δx for the spring.

The kinetic energy of the spring is given by the motion of translation energy of the mass, the translation energy of the pulley, and the rotation of the pulley.

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 + \frac{1}{2}mv^2$$

Before we combine the energies this question, let's first ask how this system will move. The spring will stretch and compress, and this will lower and raise m and the pulley, and the pulley will also rotate. At first glance, you may be tempted to assume that if the mass moves down a distance x, then the pulley should move down a distance x and the spring should be stretched a distance x. But for this system, the spring will *stretch twice as much* as m and M move down because some of the kinetic energy that goes into the pulley and mass is used to rotate the pulley rather than translate the pulley. This is the same principle behind rolling without slipping (see Chapter 7).

Let's look at the motion of the pulley. Figure 9.8 shows the translational and rotational motion of the pulley. First, consider the motion of the mass and pulley. The mass is connected to the pulley at its centre-of-mass by an inextensible rope. Whatever distance one moves, the other will move the same amount, and this motion will equal the motion of the centre-of-mass of the pulley, v_{cm}. Since the pulley is also rotating without slipping, we can connect the centre of mass motion directly to the rotation ($v_{cm} = \omega R$).

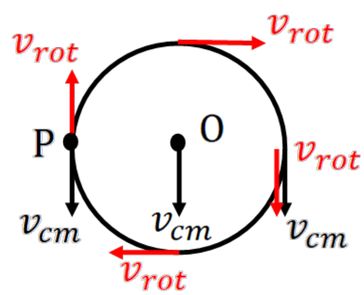


Figure 9.8: Translation and rotational motion of the pulley from Figure 9.6. The entire disk moves down with $v = v_{cm}$. But when the pulley moves, it will also rotate without slipping with $\omega = v_{cm}/R$. So at point P on the fixed side, the velocity is instantaneously zero.

Second, let's consider how the spring stretches relative to the pulley's motion. As the pulley moves down with the stretch of the spring, the pulley will rotate clockwise (see Figure 9.8). Point P is the contact point for the rotation and the net velocity there will be zero. Note that the contact point will be on the side of pulley that is fixed to the ceiling. That's because the other side with the spring is able to change in height, not the fixed side. On the side with the spring, however, the velocities from the translation and rotation add together such that pulley moves away from the spring at twice the speed of the centre of mass.

Quick Questions

1. How would the pulley rotate when it is moving upwards (spring is contracting) instead of moving downwards?
2. At what point in the simple harmonic motion is the rotation of the pulley the fastest? Does that make sense?

So this means that if the mass and pulley move x in time t, the spring stretches (or compresses) by a displacement of $2x$ in the same time. We must take into account this difference in the spring's displacement relative to the vertical displacement of the mass and pulley in our energy conservation.

The total energy is

$$\begin{aligned} E &= K + U \\ E &= \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 + \frac{1}{2}mv^2 - Mgx - mgx + \frac{1}{2}k(2x - x_0)^2 \end{aligned}$$

where the change in potential energy depends on a displacement of $2x - x_0$, because we set our reference position to x_0 and the spring stretches and compresses at twice the rate that the pulley and mass move.

For rotating without slipping, $\omega = \dfrac{v}{R}$. So we can simplify the above as:

$$
\begin{aligned}
E &= \frac{1}{2}I\left(\frac{v}{R}\right)^2 + \frac{1}{2}Mv^2 + \frac{1}{2}mv^2 - (M+m)gx + \frac{1}{2}k(2x-x_0)^2 \\
E &= \frac{1}{2}v^2\left(\frac{I}{R^2} + M + m\right) - (M+m)gx + \frac{1}{2}k(2x-x_0)^2 \\
E &= \frac{1}{2}v^2\left(\frac{1}{2}M + M + m\right) - (M+m)gx + \frac{1}{2}k(2x-x_0)^2 \quad \Longrightarrow \quad I = \tfrac{1}{2}MR^2 \\
E &= \frac{1}{2}v^2\left(\frac{3}{2}M + m\right) - (M+m)gx + \frac{1}{2}k(2x-x_0)^2
\end{aligned}
$$

Now take the derivative of the energy with respect to time.

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} = 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}v^2\left(\frac{3}{2}M + m\right) - (M+m)gx + \frac{1}{2}k(2x-x_0)^2\right] \\
0 &= \left(\frac{3}{2}M + m\right)v\frac{\mathrm{d}v}{\mathrm{d}t} - (M+m)gv + k(2x-x_0)(2v) \\
0 &= \left(\frac{3}{2}M + m\right)\frac{\mathrm{d}v}{\mathrm{d}t} - (M+m)g + 2k(2x-x_0) \quad \Longrightarrow \quad \text{eliminate } v \\
0 &= \frac{\mathrm{d}v}{\mathrm{d}t} + \left(\frac{1}{\frac{3}{2}M + m}\right)\left[4kx - (M+m)g - 2kx_0\right] \\
0 &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \left(\frac{1}{\frac{3}{2}M + m}\right)\left[4kx - (M+m)g - 2k\underbrace{\left(-\frac{g(M+m)}{2k}\right)}_{x_0}\right] \\
0 &= \frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \underbrace{\left(\frac{4k}{\frac{3}{2}M + m}\right)}_{\omega_0^2}x
\end{aligned}
$$

Note how we recover the differential equation of motion directly from energy conservation. We have it in the familiar form that we want and we can solve for the period of oscillations easily from this.

$$
\begin{aligned}
\omega_0^2 &= \frac{4k}{\frac{3}{2}M + m} \\
T &= \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{\frac{3}{2}M + m}{4k}}
\end{aligned}
$$

Try to solve the same problem using torques and forces. Be wary of your vector directions and whatever coordinate system you originally define. You should obtain the exact same solution if done properly.

9.5 Summary

Key Takeaways

This chapter applies the concepts from Chapter 8, in particular those of energy conservation, to physics problems. The main idea of energy conservation is that,

$$E = U + K = \text{constant}$$

Energy is conserved if all of the forces acting on your system are conservative forces. If there is a non-conservative force (e.g., friction), then energy will not be conserved and E will not be constant.

There are two main approaches to solving physics problems with energy conservation.

1. You can derive independent equations for E at two different times, E_1 and E_2. If you can derive an equation for the energy at two distinct times (e.g., you are given a reference speed and position), then you can solve the physics of a problem by setting $E_1 = E_2$.

2. You can derive an general equation for the energy for any given time, t. In this case, you do not have a unique solution for E from your energy equation alone. Nevertheless, since energy is conserved, it is constant with time, which means

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

   By setting the time derivative of E to zero, you will obtain a differential equation of motion from which you can use to solve the physics problem.

While going through this chapter and the practice problems below, compare how the solution from energy conservation with what you would need to do if you were applying Newton's second law instead. Consider which method you prefer and under which circumstances you would favour one over the other.

Important Equations

Energy Conservation:

$$\begin{aligned} \Delta K + \Delta U &= 0 \\ K + U &= E = \text{constant} \\ \frac{\mathrm{d}E}{\mathrm{d}t} &= 0 \end{aligned}$$

9.6 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 9-1

A roller coaster has a frictionless track in the xz-plane as shown below. The roller coaster car starts at the top of the track at rest. When the car is released, at which position(s) will it be moving with a maximum speed?

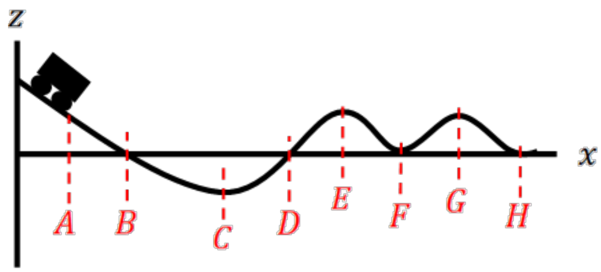


Figure 9.9: The roller coaster car travels along the track starting at position A and ending at position H.

Practice Problem 9-2

A rock of mass M is released from a height h above the ground. What is the velocity of the rock when it has fallen half the distance? Neglect air resistance.

Practice Problem 9-3

A mass m is attached to a spring with constant k and set in simple harmonic motion with an amplitude A. Find the equations for potential and kinetic energy in terms of m, k, and A. Plot these forces and their sum as a function of time from $t = 0$ to $t = T$ (the period of oscillations). For the plot, choose arbitrary values for k, m, and A.

Practice Problem 9-4

A particle of mass m slides on a frictionless wire that is bent to form a loop. The wire has a loop of radius R and the particle starts from rest at a point that is level with the midpoint of the loop as shown in the figure. What is the magnitude of the normal force on the particle when it reaches the bottom of the loop?

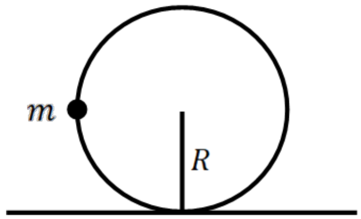


Figure 9.10: The particle of mass m travels on the loop of radius R.

Practice Problem 9-5

A small mass m forms a pendulum with an ideal rope of length L. The small mass is brought to an angle of 60° from the vertical and released from rest to strike a block right beneath the vertical as shown in the figure below. What is the speed of the particle right before it strikes the block? (Assume that the rope remains taut over the entire motion).

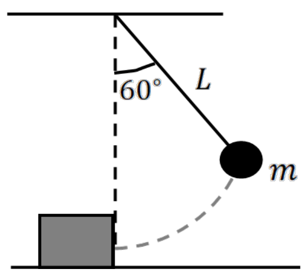


Figure 9.11: Mass striking a block

Practice Problem 9-6

A particle of mass m sits at the top of a frictionless dome of radius R. The particle starts at rest but at $t = 0$, it begins to slide off the dome. What is the speed of the particle as a function of its vertical height, y?

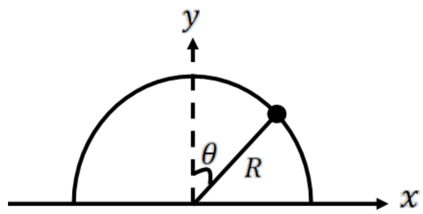


Figure 9.12: Dome and particle.

Practice Problem 9-7

A person with mass M is jumping on a trampoline that has an effective spring constant of k. The trampoline has a height of d above the ground.

a) What is the maximum height with respect to the ground that the person can reach while jumping on this trampoline?

b) How does the maximum height compare between an adult and a child jumping? Assume the child has half the mass of the adult. Does this make sense?

Practice Problem 9-8

A solid cylinder of mass M, radius R, and length L is released from rest at the top of an incline of height h. It rolls without slipping to the bottom. What is the speed of the cylinder when it reaches the bottom?

Practice Problem 9-9

See the figure below. A solid cylinder of mass M and radius R is connected to a spring with spring constant k at its centre of mass as shown in the figure. The cylinder has a moment of inertia of $\frac{1}{2}MR^2$ at its centre of mass and it can roll on the floor without slipping.

a) If the spring is displaced by x, what is the velocity of the cylinder centre of mass relative to $\dot{x}$?

b) What is the rotational angular speed of the cylinder?

c) What are the equations for rotational and translational kinetic energy? How do the rotational and translational kinetic energies compare?

d) If the cylinder is displaced a small amount from equilibrium it will oscillate. Use energy conservation to solve the differential equation of motion and find the period of oscillations.

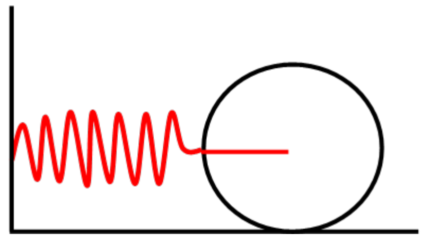


Figure 9.13: The cylinder and spring system.

10 Central Forces and Motion in Space

Learning Objectives

- Introduction to central forces
- Description of motion in space
- Effective potential of central forces
- Revisiting gravity as a central force

In this chapter, we will expand on a few concepts that have already been covered. We will discuss central forces and their application. We will also expand on the motion of objects in 3-D under central forces and apply the concepts of central forces to gravity.

10.1 Introduction to Central Forces

Consider a particle of fixed mass at a position that can be defined by the vector $\vec{r}$ with respect to an origin. We'll also define $\hat{r}$ as the unit vector in the direction of $\vec{r}$. Recall that a unit vector has a length of one such that $\hat{r} \cdot \hat{r} = 1$ and $\vec{r} = r\hat{r}$.

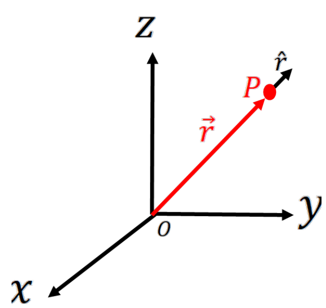


Figure 10.1: Point P is the particle. Shown in red is the vector $\vec{r}$ in the figure to the right. $\hat{r}$ is shown in black as the unit vector in the direction of $\vec{r}$.

A *central force* is defined as follows:

1. The force is directed toward or away from the origin (e.g., the force acts along $\hat{r}$)
2. The magnitude of the force depends only on the distance r.

$$\boxed{\vec{F} = f(r)\hat{r} = f(r)\frac{\vec{r}}{r}} \tag{10.1}$$

A central force is *attractive* if $f(r) < 0$ (the force points toward the origin) and repulsive if $f(r) > 0$ (the force points away from the origin). An attractive force acts to bring the particle to the origin, whereas a repulsive force acts to move the particle away from the origin. We have already discussed several such central forces in previous chapters.

Examples of central forces:

Gravity: $\vec{F} = -\dfrac{GMm}{r^2}\hat{r}$

Electrostatic force: $\vec{F} = \dfrac{kQq}{r^2}\hat{r}$

Spring Force: $\vec{F} = -kr\hat{r} \quad \Longrightarrow \quad$ often written in terms of 1-D motion (e.g., $F = -kx$)

Note that gravity and the spring force are always attractive ($f(r) < 0$). The electrostatic force can be either attractive or repulsive (depends on the charges).

10.2 Properties of Central Forces

10.2.1 Central Forces are Conservative Forces

Central forces are all *conservative forces*. In Chapter 8, we discussed conservative forces and showed that conservative forces obey,

$$\vec{\nabla} \times \vec{F} = 0$$

If a central force is defined as $\vec{F} = f(r)\hat{r}$, then we can show that $\vec{\nabla} \times \vec{F} = 0$ for all central forces. For simplicity, we will use spherical coordinates for the curl (see Appendix A.5).

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f(r) & 0 & 0 \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta}(0-0)\,\hat{r} + \frac{1}{r\sin\theta}\left(\cancelto{0}{\frac{\partial f(r)}{\partial \phi}} - 0\right)\hat{\theta} + \frac{1}{r}\left(0 - \cancelto{0}{\frac{\partial f(r)}{\partial \theta}}\right)\hat{\phi} \\
&= 0
\end{aligned}$$

Because $f(r)$ does not depend on θ or ϕ (by definition), the partial derivatives of $f(r)$ with θ and ϕ equal zero and the curl of $\vec{F}$ is zero. This means a general central force will always be conservative. Note that a conservative force may not be central. The force must still obey the two criteria in the first section to be defined as a central force.

If a central force is conservative, that means there is a potential field $U(r)$ that can describe the force where,

$$U = -\int f(r)\mathrm{d}r$$

where dr is a tiny path. Recall that for conservative fields, the change in potential energy is independent of the path. Only the initial and final points matter.

Or we can solve for the force if we know the potential of the central force:

$$\boxed{\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial r}\hat{r}} \tag{10.2}$$

since the force is radial, only the radial component of the gradient matters. For a review of the properties of conservative forces, see Chapter 8.

10.2.2 Angular Momentum is Conserved

Another feature of central forces is that angular momentum is conserved. The angular momentum for a particle is defined as:

$$\vec{L} = \vec{r} \times \vec{p}$$

where $\vec{r}$ is the radial vector from the origin to the particle and $\vec{p} = m\vec{v}$ is the momentum of that particle. If the mass of the particle is constant, the time derivative of the angular momentum is then:

$$\begin{aligned}
\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \times \vec{p}) \\
&= \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \times \vec{p} + \vec{r} \times \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \\
&= \vec{v} \times \vec{p} + \vec{r} \times \vec{F} \quad \Longrightarrow \quad \vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \text{ and } \vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} \\
&= \vec{v} \times (m\vec{v}) + \vec{r} \times (f(r)\hat{r}) \quad \Longrightarrow \quad \text{for a central force, } \vec{F} = f(r)\hat{r} \\
&= m\cancelto{0}{(\vec{v} \times \vec{v})} + f(r)\cancelto{0}{(\vec{r} \times \hat{r})} \quad \Longrightarrow \quad \text{the cross products vanish} \\
&= 0
\end{aligned}$$

So for any central force with $\vec{F} = f(r)\hat{r}$, the angular momentum of the system is conserved (assuming constant mass).

Quick Questions

1. What is the torque on a central force? Comment on this answer using the definition of torque from the central force and the definition of the net torque from Newton's second law.

10.2.3 Motion will occur on a plane

The motion of a particle under the action of a central force will take place on a 2-D plane even if the particle's position is defined in a 3-D coordinate system. Figure 10.2 shows a vector diagram of the angular momentum. The angular momentum is defined as the vector cross product of $\vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$. That means that the angular momentum vector is perpendicular to both the radial vector and the velocity vector (see diagram).

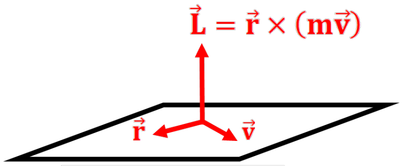


Figure 10.2: Vector diagram for the angular momentum, $\vec{L}$. Recall that $\vec{L} = \vec{r} \times (m\vec{v})$ and will therefore be perpendicular to both $\vec{r}$ and $\vec{v}$.

In the previous section, we showed that the angular momentum is constant under a central force. That includes both magnitude and direction. Since $\vec{L}$ defines the normal that is perpendicular to the plane given by $\vec{r}$ and $\vec{v}$, that plane must also be constant (otherwise the direction of $\vec{L}$ would change). So both the radius and velocity vectors are confined and motion will only occur on this 2-D plane (e.g., the plane in Figure 10.2).

10.3 Equation of Motion for a Central Force

Since motion under a central force occurs on a 2-D plane (see Chapter 10.2.3), we can simplify the motion of a particle under a central force. It is generally convenient to use polar coordinates (see Chapter 1.2).

A central force has the form of $\vec{F} = f(r)\hat{r}$. If this is the only force acting on a particle, then we can also use Newton's second law to say that $\vec{F} = m\vec{a}$ (for a constant particle mass). Thus, we can say that $f(r)\hat{r} = m\vec{a}$.

In polar coordinates, $\vec{a}$ is written as:

$$\vec{a} = \underbrace{(\ddot{r} - \dot{\theta}^2 r)}_{a_r}\hat{r} + \underbrace{(2\dot{\theta}\dot{r} + \ddot{\theta}r)}_{a_\theta}\hat{\theta}$$

where we have a radial component of the acceleration and a tangential component (azimuthal or θ component). But the central force is radial only. This is a definition of a central force. As a consequence, we can make two conclusions about the acceleration.

1) The radial acceleration is $m\vec{a}_r = f(r)\hat{r}$ because both act in the radial direction.

$$\boxed{f(r)\hat{r} = m\vec{a}_r = m(\ddot{r} - \dot{\theta}^2 r)\hat{r}} \tag{10.3}$$

2) The tangential acceleration is $\vec{a}_\theta = 0$ because central forces are only radial in direction. Setting the tangential acceleration component to zero, we get:

$$\begin{aligned} 0 &= 2\dot{\theta}\dot{r} + \ddot{\theta}r \\ 0 &= \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) \end{aligned}$$

where the latter equation can be expanded to recover the $\vec{a}_\theta$ terms. Recall that this comes from the definition of the radial vector in polar coordinates (see Chapter 1.2 for a refresher).

In the above equation, we have a time derivative equal to zero. If you have a time derivative equal to zero, that means the term in the derivative is a constant.

$$\boxed{r^2\dot{\theta} \quad = \quad \text{constant}} \tag{10.4}$$

Quick Question

1. Show that $\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta}) = 2\dot{\theta}\dot{r} + \ddot{\theta}r$ by applying the time derivative.

Note that we can also get to the conclusion that $r^2\dot{\theta}$ is constant using the angular momentum instead of Newton's second law. For the angular momentum, we have $\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times(m\vec{v})$. In polar coordinates, $\vec{v} = \dot{r}\hat{r} + \dot{\theta}r\hat{\theta}$ (Chapter 1.2), so

$$\begin{aligned}
\vec{L} &= (r\hat{r}) \times m(\dot{r}\hat{r} + \dot{\theta}r\hat{\theta}) \\
&= m\dot{r}r\cancelto{0}{(\hat{r}\times\hat{r})} + m\dot{\theta}r^2(\hat{r}\times\hat{\theta}) \\
&= m\dot{\theta}r^2\hat{k} \quad \Longrightarrow \quad \text{for cylindrical coordinates}
\end{aligned}$$

Since $\vec{L}$ is a constant, the magnitude of $\vec{L}$ is also a constant. The magnitude of $\vec{L}$ is

$$\boxed{|\vec{L}| = m\dot{\theta}r^2 = \text{constant}} \tag{10.5}$$

Since m is also a constant for our particle (assumed), we get that $\dot{\theta}r^2$ must be a constant.

10.4 Effective Potential

The effective potential is used to simplify complex problems involving central forces. For central forces, the motion occurs in a 2-D plane, so we can use plane polar coordinates to describe the motion. (Note that polar coordinates are convenient because the potential energy for the central force is radial.) In polar coordinates, the energy equations are:

$$\begin{aligned}
U &= -\int \vec{F}\cdot\mathrm{d}\vec{r} = -\int f(r)\mathrm{d}r \\
K &= \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)
\end{aligned}$$

We can use Equation 10.5 to re-write the kinetic energy in terms of the linear velocity $\dot{r}$ and the angular momentum using $\dot{\theta} = \dfrac{L}{mr^2}$.

$$\boxed{K = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2}} \tag{10.6}$$

The total energy of the system is then:

$$\begin{aligned}
E &= K + U \\
E &= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} + U(r)
\end{aligned}$$

where $U(r)$ represents the potential produced by the central force. The first term depends only on $\dot{r}$ whereas the other two terms depend only on r (m and L are constant).

$$\boxed{E = \underbrace{\frac{1}{2}m\dot{r}^2}_{\dot{r}\text{ term}} + \underbrace{\frac{1}{2}\frac{L^2}{mr^2} + U(r)}_{r\text{ terms}}} \tag{10.7}$$

This should look familiar. This is a 1-D energy equation. You have the energy entirely expressed along one coordinate axis. For example, when we looked a mass and spring, $K = \frac{1}{2}m\dot{x}^2$ and $U = \frac{1}{2}kx^2$, such that $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$, all the energy is along one axis (x).

So the energy equation for a central force can be simplified into a 1-D energy equation with an additional r term. By definition, the potential energy will depend on position alone. So we can combine the two r-terms into an *effective potential energy*, U_{eff}:

$$\boxed{U_{eff} = \frac{1}{2}\frac{L^2}{mr^2} + U(r)} \tag{10.8}$$

where first term is related to the angular momentum and is often called the centrifugal potential. The second term is the potential due to the central force itself.

Effective Potential

The effective potential energy is a mathematical description of two energy terms that each depend on the radial position only. You have a component from the central force potential and a component called the centrifugal potential. The effective potential can be considered a representative potential energy.

Combing back to our energy equation, we have:

$$\boxed{E = \frac{1}{2}m\dot{r}^2 + U_{eff}} \tag{10.9}$$

Which looks exactly like a 1-D energy problem, even though the system may be in a 3-D space and moving in a 2-D plane. By cutting back on the dimensions, we make the math much easier.

We can re-write Equation 10.9 as follows (solving for $\dot{r}^2$):

$$\dot{r}^2 = \frac{2}{m}(E - U_{eff})$$

The term $\dot{r}^2$ is always positive or zero. So if $\dot{r}^2 \geq 0$, then

$$\begin{aligned} \frac{2}{m}(E - U_{eff}) &\geq 0 \\ E &\geq U_{eff} \end{aligned}$$

So with just the effective potential, we can start to get an idea of the allowed properties in the system. The system energy must be at least equal to the effective potential.

Sample Problem 10-1

A particle of mass m and energy E moves in an inverse-cube field $f(r) = -\frac{\gamma}{r^3}$ where γ is a constant. The angular momentum of the particle is L. If $\dot{r} = 0$, **find the equation for r in terms of γ, m, and E. What kind of motion is this?**

Solution

The energy equation for our central force is $E = \frac{1}{2}m\dot{r}^2 + U_{eff}$. If $\dot{r} = 0$, then $E = U_{eff}$. The potential of the central force.

$$\begin{aligned} U(r) &= -\int f(r)\mathrm{d}r \\ U(r) &= -\int (-\frac{\gamma}{r^3})\mathrm{d}r \\ U(r) &= -\frac{\gamma}{2r^2} \end{aligned}$$

So our effective potential is:

$$\begin{aligned} U_{eff} &= \frac{1}{2}\frac{L^2}{mr^2} + U(r) \\ U_{eff} &= \frac{1}{2}\frac{L^2}{mr^2} - \frac{\gamma}{2r^2} \\ U_{eff} &= \frac{1}{2r^2}\left(\frac{L^2}{m} - \gamma\right) \end{aligned}$$

Since $U_{eff} = E$, we have:

$$E = \frac{1}{2r^2}\left(\frac{L^2}{m} - \gamma\right)$$

$$r^2 = \frac{1}{2E}\left(\frac{L^2}{m} - \gamma\right)$$

$$\boxed{r = \sqrt{\frac{1}{2E}\left(\frac{L^2}{m} - \gamma\right)}}$$

We ignore the negative case for the square-root because by definition, $r > 0$.

As for the kind of motion, this object moves in such a way that r is constant. In other words, the particle is moving in a perfect circle (circular motion).

10.5 Effective Force

If you have an effective potential, then you can define an effective force associated with that effective potential. This is not a true force, however. The only true force acting on the system is the central force. We're just treating the constant angular momentum as an additional potential term and therefore creating an effective "force" that produces it. This is why we use the term "effective", because it has an effect on the system that represents the motion, but it is not a real force. The effective force is a mathematical construct.

In general, we can get the force from the gradient of a potential, $\vec{F} = -\vec{\nabla}U$. Since our potential only depends on position r, we only care about the r component of the gradient.

$$\begin{aligned}
\vec{F}_{eff} &= -\frac{\partial U_{eff}}{\partial r}\hat{r} \\
&= -\frac{\partial}{\partial r}\left(\frac{1}{2}\frac{L^2}{mr^2} + U(r)\right)\hat{r} \\
&= -\left(-\frac{L^2}{mr^3}\right)\hat{r} - \frac{\mathrm{d}U(r)}{\mathrm{d}r}\hat{r} \\
&= \frac{L^2}{mr^3}\hat{r} - \frac{\mathrm{d}U(r)}{\mathrm{d}r}\hat{r}
\end{aligned}$$

The effective force has two terms. The first term comes from the angular momentum of the system (the centrifugal potential) and the second term comes from the central force.

Sample Problem 10-2

A modification of Earth's gravitational field is often described as $U(r) = -\frac{\gamma}{r}\left(1 + \frac{\epsilon}{r^2}\right)$ where γ and ϵ are constants. **What is the effective force associated with this potential?** Assume a particle of mass m and an angular momentum of L.

Solution

The effective force is given by $\vec{F_{eff}} = -\vec{\nabla}U_{eff}$.

$$\begin{aligned}
\vec{F}_{eff} &= -\frac{\partial U_{eff}}{\partial r}\hat{r} \\
\vec{F}_{eff} &= -\frac{\partial}{\partial r}\left[\frac{1}{2}\frac{L^2}{mr^2} + U(r)\right]\hat{r} \\
\vec{F}_{eff} &= -\frac{\partial}{\partial r}\left[\frac{1}{2}\frac{L^2}{mr^2} - \frac{\gamma}{r}\left(1 + \frac{\epsilon}{r^2}\right)\right]\hat{r} \\
\vec{F}_{eff} &= -\left(-\frac{L^2}{mr^3} + \frac{\gamma}{r^2} + \frac{3\gamma\epsilon}{r^4}\right)\hat{r} \\
\vec{F}_{eff} &= \left(\frac{L^2}{mr^3} - \frac{\gamma}{r^2} - \frac{3\gamma\epsilon}{r^4}\right)\hat{r}
\end{aligned}$$

10.6 Example: Gravity as a Central Force

The gravitational force has the form $f(r)\hat{r}$ and is therefore a central force:

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

The gravitational force also falls under a class of forces that are called inverse-square laws. Any force proportional to r^{-2} follows an inverse-square law.

What is the effective potential for a system moving under a gravitational force?

$$\begin{aligned} U_{eff} &= \frac{1}{2}\frac{L^2}{mr^2} + U(r) \\ &= \frac{1}{2}\frac{L^2}{mr^2} - \frac{GMm}{r} \end{aligned}$$

Recall that

$$U(r) = -\int \vec{F}\cdot \mathrm{d}\vec{r} = -\frac{GMm}{r}$$

relative to $U = 0$ at infinity.

1) Consider the case where $L = 0$: $L = 0$ is a special case where an object has no angular momentum. That means the object only has motion along the radial direction (e.g., recall that $L = m\dot{\theta}r^2$, so if $L = 0$, then $\dot{\theta} = 0$ and $\dot{\vec{r}} = \dot{r}\hat{r}$).

If $L = 0$, then $U_{eff} = U(r)$. That is, the effective potential is just the gravitational potential. In terms of the allowed energies, we have:

$$\begin{aligned} E &= \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} \\ E &\geq -\frac{GMm}{r} \end{aligned}$$

because $\dot{r}^2 \geq 0$.

Figure 10.3 shows the potential energy for an object moving under the gravitational force when $L = 0$. The blue curve shows U_{grav} and the shaded in area shows the allowed energies, $E > U_{grav}$. All constants are given arbitrary values for the purposes of plotting. Note that E can be positive, depending on the value of $\dot{r}$. We can only define the minimum allowed energy at each radius.
For the actual motion within this system, we need to solve the equation:

$$\begin{aligned} \dot{r}^2 &= \frac{2}{m}\left(E + \frac{GMm}{r}\right) \\ \frac{\mathrm{d}r}{\mathrm{d}t} &= \pm\sqrt{\frac{2}{m}\left(E + \frac{GMm}{r}\right)} \\ \frac{\mathrm{d}r}{\sqrt{\left(E + \frac{GMm}{r}\right)}} &= \pm\sqrt{\frac{2}{m}}\mathrm{d}t \end{aligned}$$

If you know the system energy, E, you can then solve for how the position changes with time $r(t)$ by integrating both sides.

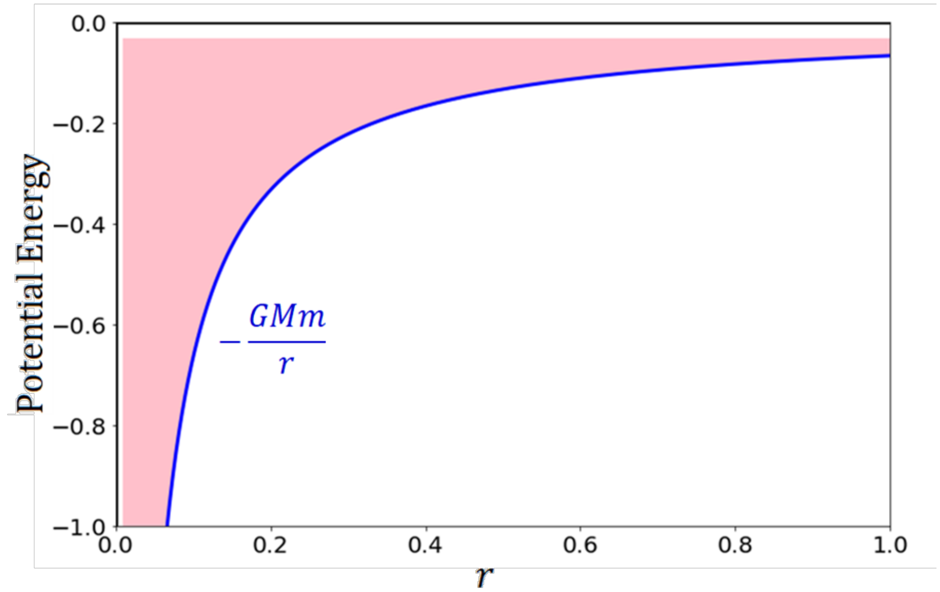


Figure 10.3: The allowed energies for a system in a gravitational potential with no angular momentum ($L = 0$). Blue curve shows the potential from gravity and the shaded in area shows the allowed values of energy.

Quick Questions

1. What is $\dot{r}$ if $E = -\frac{GMm}{r}$, which is the minimum allowed value? What does this value mean?

2) Consider the case where $L \neq 0$: If L is a non-zero, then the system has angular momentum, and that angular momentum is constant. If $L \neq 0$, then

$$U_{eff} = \frac{1}{2}\frac{L^2}{mr^2} - \frac{GMm}{r}$$

Figure 10.4 shows the effective potential (purple curve) for an object under the potential U_{eff}. The figure compares the effective potential with the centrifugal potential (red curve) and the gravitational potential (blue curve). The constants are given arbitrary values.

The effective potential still sets the minimum value of energy that a system can have. That is, we still have the condition $E \geq U_{eff}$ because $\dot{r}^2 \geq 0$. So the effective potential curve U_{eff} in Figure 10.4 shows the minimum allowed energy of the system. Note that this curve has a distinct shape with a local minimum in the potential. This shape has profound impact on how objects in this potential are going to move.

For simplicity, let's look at the condition where $\dot{r} = 0$. An object with $\dot{r} = 0$ has no radial motion. Instead, all the motion will be transverse due to the non-zero angular momentum ($L = m\dot{\theta}r^2$). Note that transverse motion describes rotation. So an object in a gravitational field will rotate or *orbit* around the source of that gravitational field.

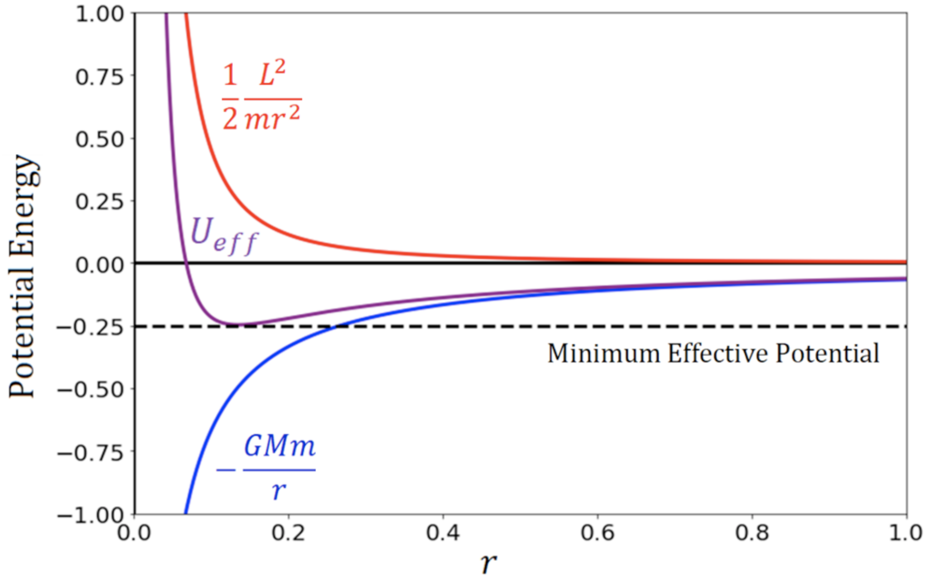


Figure 10.4: The effective potential energy for a system moving in a gravitational field. The effective potential is in purple. The potential from gravity is in blue and the centrifugal potential is in red. The functions use arbitrary constants and units for plotting.

Let's find the conditions for a stable orbit. We will substitute $\gamma = GMm$ to make the math a bit easier to read. Assuming $\dot{r} = 0$:

$$\begin{aligned}
E &= U_{eff} \\
0 &= U_{eff} - E \\
0 &= \frac{1}{2}\frac{L^2}{mr^2} - \frac{\gamma}{r} - E \quad \Longrightarrow \quad U_{eff} = \tfrac{1}{2}\frac{L^2}{mr^2} - \frac{\gamma}{r} \\
0 &= \frac{L^2}{m} - 2\gamma r - 2Er^2 \quad \Longrightarrow \quad \text{multiply by } 2r^2
\end{aligned}$$

The above equation is a quadratic equation with r, where the solution is:

$$r = \frac{-(-2\gamma) \pm \sqrt{(-2\gamma)^2 - 4(-2E)(\frac{L^2}{m})}}{2(-2E)} = \frac{\gamma \pm \sqrt{\gamma^2 + \frac{2EL^2}{m}}}{-2E}$$

where $\gamma = GMm$.

To have a real orbit, we need to have real values for r. By definition, $r \geq 0$, or r must be positive. The above equation has two positive (real) solutions for r if $E < 0$ or if the energy is negative.

Quick Question

1. Look at Figure 10.4. Assuming $E = U_{eff}$, consider the values for r that are satisfied when $E = U_{eff} = -0.1$ versus $E = U_{eff} = 0.1$. How many solutions for r do you get in each case?

Let's consider a few cases.

Case (1) $E = E_{min}$: Figure 10.4 shows that there is an energy minimum, E_{min}. A system with $E = E_{min}$ is a special case.

First, we need to find the value of E_{min}. We can do that by finding the position when the potential has a local minimum by taking the derivative of U_{eff} with respect to r.

$$\begin{aligned}
\frac{\mathrm{d}U_{eff}}{\mathrm{d}r} &= 0 \\
0 &= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{2}\frac{L^2}{mr^2} - \frac{\gamma}{r}\right) \\
0 &= -\frac{L^2}{mr^3} + \frac{\gamma}{r^2} \\
0 &= -\frac{L^2}{m} + \gamma r \\
r_{min} &= \frac{L^2}{\gamma m}
\end{aligned}$$

The energy at this minimum position is:

$$\begin{aligned}
0 &= \frac{L^2}{m} - 2\gamma r_{min} - 2E_{min} r_{min}^2 \\
0 &= \frac{L^2}{m} - 2\gamma\left(\frac{L^2}{\gamma m}\right) - 2E_{min}\left(\frac{L^2}{\gamma m}\right)^2 \\
0 &= 1 - 2 - 2E_{min}\left(\frac{L^2}{\gamma^2 m}\right) \\
E_{min} &= -\frac{1}{2}\frac{m\gamma^2}{L^2}
\end{aligned}$$

The values of r_{min} and E_{min} represent a special case where the is only one unique solution to the quadratic equation. If you sub $E = E_{min}$ into the quadratic equation for r, you will get that $r = r_{min}$ as the only solution, as expected.

Quick Question

1. Verify that $E = -\frac{1}{2}\frac{m\gamma^2}{L^2}$ gives only one unique solution for r in the quadratic equation.

When $E = E_{min}$, it means that your orbital solution has the minimum allowed potential energy and it can only orbit with a single, fixed radius. This solution describes a *circular orbit.*

Case (2) $E > E_{min}$ **and** $E < 0$: Consider Figure 10.4 for a particle with an energy of $E = -0.2$. This energy puts the system just above the minimum effective potential curve. With that energy, the allowed radii for the particle are between $r \approx 0.1$ and $r \approx 0.24$ (under the condition $E \geq U_{eff}$ in Figure 10.4). That is, for radii of $r < 0.1$ and $r > 0.24$, $E = -0.2$ would be below effective potential curve, which is not allowed.

This scenario describes a *bound elliptical orbit*. The particle can move freely from $r \approx 0.1$ to $r \approx 0.24$ and back again all with the same energy (energy is conserved). The positions of $r \approx 0.1$ and $r \approx 0.24$ are special, because there are where $\dot{r}$ is instantaneously zero, but in this case, the radial velocity does not stay zero (unlike in Case 1).

Let's say the particle starts at $t = 0$ at $r \approx 0.1$. At this instantaneous moment, $E = U_{eff}$, so $\dot{r} = 0$ ($K_r = 0$). But this is an instantaneous moment. The particle is allowed to move to larger radii (given its energy), but $E = K_r + U_{eff}$ will still be constant, so as U_{eff} drops toward larger radii, K_r will increase. As the particle approaches $r \approx 0.24$, $E \to U_{eff}$ and $K_r \to 0$ again. The particle cannot travel further radially (it does not have enough energy) and instead, it will turn around back toward the origin. Thus, the $r \approx 0.1$ and $r \approx 0.24$ points are the turnaround points in this elliptical orbit. The motion is bounded by these two limits. We'll discuss elliptical orbits in more detail in Chapter 11.

Case (3) $E > 0$: If the energy is positive, then the quadratic equation for radius:

$$r = \frac{\gamma \pm \sqrt{\gamma^2 + \frac{2EL^2}{m}}}{-2E}$$

will have one positive and one negative solution. Since negative radii are unphysical (by definition), this case describes an *unbound orbit*. Unbound orbits arise when systems have too much energy to be contained by the gravitational field. We will come back to these orbits in Chapter 11.

Quick Question

1. Convince yourself that there are two positive solutions for r if $E > E_{min}$ but $E < 0$. Set $E = -\frac{1}{4}\frac{m\gamma^2}{L^2}$ to test this assumption.
2. Convince yourself that there is only one positive solution for r if $E > 0$. Set $E = \frac{m\gamma^2}{L^2}$ to test this assumption.

Orbits and Inverse Square Laws

Gravity is a central force that follows an inverse-square law. But mathematically, any central force that obeys an inverse-square law of the form of $\vec{F} = -\frac{\gamma}{r^2}\hat{r}$, where γ is a constant, reproduces the orbital solutions discussed in this section. The properties of bound and unbound orbits can be directly linked back to the amount of energy in the system relative to the effective potential. For example, planets orbit the Sun because they have angular momentum (for the orbit), but they do not have enough energy to escape the Sun (their orbits are bound). See Chapter 11 for more details.

Sample Problem 10-3

In Sample Problem 10-2, we used a modification of Earth's gravitational field described as $U(r) = -\frac{\gamma}{r}\left(1 + \frac{\epsilon}{r^2}\right)$ where γ and ϵ are constants. A particle of mass m is in a closed orbit in this gravitational potential with an angular momentum of L. **If the angular momentum of the system is the exact value to put the particle in a circular orbit, what are the possible radii for a circular orbit?**

Solution

You have a circular orbit when $\frac{\mathrm{d}U_{eff}}{\mathrm{d}r} = 0$. For this system, $U_{eff} = \frac{L^2}{2mr^2} + U(r)$, where $U(r)$ is the potential given in the equation. If you plot the effective potential, depending on the constants, you will get a curve that looks like:

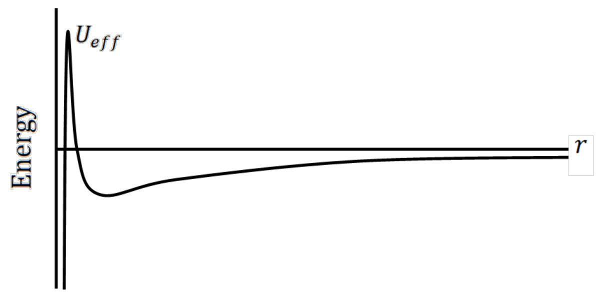


Figure 10.5: An example of U_{eff} with arbitrary constants.

So, we see that there are multiple possible places where the derivative of U_{eff} will be zero (local maxima or minima). Note that the exact the shape of the effective potential curve will depend on the constants themselves.

To get the equations for a circular orbit, let's first solve for the derivative and set that to be zero:

$$\begin{aligned}
\frac{\mathrm{d}U_{eff}}{\mathrm{d}r} &= 0 \\
0 &= \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{2}\frac{L^2}{mr^2} - \frac{\gamma}{r}\left(1 + \frac{\epsilon}{r^2}\right)\right] \\
0 &= -\frac{L^2}{mr^3} + \frac{\gamma}{r^2} + \frac{3\gamma\epsilon}{r^4} \\
0 &= -\frac{L^2 r}{m} + \gamma r^2 + 3\gamma\epsilon \quad \Longrightarrow \quad \text{multiply by } r^4
\end{aligned}$$

Let's use the quadratic equation to solve for r.

$$r = \frac{-\left(-\frac{L^2}{m}\right) \pm \sqrt{\left(-\frac{L^2}{m}\right)^2 - 4(\gamma)(3\gamma\epsilon)}}{2\gamma}$$

So there are two values of $r > 0$ for which we can have a saddle point in the effective potential.

$$r = \frac{\left(\frac{L^2}{m}\right) \pm \sqrt{\left(\frac{L^2}{m}\right)^2 - 12\gamma^2\epsilon}}{2\gamma}$$

Since these are by definition the radii at a local maxima or minima, they are the solutions for a circular orbit. But only a local minimum will produce a stable circular orbit. Recall the discussion on saddle points from Chapter 8.8.

Cora's Thoughts

Remember that effective force is the gradient of a potential, so our initial assumption was that $F_{eff} = 0$ for circular orbits. This means that our two radii are at the minimum and maximum points of potential, one being at a stable position and the other at an unstable position. We can look back to Chapter 8, to the analogy of a ball resting atop the potential energy curve to helped to describe the stability of positions. This means that the radius at the max potential will be unstable and the radius at the minimum potential will be stable.

10.7 Real World Application

Planets, comets, and asteroids have bound orbits around the Sun. That means, they do not have enough energy to escape the Sun. If an object enters the Solar System with too much energy ($E > 0$) it will not stick around. That exact event happened with the first-detected interstellar asteroid, ‘Oumuamua.

It was discovered on 19 October 2017 by Robert Weryk (a Canadian) using the University of Hawai‘i Pan-STARRS1 telescope and was noted to be moving very quickly. With more observations, it was found to have an unbound orbit. The figure below shows the orbit of ‘Oumuamua relative to the Solar System planets. The Sun's gravitational field deflected its motion, but isn't enough to keep ‘Oumuamua from escaping. This type of orbit is called a *hyperbolic orbit* (more on this in Chapter 11).

Based on its orbital properties, it was determined that ‘Oumuamua originated from outside the Solar System, making it the first detected asteroid to have come from another star. It was subsequently given the name ‘Oumuamua, which roughly means “first visitor from far away” in Hawaiian. It is also the first entry in a whole new asteroid classification system, "1I", where the "I" indicates its an interstellar object.

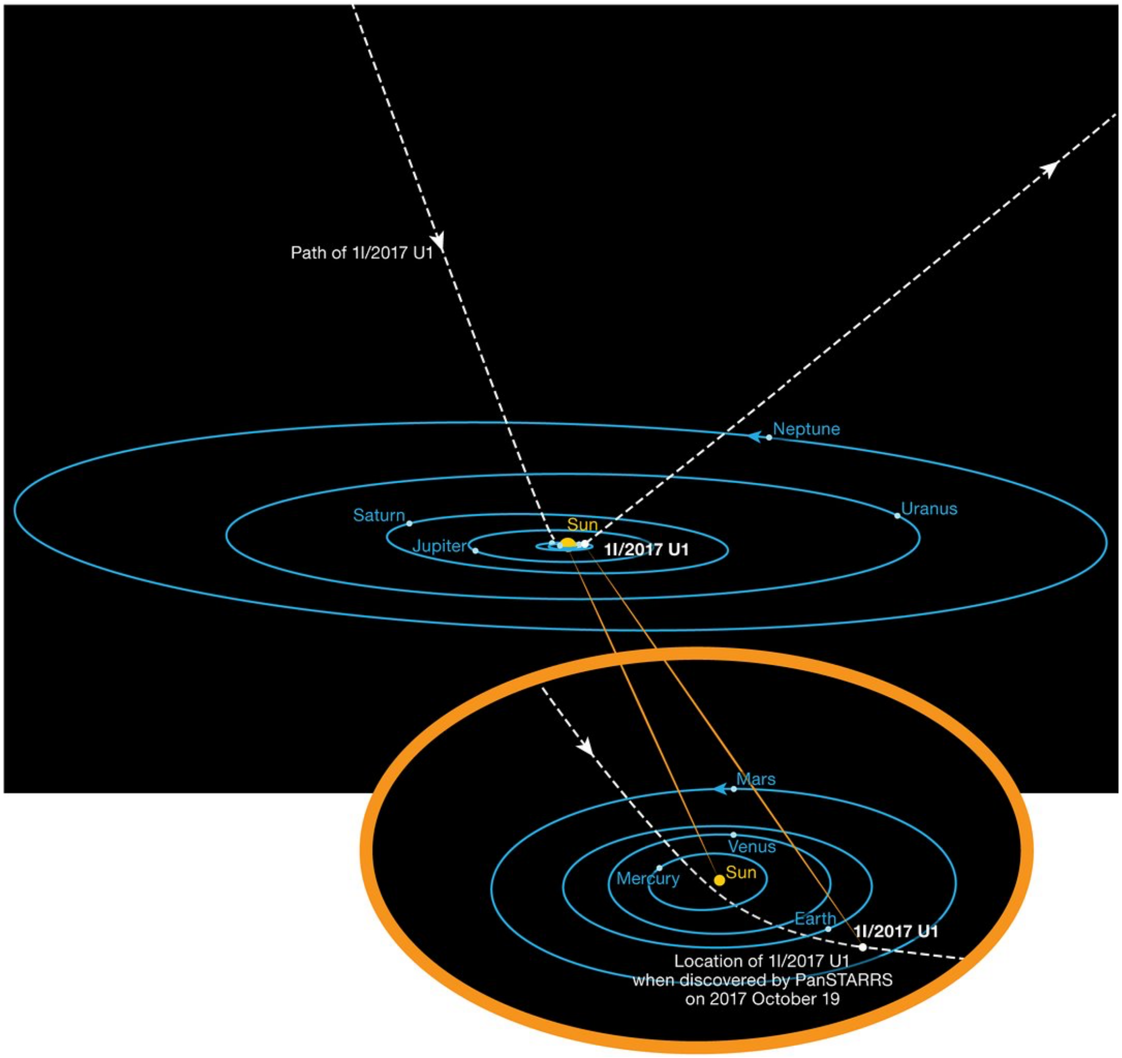


Figure 10.6: The orbit of 'Oumuamua. Image credit: ESO.

For more information:
NASA's basic information page on 'Oumuamua has some good introductory reading plus an animation of the object's orbit near its closest approach to the Sun and this NASA article highlights how much and how little we know about 'Oumuamua.

10.8 Summary

Key Takeaways

This chapter introduces the central forces, which are a class of conservative forces that only have a radial dependence.

$$\vec{F} = f(r)\hat{r}$$

All motion under a central force takes place in a 2-D plane because these forces also conserve angular momentum

$$L = m\dot{\theta}r^2 = \text{constant}$$

As a result, the motion of a particle under a central force can be simplified greatly. Since central forces are also conservative forces, they can be described by a potential energy

$$\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial r}\hat{r}$$

which only depends on position.

Using energy conservation, we can also write the kinetic energy in terms of position and a constant angular momentum.

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} + U(r)$$

We define the effective potential as the sum of the centrifugal potential (from angular momentum) and the central force potential,

$$U_{eff} = \frac{1}{2}\frac{L^2}{mr^2} + U(r)$$

If the central force potential is known, one can predict the motion of objects. Some basic cases are:

1. $\dot{r} = 0$: All motion is transverse (azimuthal) and $E = U_{eff}$.

2. $L = 0$: All motion is along $\hat{r}$ and magnitude depends only on distance.

3. $L \neq 0$ and $\dot{r} \neq 0$: There will be a mix of linear and azimuthal motion and $E > U_{eff}$ because $\dot{r}^2$ is always positive.

A key central force is gravity. Gravity follows an inverse-square law, and its effective potential has a specific shape that with a local minimum that represents a gravitational well where a particle can become bound to the central mass creating the gravitational field. We describe the motion of objects that are bound to the central mass as orbits. The shape of the orbit depends on the amount of energy.

1. For $E = E_{min}$, the orbit is circular (there is only one unique solution for radius).
2. For $E_{min} < E < 0$, the orbit is elliptical (there are two real solutions for radius).
3. $E > 0$, the orbit is unbound (there are two solutions for radius, but only one is one is physical)

Elliptical orbits are discussed more in Chapter 11.

Important Equations

Central Force:

$$\vec{F} = f(r)\hat{r} = m\vec{a}_r = m(\ddot{r} - \dot{\theta}^2 r)\hat{r}$$

Potential of a Central Force:

$$\vec{F} = -\vec{\nabla}U = -\frac{\partial U}{\partial r}\hat{r}$$

Constant Angular Momentum:

$$L = m\dot{\theta}r^2 = \text{constant}$$

Kinetic Energy:

$$K = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2}$$

Energy:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{L^2}{mr^2} + U(r)$$
$$E = \frac{1}{2}m\dot{r}^2 + U_{eff}$$

Effective Potential Energy:

$$U_{eff} = \frac{1}{2}\frac{L^2}{mr^2} + U(r)$$

Effective Force:

$$\vec{F}_{eff} = -\vec{\nabla}U_{eff}$$

Position of Saddle Point and Minimum Energy for Gravity:

$$\begin{aligned} r_{min} &= \frac{L^2}{\gamma m} \\ E_{min} &= -\frac{1}{2}\frac{m\gamma^2}{L^2} \quad \text{for } \gamma = GMm \text{ and } U = -\frac{\gamma}{r} \end{aligned}$$

10.9 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 10-1

A particle is under the influence of a central force with a central potential energy defined as:

$$U(r) = \frac{k_1}{r^3} + \frac{k_2}{r^2}$$

where k_1 and k_2 are positive constants. Find the vector equation of the force.

Practice Problem 10-2

What is the effective force on a particle moving under the influence of a central force with a potential of $U(r) = \frac{1}{r^2}$. For an angular momentum L, find the effective force.

Practice Problem 10-3

A particle of mass m moves under the influence of a central force with a potential of $U(r) = -\frac{1}{r}$. For an angular momentum L, what is the effective potential of a circular orbit? Is this a stable or unstable point?

Practice Problem 10-4

Plot the effective potential for the case where $U(r) = -\frac{m}{r^2}$ using arbitrary values for m and L. Adjust the parameters and observe the effect on the curve.

Practice Problem 10-5

A particle of mass m moves in a circular orbit of radius R under the influence of a central force with a potential of $U(r) = kmr^4$, where k is a positive constant. Find its velocity and angular momentum.

Practice Problem 10-6

Using the modification of Earth's gravitational field as described as $U(r) = -\frac{\gamma}{r}\left(1 + \frac{\epsilon}{r^2}\right)$ where γ and ϵ are constants. A particle of mass m is in a closed orbit in this gravitational potential with an angular momentum of L. What is the effective force associated with the potential?

Practice Problem 10-7

The planet Mercury is close enough to the Sun that it feels a slight perturbation in its gravitational force.Assume that Mercury feels a central force with the form of $f(r) = -\frac{\gamma}{r^2} + \epsilon r$, where γ and ϵ are constants. What is the effective potential from this force? (Remember you can have $U = 0$ at any convenient radius.)

Practice Problem 10-8

A particle of mass m and angular momentum L moves in central force field that produces a potential that can be described by $U(r) = -Ae^{-\beta r^3}$, where A and β are constants. If the particle moves in a circular orbit of radius $r = R$, what is the magnitude of angular momentum necessary to maintain this circular orbit?

Practice Problem 10-9

A particle of mass m and angular momentum of L experiences a central force that produces a potential of $U(r) = Ar^2$, where A is a positive constant.

a) Sketch the effective potential for this system. Which potential (central force potential or centrifugal potential) dominates at small radii and which one dominates at large radii? (Hint: make sure your plot matches your expectations.)

b) If the system has an angular momentum of L such that it is in a circular orbit, what is the radius of that circular orbit?

c) What is the energy of the particle?

Practice Problem 10-10

A particle of mass m and angular momentum of L moves in a spiral path of $r = A\theta^2$ under a central force. Assume that A is a positive constant.

a) Find the equation for $\dot{r}$ relative to L, m, and r. (Hint, do not have any θ or $\dot{\theta}$ terms.)

b) What is the energy equation for this system in terms of r only? Use $U(r)$ as the potential from the central force.

c) What is the potential energy from the central force?

d) What is the central force function?

Practice Problem 10-11

A particle of mass m and angular momentum of L experiences a central force that produces a potential of $U(r) = -\frac{A}{r^3}$, where A is a positive constant.

a) Sketch the effective potential for this system. Which potential (central force potential or centrifugal potential) dominates at small radii and which one dominates at large radii? Hint: make sure your plot matches your expectations.

b) At what radius do you reach a saddle point in the potential?

c) What is the effective potential at this position?

d) Is this a stable or unstable position? How do you know?

11 Orbits and Kepler's Laws

Learning Objectives

- Introduction to ellipses and elliptical orbits
- Introduction to Kepler's Laws
- Application of Kepler's laws to physics problems
- Discussion on orbital mechanics

In this chapter, we will expand on orbits and their properties within a gravitational potential that was introduced in the last chapter. When one thinks of orbits, they likely picture the planets orbiting the Sun, the Moon orbiting the Earth, or communication satellites orbiting the Earth. The force behind orbits is gravity. As a reminder, gravity is a central force (Chapter 10) that follows an inverse-square law. The consequences for an inverse-square law central force is that bound orbits will be elliptical. In this chapter, we will look at elliptical orbits in more detail and how they apply to Kepler's Laws.

Examples of Orbits

Examples of circular orbits are geostationary satellites around the Earth.

Examples of elliptical orbits are the orbits of the planets around the Sun.

A good example of a hyperbolic orbit is the interstellar asteroid, 'Oumuamua, that did a flyby of the Solar System in 2017.

11.1 Definition of an Ellipse

Figure 11.1 shows an example ellipse with several key properties labeled. An ellipse is essentially an elongated circle, where the longer of the two axes is the semi-major axis (a) and the shorter of the two axes is the semi-minor axis (b). The degree to which the circle has been stretched is called the eccentricity (or ellipticity) and is denoted by the symbol ϵ,

$$\boxed{\epsilon = \sqrt{1 - \frac{b^2}{a^2}}} \tag{11.1}$$

Figure 11.1 also shows two special points in red, which are called the foci (focus is the singular term). These two foci, denoted as f_1 and f_2, are located on the semi-major axis, each at a distance ϵa from the center of the ellipse. The foci of an ellipse define the shape. An ellipse is defined by a locus (path) of points where the total distance from any point on the locus to the two foci adds up to a constant. For example, Figure 11.1 shows a point on the ellipse that is a distance r_1 from f_1 and a distance r_2 from f_2. The shape of the ellipse is defined such that the sum of those two distances $r_1 + r_2 =$ constant for every position on the locus. For an ellipse, $r_1 + r_2 = 2a$.

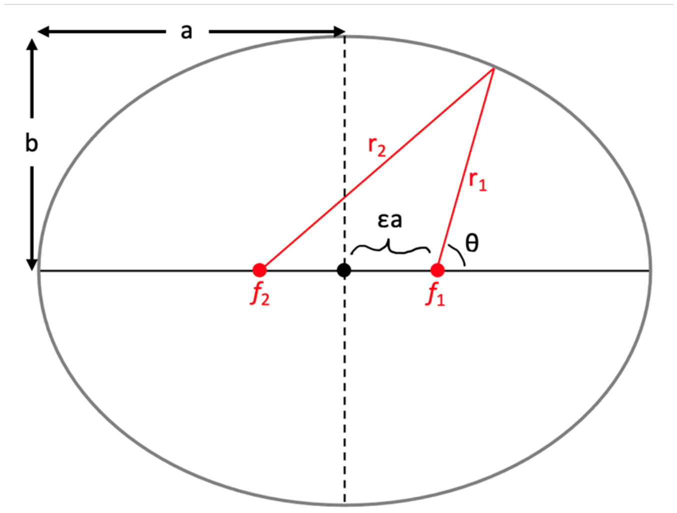


Figure 11.1: Schematic of an ellipse. The center is shown by a black dot and the two foci are shown as red dots. The semi-major axis (a) and semi-minor axis (b) are also labeled. The two foci are each a distance ϵa from the center, where ϵ is the eccentricity. The total distance between the two foci (r_1 and r_2) and any position on the ellipse sum to a constant, $r_1 + r_2 =$ constant.

The distances r_1 and r_2 in Figure 11.1 can be measured relative to a, ϵ, and a position angle, θ. Figure 11.2 shows how we can relate these properties through Pythagoras' theorem. Using the right-angle triangle in Figure 11.2, we have

$$r_2^2 = (r_1 \sin\theta)^2 + (2a\epsilon + r_1 \cos\theta)^2$$

Expanding on this, we get

$$\begin{aligned} r_2^2 &= r_1^2 \sin^2\theta + 4a^2\epsilon^2 + 4a\epsilon r_1 \cos\theta + r_1^2 \cos^2\theta \\ r_2^2 &= r_1^2 \underbrace{(\sin^2\theta + \cos^2\theta)}_{1} + 4a\epsilon(a\epsilon + r_1\cos\theta) \\ r_2^2 &= r_1^2 + 4a\epsilon(a\epsilon + r_1\cos\theta) \end{aligned}$$

Finally, we can use the property that $r_2 + r_1 = 2a$ for an ellipse.

$$\begin{aligned} (2a - r_1)^2 &= r_1^2 + 4a\epsilon(a\epsilon + r_1\cos\theta) \\ 4a^2 - 4ar_1 + r_1^2 &= r_1^2 + 4a\epsilon(a\epsilon + r_1\cos\theta) \\ -4ar_1 &= 4a\epsilon(a\epsilon + r_1\cos\theta) - 4a^2 \\ r_1 &= -a\epsilon^2 - r_1\epsilon\cos\theta + a \\ r_1(1 + \epsilon\cos\theta) &= a(1 - \epsilon^2) \\ r_1 &= \frac{a(1-\epsilon^2)}{1 + \epsilon\cos\theta} \end{aligned}$$

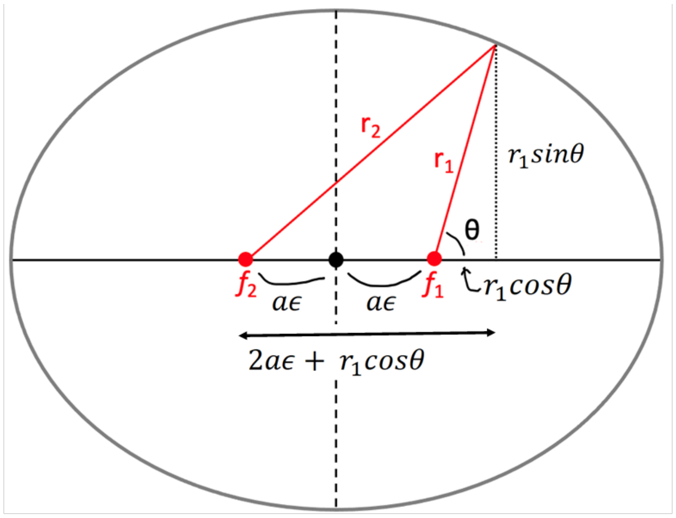


Figure 11.2: This shows the position vectors r_1 and r_2 again for the two foci, where r_1 has been broken up into two components, $r_1 \sin\theta$ and $r_1 \cos\theta$. This produces a right angle triangle with $r_2^2 = (r_1 \sin\theta)^2 + (2a\epsilon + r_1 \cos\theta)^2$ using the Pythagorean theorem.

Now we don't need to use the subscript for r. We can say that the distance to any point on the ellipse from a given focus is:

$$\boxed{r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta}} \tag{11.2}$$

Quick Questions

1. Find r when $\theta = 0, 90, 180°$. Hint, use the difference of squares to simplify.
2. For a locus point on the semi-minor axis, $r_1 = r_2$. What angle, θ, corresponds to this point? Express your answer in terms of a, b, ϵ.
3. Use the case where $r_1 = r_2$ to prove that $\epsilon = \sqrt{1 - \frac{b^2}{a^2}}$ in Equation 11.1.

For our elliptical orbit, r is the distance from one of the foci to the ellipse as a function of the angle. We define $\theta = 0$ along the semi-major axis (e.g., see Figure 11.1 for the definition of the angle). Since $-1 \leq \cos\theta \leq 1$, the distance from the focus is smallest when $\theta = 0°$ and largest when $\theta = 180°$.

Figure 11.3 shows the definition of these closest and farthest points. This shortest distance from the focus is called the pericenter (r_p) and is shown in blue. The largest distance from the focus is called the apocenter (r_a) and it is shown in purple. The pericenter and apocenter distances are defined as:

$$\boxed{r_p = a(1-\epsilon)} \tag{11.3}$$

$$\boxed{r_a = a(1+\epsilon)} \tag{11.4}$$

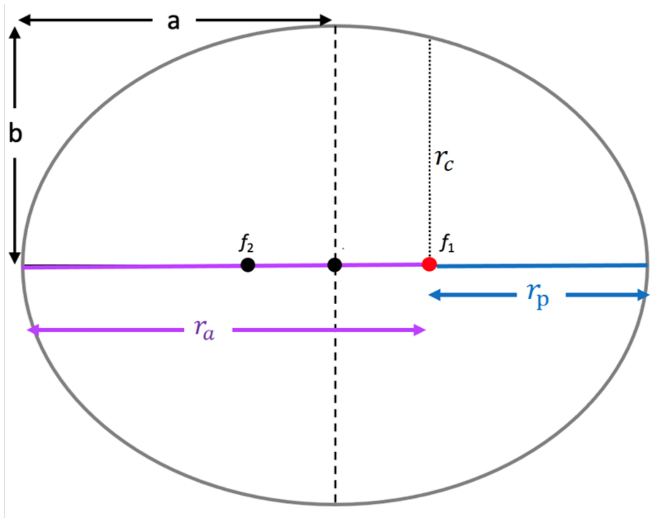


Figure 11.3: The apocenter r_a and pericenter r_p for an ellipse. Also shown is r_c the distance between the focus and locus at an angle that is perpendicular to the semi-major axis ($\theta = 90°$).

Pericenter and Apocenter

For an elliptical orbit, the pericenter and apocenter are special places where $\dot{r} = 0$. These are otherwise known as turning points. You are moving from a case of increasing radius (pericenter to apocenter) to a case of decreasing radius (apocenter to pericenter). So at these specific points, $\dot{r}$ is instantaneously zero.

If it helps, consider simple harmonic motion as an analogy. For a pendulum that is oscillating, the maximum amplitude θ_{max} occurs when $K = 0$. At that point, the system starts moving back in the opposite direction. So the maximum displacement is θ_{max} when the kinetic energy is zero. For an elliptical orbit, the apocenter and pericenter are the equivalent of our turning points and they represent the maximum and minimum distances of the orbit.

Quick Question

1. How do the apocenter and pericenter compare when $\epsilon = 0$? Does this make sense?
2. What happens to the apocenter and pericenter when $\epsilon \rightarrow 1$? Describe the shape of the ellipse. Assume ϵ gets close to 1, but doesn't reach it.

In general, we use the term *perigee* for the pericenter and *apogee* for the apocenter when talking about orbits around the Earth, and *perihelion* and *aphelion* for the pericenter and apocenter of orbits around the Sun.

11.2 Ellipses as Orbits

The previous section defines the ellipse as a geometric shape. Now we will put some physics into the elliptical orbit so we can relate the motion of an object in a gravitational field.

Gravity is a central force that follows an inverse-square law. In Chapter 10, we showed that the energy of a system under a central force has the form of

$$E = \frac{1}{2}m\dot{r}^2 + U_{eff}$$

where U_{eff} is the effective potential. Since we are interested in orbits under gravity, we can define the effective potential as

$$U_{eff} = \frac{1}{2}\frac{L^2}{mr^2} - \frac{\gamma}{r}$$

where $L = mr^2\dot{\theta}$ is the angular momentum and $\gamma = GMm$. For central forces like gravity, $L =$ constant.

Starting from these equations, we must solve for $r(\theta)$ to describe the orbit. The full derivation of this solution is given in Appendix B.1. It is a good exercise of your understanding if you can follow how we go from the two previous equations to the next equation.

Taking the equations for a central force, $r(\theta)$ is:

$$r(\theta) = \left(\frac{L^2}{m\gamma}\right)\frac{1}{1 + \sqrt{1 + \frac{2EL^2}{m\gamma^2}}\cos\theta}$$

The above equation has the same form as a general ellipse (Equation 11.2). This indicates that our solution for a central force is an ellipse. Moreover, we can define ϵ from the physics as

$$\boxed{\epsilon = \sqrt{1 + \frac{2EL^2}{m\gamma^2}}} \tag{11.5}$$

Note that for the orbit to be a true ellipse, we need $0 < \epsilon < 1$. This condition is only met if $E < 0$, which was the same conclusion that we obtained in Chapter 10 when we looked at the energy and found that r had two real solutions when $E < 0$.

In Chapter 10.6, we showed that a circular orbit occurs when the energy of the system equals the local minimum of the effective potential. The radius of this circular orbit is

$$\boxed{r_c = \frac{L^2}{m\gamma}} \tag{11.6}$$

for the gravitational force. We can also prove this definition of r_c using Newton's laws for uniform circular motion.

$$\begin{aligned}
\frac{\gamma}{r_c^2} = \frac{mv^2}{r_c} \quad \Longrightarrow \quad \gamma r_c &= mv^2 r_c^2 \\
r_c &= \frac{(mvr_c)^2}{m\gamma} \quad \Longrightarrow \quad L = mvr_c \\
r_c &= \frac{L^2}{m\gamma}
\end{aligned}$$

So r_c is the radius of a circular orbit with an angular momentum of $L = mvr_c$.

Combing Equations 11.5 and 11.6 with the equation for $r(\theta)$, we can describe the a position on the orbit as,

$$\boxed{r(\theta) = \frac{r_c}{1 + \epsilon \cos\theta}} \tag{11.7}$$

Quick Questions

1. Find the relation between r_c and a. Test this equation for $\epsilon = 0$. Does your answer make sense?

There are different kinds of orbits. For $\epsilon < 1$, the orbit is elliptical, with the special case of $\epsilon = 0$ for perfectly circular orbits. These are the only orbits we will deal with in great detail in this textbook.

For elliptical orbits, we define the pericenter and apocenter as the positions of closest and furthers distance from one of the foci. In terms of the physics of the system, we want to relate the pericenter and apocenter to r_c, because r_c contains our physics.

$$r = \frac{r_c}{1 + \epsilon \cos\theta}$$

The pericenter is the closest position and it corresponds to $\theta = 0$ and the apocenter is the furthest position when $\theta = 180°$. Putting these cases into r, we get:

$$\boxed{r_p = \frac{r_c}{1 + \epsilon}} \tag{11.8}$$

$$\boxed{r_a = \frac{r_c}{1 - \epsilon}} \tag{11.9}$$

For $\epsilon \geq 1$, the orbit is unbound. These are hyperbolic orbits ($\epsilon > 1$) or parabolic orbits ($\epsilon = 1$). The above equations for the pericenter and apocenter show that this must be true. As $\epsilon \to 1$, the apocenter distance becomes $r_a \to \infty$. That means that your furthest distance is moving so far away that the object is no longer bound to your gravitational field. If you are not bound to the gravitational field, then your system has too much energy to be contained by that field and it will just come in and go out.

Sample Problem 11-1

A satellite of mass m orbits the Earth in a circular orbit of radius r_0. One of its engines is fired briefly toward the Earth.

a) Describe how the energy of this satellite changes after the engine is fired.

b) What happens to the orbit of the satellite?

Solution

a) First, let's consider the initial energy of the satellite. Since the satellite starts in a circular orbit, we know that $\dot{r} = 0$ for the full orbit (radius does not change) and the system energy equals the minimum of the effective potential. From Chapter 10.6, the radius and energy of a circular orbit are:

$$
\begin{aligned}
r_c &= \frac{L^2}{m\gamma} \\
E_{min} &= -\frac{1}{2}\frac{m\gamma^2}{L^2}
\end{aligned}
$$

So our initial energy is $E_i = -\dfrac{m\gamma^2}{2L^2}$.

Now, let's consider what happens to the energy after the engines are fired briefly. We will first assume that the satellite moves a negligible amount, so its position vector, r is unchanged during the energy boost from the engines. We are told that the energy is directed inward toward the Earth. In other words, the energy is applied along a radial direction. Any motion along the radial direction does not change the angular momentum, because $L = \vec{r} \times \vec{p}$. The component of motion along a radial direction does not produce additional angular momentum. So L is the same before and after the energy boost. Thus, the effective potential does not change.

But, the energy boost does induce a change in the radial momentum, which means that $\dot{r} \neq 0$. If we have a radial velocity, then our energy after the boost, E_f is

$$E_f = \frac{1}{2}m\dot{r}^2 + U_{eff}$$

Since the effective potential is unaltered by the engine boost, we can re-write U_{eff} as E_i, since $E_i = U_{eff}$ before the engines fired.

$$E_f = \frac{1}{2}m\dot{r}^2 + E_i$$

Thus, the total energy has increased because kinetic energy was added to the satellite. It does not matter if the rockets move the satellite toward the Earth ($\dot{r} < 0$) or away from the Earth ($\dot{r} > 0$), the kinetic energy term is always positive so it will always add to the total energy. The final energy, E_f, must be larger than our initial energy E_i. The exact value larger depends on the radial velocity $\dot{r}$ given to the satellite by the engines. Since we are not given that quantity, all we can conclude is that the total energy of the satellite has increased due to the engines firing.

b) There are a couple of ways we can answer this question. First, we can sketch the energy diagram. Figure 11.4 shows a sketch of what the initial and final energies may look like for this satellite. Note, we are assuming that the boost in energy is sufficiently small that the satellite remains bound to the Earth.

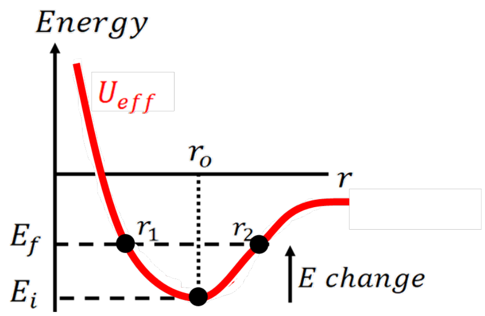


Figure 11.4: Change in energy for our perturbed satellite. The satellite starts in a circular orbit so that $E = E_i$ which is at the minimum of the effective potential. After the engines are fired, the energy increases so that $E = E_f$.

Figure 11.4 shows the initial energy, E_i at the minimum of U_{eff} and the final energy E_f which is somewhat larger. The increase in energy makes the satellite go into an elliptical orbit. We can see that the orbit is elliptical because we have two solutions for r when the energy line hits the effective potential curve (denoted by r_1 and r_2 in the figure). These values of r_1 and r_2 are the perigee and apogee positions of the satellite (with the Earth at a focus). If the engines move the rocket inward initially, the the rocket will begin moving toward its perigee position (from an initially circular orbit of r_c).

Alternatively, we could argue that the orbit is elliptical by looking at the equation for ϵ in Equation 11.5. From this equation, $\epsilon = 0$ when $E = E_{min}$ and ϵ will increase if $E > E_{min}$. An orbit is circular if $\epsilon = 0$ and elliptical for $0 < \epsilon < 1$.

11.3 Kepler's Laws

Kepler introduced three laws to describe planetary motion. These were based on careful observations by astronomer Tycho Brahe. Kepler used the systematics of these observations to determine how planets move. About 80 years later, Newton was able to explain this planetary motion using the physics of gravity.

The three laws of planetary motion are:

1. Planets move on elliptical orbits with the Sun at one focus.

2. The vector from the Sun to a planet sweeps out equal areas in equal times.

3. The square of the period of a full orbit about the Sun is proportional to the cube of the semi-major axis.

For the first law, we have shown in this chapter (and in the last chapter) that bound orbits are elliptical in a gravitational potential (we consider circular orbits to be a special case of the elliptical orbit where $r_p = r_a$). Gravity being a central force that follows an inverse-square law will naturally give rise to elliptical orbits provided that the system is bound.

Quick Question

1. The first law states that the planets have elliptical orbits with the Sun at one focus. But an ellipse by definition has two foci. What is at the other focus for planetary orbits?

For the second law, the radius vector from the Sun sweeps equal areas in equal times because of the conservation of angular momentum. Recall also that for a central force, all motion takes place in a 2-D plane (see Chapter 10). Thus, we can use plane polar coordinates to describe the motion alone. Figure 11.5 shows the area swept out by the radius vector in time Δt in polar coordinates.

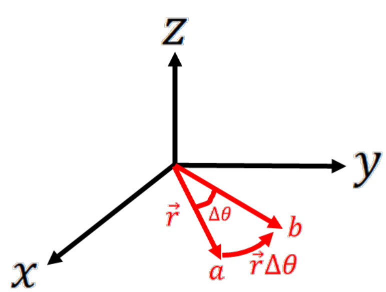


Figure 11.5: Motion in the xy plane from a central force. In time Δt, the object moves from position a to position b and sweeps out an area defined by the triangle from the origin to a and b.

In time Δt, a particle moves from position a to position b as shown in Figure 11.5. The area (from the origin) to those points is a triangle with

$$\begin{aligned}\Delta A &= \frac{1}{2}r(r\Delta\theta) = \frac{r^2\Delta\theta}{2} \\ \frac{\Delta A}{\Delta t} &= \frac{r^2}{2}\frac{\Delta\theta}{\Delta t} \quad \Longrightarrow \quad \text{divide both sides by } \Delta t\end{aligned}$$

Note that this assumes that you have small enough angles $\Delta\theta$ so that we can approximate the area as a triangle. This approximation is true for infinitesimally small times. Therefore, we can assume $\Delta t \to \mathrm{d}t$, $\Delta A \to \mathrm{d}A$, and $\Delta\theta \to \mathrm{d}\theta$.

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{r^2}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ \dot{A} &= \frac{r^2\dot{\theta}}{2} \quad \Longrightarrow \quad \text{recall } L = mr^2\dot{\theta} \\ \dot{A} &= \frac{L}{2m} \quad \Longrightarrow \quad L = \text{constant} \\ \dot{A} &= \text{constant}\end{aligned}$$

Because we have a constant angular momentum, we naturally get Kepler's second law that equal areas are swept out in equal time intervals (or $\dot{A} = \text{constant}$).

For the third law, we have:

$$\boxed{\frac{T^2}{a^3} = \text{constant}} \tag{11.10}$$

where T is the period and a is the semi-major axis. To show this is the case, we need to use the previous laws and our equations for the properties of an ellipse.

From the second law, $\dot{A}$ is a constant. So we know the speed by which we trace out an area in our ellipse. The total area of an ellipse is just $A_{tot} = \pi ab$, where a and b are the semi-major and semi-minor axes. Thus, the period can be given as:

$$T = \frac{A}{\dot{A}} = \frac{\pi ab}{L/2m} = \frac{2\pi mab}{L}$$

We also can relate the semi-major and semi-minor axes to each other. That is, $b = a\sqrt{1-\epsilon^2}$. Now we need to get $1-\epsilon^2$ in terms of a and the physics. From Equations (11.8) and (11.9)

$$r_p = \frac{r_c}{1+\epsilon}$$
$$r_a = \frac{r_c}{1-\epsilon}$$

We also know that $r_p + r_a = 2a$ for an ellipse. Therefore, we can add the above equations to give:

$$\begin{aligned}
r_p + r_a &= 2a \\
2a &= \frac{r_c}{1+\epsilon} + \frac{r_c}{1-\epsilon} \\
2a &= r_c\left(\frac{1-\epsilon}{1-\epsilon^2} + \frac{1+\epsilon}{1-\epsilon^2}\right) \\
2a &= r_c\left(\frac{2}{1-\epsilon^2}\right) \\
1-\epsilon^2 &= \frac{r_c}{a} \quad \Longrightarrow \quad r_c = \frac{L^2}{m\gamma} \\
1-\epsilon^2 &= \frac{L^2}{am\gamma}
\end{aligned}$$

Thus, we can re-write the period equation as:

$$\begin{aligned}
T &= \frac{2\pi mab}{L} \\
T &= \frac{2\pi ma(a\sqrt{1-\epsilon^2})}{L} && \Longrightarrow \quad \text{sub in for } b \\
T &= \frac{2\pi ma^2(\sqrt{\frac{L^2}{am\gamma}})}{L} && \Longrightarrow \quad \text{sub in for } 1-\epsilon^2 \\
T^2 &= \frac{4\pi^2 m^2 a^4(\frac{L^2}{am\gamma})}{L^2} && \Longrightarrow \quad \text{square both sides} \\
T^2 &= \frac{4\pi^2 ma^3}{\gamma} && \Longrightarrow \quad \text{simplify} \\
T^2 &= \frac{4\pi^2 a^3}{GM} && \Longrightarrow \quad \text{sub } \gamma = GMm
\end{aligned}$$

$$\boxed{\frac{T^2}{a^3} = \frac{4\pi^2}{GM} = \text{constant}} \tag{11.11}$$

11.4 Application of Kepler's Laws

Here we will look at a few problems that apply Kepler's Laws.

Sample Problem 11-2

Halley's comet has a period of 75 years. **What is its semi-major axis in au?** The unit of "au" is the "Astronomical Unit" and represents the distance between the Earth and the Sun (roughly 150 million km).

Solution

This question is a straight forward example of Kepler's third law. We will solve it in two ways. The first way is to just apply Kepler's third law as it is written.

$$T^2 = \frac{4\pi^2 a^3}{GM}$$

Note for comets, the orbit is around the Sun (the Sun is at one focus). So we need to set $M = M_{sun} = 2.0 \times 10^{30}$ kg. Plugging in our numbers, we get:

$$\begin{aligned}
[75 \text{ years} \times (3.154 \times 10^{7} \text{ s/year})]^2 &= \frac{4\pi^2 a^3}{(6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2})(2.0 \times 10^{30} \text{ kg})} \\
5.6 \times 10^{18} \text{ s}^2 &= (2.97 \times 10^{-19} \text{ s}^2 \text{ m}^{-3}) a^3 \\
a^3 &= 1.9 \times 10^{40} \text{ m}^3 \\
a &= 2.66 \times 10^{12} \text{ m} \\
a &= 18 \text{ au}
\end{aligned}$$

Now this is a perfectly acceptable way to solve the problem, but it involves a lot of math and plugging big numbers into a calculator. It is easy to make a mistake with that. A better way to solve this problem is to use scaling relations.

From the third law, $T^2 \propto a^3$ or $\frac{T^2}{a^3} = \text{constant}$. That means if we know T and a for one orbit, we can scale that solution to correspond to any other orbit around the same object. Consider two objects orbiting the Sun. The first object has a period T_1 and a semi-major axis a_1, the second object has a period T_2 and semi-major axis a_2. Since $T^2 \propto a^3$,

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{a_1}{a_2}\right)^3$$

This is a *scaling relation*. It is a much simpler (and faster) way to solve the same problem. As long as you have a reference system, you can scale that reference system to any other orbit that goes around the same body. A convenient reference system is the Earth. We know that it takes 1 year for the Earth to orbit the Sun and the Earth by definition is 1 au from the Sun. So we our scaling relation becomes:

$$\left(\frac{T}{1 \text{ year}}\right)^2 = \left(\frac{a}{1 \text{ au}}\right)^3$$

With this scaling relation, let's go back to our question about Halley's comet. Halley's comet has a period of 75 years.

$$\begin{aligned}
\left(\frac{75 \text{ years}}{1 \text{ year}}\right)^2 &= \left(\frac{a}{1 \text{ au}}\right)^3 \\
\left(\frac{a}{1 \text{ au}}\right) &= 75^{2/3} \\
a &= 18 \text{ au}
\end{aligned}$$

Look at how much faster it was to solve the same problem using a scaling relation. You get the same answer, but the math is much simpler. You can also make quick comparisons between systems using scaling relations, which makes these approaches to solving problems very efficient.

Note, the above scaling relation only applies to orbits around the Sun. If you change the source of the gravitational field, you need to change your reference orbit. This is because the constant of proportionality between T^2 and a^3 depends on the mass, M, that is producing the gravitational field.

Cora's Thoughts

Scaling relations, like the one used in this question, can be extremely useful. In the astrophysics field, scaling relations are often used to describe trends observed between physical properties such as size, luminosity, mass, and colour of stars and galaxies. It can be very hard to determine these properties due to the large distances, gas clouds, and other factors in space. So, scaling relations such as the Faber-Jackson Relation (FJR) are used to determine these physical properties that are otherwise difficult to obtain or compare.

Sample Problem 11-3

Halley's comet has an eccentricity of $\epsilon = 0.967$. **What are the perihelion and aphelion distances and how fast is the comet traveling at those positions?**

Solution

Perihelion and aphelion are the terms used for the pericenter and apocenter when the Sun is at the focus. From Equations (11.3) and (11.4) and the semi-major axis of 18 au for Halley's comet from the last problem, we can easily solve for r_p and r_a.

$$r_p = a(1-\epsilon) = (18 \text{ au})(1-0.967) = 0.6 \text{ au}$$
$$r_a = a(1+\epsilon) = (18 \text{ au})(1+0.967) = 35 \text{ au}$$

To get the speeds at perihelion and aphelion, let's start with the angular momentum. At these positions, the radial vectors and velocity vectors are perpendicular ($\dot{r}$ is zero). Thus,

$$L = mv_a r_a = mv_p r_p$$

which is a constant. We can also relate L to the equivalent radius for a circular orbit, r_c, and we can relate r_c to the perihelion and aphelion distances.

$$\begin{aligned} r_c &= \frac{L^2}{m\gamma} \\ r_p &= \frac{r_c}{1+\epsilon} \\ r_a &= \frac{r_c}{1-\epsilon} \end{aligned}$$

Plugging these in with the angular momentum equation,

$$\begin{aligned} mv_p r_p &= \sqrt{r_c m\gamma} \\ mv_p r_p &= \sqrt{(1+\epsilon) r_p m\gamma} \\ v_p &= \sqrt{\frac{(1+\epsilon)\gamma}{r_p m}} \\ v_p &= \sqrt{\frac{GM(1+\epsilon)}{r_p}} \end{aligned}$$

Follow the same procedure for the aphelion distance to get

$$v_a = \sqrt{\frac{GM(1-\epsilon)}{r_a}}$$

Since we are interested in speeds, we do not care about the negative signs with the square root.

If we plug in our values for M, ϵ, r_a and r_p into the above equations, we get:

$$\begin{aligned} v_p &= 55 \text{ km s}^{-1} \\ v_a &= 0.9 \text{ km s}^{-1} \end{aligned}$$

As expected, we get $v_p > v_a$.

Quick Questions

1. Prove that $v_a = \sqrt{\frac{GM(1-\epsilon)}{r_a}}$
2. Show that $v_p = 55$ km s^{-1} and $v_a = 0.9$ km s^{-1} for Halley's comet.

11.5 Real World Application

Preventing an asteroid strike on Earth may seem like a plot out of a movie, but there is ongoing research into how to do this properly. Rather than trying to blow up the asteroid, scientists have come up with different technique: alter the orbit through a kinetic impact. The basis of this plan is to slam a spacecraft into an asteroid head on so that it loses angular momentum and subsequently moves into a slightly different orbit.

The Double Asteroid Redirection Test (DART) spacecraft was launched to test this exact scenario. DART targeted a tiny asteroid called Dimorphos, which is in orbit around a larger asteroid, Didymos. The goal of this mission was to use the kinetic impact of DART to change the orbital parameters of Dimorphos.

On 26 September 2022, DART made impact on Dimorphos and successfully caused the moonlet to spiral inward into a new (smaller) orbit. Subsequent observations confirmed a new orbital period that decreased by 32 minutes (from an original length of almost 12 hours). The mission was a big success and showed that such techniques could be used to protect the Earth in future.

For more information:
The DART Mission Website has lots of information and there is also video of the impact. The Jet Propulsion Lab some information on the science and engineering behind the mission.

11.6 Summary

Key Takeaways

This chapter expands on orbits from Chapter 10, describing and defining the geometric shapes and physics of elliptical orbits. Ellipses are elongated circles of semi-major axis a, semi-minor axis b, and eccentricity ϵ. Mathematically, these shapes are defined as:

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta}$$
$$\epsilon = \sqrt{1-\frac{b^2}{a^2}}$$

where r is measured from one of the two foci of the ellipse. The closest point of an ellipse to the focus is called the pericenter and the furthest point is called the apocenter:

$$\begin{aligned} r_p &= a(1-\epsilon) \\ r_a &= a(1+\epsilon) \end{aligned}$$

For a system with only gravity acting, we derived the orbit equation in terms of the the angular momentum L and total energy E,

$$\begin{aligned} r &= \frac{L^2}{m\gamma} 1 + \epsilon \cos\theta \\ \epsilon &= \sqrt{1 + \frac{2EL^2}{m\gamma^2}} \end{aligned}$$

and we defined a radius, r_c, which represent the radius of a circular orbit for a system of angular momentum L and $E = E_{min}$.

$$r_c = \frac{L^2}{m\gamma}$$

Note that $E \geq U_{eff}$ for a gravitational field, because gravity is a central force. Thus, a key element to the shape of an orbit is the amount of energy in the system. If there is the minimum energy, the orbit is circular. As the energy increases, the orbit becomes elliptical and then unbound (parabolic or hyperbolic).

This chapter also discusses Kepler's three laws of planetary motion. These laws apply to orbits because gravity is a central force with an inverse-square law. The most applicable law is the third law, where

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} = \text{constant}$$

A useful technique when applying Kepler's third law is to use scaling relations. If you know a solution for one case of T and a, you can scale to any other case of T or a for the same gravitational field.

Important Equations

Eccentricity:

$$\begin{aligned} \epsilon &= \sqrt{1 - \frac{b^2}{a^2}} \\ \epsilon &= \sqrt{1 + \frac{2EL^2}{m\gamma^2}} \end{aligned}$$

Distance:

$$\begin{aligned} r &= \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} \\ r &= \frac{r_c}{1+\epsilon\cos\theta} \\ r_c &= \frac{L^2}{m\gamma} \end{aligned}$$

Pericenter:

$$\begin{aligned} r_p &= a(1-\epsilon) \\ r_p &= \frac{r_c}{1+\epsilon} \\ v_p &= \sqrt{\frac{GM(1+\epsilon)}{r_p}} \end{aligned}$$

Apocenter:

$$\begin{aligned} r_a &= a(1+\epsilon) \\ r_a &= \frac{r_c}{1-\epsilon} \\ v_a &= \sqrt{\frac{GM(1-\epsilon)}{r_a}} \end{aligned}$$

Kepler's 3rd Law:

$$\frac{T^2}{a^3} = \frac{4\pi^2}{GM} = \text{constant}$$

11.7 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 11-1

Find the eccentricities of the orbits of the following Solar System objects, given their aphelion and perihelion distances.

a) Mercury: aphelion = 0.4667 au, perihelion = 0.3075 au

b) Ceres: aphelion = 2.98 au, perihelion = 2.55 au

c) Halley's comet: aphelion = 35.14 au, perihelion = 0.59278 au

d) Quaoar: aphelion = 45.488 au, perihelion = 41.900 au

Practice Problem 11-2

Find the orbital periods for the Solar System objects in Problem 11-1.

Practice Problem 11-3

How much faster is Mercury at perihelion than aphelion? Use the values given in Problem 11-1.

Practice Problem 11-4

Two planets (A and B) orbit the same star. Planet A has an orbital period of 45 years and Planet B has a period of 100 years. With careful observations, the semi-major axis of Planet B is found to be 26.5 au. What is the semi-major axis of Planet A?

Practice Problem 11-5

A rogue asteroid collides with Saturn's moon Mimas and knocks Mimas into a new orbit that is exactly twice its old period of 22.5^h. Knowing its previous semi-major axis was 1.855×10^5 km, find the semi-major axis of its new orbit.

Practice Problem 11-6

Tau Ceti is a star approximately 11.9 light years from Earth and is known to have a planetary system. One of its planets, Tau Ceti-e, has an orbit with a semi-major axis ≈ 0.538 au and period ≈ 162.9 days. Use this information to estimate the mass of Tau Ceti.

Practice Problem 11-7

What is the speed of a satellite in circular orbit around the Earth with a radius of orbit of R_s?

Practice Problem 11-8

A star orbits the central black hole of a galaxy in an elliptical orbit with $a = 650$ au. If the star takes 15 years to complete one orbit, what is the mass of the central black hole relative to the Sun's mass?

Practice Problem 11-9

A satellite is orbiting the Earth in an elliptical orbit. If the eccentricity is $\epsilon = 0.4$ and its speed at apogee is v_0, what is the speed at perigee?

Practice Problem 11-10

A satellite has an elliptical orbit where it is 250km above the Earth's surface at perigee and it is traveling at a speed of 8km s^{-1} at perigee. How high above the Earth's surface is this satellite at apogee? Hint: You can assume the Earth's radius is 6370km and the mass of the Earth is 5.97×10^{24}kg.

Practice Problem 11-11

A comet has an elliptical orbit with an eccentricity of ϵ with a perihelion distance of p from the Sun. Assume the Earth has a circular orbit of radius R and the comet orbits in the plane of the Earth with $p < R$ such that the comet crosses Earth's orbit in two places.

a) Find the equation for the comet's position in its orbit as a function of its perihelion distance and an angle θ (where $\theta = 0$ at the perihelion position).

b) Find the angles where the comet crosses Earth's orbit?

c) What happens if $\epsilon = 0$?

d) What happens if $\epsilon = 1$?

12 The Lagrange Method

Learning Objectives

- Introduction to the Lagrangian and Euler-Lagrange equations
- Apply the Lagrangian method to physics problems

In this chapter, we will introduce the Lagrangian and the Euler-Lagrange method to solving physics problems. This technique represents another tool in your physics toolkit, much like how we can use energy conservation, momentum conservation, and Newton's laws to solve physics problems.

12.1 Introduction to the Lagrangian Method

Consider the case of a single particle being acted on by a conservative force in 1-D (x-axis). From the energy equations, we can say that:

$$F = -\frac{\partial U}{\partial x}$$

But from Newton's laws, we can also write the force as

$$F = \frac{\mathrm{d}p}{\mathrm{d}t} = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}\dot{x}}{\mathrm{d}t}$$

for constant mass. We can therefore relate the force to the kinetic energy, because the kinetic energy depends on the velocity, $K = \frac{1}{2}m\dot{x}^2$.

$$\begin{aligned} \frac{\mathrm{d}K}{\mathrm{d}\dot{x}} &= m\dot{x} \quad \Longrightarrow \quad \text{derivative of } K \text{ with } \dot{x} \text{ is the momentum} \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}K}{\mathrm{d}\dot{x}}\right) &= m\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = F \quad \Longrightarrow \quad \text{take the time derivative to get } m\ddot{x} \end{aligned}$$

We can combine these two force equations to say that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}K}{\mathrm{d}\dot{x}}\right) = -\frac{\mathrm{d}U}{\mathrm{d}x}$$

The above equation is a function of the kinetic and potential energies. So we define the **Lagrangian** as:

$$\boxed{\mathcal{L} = K - U} \tag{12.1}$$

where $\mathcal{L}$ is the Lagrangian, and we will use the symbol $\mathcal{L}$ for the Lagrangian to make it distinct from the angular momentum (defined as L in this text).

$\mathcal{L}$ is a function of position and velocity. That is, $\mathcal{L} = \mathcal{L}(x, \dot{x})$ in 1-D, because $K = K(\dot{x})$ and $U = U(x)$. In this case, K is not a function of position and U is not a function of velocity. As such,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= -\frac{\partial U}{\partial x} \\ \frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{\partial K}{\partial \dot{x}} \end{aligned}$$

where ∂ indicates a partial derivative. For a partial derivative, you hold all other variables constant and only take the derivative with respect to the one variable.

Therefore, we can rewrite $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}K}{\mathrm{d}\dot{x}}\right) = -\frac{\mathrm{d}U}{\mathrm{d}x}$ which we had before as:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}} \tag{12.2}$$

This is the **Euler-Lagrange equation**. The above example is for 1-D motion in the x-axis, but in practice, you can apply the arguments to represent the Euler-Lagrange equations in other (independent) coordinates. In its general form, the Euler-Lagrange equation is:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i}} \tag{12.3}$$

where x_i represents a coordinate axis (e.g., could be x, y, r, etc.). For multi-dimensional problems, you need to solve the Euler-Lagrange equations for each dimension (each axis) separately.

The Lagrangian method represents yet another way you can solve problems in physics. For the rest of the chapter, we will look at a few examples.

Quick Questions

1. Show that the units of the Euler-Lagrange equation match expectations.
2. Show that you get comparable equations for the y and z axis.

Caveats to the Lagrange Method

For the Lagrange method to work, we must treat $\dot{x}_i$ and x_i as independent variables and assume that the time dependence only enters the problem through x and $\dot{x}_i$.

12.2 Application to 1-D Problems

Sample Problem 12-1

Consider a horizontal spring-mass system on a frictionless surface. **Compare the differential equation of motion for the mass using (1) Newton's Laws, (2) Energy Conservation, and (3) the Euler-Lagrange method.**

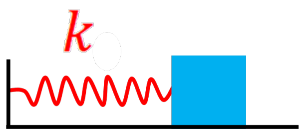


Figure 12.1: Horizontal spring mass system with the spring constant k, and mass m.

Solution

Method 1: Newton's Laws:

$$\begin{aligned}
\sum \vec{F} &= m\vec{a} \\
-kx &= m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = m\ddot{x} \quad \Longrightarrow \quad \text{net force is spring force} \\
0 &= \ddot{x} + \frac{k}{m}x
\end{aligned}$$

Method 2: Energy Conservation:

$$\begin{aligned}
E &= K + U \quad \Longrightarrow \quad \text{assume } U = 0 \text{ at equilibrium} \\
E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \quad \Longrightarrow \quad K = \tfrac{1}{2}m\dot{x}^2,\ U = \tfrac{1}{2}kx^2 \\
\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) \\
0 &= m\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} + kx\dot{x} \quad \Longrightarrow \quad \text{set } \tfrac{\mathrm{d}E}{\mathrm{d}t} = 0 \\
0 &= \ddot{x} + \frac{k}{m}x \quad \Longrightarrow \quad \text{simplify}
\end{aligned}$$

Method 3: Euler-Lagrange:

For this method, we will solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

where we can use the energy terms from previously to solve the Lagrangian,

$$\mathcal{L} = K - U = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Taking each partial differential term of the Euler-Lagrange equations separately:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{2}kx^2\right) &= -kx \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{x}}\left[\frac{1}{2}m\dot{x}^2\right]\right) &= \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = m\ddot{x} \end{aligned}$$

So now going back to the Euler-Lagrange equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\partial \mathcal{L}}{\partial x} \\ m\ddot{x} &= -kx \\ 0 &= \ddot{x} + \frac{k}{m}x \end{aligned}$$

All three methods match!

Quick Question

1. Use the Lagrangian to get the equation of motion for a vertical mass-spring system and a simple pendulum. Compare to using the other methods.

Lance's Thoughts

Thinking about a variable and its time derivative as two separate variables is a little strange to begin with, but if you're able to set up the Lagrangian system properly, a couple of quick derivatives can get you to the solution a lot faster. It might help your first few tries at it to back out of dot notation very briefly: x and v look more different than x and $\dot{x}$. Visually, it can be easier to distinguish

$$m_1a = (m_1 + m_2)v^2 + m_1gx_1 + m_2g(h - x_1)$$

from

$$m_1\ddot{x} = (m1 + m2)\dot{x} + m_1gx_1 + m_2g(h - x_1)$$

until you get used to the process and so can be easier to see position and velocity as separate variables.

Sample Problem 12-2

Consider an Atwood machine with a pulley that is a solid disk of mass M and radius R ($I = \frac{1}{2}MR^2$) and two masses, m_1 and m_2. Assume the rope is inextensible and the pulley rotates without slipping. **What is the acceleration of m_1?**

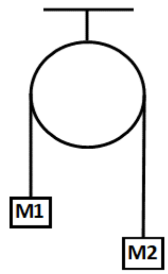


Figure 12.2: The Atwood machine for this problem.

Solution

Note that we've looked at this problem previously with energy conservation (Sample Problem 9-3). Please review that question for more details, but briefly, the potential energy of the system comes from the vertical position of the two masses and the kinetic energy is from the linear motion of the two masses and the rotational motion of the pulley. Figure 12.3 shows the definition of the position of each of the masses. Note that we have set the $y = 0$ line to be where $U = 0$.

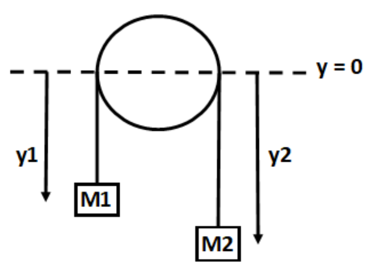


Figure 12.3: The Atwood machine with the positions of each mass.

For this system, we have:

$$\begin{aligned} U &= -m_1 g y_1 - m_2 g y_2 \\ K &= \frac{1}{2}m_1(\dot{y}_1)^2 + \frac{1}{2}m_2(\dot{y}_2)^2 + \frac{1}{2}I\omega^2 \end{aligned}$$

At first glance, this may seem like a 2-D problem, but the positions of y_1 and y_2 are not independent. As one increases, the other decreases because the rope is inextensible (does not stretch) and has a constant length. If the length of the rope is A (a constant), then $y_1 + y_2 = A$. We can then express y_2 and $\dot{y}_2$ with respect to y_1.

$$\begin{aligned} y_2 &= A - y_1 \\ \dot{y}_2 &= -\dot{y}_2 \end{aligned}$$

Moreover, since this is an ideal pulley, it rotates without slipping so we can also use $|\dot{y}_1| = R\omega$. Thus, we can re-write the energy equations as:

$$\begin{aligned} U &= -m_1 g y_1 - m_2 g(A - y_1) \\ K &= \frac{1}{2}(m_1 + m_2)(\dot{y}_1)^2 + \frac{1}{2} I \left(\frac{\dot{y}_1}{R}\right)^2 \\ &= \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{y}_1^2 \quad \Longrightarrow \quad \text{sub in } I = \tfrac{1}{2}MR^2 \end{aligned}$$

The Lagrangian (in terms of y_1 and $\dot{y}_1$ only is then:

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{y}_1^2 + m_1 g y_1 + m_2 g(A - y_1) \\ &= \frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{y}_1^2 + (m_1 - m_2) g y_1 + m_2 g A \end{aligned}$$

Note that this Lagrangian contains a constant term ($m_2 g A$). This term does not factor into the Euler-Lagrange equations because it has no dependence on position or velocity. You can essentially ignore any constant terms in the Lagrangian method.

The terms of the Euler-Lagrange equation are:

$$\frac{\partial \mathcal{L}}{\partial y_1} = \frac{\partial}{\partial y_1}\left[(m_1 - m_2) g y_1\right] = (m_1 - m_2) g$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}_1}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{y}_1}\left[\frac{1}{2}\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{y}_1^2\right]\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(m_1 + m_2 + \frac{1}{2}M\right)\dot{y}_1\right] \\ &= \left(m_1 + m_2 + \frac{1}{2}M\right)\ddot{y}_1 \end{aligned}$$

And the Euler-Lagrange equation is:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial y_1} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{y}_1}\right)\\(m_1-m_2)g &= \left(m_1+m_2+\frac{1}{2}M\right)\ddot{y}_1\\\ddot{y}_1 &= \frac{g(m_1-m_2)}{m_1+m_2+\frac{1}{2}M}\end{aligned}$$

This is exactly what we got from energy conservation (Sample Problem 9-3).

12.3 Application in 2-D

Sample Problem 12-3

Consider an object of mass m moving in an elliptical orbit under gravity in polar coordinates where the acceleration due to gravity changes with position. **Solve the Euler-Lagrange equations for this object.**

Solution

For this problem we have motion in 2-D because the object is in an elliptical orbit, which means that

$$\vec{v} = v_r\hat{r} + v_\theta\hat{\theta} = \dot{r}\hat{r} + (r\dot{\theta})\hat{\theta}$$

where $\hat{r}$ and $\hat{\theta}$ are independent dimensions.

To solve the Lagrangian, we need the sources of potential and kinetic energy. The only source of potential energy is the gravitational field and the only source of kinetic energy is the orbital motion (note that we are using the central force potential, not the effective potential here). For this system, we have:

$$\begin{aligned}U &= -\frac{GMm}{r}\\K &= \frac{1}{2}mv^2\end{aligned}$$

where the squared velocity is given by:

$$v^2 = v_r^2 + v_\theta^2 = \dot{r}^2 + (r\dot{\theta})^2$$

See Chapter 1.2 for a review of plane-polar coordinates. Thus, in polar coordinates, the energy equations are:

$$
\begin{aligned}
U &= -\frac{GMm}{r} \\
K &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)
\end{aligned}
$$

And the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{GMm}{r}$$

So the Lagrangian is really $\mathcal{L} = \mathcal{L}(r, \dot{r}, \theta, \dot{\theta})$ and each dimension must be solved separately.

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \\
\frac{\partial \mathcal{L}}{\partial \theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right)
\end{aligned}
$$

Note, however, that the Lagrangian does not have any θ dependence. That's because gravity is a radial force. As a consequence, we can say that:

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0$$

Thus, from the Euler-Lagrange equation:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right)
\end{aligned}
$$

So the derivative of the Lagrangian with angular velocity ($\frac{\partial \mathcal{L}}{\partial \dot{\theta}}$) is constant with time.

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}mr^2\dot{\theta}^2\right) = mr^2\dot{\theta} = \text{constant}$$

The term $mr^2\dot{\theta}$ is the angular momentum of the system (see Chapter 6) and we recover the condition that angular momentum is constant (as expected for a central force; Chapter 10) directly from the Lagrangian and Euler-Lagrange equations.

Now let's look at the radial terms.

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{2}mr^2\dot{\theta}^2 + \frac{GMm}{r}\right) = mr\dot{\theta}^2 - \frac{GMm}{r^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial \dot{r}}\left[\frac{1}{2}m\dot{r}^2\right]\right) = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) = m\ddot{r}$$

And the Euler-Lagrange equation is:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial r} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) \\ mr\dot{\theta}^2 - \frac{GMm}{r^2} &= m\ddot{r} \end{aligned}$$

At this point, we have two equations of motion.

$$\begin{aligned} L &= mr^2\dot{\theta} \quad \Longrightarrow \quad L \text{ is the angular momentum (and this is constant)} \\ m\ddot{r} &= mr\dot{\theta}^2 - \frac{GMm}{r^2} \quad \Longrightarrow \quad \text{the radial component} \end{aligned}$$

We can combine these equations to remove the $\dot{\theta}$ term from the radial equation.

$$\begin{aligned} m\ddot{r} &= mr\frac{L^2}{m^2r^4} - \frac{GMm}{r^2} \quad \Longrightarrow \quad \dot{\theta} = \frac{L}{mr^2} \\ \ddot{r} &= \frac{L^2}{m^2r^3} - \frac{GM}{r^2} \end{aligned}$$

We have now created a 1-D equation of motion because L, G, m, and M are all constants. The only variable is r. That means that we can solve this problem.

You will also notice that this is the effective force for a system moving under a gravitational force that we found in Chapter 10.6 and used in Chapter 11 when looking at Kepler's Law's of Planetary Motion.

12.4 Example Problem: Sphere on an Incline

The following problem includes rolling and translation motion.

Sample Problem 12-4

A sphere rolls down an incline without slipping. Assume that the moment of inertia of the sphere is $\frac{2}{5}MR^2$. **Solve the Euler-Lagrange equations for the rolling sphere?** Assume $U = 0$ at $x = 0$.

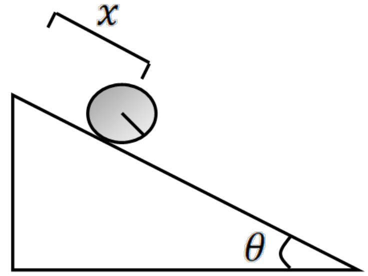


Figure 12.4: A sphere of radius R rolls down an incline of angle θ.

Solution

First, we need the potential and kinetic energies. The only source of potential energy is the change in gravitational potential energy as the sphere rolls down. For kinetic energy, we have the translation and rotation of the sphere.

Since the motion is along the incline only, we will define $\hat{x}$ to be pointing down along the incline, where $U = 0$ at $x = 0$ (the top of the incline). Figure 12.5 shows our coordinate system in terms of x. With this definition, the height of the sphere from the top of the incline as a function of x is $x \sin\theta$:

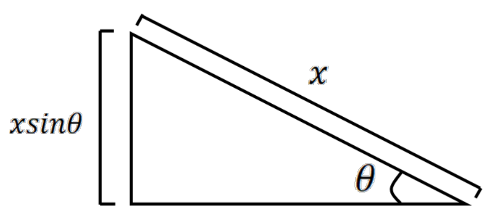


Figure 12.5: Using Pythagorean theorem we can determine the change in height of the sphere.

The potential energy is therefore:

$$\begin{aligned} U &= -Mg\Delta h \\ U &= -Mgx\sin\theta \end{aligned}$$

The potential energy is negative because U decreases as the sphere rolls down and we have defined x as positive pointing down the incline (how we defined the coordinate system).

For the kinetic energy, we have the translation and rotation motion of the sphere, which we defined in Chapter 8. The kinetic energy is:

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I\omega^2$$

For a sphere, $I = \frac{2}{5}MR^2$ (see Appendix A.4) it is rolling without slipping, which means

$$\omega = \frac{v_{cm}}{R} = \frac{\dot{x}}{R}$$

Substituting the moment of inertia equation and the rolling without slipping condition, the kinetic energy becomes:

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}\left(\frac{2}{5}\right)MR^2\left(\frac{\dot{x}}{R}\right)^2 = \frac{7}{10}M\dot{x}^2$$

Now we can solve for the Lagrangian:

$$\mathcal{L} = K - U = \frac{7}{10}M\dot{x}^2 + Mgx\sin\theta$$

Note that θ is constant and not a coordinate axis in this problem. This Lagrangian is in 1-D (all motion is along the $\hat{x}$ axis).

The Euler-Lagrange equations are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= Mg\sin\theta \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{7}{5}M\dot{x}\right) = \frac{7}{5}M\ddot{x}
\end{aligned}$$

To solve for the motion itself, we equate the two Euler-Lagrange equations:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\partial \mathcal{L}}{\partial x} \\
\frac{7}{5}M\ddot{x} &= Mg\sin\theta \\
\ddot{x} &= \frac{5}{7}g\sin\theta
\end{aligned}$$

Thus, our acceleration is proportional to gravity, with a $\sin\theta$ term due to the incline, as expected. The additional factor of $\frac{5}{7}$ is due to the fact that the sphere is rolling down the incline instead of sliding.

12.5 Challenging Problem: Particle on a Wire

For this problem, the solution is not a simple differential equation. Consider whether you would try to solve this problem using Newton's Laws or energy conservation.

Sample Problem 12-5

A particle of mass m, moves along a bent wire. The shape of the wire can be described with the equation $y = ax^4$. (Note that the wire equation represents the path the particle can move along.) **What is the equation of motion for the particle?**

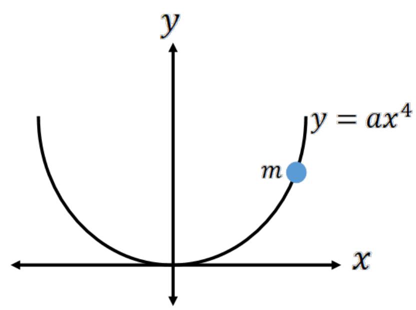


Figure 12.6: The function $y = ax^4$ is the the path that particle of mass m moves along.

Solution

To help solve this problem, we will break it down into a few parts. We will get the equation of motion when we have expressions for K and U. That means we need to describe the motion of the particle at any given time and we need to describe the position of the particle at any give time.

a) What is the velocity of the particle?

Since the particle moves in two-dimensions, x and y, we have:

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$

The position in the y-direction can be described by the equation of the wire, $y = ax^4$ so the y-velocity is the derivative with respect to time of our y-position, $\dot{y} = 4ax^3\dot{x}$. This means that:

$$\vec{v} = \dot{x}\hat{\imath} + 4ax^3\dot{x}\hat{\jmath}$$

b) Solve the Lagrangian.

We know that,

$$\mathcal{L} = K - U$$

so let's find our kinetic and potential energies. We only have translation kinetic energy from the motion of the particle,

$$K = \frac{1}{2}mv^2$$

where,

$$\begin{aligned} v^2 &= \vec{v} \cdot \vec{v} \\ v^2 &= \dot{x}^2 + \dot{y}^2 \\ v^2 &= \dot{x}^2 + (4ax^3\dot{x})^2 \\ v^2 &= \dot{x}^2 + 16a^2x^6\dot{x}^2 \\ v^2 &= \dot{x}^2(1 + 16a^2x^6) \end{aligned}$$

Therefore, we can solve for the kinetic energy in terms of x and $\dot{x}$ only.

$$K = \frac{1}{2}m\dot{x}^2(1 + 16a^2x^6)$$

The only source of potential energy is gravity,

$$\begin{aligned} U &= mgh && \Longrightarrow \quad \text{where } h = y \\ U &= mgy && \Longrightarrow \quad \text{where } U = 0 \text{ when } y = 0 \\ U &= mgax^4 \end{aligned}$$

Note that U is positive since the particle is above the $y = 0$ point, so $U > 0$. We can now solve our Lagrangian:

$$\begin{aligned} \mathcal{L} &= K - U \\ \mathcal{L} &= \frac{1}{2}m\dot{x}^2(1 + 16a^2x^6) - mgax^4 \end{aligned}$$

c) Find the differential equation of motion of the bead.

To do this let's solve the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

We can start by solving the left-side:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{1}{2}m\dot{x}^2(1+16a^2x^6) - mgax^4\right] \\
\frac{\partial \mathcal{L}}{\partial x} &= \frac{1}{2}m\dot{x}^2\left[16a^2(6x^5)\right] - 4mgax^3 \\
\frac{\partial \mathcal{L}}{\partial x} &= 48m\dot{x}^2a^2x^5 - 4mgax^3
\end{aligned}$$

Now let's solve the right-side:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{\partial}{\partial \dot{x}}\left[\frac{1}{2}m\dot{x}^2(1+16a^2x^6) - mgax^4\right] \\
\frac{\partial \mathcal{L}}{\partial \dot{x}} &= m\dot{x}(1+16a^2x^6) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left[m\dot{x}(1+16a^2x^6)\right] \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= m\ddot{x}(1+16a^2x^6) + m\dot{x}(96a^2x^5\dot{x})
\end{aligned}$$

Now we can equate the two sides and solve for the differential equation of motion:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) &= \frac{\partial \mathcal{L}}{\partial x} \\
m\ddot{x}(1+16a^2x^6) + m\dot{x}(96a^2x^5\dot{x}) &= 48m\dot{x}^2a^2x^5 - 4mgax^3 \\
m\ddot{x}(1+16a^2x^6) + m\dot{x}(48^2x^5\dot{x}) + 4mgax^3 &= 0 \\
\ddot{x}(1+16a^2x^6) + \dot{x}(48^2x^5\dot{x}) + 4gax^3 &= 0
\end{aligned}$$

The solution is a differential equation of motion, but it is difficult to solve because it is non-linear with respect to both x and $\dot{x}$. Thus, we leave the equation in this form.

Quick Questions

1. Try solving this problem using energy conservation. Hint: Remember that energy conservation uses full derivatives and not partial derivatives.
2. If you were to try Newton's law, to solve this problem, what other force is acting on the bead other than gravity (it is this force that makes applying Newton's laws difficult)?

12.6 Real-World Applications

Initially, it might seem like the Euler-Lagrange Method is unnecessarily complicated, but we're really just touching the edge of what it can be used for. Remember that this textbooks is still mostly focusing on simple, idealized problems. The real world of experimentation and research is much more complex, and many physics problems don't have simple, analytical solutions and can only be probed numerically using computers.

Lagrangian mechanics help simplify the calculations for complex or even chaotic systems where forces are hard to define or the initial conditions can drastically change the outcome (e.g., consider the motion of a double pendulum). In terms of physics research, solving problems with the Euler-Lagrange method is often more efficient when mapping the motion of stars in galactic mergers or near supermassive black holes, tracing particle collisions in accelerators, solving problems in fluid mechanics, or tracking systems of particles in thermodynamics or quantum mechanics. The Euler-Lagrange equations are a tool to break down big problems into smaller calculations.

One common application of Lagrangian mechanics is with magnetohydrodynamics (MHD), which is the study of fluids that conduct electrically. MHD is used in many branches of physics, but one example is nuclear fusion experimentation, where many experiments seek to produce energy by magnetically confining a fast-moving plasma in a torus. MHD research must solving various equations such as the equation of state, mass continuity, Faraday's law, and Ohm's law simultaneously for the entire system, and these equations are usually non-linear with time. As such, Lagrangian mechanics are often employed to simplify the problem.

For more information:
Wikipedia webpage on MHD, listing various forms and equations. Science article on some recent nuclear fusion experiment designs.

12.7 Summary on the Lagrange Method

Key Takeaways

We have shown that the Lagrange method involves two main steps. First, you define the Lagrangian:

$$\boxed{\mathcal{L} = K - U}$$

Second, you solve the Euler-Lagrange equations. For example, for motion in the x-axis only:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}}$$

The Euler-Lagrange equations and the Lagrangian hold under very general circumstances. You can apply these equations to n-dimensional systems (e.g., you do not need to be restricted to 3 dimensions). Systems in n-dimension are often called Euclidean space. In broad terms, you can express the Euler-Lagrange equations as:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i}}$$

where x_i represents a coordinate (and these do not need to be Cartesian coordinates).

Strategies for employing the Lagrange method successfully.

1. Choose your coordinate system carefully. You want a coordinate system that simplifies the problem.

2. Write out K and U and $\mathcal{L} = K - U$. Note that the physics is just in $\mathcal{L} = K - U$, so once you have the Lagrangian, you have finished with the physics.

3. Solve the Euler-Lagrange equations for each coordinate system separately. Note that a constant in the potential (e.g., $U \to U(x_i) + U_0$) does not affect the solution. So you do not care about constant values for the potential.

4. If $\mathcal{L}$ does not depend on a position coordinate (e.g., x_i), then $\frac{\partial \mathcal{L}}{\partial x_i} = 0$ and $\frac{\partial \mathcal{L}}{\partial \dot{x}_i}$ is a constant. See for example the Gravity example in Section 12.3.

5. Once you have $\mathcal{L}$, you can obtain the equation of motion using the Euler-Lagrange equations.

Important Equations

Lagrangian:

$$\mathcal{L} = K - U$$

Euler-Lagrange Equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}\right) = \frac{\partial \mathcal{L}}{\partial x_i}$$

12.8 Practice Problems

See Appendix C for answers to the practice problems.

Practice Problem 12-1

Use the Euler-Lagrange method to find the differential equation of motion for a vertical spring and confirm you get the same answer as Chapter 3.

Practice Problem 12-2

Use the Euler-Lagrange method to find the differential equation for a physical pendulum that is constructed from a rod of length L and mass M that is pivoted at one end. Confirm that you get the same answer as Chapter 7.

Practice Problem 12-3

A mass M sits at the top of a frictionless incline of angle θ and slides down the plane. Use the Euler-Lagrange equations to find the differential equation of motion.

Practice Problem 12-4

An ideal string is wrapped around a disk of mass M and radius R so that it unwinds as the disk falls. If the moment of inertia of the disk is $\frac{1}{2}MR^2$, what is the Lagrangian for this motion? Assume that the disk moves without slipping and only gravity affects this system. Assume $U = 0$ at $y = 0$.

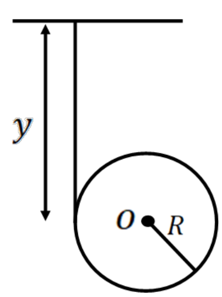


Figure 12.7: A disk hangs from the ceiling.

Practice Problem 12-5

Two identical blocks of mass m are connected by an ideal rope. One block is placed on a smooth horizontal table and the other block hangs over the edge. When the second hanging block is released, it pulls on the first block. Find the Euler-Lagrange equations while the first block remains on the table. Assume $U = 0$ at the height of the table.

Practice Problem 12-6

A block of mass m is attached to a spring with constant k on an incline as shown in the figure. If the block is displaced a small value x from the equilibrium position,

a) What is the kinetic energy of the mass?

b) What is the potential energy of the mass?

c) What is the Lagrangian of the system?

d) What are the Euler-Lagrange equations?

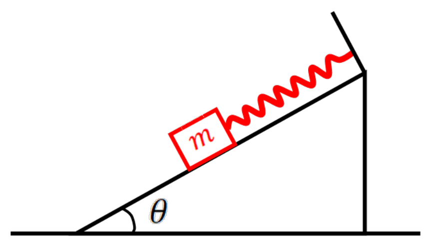


Figure 12.8: Mass and spring on an incline

Practice Problem 12-7

Two masses are joined by an ideal rope across an ideal pulley as shown. Mass M_2 hangs over the edge and mass M_1 sits on a frictionless incline. Use the Euler-Lagrange equations to find the differential equation of motion.

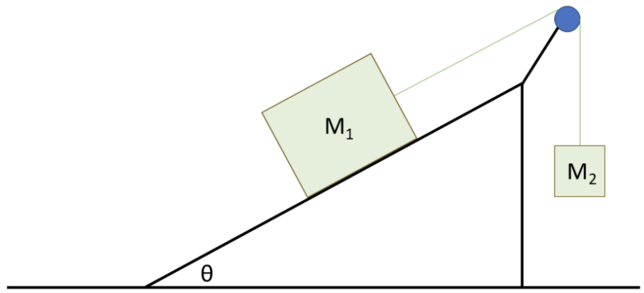


Figure 12.9: Two masses attached via an ideal pulley.

Practice Problem 12-8

A circular disk of mass M and radius R hangs from a pivot point that is displaced from the centre of mass by a distance $s = R/2$ as shown.

a) What is the Lagrangian for this system?

b) Check that this answer makes sense by applying the Euler-Lagrange equations and showing that you get a standard differential equation of motion.

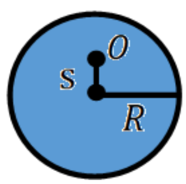


Figure 12.10: A disk as a physical pendulum.

Practice Problem 12-9

A snowboarder of mass m is sliding on a 2-D frictionless half-pipe that has a cycloid shape described by:

$$\begin{aligned} x &= \frac{a}{4}(2\theta + \sin 2\theta) \\ y &= \frac{a}{4}(1 - \cos 2\theta) \end{aligned}$$

where a is a constant and $0 < \theta < \pi$. See the figure below for a sketch of what this looks like.

a) If gravity is the only force on the snowboarder, find the potential energy. Express this in terms of θ and $\dot{\theta}$.

b) What is the kinetic energy of the snowboarder? Express this in terms of θ and $\dot{\theta}$.

c) What is the Lagrangian of the system? Express this in terms of θ and $\dot{\theta}$.

d) Use the Euler-Lagrange equations to find the differential equation of motion. Do not solve this equation.

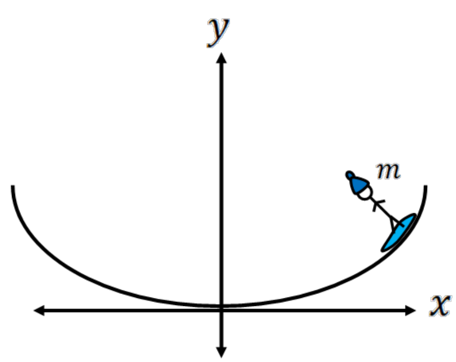


Figure 12.11: A snowboarder on a cycloid-shaped half-pipe.

Practice Problem 12-10

See the figure below. A mass M hangs from a spring that is allowed to expand/contract and rotate. Assume that the potential energy from gravity is zero at the pivot point.

a) Describe how the system would move if it is displaced a small distance x along the axis of the spring and an angle θ from the vertical.

b) What is the potential energy of this system?

c) What is the kinetic energy of this system?

d) Use the Lagrange method to find the differential equation of motion for this system? Note: you must solve the x and θ terms separately. Do not solve the differential equation.

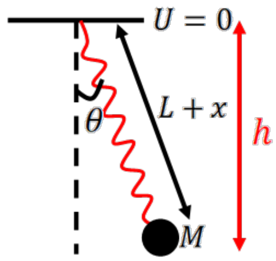


Figure 12.12: Spring pendulum for the problem.

A Resources

This Appendix contains various formulas and constants that may be used throughout this text or needed to solve problems.

A.1 Constants

Acceleration due to gravity at Earth's surface	g	9.81 m s^{-2}
Gravitational constant	G	6.674×10^{-11} m^3 kg^{-1} s^{-2}
Speed of light (vacuum)	c	2.998×10^8 m s^{-1}
Speed of sound in air at 20 °C	c_s	343 m s^{-1}
Mass of Earth	M_E	5.98×10^{24} kg
Mass of Sun	M_S	1.99×10^{30} kg
Mass of Moon	M_M	7.36×10^{22} kg
Mean Earth orbit (astronomical unit)	r_E	1.50×10^{11} m
Earth radius	R_E	6.37×10^6 m
Sun radius	R_S	6.96×10^8 m
Moon radius	R_M	1.74×10^6 m
Planck constant	h	6.63×10^{-34} J s
Boltzmann constant	k_B	1.38×10^{-23} J K^{-1}
Permittivity of Free Space	ε_0	8.854×10^{-12} C V^{-1} m^{-1}
Permeability of Free Space	μ_0	$4\pi \times 10^{-7}$ T m A^{-1}
Elementary charge	$\|e\|$	1.6×10^{-19} C
Electron mass	m_e	9.11×10^{-31} kg
Proton mass	m_p	1.67×10^{-27} kg
Coulomb constant	$k = \frac{1}{4\pi\varepsilon_0}$	8.99×10^9 N m^2 C^{-2}

Numerical values may also be presented with prefixes. For example, km corresponds to kilometer or 1000 m.

10^{-3}	milli	m	10^3	kilo	k
10^{-6}	micro	μ	10^6	mega	M
10^{-9}	nano	n	10^9	giga	G
10^{-12}	pico	p	10^{12}	tera	T

A.2 Math Identities

$$\vec{a}\cdot\vec{b} = ab\cos\theta = a_x b_x + a_y b_y + a_z b_z$$

$$\vec{a}\times\vec{b} = (a_y b_z - a_z b_y)\hat{\imath} + (a_z b_x - a_x b_z)\hat{\jmath} + (a_x b_y - a_y b_x)\hat{k}, \quad |\vec{a}\times\vec{b}| = ab\sin\phi$$

$$\int \frac{\mathrm{d}x}{(x^2+a^2)^{3/2}} = \frac{1}{a^2}\frac{x}{\sqrt{x^2+a^2}}, \qquad \int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right), \qquad \int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \ln(x+\sqrt{x^2+a^2}), \qquad \int \frac{x\mathrm{d}x}{(x^2+a^2)^{3/2}} = \frac{-1}{\sqrt{x^2+a^2}}, \qquad \int \frac{x\mathrm{d}x}{\sqrt{x^2+a^2}} = \sqrt{x^2+a^2}$$

$$\sin(\theta\pm\gamma) = \sin\theta\cos\gamma \pm \cos\theta\sin\gamma, \qquad \cos(\theta\pm\gamma) = \cos\theta\cos\gamma \mp \sin\theta\sin\gamma$$

$$\sin\alpha \pm \sin\beta = 2\sin\left(\frac{\alpha\pm\beta}{2}\right)\cos\left(\frac{\alpha\mp\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\ln(ab) = \ln(a)+\ln(b), \qquad \ln\left(\frac{a}{b}\right) = \ln(a)-\ln(b)$$

A.3 Common Approximations

See Chapter 1.6 and Appendix B for details on how functions can be approximated.

Common Taylor Series approximations for values around $x \approx 0$. Note that angles must be in units of radians for these approximations to be applicable:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots \qquad \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + x^6 + \cdots \qquad \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \qquad \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

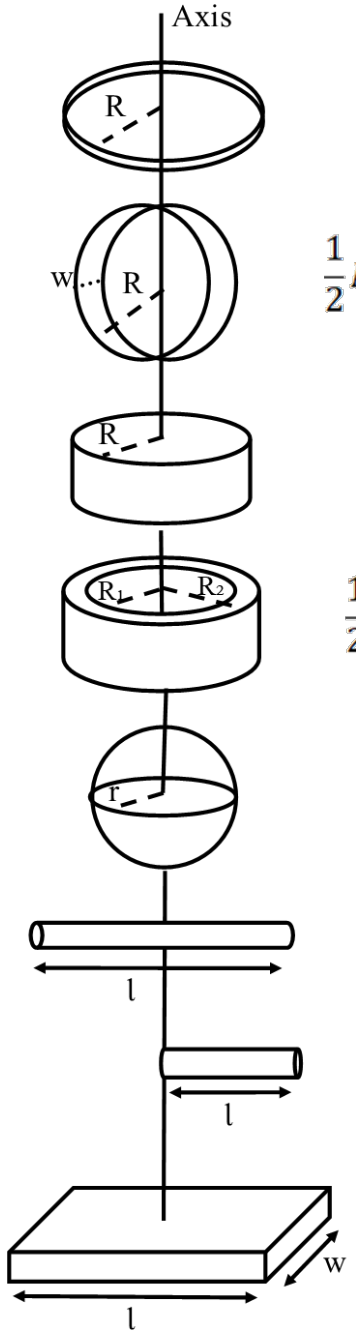
$$\frac{1}{\sqrt{1-x}} = 1 + \frac{x}{2} + \frac{3x^2}{8} + \frac{5x^3}{16} + \cdots \qquad \frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \frac{5x^3}{16} + \cdots$$

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{5x^6}{16} + \cdots \qquad \frac{1}{\sqrt{1+x^2}} = 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16} + \cdots$$

A.4 Moment of Inertia

The Moment of Inertia, I, represents how the mass of a system is distributed as a function of position and describes how efficiently the system rotates. Here is a chart of basic shapes and their moments of inertia. For information on how to calculate Moments of Inertia see 7.2.2.

Object	Location of Axis	Moment of Inertia
Thin Hoop Radius R	Through centre	MR^2
Thin Hoop, Radius R, Width w	Through central diameter	$\frac{1}{2}MR^2 + \frac{1}{12}Mw^2$
Solid Cylinder, Radius R	Through centre	$\frac{1}{2}MR^2$
Hollow Cylinder, Inner radius R_1, Outer radius R_2	Through centre	$\frac{1}{2}M(R_1^2 + R_2^2)$
Uniform Sphere, Radius r	Through centre	$\frac{2}{5}Mr^2$
Long Uniform Rod, Length l	Through centre	$\frac{1}{12}Ml^2$
Long Uniform Rod, Length l	Through end	$\frac{1}{3}Ml^2$
Rectangular Thin Plate, Length l, Width w	Through centre	$\frac{1}{12}M(l^2 + w^2)$



A.5 Vector Differential Operators

This section gives the full coordinate transformations for the gradient, divergence, and curl in 3-D. The following is a brief explanation of those coordinate transformations.

A.5.1 General Coordinates

In general, consider a 3-D coordinate system c_1, c_2, and c_3 with orthogonal unit vectors defined as $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_2$. Note that c_1, c_2, and c_3 are merely stand-ins for x, y, z or r, θ, ϕ.

In this general 3-D coordinate system, a line element would be

$$\mathrm{d}s = \langle h_1 \mathrm{d}c_1, h_2 \mathrm{d}c_2, h_3 \mathrm{d}c_3 \rangle$$

where h_1, h_2, and h_3 are scale factors that may need to be applied to each coordinate (the value of these scale factors depends on the coordinate transformation - more on this below).

Because of these scale factors, the gradient, divergence, and curl transformations will be a bit different in each coordinate system. In the general form, these functions are:

The gradient of a function f is then defined as:

$$\vec{\nabla} f = \left(\frac{1}{h_1} \frac{\partial f}{\partial c_1} \right) \hat{e}_1 + \left(\frac{1}{h_2} \frac{\partial f}{\partial c_2} \right) \hat{e}_2 + \left(\frac{1}{h_3} \frac{\partial f}{\partial c_3} \right) \hat{e}_3$$

The divergence of a vector $\vec{A}$ is:

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial c_1} (h_2 h_3 A_1) + \frac{\partial}{\partial c_2} (h_1 h_3 A_2) + \frac{\partial}{\partial c_3} (h_1 h_2 A_3) \right]$$

and the curl of a vector $\vec{A}$ is:

$$\vec{\nabla} \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial c_1} & \frac{\partial}{\partial c_2} & \frac{\partial}{\partial c_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

A.5.2 Cartesian Coordinates

In the Cartesian system, we have $c_1 = x$, $c_2 = y$, and $c_3 = z$ and $h_1 = 1$, $h_2 = 1$, and $h_3 = 1$. As a result, we have the following for Cartesian Coordinates:

$$\vec{\nabla} f = \frac{\partial f}{\partial x} \hat{x} + \frac{\partial f}{\partial y} \hat{y} + \frac{\partial f}{\partial z} \hat{z}$$

$$\vec{\nabla} \cdot \vec{A} = \left[\frac{\partial}{\partial x} (A_1) + \frac{\partial}{\partial y} (A_2) + \frac{\partial}{\partial z} (A_3) \right]$$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

where $\hat{x}$ is used for $\hat{\imath}$, $\hat{y}$ is used for $\hat{\jmath}$, and $\hat{z}$ is used for $\hat{k}$.

A.5.3 Cylindrical Coordinates

In cylindrical coordinates, $c_1 = r$, $c_2 = \theta$, and $c_3 = z$, where

$$x = r\cos\phi \qquad y = r\sin\phi$$

For cylindrical coordinates, $h_1 = 1$, $h_2 = r$, and $h_3 = 1$. Note, these terms should look familiar. That is, for a cylinder, a tiny section of volume is given by:

$$dV = r\ \mathrm{d}r\mathrm{d}\theta\mathrm{d}z$$

Based on the above, we get the following for cylindrical coordinates.

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial r}\right)\hat{r} + \left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right)\hat{\theta} + \left(\frac{\partial f}{\partial z}\right)\hat{z}$$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_1) + \frac{\partial}{\partial \theta}(A_2) + \frac{\partial}{\partial z}(rA_3)\right]$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_1 & rA_2 & A_3 \end{vmatrix}$$

A.5.4 Spherical Coordinates

In spherical coordinates, $c_1 = r$, $c_2 = \theta$, and $c_3 = \phi$, where

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

For spherical coordinates, $h_1 = 1$, $h_2 = r$, and $h_3 = r\sin\theta$. Note, these terms should look familiar. That is, for a sphere, a tiny section of volume is given by:

$$dV = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$$

Based on the above, we get the following for spherical coordinates.

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial r}\right)\hat{r} + \left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right)\hat{\theta} + \left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\right)\hat{\phi}$$

$$\vec{\nabla}\cdot\vec{A} \quad = \quad \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta A_1) + \frac{\partial}{\partial \theta}(r\sin\theta A_2) + \frac{\partial}{\partial \phi}(rA_3)\right]$$

$$\vec{\nabla}\times\vec{A} \quad = \quad \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ A_1 & rA_2 & r\sin\theta A_3 \end{vmatrix}$$

B Derivations and Approximations

B.1 Derivation of Elliptical Orbits

Starting from the energy equation for a central force (Chapter 10), we have

$$E = \frac{1}{2}m\dot{r}^2 + U_{eff}$$

where U_{eff} is the effective potential. Re-arranging the equation gives:

$$\dot{r}^2 = \frac{2}{m}(E - U_{eff}) \tag{B.1}$$

where $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}$.

We want to get $r(\theta)$ to describe the orbit. To do this, we will use the ratio of $\dot{r}$ and $\dot{\theta}$,

$$\frac{\dot{r}}{\dot{\theta}} = \frac{\mathrm{d}r/\mathrm{d}t}{\mathrm{d}\theta/\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta} \tag{B.2}$$

So we need to get an equation for $\dot{\theta}$.

For $\dot{\theta}$, we can use the conservation of angular momentum. Recall that central forces conserve angular momentum by definition (see Chapter 10). The magnitude of angular momentum is:

$$L = mr^2\dot{\theta} \quad \Longrightarrow \quad \dot{\theta} = \frac{L}{mr^2} \tag{B.3}$$

with $L =$ constant.

We can then combine the $\dot{r}$ and $\dot{\theta}$ equations as follows:

$$\begin{aligned}
\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= \frac{\dot{r}^2}{\dot{\theta}^2} && \Longrightarrow \quad \text{from Equation B.2} \\
\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= \frac{\frac{2}{m}(E - U_{eff})}{\left(\frac{L}{mr^2}\right)^2} && \Longrightarrow \quad \text{sub in Equations B.1 and B.3} \\
\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= \frac{2}{m}\frac{(E - U_{eff})}{L^2}(m^2r^4) && \\
\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 &= \frac{2m(E - U_{eff})}{L^2} && \Longrightarrow \quad \text{bring all } r \text{ terms to the other side}
\end{aligned}$$

$$\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \frac{2m(E-U_{eff})}{L^2} \implies \text{simplify the LHS of the equation} \tag{B.4}$$

Equation B.4 applies to any effective potential. Since we're talking about orbits, the central force is gravity. That makes the effective potential:

$$U_{eff} = \frac{L^2}{2mr^2} - \frac{\gamma}{r} \tag{B.5}$$

where $\gamma = GMm$ for simplicity. Substituting Equation B.5 into Equation B.4 gives,

$$\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \frac{2mE}{L^2} - \left(\frac{2m}{L^2}\right)\left(\frac{L^2}{2mr^2}\right) + \left(\frac{2m}{L^2}\right)\left(\frac{\gamma}{r}\right)$$

$$\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \frac{2mE}{L^2} - \frac{1}{r^2} + \frac{2m\gamma}{rL^2} \implies \text{simplify} \tag{B.6}$$

To solve Equation B.6, there is a substitution trick we can use to help simplify the problem. Instead of solving for r, we're going to solve for $y = \frac{1}{r}$. The reason for this substitution is that

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} \implies \text{remember that } y = \tfrac{1}{r}$$

Thus, we can write Equation B.6 in terms of $y = \frac{1}{r}$ instead of r.

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = \frac{2mE}{L^2} - y^2 + \frac{2m\gamma}{L^2}y \implies \text{sub } y = \tfrac{1}{r} \text{ into Equations B.6} \tag{B.7}$$

Already this looks simpler. We can simplify more, however, by "completing the square",

$$-\left(y - \frac{m\gamma}{L^2}\right)^2 = -y^2 + \frac{2m\gamma}{L^2}y - \left(\frac{m\gamma}{L^2}\right)^2 \tag{B.8}$$

Note that the first two terms on the righthand side of Equation 8 are present in the righthand side of Equation B.7. From Equation B.8, we can say that

$$-y^2 + \frac{2m\gamma}{L^2} = -\left(y - \frac{m\gamma}{L^2}\right)^2 + \left(\frac{m\gamma}{L^2}\right)^2 \tag{B.9}$$

And we can substitute Equation B.9 into Equation B.7 to give:

$$\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 = \frac{2mE}{L^2} - \left(y - \frac{m\gamma}{L^2}\right)^2 + \left(\frac{m\gamma}{L^2}\right)^2 \tag{B.10}$$

With Equation B.10, we have an equation with one y term and two constant terms. To make the equation a bit simpler, we will introduce a couple more substitutions for the constants.

First, we will use $z = y - \frac{m\gamma}{L^2}$. Note that $\frac{\mathrm{d}z}{\mathrm{d}\theta} = \frac{\mathrm{d}y}{\mathrm{d}\theta}$ because $\frac{m}{\gamma/L^2}$ is a constant.

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 = -z^2 + \frac{2mE}{L^2} + \left(\frac{m\gamma}{L^2}\right)^2 \tag{B.11}$$

Second, we will set the substitute the sum of the remaining constants as B^2,

$$B^2 = \frac{2mE}{L^2} + \left(\frac{m\gamma}{L^2}\right)^2 \tag{B.12}$$

$$B^2 = \left(\frac{m\gamma}{L^2}\right)^2 \left(\frac{2EL^2}{m\gamma^2} + 1\right)$$

$$B = \frac{m\gamma}{L^2}\sqrt{1 + \frac{2EL^2}{m\gamma^2}} \tag{B.13}$$

Substituting Equation B.12 into Equation B.13, we get:

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 = -z^2 + B^2$$

Now we have a fairly straightforward integral to solve. The solution to this problem is available in Appendix A.

$$\begin{aligned}
\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 &= -z^2 + B^2 \\
\frac{\mathrm{d}z}{\mathrm{d}\theta} &= \sqrt{B^2 - z^2} \\
\frac{\mathrm{d}z}{\sqrt{B^2 - z^2}} &= \mathrm{d}\theta \\
\int \frac{\mathrm{d}z}{\sqrt{B^2 - z^2}} &= \int \mathrm{d}\theta \\
\sin^{-1}\left(\frac{z}{B}\right) &= \theta + C \quad \Longrightarrow \quad C \text{ is a constant of integration}
\end{aligned}$$

We can select the initial conditions such that the initial angle is anything we want because the motion is periodic. For ellipses, it helps to have the angle in terms of cos instead of sin, so we can set $C = \frac{\pi}{2}$. Thus, we get the solution:

$$z = B\sin\left(\theta + \frac{\pi}{2}\right) = B\cos\theta \tag{B.14}$$

Now we need to put back all of those substitutions that we made!

1. $z = y - \frac{m\gamma}{L^2} \quad \Longrightarrow \quad y - \frac{m\gamma}{L^2} = B\cos\theta$

2. $y = \frac{1}{r} \quad \Longrightarrow \quad \frac{1}{r} - \frac{m\gamma}{L^2} = B\cos\theta$

3. From Equation B.13: $B = \frac{m\gamma}{L^2}\sqrt{1+\frac{2EL^2}{m\gamma^2}} \implies$

$$\begin{aligned}
\frac{1}{r} - \frac{m\gamma}{L^2} &= \left(\frac{m\gamma}{L^2}\sqrt{1+\frac{2EL^2}{m\gamma^2}}\right)\cos\theta \\
\frac{1}{r} &= \frac{m\gamma}{L^2} + \left(\frac{m\gamma}{L^2}\sqrt{1+\frac{2EL^2}{m\gamma^2}}\right)\cos\theta \\
\frac{1}{r} &= \frac{m\gamma}{L^2}\left(1+\sqrt{1+\frac{2EL^2}{m\gamma^2}}\cos\theta\right)
\end{aligned} \tag{B.15}$$

We now define the eccentricity of the orbit, ϵ as:

$$\epsilon = \sqrt{1+\frac{2EL^2}{m\gamma^2}} \tag{B.16}$$

Substitute Equation B.16 into Equation B.15:

$$\frac{1}{r} = \frac{m\gamma}{L^2}(1+\epsilon\cos\theta) \tag{B.17}$$

Finally, let's solve for r.

$$r = \left(\frac{L^2}{m\gamma}\right)\frac{1}{1+\epsilon\cos\theta} \tag{B.18}$$

We now have an equation of $r(\theta)$ that defines our orbit. You will note that Equation B.18 has a similar structure to the equation for a generic ellipse (Equation 11.2), but with different constants out front.

If $\epsilon = 0$, then we have a circular orbit with a radius of

$$\boxed{r_c = \frac{L^2}{m\gamma}} \tag{B.19}$$

We define r_c as the radius of a circular orbit for a given angular momentum, L. See also Chapter 11.2 for the derivation of r_c using Newton's laws.

So substituting in r_c, we recover Equation 11.7.

$$r = \frac{r_c}{1+\epsilon\cos\theta} \tag{B.20}$$

B.2 Approximations

Many times, mathematical functions can be simplified by using limits and making reasonable approximations. There are many ways to simplify a function. Here, we will look at a few of them.

B.2.1 Binomial Approximation

For simple power functions, such as $f(x) = (1+x)^a$, you can apply the binomial or linear expansion to approximate their value. The approximation works as follows:

$$f(x) \approx f(x_0) + \frac{\mathrm{d}f(x_0)}{\mathrm{d}x}(x - x_0)$$

where x_0 is the point about which you are measuring x. That is, you are expanding the series about the point where $x = x_0$.

At first glance, this approximation seems reasonable. The approximation states that the value of a function at position x near position x_0 can be approximated by the value at position x_0 with a modification given by the slope at x_0 and the distance between x_0 and x. For values of $x \approx x_0$, this approximation should be reasonable.

For the function, $f(x) = (1+x)^a$ with $x_0 = 0$, the binomial approximation would be,

$$\begin{aligned} f(x) &\approx f(0) + \frac{\mathrm{d}f(0)}{\mathrm{d}x}(x) \\ f(x) &\approx 1 + ax \end{aligned}$$

In this case, the value of x must be close to zero for the approximation to be valid. If x is not close to zero, then we need to shift x_0.

B.2.2 Taylor Series

For more general functions, you can approximate the function using its Taylor series,

$$f(x) = f(x_0) + \frac{\mathrm{d}f(x_0)}{\mathrm{d}x}(x - x_0) + \frac{1}{2}\frac{\mathrm{d}^2 f(x_0)}{\mathrm{d}x^2}(x - x_0)^2 + \frac{1}{3!}\frac{\mathrm{d}^3 f(x_0)}{\mathrm{d}x^3}(x - x_0)^3 + \cdots + \frac{1}{n!}\frac{\mathrm{d}^n f(x_0)}{\mathrm{d}x^n}(x - x_0)^n$$

where $x_0 = 0$ has the same meaning as before.

For example, the Taylor series expansion of $\sqrt{1+x}$ (where $x_0 = 0$) is equal to:

$$\begin{aligned} \sqrt{1+x} &= 1 + \frac{1}{2}(x) + \frac{1}{2!}\left(-\frac{1}{4}\right)x^2 + \frac{1}{3!}\left(\frac{3}{8}\right)x^3 + \cdots \\ \sqrt{1+x} &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots \end{aligned}$$

If x is small (x is close to zero), then x^2 and x^3 are very small and don't change the result much. For example, if $x = 0.1$, then $x^3 = 0.001$ and $\frac{1}{16}x^3 = 6.25 \times 10^{-5}$. That term is much smaller than 1 and all subsequent terms will be even smaller, so they are negligible. That means, for small values of x, we can approximate $\sqrt{1+x}$ as:

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x$$

It is key that x is small (for $x_0 = 0$). If x is larger, then the higher order terms are more significant and cannot be considered negligible.

The Taylor series for the $\sin x$ and $\cos x$ (with $x_0 = 0$) are:

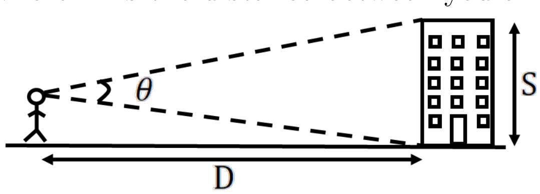
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Note that x must be in radians for the Taylor series approximation to hold. You cannot use x in degree. If x (in radians) is small, then the higher order terms again become negligible and $\sin x \approx x$ and $\cos x \approx 1 - \frac{x^2}{2}$.

There are several functions with well established Taylor series approximations based on the above definition. Appendix A.3 lists the expansions for many common equations.

Small Angle Approximation

The small angle approximation is an application of the Taylor Series expansion. It states that the separation s between two points subtended by an angle is $s = D \tan\theta \approx D\theta$, where D is the distance between you and the points (see figure below for definitions).



You will use the small angle assumption many times in this course and in other courses. As an astronomer, I use the small angle approximation in my research all the time. For example, large objects in space (e.g., diameter of a crater on the moon, radius of a planet-forming disk around another star, distance between two interacting galaxies) subtend very tiny angles because they are so far away (θ is small because D is large. We also use this approximation in optics with interference and diffraction patterns, where the distance to the first fringes correspond to small angle differences from the normal.

Quick Questions

1. Use a calculator to verify that $\sin x \approx x$ and $\cos x \approx 1 - \frac{x^2}{2}$ for small angles.
2. Try plotting both functions and see at which angles the approximations break down. Practice using a programming language like python if you can.
3. You have a telescope and lens that can resolve (separate) objects that subtend angles of at least 0.0003°. Could you resolve a crater that is 1 km in diameter on the Moon with this telescope? Assume the Moon is 300,000 km away.

C Solutions to Problems

C.1 Calculus and Vectors

Solutions to Practice Problems from Chapter 1.9

Calculus and Vectors Solutions:

Problem 1-1:

$v = C\tau(1 - e^{-t/\tau}) + v_0$

Problem 1-2:

23c, $-7c\hat{k}$

Problem 1-3:

$\cos\theta = \dfrac{2k}{k^2+1}$

Problem 1-4:

a) -48, b) 10, c) $9c - 9 - 2c^2$, d) $c = 3, \frac{3}{2}$

Problem 1-5:

a) $4\hat{\imath} - 26\hat{\jmath} - 57\hat{k}$

b) $(-45s - 6)\hat{\imath} + 42s\hat{\jmath} + (15s - 12)\hat{k}$

c) $(45s + 6)\hat{\imath} - 42s\hat{\jmath} + (-15s + 12)\hat{k}$

d) $(27s + 6)\hat{\imath} + 75s\hat{\jmath} + (-9s - 27)\hat{k}$

Problem 1-6:

a) $4\hat{\imath} - 7\hat{\jmath} - \hat{k}$

b) $-7\hat{\imath} + 4\hat{\jmath} - \hat{k}$

c) $7\hat{\imath} - 4\hat{\jmath} + \hat{k}$

Problem 1-7:

a) $(5, \tan^{-1}(\frac{4}{3}))$

b) $\theta = \frac{\pi}{4}$ or $\theta = 45°$

c) $r = \dfrac{\sin\theta - \cos\theta}{\cos^2\theta}$

Problem 1-8:

$\vec{v} = 2e^{2t}\hat{r} + 2te^{2t}\hat{\theta}$, $\vec{a} = (4e^{2t} - 2t^2e^{2t})\hat{r} + (2e^{2t} + 8te^{2t})\hat{\theta}$

Problem 1-9:

$\theta = \dfrac{\pi}{4}$

Problem 1-10:

a) $\vec{v} = kr\hat{r} + cr\hat{\theta}$

b) $\vec{a} = (k^2 - c^2)r(t)\hat{r} + (2kc)r(t)\hat{\theta}$

c) $\cos\phi = \dfrac{k}{\sqrt{k^2+c^2}} = constant$

Problem 1-11:

a) x, b) $1 + x + x^2$, c) $-x - \frac{1}{2}x^2$

Problem 1-12:

a) 1, b) $1 + 5x$, c) $\ln 2 + \frac{3}{2}x$

C.2 Newtonian Review

Solutions to Practice Problems from Chapter 2.8

Review Newtonian Solutions:

Problem 2-1:

$m_1 : T - m_1 g = m_1\ddot{y}_1$

$m_2 : T - m_2 g = m_2\ddot{y}_2$

$pulley : F - m_p g - 2T = m_p a_F = 0$

Problem 2-2:

$v_{min} = \sqrt{\dfrac{2g}{\sin^2\theta}(h - y_0)}$

Problem 2-4:

$v = \left(\frac{F_0}{m}\right) t - \left(\frac{\alpha}{3m}\right) t^3 + v_0$

Problem 2-5:

a) $v = -\frac{k}{2}x^2 + v_0$

b) $x(t) = \sqrt{\frac{2v_0}{k}} \frac{\left(e^{\sqrt{2kv_0}t} - 1\right)}{\left(1 + e^{\sqrt{2kv_0}t}\right)}$

Problem 2-6:

a) $\sum F = -bmv + mg$, where down is positive

b) $\frac{g}{b}$

c) $y = \frac{g}{b^2}e^{-bt} + \frac{g}{b}t - \frac{g}{b^2}$

Problem 2-7:

b) $\theta = \tan^{-1}\mu$

c) $F_{ext} = \frac{\mu M g}{\sqrt{1 + \mu^2}}$

Problem 2-8:

b) $(m - \rho V)g - \alpha v = m\frac{\mathrm{d}v}{\mathrm{d}t}$

c) $v(t) = \frac{(m - \rho V)g}{\alpha}\left(1 - e^{-\alpha t/m}\right)$

d) $v = \frac{(m - \rho V)g}{\alpha}$

Problem 2-9:

b) $t_r = \frac{m_2 v_0}{\mu g m_1}$

c) $v_r = \frac{m_1 v_0}{m_2 + m_1}$

Problem 2-10:

b) lower pulley goes down, M goes up, M_2 falls faster than M_1.

c) $-\frac{1}{2}(\ddot{y}_1 + \ddot{y}_2)$, where $\ddot{y}_1$ is the acceleration of M_1 and $\ddot{y}_2$ is the acceleration of M_2.

d) $\ddot{y}_2 = \frac{21}{29}g$

C.3 Simple Harmonic Motion

Solutions to Practice Problems from Chapter 3.9

Simple Harmonic Motion Solutions:

Problem 3-1:

Quarter the mass ($m/4$)

Problem 3-2:

Looking up the gravitational acceleration for each planet, some examples:

Mercury: 0.375 m, Earth: 1 m, Mars: 0.376 m, Jupiter: 2.5 m, Neptune: 1.13 m

Problem 3-3:

a) $\ddot{x} = \dfrac{3g}{L}x = 0$

c) $L = 0.745$ m

Problem 3-4:

$\omega_0 = \sqrt{\dfrac{(k_1 + k_2)}{m}}$

Problem 3-5:

a) $T = 2\pi\sqrt{\dfrac{L}{g}}$

b) $T = 2\pi\sqrt{\dfrac{L}{(g + \ddot{y})}}$

Problem 3-6:

a) $x = A\cos\left(t\sqrt{\dfrac{k}{m} + \dfrac{g}{L}}\right)$

$v = -A\sqrt{\dfrac{k}{m} + \dfrac{g}{L}}\sin\left(t\sqrt{\dfrac{k}{m} + \dfrac{g}{L}}\right)$

$a = -A\left(\dfrac{k}{m} + \dfrac{g}{L}\right)\cos\left(t\sqrt{\dfrac{k}{m} + \dfrac{g}{L}}\right)$

Problem 3-7:

b) $x_0 = -\dfrac{mg\sin\theta}{k}$

c) $0 = \ddot{x} + \dfrac{k}{m}(x - x_0)$

d) $T = 2\pi\sqrt{\dfrac{m}{k}}$

Problem 3-8:

b) $\ddot{x} + \dfrac{4k}{m}x = 0$

c) $\omega_0 = \sqrt{\dfrac{4k}{m}}$, $T = 2\pi\sqrt{\dfrac{m}{4k}}$,

d) $x = A\cos\left(t\sqrt{\dfrac{4k}{m}}\right)$, $v = -A\sqrt{\dfrac{4k}{m}}\sin\left(t\sqrt{\dfrac{4k}{m}}\right)$, $a = -A\dfrac{4k}{m}\cos\left(t\sqrt{\dfrac{k}{m}}\right)$

Problem 3-10:

b) $0 = \ddot{x} + \left(\dfrac{k_1k_2 + k_3(k_2 + k_1)}{m(k_1 + k_2)}\right)x$

c) $T = 2\pi\sqrt{\dfrac{m(k_1 + k_2)}{k_1k_2 + k_3(k_1 + k_2)}}$

C.4 Introduction to Non-Inertial and Rotating Frames

Solutions to Practice Problems from Chapter 4.8

Introduction to Non-Inertial and Rotating Frames Solutions:

Problem 4-1:

a) The mass m is inside the accelerating frame, so the inertial frame sees gravity pointing downward and tension pointing upward and to the positive x-direction (the direction of the acceleration).

b) The mass m is inside the accelerating frame, so the non-inertial frame sees gravity pointing downward and tension pointing upward and to the positive x-direction (the direction of the acceleration) and a fictitious force opposite to the acceleration.

Problem 4-2:

a) 515N [down]

b) 860N [down]

Problem 4-3:

$\theta = 26.6$ degrees

Problem 4-4:

a) $a = g \tan\theta$, b) $g_{eff} = g/\cos\theta$

Problem 4-5:

a) higher (fictitious force points down the incline)

b) $g_{eff} = 1.05g$

Problem 4-6:

$T_{acc} = 0.97T_0$

Problem 4-7:

$a_{cent} = \frac{v_0^2}{R}$, $a_{Cor} = 0$

Problem 4-8:

a) No fictitious forces act (assuming the very center of the room is the rotation axis)

b) Centrifugal force

c) Centrifugal and Coriolis forces

Problem 4-9:

a) $F_{fic} = -ma\hat{z}$

b) $g_{eff} = -9.8\text{m s}^{-2}\hat{z}$

c) $v_{min} = 15.65\text{m s}^{-1}$ and $t = 0.64\text{s}$

d) No forces

e) $v_{min} = 15.65\text{m s}^{-1}$ and $t = 0.64\text{s}$

f) Inertial: horizontal and Non-Inertial: parabolic

C.5 Applications of Non-Inertial and Rotating Frames

Solutions to Practice Problems from Chapter 5.8

Applications of Non-Inertial and Rotating Frames Solutions:

Problem 5-1:

a) F_{Cor}

b) F_{cent}

c) F_{cent} and F_{az}

d) F_{cent} and F_{Cor}

e) F_{cent}, F_{Cor}, F_{az} and F_{trans}

Problem 5-2:

a) $8.84\omega^2\hat{r}$

b) 4.7 s^{-1} or 0.75 revolutions per second

Problem 5-3:

The azimuthal force F_{az} points in the positive y'-direction and the centrifugal force F_{cent} points in the positive x'-direction.

Problem 5-3:

a) azimuthal and centrifugal forces

b) 0.014 s^{-2}

c) $F_{cent} = 343$ N, $F_{az} = 12$ N

Problem 5-5:

a) $-\hat{x}$-direction (west)

b) 0.0005

Problem 5-6:

$\theta = 35\,\text{deg}$

Problem 5-7:

East

Problem 5-8:

a) $\vec{F}_{cent} = m\omega^2 x'\hat{x}'$ and $\vec{F}_{Cor} = -2m\omega\dot{x}'\hat{y}'$

b) $\vec{F}_I = 2m\omega\dot{x}'\hat{y}'$

c) The centrifugal force F_{cent} points parallel to the x prime axis, the Coriolis force F_{Cor} is anti-parallel to the y prime axis, and the inertial force F_I is parallel to the y prime axis. the system is rotating in the counter-clockwise direction. [Reaction force between the bead and rod]

d) $x(t) = \frac{L}{2}\left[e^{\omega t} - e^{-\omega t}\right]$

C.6 Momentum and Variable Mass

Solutions to Practice Problems from Chapter 6.8

Momentum and Variable Mass Solutions:

Problem 6-1:

$\vec{v}_{cm} = \hat{\imath} + \left(\frac{2}{3}\right)\hat{\jmath} + \left(\frac{1}{3}\right)\hat{k}$

Problem 6-2:

a) $\frac{M_1}{M_2}v_1\hat{\imath}$

b) $\frac{M_1}{M_1+M_2}v_1\hat{\imath}$

Problem 6-3:

a) $8\hat{\imath}$

b) $4\hat{\imath}\ + \hat{\jmath}$

Problem 6-4:

a) $\vec{I} = -325$ N s

b) $|\vec{F}| = 1625$ N

Problem 6-5:

$\vec{u} = \dfrac{m}{M}\dfrac{v}{2}\hat{\imath}$

Problem 6-6:

$|\vec{v}| = v_0\sqrt{4\cos^2\theta + \frac{1}{4}}$

Problem 6-7:

a) $\vec{v}_f = \dfrac{m_p}{(m_p + M)}\vec{v}_0$

b) $A = \dfrac{m_p}{(m_p + M)} v_0 \sqrt{\dfrac{L}{g}}$

c) $\theta_{max} = \dfrac{m_p}{(m_p + M)} v_0 \sqrt{\dfrac{1}{Lg}}$

d) 31.91 m s^{-1}

Problem 6-7:

0.375 m s^{-1}

Problem 6-9:

a) $\vec{v}_{cm} = \dfrac{m_1 \vec{u}}{(m_1 + m_2)}$

b) $\dfrac{\mathrm{d}v}{\mathrm{d}t} = -\dfrac{k}{(m_1 + m_2)} x$

c) $A = \sqrt{\dfrac{m_1^2 u^2}{k(m_1 + m_2)}}$

Problem 6-10:

a) $M + \lambda x$

b) $v = \dfrac{M v_0}{M + \lambda x}$

c) $T = \lambda v^2$

Problem 6-11:

b) $M = \dfrac{v_0}{v} M_0$

c) $v = \sqrt{\dfrac{v_0^2 M_0}{2\rho A v_0 t + M_0}}$

C.7 Torques and Angular Momentum

Solutions to Practice Problems from Chapter 7.9

Torques and Angular Momentum Solutions:

Problem 7-1:

a) $\frac{3}{2}MR^2$

b) $\frac{1}{3}M_RL^2 + \frac{1}{2}M_SR^2 + M_S(L+R)^2$

c) $\frac{1}{3}M_RL^2 + \frac{1}{2}M_P(\ell^2 + w^2) + M_P(L + \frac{1}{2}\ell)^2$

d) $\frac{1}{12}M_RL^2 + \frac{1}{2}M_C(R_1^2 + R_2^2) + M_C(\frac{1}{2}L + R_2)^2$

Problem 7-2:

5.6×10^{14} times faster

Problem 7-3:

a) $\vec{\tau} = rF$ [into the page]

b) $\vec{\tau} = rF\sin\theta$ [into the page]

c) $\vec{\tau} = Rmg$ [into the page]

d) $\vec{\tau} = R(m_1 - m_2)g$ [out of the page]

Problem 7-4:

a) $\vec{L}_i = \dfrac{\ell m v_0}{2}$ and $\vec{L}_f = \left[\frac{1}{12}M\ell^2 + \frac{1}{4}m\ell^2\right]\omega$

b) $\omega = \dfrac{6mv_0}{M\ell + 3m\ell}$

Problem 7-5:

$h = \dfrac{L}{2\sqrt{2}}$

Problem 7-6:

a) $I_0 = \frac{1}{2}MR^2 + ms^2$

b) $T = 2\pi\sqrt{\dfrac{R^2 + 2s^2}{2gs}}$

c) $s = \dfrac{R}{\sqrt{2}}$

Problem 7-7:

$\omega_0 = 4.1\ \mathrm{s}^{-1}$

Problem 7-8:

a) $h = \dfrac{3L + R}{4}$

b) $I = \frac{1}{3}ML^2 + \frac{2}{5}MR^2 + M(L + \frac{1}{2}R)^2$

c) $T = 2\pi\sqrt{\dfrac{\frac{2}{3}ML^2 + \frac{4}{5}MR^2 + 2M(L + \frac{1}{2}R)^2}{Mg(3L + R)}}$

Problem 7-9:

b) $\tau = RF_f = I\alpha$

c) $\ddot{x} = \frac{2}{3}g\sin\theta$

d) $\theta = \tan^{-1}(3\mu)$

C.8 Work and Energy

Solutions to Practice Problems from Chapter 8.10

Work and Energy Solutions:

Problem 8-1:

a) mgh

b) 1.7×10^5 J

c) 28.7 J

Problem 8-2:

a) $6.18 \times 10^5\ \mathrm{m\ s}^{-1}$

b) $4.2 \times 10^4\ \mathrm{m\ s}^{-1}$

c) $7.7 \times 10^3\ \mathrm{m\ s}^{-1}$

Problem 8-3:

a) no, b) no, c) yes, d) yes

Problem 8-4:

Only if $2b = c$

Problem 8-5:

Only if $c = -1$

Problem 8-6:

a) $\vec{F} = 2\hat{\imath} + 6y\hat{\jmath} + 8z\hat{k}$

b) $\vec{F} = 2xy^2\hat{\imath} + 2x^2y\hat{\jmath} + 3z^2\hat{k}$

Problem 8-7:

$x = \pm\sqrt{\frac{A}{B}}$

Problem 8-8:

$v = \sqrt{\frac{2}{m}(a+b+c)}$

Problem 8-9:

a) 0.85 J, b) 11.54 J, c) 12.39 J, d) 31.20 J, e) 32.89 m/s

Problem 8-10:

a) $W_1 = 1$ and $W_2 = -1$

b) $W_1 = 1$ and $W_2 = 0$

c) $\vec{F}_1$ gives the same work for two different paths. $\vec{F}_2$ gives different work for two different paths. $\vec{F}_1$ is a conservative force. $\vec{F}_2$ is not a conservative force.

d) F_1 is conservative. F_2 is not conservative.

C.9 Applications of Energy Conservation

Solutions to Practice Problems from Chapter 9.6

Applications of Energy Conservation Solutions:

Problem 9-1:

Position C

Problem 9-2:

$v = \sqrt{gh}$

Problem 9-3:

$U = \frac{1}{2}kA^2\cos^2\left(\sqrt{\frac{k}{m}}t\right)$

$K = \frac{1}{2}kA^2 \sin^2\left(\sqrt{\frac{k}{m}}t\right)$

Problem 9-4:

$N = 3m$

Problem 9-5:

$v = \sqrt{gL}$

Problem 9-6:

$v = \sqrt{2g(R-y)}$

Problem 9-7:

a) $h = \dfrac{1}{2}\dfrac{kd^2}{mg}$

b) The adult can only jump half as high as the child.

Problem 9-8:

$v = \sqrt{\frac{4}{3}gh}$

Problem 9-9:

a) $v_{cm} = \dfrac{\mathrm{d}x}{\mathrm{d}t}$

b) $\omega = \dfrac{v_{cm}}{R}$

c) $\dfrac{k_{rot}}{K_{trans}} = \dfrac{1}{2}$

d) $T = 2\pi\sqrt{\dfrac{3m}{2k}}$

C.10 Central Forces and Motion in Space

Solutions to Practice Problems from Chapter 10.9

Central Forces and Motion in Space Solutions:

Problem 10-1:

$f(r) = \left(\dfrac{3k_1}{r^4} + \dfrac{2k_2}{r^3}\right)\hat{r}$

Problem 10-2:

$$F_{eff} = \frac{L^2 + 2m}{mr^3}\hat{r}$$

Problem 10-3:

$U_{eff,max} = -\dfrac{m}{2L}$, stable

Problem 10-5:

$L = 2m\sqrt{k}R^3$, $v = 2\sqrt{k}R^2$

Problem 10-6:

$$F_{eff} = \frac{L^2}{mr^3} - \frac{\gamma}{r^2} - \frac{3\gamma\epsilon}{r^4}$$

Problem 10-2:

$$U_{eff} = \frac{1}{2}\frac{L^2}{mr^2} + -\frac{\gamma}{r} - \frac{1}{2}\epsilon r^2$$

Problem 10-3:

$$L = \sqrt{3A\beta mR^5 e^{\beta R^3}}$$

Problem 10-4:

a) For small values of r, $\frac{L^2}{mr^2} >> Ar^2$, so the centrifugal potential dominates U_{eff}. For large values of r, $\frac{L^2}{mr^2} << Ar^2$, so the central force potential dominates U_{eff}.

b) $r = \left(\dfrac{L^2}{2Am}\right)^{1/4}$

c) $E = \sqrt{\dfrac{2AL^2}{m}}$

Problem 10-5:

a) $\dot{r} = 2\sqrt{A}\dfrac{L}{mr^{3/2}}$

b) $E = \dfrac{L^2}{2m}\left[\dfrac{4A}{r^3} + \dfrac{1}{r^2}\right] + U(r)$

c) $U(r) = E - \dfrac{L^2}{2m}\left[\dfrac{4A}{r^3} + \dfrac{1}{r^2}\right]$

d) $F_r = -\dfrac{L^2}{m}\left(\dfrac{6A}{r^4} + \dfrac{1}{r^3}\right)$

Problem 10-6:

a) For small values of r, $\frac{1}{r^3} \gg \frac{1}{r^2}$. For large values of r, $\frac{1}{r^3} \ll \frac{1}{r^2}$.

b) $r = \dfrac{3Am}{L^2}$

c) $U_{eff}(r = r_0) = \dfrac{1}{54}\left(\dfrac{L^6}{A^2m^2}\right)$

d) Unstable

C.11 Orbits and Kepler's Laws

Solutions to Practice Problems from Chapter 11.7

Orbits and Kepler's Laws Solutions:

Problem 11-1:

a) 0.206, b) 0.078, c) 0.967, d) 0.041

Problem 11-2:

a) 0.24 yr, b) 4.6 yr, c) 75.5 yr, d) 289 yr

Problem 11-3:

$v_p = 1.52v_a$

Problem 11-4:

$a_A = 16$ au

Problem 11-5:

$a_f = 2.945 \times 10^5$ km

Problem 11-6:

$v = \sqrt{\frac{GM}{R_s}}$

Problem 11-7:

$M_{BH} = 1.2 \times 10^6 M_{sun}$

Problem 11-8:

$v_p = 2.3v_0$

Problem 11-9:

Height = 1140 km

Problem 11-10:

a) $r = \frac{p(1+\epsilon)}{1+\epsilon\cos\theta}$

b) $\cos\theta = \frac{p(1+\epsilon)-R}{R\epsilon}$

c) If $\epsilon = 0$, then the comet only crosses the earths orbit if $p = R$

d) If $\epsilon = 1$, then $\cos\theta = \frac{2p-R}{R}$. From symmetry, there are two places of crossing ($\pm\theta$).

C.12 The Lagrange Method

Solutions to Practice Problems from Chapter 12.8

The Lagrange Method Solutions:

Problem 12-1:

$\ddot{x} = \frac{k}{m}(x - x_0)$

Problem 12-2:

$\ddot{\theta} = \frac{3g}{4L}\theta$

Problem 12-3:

$\ddot{x} = g\sin\theta$

Problem 12-4:

$\mathcal{L} = \frac{3}{4}M\dot{y}^2 + Mgy$

Problem 12-5:

$\mathcal{L} = K – U = m\dot{y}^2 + mgy$

$\frac{\partial\mathcal{L}}{\partial y} = mg$

$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{y}} = \frac{\mathrm{d}}{\mathrm{d}t}(2m\dot{y}) = 2m\ddot{y}$

Problem 12-6:

a) $K = \frac{1}{2}m\dot{x}^2$

b) $U_{spring} = \frac{1}{2}kx^2$ and $U_{grav} = mgx\sin\theta$

c) $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - mgx\sin\theta$

d) $\dfrac{\partial \mathcal{L}}{\partial \theta} = -kx - mg\sin\theta$ and $\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \mathcal{L}}{\partial \dot{x}} = m\ddot{x}$

Problem 12-7:

$\ddot{x} = \dfrac{(M\sin\theta - m)g}{M+m}$

Problem 12-8:

a) $\mathcal{L} = \frac{3}{8}MR\dot{\theta}^2 - \frac{1}{4}MgR\theta^2$

b) $0 = \ddot{\theta} + \frac{2g}{3R}\theta$

Problem 12-9:

a) $U_{grav} = mg\frac{a}{4}(1-\cos 2\theta)$

b) $K = \frac{ma^2}{4}\dot{\theta}^2\left[1+\cos 2\theta\right]$

c) $\mathcal{L} = \frac{ma^2}{4}\dot{\theta}^2\left[1+\cos 2\theta\right] + mg\frac{a}{4}\cos 2\theta$

d) $0 = a\ddot{\theta}(1+\cos 2\theta) + \sin 2\theta(g - a\dot{\theta}^2)$

Problem 12-10:

a) The mass will swing from the motion of the pendulum as well as oscillate up and down along the axis of the spring due to the motion of the spring.

b) $U = \frac{1}{2}kx^2 - Mg(L+x)\cos\theta$

c) $K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}M(L+x)^2\dot{\theta}^2$

d) $M\ddot{x} = M(L+x)\dot{\theta}^2 - kx + Mg\cos\theta$ and $M(L+x)^2\ddot{\theta} + 2M(L+x)\dot{x}\dot{\theta} = -Mg(L+x)\sin\theta$